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Gerstenhaber Algebras and BV-Algebras in Poisson Geometry

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Abstract: The purpose of this paper is to establish an explicit correspondence between various geometric structures on a vector bundle with some well-known algebraic structures such as Gerstenhaber algebras and BV-algebras. Some applications are discussed. In particular, we find an explicit connection between the Koszul–Brylinski operator and the modular class of a Poisson manifold. As a consequence, we prove that Poisson homology is isomorphic to Poisson cohomology for unimodular Poisson structures.

1. Introduction

BV-algebras arise from the BRST theory of topological field theory [30]. Recently, there has been a great deal of interest in these algebras in connection with various subjects such as operads and string theory [7, 8, 11, 18, 22, 25, 26, 31].

Let us first recall various relevant definitions, following the terminology of [14].

A *Gerstenhaber algebra* consists of a triple $(\mathcal{A} = \oplus_{i\in\mathbb{Z}}\mathcal{A}^i, \wedge, [\cdot,\cdot])$ such that $(\mathcal{A}, \wedge)$ is a graded commutative associative algebra, and $(\mathcal{A} = \oplus_{i\in\mathbb{Z}}\mathcal{A}^{(i)}, [\cdot,\cdot])$, with $\mathcal{A}^{(i)} = \mathcal{A}^{i+1}$, is a graded Lie algebra, and $[a,\cdot]$, for each $a \in \mathcal{A}^{(i)}$ is a derivation with respect to $\wedge$ with degree i.

An operator D of degree -1 is said to generate the Gerstenhaber algebra bracket if for every $a \in \mathcal{A}^{|a|}$ and $b \in \mathcal{A}$,

$$[a,b] = (-1)^{|a|}(D(a\wedge b) - Da\wedge b - (-1)^{|a|}a\wedge Db). \tag{1}$$

A Gerstenhaber algebra is said to be *exact* if there is an operator D of square zero generating the bracket. In this case, D is called a *generating operator*. An exact Gerstenhaber algebra is also called a *Batalin–Vilkovisky algebra* (or *BV-algebra* in short).

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A *differential Gerstenhaber algebra* is a Gerstenhaber algebra equipped with a differential d, which is a derivation of degree 1 with respect to $\wedge$ and $d^2 = 0$. It is called a *strong differential Gerstenhaber algebra* if, in addition, d is a derivation of the graded Lie bracket.

Kosmann–Schwarzbach noted [13] that these algebra structures had also appeared in Koszul's work [17] in 1985 in his study of Poisson manifolds. In fact they are connected with a certain differential structure on vector bundles, called *Lie algebroids* by Pradines [23]. Let us recall for the benefit of the reader the definition of a Lie algebroid [23, 24].

Definition 1.1. *A Lie algebroid is a vector bundle A over M together with a Lie algebra structure on the space $\Gamma(A)$ of smooth sections of A, and a bundle map $a : A \to TP$ (called the anchor), extended to a map between sections of these bundles, such that*

(i) $a([X, Y]) = [a(X),\ a(Y)]$*; and*
(ii) $[X, fY] = f[X, Y] + (a(X)f)Y$

for any smooth sections X and Y of A and any smooth function f on M.

Among many examples of Lie algebroids are the usual Lie algebras, the tangent bundle of a manifold, and an integrable distribution over a manifold (see [20]). In recent years, Lie algebroids have become increasingly interesting in Poisson geometry. One main reason for this is given by the following example. Let P be a Poisson manifold with Poisson tensor π. Then T^*P carries a natural Lie algebroid structure, called the cotangent Lie algebroid of the Poisson manifold P [4]. The anchor map $\pi^\# : T^*P \to TP$ is defined by

$$\pi^\# :\ T_p^*P \longrightarrow T_pP :\ \pi^\#(\xi)(\eta) = \pi(\xi, \eta), \quad \forall \xi, \eta \in T_p^*P \tag{2}$$

and the Lie bracket of 1-forms α and β is given by

$$[\alpha, \beta] = L_{\pi^\#(\alpha)}\beta \ - \ L_{\pi^\#(\beta)}\alpha \ - \ d\pi(\alpha, \beta). \tag{3}$$

In [13], Kosmann–Schwarzbach constructed various examples of strong differential Gerstenhaber algebras and BV-algebras in connection with Lie algebroids. Motivated by her work, in this paper we will study the relation between these algebra structures and some of the well-known geometric structures in Poisson geometry. More precisely, we will investigate the following question. Let A be a vector bundle of rank n over the base M, and let $\mathcal{A} = \oplus_{0\le k\le n}\Gamma(\wedge^k A)$ be its corresponding exterior algebra. It is graded commutative. The question is:

> What additional structure on A will make $\mathcal{A}$ into a Gerstenhaber algebra, a strong differential Gerstenhaber algebra, or an exact Gerstenhaber algebra (or a BV-algebra)?

The answer is surprisingly simple. Gerstenhaber algebras and strong differential Gerstenhaber algebras correspond exactly to the structures of Lie algebroids and Lie bialgebroids (see Sect. 2 for the definition), respectively, as already indicated in [13]. And an exact Gerstenhaber algebra structure corresponds to a Lie algebroid A together with a flat A-connection on its canonical line bundle $\wedge^n A$. This fact was already implicitly contained in Koszul's work [17] although he only treated the case of multivector fields. However, the formulas (9) and (14) below establishing the explicit correspondence seem to be new. Below is a table of the correspondence.

Structures on algebra $\mathcal{A}$		Structures on the vector bundle A
Gerstenhaber algebras	$\leftrightarrow$	Lie algebroids
Strong differential Gerstenhaber algebras	$\leftrightarrow$	Lie bialgebroids
Exact Gerstenhaber algebras (BV-algebras)	$\leftrightarrow$	Lie algebroids with a flat A-connection on $\wedge^n A$

The content above occupies Sects. 2 and 3. Sect. 4 is devoted to applications. In particular, we find an explicit connection between the Koszul–Brylinski operator on a Poisson manifold with its modular class. As a consequence, we prove that Poisson homology is isomorphic to Poisson cohomology for unimodular Poisson structures (see [3, 28] for the definition).

As another application, we define Lie algebroid homology as the homology group of the complex: $D : \Gamma(\wedge^* A) \longrightarrow \Gamma(\wedge^{*-1} A)$ for a generating operator D. Since a generating operator depends on the choice of a flat A-connection on the line bundle $\wedge^n A$, in general this homology depends on the choice of such a connection ∇. When two connections are homotopic (see Sect. 4 for the precise definition), their corresponding homology groups are isomorphic. So a given Lie algebroid has homologies which are in fact parameterized by the first Lie algebroid cohomology $H^1(A, \mathbb{R})$. When A is a Lie algebra and ∇ is the trivial connection, this reduces to the usual Lie algebra homology with trivial coefficients. On the other hand, Poisson homology can also be considered as a special case of Lie algebroid homology, where A is taken as the cotangent Lie algebroid of a Poisson manifold.

We note that in a recent paper [5], Evens, Lu and Weinstein have also established a connection between Poisson homology and the modular class of Poisson manifolds. Some results in the paper have recently been generalized by Huebschmann to the algebraic context of Lie-Rinehart algebras [9, 10].

2. Gerstenhaber Algebras and Differential Gerstenhaber Algebras

In this section, we will treat Gerstenhaber algebras and differential Gerstenhaber algebras arising from a vector bundle.

Again, let A be a vector bundle of rank n over M, and let $\mathcal{A} = \oplus_{0\leq k\leq n}\Gamma(\wedge^k A)$. The following proposition establishes a one-one correspondence between Gerstenhaber algebra structures on $\mathcal{A}$ and Lie algebroid structures on the underlying vector bundle A.

Proposition 2.1. *$\mathcal{A}$ is a Gerstenhaber algebra iff A is a Lie algebroid.*

This is a well-known result (see [6, 16, 21]). For completeness, we sketch a proof below.

Proof. Suppose that there is a graded Lie bracket $[\cdot,\cdot]$ that makes $\mathcal{A}$ into a Gerstenhaber algebra. It is clear that $(\Gamma(A), [\cdot,\cdot])$ is a Lie algebra. Second, for any $X \in \Gamma(A)$ and $f, g \in C^\infty(M)$, it follows from the derivation property that

$$[X, fg] = [X, f]g + f[X, g].$$

Hence, $[X, \cdot]$ defines a vector field on M, which will be denoted by $a(X)$. It is easy to see that a is in fact induced by a bundle map from A to TP. By applying the graded Jacobi identity, we find that

$$a([X, Y]) = [a(X), a(Y)].$$

Finally, again from the derivation property, it follows that

$$[X, fY] = (a(X)f)Y + f[X, Y].$$

This shows that A is indeed a Lie algebroid.

Conversely, given a Lie algebroid A, it is easy to check that $\mathcal{A} = \oplus_{0\leq k\leq n}\Gamma(\wedge^k A)$ forms a Gerstenhaber algebra (see [13, 21]). □

The following lemma gives another characterization of a Lie algebroid, which should be of interest itself. Recall that a *differential graded algebra* is a graded commutative associative algebra equipped with a differential d, which is a derivation of degree 1 and of square zero.

Lemma 2.2 ([16, 12]). *Given a vector bundle A over M, A is a Lie algebroid iff $\oplus_k\Gamma(\wedge^k A^*)$ is a differential graded algebra.*

Proof. Given a Lie algebroid A, it is known that $\oplus_k\Gamma(\wedge^k A^*)$ admits a differential d that makes it into a differential graded algebra [16]. Here, $d : \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^{k+1} A^*)$ is simply the differential defining the Lie algebroid cohomology ([20, 21, 29]):

$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}a(X_i)(\omega(X_1, \overset{\wedge}{\ldots}, X_{k+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \overset{\wedge}{\ldots}\overset{\wedge}{\ldots}, X_{k+1}), \end{aligned} \tag{4}$$

for $\omega \in \Gamma(\wedge^k A^*)$, $X_i \in \Gamma A,\ 1 \leq i \leq k+1$.

Conversely, if $\oplus_k\Gamma(\wedge^k A^*)$ is a differential graded algebra with differential d, then the equations:

$$a(X)f = \langle df, X\rangle, \quad \text{and} \tag{5}$$

$$\langle [X, Y], \theta\rangle = a(X)(\theta \cdot Y) - a(Y)(\theta \cdot X) - (d\theta)(X, Y) \tag{6}$$

$\forall f \in C^\infty(M), X, Y \in \Gamma(A)$, and $\theta \in \Gamma(A^*)$, define a Lie algebroid structure on A. □

Remark. The lemma above is essentially Proposition 6.1 of [16]. Equation (6) is Formula (6.6) in [16].

Recall that a *Lie bialgebroid* [13, 21] is a dual pair (A, A^*) of vector bundles equipped with Lie algebroid structures such that the differential d_*, induced from the Lie algebroid structure on A^* as defined by Eq. (4), is a derivation of the Lie bracket on $\Gamma(A)$, i.e.,

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y], \quad \forall X, Y \in \Gamma(A). \tag{7}$$

The following result is due to Kosmann–Schwarzbach [13].

Proposition 2.3. *$\mathcal{A}$ is a strong differential Gerstenhaber algebra iff (A, A^*) is a Lie bialgebroid.*

Proof. Assume that $\mathcal{A}$ is a strong differential Gerstenhaber algebra. Then, A^* is a Lie algebroid according to Lemma 2.2. Moreover, the derivation property of the differential with respect to the Lie bracket on $\Gamma(A)$ implies that (A^*, A) is a Lie bialgebroid. This is equivalent to that (A, A^*) is a Lie bialgebroid by duality [21]. Conversely, it is straightforward to see, for a given Lie bialgebroid (A, A^*), that $\mathcal{A}$ is a strong differential Gerstenhaber algebra (see [13]). □

Example 2.4. Let P be a Poisson manifold with Poisson tensor π. Let $A = TP$ with the standard Lie algebroid structure. It is well known that the space of multivector fields $\mathcal{A} = \oplus_k \Gamma(\wedge^k TP)$ has a Gerstenhaber algebra structure, where the graded Lie bracket is called the Schouten bracket.

In 1977, Lichnerowicz introduced a differential $d = [\pi, \cdot] : \Gamma(\wedge^k TP) \longrightarrow \Gamma(\wedge^{k+1} TP)$, which he used to define the Poisson cohomology [19]. It is obvious that $\mathcal{A}$ becomes a strong differential Gerstenhaber algebra, so it corresponds to a Lie bialgebroid structure on (TP, T^*P) according to Proposition 2.3. It is not surprising that this Lie bialgebroid is just the standard Lie bialgebroid associated to a Poisson manifold [21], where the Lie algebroid structure on T^*P is defined as in the introduction (see Eqs. (2) and (3)). It is, however, quite amazing that the Lie algebroid structure on T^*P was not known until the middle of 1980's (see [15] for the references) and the Lie bialgebroid structure comes much later! For the Lie algebroid T^*P, the associated differential on $\oplus \Gamma(\wedge^* TP)$ is the Lichnerowicz differential $d = [\pi, \cdot]$. This property was proved, independently by Bhaskara and Viswanath [1], and Kosmann–Schwarzbach and Magri [16].

3. Exact Gerstenhaber Algebras

In this section, we will move to exact Gerstenhaber algebras arising from a vector bundle. Let $A \longrightarrow M$ be a Lie algebroid with anchor a and $E \longrightarrow M$ a vector bundle over M. By an A-connection on E, we mean an $\mathbb{R}$-linear map:

$$\begin{aligned} \Gamma(A) \otimes \Gamma(E) &\longrightarrow \Gamma(E), \\ X \otimes s &\longrightarrow \nabla_X s, \end{aligned}$$

satisfying the axioms resembling those of the usual linear connections, i.e., $\forall f \in C^\infty(M),\ X \in \Gamma(A), s \in \Gamma(E)$,

$$\begin{aligned} &\nabla_{fX} s = f \nabla_X s; \\ &\nabla_X (fs) = (a(X) f) s + f \nabla_X s. \end{aligned}$$

Similarly, the curvature R of an A-connection ∇ is the element in $\Gamma(\wedge^2 A^*) \otimes End(E)$ defined by

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},\ \ \forall X, Y \in \Gamma(A). \tag{8}$$

Given a Lie algebroid A of rank n and an A-connection ∇ on the line bundle $\wedge^n A$, we define a differential operator $D : \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^{k-1} A)$ as follows. Let U be any section in $\Gamma(\wedge^k A)$ and write, locally, $U = \omega \lrcorner \Lambda$, where $\omega \in \Gamma(\wedge^{n-k} A^*)$ and $\Lambda \in \Gamma(\wedge^n A)$. Set, for each $m \in M$,

$$DU|_m = -(-1)^{|\omega|}(d\omega \lrcorner \Lambda + \sum_{i=1}^{n} (\alpha_i \wedge \omega) \lrcorner \nabla_{X_i} \Lambda), \tag{9}$$

where $X_1, \cdots, X_n$ is a basis of $A|_m$ and $\alpha_1, \cdots, \alpha_n$ its dual basis in $A^*|_m$. Clearly, this definition is independent of the choice of the basis.

Remark. We would like to make a remark on the notation. Let E be a vector bundle over M. Assume that $V \in \Gamma(\wedge^k E)$ and $\theta \in \Gamma(\wedge^l E^*)$ with $k \geq l$. Then, by $\theta \lrcorner V$ we denote the section of $\wedge^{k-l} E$ given by

$$(\theta \lrcorner V)(\omega) = V(\theta \wedge \omega), \quad \forall \omega \in \Gamma(\wedge^{k-l} E^*).$$

We will stick to this convention in the sequel no matter whether E is A itself or its dual A^*.

Proposition 3.1. *The operator* D *is well defined and*

$$D^2 U = -R \lrcorner U,$$

where $R \in \Gamma(\wedge^2 A^*)$ *is the curvature of the connection* ∇ *(note that* $EndE$ *is a trivial line bundle).*

Proof. Assume that f is any locally nonzero function on M, and $U = f\omega \lrcorner \frac{1}{f}\Lambda$. Then,

$$\begin{aligned}
& d(f\omega) \lrcorner \frac{1}{f}\Lambda + \sum_i (\alpha_i \wedge f\omega) \lrcorner \nabla_{X_i}(\frac{1}{f}\Lambda) \\
= & \frac{1}{f}(df \wedge \omega) \lrcorner \Lambda + d\omega \lrcorner \Lambda + \sum_i f(aX_i)(\frac{1}{f})(\alpha_i \wedge \omega) \lrcorner \Lambda \\
& + \sum_i (\alpha_i \wedge \omega) \lrcorner \nabla_{X_i}\Lambda \\
= & \frac{1}{f}(df \wedge \omega) \lrcorner \Lambda + f(d(\frac{1}{f}) \wedge \omega) \lrcorner \Lambda + d\omega \lrcorner \Lambda + \sum_i (\alpha_i \wedge \omega) \lrcorner \nabla_{X_i}\Lambda \\
= & d\omega \lrcorner \Lambda + \sum_i (\alpha_i \wedge \omega) \lrcorner \nabla_{X_i}\Lambda.
\end{aligned}$$

This shows that D is well-defined. For the second part, we have

$$\begin{aligned}
D^2 U = & -(-1)^{|\omega|} D(d\omega \lrcorner \Lambda + \sum_{i=1}^n (\alpha_i \wedge \omega) \lrcorner \nabla_{X_i}\Lambda) \\
= & -(\sum_i (\alpha_i \wedge d\omega) \lrcorner \nabla_{X_i}\Lambda + \sum_i d(\alpha_i \wedge \omega) \lrcorner \nabla_{X_i}\Lambda \\
& + \sum_{j,i} (\alpha_j \wedge \alpha_i \wedge \omega) \lrcorner \nabla_{X_j}\nabla_{X_i}\Lambda) \\
= & -(\sum_i (d\alpha_i \wedge \omega) \lrcorner \nabla_{X_i}\Lambda + \sum_{j,i} (\alpha_j \wedge \alpha_i \wedge \omega) \lrcorner \nabla_{X_j}\nabla_{X_i}\Lambda) \\
= & -[\sum_i \omega \lrcorner (d\alpha_i \lrcorner \nabla_{X_i}\Lambda) + \sum_{j,i} \omega \lrcorner (\alpha_i \wedge \alpha_j \lrcorner \nabla_{X_i}\nabla_{X_j}\Lambda)].
\end{aligned}$$

The conclusion thus follows from the following lemma.

Lemma 3.2.

$$\sum_i d\alpha_i \lrcorner \nabla_{X_i}\Lambda + \sum_{j,i}(\alpha_i \wedge \alpha_j) \lrcorner \nabla_{X_i}\nabla_{X_j}\Lambda = -R \lrcorner \Lambda.$$

Proof. It is a straightforward verification, and is left to the readers. □

Proposition 3.3. *Let $D : \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^{k-1} A)$ be the operator as defined in Equation (9). Then, D generates the Gerstenhaber algebra bracket on $\oplus_k \Gamma(\wedge^k A)$, i.e, for any $U \in \Gamma(\wedge^u A)$ and $V \in \Gamma(\wedge^v A)$,*

$$[U, V] = (-1)^u (D(U \wedge V) - DU \wedge V - (-1)^u U \wedge DV). \tag{10}$$

We need a couple of lemmas before proving this proposition.

Lemma 3.4. *For any $U \in \Gamma(\wedge^u A),\ V \in \Gamma(\wedge^v A)$ and $\theta \in \Gamma(\wedge^{u+v-1} A^*)$,*

$$[U, V] \lrcorner \theta = (-1)^{(u-1)(v-1)} U \lrcorner d(V \lrcorner \theta) - V \lrcorner d(U \lrcorner \theta) - (-1)^{u+1}(U \wedge V) \lrcorner d\theta. \tag{11}$$

Proof. See Eq. (1.16) in [27]. □

Lemma 3.5. *For any $U \in \Gamma(\wedge^u A)$ and $\theta \in \Gamma(\wedge^{u-1} A^*)$,*

$$\theta \lrcorner DU = (-1)^{|\theta|} D(\theta \lrcorner U) + d\theta \lrcorner U. \tag{12}$$

Proof. Assume that $U = \omega \lrcorner \Lambda$. Then $\theta \lrcorner U = (\omega \wedge \theta) \lrcorner \Lambda$, and therefore,

$$\begin{aligned}
&D(\theta \lrcorner U)\\
&= -(-1)^{|\omega|+|\theta|}(d(\omega \wedge \theta) \lrcorner \Lambda + \sum_i (\alpha_i \wedge \omega \wedge \theta) \lrcorner \nabla_{X_i}\Lambda)\\
&= -(-1)^{|\omega|+|\theta|}[(d\omega \wedge \theta) \lrcorner \Lambda + (-1)^{|\omega|}(\omega \wedge d\theta) \lrcorner \Lambda + \sum_i (\alpha_i \wedge \omega \wedge \theta) \lrcorner \nabla_{X_i}\Lambda]\\
&= (-1)^{|\theta|}(\theta \lrcorner DU) - (-1)^{|\theta|} d\theta \lrcorner U.
\end{aligned}$$
□

Proof of Proposition 3.3. For any $U \in \Gamma(\wedge^u A)$, $V \in \Gamma(\wedge^v A)$ and $\theta \in \Gamma(\wedge^{u+v-1} A^*)$, using Eq. (12), we have

$$\theta \lrcorner D(U \wedge V) = (-1)^{|\theta|} D(\theta \lrcorner (U \wedge V)) + d\theta \lrcorner (U \wedge V).$$

On the other hand, we have

$$\begin{aligned}
\theta \lrcorner (U \wedge DV) &= (U \lrcorner \theta) \lrcorner DV\\
&= (-1)^{|\theta|-u} D((U \lrcorner \theta) \lrcorner V) + d(U \lrcorner \theta) \lrcorner V,
\end{aligned}$$

and

$$\begin{aligned}
&\theta \lrcorner (DU \wedge V)\\
&= (-1)^{(u-1)v} \theta \lrcorner (V \wedge DU)\\
&= (-1)^{(u-1)v}[(-1)^{|\theta|-v} D((V \lrcorner \theta) \lrcorner U) + d(V \lrcorner \theta) \lrcorner U]\\
&= (-1)^{uv+|\theta|} D((V \lrcorner \theta) \lrcorner U) + (-1)^{(u-1)v} d(V \lrcorner \theta) \lrcorner U).
\end{aligned}$$

It thus follows that

$$\begin{aligned}
&\theta \lrcorner [(-1)^u D(U \wedge V) - (-1)^u DU \wedge V - U \wedge DV] \\
&= (-1)^{|\theta|+u} D(\theta \lrcorner (U \wedge V)) - (-1)^{uv+|\theta|+u} D((V \lrcorner \theta) \lrcorner U) \\
&\quad -(-1)^{|\theta|-u} D((U \lrcorner \theta) \lrcorner V) \\
&\quad +(-1)^u d\theta \lrcorner (U \wedge V) + (-1)^{(u-1)(v-1)} d(V \lrcorner \theta) \lrcorner U - d(U \lrcorner \theta) \lrcorner V.
\end{aligned}$$

The conclusion thus follows from Eq. (11) and the identity:

$$\theta \lrcorner (U \wedge V) = (U \lrcorner \theta) \lrcorner V + (-1)^{uv} (V \lrcorner \theta) \lrcorner U. \qquad \square \tag{13}$$

Proposition 3.3 describes a construction from an A-connection to an operator D generating the Gerstenhaber algebra bracket. This construction is in fact reversible. Namely, the connection ∇ can also be recovered from the operator D. More precisely, we have

Proposition 3.6. *Suppose that $D : \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^{k-1} A)$ is the operator corresponding to an A-connection ∇ on $\wedge^n A$. Then, for any $X \in \Gamma(A)$ and $\Lambda \in \Gamma(\wedge^n A)$,*

$$\nabla_X \Lambda = -X \wedge D\Lambda. \tag{14}$$

Proof. By definition, $D\Lambda = -\alpha_i \lrcorner \nabla_{X_i} \Lambda$. Hence,

$$\begin{aligned}
-X \wedge D\Lambda &= \sum_i X \wedge (\alpha_i \lrcorner \nabla_{X_i} \Lambda) \\
&= \sum_i \alpha_i(X) \nabla_{X_i} \Lambda \\
&= \nabla_X \Lambda,
\end{aligned}$$

where the last equality uses the identity: $X = \sum_i \alpha_i(X) X_i$, and the second equality follows from the following simple fact in linear algebra:

Lemma 3.7. *Let V be a vector space of dimension n, $X \in V$, $\alpha \in V^*$ and $\Lambda \in \wedge^n V$. Then,*

$$X \wedge (\alpha \lrcorner \Lambda) = \alpha(X) \Lambda.$$

$\square$

Now we are ready to prove the main theorem of the section.

Theorem 3.8. *Let A be a Lie algebroid with anchor a, and $\mathcal{A} = \oplus_k \Gamma(\wedge^k A)$ its corresponding Gerstenhaber algebra. There is a one-to-one correspondence between A-connections on the line bundle $\wedge^n A$ and linear operators D generating the Gerstenhaber algebra bracket on $\mathcal{A}$. Under this correspondence, flat connections correspond to operators of square zero.*

Proof. It remains to prove that Eq. (14) indeed defines an A-connection on $\wedge^n A$ if D is an operator generating the Gerstenhaber algebra bracket.

First, it is clear that, with this definition, $\nabla_{fX} \Lambda = f \nabla_X \Lambda$ for any $f \in C^\infty(M)$. To prove that it satisfies the second axiom of a connection, we observe that for any $f \in C^\infty(M)$, and $\Lambda \in \Gamma(\wedge^n A)$,

$$\begin{aligned} D(f\Lambda) &= (Df)\Lambda + fD\Lambda + [f,\Lambda] \\ &= fD\Lambda + [f,\Lambda]. \end{aligned}$$

Hence,

$$\begin{aligned} \nabla_X(f\Lambda) &= -X\wedge D(f\Lambda) \\ &= -X\wedge(fD\Lambda + [f,\Lambda]) \\ &= f\nabla_X\Lambda - X\wedge[f,\Lambda]. \end{aligned}$$

On the other hand, using the property of Gerstenhaber algebras,

$$\begin{aligned} [f, X\wedge\Lambda] &= [f,X]\wedge\Lambda + (-1)X\wedge[f,\Lambda] \\ &= -(a(X)f)\Lambda - X\wedge[f,\Lambda]. \end{aligned}$$

Thus, $X\wedge[f,\Lambda] = -(a(X)f)\Lambda$. Hence, $\nabla_X(f\Lambda) = f\nabla_X\Lambda + (a(X)f)\Lambda$. □

A flat A-connection always exists on the line bundle $\wedge^n A$. To see this, note that $\wedge^n A\otimes\wedge^n A$ is a trivial line bundle, which always admits a flat connection. So the "square root" of this connection (see Proposition 4.3 in [5]) is a flat connection we need. Therefore, for a given Lie algebroid, there always exists an operator of degree -1 and of square zero generating the corresponding Gerstenhaber algebra. Such an operator is called a *generating operator*.

Any A-connection ∇ on A induces an A-connection on the line bundle $\wedge^n A$. Therefore, it corresponds to a linear operator D generating the Gerstenhaber algebra $\mathcal{A}$. In particular, if it is torsion free, i.e.,

$$\nabla_X Y - \nabla_Y X = [X,Y], \quad \forall X,Y\in\Gamma(A),$$

D possesses a simpler expression. Note that ∇ induces an A-connection on the exterior power $\wedge^k A$ and the dual bundle A^* as well. We will denote them by the same symbol ∇.

Proposition 3.9. *Suppose that ∇ is a torsion free A-connection on A. Let D : $\Gamma(\wedge^* A)\longrightarrow\Gamma(\wedge^{*-1}A)$ be its induced operator. Then, for any $U\in\Gamma(\wedge^u A)$,*

$$DU|_m = -\sum_i \alpha_i \lrcorner \nabla_{X_i} U, \tag{15}$$

where $X_1,\cdots,X_n$ is a basis of $A|_m$ and $\alpha_1,\cdots,\alpha_n$ the dual basis of $A^*|_m$.

Proof. Assume that $U=\omega\lrcorner\Lambda$ for some $\Lambda\in\Gamma(\wedge^n A)$ and $\omega\in\Gamma(\wedge^{n-u}A^*)$. Then,

$$\begin{aligned} \sum_i \alpha_i\lrcorner\nabla_{X_i}(\omega\lrcorner\Lambda) &= \sum_i \alpha_i\lrcorner[\nabla_{X_i}\omega\lrcorner\Lambda + \omega\lrcorner\nabla_{X_i}\Lambda] \\ &= \sum_i[(\nabla_{X_i}\omega\wedge\alpha_i)\lrcorner\Lambda + (\omega\wedge\alpha_i)\lrcorner\nabla_{X_i}\Lambda] \\ &= (-1)^{|\omega|}(\sum_i(\alpha_i\wedge\nabla_{X_i}\omega)\lrcorner\Lambda + \sum_i(\alpha_i\wedge\omega)\lrcorner\nabla_{X_i}\Lambda). \end{aligned}$$

The conclusion thus follows from the following

Lemma 3.10. *For any* $\omega \in \Gamma(\wedge^{|\omega|} A^*)$,

$$d\omega = \sum_i \alpha_i \wedge \nabla_{X_i} \omega.$$

Proof. Define an operator $\delta : \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^{k+1} A^*)$, for all $0 \leq k \leq n$, by

$$\delta\omega = \sum_i \alpha_i \wedge \nabla_{X_i} \omega.$$

It is simple to check that δ is a graded derivation with respect to the wedge product, i.e.,

$$\delta(\omega \wedge \theta) = \delta\omega \wedge \theta + (-1)^{|\omega|} \omega \wedge \delta\theta.$$

For any $f \in C^\infty(M)$,

$$\delta f = \sum_i \alpha_i \nabla_{X_i} f = \sum_i [a(X_i) f] \alpha_i = df.$$

For any $\theta \in \Gamma(A^*)$ and $X,\ Y \in \Gamma(A)$,

$$\begin{aligned}
&(\delta\theta)(X, Y) \\
&= \sum_i (\alpha_i \wedge \nabla_{X_i} \theta)(X, Y) \\
&= \sum_i [\alpha_i(X)(\nabla_{X_i}\theta)(Y) - \alpha_i(Y)(\nabla_{X_i}\theta)(X)] \\
&= \sum_i \alpha_i(X)(\nabla_{X_i}(\theta \cdot Y) - \theta \cdot \nabla_{X_i} Y) - \sum_i \alpha_i(Y)(\nabla_{X_i}(\theta \cdot X) - \theta \cdot \nabla_{X_i} X) \\
&= \sum_i \alpha_i(X)(a(X_i)(\theta \cdot Y) - \theta \cdot \nabla_{X_i} Y) - \sum_i \alpha_i(Y)(a(X_i)(\theta \cdot X) - \theta \cdot \nabla_{X_i} X) \\
&= a(X)(\theta \cdot Y) - \theta \cdot \nabla_X Y - a(Y)(\theta \cdot X) + \theta \cdot \nabla_Y X \\
&= a(X)(\theta \cdot Y) - a(Y)(\theta \cdot X) - \theta \cdot (\nabla_X Y - \nabla_Y X) \\
&= a(X)(\theta \cdot Y) - a(Y)(\theta \cdot X) - \theta \cdot [X, Y] \\
&= d\theta(X, Y).
\end{aligned}$$

Therefore, δ coincides with the exterior derivative d, since $\oplus_k \Gamma(\wedge^k A^*)$ is generated by $\Gamma(A^*)$ over the module $C^\infty(M)$. □

Remark. (1) Theorem 3.8 was proved by Koszul [17] for the case of the tangent bundle Lie algebroid TP. In fact, his result was the main motivation of the present work. However, Koszul used an indirect argument instead of using Eqs. (9) and (14). We will see more applications of these equations in the next section.

(2) A flat A-connection on a vector bundle E is also called a representation of the Lie algebroid by Mackenzie [20, 5].

We end this section by introducing the notion of generalized divergence. Let ∇ be a flat A-connection on $\wedge^n A$, and D its corresponding generating operator. For any section $X \in \Gamma(A)$, we use $div_\nabla X$ to denote the function DX. When $A = TP$ with the usual Lie algebroid structure and ∇ is the flat connection induced by a volume, DX is the divergence in the ordinary sense. So DX can indeed be considered as a generalized divergence.

The following proposition gives a simple geometric characterization for the divergence of a section $X \in \Gamma(A)$.

Proposition 3.11. *For any $X \in \Gamma(A)$ and $\Lambda \in \Gamma(\wedge^n A)$,*

$$L_X \Lambda - \nabla_X \Lambda = (div_\nabla X)\Lambda.$$

In other words, the function $div_\nabla X$ is the multiplier corresponding to the endomorphism $L_X - \nabla_X$ of the line bundle $\wedge^n A$.

Proof. Assume that $X = \omega \lrcorner \Lambda$ for some $\omega \in \Gamma(\wedge^{n-1} A^*)$. Then,

$$DX = -(-1)^{|\omega|}(d\omega \lrcorner \Lambda + \sum_{i=1}^{n} (\alpha_i \wedge \omega) \lrcorner \nabla_{X_i} \Lambda).$$

Now

$$\begin{aligned} \sum_{i=1}^{n} ((\alpha_i \wedge \omega) \lrcorner \nabla_{X_i} \Lambda)\Lambda &= \sum_{i=1}^{n} ((\alpha_i \wedge \omega) \lrcorner \Lambda)\nabla_{X_i} \Lambda \\ &= \sum_i (-1)^{|\omega|}((\omega \wedge \alpha_i) \lrcorner \Lambda)\nabla_{X_i} \Lambda \\ &= \sum_i (-1)^{|\omega|}(\alpha_i \lrcorner X)\nabla_{X_i} \Lambda \\ &= \sum_i (-1)^{|\omega|} X(\alpha_i)\nabla_{X_i} \Lambda \\ &= (-1)^{|\omega|}\nabla_X \Lambda. \end{aligned}$$

Let $\theta \in \Gamma(\wedge^n A^*)$ be the dual element of Λ. It follows from Eq. (11) that

$$[X, \Lambda] \lrcorner \theta = -\Lambda \lrcorner d(X \lrcorner \theta).$$

It is simple to see that $X \lrcorner \theta = (\omega \lrcorner \Lambda) \lrcorner \theta = (-1)^{|\omega|(n-|\omega|)}\omega$. Since $n = |\omega| - 1$, then $X \lrcorner \theta = (-1)^{|\omega|}\omega$, and $[X, \Lambda] \lrcorner \theta = -(-1)^{|\omega|} d\omega \lrcorner \Lambda$. Hence, $(DX)\Lambda = [X, \Lambda] - \nabla_X \Lambda = L_X \Lambda - \nabla_X \Lambda$. This concludes the proof of the proposition. □

4. Lie Algebroid Homology

Let A be a Lie algebroid, and ∇ a flat A-connection on the line bundle $\wedge^n A$. Let D be its corresponding generating operator and $\partial = (-1)^{n-k} D : \Gamma(\wedge^k A) \longrightarrow \Gamma(\wedge^{k-1} A)$ (the reason for choosing this sign in the definition of ∂ will become clear later (see Eq. (17))). Then $\partial^2 = 0$, and we obtain a chain complex. Let $H_*(A, \nabla)$ denote its homology:

$$H_*(A, \nabla) = ker\partial / Im\partial.$$

Since D is a derivation with respect to $[\cdot, \cdot]$, immediately we have

Proposition 4.1. *The Schouten bracket passes to the homology $H_*(A,\nabla)$.*

Since this homology depends on the choice of the connection ∇, it is natural to ask how $H_*(A,\nabla)$ changes according to the connection ∇.

Proposition 4.2. *Suppose that both $\tilde{D}$ and D are operators generating the Gerstenhaber bracket on $\mathcal{A}$. Then $D-\tilde{D} = i_\alpha$ for some $\alpha \in \Gamma(A^*)$. And $\tilde{D}^2 - D^2 = -i_{d\alpha}$. In particular, if $\tilde{D}^2 = D^2 = 0$, then $\alpha \in \Gamma(A^*)$ is closed.*

Proof. Let $\tilde{\nabla}$ and ∇ be the A-connections on $\wedge^n A$ corresponding to $\tilde{D}$ and D respectively. Then there exists $\alpha \in \Gamma(A^*)$ such that

$$\tilde{\nabla}_X s = \nabla_X s + <\alpha, X> s, \quad \forall s \in \Gamma(\wedge^n A).$$

It follows from a direct verification that $\tilde{D} = D - i_\alpha$. According to Proposition 3.1, we have $\tilde{D}^2 U - D^2 U = -(\tilde{R} - R) \,\lrcorner\, U$, where $\tilde{R}$ and R are the curvatures of $\tilde{\nabla}$ and ∇, respectively. Finally, it is routine to check that $\tilde{R} - R = d\alpha$. □

Definition 4.3. *A-connections ∇_1 and ∇_2 are said to be homotopic if they differ by an exact form in $\Gamma(A^*)$. Similarly two generating operators D_1 and D_2 are said to be homotopic if they differ by an exact form, i.e., $D_1 - D_2 = i_\alpha$ for some exact form $\alpha \in \Gamma(A^*)$.*

The following result is thus immediate.

Proposition 4.4. *Let ∇_1 and ∇_2 be two flat A-connections on the canonical line bundle $\wedge^n A$, and D_1 and D_2 their corresponding generating operators. If ∇_1 and ∇_2 are homotopic (or equivalently D_1 and D_2 are homotopic), then,*

$$H_*(A,\nabla_1) \cong H_*(A,\nabla_2). \tag{16}$$

Now let us assume that $\wedge^n A$ is a trivial bundle, so there exists a nowhere vanishing volume $\Lambda \in \Gamma(\wedge^n A)$. This volume induces a flat A-connection ∇_0 on $\wedge^n A$ simply by $(\nabla_0)_X \Lambda = 0$ for all $X \in \Gamma(A)$. Let D_0 be its corresponding generating operator. Note that Λ being horizontal is equivalent to the condition:

$$D_0 \Lambda = 0.$$

Suppose that Λ' is another nonvanishing volume, and ∇' its corresponding flat connection on $\wedge^n A$. Assume that $\Lambda' = f\Lambda$ for some positive $f \in C^\infty(M)$. Then, it is easy to see that

$$\nabla'_X s = (\nabla_0)_X s - < d\ln f, X > s.$$

In other words, their corresponding generating operators are homotopic .

Let us now fix such a volume $\Lambda \in \Gamma(\wedge^n A)$. Define a $*$-operator from $\Gamma(\wedge^k A^*)$ to $\Gamma(\wedge^{n-k} A)$ by

$$*\omega = \omega \,\lrcorner\, \Lambda.$$

Clearly $*$ is an isomorphism. The following proposition follows immediately from the definition of $*$.

Proposition 4.5. *The operator $\partial_0 = (-1)^{n-k} D_0$ equals to $- * \circ d \circ *^{-1}$. That is,*

$$\partial_0 = - * \circ d \circ *^{-1}. \tag{17}$$

Here d is the Lie algebroid cohomology differential (see Eq. (4)). Thus, as a consequence, we have

Theorem 4.6. *Let ∇_0 be an A-connection on $\wedge^n A$ which admits a global nowhere vanishing horizontal section $\Lambda \in \Gamma(\wedge^n A)$. Then*

$$H_*(A, \nabla_0) \cong H^{n-*}(A, \mathbb{R}).$$

Remark. From the discussion above we see that there is a family of Lie algebroid homologies parameterized by the first Lie algebroid cohomology. In the case that the line bundle $\wedge^n A$ is trivial, one of these homologies is isomorphic to the Lie algebroid cohomology with trivial coefficients, and the rest of them can be considered as Lie algebroid cohomology with twisted coefficients. In general, these are special cases of Lie algebroid cohomology with general coefficients in a line bundle (see [20, 5]).

We will discuss two special cases below.

Let $\mathfrak{g}$ be an n-dimensional Lie algebra. Then $\wedge^n \mathfrak{g}$ is one-dimensional, so it admits a trivial $\mathfrak{g}$-connection, which in turn induces a generating operator $D_0 : \wedge^* \mathfrak{g} \longrightarrow \wedge^{*-1}\mathfrak{g}$. On the other hand, there exists another standard operator $D : \wedge^* \mathfrak{g} \longrightarrow \wedge^{*-1}\mathfrak{g}$, namely the dual of the Lie algebra cohomology differential. In general, D and D_0 are different. In fact, it is easy to check that $D - D_0 = i_\alpha$, where α is the modular character of the Lie algebra. In particular, when $\mathfrak{g}$ is a unimodular Lie algebra, the Lie algebra homology is isomorphic to Lie algebra cohomology, a well-known result.

Another interesting case, which does not seem trivial, is when A is the cotangent Lie algebroid T^*P of a Poisson manifold P (see Eqs. (2) and (3)). In this case, $\Gamma(\wedge^k T^*P) = \Omega^k(P)$. There is a well known operator $D : \Omega^k(P) \longrightarrow \Omega^{k-1}(P)$ due to Koszul [17] and Brylinski [2], given by $D = [i_\pi, d]$. The corresponding homology is called Poisson homology, and is denoted by $H_*(P, \pi)$. It was shown in [17] that D indeed generates the Gerstenhaber bracket on $\Omega^*(P)$ induced from the cotangent Lie algebroid of P. Therefore, it corresponds to a flat Lie algebroid connection on $\wedge^n T^*P$. According to Eq. (14), this connection has the form:

$$\nabla_\theta \Omega = -\theta \wedge D\Omega = \theta \wedge d(\pi \lrcorner \Omega), \tag{18}$$

for any $\theta \in \Omega^1(P)$ and $\Omega \in \Omega^n(P)$. We note that a similar formula was also discovered independently, by Evens–Lu–Weinstein [5].

The Koszul–Brylinski operator D is intimately related to the so called modular class of the Poisson manifold, a classical analogue of the modular form of a von Neumann algebra, which was introduced recently by Weinstein [28], and independently by Brylinski and Zuckerman [3]. Let us briefly recall its definition below. For simplicity, we assume that P is orientable with a volume form Ω. The modular vector field ν_Ω is the vector field defined by

$$f \longrightarrow (L_{X_f}\Omega)/\Omega, \quad \forall f \in C^\infty(P).$$

It is easily shown that the above map satisfies the Leibniz rule, so it indeed defines a vector field on P. It can also be shown that ν_Ω preserves the Poisson structure, and in other words it is a Poisson vector field. When we change the volume Ω, the corresponding modular vector fields differ by a hamiltonian vector field. Therefore it defines an element in the first Poisson cohomology $H^1_\pi(P)$, which is called the modular class of the Poisson manifold. A Poisson manifold is called *unimodular* if its modular class vanishes. In fact, the modular class can be defined for any Poisson manifold by just replacing the volume form by a positive density. We refer the interested reader to [28] for more detail.

Now let P be an orientable Poisson manifold with volume form Ω, and let D_0 be its corresponding generating operator as in the observation preceding Proposition 4.5. The following result follows immediately from a direct verification.

Proposition 4.7. *Let D be the Koszul–Brylinski operator of a Poisson manifold P. Then $D - D_0 = i_{\nu_\Omega}$, where ν_Ω is the modular vector field corresponding to the volume Ω.*

As an immediate consequence, we have

Theorem 4.8. *If P is an orientable unimodular Poisson manifold, then*

$$H_*(P, \pi) \cong H_\pi^{n-*}(P).$$

In particular, this result holds for any symplectic manifold, which was first proved by Brylinski [2].

Remark. The above situation can be generalized to the case of triangular Lie bialgebroids. Let A be a Lie algebroid with anchor a. A triangular r-matrix is a section π in $\Gamma(\wedge^2 A)$ satisfying the condition $[\pi, \pi] = 0$. One may think that this is a sort of generalized “Poisson structure" on the generalized manifold A. In this case, A^* is equipped with a Lie algebroid structure with the anchor $a\circ\pi^\#$ and the Lie bracket as defined by an equation identical to the one defining the Lie bracket on one-forms of a Poisson manifold.

Similarly, $D = [i_\pi, d] : \Gamma(\wedge^k A^*) \longrightarrow \Gamma(\wedge^{k-1} A^*)$ is an operator of square zero and generates the Gerstenhaber bracket $[\cdot, \cdot]$ on $\oplus_k \Gamma(\wedge^k A^*)$. A form of top degree $\Omega \in \Gamma(\wedge^n A^*)$ satisfies the condition $D\Omega = 0$ iff $\pi \lrcorner \Omega \in \Gamma(\wedge^{n-2} A^*)$ is closed. If there exists such a nowhere vanishing form, the homology $H_*(A, \nabla)$ is then isomorphic to the cohomology $H^{n-*}(A, \mathbb{R})$.

5. Discussions

We end this paper by a list of open questions.

Question 1. In the above remark, is the condition that $\pi \lrcorner \Omega \in \Gamma(\wedge^{n-2} A^*)$ is closed equivalent to the Lie algebroid A^* being unimodular?

Question 2. For a general Lie algebroid A, does there exist a canonical generating operator corresponding to the modular class of the Lie algebroid in analogue to the case of cotangent Lie algebroid of a Poisson manifold (see Proposition 4.7)?

Question 3. For a Poisson manifold P, there is a family of the homologies parameterized by the first Poisson cohomology $H^1_\pi(P)$. What is the meaning of the rest of the homologies besides the Poisson homology?

Question 4. Suppose that (A, A^*) is a Lie bialgebroid and ∇ a flat A-connection on $\wedge^n A$. Then $(\Gamma(\wedge^* A), \wedge, d_*, [,], D)$ is a strong differential BV-algebra. It is clear that $d_* D + D d_*$ is a derivation with respect to both $\wedge$ and $[,]$. When is $d_* D + D d_*$ inner and in particular, when is $d_* D + D d_* = 0$?

For the Lie bialgebroid (T^*P, TP) of a Poisson manifold, we may take the connection ∇ as in Eq. (18). Then d_* is the usual de-Rham differential and D is the Koszul–Brylinski operator. Thus, $d_* D + D d_*$ is automatically zero, which gives rise to the Brylinski double complex [2]. On the other hand, if we switch the order and consider the Lie bialgebroid (TP, T^*P) for a Poisson manifold P with a volume, then $\mathcal{A} = \oplus_k \Gamma(\wedge^k A)$ is the space of multivector fields. In this case, $d_* = [\pi, \cdot]$ is the Lichnerowicz Poisson cohomology differential, and $D = -(-1)^{n-k} * \circ d \circ *^{-1}$. Here $*$ is the isomorphism between the space of multivector fields and that of differential forms induced by the volume element. Then $d_* D + D d_* = L_X$, where X is the modular vector field of the Poisson manifold (see p. 265 of [17]). So it vanishes iff P is unimodular.

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