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On the Incompressible Fluid Limit and the Vortex Motion Law of the Nonlinear Schrödinger Equation

F.-H. Lin[1], **J. X. Xin**[2]

[1] Courant Institute, New York University, 251 Mercer Street, New York, NY 10012, USA
[2] Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

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Abstract: The nonlinear Schrödinger equation (NLS) has been a fundamental model for understanding vortex motion in superfluids. The vortex motion law has been formally derived on various physical grounds and has been around for almost half a century. We study the nonlinear Schrödinger equation in the incompressible fluid limit on a bounded domain with Dirichlet or Neumann boundary condition. The initial condition contains any finite number of degree ± 1 vortices. We prove that the NLS linear momentum weakly converges to a solution of the incompressible Euler equation away from the vortices. If the initial NLS energy is almost minimizing, we show that the vortex motion obeys the classical Kirchhoff law for fluid point vortices. Similar results hold for the entire plane and periodic cases, and a related complex Ginzburg–Landau equation. We treat as well the semi-classical (WKB) limit of NLS in the presence of vortices. In this limit, sound waves propagate through steady vortices.

1. Introduction

We study the two dimensional nonlinear Schrödinger (NLS) equation:

$$iu_{\epsilon,t} = \Delta_x u_\epsilon + \epsilon^{-2}(1 - |u_\epsilon|^2)u_\epsilon, \quad x \in \Omega, \tag{1.1}$$

where $u_\epsilon = u_\epsilon(t, x)$ is a complex valued function defined for each $t > 0$; ϵ a small positive parameter; $x = (x_1, x_2) \in \Omega$, a simply connected bounded domain with smooth boundary in $\mathbb{R}^2$; $\Delta = \partial_{x_1x_1} + \partial_{x_2x_2}$ denotes the two-dimensional Laplacian. The NLS (1.1) has been proposed and studied as the fundamental equation for understanding superfluids, see Ginzburg and Pitaevskii [14], Landau and Lifschitz [19], Donnelly [9], Frisch, Pomeau and Rica [13], Josserand and Pomeau [18], and many others.

We shall consider (1.1) with the prescribed Dirichlet boundary condition:

$$u|_{\partial\Omega} = g(x), \; |g| = 1, \; \deg(g, \partial\Omega) = \pm n, \tag{1.2}$$

where n is a given positive integer, and the zero Neumann boundary condition:

$$u_\nu|_{\partial\Omega} = 0, \tag{1.3}$$

ν the normal direction. Our method is general enough that we can handle the entire plane case ($\Omega = \mathbb{R}^2$) and the periodic case too.

We will see that as $\epsilon \downarrow 0$, the Dirichlet boundary condition corresponds to applying a tangential force at the boundary so that the tangential fluid velocity is $g \wedge g_\tau$, τ the tangential unit direction. The Neumann boundary condition corresponds to zero normal fluid velocity (no fluid penetration) at the boundary. For ease of presentation, we shall work with the Dirichlet case first, then comment on all necessary modifications in the proof to reach a similar conclusion for the Neumann case. Subsequently, we also remark on the entire plane and periodic cases.

The NLS (1.1) preserves the total energy:

$$E_\epsilon(u_\epsilon) = \int_\Omega e_\epsilon(u_\epsilon) \equiv \int_\Omega \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2}, \tag{1.4}$$

and admits vortices in solutions, which are points where $|u_\epsilon|$ becomes zero and the phase of u_ϵ or $\frac{u_\epsilon}{|u_\epsilon|}$ has singularities. These points are the locations of regular fluids, which are surrounded by superfluids. If there are n degree one point vortices in the solution, the energy $E_\epsilon(u_\epsilon)$ has the asymptotic expression:

$$E_\epsilon(u_\epsilon)(t) = E_\epsilon(u_\epsilon)(0) = n\pi \log \frac{1}{\epsilon} + O(1). \tag{1.5}$$

So we shall consider initial data $u_\epsilon(0,x) = u^0_\epsilon(x)$ with n degree one vortices, and belonging to $H^2(\Omega)$ for each $\epsilon > 0$ so that (1.5) holds. With initial and boundary data (1.5) and (1.2), it is well-known [3] that the defocusing NLS (1.1) is globally well-posed in $C(R^+, H^2) \cap C^1(R^+, L^2)$ for each $\epsilon > 0$. Our goal is to analyze the limiting behavior of solutions as $\epsilon \downarrow 0$.

The systematic matched asymptotic derivation of the limiting vortex motion law was carried out by Neu [28] for $\Omega = \mathbb{R}^2$. The motion law is the classical Kirchhoff law for fluid point vortices [1], and was known to Onsager [30] in 1949. The connection between Schrödinger equations and the classical fluid mechanics was already noted in 1927 by Madelung [26], which applies to NLS (1.1) as well. Along this line, there have been over the years many formal derivations of Kirchhoff law based on Madelung's fluid mechanical formulation, see Creswick and Morrison [7], Ercolani and Montgomery [11], among others. Madelung's idea was to identify $|u|^2$ as the fluid density ρ, and $\nabla\theta = \nabla \arg u$, as the fluid velocity v. Then he defined the linear momentum $p = \rho\nabla\theta$. In the new variables (ρ, v), the NLS (1.1) becomes:

$$\rho_t - 2\nabla \cdot p = 0, \tag{1.6}$$

$$p_t - 2\nabla \cdot (\rho v \otimes v) = -\nabla P(\rho) - \frac{1}{2}\nabla \cdot (\rho \mathrm{Hess}(\log \rho)), \tag{1.7}$$

where $P = \frac{1}{2\epsilon^2}(1-\rho^2)$ is the pressure, and Hess denotes the Hessian. Madelung's formulation of course relies on the assumption that the amplitude of u is not zero and the phase θ is not singular, otherwise the transform is not well-defined and (1.6)–(1.7) gets singular even though NLS itself is still regular. When we are studying solutions with vortices, this singular case is however just what we have to deal with, and so an alternative intepretation of the fluid formalism related to but different from Madelung's transform must be used instead. In view of the energy functional (1.4), ρ is close to one almost everywhere as $\epsilon \downarrow 0$, and (1.6) implies formally that $\nabla \cdot v = 0$, provided v converges. Hence the limiting problem we are considering is an incompressible fluid limit involving vortices. We also see that the Neumann boundary condition (1.3) says that $\theta_\nu = v \cdot \nu = 0$, if we write $u = \rho^{1/2} e^{i\theta}$ and assume that vortices are away from the boundary (so $\rho \sim 1$). Hence (1.3) reduces to the zero normal velocity boundary condition for ideal classical fluids.

Let us mention that a modified Madelung's transform has been utilized in the study of the semi-classical limit (WKB limit) of NLS:

$$iu_t^\epsilon = \epsilon \Delta_x u^\epsilon + \epsilon^{-1} |u^\epsilon|^2 u^\epsilon, \tag{1.8}$$

with initial data: $u(0, x) = a_0(x) e^{iS_0(x)/\epsilon}$. Grenier [15] showed in particular that for a_0 and S_0 in $H^s(\mathbb{R}^d)$, $s > 2 + d/2$, solutions u^ϵ exist on a small time interval $[0, T]$, T independent of ϵ. Moreover, $u^\epsilon = a(t, x, \epsilon) e^{iS(t,x,\epsilon)/\epsilon}$, with a and S in $L^\infty([0, T]; H^s)$ uniformly in ϵ, and $(\rho, \nabla S)$ converge to the solution (ρ, v) of the isentropic compressible Euler equation:

$$\begin{aligned} &\rho_t + \nabla \cdot (\rho v) = 0, \\ &v_t + \nabla \left(\frac{|v|^2}{2} + \rho\right) = 0. \end{aligned} \tag{1.9}$$

In one space dimension, using integrable machinery, Jin, Levermore and McLaughlin [17] obtained the above convergence results globally in time. These works on the compressible fluid limit treated only the regime of smooth phase functions, and there are no vortices involved.

Since the formation of vortices, their motion, and the resulting drag force are of tremendous physical significance in superfluids, [13, 18], it has been a longstanding fundamental problem to understand how to rigorously pass to the classical fluid limit in the presence of vortices.

Our approach begins with writing the conservation laws of NLS in the form of fluid dynamic representation. However, in contrast to all earlier applications of the Madelung transform, we avoid making explicit use of the phase variable θ and do not work with (1.6)–(1.7). The conservation laws of NLS are put into the form:

- Conservation of mass:

$$\partial_t |u^\epsilon|^2 = 2 \nabla \cdot p(u_\epsilon), \tag{1.10}$$

  where in vector notation $p(u_\epsilon) = u_\epsilon \wedge \nabla u_\epsilon$, the linear momentum.
- Conservation of linear momentum:

$$\partial_t p(u_\epsilon) = 2 \operatorname{div} (\nabla u_\epsilon \otimes \nabla u_\epsilon) - \nabla P_\epsilon, \tag{1.11}$$

  where:

$$P_\epsilon = |\nabla u_\epsilon|^2 + u_\epsilon \cdot \Delta u_\epsilon - \frac{|u_\epsilon|^4 - 1}{2\epsilon^2}, \tag{1.12}$$

  is the pressure.
- Conservation of energy:

$$\partial_t e_\epsilon(u_\epsilon) = \operatorname{div} (u_{\epsilon,t} \nabla u_\epsilon). \tag{1.13}$$

Then we study convergence of various terms in (1.10)–(1.11) using the above three conservation laws (in particular the projection of (1.11) onto divergence free fields), and perform various circulation calculations involving the linear momentum p and its first moments. We show that vortices do not move on the slower time scale $t \sim O(\lambda_\epsilon)$, $\lambda_\epsilon \to 0$ as $\epsilon \to 0$, and they move continuously on the scale $t \sim O(1)$. With precise characterization of weak limits of linear momentum p, we are able to show that p converges locally in space to v, the solution of the two-dimensional incompressible Euler equation away from the n continuously moving point vortices, and moreover, v is curl-free. That v is curl-free away from vortices agrees with the physical picture that superfluids are potential flows [19]. Finally, the motion law of point vortices (the Kirchhoff law) follows from the limiting linear momentum equation. Our main results are:

Theorem 1.1 (*Weak convergence and fluid limit***).** *Let us consider NLS (1.1) with Dirichlet boundary condition (1.2), and initial energy (1.5) with n degree $n_j = \pm 1$ vortices. Then as $\epsilon \downarrow 0$, the energy density $e_\epsilon(u_\epsilon)$ concentrates as Radon measures in $\mathcal{M}(\Omega)$ for any fixed time $t \geq 0$:*

$$\frac{e(u_\epsilon)dx}{\pi n \log \frac{1}{\epsilon}} \rightharpoonup \sum_{j=1}^{n} \delta_{a_j(t)},$$

and vortices of u_ϵ converge to $a_j(t)$ moving continuously in time of $t \sim O(1)$ (or $t \in [0,T]$, T any fixed constant) as $\epsilon \downarrow 0$. Vortices of u_ϵ do not move on any slower time scale $t \sim O(\lambda_\epsilon) = o(1)$ (or $t = \lambda_\epsilon \tau$, $\tau \in [0,T]$, T any fixed positive constant, and $\lambda_\epsilon \to 0$) as $\epsilon \downarrow 0$. Moreover on the time scale $t \sim O(1)$, the linear momentum $p(u_\epsilon)$ converges weakly in $L^1([0,T]; L^1_{loc}(\Omega_a))$ to a solution v of the incompressible Euler equation:

$$v_t = 2v \cdot \nabla v - 2\nabla P, \quad \operatorname{div} v = 0, \quad x \in \Omega_a \equiv \{\Omega \backslash (a_1(t), \cdots, a_n(t))\}$$

with boundary condition: $v \cdot \tau = g \wedge g_\tau$, τ the unit tangential vector on $\partial\Omega$. The function v is precisely characterized as:

$$v = \nabla(\Theta_a + h_a),$$

where

$$\Theta_a = \sum_{j=1}^{n} \arg \left(\frac{x - a_j(t)}{|x - a_j(t)|} \right)^{n_j},$$

and h_a is harmonic on Ω satisfying the boundary condition: $h_{a,\tau} = -\Theta_{a,\tau} + g \wedge g_\tau$, on $\partial\Omega$. So h is unique up to an additive constant. The total pressure $2P$ is a single-valued function on Ω, and is smooth on Ω_a. The quadratic tensor product weakly converges as:

$$\nabla u_\epsilon \otimes \nabla u_\epsilon \rightharpoonup v \otimes v + \mu, \qquad \mathcal{M}(\Omega_a), \tag{1.14}$$

where μ is a symmetric tensorial Radon defect measure of finite mass over Ω; and $\operatorname{div}(\mu) = \nabla P_\mu$ *on Ω_a, where P_μ is a well-defined distribution function on Ω_a.*

Theorem 1.2 (*Vortex motion law***).** *Consider the same assumptions as in Theorem 1.1, and in addition assume that the initial NLS energy is almost minimizing, namely*

$$E_\epsilon(u_\epsilon)(0) = n\pi \log \frac{1}{\epsilon} + \pi W(a(0)) + o(1),$$

as ϵ goes to zero. Let $H_j = H_j(a)$, $a = (a_1, \cdots, a_n)$, denote the smooth part of $\Theta_a + h_a$ near each vortex, and define the renormalized energy function as:

$$\nabla_{a_j} W(a) = 2n_j \left(-\frac{\partial H_j}{\partial x_2}(a_j), \frac{\partial H_j}{\partial x_1}(a_j) \right),$$

$j = 1, \cdots, n$. The vortex motion obeys the classical Kirchhoff law:

$$a_j'(t) = n_j J \nabla_{a_j} W(a) = -2\nabla H_j(a),$$

$j = 1, \cdots, n$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$W(a) = -\sum_{l \neq j} n_l n_j \log |a_l - a_j| + \text{boundary contributions}.$$

We remark that the total initial NLS energy $E_\epsilon(u_\epsilon)$ in (1.5) can be decomposed into a sum of three parts: the vortex self-energy $n\pi \log \frac{1}{\epsilon}$, the Kirchhoff energy $\pi W(a(0))$, and the remaining $O(1)$ excessive energy in general. The Kirchhoff energy facilitates the vortex motion. The remaining energy creates the defect measure μ. The total pressure consists of the contribution from the original NLS pressure and the contribution from the defect measure (the defect pressure). If the excessive energy is absent, or in other words the initial energy satisfies:

$$E_\epsilon(u_\epsilon)(0) = n\pi \log \frac{1}{\epsilon} + \pi W(a(0)) + o(1), \tag{1.15}$$

which also means that u_ϵ is almost energy minimizing for the given vortex locations, the linear momentum $p(u_\epsilon)$ converges strongly in $L^1([0, T]; L^1_{loc}(\Omega_a))$ and the defect measure $\mu = 0$. In general, with $O(1)$ excessive energy, to prove the same motion law requires further information on μ; either that the divergence of the defect measure μ is a gradient of a distribution on the entire domain Ω (i.e. is globally curl-free as a distribution) or that the support of μ is away from the vicinities of vortex locations. Physically the excessive energy is carried by sound waves (time dependent phase waves), see the discussion of the WKB limit in Sect. 7. It is conceivable that vortices still move according to Kirchhoff law when sound waves have propagated away from them, either absorbed by the vortex cores or the physical boundary. Otherwise, sound waves may modify the motion of vortices by creating oscillations, [13]. It is very interesting to understand the vortex sound interaction (Nore et al. [29]) in terms of the structure of the defect measure μ based on our results here.

Due to the local nature of our method, we are able to prove the same theorems for the zero Neumann case (1.3), with the modification that the boundary condition is instead $v \cdot \nu = 0$, and $h_{a,\nu} = -\Theta_{a,\nu}$. Similar results are established for the entire plane and the periodic cases, as long as the sum of vortex degrees is zero and the total energy obeys (1.5). Our results on the Dirichlet and Neumann cases easily extend to the situation where there are $2k + n$ vortices in a bounded domain, $n + k$ being of degree +1, and k of degree -1. Due to the possibility of finite time vortex collisions in Kirchhoff law in the case of signed vortices [27], the results are meant for any time before any two vortices come together.

It is remarkable that NLS vortices obey the Kirchhoff law in the incompressible fluid limit, considering that the ± 1 vortices are only known to be dynamically marginally stable in the spectral sense, see Weinstein and Xin [32]. For this reason, it seems impossible to prove the validity of the motion law for the above mentioned initial and boundary conditions by attempting to justify the matched asymptotic derivation of Neu [28] which relied on linearization about vortices. The fluid dynamic approach developed here has been extended by the authors [25] to establish the vortex motion laws of the analogous nonlinear wave (NLW) equation, and the nonlinear heat (NLH) equation. In NLW and NLH, Euler-like equations also appear and lead to the motion laws. Under a similar energy almost minimizing assumption (1.15), the NLW vortex motion law is: $a_j'' = -n_j \nabla_{a_j} W$, on the time scale $t \sim O(\log^{\frac{1}{2}} \frac{1}{\epsilon})$.

During the preparation of this paper, we learned of Colliander and Jerrard [5] on the periodic case of NLS. They showed the motion law under the energy almost minimizing assumption, however, did not study the defect measure and the general fluid limit.

The rest of the paper is organized as follows. In Sect. 2, we state and prove energy concentration, and show its direct consequences on convergence of linear momentum away from vortices and basic energy type bounds. In Sect. 3, we study mobility and continuity of vortex locations based on linear momentum equation and subsequently refine the form of weak limit of solutions based on conservation of mass. We also prove a key energy estimate which is used later to control the defect measure. In Sect. 4, we show using all results in previous sections that the NLS linear momentum converges to a solution of the two dimensional incompressible Euler equation away from vortices. The Kirchhoff law then follows from the limiting linear momentum equation under the energy minimizing assumption. In Sect. 5, we comment on all necessary modifications to establish the similar results for the zero Neumann case, as well as the entire plane and periodic cases. In Sect. 6, we apply our method to show the vortex motion law for a related complex Ginzburg–Landau (CGL) equation. Besides the interest of CGL vortices in its own right, this result provides another proof of NLS vortex motion law by passing the CGL to NLS limit. In Sect. 7, we study the semi-classical (WKB) limit of NLS. Due to the slow time scale $O(\epsilon)$, vortices do not move, and the regular part of the phase function of the solution satisfies the linear wave equation, indicating the propagation of sound waves through vortices.

2. Energy Concentration and Basic Weak Limits

In this section, we present weak convergence results on two basic physical quantities: the energy $e(u_\epsilon)$ and the linear momentum $p(u_\epsilon)$. Consequently, we deduce the weak convergence of the curl of $p(u_\epsilon)$. The one half curl of $p(u_\epsilon)$ is equal to the Jacobian of the map u_ϵ, hence it will be denoted by $J_{ac}(u_\epsilon)$, and it is also known as vorticity. All the results follow from energy concentration and energy comparisons, and are independent of dynamics.

Lemma 2.1. *Suppose u_{ϵ_k} is a sequence of H^1-maps from Ω into $\mathbb{C}$ (the complex plane) satisfying the Dirichlet boundary condition $u_{\epsilon_k}|_{\partial\Omega} = g$. Suppose also that for a positive ϵ_k independent constant C_0 the energy satisfies:*

$$E_{\epsilon_k}(u_{\epsilon_k}) = \int_\Omega e_{\epsilon_k}(u_{\epsilon_k}) \equiv \int_\Omega \frac{1}{2}|\nabla u_{\epsilon_k}|^2 + \frac{(1-|u_{\epsilon_k}|^2)^2}{4\epsilon_k^2} \leq \pi n \log \frac{1}{\epsilon_k} + C_0 .$$

Then taking a subsequence in ϵ_k if necessary, we have as $\epsilon = \epsilon_k \downarrow 0$ that

$$\frac{e_\epsilon(u_\epsilon)dx}{\pi n \log \frac{1}{\epsilon}} \rightharpoonup \sum_{j=1}^{n} \delta_{a_j}, \tag{2.1}$$

as Radon measures. Moreover,

$$\min\{|a_l - a_j|,\ \text{dist}(a_l, \partial\Omega),\ l, j = 1, \cdots, n,\ l \neq j\} \geq \delta_0(g, \Omega, C_0) > 0.$$

Proof. This lemma is same as Proposition 1 of Lin [23], where the earlier structure theorem of Lin [20] (Theorem 2.4) is extended to show that there are small positive numbers ϵ_0 and α_0 such that for $\epsilon_k \in (0, \epsilon_0)$, there are n distinct balls B_j's with radii $\epsilon_k^{\alpha_j}$, $\alpha_j \in [\alpha_0, 1/2]$, which contain vortices of degrees ± 1. In other words, vortex locations are known up to an error of $O(\epsilon_k^{\alpha_j})$. □

Lemma 2.2. *Under the assumptions of Lemma 2.1, we have up to a subsequence if necessary:*

$$u_\epsilon \rightharpoonup \prod_{j=1}^{n} \left(\frac{x - a_j}{|x - a_j|} \right)^{n_j} e^{ih_a(x)} \equiv u_a, \tag{2.2}$$

$n_j = \pm 1$, *weakly in* $H^1_{loc}(\overline{\Omega} \backslash \{a_1, \cdots, a_n\}) \equiv H^1_{loc}(\Omega_a)$ *for some* $h_a \in H^1(\Omega)$. *Moreover,*

$$\int_\Omega |\nabla h_a|^2 \leq C_1, \tag{2.3}$$

$$\int_\Omega \frac{(1 - |u_\epsilon|^2)^2}{\epsilon^2} \leq C_1, \tag{2.4}$$

$$\int_\Omega |\nabla |u_\epsilon||^2 \leq C_1, \tag{2.5}$$

for a positive constant C_1, *uniformly in* ϵ.

Proof. These results follow from energy comparisons. For the weak convergence (2.2) and inequality (2.3), see the general convergence theorem of [20] and also Proposition 2 of [23]. The inequality (2.4) is shown in Lecture 1 of [21]. For (2.5), we use the fact that $\nabla|u_\epsilon| = 0$, $a.e.$ on the set $\{x \in \Omega : |u_\epsilon| = 0\}$, and write $u_\epsilon = |u_\epsilon| e^{iH_\epsilon}$ whenever $|u_\epsilon| \neq 0$. Substituting this expression into the total energy, which is uniformly bounded away from the set $\{x \in \Omega : |u_\epsilon| = 0\}$, gives (2.5). Intuitively, the singular part of energy that contributes to $n\pi \log \frac{1}{\epsilon}$ comes from the singular part of the phase of u^ϵ (the sum of vortex phases). The above three inequalities are valid since they either involve only the amplitude $|u^\epsilon|$ or the regular part of the phase h_a. □

Remark 2.1. Under the same assumptions as in Lemma 2.1, the renormalized energy is defined as (γ a universal constant):

$$W = W(a_1, \cdots, a_n) = \lim_{r \downarrow 0} \left[\frac{1}{2\pi} \int_{\Omega \backslash \bigcup_{j=1}^{n} B_r(a_j)} |\nabla u_a|^2 - n \log 1/r \right] + \gamma n, \tag{2.6}$$

see Bethuel, Brezis and Hélein [2]. Here u_a is a harmonic map of the form (2.2). The W function has the properties that: $W \to +\infty$ if some a_j reaches the boundary $\partial\Omega$ or $a_j = a_l$ for some $j \neq l$; otherwise, it is locally analytic in a. Due to γn, $W(a)$ is also local energy minimizing.

Lemma 2.3. *Under the same assumptions as Lemma 2.1, the linear momentum $p(u_\epsilon)$ is uniformly bounded in $L^1_{loc}(\Omega_a)$, and up to a subsequence if necessary:*

$$p(u_\epsilon) \rightharpoonup v = \nabla\Theta_a + \nabla h_a, \tag{2.7}$$

in $L^1_{loc}(\Omega_a)$, where

$$\Theta_a = \sum_{j=1}^{n} \arg\left(\frac{x-a_j}{|x-a_j|}\right)^{n_j}. \tag{2.8}$$

Moreover,

$$2J_{ac}(u_\epsilon)\,dx = \text{curl}\,(p(u_\epsilon))\,dx \rightharpoonup 0, \tag{2.9}$$

in the sense of bounded measures $\mathcal{M}(\Omega_a)$.

Proof. We see from Lemmas 2.1 and 2.2 that $p(u_\epsilon)$ is uniformly bounded in L^1 away from vortices $\{a_1, \cdots, a_n\}$. Since ∇u_ϵ is weakly compact in $H^1(\Omega_a)$, and u_ϵ compact in $L^2(\Omega_a)$, we have:

$$p(u_\epsilon) = u_\epsilon \wedge \nabla u_\epsilon \rightharpoonup v = \nabla\Theta_a + \nabla h_a,$$

in $L^1_{loc}(\Omega_a)$. Noticing that v is a gradient of an H^1 function, we have by taking the curl of $p(u_\epsilon)$ and the weak continuity of Jacobians with respect to H^1 weak convergence that

$$2J_{ac}(u_\epsilon)\,dx \;=\; \text{curl}\,p(u_\epsilon)\,dx \;\rightharpoonup 0, \tag{2.10}$$

in $\mathcal{M}(\Omega_a)$. Note that $J_{ac}(u_\epsilon) \in L^1_{loc}(\Omega_a)$. The proof is complete. $\square$

Lemma 2.4. *The linear momentum $p(u_\epsilon) \in L^1(\Omega)$ uniformly in ϵ. Let $\varphi \in C_0^\infty(\Omega)$, $\varphi = x_1$ for $x \in B_{R/2}(a_j)$, $\varphi = 0$, for $x \notin B_R(a_j)$, where $R \in (0, \delta_0)$. Then we have with $a_j = (\xi_j, \eta_j)$:*

$$\int_{B_R(a_j)} \nabla^\perp\varphi \cdot p(u_\epsilon) \to 2\pi\xi_j. \tag{2.11}$$

A similar convergence holds with x_2 in place of x_1, η_j in place of ξ_j.

Proof. The integral in (2.11) is the projection of the linear momentum onto a divergence free field. We have from Lemma 2.2 that $|u_\epsilon| \in H^1(\Omega)$, uniformly in ϵ. Hence $|u_\epsilon| \in L^q(\Omega)$, uniformly in ϵ, for any $q < \infty$ by the Gagliardo–Nirenberg inequality. We shall establish that $\nabla u^\epsilon \in L^{p'}(\Omega)$, uniformly in ϵ, for $p' \in [1, 2)$. Given this fact, $p(u_\epsilon) = u_\epsilon \wedge \nabla u_\epsilon \in L^r(\Omega)$, uniformly in ϵ for any $r \in [1, 2)$. This and Lemma 2.3 imply that:

$$\begin{aligned}\int_{B_R} \nabla^\perp\varphi \cdot p(u_\epsilon) &\to \int_{B_R} \nabla^\perp\varphi \cdot (\nabla\Theta_a + h_a)\\ &= \int_{B_R} \nabla^\perp\varphi \cdot \nabla\theta_j\\ &= \int_{B_{\epsilon'}(a_j)} \nabla^\perp\varphi \cdot \nabla\theta_j + \int_{\partial B_{\epsilon'}} x_1 \partial_\tau \theta_j,\end{aligned}$$

where $B_{\epsilon'}$ is a small ball of radius ϵ' about a_j, and ∂_τ is the tangential derivative. The first integral clearly goes to zero as $\epsilon' \to 0$, and the second integral goes to $2\pi\xi_j$ by a direct calculation. The convergence (2.11) follows.

Now we show that $\nabla u^\epsilon \in L^{p'}(\Omega)$, uniformly in ϵ, for $p' \in [1, 2)$, by an energy argument. It is sufficient to consider a finite neighborhood of a single, say plus one, vortex. Without loss of generality, we can assume that the essential zero of u^ϵ is inside $B(0, \epsilon^\alpha)$, for some $\alpha \in (1/4, 1/2)$, and that $B(0, 1)$ is inside Ω and contains the essential zero. We have then from Lin [20]:

$$\begin{aligned} &E_\epsilon(u_\epsilon, B(0, 1)) \le \pi \log \frac{1}{\epsilon} + C_1, \\ &\epsilon^\alpha \int_{\partial B(0,\epsilon^\alpha)} e_\epsilon(u_\epsilon) \le C_2(\alpha, C_1), \\ &deg(u_\epsilon/|u_\epsilon|, \partial B(0, \epsilon^\alpha)) = 1. \end{aligned} \tag{2.12}$$

It follows from (2.12) that there exists a $\theta_\epsilon \in (1/4, 1/2)$, and a constant $\epsilon_0(C_1)$ such that if $\epsilon \le \epsilon_0(C_1)$:

$$\int_{B(0,1)\setminus B(0,\theta_\epsilon)} e_\epsilon(u_\epsilon) \ge \pi \log \frac{1}{\theta_\epsilon} - C_0\epsilon. \tag{2.13}$$

In fact, there exists $\theta_\epsilon \in (1/4, 1/2)$ such that $u_\epsilon \rightharpoonup e^{i(\Theta+h)}$, in $H^1_{loc}(B(0, 1)\setminus 0)$; $u_\epsilon \rightharpoonup e^{i(\Theta+h)}$ in $H^1(\partial B(0, \theta_\epsilon))$; $\theta_\epsilon \int_{\partial B(0,\theta_\epsilon)} e_\epsilon(u_\epsilon) \le C(C_1)$. So $\int_{B(0,1)\setminus B(0,\theta_\epsilon)} e_\epsilon(u_\epsilon) \le C$. Now as in Lin [20], replace u_ϵ by the minimizer $\tilde{u}_\epsilon$ of the energy $\int_{B(0,1)\setminus B(0,\theta_\epsilon)} e_\epsilon(u_\epsilon)$ subject to the Dirichlet boundary condition $\tilde{u}_\epsilon = u_\epsilon$, on $\partial B(0, \theta_\epsilon)$, and zero Neumann on $\partial B(0, 1)$. Such a minimizer satisfies $|\tilde{u}_\epsilon| \ge 1/2$ on $B(0, 1)\setminus B(0, \theta_\epsilon)$ and that:

$$\int_{B(0,1)\setminus B(0,\theta_\epsilon)} e_\epsilon(\tilde{u}_\epsilon) \ge \pi \log \frac{1}{\theta_\epsilon} - C_0\epsilon, \tag{2.14}$$

proving (2.13).

Combining (2.13) and (2.12), we have:

$$\int_{B(0,\theta_\epsilon)} e_\epsilon(u_\epsilon) \le \pi \log \frac{\theta_\epsilon}{\epsilon} + C_1 + C_0\epsilon. \tag{2.15}$$

Now we iterate (2.15) to a sequence of balls $B(0, r_\epsilon^{(n)})$, $r_\epsilon^{(n)} = \theta_\epsilon^{(1)} \cdots \theta_\epsilon^{(n-1)}$, $\theta_\epsilon^{(1)} = \theta_\epsilon$, and $\theta_\epsilon^{(j)}$'s $\in (1/4, 1/2)$, $n = 1, 2, \cdots, N$, where N is such that $r_\epsilon^{(N)} \ge 2\epsilon^\alpha$. At each n, the lower energy bound on the annuli becomes:

$$\int_{B(0,r_\epsilon^{(n)})\setminus B(0,r_\epsilon^{(n+1)})} e_\epsilon(u_\epsilon) \ge \pi \log \frac{1}{\theta_\epsilon^{(n+1)}} - C_0 \frac{\epsilon}{r_\epsilon^{(n)}}, \tag{2.16}$$

and the upper bound is:

$$\int_{B(0,r_\epsilon^{(n)})} e_\epsilon(u_\epsilon) \le \pi \log \frac{r_\epsilon^{(n)}}{\epsilon} + C_1 + \epsilon C_0(1 + \sum_{j=1}^{n} 1/r_\epsilon^{(j)}). \tag{2.17}$$

The sum of the second term in (2.17) is bounded by a geometric sum from above since $\theta_\epsilon^{(j)} \in (1/4, 1/2)$, and its upper bound is const. $\epsilon^{-\alpha}$. Hence the energy upper bound finally is:

$$\int_{B(0,r)} e_\epsilon(u_\epsilon) \leq \pi \log \frac{r}{\epsilon} + C_1 + C_3\epsilon^{1-\alpha} \leq \pi \log \frac{r}{\epsilon} + C_1 + 2C_0, \tag{2.18}$$

for small ϵ, and $r \in (2\epsilon^\alpha, 1)$.

With a similar argument via the energy minimizer, we also have:

$$\int_{B(0,r')} e_\epsilon(u_\epsilon) \geq \pi \log \frac{r'}{\epsilon} - C_4, \tag{2.19}$$

for any $r' \in (2\epsilon^\alpha, 1)$. Combining (2.18) and (2.19), we infer that for $r \geq 2\epsilon^\alpha$:

$$\int_{B(0,2r)\setminus B(0,r)} e_\epsilon(u_\epsilon) \leq C_5. \tag{2.20}$$

Now we bound for any $p' \in [1,2)$ ($2^{N+1}\epsilon^\alpha \in (1/2, 2/3)$) using the Hölder inequality:

$$\begin{aligned}\int_{B(0,1/2)} |\nabla u_\epsilon|^{p'} &\leq \int_{B(0,2\epsilon^\alpha)} |\nabla u_\epsilon|^{p'} + \sum_{j=1}^{N} \int_{B(0,2^{j+1}\epsilon^\alpha)\setminus B(0,2^j\epsilon^\alpha))} |\nabla u_\epsilon|^{p'} \\ &\leq \left(2\int_{B(0,2\epsilon^\alpha)} e_\epsilon(u_\epsilon)\right)^{p'/2} c_{p'}\,\epsilon^{(2-p')\alpha} \\ &+ \sum_{j=1}^{N} c(p', C_5)(|B(0,2^{j+1}\epsilon^\alpha)\setminus B(0,2^j\epsilon^\alpha)|)^{(2-p')/2} \\ &\leq o(1) + c(p', C_5)(3\pi)^{(2-p')/2}\sum_{j=1}^{N}(2^j\epsilon^\alpha)^{2-p'} \leq C_6(p', C_5). \end{aligned} \tag{2.21}$$

The proof is complete. □

3. Mobility and Continuity of Vortex Motion

In the previous section, we obtained in Lemma 2.2 the weak limit of solutions based on the energy consideration. Due to conservation of energy, Lemma 2.2 applies to each time slice of evolution, and so Lemma 2.2 holds with $a_j = a_j(t)$, and $h_a = h_a(t,x)$. In this section, we shall utilize the conservation of linear momentum to show the mobility and continuity of vortex motion. With the additional help of conservation of mass, we also refine the weak limit of solution u_ϵ in that we find out how the function h depends on vortex locations $a_j's$, and that it is harmonic in space. Subsequently, we also prove a key energy estimate for the later analysis of the defect measure.

Proposition 3.1. *The vortices in u_ϵ do not move in any slower time scale $t \sim o(1)$, as $\epsilon \to 0$. On the time scale $t \sim O(1)$, the vortex locations $a_{\epsilon,j}(t)$ are uniformly continuous in t as $\epsilon \to 0$.*

Proof. By Lemma 2.1:

$$u_\epsilon(0,x) \rightharpoonup \prod_{j=1}^{n} \frac{x - a_j^0}{|x - a_j^0|} e^{ih_0(x)},$$

in $H^1_{loc}(\Omega_{a_0})$ with $\|h_0\|_{H^1(\Omega)} \le C_0$. Let $R > 0$ be a small number, $R \ll \frac{1}{4}R_0$, where

$$R_0 = \min\{|a_l - a_j|,\ \text{dist}(a_l, \partial\Omega),\ l, j = 1, \cdots, n,\ l \neq j\}.$$

Due to energy conservation, the number R_0 remains positive for all time. Let t_ϵ be such that $\forall t \in [0, t_\epsilon)$, $u_\epsilon(t, x)$ has vortices inside $\cup_{l=1}^n B_{R/4}(a_l^0)$, and t_ϵ is the maximum time with this property. In other words, for some j, $a_{\epsilon,j}(t_\epsilon) \in \partial B_{R/4}(a_j^0)$. By the H^1 continuity of u_ϵ in time for each $\epsilon > 0$, such $t_\epsilon > 0$ exists. We prove that $\liminf_{\epsilon\to 0^+} t_\epsilon > 0$.

Suppose otherwise, at least for a subsequence of ϵ, still denoted the same, $t_\epsilon \to 0$. Write $v_\epsilon(t, x) = u_\epsilon(x, t_\epsilon t)$, then the NLS for v_ϵ becomes

$$iv_{\epsilon,t} = t_\epsilon \Delta v_\epsilon + \frac{t_\epsilon}{\epsilon^2}(1 - |v_\epsilon|^2)v_\epsilon,$$

and the linear momentum equation:

$$\partial_t p(v_\epsilon) = 2t_\epsilon\, \text{div}\,(\nabla v_\epsilon \otimes \nabla v_\epsilon) - \nabla(t_\epsilon P_\epsilon). \tag{3.1}$$

The vortices of v_ϵ lie in $\cup_{l=1}^n B_{R/4}(a_l^0)$ for all $t \in [0, 1)$, and at $t = 1$, one of the vortices, say $a_{\epsilon,j}(1)$, reaches $\cup_{l=1}^n \partial B_{R/4}(a_l^0)$. The vortex locations are well-defined up to a small error of $O(\epsilon^{\alpha_0})$. With no loss of generality, let us assume that $a_{\epsilon,j}(0) = 0$. Let $\varphi \in C_0^\infty(B_{R_0/2})$, and $\varphi = x_1$ for $x \in B_{R_0/4}$. Multiplying both sides of (3.1) by $\nabla^\perp\varphi$ and integrating over $B_{R_0/2} \times [0, 1]$, we obtain with integration by parts:

$$\int_{\partial B_{R_0/2}} \nabla^\perp\varphi \cdot p(u_\epsilon)|_0^1 = -2t_\epsilon \int_0^1 dt \int_{\partial B_{R_0/2}} (\nabla u_\epsilon \otimes \nabla u_\epsilon) : \nabla\nabla^\perp \varphi. \tag{3.2}$$

The right side integral is in fact over $B_{R_0/2} \backslash B_{R_0/4}$, hence is uniformly bounded by a constant C independent of ϵ. Passing $\epsilon \downarrow 0$, by Lemma 2.4, the left hand side converges to $2\pi(\xi_j(1) - \xi_j(0))$. Since $t_\epsilon \to 0$, $\xi_j(1) = \xi_j(0)$. Similarly, $\eta_j(1) = \eta_j(0)$, contradicting the assumption that a_j travels a distance $R/4$ at $t = 1$.

Hence t_ϵ is bounded away from zero uniformly in ϵ. Since R can be any small number, we have proved that vortices $a_{\epsilon,l}(t)$, $l = 1, \cdots, n$ are uniformly continuous in t, or the limiting locations $a_l(t)$ are continuous in t. As a byproduct, we have also shown that vortices in u_ϵ do not move on any slow time scale $t \sim o(1)$ as $\epsilon \to 0$. □

Replacing t_ϵ by $t = O(1)$ in the above proof, we in fact have shown that:

Corollary 3.1. *On the time scale $t \sim O(1)$, the limiting vortex locations $a_l(t)$, are Lipschitz continuous, where $l = 1, \cdots, n$.*

Now let us characterize the function $h_a = h_a(t, x)$ in:

Proposition 3.2. *The function $h_a(t, x)$ in the weak limit (2.2) of Lemma 2.1 satisfies:*

$$\begin{aligned} &\Delta h_a = 0, \quad x \in \Omega, \\ &h_{a,\tau} = -\Theta_{a,\tau} + g \wedge g_\tau, \quad x \in \partial\Omega, \end{aligned} \tag{3.3}$$

where Θ_a is given in (2.8). So h_a is unique up to an additive constant, and depends on time via vortex locations $a_j(t)$.

Proof. By Lemma 2.3 and dominated convergence, for any function $\psi_1(x) \in C_0^\infty(\Omega_a)$ and $\varphi(t) \in C_0^\infty((0,T))$, we have:

$$\lim_{\epsilon\to 0}\int_0^T \varphi(t)\int_{\Omega_a} p(u_\epsilon)\psi_1(x) = \int_0^T \varphi(t)\int_{\Omega_a} \nabla(\Theta_a + h_a)\psi_1(x). \tag{3.4}$$

In addition, using the mass conservation law (1.10), we also have:

$$\begin{aligned}\int_0^T \varphi(t)\int_{\Omega_a} p(u^\epsilon)\cdot\nabla\psi_1(x) &= \frac{1}{2}\int_0^T \varphi_t(t)\int_{\Omega_a} |u_\epsilon|^2\psi_1(x)\\ &\to \frac{1}{2}\int_{\Omega_a}\psi_1(x)\int_0^T \varphi_t(t) = 0,\end{aligned} \tag{3.5}$$

where the convergence is due to (2.4) of Lemma 2.2. It follows that the weak limit of $p(u_\epsilon)$ is divergence free. It follows that h_a is a harmonic function on Ω_a and is also $H^1(\Omega)$ by Lemma 2.2. Thus h_a can have at worst removable singularities and is a harmonic function on the whole domain Ω. The function h_a then has a well-defined boundary value, which we identify next.

Let $\psi = \psi(t,x)$ be a compactly supported function in a small region Ω' near the boundary $\partial\Omega$; for each t, $\text{supp}\{\psi\}\cap\partial\Omega$ contains a finite curve; ψ is also compactly supported inside the time interval $[0,T]$, $T > 0$. Note that near the boundary, there are no vortices, hence Θ_a is a single valued function. Let us calculate:

$$\begin{aligned}\int_{\partial\Omega'}\psi p(u_a)\cdot\tau ds &= \oint_{\partial\Omega'}\psi p(u_a)\cdot d\vec{l} = \int_{\Omega'}\text{curl}\,(\psi p(u_a)) = \int_{\Omega'}\nabla\psi\wedge p(u_a)\\ &= \lim_{\epsilon\downarrow 0}\int_{\Omega'}\nabla\psi\wedge p(u_\epsilon) = \lim_{\epsilon\downarrow 0}[\int_{\Omega'}\text{curl}\,(\psi p(u_\epsilon)) - \int_{\Omega'}\psi\text{curl}\,p(u_\epsilon)]\\ &= \lim_{\epsilon\downarrow 0}\oint_{\partial\Omega'}\psi p(u_\epsilon)\cdot d\vec{l} = \int_{\partial\Omega'}\psi(g\wedge g_\tau)ds,\end{aligned} \tag{3.6}$$

implying that: $p(u_a) = \partial_\tau(\Theta_a + h_a) = g\wedge g_\tau$, on the boundary $\partial\Omega$ for all $t \geq 0$. Hence the harmonic function h_a is uniquely determined up to an additive constant, due to integrating the tangential derivative once along the boundary to recover the related Dirichlet boundary data. Prescribing the boundary map g with certain degree for NLS implies a boundary force along the tangential direction for the limiting fluid motion. We complete the proof. □

Proposition 3.3. *Let $t > 0$ and $u_\epsilon = u_\epsilon(t,x)$ be as in Lemma 2.1, with vortex locations $(a_1, a_2, \cdots, a_n)$. If for some $\omega_0 > 0$:*

$$\limsup_{\epsilon\to 0}\left(E_\epsilon(u_\epsilon) - \pi n\log\frac{1}{\epsilon}\right) \leq \pi W(a) + \omega_0,$$

then for any $r > 0$, there is a constant C independent of ϵ and r such that for any $t > 0$:

$$\limsup_{\epsilon\to 0}\left\|\frac{p_\epsilon(u_\epsilon)}{|u_\epsilon|} - v\right\|^2_{L^2(\Omega\setminus U_{j=1}^n B_r(a_j))} \leq C\omega_0, \tag{3.7}$$

$$\limsup_{\epsilon\to 0}\|\nabla|u_\epsilon|\|^2_{L^2(\Omega\setminus U_{j=1}^n B_r(a_j))} \leq C\omega_0. \tag{3.8}$$

Proof. We first let $\epsilon_k \to 0$ such that

$$\limsup_{\epsilon\to 0} \| \nabla |u_\epsilon| \|^2_{L^2(\Omega\setminus U^n_{j=1}B_r(a_j))} = \limsup_{\epsilon_k\to 0} \| \nabla |u_{\epsilon_k}| \|^2_{L^2(\Omega\setminus U^n_{j=1}B_r(a_j))}.$$

By Lemma 2.2, we can assume without loss of generality that

$$u_{\epsilon_k} \overset{H^1_{loc}(\Omega_a)}{\rightharpoonup} e^{i(\Theta_a+h)},$$

for some $h \in H^1(\Omega)$. Here $e^{i\Theta_a} = \prod_{j=1}^n \frac{x-a_j}{|x-a_j|}$. Hence

$$\frac{p_{\epsilon_k}(u_{\epsilon_k})}{|u_{\epsilon_k}|} \overset{L^2_{loc}(\Omega_a)}{\rightharpoonup} \nabla(\Theta_a + h).$$

For any $\rho > 0$, then

$$\begin{aligned} & E_{\epsilon_k}(u_{\epsilon_k}, \Omega\setminus U^n_{j=1}B_\rho(a_j)) \\ \equiv\ & \frac{1}{2}\int_{\Omega\setminus U^n_{j=1}B_\rho(a_j)} \left[|\nabla|u_{\epsilon_k}||^2 + \left|\frac{p_{\epsilon_k}(u_{\epsilon_k})}{|u_{\epsilon_k}|}\right|^2 + \frac{1}{2\epsilon_k^2}(1-|u_{\epsilon_k}|^2)^2 \right] \\ \geq\ & \frac{1}{2}\int_{\Omega\setminus U^n_{j=1}B_\rho(a_j)} |\nabla|u_{\epsilon_k}||^2 + \left|\frac{p_{\epsilon_k}(u_{\epsilon_k})}{|u_{\epsilon_k}|} - \nabla(\Theta_a+h)\right|^2 \\ +\ & \frac{1}{2}\int_{\Omega\setminus U^n_{j=1}B_\rho(a_j)} |\nabla(\Theta_a+h)|^2\, dx + o_{\epsilon_k}(1), \end{aligned} \tag{3.9}$$

here $o_{\epsilon_k}(1) \to 0$ as $k \to \infty$. Next, we let $u_{\epsilon_k}(h,\rho)$ be such that $u_{\epsilon_k}(h,\rho) = e^{i(\Theta_a+h)}$ on $\Omega\setminus U^n_{j=1}B_\rho(a_j)$; and on each $B_\rho(a_j)$, $u_{\epsilon_k}(h,\rho)$ is a minimizer of E_{ϵ_k} on each $B_\rho(a_j)$ with boundary value $e^{i(\Theta_a+h)}$. We choose $\rho \in (\frac{r}{2}, r)$ so that $u_{\epsilon_k}|_{\partial B_\rho} \rightharpoonup e^{(\Theta_a+h)}$ in $H^1(\partial B_\rho(a_j))$ for $j = 1,\cdots,n$, by taking the subsequence of ϵ_k as needed. Then it is easy to see by a simple comparison that for $j = 1,\cdots,n$:

$$E_{\epsilon_k}(u_{\epsilon_k}, B_\rho(a_j)) \geq E(u_{\epsilon_k}(h,\rho), B_\rho(a_j)) + o(\rho,\epsilon_k),$$

here $o(\rho,\epsilon_k) \to 0$ as $k \to \infty$. Therefore

$$\begin{aligned} \pi W(a) + o_{\epsilon_k}(1) &\leq E_{\epsilon_k}(u_{\epsilon_k}(h,\rho)) - \pi n \log\frac{1}{\epsilon_k} \\ &\leq E_{\epsilon_k}(u_{\epsilon_k}) - n\pi \log\frac{1}{\epsilon_k} + o(\rho,\epsilon_k) - \frac{1}{2}\int_{\Omega\setminus U^n_{j=1}B_\rho(a_j)} |\nabla|u_{\epsilon_k}||^2 dx \\ &\quad - \frac{1}{2}\int_{\Omega\setminus U^n_{j=1}B_\rho(a_j)} \left|\frac{p_{\epsilon_k}(u_{\epsilon_k})}{|u_{\epsilon_k}|} - \nabla(\Theta_a+h)\right|^2 dx. \end{aligned} \tag{3.10}$$

Since $E_\epsilon(u_{\epsilon_k}) - \pi n \log\frac{1}{\epsilon_k} \leq \pi W(a) + w_0$, we thus conclude that

$$\lim_{\epsilon_k\to 0} \int_{\Omega\setminus U^n_{j=1}B_r(a_j)} |\nabla|u_{\epsilon_k}||^2 \leq 2w_0, \tag{3.11}$$

which implies (3.8) and that

$$\limsup_{\epsilon_k\to 0}\int_{\Omega\setminus U_{j=1}^n B_r(a_j)}\left|\frac{p_{\epsilon_k}(u_{\epsilon_k})}{|u_{\epsilon_k}|}-\nabla(\Theta_a+h)\right|^2\le 2w_0. \tag{3.12}$$

We observe now if $\epsilon_k\to 0$ is so that

$$\lim_{\epsilon_k\to 0}\int_{\Omega\setminus U_{j=1}^n B_r(a_j)}\left|\frac{p_{\epsilon_k}(u_{\epsilon_k})}{|u_{\epsilon_k}|}-v\right|^2 dx,$$

is the left-hand side of (3.7), then by (3.12):

$$\limsup_{\epsilon\to 0}\int_{\Omega\setminus U_{j=1}^n B_r(a_j)}\left|\frac{p_{\epsilon}(u_{\epsilon})}{|u_{\epsilon}|}-v\right|^2 dx\le 4w_0+2\int_{\Omega\setminus U_{j=1}^n B_r(a_j)}|\nabla h-\nabla h_a|^2. \tag{3.13}$$

Here $v=\nabla(\Theta_a+h_a)$.

Now we show that

$$\int_{\Omega\setminus U_{j=1}^n B_r(a_j)}|\nabla h-\nabla h_a|^2\le w_0.$$

To do this, we observe that for a $\rho>0$ with

$$\int_{\partial B_\rho}|\nabla h|^2\le\frac{2}{\rho}\int_{B_{2\rho}\setminus B_{\rho/2}}|\nabla h|^2dx\le\frac{C}{\rho},$$

we have

$$E_\epsilon(u_\epsilon(h,\rho),U_{j=1}^n B_\rho(a_j))\ge\pi n\log\frac{\rho}{\epsilon}+\gamma n+o(\rho,\epsilon).$$

This follows from an easy energy estimate, see [22]. Here $o(\rho,\epsilon)\to 0$ as $\epsilon\to 0^+$. This implies in turn that

$$E_\epsilon(u_\epsilon(h,\rho),\Omega\setminus U_{j=1}^n B_\rho(a_j))=\frac{1}{2}\int_{\Omega\setminus U_{j=1}^n B_\rho(a_j)}|\nabla(\Theta_a+h)|^2$$
$$\le\pi W(a)-\gamma n+w_0+o(\rho,\epsilon)+n\pi\log\frac{1}{\rho}. \tag{3.14}$$

On the other hand, we have:

$$\frac{1}{2}\int_{\Omega\setminus U_{j=1}^n B_\rho(a_j)}|\nabla(\Theta_a+h_a)|^2=n\pi\log\frac{1}{\rho}+\pi W(a)-\gamma n+o(\rho), \tag{3.15}$$

where $o(\rho)\to 0^+$, as $\rho\to 0$. We also note for any $h\in H^1(\Omega)$:

$$\int_{\Omega\setminus U_{j=1}^n B_\rho(a_j)}|\nabla(\Theta_a+h)|^2dx=\int_{\Omega\setminus U_{j=1}^n B_\rho(a_j)}|\nabla\Theta_a|^2+|\nabla h|^2$$
$$+2\int_{\partial\Omega}\frac{\partial\Theta_a}{\partial\nu}\cdot h-2\sum_{j=1}^n\int_{\partial B_\rho(a_j)}(h-\overline{h})\frac{\partial\Theta_a}{\partial n}, \tag{3.16}$$

where the last term is bounded by const. $\sum_{j=1}^{n} \rho \int_{\partial B_\rho} |\nabla h|$, which goes to zero as $\rho \to 0$. By sending $\epsilon \to 0$, then $\rho \to 0$, we therefore obtain by combining (3.14), (3.15), and (3.16) that

$$\int_\Omega |\nabla h|^2 \leq \int_\Omega |\nabla h_a|^2 + w_0. \tag{3.17}$$

Inequality (3.17), along with the fact that h_a is harmonic, and $h|_{\partial\Omega} = h_a|_{\partial\Omega}$, yields $\int_\Omega |\nabla(h - h_a)|^2 \leq w_0$. The proof is complete. $\square$

We end this section with an interesting conjugation property of the regular part of the vortex phase in terms of the renormalized energy function W. Near each vortex a_j, write the weak limit as $e^{i\arg(x-a_j)+iH_j}$, where H_j is harmonic. Then:

Lemma 3.1.

$$\nabla_{a_j} W(a) = 2n_j \left(-\frac{\partial H_j}{\partial x_2}(a_j), \frac{\partial H_j}{\partial x_1}(a_j)\right). \tag{3.18}$$

For a proof, see [2] (Theorem 8.3).

4. Convergence to Incompressible Euler Equation and Vortex Motion Law

In this section, we use continuity of vortices, the weak convergence and the precise form of the weak limit discussed in the previous sections to pass the linear momentum equation (1.11) to the incompressible limit on the punctured domain Ω_a, and show that the limiting equation is the two dimensional Euler equation. We show properties of defect measures and total pressure P to finish proving Theorem 1.1. We then establish the Kirchhoff law for vortex motion based on the limiting projected linear momentum equation. Finally, we show strong convergence of the linear momentum under the initial energy almost minimizing assumption.

Let us write the linear momentum equation in component form:

$$p_m(u_\epsilon)_t = 2(u_{\epsilon,x_m} \cdot u_{\epsilon,x_j})_{x_j} - P_{x_m}, \quad m = 1, 2. \tag{4.1}$$

Direct calculation shows that if $|u_\epsilon| > 0$ then

$$u_{\epsilon,x_m} = \frac{p_m(u_\epsilon)}{|u_\epsilon|}\frac{iu_\epsilon}{|u_\epsilon|} + |u_\epsilon|_{x_m}\frac{u_\epsilon}{|u_\epsilon|}. \tag{4.2}$$

Note that $|\nabla u_\epsilon| = 0, a.e$, on the set $\{|u_\epsilon| = 0\}$. Hence, we only need to consider the set $\{|u^\epsilon| > 0\}$. It follows from (4.2) that

$$\begin{aligned} u_{\epsilon,x_m} \cdot u_{\epsilon,x_j} &= \frac{p_m(u_\epsilon) \cdot p_j(u_\epsilon)}{|u_\epsilon|^2} + |u_\epsilon|_{x_m}|u_\epsilon|_{x_j} \\ &= \left(\frac{p_m(u_\epsilon)}{|u_\epsilon|} - v_m\right)\left(\frac{p_j(u_\epsilon)}{|u_\epsilon|} - v_j\right) + |u_\epsilon|_{x_m}|u_\epsilon|_{x_j} \\ &\quad + v_m\frac{p_j(u_\epsilon)}{|u_\epsilon|} + v_j\frac{p_m(u_\epsilon)}{|u_\epsilon|} - v_m v_j. \end{aligned} \tag{4.3}$$

Note that $\||u_\epsilon|^{-1}p(u_\epsilon)\|_{L^2_{loc}(\Omega_a)} \leq C$, for a positive constant independent of ϵ, and $t \in [0,T]$. Hence $|u_\epsilon|^{-1}p(u_\epsilon)$ is weakly compact in $L^2(\Omega_a \times [0,T])$. Since $|u_\epsilon| \to 1$

in $L^2(\Omega_a \times [0,T])$, the weak $L^1(\Omega_a \times [0,T])$ limit of $p(u_\epsilon)$ equal to $v = \nabla(\Theta_a + h_a)$ coincides with the weak $L^2(\Omega_a \times [0,T])$ limit of $|u_\epsilon|^{-1}p(u_\epsilon)$. It follows that

$$v_m \frac{p_j(u_\epsilon)}{|u_\epsilon|} \rightharpoonup v_m v_j, \quad v_j \frac{p_m(u_\epsilon)}{|u_\epsilon|} \rightharpoonup v_j v_m \tag{4.4}$$

in $L^2(\Omega_a \times [0,T])$.

The product terms

$$\left(\frac{p_m(u_\epsilon)}{|u_\epsilon|} - v_m\right)\left(\frac{p_j(u_\epsilon)}{|u_\epsilon|} - v_j\right) + |u_\epsilon|_{x_m}|u_\epsilon|_{x_j} \rightharpoonup \mu_{m,j}, \tag{4.5}$$

as measures to a symmetric tensorial measure $\mu_{m,j} \in \mathcal{M}(\Omega_a)$. We prove:

Proposition 4.1. *The defect measure $\mu = (\mu_{m,j})$ is a finite mass Radon measure on the domain Ω. Its divergence* $\mathrm{div}(\mu_{m,j})$ *is curl free in the sense of a distribution, and can be written into ∇P_μ on Ω_a, where P_μ is a distribution function well-defined on the entire domain Ω_a. The weak limit v is a solution of the incompressible Euler equation:*

$$v_t = 2v \cdot \nabla v - 2\nabla P, \quad \mathrm{div}\, v = 0, \quad \forall\, x \in \Omega_a,$$

where the total pressure $2P$ is a single-valued function, and smooth in Ω_a.

Proof. That the defect measure $\mu \geq 0$ is a finite mass Radon measure on the entire domain Ω follows from Proposition 3.3. Let us take $\psi \in (C_0^\infty(\Omega_a \times [0,T)))^2$, $\mathrm{div}\psi = 0$, form the inner product of ψ with both sides of the linear momentum equation (1.11), and integrate by parts to get

$$\int \psi(0,x)p(u_\epsilon^0) + \int\int \psi_t \cdot p(u_\epsilon) - 2(\nabla u_\epsilon \otimes \nabla u_\epsilon) : \nabla\psi = 0.$$

Passing to the limit, we get

$$\int \psi(0,x)v^0 + \int\int \psi_t v - 2(v \otimes v + \mu) : \nabla\psi = 0. \tag{4.6}$$

In particular, we choose ψ to be of the form:

$$\psi = \alpha(t)(-\varphi_{x_2}, \varphi_{x_1}) = \alpha(t)\nabla^\perp\varphi, \tag{4.7}$$

where $\varphi \in C_0^\infty(\Omega_a)$, $\alpha(0) = 0$. Then due to v being curl free on Ω_a, (4.6) reduces to

$$\int\int \alpha(t)\mu : \nabla\nabla^\perp\varphi = 0, \tag{4.8}$$

which means that the weak divergence of the measure μ is a weak gradient away from vortices, hence can be written locally into a gradient of another distribution, by an approximation argument. We denote $div\, \mu = \nabla P_\mu$, P_μ is a local distribution for now. It follows that (4.6) reduces to

$$\int \psi(0,x)v^0 + \int\int \psi_t v - 2(v \otimes v) : \nabla\psi = 0. \tag{4.9}$$

Since v is harmonic in x and Lipschitz continuous in time, it is easy to bootstrap on (4.9) to show that v is smooth in $(x,t) \in \Omega_a \times (0,T)$. We can now write (4.9) into the strong form of the Euler equation:

$$v_t = 2v \cdot \nabla v - 2\nabla P, \quad x \in \Omega_a, \quad \operatorname{div} v = 0,\ v(0,x) = v^0(x) \tag{4.10}$$

for some function $2P$ locally defined on $\Omega_a \times (0,T)$. Taking the divergence of (4.10) gives $\Delta P = \operatorname{div}(v \cdot \nabla v)$. That v is harmonic in Ω_a then implies that P is smooth in Ω_a.

Using (4.10), we see that the integral around each vortex:

$$-\int_{\partial B_R(a_j)} \frac{\partial P}{\partial \theta} = \frac{1}{2} \oint_{\partial B_R(a_j)} v_t \cdot d\vec{l} - \oint_{\partial B_R(a_j)} v \cdot \nabla v \cdot d\vec{l}.$$

By the form of weak limit v, the circulation of v_t is zero. The circulation of the $v \cdot \nabla v$ term is also zero by a direct calculation with $v = \nabla(\Theta_a + h_a)$. First we note that $\operatorname{curl}(v \cdot \nabla v) = v \cdot \nabla \operatorname{curl} v = 0$, $x \in \Omega_a$. Hence it is enough to calculate the circulation on a very small circle around a_j and show that it goes to zero as the radius of the circle goes to zero. Let $a_j = (\xi_j, \eta_j)$, and $x = (\xi, \eta)$. Let us write $H = \Theta_a + h_a = \arg \frac{x-a_j}{|x-a_j|} + H_j$ and so

$$\begin{aligned} H_\xi &= (H_j)_\xi + \frac{-(\eta-\eta_j)}{(\xi-\xi_j)^2 + (\eta-\eta_j)^2}, \\ H_\eta &= (H_j)_\eta + \frac{(\xi-\xi_j)}{(\xi-\xi_j)^2 + (\eta-\eta_j)^2}, \end{aligned} \tag{4.11}$$

and below we denote $\nabla H_j = (I, II)$. Noticing that $I_\xi + II_\eta = \Delta H_j = 0$, we have

$$\begin{aligned} &\oint_{\partial B_R(a_j)} v \cdot \nabla v \cdot d\vec{l} = \int_0^{2\pi} R[v \cdot \nabla v_1(-\sin\theta) + v \cdot \nabla v_2 \cos\theta] d\theta \\ &= \int_0^{2\pi} R[(I - R^{-1}\sin\theta)(I_\xi + 2R^{-2}\sin\theta\cos\theta)(-\sin\theta) \\ &\quad +(II + R^{-1}\cos\theta)(I_\eta - R^{-2}\cos 2\theta)(-\sin\theta)] \\ &+ \int_0^{2\pi} R[(I - R^{-1}\sin\theta)(II_\xi - R^{-2}\cos 2\theta)\cos\theta \\ &\quad +(II + R^{-1}\cos\theta)(II_\eta - R^{-2}\sin 2\theta)\cos\theta] d\theta \\ &= \int_0^{2\pi} [I_\xi \sin^2\theta + II_\eta \cos^2\theta] d\theta + O(R) \\ &= \pi(I_\xi(a_j) + II_\eta(a_j)) + O(R) = O(R) \to 0. \end{aligned} \tag{4.12}$$

Thus the total pressure $2P$ is a well-defined single-valued function over the whole domain Ω. It consists of the defect pressure from μ and the contribution from the original NLS pressure.

Finally, we show that the defect pressure P_μ is a well-defined distribution on Ω. For $\psi = \psi(r)$, supported in the annulus $B_R(a_j(s)) \backslash B_{R/2}(a_j(s)) = B_R \backslash B_{R/2}$, it follows from the linear momentum equation for t near s that

$$\frac{d}{dt} \int_{B_R \backslash B_{R/2}} p(u_\epsilon)(\psi\tau) = -2 \int_{B_R \backslash B_{R/2}} \nabla u_\epsilon \otimes \nabla u_\epsilon : \nabla(\psi\tau),$$

where the NLS pressure has zero circulation and is removed. Passing $\epsilon \downarrow 0$ and using the fact that $v \cdot \nabla v$ has zero circulation as proved above, we have

$$0 = \int_{B_R \setminus B_{R/2}} (\mu + v \otimes v) : \nabla(\psi\tau) = -\int_{B_R \setminus B_{R/2}} (div\, \mu + v \cdot \nabla v) \cdot (\psi\tau)$$
$$= -\int_{R/2}^{R} dr\, \varphi(r) \int_{\partial B_r} \frac{\partial P_\mu}{\partial \theta},$$

implying that $\int_{\partial B_r} \frac{\partial P_\mu}{\partial \theta} = 0$ for any $r > 0$, hence P_μ is a well-defined distribution on Ω_a. The proof of the proposition and also that of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Let us consider the time interval $[t, t + k]$, with k small, and the ball $B_R = B_R(a_j(t))$ inside the annulus $B_{R_0/2}$ as in the proof of Proposition 3.1. The number R is much smaller than R_0 and is large enough to contain $a_j(s)$, $s \in [t, t + k]$. For example, $R = Ck$, for a suitable constant C depending on the Lipschitz constant of a_j. Proceeding as in Proposition 3.1, with $\varphi = x_1$ in $B_R(a_j(t))$ and supported inside $B_{R_0/2}$, we find:

$$\int_{B_{R_0/2}} \nabla^\perp \varphi\, p(u_\epsilon)|_t^{t+k}$$
$$= -2 \int_t^{t+k} ds \int_{B_{R_0/2} \setminus B_R} (\nabla u_\epsilon \otimes \nabla u_\epsilon) : \nabla \nabla^\perp \varphi$$
$$\to 2 \int_t^{t+k} ds \int_{B_{R_0/2} \setminus B_R} -(\mu + v \otimes v) : \nabla\nabla^\perp \varphi. \tag{4.13}$$

Here $\mu \in \mathcal{M}(\Omega)$ and $v \otimes v \notin L^1(\Omega)$. As in Proposition 3.1, the left hand side of (4.13) converges to $2\pi(\xi_j(t + k) - \xi_j(t))$.

For the right-hand side, we calculate the second term in (4.13):

$$\int_s^{s+k} ds \int_{B_{R_0/2}(a_j(s)) \setminus B_R(a_j(s))} -(v \otimes v) : \nabla\nabla^\perp \varphi$$
$$= \int_s^{s+k} ds \int_{B_{R_0/2}(a_j(s)) \setminus B_R(a_j(s))} v \cdot \nabla v \cdot \nabla^\perp \varphi$$
$$- \int_s^{s+k} ds \int_{\partial B_R(a_j(s))} (v \otimes v) : (\nu \otimes n^\perp)$$
$$= \int_s^{s+k} ds \int_{\partial B_R(a_j(s))} (v \cdot \nabla v \cdot \nu^\perp)(n \cdot x)$$
$$+ \int_s^{s+k} ds \int_{\partial B_R(a_j(s))} -(v \otimes v) : (\nu \otimes n^\perp), \tag{4.14}$$

where $n = (1, 0)$ and ν is the normal direction at $\partial B_R(a_j(s))$.

Let us calculate the inner part of the first integral of the right-hand side of (4.14) as follows:

$$\begin{aligned}
&\int_0^{2\pi} (\xi_j(t)R + R^2\cos\theta)[v\cdot\nabla v_1(-\sin\theta) + v\cdot\nabla v_2\cos\theta]d\theta \\
&= \int_0^{2\pi} (\xi_j(t)R + R^2\cos\theta)[(I - R^{-1}\sin\theta)(I_\xi + 2R^{-2}\sin\theta\cos\theta)(-\sin\theta) \\
&\quad +(II + R^{-1}\cos\theta)(I_\eta - R^{-2}\cos 2\theta)(-\sin\theta)]\, d\theta \\
&+ \int_0^{2\pi} (\xi_j(t)R + R^2\cos\theta)[(I - R^{-1}\sin\theta)(II_\xi - R^{-2}\cos 2\theta)\cos\theta \\
&\quad +(II + R^{-1}\cos\theta)(II_\eta - R^{-2}\sin 2\theta)\cos\theta]d\theta \\
&= -I\int_0^{2\pi} 2(\sin\theta\cos\theta)^2 d\theta - I\int_0^{2\pi}\cos^2\theta\cos 2\theta d\theta + O(R) \\
&= -I\int_0^{2\pi}\cos^2\theta = -\pi I.
\end{aligned} \tag{4.15}$$

Similarly, the inner part of the second integral of the right hand side of (4.14) also contributes $-\pi I$. Therefore dividing by k and letting $k \to 0$, we have from (4.13)–(4.15) that $\xi_j' = -2H_{j,\xi} + f_{j,1}(\mu)$. With a similar equation for η_j, we conclude that

$$a_j' = -2\nabla H_j + f_j(\mu), \tag{4.16}$$

where $f_j(\mu)$ is a possible correction due to the defect measure μ. Using the conjugation of H_j with the renormalized energy, we rewrite (4.16) into

$$a_j' = n_j J\nabla_{a_j} W(a) + f_j(\mu), \tag{4.17}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$W(a) = -\sum_{l\neq j} n_l n_j \log|a_l - a_j| + \text{boundary contributions}.$$

The Kirchhoff law follows if $f_j(\mu) = 0$, which we show below under the energy almost minimizing assumption.

Since the Kirchhoff law may encounter finite time collapse for signed vortices, the validity established here applies also to any time prior to the collapse in the signed vortex situation. □

Proposition 4.2. *Under the almost minimizing initial energy assumption, we have*

$$\frac{p(u_\epsilon)}{|u_\epsilon|} - v \to 0,\ \nabla|u_\epsilon| \to 0,$$

in $L^2(\Omega_a)$, and the defect measure $\mu = 0$. The Kirchhoff law holds.

Proof. For simplicity, let us consider vortices of the same sign plus one. Let $\tilde{a}_{j,t} = J\nabla_{\tilde{a}_j} W(\tilde{a})$, $\tilde{a}(0) = a(0)$; and define

$$m(t) = \sum_{j=1}^{n} |a_j(t) - \tilde{a}_j(t)|.$$

Take a small time interval $t \in [0, t_\delta]$ so that $|m(t)| \leq \delta$, with δ a small number to be selected. Lipschitz continuity of m implies that it is differentiable a.e. in t. We have

$$\begin{aligned} m'(t) &\leq \sum_{j=1}^{n} |a_j'(t) - \tilde{a}_j'(t)| \\ &\leq \sum_{j=1}^{n} |a_j'(t) - J\nabla_{a_j} W(a)| + \sum_{j=1}^{n} |J\nabla_{a_j} W(a) - J\nabla_{\tilde{a}_j} W(\tilde{a})| \\ &\leq \sum_{j=1}^{n} |a_j'(t) - J\nabla_{a_j} W(a)| + Cm(t). \end{aligned} \tag{4.18}$$

As before, consider the time interval $[t, t+k]$, with k small, and the ball $B_R = B_R(a_j(t))$ inside $B_{R_0/2}$. Proceeding as before, we find

$$\begin{aligned} LHS &= \int_{B_{R_0/2}} \nabla^\perp \varphi \, p(u_\epsilon)|_t^{t+k} \\ &= -2 \int_t^{t+k} ds \int_{B_{R_0/2} \setminus B_R} (\nabla u_\epsilon \otimes \nabla u_\epsilon) : \nabla\nabla^\perp \varphi \\ &= -2 \int_t^{t+k} ds \int_{B_{R_0/2} \setminus B_R} \left(v \otimes \frac{p(u_\epsilon)}{|u_\epsilon|} + \left[v \otimes \frac{p(u_\epsilon)}{|u_\epsilon|} \right]^T - v \otimes v \right) : \nabla\nabla^\perp \varphi \\ &\quad + (-2) \int_t^{t+k} ds \int_{B_{R_0/2} \setminus B_R} [\left(\frac{p(u_\epsilon)}{|u_\epsilon|} - v \right) \\ &\quad \otimes \left(\frac{p(u_\epsilon)}{|u_\epsilon|} - v \right) + \nabla |u_\epsilon| \otimes \nabla |u_\epsilon|] : \nabla\nabla^\perp \varphi \\ &= RHS_1 + RHS_2. \end{aligned} \tag{4.19}$$

Now the almost minimizing energy assumption gives:

$$\begin{aligned} E(u_\epsilon) &= n\pi \log \frac{1}{\epsilon} + W(a(0)) + o(1) \\ &= n\pi \log \frac{1}{\epsilon} + W(\tilde{a}(t)) + o(1) \\ &\leq n\pi \log \frac{1}{\epsilon} + W(a(t)) + Cm(t) + o(1). \end{aligned} \tag{4.20}$$

Selecting $\delta C \leq \omega_0 \in (0,1)$, we infer from Proposition 3.3 that for all $t \in (0, t_\delta)$:

$$\limsup_{\epsilon \to 0} \left\| \frac{p(u_\epsilon)}{|u_\epsilon|} - v \right\|_{L^2(B_{R_0/2} \setminus B_R)} \leq C_1 m(t),$$

and

$$\limsup_{\epsilon\to 0} \|\nabla|u_\epsilon|\|_{L^2(B_{R_0/2}\setminus B_R)} \le C_1 m(t). \tag{4.21}$$

Passing $\epsilon \to 0$ in (4.19), then dividing and sending $k \downarrow 0$, we get ($a = (\xi, \eta)$):

$$LHS \to 2\pi\xi_j'(t), \quad RHS_1 \to 2\pi J W_{\xi_j}(a(t)).$$

In view of (4.21), we have from (4.19) that $|\xi_j'(t) - JW_{\xi_j}(a)| \le C_2 m(t)$. With a similar estimate on $\eta_j(t)$, we get $|a_j'(t) - J\nabla_{a_j} W(a)| \le C_2 m(t)$. It follows that $m'(t) \le Cm(t)$, with $m(0) = 0$, hence $m(t) = 0$ for all $t \in [0, t_\delta]$. Induction in time shows $a(t) \equiv \tilde{a}$ for all $t \ge 0$. Hence the Kirchhoff law holds with strong convergence of p_ϵ and $\nabla|u_\epsilon|$. The proof is complete. □

5. Zero Neumann and Other Boundary Conditions

In this section, we comment on all necessary modifications in the proofs of previous sections to establish similar results for the zero Neumann case, the entire space case, and the periodic case.

For the Neumann boundary case, the h_a in the weak limit is harmonic and satisfies the boundary condition: $h_{a,\nu} = -\Theta_{a,\nu}$. The resulting renormalized energy W goes to $-\infty$ if one of the vortices goes near $\partial\Omega$. To establish a uniform bound on W, we proceed by first showing the vortex continuous motion in time, then using the dynamical law to deduce that the renormalized energy is conserved. Thus the vortices never come close to each other or to the boundary $\partial\Omega$ since initially W is finite. The energy arguments can be modified as in Lin [22] and [23]. What remains is the treatment of the boundary value of h_a.

Let us derive the Neumann boundary condition on h_a. First, near the boundary $\partial\Omega$, there are no vortices by induction in time. So we can write $u^\epsilon = \rho^\epsilon e^{iH^\epsilon}$, where both ρ^ϵ and H^ϵ are real functions. Direct calculation shows:

$$\begin{aligned} p(u^\epsilon) &= (\rho^\epsilon)^2 \nabla H^\epsilon, \\ p(u^\epsilon)\cdot\nu &= (\rho^\epsilon)^2 H^\epsilon_\nu, \qquad x \in \partial\Omega. \end{aligned} \tag{5.1}$$

Similarly

$$u^\epsilon_\nu = (\rho^\epsilon_\nu + iH^\epsilon_\nu)e^{iH^\epsilon},$$

and so zero Neumann boundary condition (1.3) says

$$\rho^\epsilon_\nu = 0, \; H^\epsilon_\nu = 0, \quad \partial\Omega, \tag{5.2}$$

implying in view of (5.1):

$$p(u^\epsilon)\cdot\nu = 0, \quad \partial\Omega, \; \forall\, \epsilon > 0. \tag{5.3}$$

Let $\psi = \psi(t, x)$ be a compactly supported function in a small region near the boundary; for each t, supp $\{\psi\} \cap \partial\Omega$ contains a finite curve; ψ is also compactly supported inside the time interval $[0, T]$, $T > 0$.

Due to div $p(u_a) = 0$ on Ω_a, we have using (5.3) and mass conservation:

$$\int_{\partial\Omega} \psi\, p(u_a)\cdot\nu = \int_{\Omega_a} p(u_a)\cdot\nabla\psi = \lim_{\epsilon\to 0}\int_{\Omega_a} p(u_\epsilon)\cdot\nabla\psi$$
$$= -\lim_{\epsilon\to 0}\int_{\Omega_a} \operatorname{div} p(u^\epsilon)\psi = -\frac{1}{2}\lim_{\epsilon\to 0}\int_{\Omega_a} |u^\epsilon|_t^2\psi, \tag{5.4}$$

which upon integration over $[0,T]$ and integration by parts gives

$$\int_0^T\int_{\partial\Omega} \psi\, p(u_a)\cdot\nu = \frac{1}{2}\lim_{\epsilon\to 0}\int_0^T\int_{\Omega_a} |u^\epsilon|^2\psi_t = 0. \tag{5.5}$$

It follows from arbitrariness of ψ and smoothness of $p(u_a)$ that $p(u_a)\cdot\nu = 0$ on $\partial\Omega$, which is just the desired boundary condition $h_{a,\nu} = -\Theta_{a,\nu}$. Physically, h_a plays the role of correcting Θ_a on the boundary so that there is no flow into the wall.

Let us turn to the entire space $\mathbb{R}^2$ case and the periodic case. For these two cases, we assume that the sum of degrees $\sum_{j=1}^n n_j = 0$ (zero sum condition). Under this condition and that $u_\epsilon(0,x)$ converges to a constant $e^{i\theta_0}$ at $x=\infty$ sufficiently fast, the total energy E_ϵ on $\mathbb{R}^2$ remains the same asymptotic expression $n\pi\log\frac{1}{\epsilon} + O(1)$. Otherwise, the energy is infinite, and one has to look at the energy distribution over finite domains to locate vortices. The analogous problem on $\mathbb{R}^2$ with infinite initial energy has been solved recently for the Ginzburg–Landau equation in Lin and Xin [24]. When the sum of vortex degrees is zero, the harmonic function h_a having a finite L^2 gradient over $\mathbb{R}^2$ is a constant. The renormalized energy simplifies to $W_{\mathbb{R}^2} = -\sum_{l\neq j} n_l n_j \log|a_l - a_j|$, free of boundary contributions. The zero sum condition is needed in the periodic case in order to maintain the boundary condition for solutions containing vortices. The renormalized energy is similar: $W_{per} = -\sum_{l\neq j} n_l n_j G(a_l - a_j)$, with G the periodic Green's function for the Laplacian on the two dimensional torus ($\Delta G = 2\pi(\delta_0 - 1)$).

6. Vortex Motion Law of a CGL

In this section, we apply our method to establish the vortex motion law of a related complex Ginzburg–Landau (CGL) equation:

$$\frac{\delta}{\log\frac{1}{\epsilon}} u_{\epsilon,t} + i u_{\epsilon,t} = \Delta u_\epsilon + \epsilon^{-2}(1-|u_\epsilon|^2)u_\epsilon, \tag{6.1}$$

where $\delta > 0$ is a fixed positive number. We shall only consider the Dirichlet boundary condition (1.2), with extensions to other boundary conditions the same as remarked in the last section.

The energy conservation is

$$\frac{d}{dt}\int_\Omega e_\epsilon(u_\epsilon)(t,x)\,dx = -\frac{\delta}{\log\frac{1}{\epsilon}}\int_\Omega |u_{\epsilon,t}|^2, \tag{6.2}$$

which implies via Lemma 2.1:

$$\int_0^T\int_\Omega \frac{\delta u_{\epsilon,t}^2}{\log\frac{1}{\epsilon}}\,dx \le C_0, \tag{6.3}$$

and energy concentration:

$$\mu_\epsilon(t,x) = \frac{e_\epsilon(u_\epsilon)(t,x)\,dx}{\pi \log \frac{1}{\epsilon}} \rightharpoonup \mu(t,x) = \sum_{j=1}^{n} \delta_{a_j(t)}. \tag{6.4}$$

It follows from (6.3) that $a_j(t)$ are Lipschitz continuous in t for any $\delta > 0$, see [20, 22] for details.

The conservation of mass is now

$$\partial_t |u_\epsilon|^2 = 2\mathrm{div} p(u_\epsilon) - \frac{2\delta}{\log \frac{1}{\epsilon}} u_\epsilon \wedge u_{\epsilon,t}, \tag{6.5}$$

and the conservation of linear momentum is

$$\partial_t p(u_\epsilon) = 2\mathrm{div}\,(\nabla u_\epsilon \otimes \nabla u_\epsilon) - \nabla P_\epsilon - \frac{2\delta}{\log \frac{1}{\epsilon}} u_{\epsilon,t} \cdot \nabla u_\epsilon, \tag{6.6}$$

with the pressure

$$P_\epsilon = |\nabla u_\epsilon|^2 + u_\epsilon \cdot \Delta u_\epsilon - \frac{|u_\epsilon|^4 - 1}{2\epsilon^2} - \frac{\delta}{\log \frac{1}{\epsilon}} u_\epsilon \cdot u_{\epsilon,t}. \tag{6.7}$$

We observe that

$$\frac{\delta}{\log \frac{1}{\epsilon}} u_\epsilon \wedge u_{\epsilon,t} \to 0, \quad L^1([0,T]; L^1(\Omega)),$$

by (6.3), and similarly

$$\frac{\delta}{\log \frac{1}{\epsilon}} u_{\epsilon,t} \cdot \nabla u_\epsilon \to 0, \quad L^1([0,T]; L^1(\Omega_a)).$$

Using the same arguments as before for NLS, we deduce that $p(u_\epsilon) \rightharpoonup v$ satisfying the Euler equation on Ω_a; moreover, the vortices $a_j(t)$ obey the same Kirchhoff law as in Theorem 1.2. Since the results are independent of δ, we have as a byproduct another proof of continuity and the dynamical law for NLS vortices sending $\delta \downarrow 0$.

7. Semiclassical Limit of NLS

In this section, we consider the semiclassical (WKB) limit of NLS:

$$\epsilon i v_{\epsilon,t} = \epsilon^2 \Delta v_\epsilon + (1 - |v_\epsilon|^2) v_\epsilon, \tag{7.1}$$

with the Dirichlet boundary condition (1.2) and initial data satisfying (1.5). The case when there are no vortices in solutions (uniformly bounded energy as $\epsilon \downarrow 0$), has been studied in Colin and Soyeur [4]. Here we are concerned with the case when there are vortices. We show:

Theorem 7.1. *Suppose that the initial data*

$$v_\epsilon(0,x) \rightharpoonup \prod_{j=1}^{n} \frac{x-a_j}{|x-a_j|} e^{ih(x)},$$

weakly in $H^1(\Omega_a)$, $h(x) \in H^1(\Omega)$, and that $\frac{|v_\epsilon(0,x)|-1}{\epsilon} \to 0$ in $L^2(\Omega')$, for any compact subset Ω' of Ω_a. Then there is no vortex motion at a later time and

$$v_\epsilon(t,x) \rightharpoonup \prod_{j=1}^{n} \frac{x-a_j}{|x-a_j|} e^{ih(t,x)}, \tag{7.2}$$

where the phase function $h(t,x) \in H^1(\Omega)$ and is the weak solution of the finite energy of the following initial-boundary value problem of the linear wave equation:

$$\begin{aligned} &h_{tt} - 2\Delta h = 0, \quad x \in \Omega,\\ &h(t,x) = h(x), \quad x \in \partial\Omega,\\ &h(0,x) = h(x), \quad h_t(0,x) = 0. \end{aligned} \tag{7.3}$$

Proof. By Proposition 3.1 ($t_\epsilon = \epsilon$), we know that vortices do not move on this slow WKB time scale. By Lemma 2.1 and Lemma 2.2:

$$v_\epsilon(t,x) \rightharpoonup \prod_{j=1}^{n} \frac{x-a_j}{|x-a_j|} e^{ih(t,x)}, \tag{7.4}$$

where $h(t,x) \in H^1(\Omega)$ for each time t. The conservation of mass is now

$$\left(\frac{1-|v_\epsilon|^2}{\epsilon}\right)_t + 2\mathrm{div}(p(v_\epsilon)) = 0, \tag{7.5}$$

and the conservation of energy implies

$$\int_{\Omega'} |\nabla v_\epsilon|^2 + \int_\Omega \frac{(1-|v_\epsilon|^2)^2}{2\epsilon^2} \le C_0, \tag{7.6}$$

where Ω' is a compact subset of Ω_a, C_0 a positive constant independent of ϵ. It follows that v_ϵ is bounded in $L^\infty([0,T]; H^1(\Omega'))$; $v_{\epsilon,t}$ bounded in $L^\infty([0,T]; H^{-1}(\Omega'))$ in view of (7.6) and (7.1); and $\frac{(1-|v_\epsilon|^2)}{\epsilon}$ bounded in $L^\infty([0,T], L^2)$. So v_ϵ is strongly compact in $C([0,T], L^2(\Omega'))$ and weakly compact in $L^\infty([0,T], H^1(\Omega'))$. Up to a subsequence if necessary: $v_\epsilon \to v$ strongly in $L^\infty([0,T]; L^2(\Omega'))$ and weakly in $L^\infty([0,T]; H^1(\Omega'))$. In the meantime, (7.5) gives:

$$\frac{1-|v_\epsilon|^2}{\epsilon} = -2\int_0^t \mathrm{div}\, p(v_\epsilon)(t')\, dt' + \frac{1-|v_\epsilon(0,x)|^2}{\epsilon} \rightharpoonup -2\int_0^t \mathrm{div}\, p(v)(t')\, dt', \tag{7.7}$$

in the sense of the distribution on Ω'. This then allows us to pass $\epsilon \downarrow 0$ in (7.1) and obtain

$$iv_t = -2v\int_0^t \mathrm{div}\, p(v)(t')\, dt',$$

in the distribution sense on Ω'. Also $|v| = 1$, $\lim_{t\downarrow 0^+} v(t,x) = h(x)$ in $L^2(\Omega')$, and $\lim_{t\downarrow 0^+} v_t(t,x) = 0$, in $H^{-1}(\Omega')$. Writing $v = e^{iH}$ shows

$$H_t - 2\int_0^t \Delta H(t')dt' = 0, \quad x \in \Omega'$$

and further letting $H = \Theta_a + h(t,x)$, with Θ_a harmonic on Ω', yields

$$h_t - 2\int_0^t \Delta h(t')dt' = 0, \quad x \in \Omega', \tag{7.8}$$

or by arbitrariness of Ω':

$$h_{tt} - 2\Delta h = 0, \quad \mathcal{D}'(\Omega_a \times [0,T]). \tag{7.9}$$

It follows that h is a distribution solution of the linear wave equation on Ω_a. The boundary data $h(t,x) = h(x)$, $x \in \partial\Omega$, follows from $v_\epsilon \to v$ in H^s, $s \in (1/2, 1)$, near the boundary and the standard trace imbedding. Finally, $h(t,x) \in H^1(\Omega)$ implies that h is the unique weak solution of (7.3) with finite total energy. The proof is complete. $\square$

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