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Domain Branching in Uniaxial Ferromagnets: A Scaling Law for the Minimum Energy

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Abstract: We address the branching of magnetic domains in a uniaxial ferromagnet. Our thesis is that branching is required by energy minimization. To show this, we consider the nonlocal, nonconvex variational problem of micromagnetics. We identify the scaling law of the minimum energy by proving a rigorous lower bound which matches the already-known upper bound. We further show that any domain pattern achieving this scaling law must have average width of order $L^{2/3}$, where L is the length of the magnet in the easy direction. Finally we argue that branching is required, by considering the constrained variational problem in which branching is prohibited and the domain structure is invariant in the easy direction. Its scaling law is different.

1. Introduction

This paper is motivated by the phenomenon of domain branching in strongly uniaxial ferromagnets. We consider the micromagnetic energy of such a ferromagnet – a certain nonlocal, nonconvex variational problem. Our results concern the scaling of the minimum energy, and the geometry of domain structures which achieve this scaling law.

Physically, we consider a uniaxial ferromagnet below its Curie temperature. The spontaneous magnetization is represented by a unit vector field **m**. The material forms magnetic domains, where **m** varies smoothly, separated by domain walls where **m** is nearly discontinuous. Since the ferromagnet is uniaxial there is a single preferred direction for **m**, the “easy axis”; the plane perpendicular to the easy axis is known as the basal plane. The magnetic domains in such a material run roughly parallel to the easy axis, refining their length scale by branching near a boundary parallel to the basal plane. Figure 1 gives a two-dimensional cartoon of this phenomenon; in truth the branching is three-dimensional, more like the left half of Fig. 2. Our results indicate that such branching is required for a domain pattern to approach the minimum energy.

This paper is restricted to uniaxial materials. However, domain branching occurs in other settings as well – for example in iron, which is cubic, see e.g. [5] and [16]. There the conventional explanation involves magnetoelastic coupling. It would be interesting to examine that explanation from the perspective of this article.

Mathematically, we consider a slightly simplified version of micromagnetics. The main simplification is that we treat the domain walls as sharp interfaces. This is convenient because it permits us to focus on the morphology of domain structure, without having to resolve simultaneously the internal structure of the walls. See [1, 3, 5], and [14] for discussions of micromagnetics, and [2] and [4] for additional discussion of the sharp-interface reduction. A further simplification is the use of periodic boundary conditions, which lets us define negative Sobolev norms through Fourier series. Neither simplification is essential to our analysis (see Remarks 1.1 and 4.5).

The spatial domain for our micromagnetic variational problem is

$$(x, y, z) \in \Omega = (-L, L) \times Q,$$

where the x-line is the easy axis. The cross section Q is the unit square $Q = (0, 1] \times (0, 1]$, and we impose periodicity in y and z. A magnetization field must have unit magnitude on Ω and must vanish off Ω; thus the set $\mathcal{A}$ of admissible magnetizations is:

$$\mathcal{A} := \Bigg\{ \mathbf{m}(x, y, z) = (m_1, m_2, m_3) \;\Bigg|\; |\mathbf{m}| = 1 \quad \text{for } |x| < L, \\ \mathbf{m} = 0 \text{ for } |x| > L, \mathbf{m} \text{ is periodic in } (y, z) \Bigg\}.$$

The micromagnetic energy $E = E_a + E_f + E_s$ is the sum of three contributions:

- The anisotropy energy, which favors magnetizations parallel to the x-axis by penalizing (m_2, m_3):
$$E_a(m) = \alpha \int_\Omega m_2^2 + m_3^2 \; dx\, dy\, dz.$$
- The field energy, a nonlocal term which favors div$\mathbf{m} = 0$ in the sense of distributions (i.e. div $\mathbf{m} = 0$ in Ω and $m_1 = 0$ at $x = \pm L$):
$$E_f(\mathbf{m}) = \beta \int_{\mathbf{R} \times Q} |\nabla u|^2 \; dx\, dy\, dz, \quad \text{where } u \text{ is periodic and satisfies}$$
$$\int_{\mathbf{R} \times Q} \nabla u \cdot \nabla \zeta \; dx\, dy\, dz = \int_\Omega \mathbf{m} \cdot \nabla \zeta \; dx\, dy\, dz \quad \text{for all periodic } \zeta.$$
Notice that ∇u is the L^2-projection of $\mathbf{m}$ on the space of periodic gradient fields (Helmholtz projection).
- The interfacial energy, which prefers fewer domain walls:
$$E_s(m) = \varepsilon \int_\Omega |\nabla \mathbf{m}| \; dx\, dy\, dz.$$
We are abusing notation slightly: $\mathbf{m}$ is permitted to have discontinuities across surfaces, i.e. $\nabla \mathbf{m}$ is a measure, and the interfacial energy is its total variation in Ω. In other words, the energy of a domain wall is the surface integral of the jump in $\mathbf{m}$. In practice $\mathbf{m}$ jumps from about $(1, 0, 0)$ to about $(-1, 0, 0)$ across any domain wall, so E_s is about 2ε times the total area of the domain walls.

In summary, the variational problem we consider is

$$\min_{\mathbf{m}\in\mathcal{A}} \alpha \int_\Omega m_2^2 + m_3^2 \; dx\, dy\, dz + \varepsilon \int_\Omega |\nabla \mathbf{m}| \; dx\, dy\, dz + \beta \int_{\mathbf{R}\times Q} |\nabla u|^2 \; dx\, dy\, dz, \tag{1.1}$$

where ∇u is the Helmholtz projection of **m**.

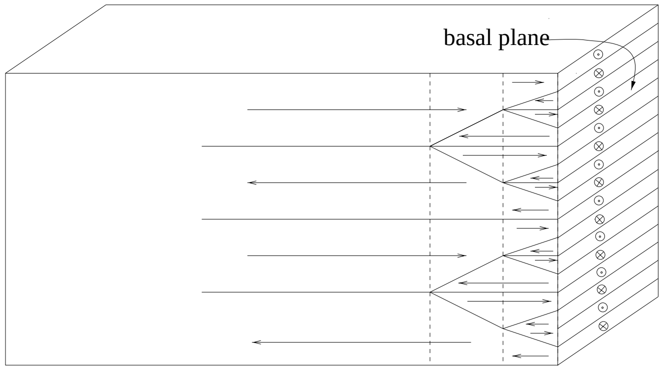

Fig. 1. Schematic of a two-dimensional domain pattern achieving the optimal scaling law

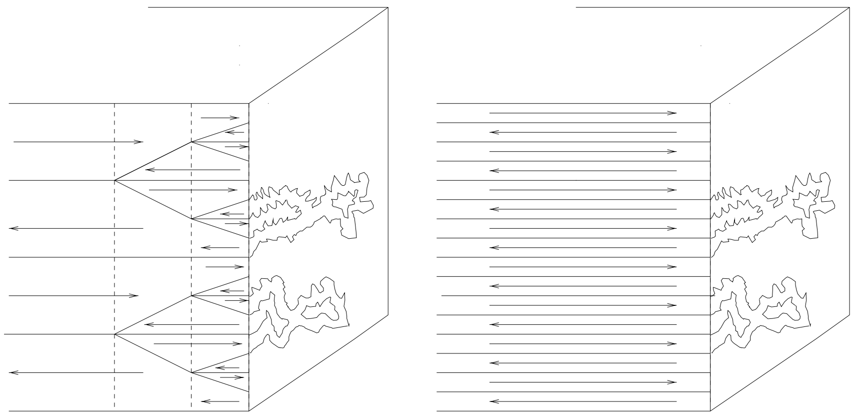

Fig. 2. *Left*. Schematic of the domain patterns actually seen in uniaxial ferromagnets. They branch three-dimensionally rather than as shown in Fig. 1. *Right*. A domain pattern with **m** independent of x. Regardless of its complexity, such a pattern has far from minimum energy, by (1.7)

Here is the intuition why low-energy magnetization fields display spatial patterns. The penalization of (m_2, m_3) by E_a, taken in conjunction with the constraint $|\mathbf{m}| = 1$, favors magnetizations of $(1, 0, 0)$ and $(-1, 0, 0)$. On the other hand, E_f favors $m_1 = 0$ at the basal planes $x = \pm L$. The penalization of m_1 at the basal planes is "soft", i.e. the field energy can be made small by having m_1 oscillate with mean value zero; on the other hand the penalization of (m_2, m_3) is "hard", more like the constraint $|\mathbf{m}| = 1$. Therefore the combined effect of E_a and E_f is to force **m** to oscillate rapidly in the (y, z) plane between $(1, 0, 0)$ and $(-1, 0, 0)$. The minimum value of $E_a + E_f$ alone is zero, but minimizing sequences do not converge strongly and the minimum is not attained. The inclusion of E_s penalizes these oscillations and restores existence of minimizers. The competition between $E_a + E_f$ and E_s should select the length scale of the oscillation, and the spatial patterns of minimizers. See e.g. [4, 10, 11, 12, 13] for discussions of pattern formation in closely related settings.

Examining this argument more closely, it is easy to understand why magnetic domains should branch. The incentive to oscillate in the (y, z) plane between $(1, 0, 0)$ and $(-1, 0, 0)$ comes from the penalization of m_1 near the basal planes $x = \pm L$. Moving away from these planes, the regularizing effect of E_s is increasingly dominant, leading to coarser domains in the interior of the sample – hence the domain branching.

Our rigorous results reveal a quantitative agreement between an experimentally observed scaling law and the predictions of micromagnetics. The experiments show that the typical domain size ℓ away from the basal planes scales as

$$\ell \sim L^{\frac{2}{3}} \tag{1.2}$$

when the thickness of the magnet L is sufficiently large [6, 7]. Our analysis considers the micromagnetic model in the parameter regime when

$$\frac{\varepsilon}{\gamma L} \ll 1 \tag{1.3}$$

and

$$\frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} \ll 1, \tag{1.4}$$

where $\gamma = \min\{\alpha, \beta\}$ (we will discuss these conditions shortly). One of our results can be interpreted as follows (see Remark 4.2): in the regime (1.3)–(1.4), the typical domain size ℓ^* of any low-energy magnetization $\mathbf{m}^*$ behaves as

$$\ell^* \sim \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}}, \tag{1.5}$$

in agreement with (1.2).

By a “low-energy magnetization” we mean any $\mathbf{m}^*$ whose energy is within a specified factor of the absolute minimum energy. We do not require that $\mathbf{m}^*$ achieve the absolute minimum, nor even a local minimum. This notion is useful because we know the minimum energy within a constant factor. In fact, we shall prove that in the parameter regime (1.3)–(1.4), the minimum energy E_0 scales as

$$E_0 \sim \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}} \tag{1.6}$$

(Theorem 2.1).

Our analysis also suggests that for any low-energy $\mathbf{m}^*$, the typical domain size must decrease near the basal planes. What we actually show is this: when the admissible magnetizations $\mathbf{m}(x, y, z)$ are restricted to be independent of $x \in (-L, L)$, the revised minimum energy E_1 scales differently:

$$E_1 \sim \gamma^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} L^{\frac{1}{2}} \tag{1.7}$$

(Theorem 2.2). Notice that E_1 is much larger than E_0, as a consequence of (1.3).

Now a word about the meaning of our parameter restrictions (1.3), (1.4). Condition (1.3) assures that a branched domain pattern achieves lower energy than an unbranched one; put differently, it is the condition that (1.6) be smaller than (1.7). When ε and γ are fixed, (1.3) requires that L be sufficiently large, and indeed branching is only observed experimentally when L is large enough. Turning to the other restriction: (1.4) assures that our (artificial) periodic boundary conditions do not interfere with the branching; put differently, it is the condition that length scale ℓ^* defined by (1.5) fits in the period cell Q. Condition (1.3) is dimensionless but (1.4) is dimensional. If Q were replaced by a $w \times w$ unit cell, (1.3) would not change but (1.4) would become

$$\frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} \ll w,$$

cf. Remark 4.3.

We offer some remarks concerning the nature of our analysis. It seems hopeless to solve the nonconvex variational problem of micromagnetics explicitly. It also seems difficult to use the Euler–Lagrange equation, which should anyway have many solutions, most of them unstable. Instead, we focus on *estimating the minimum energy*. Thus our main results concern how the minimum energy scales with the parameters of the problem. The importance of lower bounds with optimal scaling was first noticed in a related but different setting by Kohn and Müller [10, 11, 12].

The upper bounds implicit in the scalings (1.6) and (1.7) are already known. They are consequences of various constructions (domain patterns) – which, in the case of (1.6), mimic the observed phenomenon of domain branching, see e.g. [4, 6, 8, 9, 15, 16], and [17].

The lower bound implicit in (1.6) is one of our main results. It rules out the possibility that some as-yet-undiscovered domain pattern could achieve a better scaling law. And it suggests that energy minimizers should resemble the domain patterns used to prove the upper bounds, since their energies are similar. The lower bound implicit in (1.7) is another of our main results. It indicates that branching is required to achieve the optimal scaling. More precisely: when $\varepsilon \ll \gamma L$, a domain structure which is independent of x is *far* from being a minimizer, regardless of its complexity. The estimate (1.5) concerning the average domain length scale is a third main result. It results from a sort of "equipartition" of the energy into two separate parts, $E_a + E_f$ and E_s, each scaling the same way, for any magnetization with nearly minimum energy (Proposition 4.1).

The arguments in this paper make no use of any Euler-Lagrange equation or first variation. This has the advantage of robustness: our results apply to *any* magnetization $\mathbf{m}$ whose energy $E(\mathbf{m})$ scales like the minimal energy E_0 in the parameters ε, γ, L – regardless of how it might be reached, and regardless of whether it minimizes the energy locally or globally. However the use of energy-based arguments also has some disadvantages. For example, such methods cannot give pointwise conclusions, since changing $\mathbf{m}$ on a set of small measure (and small perimeter) has little effect on $E(\mathbf{m})$. Thus, we are unable to estimate the length scale of domains in the basal planes $x = \pm L$.

A two-dimensional reduction of our problem was studied by Choksi and Kohn in [4]. That work considered only magnetizations $\mathbf{m}(x, y, z)$ which were independent of z. It proved the lower bound (1.6) and a "matching" upper bound, drawing for the latter from ideas of Hubert [6] and Privorotskii [17]. The three-dimensional lower bound proved here is more difficult than its two-dimensional reduction, because the class of admissible patterns is much larger. In fact, observed domain patterns in uniaxial materials such as cobalt and magnetoplumbite are fully three-dimensional, resembling the left half of Fig. 2 (see for example [3, 5], or [7]). One might imagine that such three-dimensional structures achieve a better scaling law. Our analysis shows they do not. This does not contradict the experimental observations. The three-dimensional structures may be local minima, or they may achieve a better value of the constant in front of $\gamma^{1/3}\varepsilon^{2/3}L^{1/3}$.

Our approach to the lower bound is quite different from the one in [4], and in our opinion more natural. The argument of [4] made extensive use of stream functions. Here we use instead a special interpolation inequality, Lemma 2.3, involving the BV norm, the L^∞ norm, and a certain negative norm. The BV norm controls E_s, while the negative norm is related to E_f. It seems entirely natural that the competition between $E_a + E_f$ and E_s should be captured by a scale-invariant interpolation inequality.

There is an analogy between the structure of magnetic domains in a uniaxial ferromagnet and the structure of normal and superconducting flux domains in the intermediate state of a type-I superconductor. We shall explain elsewhere how our results apply in that setting.

Remark 1.1. This paper uses a sharp-interface reduction of micromagnetics. Without this simplification our energy (1.1) would be replaced by the standard micromagnetic energy

$$\alpha \int_\Omega m_2^2 + m_3^2 \; dx\,dy\,dz + \delta\beta \int_\Omega |\nabla \mathbf{m}|^2 \; dx\,dy\,dz + \beta \int_{\mathbf{R}^3} |\nabla u|^2 \; dx\,dy\,dz, \quad (1.8)$$

see e.g. [4]. Bounds of the type presented here can also be proved for (1.8); the role of the surface tension ε is played by $\sqrt{\alpha\beta\delta}$. The argument requires some additional work and will be presented elsewhere. The fact that (1.1) and (1.8) have similar scaling laws provides additional support for the sharp–interface reduction which links the two problems.

2. Bounds on the Minimum Energy

Let E_0 denote the minimum energy among all admissible magnetizations $m(x, y, z)$ and E_1 the minimum energy among all admissible magnetizations $m(x, y, z)$ which are independent of x for $|x| < L$. Let $\gamma := \min\{\alpha, \beta\}$. Our main results are the following two theorems.

Theorem 2.1. *There exist universal constants* $0 < c_0 < C_0 < \infty$ *such that if*

$$\frac{\varepsilon}{\gamma L} < 1 \quad \textit{and} \quad \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} < 1, \quad (2.1)$$

then

$$c_0 \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}} \le E_0 \le C_0 \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}}.$$

Theorem 2.2. *There exist universal constants* $0 < c_0' < C_0' < \infty$ *such if*

$$\frac{\varepsilon}{\gamma L} < 1 \quad \textit{and} \quad \frac{\varepsilon^{\frac{1}{2}} L^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}} < 1, \quad (2.2)$$

then

$$c_0' \gamma^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} L^{\frac{1}{2}} \le E_1 \le C_0' \gamma^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} L^{\frac{1}{2}}.$$

Theorem 2.1 identifies the scaling law of the minimum energy. Theorem 2.2 shows that if the domain structure is restricted to be independent of x then the scaling law is different. We view this as an indication that refinement is required for energy minimization. Notice that the hypothesis of Theorem 2.2 is slightly less restrictive than that of Theorem 2.1, since for $\varepsilon/\gamma L < 1$, (2.1) implies (2.2).

The proofs of Theorems 2.1 and 2.2 rely on the following interpolation inequalities for periodic functions on $\mathbf{R}^2$ with unit cell $Q = (0,1] \times (0,1]$. Roughly speaking, the first inequality interpolates between BV, L^∞ and H^{-1}, the second between BV, L^∞ and $H^{-1/2}$.

Lemma 2.3. *Let $f \in BV(Q) \cap L^\infty(Q)$. Then there exists a constant c_1 such that for all positive integers N,*

$$a) \int_Q |f|^2 \, dx\, dy \le c_1 \left\{ \frac{1}{N} \sup_Q |f| \int_Q |\nabla f| \, dx\, dy + \sum_{\mathbf{n}\in\mathbf{Z}^2} \min\left\{1, \frac{N^2}{|\mathbf{n}|^2}\right\} |f_\mathbf{n}|^2 \right\}, \tag{2.3}$$

and

$$b) \int_Q |f|^2 \, dx\, dy \le c_1 \left\{ \frac{1}{N} \sup_Q |f| \int_Q |\nabla f| \, dx\, dy + \sum_{\mathbf{n}\in\mathbf{Z}^2} \min\left\{1, \frac{N}{|\mathbf{n}|}\right\} |f_\mathbf{n}|^2 \right\}, \tag{2.4}$$

where for $\mathbf{n} \in \mathbf{Z}^2$, $f_\mathbf{n}$ denotes the $\mathbf{n}^{\text{th}}$ Fourier coefficient of f.

The role of these inequalities is roughly as follows. Our goal is to show that the total energy is not too small. If the total surface energy is small, then the surface energy of a generic cross-section should also be small. The interpolation inequalities will let us conclude that a negative norm of the cross-sectional domain structure must be large. To conclude the argument we must bound these negative cross-sectional norms in terms of the energy. The argument uses a second lemma:

Lemma 2.4. *Let $\lambda \in \mathbf{R}$ and $g \in L^2(\mathbf{R},\mathbf{R})$ such that $g(x) = 0$ for a.e. x with $|x| > L$. Then there exists a constant $c_2 > 0$ such that*

$$\int_{-\infty}^{\infty} \frac{\xi^2}{\xi^2 + \lambda^2} |\widehat{g}(\xi)|^2 d\xi \ge \frac{c_2}{1 + (\lambda L)^2} \int_{-L}^{L} |g(x)|^2 \, dx.$$

Lemmas 2.3 and 2.4 will be proved in Sect. 3. We now apply them to demonstrate the theorems.

Proof of Theorem 2.1. We can dispense with the upper bound quickly: it is a consequence of the two-dimensional constructions presented in [4], following [6] and [17]. The paper [4] discusses two different domain structures: one works in the limit of high anisotropy, the other works in the limit of low anisotropy. Each magnetization can be viewed as a three-dimensional test function $\mathbf{m}(x,y,z)$ which happens to be independent of z. Propositions 2.1 and 3.1 of [4] persist to the three-dimensional setting and give the desired upper bound $E_0 \le C_0 \gamma^{1/3} \varepsilon^{2/3} L^{1/3}$.

We turn now to the lower bound. It suffices to consider the case when α and β are both replaced with $\gamma := \min\{\alpha, \beta\}$; in effect we shall show that

$$\begin{aligned} \tilde{E}(\mathbf{m}) = \gamma \int_\Omega m_2^2 + m_3^2 \, dx\, dy\, dz + \varepsilon \int_\Omega |\nabla \mathbf{m}| \, dx\, dy\, dz \\ + \frac{\gamma}{2} \int_{\mathbf{R}\times Q} |\nabla u|^2 \, dx\, dy\, dz \ge c_0 \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}} \end{aligned} \tag{2.5}$$

whenever ε, γ, and L satisfy (2.1). (It is more convenient for the following to have $\gamma/2$ instead of γ as the coefficient of the nonlocal term; of course this is immaterial to our claim).

We use $\widehat{\mathbf{m}}$ to denote both the discrete Fourier series in $(y,z)\in Q$,

$$\widehat{\mathbf{m}}(x,\mathbf{n}) = \int_Q e^{-2\pi i(yn_1+zn_2)}\mathbf{m}(x,y,z)\,dy\,dz,$$

and the discrete Fourier series in $(y,z)\in Q$ combined with the Fourier transform in $x\in\mathbf{R}$,

$$\widehat{\mathbf{m}}(\xi,\mathbf{n}) = \int_{-\infty}^{\infty} e^{-2\pi i\xi x}\mathbf{m}(x,\mathbf{n})\,dx,$$

where $\mathbf{n}\in\mathbf{Z}^2$ and $\xi\in\mathbf{R}$. We make use of the latter to re-express the sum of anisotropy plus demagnetization energy:

$$\begin{aligned}
&\gamma\int_\Omega m_2^2+m_3^2\,dx\,dy\,dz+\frac{\gamma}{2}\int_{\mathbf{R}\times Q}|\nabla u|^2\,dx\,dy\,dz\\
&=\gamma\sum_{\mathbf{n}}\int_{-\infty}^{\infty}|\widehat{m}_2(\xi,\mathbf{n})|^2+|\widehat{m}_3(\xi,\mathbf{n})|^2d\xi+\frac{\gamma}{2}\sum_{\mathbf{n}}\int_{-\infty}^{\infty}\frac{|(\xi,n_1,n_2)\cdot\widehat{\mathbf{m}}(\xi,\mathbf{n})|^2}{\xi^2+|\mathbf{n}|^2}d\xi.
\end{aligned}$$

Using

$$\begin{aligned}
|(\xi,n_1,n_2)\cdot\widehat{\mathbf{m}}(\xi,\mathbf{n})|^2 &\geq \frac{1}{2}\left|\xi\widehat{m}_1(\xi,\mathbf{n})\right|^2-\left|\mathbf{n}\cdot(\widehat{m}_2(\xi,\mathbf{n}),\widehat{m}_3(\xi,\mathbf{n}))\right|^2\\
&\geq \frac{1}{2}\xi^2|\widehat{m}_1(\xi,\mathbf{n})|^2-|\mathbf{n}|^2\left(|\widehat{m}_2(\xi,\mathbf{n})|^2+|\widehat{m}_3(\xi,\mathbf{n})|^2\right),
\end{aligned}$$

we obtain the following estimate:

$$\begin{aligned}
&\gamma\int_\Omega m_2^2+m_3^2\,dx\,dy\,dz+\frac{\gamma}{2}\int_{\mathbf{R}\times Q}|\nabla u|^2\,dx\,dy\,dz\\
&\geq\frac{\gamma}{2}\sum_{\mathbf{n}}\int_{-\infty}^{\infty}|\widehat{m}_2(\xi,\mathbf{n})|^2+|\widehat{m}_3(\xi,\mathbf{n})|^2d\xi+\frac{\gamma}{4}\sum_{\mathbf{n}}\int_{-\infty}^{\infty}\frac{\xi^2}{\xi^2+|\mathbf{n}|^2}|\widehat{m}_1(\xi,\mathbf{n})|^2d\xi.
\end{aligned}$$

From Lemma 2.4 with $\lambda = |\mathbf{n}|$ and $g(x) = \widehat{m}_1(x,\mathbf{n})$, we infer that

$$\begin{aligned}
&\gamma\int_\Omega m_2^2+m_3^2\,dx\,dy\,dz+\frac{\gamma}{2}\int_{\mathbf{R}\times Q}|\nabla u|^2\,dx\,dy\,dz\\
&\geq\frac{\gamma}{2}\int_\Omega m_2^2+m_3^2\,dx\,dy\,dz+\gamma\sum_{\mathbf{n}}\frac{c}{1+(|\mathbf{n}|L)^2}\int_{-L}^{L}|\widehat{m}_1(x,\mathbf{n})|^2\,dx.
\end{aligned}$$

Here and in the following, c denotes a positive, universal constant which may change from line to line. Combining the last term on the right hand side with the total variation in y,z of $m_1(x)$, and using Lemma 2.3 a), we obtain

$$
\begin{aligned}
&\int_{-L}^{L} \left\{ \varepsilon \int_Q |\nabla m_1(x)|\, dy\, dz + \gamma \sum_{\mathbf{n}} \frac{c}{1+(|\mathbf{n}|L)^2} |\widehat{m}_1(x,\mathbf{n})|^2 \right\} dx \\
\geq\ & \int_{-L}^{L} \frac{\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}}{L^{\frac{2}{3}}} \left\{ \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} \int_Q |\nabla m_1(x)|\, dy\, dz \right. \\
&\qquad \left. + c \frac{\gamma^{\frac{2}{3}} L^{\frac{2}{3}}}{\varepsilon^{\frac{2}{3}}} \sum_{\mathbf{n}} \min\left\{1, \frac{1}{(|\mathbf{n}|L)^2}\right\} |\widehat{m}_1(x,\mathbf{n})|^2 \right\} dx \qquad (2.6) \\
\geq\ & \int_{-L}^{L} \frac{\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}}{L^{\frac{2}{3}}} \left\{ \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} \int_Q |\nabla m_1(x)|\, dy\, dz \right. \\
&\qquad \left. + c \sum_{\mathbf{n}} \min\left\{1, \frac{\gamma^{\frac{2}{3}}}{\varepsilon^{\frac{2}{3}} L^{\frac{4}{3}}} \frac{1}{|\mathbf{n}|^2}\right\} |\widehat{m}_1(x,\mathbf{n})|^2 \right\} dx \qquad (2.7) \\
\geq\ & \int_{-L}^{L} \left\{ \frac{c\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}}{L^{\frac{2}{3}}} \int_Q m_1^2\, dy\, dz \right\} dx. \qquad (2.8)
\end{aligned}
$$

From (2.6) to (2.7) , we used $L > \varepsilon/\gamma$, the first part of our hypothesis (2.1). We passed from (2.7) to line (2.8) by applying Lemma 2.3, with the choice $N \sim \gamma^{\frac{1}{3}}\varepsilon^{-\frac{1}{3}}L^{-\frac{2}{3}}$, which is possible because of the second part of our hypothesis (2.1). Collecting our results, we see that

$$
\tilde{E}(\mathbf{m}) \geq \frac{c\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}}{L^{\frac{2}{3}}} \int_\Omega m_1^2\, dx\, dy\, dz + \frac{\gamma}{2} \int_\Omega m_2^2 + m_3^2\, dx\, dy\, dz,
$$

using the fact that $\int_Q |\nabla m_1|\, dy\, dz \leq \int_Q |\nabla \mathbf{m}|\, dy\, dz$. We may suppose that $c < \frac{1}{2}$, so that our hypotheses $\varepsilon/\gamma L < 1$ implies

$$
\frac{\gamma}{2} > \frac{c\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}}{L^{\frac{2}{3}}},
$$

so that

$$
\tilde{E}(\mathbf{m}) \geq \frac{c\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}}{L^{\frac{2}{3}}} \int_\Omega m_1^2 + m_2^2 + m_3^2\, dx\, dy\, dz = c\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}L^{\frac{1}{3}}. \qquad \square
$$

Proof of Theorem 2.2. The upper bound follows directly from the classical structures of Kittel and Landau-Lifshitz, see [4] for details. For the lower bound, we proceed as for Theorem 2.1. We assume throughout that $\mathbf{m} = \mathbf{m}(y,z)$ is independent of x for $|x| < L$ (and vanishes for $|x| > L$).

From the proof of Theorem 2.1, we have for the sum of anisotropy and demagnetization energy

$$
\begin{aligned}
&\gamma \int_\Omega m_2^2 + m_3^2\, dx\, dy\, dz + \frac{\gamma}{2} \int_{\mathbf{R}\times Q} |\nabla u|^2\, dx\, dy\, dz \\
\geq\ & \frac{\gamma}{2} \sum_{\mathbf{n}} \int_{-\infty}^{\infty} |\widehat{m}_2(\xi,\mathbf{n})|^2 + |\widehat{m}_3(\xi,\mathbf{n})|^2 d\xi + \frac{\gamma}{4} \sum_{\mathbf{n}} \int_{-\infty}^{\infty} \frac{\xi^2}{\xi^2 + |\mathbf{n}|^2} |\widehat{m}_1(\xi,\mathbf{n})|^2 d\xi.
\end{aligned}
$$

But this time,

$$\widehat{m}_1(\xi, \mathbf{n}) = \frac{\sin(2\pi\xi L)}{\pi\xi}\widehat{m}_1(\mathbf{n}),$$

where $\widehat{m}_1(\mathbf{n})$ denotes the Fourier series of the restriction of m_1 to $|x| < L$, viewed as a function of the two variables (y, z). Hence

$$\gamma \int_\Omega m_2^2 + m_3^2 \, dx\, dy\, dz + \frac{\gamma}{2} \int_{\mathbf{R}\times Q} |\nabla u|^2 \, dx\, dy\, dz$$
$$\geq \frac{\gamma}{2} \sum_{\mathbf{n}} \int_{-\infty}^{\infty} |\widehat{m}_2(\xi, \mathbf{n})|^2 + |\widehat{m}_3(\xi, \mathbf{n})|^2 d\xi + \frac{\gamma}{4\pi^2} \sum_{\mathbf{n}} \int_{-\infty}^{\infty} \frac{\sin(2\pi\xi L)^2}{\xi^2 + |\mathbf{n}|^2} d\xi |\widehat{m}_1(\mathbf{n})|^2.$$

The role of Lemma 2.4 is now played by the following estimate:

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin^2(2\pi\xi L)}{\xi^2 + |\mathbf{n}|^2} d\xi = \frac{L}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin^2(2\pi\xi)}{\xi^2 + |L\mathbf{n}|^2} d\xi \geq cL \min\left\{1, \frac{1}{L|\mathbf{n}|}\right\}.$$

We obtain

$$\gamma \int_\Omega m_2^2 + m_3^2 \, dx\, dy\, dz + \frac{\gamma}{2} \int_{\mathbf{R}\times Q} |\nabla u|^2 \, dx\, dy\, dz$$
$$\geq \gamma L \int_Q m_2^2 + m_3^2 \, dy\, dz + c\gamma L \sum_{\mathbf{n}} \min\left\{1, \frac{1}{L|\mathbf{n}|}\right\} |\widehat{m}_1(\mathbf{n})|^2.$$

As before, we combine the last term on the right-hand side with the m_1-part of the surface energy and use Lemma 2.3, but now part (b):

$$L\left\{\varepsilon \int_Q |\nabla m_1(x)| \, dy\, dz + c\gamma \sum_{\mathbf{n}} \min\left\{1, \frac{1}{L|\mathbf{n}|}\right\} |\widehat{m}_1(\mathbf{n})|^2\right\}$$
$$\geq \gamma^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}L^{\frac{1}{2}} \left\{\frac{\varepsilon^{\frac{1}{2}}L^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}} \int_Q |\nabla m_1| \, dy\, dz + \frac{c\gamma^{\frac{1}{2}}L^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}} \sum_{\mathbf{n}} \min\left\{1, \frac{1}{L|\mathbf{n}|}\right\} |\widehat{m}_1(\mathbf{n})|^2\right\} \quad (2.9)$$
$$\geq \gamma^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}L^{\frac{1}{2}} \left\{\frac{\varepsilon^{\frac{1}{2}}L^{\frac{1}{2}}}{\gamma^{\frac{1}{2}}} \int_Q |\nabla m_1| \, dy\, dz + c \sum_{\mathbf{n}} \min\left\{1, \frac{\gamma^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}L^{\frac{1}{2}}} \frac{1}{|\mathbf{n}|}\right\} |\widehat{m}_1(\mathbf{n})|^2\right\} \quad (2.10)$$
$$\geq c\gamma^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}L^{\frac{1}{2}} \int_Q m_1^2 \, dy\, dz. \quad (2.11)$$

From (2.9) to (2.10), we used the first part of our hypothesis (2.2), $L > \epsilon/\gamma$. We passed from (2.10) to (2.11) by applying Lemma 2.3, with the choice $N \sim \gamma^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}}L^{-\frac{1}{2}}$, which is possible because of the second part of our hypothesis (2.2). Collecting terms, we see that

$$\tilde{E}(\mathbf{m}) \geq c\gamma^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}L^{\frac{1}{2}} \int_Q m_1^2 \, dy\, dz + \gamma L \int_Q m_2^2 + m_3^2 \, dy\, dz.$$

From this point on the argument is identical to that of Theorem 2.1. □

3. Proofs of the Lemmas

This section gives elementary and self-contained proofs of the crucial inequalities stated as Lemmas 2.3 and 2.4.

Lemma 2.3 interpolates, loosely speaking, between the BV norm $\int |\nabla u|$, the L^∞ norm, and the negative Sobolev norms $\|u\|_{H^{-1}}$ or $\|u\|_{H^{-1/2}}$. It is natural to think there should be an alternative, more functional analytic argument using interpolation theory. We do not give such an argument, but we describe in Remark 3.1 how interpolation theory leads to closely related results.

Proof of Lemma 2.3. Fix an integer $N > 0$. We shall construct a linear operator $T_N : L^2(Q) \to L^2(Q)$ satisfying

$$\int_Q |f - T_N f|^2 \, dx \, dy \leq C_1 \frac{1}{N} \sup_Q |f| \int_Q |\nabla f| \, dx \, dy, \tag{3.1}$$

$$\int_Q |T_N f|^2 \leq C_2 \sum_{\mathbf{n} \in \mathbf{Z}^2} \min \left\{ 1, \frac{N^2}{|\mathbf{n}|^2} \right\} |f_{\mathbf{n}}|^2, \tag{3.2}$$

and

$$\int_Q |T_N f|^2 \leq C_2 \sum_{\mathbf{n} \in \mathbf{Z}^2} \min \left\{ 1, \frac{N}{|\mathbf{n}|} \right\} |f_{\mathbf{n}}|^2 \tag{3.3}$$

for constants C_1, C_2 independent of N. To this end, fix $\eta \in C_0^\infty(Q)$ such that $\int_Q \eta = 1$. The constants C_i will depend only on the choice of η. We define a partition of Q into N^2 subsquares $Q_{\mathbf{m}}$ where $\mathbf{m} \in \{0, \dots, N-1\}^2$, namely

$$Q_{\mathbf{m}} := \frac{1}{N}(\mathbf{m} + Q).$$

Next we define $\eta_{\mathbf{m}}$ with support in $Q_{\mathbf{m}}$ by

$$\frac{1}{N^2} \eta_{\mathbf{m}} \left(\frac{1}{N}(\mathbf{m} + \mathbf{x}) \right) = \eta(\mathbf{x}).$$

Finally we define the operator T_N by

$$T_N f := \int_{Q_{\mathbf{m}}} f \eta_{\mathbf{m}} d\mathbf{x} \text{ on each square } Q_{\mathbf{m}}.$$

To prove (3.1), it suffices to prove the inequality

$$\int_{Q_{\mathbf{m}}} \left| f - \int_Q f \eta_{\mathbf{m}} \, dx \, dy \right|^2 dx \, dy \leq C_1 \frac{1}{N} \sup_{Q_{\mathbf{m}}} |f| \int_{Q_{\mathbf{m}}} |\nabla f| \, dx \, dy,$$

for all $\mathbf{m}$ and then sum over the squares $Q_{\mathbf{m}}$. The above inequality is simply a rescaled version of

$$\int_Q \left| f - \int_Q f \eta \, dx \, dy \right|^2 dx \, dy \leq C_1 \sup_Q |f| \int_Q |\nabla f| \, dx \, dy.$$

This estimate follows from the Poincaré-type inequality

$$\int_Q \left| f - \int_Q f \eta \, dx \, dy \right| dx \, dy \leq C \int_Q |\nabla f| \, dx \, dy. \tag{3.4}$$

Proving (3.4) is equivalent to proving

$$\int_Q |f|\,dx\,dy \le C\int_Q |\nabla f|\,dx\,dy \text{ for all } f\in L^2(Q) \text{ s.t. } \int_Q f\eta = 0,$$

which easily follows from the compactness of the embedding of BV into L^1.

We now prove (3.2) and (3.3). These inequalities follow respectively from

$$\sum_{\mathbf{n}\in\mathbf{Z}^2} \max\left\{1, \frac{|\mathbf{n}|^2}{N^2}\right\} |(T_N^* f)_{\mathbf{n}}|^2 \le C\int_Q |f|^2, \tag{3.5}$$

and

$$\sum_{\mathbf{n}\in\mathbf{Z}^2} \max\left\{1, \frac{|\mathbf{n}|}{N}\right\} |(T_N^* f)_{\mathbf{n}}|^2 \le C\int_Q |f|^2, \tag{3.6}$$

where T_N^* denotes the adjoint of T_N, which is easily seen to be

$$T_N^* f = \sum_{\mathbf{m}} \left(\int_{Q_{\mathbf{m}}} f\,d\mathbf{x}\right)\eta_{\mathbf{m}}.$$

To see that (3.5) implies (3.2), we argue as follows (where the bar denotes complex conjugation):

$$\begin{aligned}\left(\sum_{\mathbf{n}\in\mathbf{Z}^2} \min\left\{1, \frac{N^2}{|\mathbf{n}|^2}\right\} |f_{\mathbf{n}}|^2\right)^{1/2} &= \sup_{\{\phi_{\mathbf{n}}\}\in l^2} \frac{\sum_{\mathbf{n}\in\mathbf{Z}^2} \min\left\{1, \frac{N}{|\mathbf{n}|}\right\} f_{\mathbf{n}}\overline{\phi_{\mathbf{n}}}}{\left(\sum_{\mathbf{n}\in\mathbf{Z}^2} |\phi_{\mathbf{n}}|^2\right)^{1/2}} \\ &\ge \sup_{\zeta\in L^2(Q)} \frac{\sum_{\mathbf{n}\in\mathbf{Z}^2} f_{\mathbf{n}}\overline{(T_N^*\zeta)_{\mathbf{n}}}}{\left(\sum_{\mathbf{n}\in\mathbf{Z}^2} \max\left\{1, \frac{|\mathbf{n}|^2}{N^2}\right\} |(T_N^*\zeta)_{\mathbf{n}}|^2\right)^{1/2}} \\ &\overset{(3.5)}{\ge} C \sup_{\zeta\in L^2(Q)} \frac{\int_Q (T_N f)\overline{\zeta}}{\left(\int_Q |\zeta|^2\right)^{1/2}} \\ &= C\left(\int_Q |T_N f|^2\right)^{1/2}.\end{aligned}$$

The argument that (3.6) implies (3.3) is similar.

It remains to prove (3.5) and (3.6). Concerning the former, we have

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbf{Z}^2}\max\left\{1,\frac{|\mathbf{n}|^2}{N^2}\right\}|(T_N^*f)_{\mathbf{n}}|^2 &\le \sum_{\mathbf{n}\in\mathbf{Z}^2}\frac{|\mathbf{n}|^2}{N^2}|(T_N^*f)_{\mathbf{n}}|^2+\sum_{\mathbf{n}\in\mathbf{Z}^2}|(T_N^*f)_{\mathbf{n}}|^2 \\ &= \frac{1}{4\pi^2N^2}\int_Q|\nabla(T_N^*f)|^2d\mathbf{x}+\int_Q|T_N^*f|^2d\mathbf{x} \\ &= \sum_{\mathbf{m}}\left(\int_{Q_{\mathbf{m}}}f\right)^2\int_{Q_{\mathbf{m}}}\left(\frac{1}{4\pi^2N^2}|\nabla\eta_{\mathbf{m}}|^2+\eta_{\mathbf{m}}^2\right)d\mathbf{x} \\ &= \sum_{\mathbf{m}}\left(\int_{Q_{\mathbf{m}}}f\right)^2N^2\int_Q\left(\frac{1}{4\pi^2}|\nabla\eta|^2+\eta^2\right)d\mathbf{x} \\ &\le C\int_Q\left(\frac{1}{4\pi^2}|\nabla\eta|^2+\eta^2\right)d\mathbf{x}\cdot\int_Q|f|^2d\mathbf{x}.\end{aligned} \tag{3.7}$$

Turning to (3.6), we have

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbf{Z}^2}\max\left\{1,\frac{|\mathbf{n}|}{N}\right\}|(T_N^*f)_{\mathbf{n}}|^2 &\le \sum_{|\mathbf{n}|>N}\frac{|\mathbf{n}|^2}{N^2}|(T_N^*f)_{\mathbf{n}}|^2+\sum_{|\mathbf{n}|\le N}|(T_N^*f)_{\mathbf{n}}|^2 \\ &\le \sum_{\mathbf{n}\in\mathbf{Z}^2}\frac{|\mathbf{n}|^2}{N^2}|(T_N^*f)_{\mathbf{n}}|^2+\sum_{\mathbf{n}\in\mathbf{Z}^2}|(T_N^*f)_{\mathbf{n}}|^2,\end{aligned}$$

so it suffices to follow (3.7). □

Remark 3.1. We are grateful to Luc Tartar for showing us how inequalities similar to (2.3) and (2.4) arise as special cases of general interpolation theorems. We discuss only the simplest case, involving functions on all $\mathbf{R}^n$, drawing from Sects. 2.3–2.5 of [18]. The Besov spaces $B^s_{p,q}$ can be defined by interpolation of Sobolev spaces:

$$B^s_{p,q}=\left(W^{s_0}_p,W^{s_1}_p\right)_{\theta,q},$$

where $s_0\ne s_1$, $0<\theta<1$, and $s=\theta s_0+(1-\theta)s_1$. To interpret the right hand side, we note that when s is a nonnegative integer, W^s_p is the space of functions on all $\mathbf{R}^n$ whose derivatives of order up to s are in L^p. The L^2-based spaces H^s arise as special cases:

$$H^s=B^s_{2,2}.$$

In particular,

$$H^{1/2}=B^{1/2}_{2,2}=\left(W^0_2,W^1_2\right)_{1/2,2}=\left(L^2,H^1\right)_{1/2,2},$$

whereas

$$B^{1/2}_{2,\infty}=\left(W^0_2,W^1_2\right)_{1/2,\infty}=\left(L^2,H^1\right)_{1/2,\infty}$$

is a little smaller than $H^{1/2}$. (See (3.11) for the dependence of $B^{1/2}_{2,q}$ on q.)

The Besov spaces satisfy the general interpolation relation

$$\left(B^{s_0}_{p,q_0},B^{s_1}_{p,q_1}\right)_{\theta,q}=B^s_{p,q}$$

with $s = \theta s_0 + (1-\theta)s_1$, for any $-\infty < s_0 \neq s_1 < \infty$, $1 < p < \infty$, $1 \leq q_0, q_1, q \leq \infty$, and $0 < \theta < 1$. This implies

$$\|f\|_{B^s_{p,q}} \leq C\|f\|^{\theta}_{B^{s_0}_{p,q_0}} \|f\|^{1-\theta}_{B^{s_1}_{p,q_1}}.$$

We obtain

$$\|f\|_{L^2} \leq C\|f\|^{1/3}_{H^{-1}} \|f\|^{2/3}_{B^{1/2}_{2,\infty}}$$

and

$$\|f\|_{L^2} \leq C\|f\|^{1/2}_{H^{-1/2}} \|f\|^{1/2}_{B^{1/2}_{2,\infty}}$$

as special cases.

The link with (2.3)–(2.4) lies in the relation

$$\|f\|_{B^{1/2}_{2,\infty}} \leq C \left\{ \|f\|_{L^2} + \|f\|^{1/2}_{L^\infty} \left(\int |\nabla f| d\mathbf{x} \right)^{1/2} \right\}. \tag{3.8}$$

Combined with the interpolation inequalities it gives

$$\|f\|_{L^2} \leq C\|f\|^{1/3}_{H^{-1}} \left\{ \|f\|^{2/3}_{L^2} + \|f\|^{1/3}_{L^\infty} \left(\int |\nabla f| d\mathbf{x} \right)^{1/3} \right\} \tag{3.9}$$

and

$$\|f\|_{L^2} \leq C\|f\|^{1/2}_{H^{-1/2}} \left\{ \|f\|^{1/2}_{L^2} + \|f\|^{1/4}_{L^\infty} \left(\int |\nabla f| d\mathbf{x} \right)^{1/4} \right\}. \tag{3.10}$$

The analogy should now be clear: (2.3) and (2.4) interpolate additively between the L^∞, BV, and H^{-1} or $H^{-1/2}$ norms (for periodic functions), while (3.9) and (3.10) interpolate multiplicatively between the same norms (for functions on $\mathbf{R}^n$).

It remains to justify (3.8). We start from the fact that the $B^{1/2}_{2,q}$ norm is equivalent to

$$\|f\|_{L^2} + \left(\int_{|\mathbf{h}|<1} |\mathbf{h}|^{-q/2} \|\Delta_{\mathbf{h}} f\|^q_{L^2} d\mathbf{h} \right)^{1/q}, \tag{3.11}$$

with the notation $\Delta_{\mathbf{h}} f(\mathbf{x}) = f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})$. Thus the norm on $B^{1/2}_{2,\infty}$ is equivalent to

$$\|f\|_{L^2} + \sup_{0<|\mathbf{h}|<1} |\mathbf{h}|^{-1/2} \|\Delta_{\mathbf{h}} f\|_{L^2}. \tag{3.12}$$

One easily verifies that

$$\|\Delta_{\mathbf{h}} f\|_{L^1} = \int |f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})| d\mathbf{x} \leq C|\mathbf{h}| \int |\nabla f| d\mathbf{x}$$

for any $\mathbf{h}$, using the fundamental theorem of calculus. We also have

$$\|\Delta_{\mathbf{h}} f\|_{L^\infty} = \sup_{\mathbf{x}} |f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})| \leq 2\|f\|_{L^\infty}.$$

Interpolating between these two results gives

$$\|\Delta_{\mathbf{h}} f\|_{L^2} \leq C|\mathbf{h}|^{1/2} \|f\|^{1/2}_{L^\infty} \left(\int |\nabla f| d\mathbf{x} \right)^{1/2},$$

which when combined with (3.12) gives (3.8).

We turn now to the second basic lemma.

Proof of Lemma 2.4. By rescaling, it suffices to prove that for g with support in $[-1,1]$, we have

$$\int_{-\infty}^{\infty} \frac{\xi^2}{\xi^2+\lambda^2} |\widehat{g}(\xi)|^2 d\xi \geq \frac{c_2}{1+\lambda^2} \int_{-1}^{1} |g(x)|^2 \, dx. \tag{3.13}$$

To this end, we note that by Holder's inequality

$$|\widehat{g}(\xi)|^2 \leq 2 \int_{-1}^{1} |g(x)|^2 \, dx. \tag{3.14}$$

Working on the left hand side of (3.13), we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\xi^2}{\xi^2+\lambda^2} |\widehat{g}(\xi)|^2 d\xi &\geq \int_{|\xi|>1/8} \frac{\xi^2}{\xi^2+\lambda^2} |\widehat{g}(\xi)|^2 d\xi \\ &\geq \frac{1}{1+(8\lambda)^2} \left(\int_{-\infty}^{\infty} |\widehat{g}(\xi)|^2 d\xi - \int_{|\xi|<1/8} |\widehat{g}(\xi)|^2 d\xi \right), \end{aligned} \tag{3.15}$$

using the fact that when $|\xi| > 1/8$, $\frac{\xi^2}{\xi^2+\lambda^2} \geq \frac{1}{1+(8\lambda)^2}$. Now by (3.14), we have

$$\int_{|\xi|<1/8} |\widehat{g}(\xi)|^2 d\xi \leq \frac{1}{2} \int_{-1}^{1} |g(x)|^2 \, dx,$$

and hence (3.13) follows from (3.15) and Plancherel's theorem. □

4. Refinements and Remarks

Theorem 2.1 says that the energy of any admissible magnetization **m** is at least $c_0 \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}}$. This bound arises through a competition between the field energy (which prefers $\mathbf{m} \cdot \mathbf{n} = 0$ at $x = \pm L$), the anisotropy energy (which prefers $\mathbf{m} = (\pm 1, 0, 0)$), and the surface energy (which prefers fewer domain walls). It is natural to guess that the burden of energy minimization must be shared by these three terms. However, there is a possibility of trade-off between field energy and anisotropy, since the optimal scaling can be reached with field energy identically zero or with anisotropy energy identically zero [4]. So the true competition is between field+anisotropy energy and surface energy. The following results confirm this intuition, by showing that our lower bound holds separately for each term.

Proposition 4.1. *For any $G > 0$, consider admissible magnetization fields* **m** *satisfying*

$$E(\mathbf{m}) \leq G \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}}. \tag{4.1}$$

Let c_0 denote the constant in the lower bound of Theorem 2.1. There exists a universal constant $\mu > 0$ such that the following statements are true:

a) *If*

$$\frac{\varepsilon}{\gamma L} < 1 \quad \textit{and} \quad \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} < 1, \tag{4.2}$$

then

$$\gamma \left(\int_{\Omega} m_2^2 + m_3^2 \, dx \, dy \, dz + \int_{\mathbf{R} \times Q} |\nabla u|^2 \, dx \, dy \, dz \right) \geq \mu c_0 \left(\frac{c_0}{G} \right)^2 \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}}.$$

b) *If*

$$\frac{\varepsilon}{\gamma L} < \mu \left(\frac{c_0}{G} \right)^{\frac{3}{2}} \quad \textit{and} \quad \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} < \mu \left(\frac{c_0}{G} \right)^{\frac{1}{2}}, \tag{4.3}$$

then

$$\varepsilon \int_{\Omega} |\nabla m| \, dx \, dy \, dz \geq \mu c_0 \left(\frac{c_0}{G} \right)^{\frac{1}{2}} \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}}.$$

Remark 4.2. Observe that

$$\ell^{-1} := \frac{1}{L} \int_{\Omega} |\nabla \mathbf{m}| \, dx \, dy \, dz \tag{4.4}$$

can be interpreted as the inverse of the average size of the magnetic domains. The upper bound in Theorem 2.1 and the lower bound in Proposition 4.1 b) show the existence of universal constants $0 < c_3 < C_3 < \infty$ such that the ℓ^* corresponding to the minimizer $\mathbf{m}^*$ of E satisfies

$$c_3 \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} \leq \ell^* \leq C_3 \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}}, \tag{4.5}$$

provided that

$$\frac{\varepsilon}{\gamma L} < c_3 \quad \text{and} \quad \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} < c_3.$$

Moreover, this conclusion is not restricted to the minimizer: for any magnetization $\mathbf{m}$ achieving the optimal scaling law, the associated ℓ satisfies an estimate of the form (4.5).

Remark 4.3. The attentive reader may be disturbed by the apparent disregard for dimensions in (4.3). Indeed, in the first inequality of (4.3) μ appears to be dimensionless while in the second it appears to have dimensions of length. This arises because two of the physical dimensions of Ω were set equal to 1 from the start, and hence certain units of length have become invisible. If the y and z sides of Ω had length w, then the second assumption of (4.2) would become

$$\frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} < w,$$

and the second inequality in (4.3) would become

$$\frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} < w\mu \left(\frac{c_0}{G} \right)^{\frac{1}{2}}.$$

The conclusion of Theorem 2.1 becomes $E_0 \sim w^2\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}L^{\frac{1}{3}}$, and that of Theorem 2.2 becomes $E_1 \sim w^2\gamma^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}L^{\frac{1}{2}}$; these expressions now truly have the dimensions of energy. The definition of ℓ must change, of course, to $\ell^{-1} = \frac{1}{w^2L}\int_\Omega |\nabla\mathbf{m}|\,dx\,dy\,dz$. Thus the estimate for ℓ, (4.5), remains unaltered.

Proof of Proposition 4.1. We start with a). By assumption (4.1) we have

$$\begin{aligned} &\gamma\left(\int_\Omega m_2^2 + m_3^2\,dx\,dy\,dz + \int_{\mathbf{R}\times Q}|\nabla u|^2\,dx\,dy\,dz\right) + \varepsilon\int_\Omega|\nabla\mathbf{m}|\,dx\,dy\,dz \\ &\le G\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}L^{\frac{1}{3}}. \end{aligned} \tag{4.6}$$

On the other hand, we have for any $R > 1$ according to Theorem 2.1 (more precisely, according to Eq. (2.5))

$$\begin{aligned} &R\gamma\left(\int_\Omega m_2^2 + m_3^2\,dx\,dy\,dz + \int_{\mathbf{R}\times Q}|\nabla u|^2\,dx\,dy\,dz\right) + \varepsilon\int_\Omega|\nabla\mathbf{m}|\,dx\,dy\,dz \\ &\ge c_0(R\gamma)^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}L^{\frac{1}{3}}, \end{aligned} \tag{4.7}$$

since (4.2) ensures that

$$\frac{\varepsilon}{(R\gamma)L} < 1 \quad\text{and}\quad \frac{\varepsilon^{\frac{1}{3}}L^{\frac{2}{3}}}{(R\gamma)^{\frac{1}{3}}} < 1.$$

We subtract (4.6) from (4.7) and divide by $R-1$ to get

$$\gamma\left(\int_\Omega m_2^2 + m_3^2\,dx\,dy\,dz + \int_{\mathbf{R}\times Q}|\nabla u|^2\,dx\,dy\,dz\right) \ge \frac{c_0R^{\frac{1}{3}} - G}{R-1}\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}L^{\frac{1}{3}}.$$

A non-optimal but convenient choice of R is

$$R = \left(2\frac{G}{c_0}\right)^3,$$

which leads to

$$\gamma\left(\int_\Omega m_2^2 + m_3^2\,dx\,dy\,dz + \int_{\mathbf{R}\times Q}|\nabla u|^2\,dx\,dy\,dz\right) \ge 2^{-3}c_0\left(\frac{c_0}{G}\right)^2\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}L^{\frac{1}{3}}.$$

Now we tackle b). By assumption (4.1) we have

$$\begin{aligned} &\gamma\left(\int_\Omega m_2^2 + m_3^2\,dx\,dy\,dz + \int_{\mathbf{R}\times Q}|\nabla u|^2\,dx\,dy\,dz\right) + \varepsilon\int_\Omega|\nabla\mathbf{m}|\,dx\,dy\,dz \\ &\le G\gamma^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}L^{\frac{1}{3}}. \end{aligned} \tag{4.8}$$

On the other hand, we have for any $R > 1$ according to Theorem 2.1 (more precisely (2.5))

$$\begin{aligned} &\gamma\left(\int_\Omega m_2^2 + m_3^2\,dx\,dy\,dz + \int_{\mathbf{R}\times Q}|\nabla u|^2\,dx\,dy\,dz\right) + R\varepsilon\int_\Omega|\nabla\mathbf{m}|\,dx\,dy\,dz \\ &\ge c_0\gamma^{\frac{1}{3}}(R\varepsilon)^{\frac{2}{3}}L^{\frac{1}{3}}, \end{aligned} \tag{4.9}$$

provided that

$$\frac{R\varepsilon}{\gamma L} < 1 \quad \text{and} \quad \frac{(R\varepsilon)^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} < 1.$$

As before, we subtract (4.8) from (4.9) and divide by $R - 1$ to get

$$\varepsilon \int_\Omega |\nabla \mathbf{m}| \, dx \, dy \, dz \geq \frac{c_0 R^{\frac{2}{3}} - G}{R - 1} \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}}.$$

A non-optimal but convenient choice of R is

$$R = \left(2 \frac{G}{c_0} \right)^{\frac{3}{2}}.$$

Then we obtain

$$\varepsilon \int_\Omega |\nabla \mathbf{m}| \, dx \, dy \, dz \geq 2^{-\frac{3}{2}} c_0 \left(\frac{c_0}{G} \right)^{\frac{1}{2}} \gamma^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} L^{\frac{1}{3}}$$

under the condition that

$$\frac{\varepsilon}{\gamma L} < 2^{-\frac{3}{2}} \left(\frac{c_0}{G} \right)^{\frac{3}{2}} \quad \text{and} \quad \frac{\varepsilon^{\frac{1}{3}} L^{\frac{2}{3}}}{\gamma^{\frac{1}{3}}} < 2^{-\frac{1}{2}} \left(\frac{c_0}{G} \right)^{\frac{1}{2}}. \qquad \square$$

Remark 4.4. Proposition 4.1 refines Theorem 2.1. Analogous refinements of Theorem 2.2 are also valid, using essentially the same arguments.

Remark 4.5. As mentioned in the Introduction, our method is not restricted to the case when **m** is periodic in y and z. The same method can be used in the more physical setting when **m** vanishes outside Ω. Of course the field energy is then $\int_{\mathbf{R}^3} |\nabla u|^2$ rather than $\int_{\mathbf{R} \times Q} |\nabla u|^2$. The obvious analogues of Theorems 2.1, 2.2 and Proposition 4.1 remain valid; the proofs are obtained, in essence, by replacing the discrete Fourier series in y, z with the continuous Fourier transform. For example, in proving the analogue of Theorem 2.1 one uses the following analogue of our first interpolation inequality (2.3): if $f \in BV(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$ with $f = 0$ outside Q, then

$$\int_Q |f|^2 \, dx \, dy \leq c_1 \left\{ \frac{1}{N} \sup_Q |f| \int_Q |\nabla f| \, dx \, dy + \int_{\mathbf{R}^2} \min \left\{ 1, \frac{N^2}{|\xi|^2} \right\} |\widehat{f}(\xi)|^2 d\xi \right\}$$

for a suitable constant c_1. The proof is very similar to that of (2.3).

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