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Propagation of Smoothness and the Rate of Exponential Convergence to Equilibrium for a Spatially Homogeneous Maxwellian Gas

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Abstract: We prove an inequality for the gain term in the Boltzmann equation for Maxwellian molecules that implies a uniform bound on Sobolev norms of the solution, provided the initial data has a finite norm in the corresponding Sobolev space. We then prove a *sharp* bound on the rate of exponential convergence to equilibrium in a weak norm. These results are then combined, using interpolation inequalities, to obtain the optimal rate of exponential convergence in the strong L^1 norm, as well as various Sobolev norms. These results are the first showing that the spectral gap in the linearized collision operator actually does govern the rate of approach to equilibrium for the full non-linear Boltzmann equation, even for initial data that is far from equilibrium.

1. Introduction

This paper concerns the large time behavior of solutions of the Boltzmann equation for Maxwellian molecules in the case of spatially homogeneous initial data:

$$\frac{\partial}{\partial t} f(\mathbf{v}, t) = Q(f, f)(\mathbf{v}, t). \tag{1.1}$$

Here, $f(\mathbf{v}, t)$ is the probability density for the velocity space distribution of the molecules at time t, and Q, which represents the effects of binary collisions, has the form:

$$Q(f, g)(\mathbf{v}) \int_{\mathbf{R}^3 \times S^2} B\left(q, \mathbf{q}\cdot\mathbf{n}/q\right) \left[f(\mathbf{v}_1)g(\mathbf{w_1}) - f(\mathbf{v})g(\mathbf{w})\right] \mathrm{d}\mathbf{w}\mathrm{d}\mathbf{n}. \tag{1.2}$$

In expression (1.2), $\mathbf{n}$ is a unit vector, and $d\mathbf{n}$ denotes *normalized* surface measure on the unit sphere S^2. Moreover $\mathbf{q} = \mathbf{v} - \mathbf{w}$ is the relative velocity, and in $\mathbf{q}\cdot\mathbf{n}$, the dot denotes the usual inner product. The vector $\mathbf{n}$ parameterizes the set of all kinematicly possible (i.e., those conserving energy and momentum) post–collisional velocities ($\mathbf{v}_1$ and $\mathbf{w}_1$) by

$$\begin{aligned}\mathbf{v}_1 &= \frac{1}{2}(\mathbf{v} + \mathbf{w} + q\mathbf{n}),\\ \mathbf{w}_1 &= \frac{1}{2}(\mathbf{v} + \mathbf{w} - q\mathbf{n}).\end{aligned} \tag{1.3}$$

The relative likelihood of these kinematicly possible outcomes depends of course on the nature of the interaction between the molecules, and this is taken into account in the rate function B. Maxwell found that when this interaction is through an r^{-5} force law, B depends only on the scattering angle θ in $\cos\theta = \mathbf{q}\cdot\mathbf{n}/q$, and not on q itself. By a *Boltzmann equation for Maxwellian molecules*, we mean throughout this paper one in which the rate function B has this simple form $B(\cos\theta)$. We shall further suppose during most of our analysis that B is integrable:

$$\int_{-1}^{1} B(u)\mathrm{d}u := b < \infty. \tag{1.4}$$

This condition *is not* satisfied for the actual rate function Maxwell considered, i.e., that one corresponding to an r^{-5} force law. In this case, the integral above diverges due to a singularity at $u = 1$, i.e., for small angle collisions. The standard strategy is to "cut off" these small angle collisions so that B becomes integrable, and then to seek estimates, that are independent of the cut-off.

When (1.4) does hold, one can split $Q(f,f)$ into its "gain" and "loss" terms $Q(f,f) = Q^{+}(f,f) - Q^{-}(f,f)$. One easily sees (since f is a probability density) that $Q^{-}(f,f)(\mathbf{v}) = bf(\mathbf{v})$ so that the Boltzmann equation can be rewritten

$$\frac{\partial}{\partial t} f(\mathbf{v},t) + bf(\mathbf{v},t) = Q^{+}(f,f)(\mathbf{v},t). \tag{1.5}$$

This equation has been extensively investigated, and much is known. In particular, existence and uniqueness have been established, and moreover, it has been shown that, given enough moments for the initial density, the convergence is exponential in the strong L^1 norm [Ar88, We93]. However, existing results provide little or no information on what the rate of this exponential convergence might be. This is significant for the following reasons.

The unit time scale relevant for Eq. (1.1) is the mean time between collisions. This time scale is much, much shorter than the time scale governing macroscopic transport phenomena, so that it is commonly believed that (1.1) governs the rate of approach to *local* equilibrium even in *non-homogeneous* settings. There is a natural conjecture as to what this rate should be, which one obtains by linearizing the collision kernel $Q(f,f)$.

That is, let $M_f(\mathbf{v})$ be the Maxwellian density

$$\begin{aligned} M_f(\mathbf{v}) &= (6\pi T)^{-3/2}\exp(-|\mathbf{v}-\mathbf{u}|^2/6T),\\ \mathbf{u} &= \int_{\mathbb{R}^3} \mathbf{v} f(\mathbf{v})\mathrm{d}^3 v,\\ 3T &= \int_{\mathbb{R}^3} |\mathbf{v}-\mathbf{u}|^2 f(\mathbf{v})\mathrm{d}^3 v. \end{aligned} \tag{1.6}$$

Then, M_f is the equilibrium solution of (1.1) towards which $f(\cdot,t)$ tends, and is, of course, independent of t: $M_{f(\cdot,t)} = M_{f(\cdot,0)}$ for all t since the temperature T and bulk velocity $\mathbf{u}$ in (1.6) are conserved.

Without loss of generality, we may suppose that our initial data is such that $T = 1$ and $\mathbf{u} = 0$, and we shall simply write M for M_f in this case.

At this point losing generality (for the moment), suppose that the density f has the form

$$f(\mathbf{v},t) = M(\mathbf{v})(1+\epsilon h(\mathbf{v},t)) \tag{1.7}$$

for some function h with

$$\int_{\mathbb{R}^3} |h(\mathbf{v},0)|^2 M(\mathbf{v})\mathrm{d}v = 1 \tag{1.8}$$

and some small number ϵ. Inserting this in (1.1), one obtains

$$\frac{\partial}{\partial t} h(\mathbf{v},t) = \mathcal{L}h(\mathbf{v},t) + \epsilon\frac{1}{M(\mathbf{v})} Q(Mh,Mh)(\mathbf{v}). \tag{1.9}$$

Here, $\mathcal{L}$ is the linearized collision operator:

$$\mathcal{L}h(\mathbf{v}) = \frac{1}{M(\mathbf{v})}\left(Q^+\big(M,Mh\big)(\mathbf{v}) + Q^+\big(Mh,M\big)(\mathbf{v})\right) - \int_{\mathbb{R}^3} Mh(\mathbf{v})\mathrm{d}v - h(\mathbf{v}). \tag{1.10}$$

Observe the $\mathcal{L}$ is self-adjoint on the Hilbert space $\mathcal{H}$ with norm

$$\|h\|^2_{\mathcal{H}} = \int_{\mathbb{R}^3} |h(\mathbf{v})|^2 M(\mathbf{v})\mathrm{d}^3 v\ . \tag{1.11}$$

The natural conjecture is that *the spectral properties of $\mathcal{L}$ govern the rate of approach to equilibrium in L^1 for solutions of (1.1).*

Now, the spectrum of $\mathcal{L}$ has been computed [WU70], and the following facts are well known: $\mathcal{L}$ is negative semi–definite on $\mathcal{H}$ with a five dimensional null space due to the conservation of total probability, bulk momentum $\mathbf{u}$, and temperature T. The remaining eigenvalues are discrete and strictly negative, and, in particular, let λ_1 denote the *absolute value* of the first of these eigenvalues when they are arranged in order of increasing magnitudes. Thus λ_1 is the "spectral gap" of the linearized collision operator.

A concise statement of one of our main results is the following:

Theorem 1.1. *Let $f_0(\mathbf{v})$ be initial data for (1.1) with Maxwellian collisions. Suppose that the bulk velocity $\mathbf{u} = 0$, and the temperature $T = 1$. Let $\epsilon > 0$ be given. Then there is a number n depending only on ϵ so that whenever*

$$\int_{\mathbb{R}^3} |\mathbf{v}|^{2n} f_0(\mathbf{v})\mathrm{d}^3 v + \int_{\mathbb{R}^3} |\xi|^{2n} |\widehat{f_0}(\xi)|^2 \mathrm{d}^3\xi < \infty$$

then it holds that

$$\|f(\cdot,t) - M\|_{L^1} \le C_\epsilon e^{-(1-\epsilon)\lambda_1 t}\ .$$

Here, λ_1 is the spectral gap of the linearized collision operator, $\widehat{f_0}$ denotes the Fourier transform of f_0, and C_ϵ is computable in terms of the integral specified above.

This result will be reformulated in more detail later in the paper, where in particular, we shall specify the relation between n, ϵ and C_ϵ. Here in the introduction, we wish to focus on a few key points.

First, transport coefficients for a rarefied gas, i.e., the bulk and thermal diffusivity, may be calculated in terms of the eigenvalues of $\mathcal{L}$ – assuming that this operator really does control the trend toward *local equilibrium*. This is not yet proved.

In fact, until now, it had not even been proved that $\mathcal{L}$ governs the rate of approach to equilibrium in the spatially homogeneous case for initial data far from equilibrium.

Cercignani, Lampis and Sgarra [CLS88] have proven an inequality for the non-linear term in (1.9) which shows this for initial data that is in a sufficiently small neighborhood of equilibrium, but for reasons we will now explain, the treatment of initial data that is far from equilibrium is delicate.

This may be surprising to those who are encountering the problem for the first time. After all, we have said above that it is known that $\|f(\cdot,t) - M\|_{L^1}$ tends to zero exponentially at *some* rate, so what can prevent it from eventually entering a small neighborhood of M in which $\mathcal{L}$ dominates the remaining evolution, with its spectrum governing the asymptotic speed of convergence?

The answer lies with the meaning of "small neighborhood". The operator $\mathcal{L}$ is self-adjoint on the Hilbert space $\mathcal{H}$, and the requirement on f is that if we write $f = M(1+h)$, then $\|h\|_{\mathcal{H}} < \epsilon_0$ for some sufficiently small number ϵ_0. Stated in terms of f, this is a requirement that

$$\int_{\mathbb{R}^3} \frac{|f(\mathbf{v},t) - M(\mathbf{v})|^2}{M(\mathbf{v})} \mathrm{d}^3 v < \epsilon_0 \tag{1.12}$$

should be sufficiently small. This requires more control on the tail of the distribution $f(\mathbf{v},t)$, uniformly in t, than is available. If it were known that for some value r with $1/2 < r < 1$,

$$\sup_{t>0} \int_{\mathbb{R}^3} |f(\mathbf{v},t)|^2 M^{-r}(\mathbf{v}) \mathrm{d}^3 v < C, \tag{1.13}$$

then the eventual validity of (1.12) would follow from the decay of $\|f(\cdot,t) - M\|_{L^1}$. However, it remains an open problem to establish (1.13) for any reasonably general class of initial data – even, say, for initial data with compact support. This is true despite the fact that *each individual moment* of f will remain bounded, uniformly in time, in terms of the initial value of that moment. In short, the lack of sufficient control on the tails of the distribution $f(\mathbf{v},t)$, uniformly in time, is a significant obstacle in the way of establishing the relevance of the spectrum of $\mathcal{L}$ in $\mathcal{H}$ to the rate of convergence to equilibrium for (1.1).

We overcome this obstacle here by establishing a propagation of smoothness result for (1.1). This follows from an inequality on the gain term Q^+ which is of independent interest, and indeed has already been applied in another problem in [CELMR96]. To state the result concisely, we introduce the Sobolev space norms $\|\cdot\|_{\mathrm{H}^k}$ by

$$\|f\|_{H^k}^2 = \int_{\mathbb{R}^3} |\widehat{f}(\xi)|^2 |\xi|^{2k} \mathrm{d}^3\xi$$

for all $k \geq 0$. Our convention for the Fourier transform is that

$$\widehat{f}(\xi,t) = \int_{\mathbb{R}^3} f(\mathbf{v},t) e^{-i\mathbf{v}\cdot\xi} \mathrm{d}^3 v.$$

We recall that the entropy of f, $H(f)$, is defined by

$$H(f) = -\int_{\mathbb{R}^3} f(\mathbf{v}) \ln f(\mathbf{v}) \mathrm{d}^3 v.$$

The key inequality enabling us to bound the H^m norm of solutions of (1.1) uniformly in time is the following:

Theorem 1.2. *Let f be any probability density on $\mathbb{R}$ with unit variance, and $\|f\|_{\mathrm{H}^m}$ finite. Then, there are universal constants $C_m < \infty$ and $K_m > 0$ so that for all such f,*

$$\|\mathcal{Q}^+(f)\|^2_{\mathrm{H}^m} \leq (1/2)\|f\|^2_{\mathrm{H}^m} + C_m \tag{1.14}$$

whenever

$$H(M_f) - H(f) \leq K_m.$$

We shall later reformulate this inequality in more detail, and in fact a better form, with explicit determination of the constants. For now we observe that as soon as $H(M_f) - H(f) \leq K_m$ holds at some time t_0, it holds at all succeeding times since $H(f)$ is strictly increasing for non-equilibrium solutions of (1.1). Then if we define $\phi(t)$ by

$$\phi(t) = \|f(\cdot,t)\|^2_{\mathrm{H}^m}$$

it easily follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq -\frac{1}{2}\phi(t) + K \qquad \text{for all } t \geq t_0$$

with the consequence that

$$\phi(t) \leq \max\{\phi(t_0), 2K\} \qquad\qquad \text{for all } t \geq t_0.$$

Hence, once one has a bound on t_0, which can be obtained from entropy production bounds [CC94], it is a simple matter to bound $\phi(t_0)$ in terms of $\phi(0)$. In this way, we obtain uniform bounds on the H^m norm of solutions of (1.1).

We shall apply this by using an interpolation inequality to bound $\|f(\cdot,t) - M_f(\cdot)\|_{L^1}$ by the geometric mean of weak norm bound on $f(\cdot,t) - M_f(\cdot)$, which decays at the required rate, and the H^m bound on this quantity which stays bounded above uniformly in t. Since, for n large enough, we shall be able to take *arbitrarily little* of the H^m norm in our geometric mean, this leads to Theorem 1.1.

Clearly then, a crucial role is played by this weak norm convergence, which is obtained by further pushing the development of a recent method for obtaining exponential convergence for Maxwellian molecules in certain weak norms [GTW95]. To show that the convergence in these weak norms is taking place at the rate suggested by the spectral gap in the linearized collision operator $\mathcal{L}$, we must work with a particular choice of these norms, outside the range originally considered.

Namely, define the norm $|\!|\!| \cdot |\!|\!|$ by

$$|\!|\!| g |\!|\!| = \sup_{\xi\in\mathbb{R}^3} \frac{|\widehat{g}(\xi)|}{|\xi|^4}. \tag{1.15}$$

This norm is well defined and finite on the space of integrable functions g such that $\int_{\mathbb{R}^3} |\mathbf{v}|^4 |g(\mathbf{v})| \mathrm{d}^3 v < \infty$, and $\int_{\mathbb{R}^3} P(\mathbf{v})|g(\mathbf{v})|\mathrm{d}^3 v = 0$ whenever $P(\mathbf{v})$ is a polynomial of total degree three or less in the components of $\mathbf{v}$. This space does not include $f(\cdot,t)$, or any probability density for that matter, but it does include $f(\cdot,t) - M_f(\cdot) - S(\cdot,t)$, where S is a subtraction term taking care of the first, second and third moments. Because of known results on the explicit exponential convergence of all of the moments of f to those of M_f, it will be easy to show that $\|S(t)\|_{L^1}$ converges to zero at faster than the required rate, and that $\|S(t)\|_{\mathrm{H}^m}$ remains bounded uniformly in time. Thus, as far as either the L^1 norm or the H^m norm are concerned, both $f(\cdot,t) - M_f(\cdot) - S(\cdot,t)$ and $f(\cdot,t) - M_f(\cdot)$ have the same decay and boundedness properties. Concerning the former, we have the following theorem:

Theorem 1.3. *Let f_0 be a probability density with $\int_{\mathbb{R}^3} |\mathbf{v}|^4 f_0(\mathbf{v}) \mathrm{d}^3 v < K$, and let $\epsilon > 0$ be given. Then there are constants B and C and a function $S(\cdot, t)$ such that*

$$\|\!|f(\cdot,t) - M_f(\cdot) - S(\cdot,t)\|\!| \leq Bte^{-t(1-\epsilon)\lambda_1} \|\!|f(\cdot,0) - M_f(\cdot) - S(\cdot,0)\|\!|$$

for all $t \geq 0$, and with $\|\!|f(\cdot,0) - M_f(\cdot) - S(\cdot,0)\|\!| < \infty$, such that for all m,

$$e^{t\lambda_1}\left(\|S(\cdot,t)\|_{L^1} + \|S(\cdot,t)\|_{\mathrm{H}^m}\right) \leq C, \quad \text{for all } t \geq 0.$$

Here λ_1 is the spectral gap of the linearized collision operator.

Again, a more explicit version will be provided later. To combine the second and third theorem to prove the first, it is only necessary to use an interpolation inequality of the form

$$\|f - M_f - S\|_{L^1} \leq C_\epsilon \|\!|f - M_f - S\|\!|^{1-\epsilon} \|f - M_f - S\|^{\epsilon}_{\mathrm{H}^m} \qquad (1.16)$$

which holds for any $\epsilon > 0$ provided f_0, and hence $f(\cdot,t) - M_f(\cdot) - S(\cdot,t)$ has sufficiently many moments and belongs to H^m for m sufficiently large. Theorem 1.2 (and part of Theorem 1.3) says that the $\|\cdot\|_{\mathrm{H}^m}$ terms stay bounded, and Theorem 1.3 says that the other norm is decaying at the desired rate.

The methods will actually yield more: we can also prove convergence in Sobolev norms for sufficiently smooth and rapidly decaying initial data, again at the exponential rate given by the spectral gap in the linearized collision operator.

We now briefly discuss related results in the literature. The result most closely related to Theorem 1.2 is the estimate of Lions [Li94]. In particular, in presence of smooth kernels B that vanish for small and large relative velocities, uniformly in the argument $\mathbf{q} \cdot \mathbf{n}/q$, the gain term has been shown to possess a regularizing effect

$$\|Q^+(f,g)\|_{H^1} \leq C\|f\|_{L^1}\|g\|_{L^2}. \qquad (1.17)$$

The main application of the above result was to prove propagation of strong L^1-compactness for renormalized solutions of the Boltzmann equation, and to prove that the weak solutions are strong, if any strong solution exists. For this purpose, the gain term is modified to have regular kernels, being the passage to the limit based on the averaging lemma.

The paper by Wennberg, [We94], gives a simplified proof of this result, using a different representation of the gain term due to Carleman [Ca57], and Radon transform estimates. Furthermore, he was able to prove a similar inequality for smooth B that *aren't* compactly supported, including the case of hard spheres, provided f and g posses sufficient additional L^p regularity and have sufficiently many moments, with norms on the right side reflecting these requirements.

As application of this, Wennberg proves for the spatially homogeneous Boltzmann equation with hard sphere collisions that if the initial data f_0 satisfies $f_0(\mathbf{v})(1+|\mathbf{v}|^2)^{1/2} \in L^1 \cap L^p$ with $p > 6$, and if $f_0 \in H^1$, then the same holds for the solutions, uniformly in time. The argument does not provide propagation of regularity in H^k, for $k > 1$.

There are few other results on propagation of smoothness for the Boltzmann equation, all of them obtained in recent years. These results are concerned with certain generalizations of the Fisher information, which, up to a constant is the square of H^1 norm of the *square root* of the density f. McKean [McK66] showed that this quantity was *monotonically decreasing* for solutions of the Kac equation. This monotonicity is possible because of the uncertainty principle which says that among all densities with given variance, Maxwellians have the least Fisher information, i.e.:

$$\|\nabla\sqrt{f}\|_{\mathrm{H}^1} \geq \|\nabla\sqrt{M_f}\|_{\mathrm{H}^1}.$$

This extremal property of Maxwellians does not hold for the Sobolev norms considered here; that is, one easily sees that there are densities f for which

$$\|f\|_{\mathrm{H}^1} \leq \|M_f\|_{\mathrm{H}^1}$$

and hence such a monotonicity property is impossible.

Moreover it is not clear how far such monotonicity results can be extended from the Kac model to the Boltzmann equation. Carlen and Carvalho [CC92] showed that the Fisher information is decreasing for the Boltzmann equation in the case of constant B, Toscani [To92] showed this for Maxwellian molecules in two dimensions, and Bobylev and Toscani [BT92] also in three dimensions with certain symmetries effectively reducing the dimension to two.

As far as the higher regularity of solutions is concerned, natural analogs of the Fisher information involving higher derivatives were recently studied by Gabetta [Ga95] and by Lions and Toscani [LT95]. They developed methods using these quantities to control the convergence towards the Gaussian density in the central limit theorem of probability theory as measured by these Sobolev norm like functionals.

The methods of Lions and Toscani have been extended to the Kac equation by Gabetta and Pareschi [GP94] to prove propagation of regularity and convergence to equilibrium in various norms of the solution. Subsequently Toscani [To96] with the same tools obtained analogous results for the solution to the Boltzmann equation for Maxwell pseudomolecules, both in plane geometry and in the axially symmetric case. The key of his proof relies in the fact that, as already mentioned, in these cases Fisher information has been shown to be a nonincreasing Lyapunov functional [BT92].

Concerning Theorem 1.1, the rate at which the solution to the Boltzmann equation approaches equilibrium has been extensively studied starting from the fifties, when Ikenberry and Truesdell [IT56] proved that all moments of the solution to the spatially homogeneous Maxwell gas, that exist initially, converge exponentially to the corresponding ones of the equilibrium distribution.

For intermolecular forces harder than Maxwellian ones, and in the presence of a cut-off, Arkeryd [Ar88] obtained stability results in L^1. These results were extended to pseudo–Maxwellian molecules by Wennberg [We93]. Here the method of proof is based on the spectral theory of the linearized collision operator, and gives exponential convergence to equilibrium, provided the initial data belong to an appropriately small neighborhood of the equilibrium itself. However, in these proofs, one uses the spectrum of the linearized operator *not* in its natural space, as discussed above, but in certain polynomially weighted L^1 spaces. Here, it is not possible to explicitly compute the spectrum, and one must resort to compactness arguments to prove the existence of a spectral gap *in the spaces considered*. Hence, such an approach, while fully successful in establishing exponential convergence, gives no information as to what the exponential rate might be.

The exponential convergence towards equilibrium has been obtained by Gabetta, Toscani and Wennberg [GBT95], for the Kac model and for Maxwellian molecules in a metric equivalent to the weak-$\star$ convergence of measures, closely related to the norm $|\!|\!|\cdot|\!|\!|$ considered here. In fact, they used a norm $|\!|\!|\cdot|\!|\!|_\alpha$ which in definition differs from the norm $|\!|\!|\cdot|\!|\!|$ in that they divided by $|\xi|^{2+\alpha}$, $\alpha > 0$ (but small) instead of $|\xi|^4$. The basic tool in [GBT95] is a Fourier transformed version of the Boltzmann equation for Maxwellian molecules, due to Bobylev [B88]. While this method gives exponential convergence in a very weak norm, it has the considerable advantage of doing so *with an explicitly computable rate*.

By taking $\alpha = 2$, we need to introduce extra subtraction terms (the function S in Theorem 1.3 becomes more complicated), but having done this, the approach can be extended to pick off the sharp behavior that we seek. We shall explain why this works in the course of proving Theorem 1.3.

We shall begin the paper by first carrying out the program in the much simpler case of the Kac model. This not only adds considerable clarity, but the results for the Kac model are interesting in their own right. Indeed, McKean proved the strong L^1 convergence to equilibrium at an exponential rate

$$\|f(\cdot,t) - M\|_{L^1} \le Ce^{-\lambda t}$$

with $\lambda \approx 0.016$. He conjectured that the true rate should be given by $\lambda = 1/4$, which is the spectral gap in the linearized collision operator for the Kac model. We shall prove this conjecture here. Our result improves his bound on the rate of decay by more than an order of magnitude.

The structure of the paper is as follows: In Sect. 2, we introduce the Kac model, and prove the analog of Theorem 1.2 in this context. In Sect. 3 we prove the analog of Theorem 1.3 in this context. At this point we need the interpolation inequalities. So we prove them in Sect. 4, in a general form suitable for both the Kac Model and the Boltzmann equation. Then, in Sect. 5, we prove the analog of Theorem 1.1 for the Kac model, and prove a conjecture of McKean. Sect. 6 then presents some geometric lemmas needed for our analysis of the Boltzmann equation. These are applied in Sect. 7 to prove Theorem 1.2. Next in Sect. 8 we prove Theorem 1.3, and finally, in Sect. 9, Theorem 1.1.

2. Propagation of Smoothness for the Kac Equation

The Kac equation is a caricature of the Boltzmann equation introduced by Kac, and reduced to its essentials by McKean. It models a gas of one dimensional particles with collisions that conserve energy but not momentum (or else, in one dimension, the number of conserved quantities would equal the number of degrees of freedom). Thus, all of the kinematicly possible collisions $(v,w) \to (v',w')$ are given by

$$v' = v\cos\theta + w\sin\theta \qquad \text{and} \qquad w' = -v\sin\theta + w\cos\theta \tag{2.1}$$

for $0 \le \theta < 2\pi$. We could introduce a weight $B(\cos\theta)$ favoring some collisions over others, as in [De94], but we shall follow McKean and simply take B to be constant. Then the gain term in the Kac model collision kernel is

$$\mathcal{Q}^+(f) = \frac{B}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} f(v')f(w')\mathrm{d}w\mathrm{d}\theta, \tag{2.2}$$

the loss term is

$$\mathcal{Q}^-(f) = \frac{B}{2\pi}\int_0^{2\pi}\int_{\mathbb{R}} f(v)f(w)\mathrm{d}w\mathrm{d}\theta \; = Bf(v),$$

and hence the equation itself is

$$\frac{\partial f(v,t)}{\partial t} + f(v,t) = \mathcal{Q}^+(f) \tag{2.3}$$

where, after a rescaling of time, we have taken $B = 1$.

Further shifting the frame of reference and rescaling, we may freely suppose that

$$M_f(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}.$$

Then, as in McKean's paper [Mk66], one linearizes about $M = M_f$ by writing

$$f = M(1 + h)$$

and finds that

$$\frac{\partial h(v,t)}{\partial t} = \mathcal{L}h(v,t) + O(h^2) \tag{2.4}$$

where $\mathcal{L}$, the linearized collision operator is given by

$$\mathcal{L}h(v) = \frac{1}{\pi M(v)} \int_0^{2\pi} \int_{\mathbb{R}} M(v')M(w')\big[h(v') + h(w') - h(v) - h(w)\big] \mathrm{d}w\mathrm{d}\theta. \tag{2.5}$$

As McKean observed, the Hermite polynomials are a complete set of eigenfunctions for $\mathcal{L}$ (which is an average over Mehler kernels). All of the odd Hermite polynomials [Mk73] have eigenvalue -1. The null space of $\mathcal{L}$ consists of the span of the first two even such polynomials, $h_0(v) = 1$ and $h_1(v) = 1 - v^2$. Let $h_{2k}(v)$ be the normalised Hermite polynomial of degree $2k$. Since the leading coefficient is a multiple of v^{2k}, we need only apply $\mathcal{L}$ to v^{2k} to determine the corresponding eigenvalue. Doing so, one has repeated McKean's calculation that

$$\mathcal{L}h_{2k}(v) = \frac{1}{\pi} \int_0^{2\pi} \sin^{2k}(\theta)\mathrm{d}\theta - 1$$

for all $k \geq 1$. The largest of these eigenvalues, λ_1, is given by $\lambda_1 = -1/4$ and corresponds to $h_4(v)$.

Our goal in the next few sections is to show that for any $\epsilon > 0$, there is a constant C_ϵ so that for all sufficiently smooth and rapidly decaying initial data $f_0(v)$, the corresponding solution $f(v,t)$ of the Kac equation satisfies

$$\|f(\cdot,t) - M(\cdot)\|_{L^1} \leq C_\epsilon e^{-(1-\epsilon)\lambda_1 t}. \tag{2.7}$$

As indicated in the introduction, the first step will be to show that the smoothness of the initial data is propagated so that we have bounds on the smoothness uniform in time. To do this, we prove the analog of Theorem 1.2 for the Kac equation gain term.

Theorem 2.1 (*Smoothness bound on the gain term for Kac equation***).** *Let f be any probability density on $\mathbb{R}$ with unit variance, and $\|f\|_{\mathrm{H}^m}$ finite. Then, whenever*

$$\|f - M_f\|_1 \leq (1/2)^{\frac{m+1}{2}},$$

$$\|\mathcal{Q}^+(f)\|^2_{\mathrm{H}^m} \leq C_m F_m\left(\|f - M_f\|_1\right) \|f\|^2_{\mathrm{H}^m} + K_m, \tag{2.8}$$

where

$$C_m = \frac{4}{(1 - \frac{1}{2}^{2/(m+1)})^{m+1/2}}, \tag{2.9}$$

$$K_m = \frac{8}{2m+1}\left[\frac{2}{e}(2m+1)(m+1)\right]^{2m+1} + 4\left\{\left\|\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right\|_{H^m} + 4\left\|\frac{1}{\sqrt{\pi}}e^{-x^2}\right\|_{H^m}\right\}, \tag{2.10}$$

and

$$F_m(x) = \frac{x^{1/(m+1)}}{\pi} + x^2, \quad x \geq 0. \tag{2.11}$$

Proof. We shall break the integral defining $\|\mathcal{Q}^+(f)\|^2_{\mathrm{H}^m}$ into several pieces. We first consider those angles θ for which either $\cos\theta$ or $\sin\theta$ is small. Fix $\epsilon > 0$, and define

$$A_\epsilon = \{\theta \mid |\theta - k\pi/2| \leq \epsilon, k = 1,2,3,4 \quad \textit{and} \quad 0 \leq \theta \leq 2\pi\}$$

and let A^c_ϵ be its complement in $[0, 2\pi]$. Then, by Jensen's inequality,

$$\begin{aligned} &\|\mathcal{Q}^+(f)\|^2_{\mathrm{H}^m} \leq \\ &\frac{1}{2\pi}\int_{A_\epsilon}\int_{\mathbb{R}} |\widehat{f}(\xi\cos\theta)|^2|\widehat{f}(\xi\sin\theta)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta + \\ &+\frac{1}{2\pi}\int_{A^c_\epsilon}\int_{\mathbb{R}} |\widehat{f}(\xi\cos\theta)|^2|\widehat{f}(\xi\sin\theta)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta. \end{aligned} \tag{2.12}$$

The integral over A_ϵ has four parts. Consider the one with $|\theta| \leq \epsilon$, on which $\cos\theta \geq \sqrt{1-\epsilon^2}$. Then with $\eta = (\cos\theta)\xi$,

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\int_{\mathbb{R}} |\widehat{f}(\xi\cos\theta)|^2|\widehat{f}(\xi\sin\theta)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta \leq \\ &\left(\frac{1}{1-\epsilon^2}\right)^{m+1/2}\frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\int_{\mathbb{R}} |\widehat{f}(\xi)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta = \\ &\left(\frac{1}{1-\epsilon^2}\right)^{m+1/2}\frac{\epsilon}{\pi}\|f\|^2_{\mathrm{H}^m}, \end{aligned}$$

where we have used the fact that

$$\sup_\xi |\widehat{f}(\xi\sin\theta)| \leq 1.$$

There are four contributions of this type, and hence

$$\frac{1}{2\pi}\int_{A_\epsilon}\int_{\mathbb{R}} |\widehat{f}(\xi\cos\theta)|^2|\widehat{f}(\xi\sin\theta)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta \leq 4\left(\frac{1}{1-\epsilon^2}\right)^{m+1/2}\frac{\epsilon}{\pi}\|f\|^2_{\mathrm{H}^m}. \tag{2.13}$$

Let us set $c = \|f - M_f\|_1$. Then

$$|\widehat{f}(\xi)| \leq |\widehat{f}(\xi) - \widehat{M}_f(\xi)| + \widehat{M}_f(\xi) \leq c + e^{-\xi^2/2}. \tag{2.14}$$

On A^c_ϵ we split the integration into the two parts where $|\xi| > R$, and $|\xi| \leq R$, for some $R > 0$ to be fixed later. On the latter region, using inequality (2.14) we obtain

$$
\begin{gathered}
\int_{A_\epsilon^c}\int_{|\xi|\le R} |\widehat{f}(\xi\cos\theta)|^2|\widehat{f}(\xi\sin\theta)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta \le \\
\int_{A_\epsilon^c}\int_{|\xi|\le R}\left(2c^2+2e^{-\xi^2\cos^2\theta}\right)\left(2c^2+2e^{-\xi^2\sin^2\theta}\right)|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta = \\
4\int_{A_\epsilon^c}\int_{|\xi|\le R}\left(c^4+c^2e^{-\xi^2\sin^2\theta}+c^2e^{-\xi^2\cos^2\theta}+e^{-\xi^2}\right)|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta.
\end{gathered} \tag{2.15}
$$

From now on, let us fix $\epsilon^2 \le 1/2$. Then, on the set A_ϵ^c, or $\sin^2\theta \ge 1/2$, or $\cos^2\theta \ge 1/2$. Hence, by (2.15)

$$
\begin{gathered}
\frac{1}{2\pi}\int_{A_\epsilon^c}\int_{|\xi|\le R} |\widehat{f}(\xi\cos\theta)|^2|\widehat{f}(\xi\sin\theta)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta \le \\
4\left(c^4+c^2\right)\int_{|\xi|\le R}|\xi|^{2m}\mathrm{d}\xi + 4c^2\left\|\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right\|_{H^m} + 4\left\|\frac{1}{\sqrt{\pi}}e^{-x^2}\right\|_{H^m} \le \\
\frac{8}{2m+1}c^2R^{2m+1} + 4\left\{\left\|\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right\|_{H^m} + 4\left\|\frac{1}{\sqrt{\pi}}e^{-x^2}\right\|_{H^m}\right\}.
\end{gathered} \tag{2.16}
$$

To handle the integration over the remaining region, we use the fact that

$$
\exp\left\{-\frac{\xi^2}{2}\epsilon^2\right\} \le c
$$

if $|\xi| \ge -\frac{1}{\epsilon}\log c^2$. Thus, if we choose $R = R(c,\epsilon) - \frac{1}{\epsilon}\log c^2$, by (2.14) we conclude that, if $\xi \ge R$, or $|\widehat{f}(\xi\sin\theta)| \le 2c$, or $|\widehat{f}(\xi\cos\theta)| \le 2c$. So we obtain

$$
\begin{gathered}
\frac{1}{2\pi}\int_{A_\epsilon^c}\int_{|\xi|>R} |\widehat{f}(\xi\cos\theta)|^2|\widehat{f}(\xi\sin\theta)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta \le \\
\frac{4c^2}{(1-\epsilon^2)^{m+1/2}}\|f\|^2_{H^m}.
\end{gathered} \tag{2.17}
$$

Let us fix $c = \epsilon^{(m+1)}$. Then

$$
\sup_{c\le 1} c^2R(c, c^{1/(m+1)}) = \left[\frac{2}{e}(2m+1)(m+1)\right]^{2m+1}, \tag{2.18}
$$

and, grouping inequalities (2.13), (2.16) and (2.17) we obtain

$$
\begin{gathered}
\|\mathcal{Q}^+(f)\|^2_{\mathrm{H}^m} \le \frac{4}{(1-c^{2/(m+1)})^{m+1/2}}\left[\frac{c^{1/(m+1)}}{\pi}+c^2\right]\|f\|^2_{H^m} + \\
\frac{8}{2m+1}\left[\frac{1}{e}(2m+1)(m+1)\right]^{2m+1} + 4\left\{\left\|\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right\|_{H^m} + 4\left\|\frac{1}{\sqrt{\pi}}e^{-x^2}\right\|_{H^m}\right\}.
\end{gathered} \tag{2.19}
$$

This proves the theorem, and gives at the same time the explicit form of the function $F_m(\cdot)$ and of the constants C_m and K_m. □

Several useful variants of inequality (2.8) are easily established. In particular, this result becomes more useful if we replace the L^1 norm, which is not monotonic, with the relative entropy, which is. This is easily done using the Kullback–Cszilar inequality which says that

$$\|f - M_f\|_{L^1}^2 \leq 2\big(H(M_f) - H(f)\big)$$

and thus we can take

$$c^2 = \frac{1}{2}\left(H(M_f) - H(f)\right)$$

in inequality (2.14). We have

Theorem 2.2. *Smoothness bound on the gain term for Kac equation, entropic version. Let f be any density on $\mathbb{R}^3$ with $\|f\|_{\mathrm{H}^m}$ finite. Then, whenever*

$$\begin{gathered} H(M_f) - H(f) \leq (1/2)^{m+2}, \\ \|\mathcal{Q}^+(f)\|_{\mathrm{H}^m}^2 \leq C_m G_m\left(H(M_f) - H(f)\right)\|f\|_{\mathrm{H}^m}^2 + K_m, \end{gathered} \tag{2.20}$$

where C_m and K_m are defined by (2.9) and (2.10) respectively, and

$$G_m(x) = \frac{1}{\pi}\left(\frac{x}{\sqrt{2}}\right)^{1/2(m+1)} + \frac{1}{2}x, \quad x \geq 0. \tag{2.21}$$

The next version is given in terms of the Fisher information, and is different in that it does not require any “smallness” condition to apply.

The main point is the determination of a bound on the decay of $|\widehat{f}(\xi)|$ in terms of the Fisher information $I(f)$. For the Kac equation, as well for Maxwell pseudomolecules and for certain rate functions $b(\cos\theta)$, Fisher information is known to be non-increasing in time when evaluated along the solution [Mk66, BT92].

The result that follows is independent of the dimension, even regarding the constants, so we prove it on $\mathbb{R}^n$.

The Fisher information $I(f)$ is defined by

$$I(f) = 4\int_{\mathbb{R}^n} |\nabla\sqrt{f(v)}|^2\, dv \int_{\mathbb{R}^n} |\nabla \log f(v)|^2 f(v)\, dv. \tag{2.22}$$

We have

Lemma 2.3. *For any probability density f on $\mathbb{R}^n$ with $I(f)$ finite,*

$$|\widehat{f}(\xi)| \leq \frac{\sqrt{I(f)}}{|\xi|}. \tag{2.23}$$

Proof. As in the standard proof of the Riemann–Lebesgue Lemma, write

$$\widehat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^n}\big(f(v) - f(v + (\pi/|\xi|^2)\xi)\big)e^{i\xi\cdot v}\mathrm{d}v,$$

so that

$$|\widehat{f}(\xi)| \leq \frac{1}{2}\int_{\mathbb{R}^n}|f(v) - f(v + (\pi/|\xi|^2)\xi)|\mathrm{d}v.$$

Next, write

$$
\begin{aligned}
&|\big(f(v) - f(v + (\pi/|\xi|^2)\xi)\big)| = \\
&\big(\sqrt{f(v)} + \sqrt{f(v + (\pi/|\xi|^2)\xi)}\big)|\big(\sqrt{f(v)} - \sqrt{f(v + (\pi/|\xi|^2)\xi)}\big)| \leq \\
&\big(\sqrt{f(v)} + \sqrt{f(v + (\pi/|\xi|^2)\xi)}\big)(\pi/|\xi|)\int_0^1 |\nabla \sqrt{f(v + t(\pi/|\xi|^2)\xi)}|\, dt.
\end{aligned}
$$

Inserting this into the integral over v, and applying the Schwarz inequality, and then the Minkowski inequlity twice, one easily gets the stated result. □

Theorem 2.4. *(Smoothness bound on the gain term for Kac equation, Fisher information version.) Let f be any density on $\mathbb{R}$ with $\|f\|_{\mathrm{H}^m}$ finite. Then there is a constant $K_m(I(f))$ so that*

$$
\|\mathcal{Q}_+(f)\|^2_{\mathrm{H}^m} \leq \frac{1}{2}\|f\|^2_{\mathrm{H}^m} + K_m(I(f)). \tag{2.24}
$$

Proof. We proceed as in the proof of Theorem 2.1, except that we use $I(f_0)$ to control the size of $|\widehat{f}(\xi)|$ for large ξ. In consequence of Lemma 2.3, inequality (2.17) can now be substituted by

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{A_\epsilon^c}\int_{|\xi|>R} |\widehat{f}(\xi\cos\theta)|^2|\widehat{f}(\xi\sin\theta)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\theta \leq \\
&\qquad \frac{1}{(1-\epsilon^2)^{m+1/2}}\frac{I(f)}{R^2\epsilon^2}\|f\|^2_{H^m}.
\end{aligned} \tag{2.25}
$$

Grouping inequalities (2.13), (2.16) and (2.25) we obtain

$$
\begin{aligned}
&\|\mathcal{Q}^+(f)\|^2_{\mathrm{H}^m} \leq \frac{1}{(1-\epsilon^2)^{m+1/2}}\left[\frac{4\epsilon}{\pi} + \frac{I^2(f)}{R^2\epsilon^2}\right]\|f\|^2_{H^m} + \\
&\frac{8}{2m+1}R^{2m+1} + 4\left\{\left\|\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right\|_{H^m} + 4\left\|\frac{1}{\sqrt{\pi}}e^{-x^2}\right\|_{H^m}\right\}.
\end{aligned} \tag{2.26}
$$

Now, choose $R = 2\sqrt{I(f)}/\epsilon$, and then $\epsilon = \epsilon(I(f))$ to satisfy

$$
\frac{1}{(1-\epsilon^2)^{m+1/2}}\left[\frac{4\epsilon}{\pi} + \frac{1}{4}\right] = \frac{1}{2}
$$

and the result follows. □

3. Optimal Exponential Convergence in the $\vert\!\vert\!\vert \cdot \vert\!\vert\!\vert$ Norm for the Kac Equation

We present here, in the spirit of the proof of [GTW95], a new estimate for the rapid convergence towards equilibrium when sufficiently many moments exist initially. The new feature is that the *best possible* rate is obtained when the fourth moment exists.

Theorem 3.1. *Let f_0 be a probability density with $\int_{\mathbb{R}}|v|^4|f_0(v)\mathrm{d}v < K$. Then there are constants B and C and a function $S(\cdot,t)$ such that*

$$
\vert\!\vert\!\vert f(\cdot,t) - M_f(\cdot) - S(\cdot,t)\vert\!\vert\!\vert \leq Bte^{-t\lambda_1}\vert\!\vert\!\vert f(\cdot,0) - M_f(\cdot) - S(\cdot,0)\vert\!\vert\!\vert
$$

for all $t \geq 0$ and with $\vert\!\vert\!\vert f(\cdot,0) - M_f(\cdot) - S(\cdot,0)\vert\!\vert\!\vert < \infty$, such that for all m,

$$
e^{t\lambda_1}\big(\|S(\cdot,t)\|_{L^1} + \|S(\cdot,t)\|_{\mathrm{H}^m}\big) \leq C, \quad \text{for all } t \geq 0.
$$

Here, $\lambda_1 = 1/4$ is the spectral gap in the linearized collision operator for the Kac model.

Proof. First, we need some preliminary bounds on the evolution of the moments to control the subtraction term S. These are simple analogs for the Kac model of certain bounds proved for the Boltzmann equation by Ikenberry and Truesdell [IT55]. Note in particular that the moments satisfy equations that are independent of the particular solution f. We will use this fact instead of direct calculation in the case of Maxwellian molecules.

For any natural number k, let us denote $m_k(t) = \int_{\mathbf{R}} v^k f(v,t)\,dv$, and let us put $m_k = m_k(0)$. An easy computation shows that,

$$\int_{\mathbf{R}} v^k Q^+(f,f)(v)\,dv$$

$$\int_{\mathbf{R}^2\times[0,2\pi]} \frac{1}{2\pi}(v\cos\theta + w\sin\theta)^k f(v)f(w)\,dvdwd\theta.$$

Hence,

$$\int_{\mathbf{R}} v^{2k+1} Q^+(f,f)(v)\,dv = 0,$$

which implies

$$m_{2k+1}(t) = m_{2k+1}e^{-t},$$

and, if $k = 4$, owing to the conservation of the energy ($m_2(t) = m_2$), we obtain that $m_4(t)$ satisfies

$$\frac{d}{dt}\left[m_4(t) - 3m_2^2\right] = -\frac{1}{4}\left[m_4(t) - 3m_2^2\right]$$

so that

$$m_4(t) = 3m_2^2 + e^{-(1/4)t}\left[m_4 - 3m_2^2\right].$$

Let M denote the Maxwellian distribution with the same mass and temperature of f, and let us put

$$\phi(\xi,t) = \widehat{f}(\xi,t) - \widehat{M}(\xi).$$

Now, we can't divide by $|\xi|^4$ and expect a finite supremum norm because of the possibly non-vanishing first and third moments of f. Thus we introduce a subtraction term to cancel these out. This is built by taking the third order Taylor polynomial of ϕ, and multiplying by a cut-off function as follows:

Since f_0 has four finite moments, taking a Taylor expansion of ϕ up to the fourth order, and using the above bounds, one gets

$$\phi(\xi,t) = \left[-im_1\xi + im_3\frac{\xi^3}{3!}\right]e^{-t} + \frac{\xi^4}{4!}\left[m_4 - 3m_2^2\right]e^{-(1/4)t} + o(\xi^4).$$

Define $X(\xi) = \xi$ if $|\xi| \le 1$, and $X(\xi) = 0$ otherwise, and let

$$\widehat{S}(\xi,t) = \left[-im_1X(\xi) + im_3\frac{X(\xi)^3}{3!}\right]e^{-t} + \frac{X(\xi)^4}{4!}\left[m_4 - 3m_2^2\right]e^{-(1/4)t}.$$

$$\phi_1(\xi,t) = \phi(\xi,t) - \widehat{S}(\xi,t).$$

Consider that

$$\frac{\partial}{\partial t}\widehat{S}(\xi,t) + \widehat{S}(\xi,t)\frac{3}{4}\frac{X(\xi)^4}{4!}\left[m_4 - 3m_2^2\right]e^{-(1/4)t}.$$

Hence, ϕ_1 satisfies

$$\frac{\partial}{\partial t}\phi_1 + \phi_1\widehat{Q}^+(\phi_1,\widehat{f}) + \widehat{Q}^+(\widehat{M},\phi_1) + \widehat{Q}^+(\widehat{S},\widehat{f}) + \widehat{Q}^+(\widehat{M},\widehat{S}) - \frac{3}{4}\frac{X(\xi)^4}{4!}\left[m_4 - 3m_2^2\right]e^{-(1/4)t}.$$

Now,

$$\widehat{Q}^+(\widehat{S},\widehat{f})(\xi,t) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\left[-im_1X(\xi\cos\theta) + im_3\frac{X(\xi\cos\theta)^3}{3!}\right]e^{-t} + \frac{X(\xi\cos\theta)^4}{4!}\left[m_4 - 3m_2^2\right]e^{-(1/4)t}\right\}\widehat{f}(\xi\sin\theta)\,d\theta.$$

On the other hand, for all $k \in \mathbf{N}$,

$$\int_0^{2\pi}\xi^k\sin^k\theta X(\xi\cos\theta)\,d\theta = 0,$$

$$\int_0^{2\pi}\xi^k\sin^k\theta X(\xi\cos\theta)^3\,d\theta = 0.$$

This implies that

$$\frac{1}{2\pi}\int_0^{2\pi} im_1X(\xi\cos\theta)\left[1 - im_1\xi e^{-t} - m_2\frac{\xi^2}{2}\right]\,d\theta = 0$$

and, since for a certain $\bar{\xi}$

$$\left|\widehat{f}(\xi,t) - 1 + im_1\xi e^{-t} + m_2\frac{\xi^2}{2}\right| \leq \frac{|\xi|^3}{3}\left|\widehat{f}'''(\bar{\xi},t)\right|$$

and by Hölder inequality

$$\left|\widehat{f}'''(\bar{\xi},t)\right| \leq \int_{\mathbf{R}}|v|^3 f(v,t)\,dv \leq \left[m_4 - 3m_2^2\right]^{3/4}$$

we obtain

$$\frac{1}{2\pi}\left|\int_0^{2\pi} im_1X(\xi\cos\theta)\widehat{f}(\xi\sin\theta,t)\,d\theta\right| \leq |m_1|\left[m_4 - 3m_2^2\right]^{3/4}\frac{|\xi|^4}{3!}.$$

Using the same method, and recalling that

$$\left|\widehat{f}'(\bar{\xi},t)\right| \leq \int_{\mathbf{R}}|v| f(v,t)\,dv \leq m_2^{1/2} = 1,$$

we obtain

$$\frac{1}{2\pi}\left|\int_0^{2\pi} im_3\frac{X(\xi\cos\theta)}{3!}\widehat{f}(\xi\sin\theta,t)\,d\theta\right| \leq |m_3|\frac{|\xi|^4}{3!}.$$

The exact same bound can be derived for $\widehat{Q}^+(\widehat{M}, \widehat{S})$. Thus

$$|\widehat{Q}^+(\widehat{S}, \widehat{f})(\xi, t)| + |\widehat{Q}^+(\widehat{M}, \widehat{S})| \le$$
$$2e^{-t}\left[|m_1|\left[m_4 - 3m_2^2\right]^{3/4} + |m_3|\right]\frac{|\xi|^4}{3!} + 2e^{-(1/4)t}\left[m_4 - 3m_2^2\right]\frac{|\xi|^4}{4!}.$$

We can simplify this using the fact that $|m_k| \le m_n^{k/n}$ for all even $n > k$. The bound becomes

$$|\widehat{Q}^+(\widehat{S}, \widehat{f})(\xi, t)| + |\widehat{Q}^+(\widehat{M}, \widehat{S})| \le \left(4m_4 e^{-t} + [m_4 - 3m_2^2]e^{-t/4}\right)\frac{|\xi|^4}{3!}.$$

Hence

$$\left|\frac{\partial}{\partial t}\phi_1 + \phi_1\right| \le |\widehat{Q}^+(\phi_1, \widehat{f}) + \widehat{Q}^+(\widehat{M}, \phi_1)| + \left[ce^{-t} + de^{-(1/4)t}\right]|\xi|^4$$

with constants c and d which are given explicitly just above.

Define $\phi_2(\xi, t) = \phi_1(\xi, t)/|\xi|^4$. Then, ϕ_2 satisfies

$$\left|\frac{\partial}{\partial t}\phi_2 + \phi_2\right| \le \frac{1}{|\xi|^4}|\widehat{Q}^+(\phi_1, \widehat{f}) + \widehat{Q}^+(\widehat{M}, \phi_1)| + \left[ce^{-t} + de^{-(1/4)t}\right].$$

Now it remains to estimate

$$\frac{1}{|\xi|^4}|\widehat{Q}^+(\phi_1, \widehat{f}) + \widehat{Q}^+(\widehat{M}, \phi_1)|\ ,$$

and to show how it is controlled by the spectral gap.

Consider first the term $|\widehat{Q}^+(\widehat{M}, \phi_1)|/|\xi|^4$. Clearly

$$|\widehat{Q}^+(\widehat{M}, \phi_1)|/|\xi|^4 \le$$
$$\frac{1}{2\pi}\int_0^{2\pi} |\widehat{M}((\cos\theta)\xi)\phi_2((\sin\theta)\xi)|(\sin\theta)^4 \mathrm{d}\theta \le$$
$$\|\widehat{M}\|_{L^\infty}\|\!|\phi_2|\!\|\frac{1}{2\pi}\int_0^{2\pi}(\sin\theta)^4\mathrm{d}\theta.$$

Now, $\|\widehat{M}\|_{L^\infty} = 1$, and since the same is true for $\widehat{f}$, we get the same bound for the other term. The value of the integral is $3/8$, and so we have

$$\frac{\partial}{\partial t}\|\!|\phi_2|\!\| + \frac{1}{4}\|\!|\phi_2|\!\| \le \left[ce^{-t} + de^{-(1/4)t}\right]$$

which is

$$\frac{\partial}{\partial t}\left(e^{(1/4)t}\|\!|\phi_2|\!\|\right) \le \left[ce^{-(3/4)t} + d\right]$$

so that upon integration we have

$$e^{(1/4)t}\|\!|\phi_2(t)|\!\| \le (4/3)c + dt.$$

This proves the bound, and makes it a simple matter to reckon the constants. □

This proves the theorem, but it will be helpful for what follows to make the connection with $\mathcal{L}$ more explicit. To see the role of $\mathcal{L}$ explicitly, note that the bound we get for $\widehat{Q}^{+}(\phi_1, \widehat{f}) + \widehat{Q}^{+}(\widehat{M}, \phi_1)$, is, as we have pointed out, the same as the bound we would get for $\widehat{Q}^{+}(\phi_1, \widehat{M}) + \widehat{Q}^{+}(\widehat{M}, \phi_1)$. But if we define h by $\phi_1 = Mh$, then

$$\widehat{Q}^{+}(\phi_1, \widehat{M}) + \widehat{Q}^{+}(\widehat{M}, \phi_1) - \phi_1 = \widehat{M\mathcal{L}h}$$

since $\int_{\mathbb{R}} Mh(\mathbf{v})\mathrm{d}v = 0$. This is the reason that the constant we get is the spectral gap – which is naturally best possible. For the same reason we will see that the spectral gap controls the approach to equilibrium also for Maxwellian molecules. There, however, it will be more convenient to rely on a further development of the above explanation, instead of on direct calculation. Hence Sect. 8 will shed light on why this approach did yield the optimal bound.

4. Interpolation Inequalities

This section contains the several interpolation inequalities that we shall use to extract strong convergence estimates from weak convergence estimates. The first result shows that $\|\!| \cdot |\!\|_\alpha$ and arbitrarily little $\|\cdot\|_{\mathrm{H}^m}$ control $\|\cdot\|_{\mathrm{H}^k}$ for m sufficiently larger than k.

Theorem 4.1. *Let $k \geq 0$ and $\alpha, \beta, r > 0$, $0 < r < 1$, be given. Then*

$$\|f\|_{\mathrm{H}^k} \leq C(r,\beta) \|\!|f|\!\|_\alpha^{2(1-r)} \left(\|f\|_{\mathrm{H}^M}^{2r} + \|f\|_{\mathrm{H}^N}^{2r} \right)$$

with

$$M = \frac{k + (2+\alpha)(1-r)}{r}, \qquad N = M + \frac{(1-r)(n+\beta)}{2r},$$

$$C(r,\beta) = \left(|B^n|(1 + n/\beta) \right)^{1-r},$$

and where $|B^n|$ denotes the volume of the unit ball in $\mathbb{R}^n$.

Proof. For any $p > 0$, and any r with $0 < r < 1$,

$$\|f\|_{\mathrm{H}^k}^2 = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\xi|^{2k} \mathrm{d}^n\xi =$$

$$\int_{\mathbb{R}^n} \frac{|\widehat{f}(\xi)|^{2-2r}}{|\xi|^{(2+\alpha)(2-2r)}} |\widehat{f}(\xi)|^{2r} |\xi|^{2k+(2+\alpha)(2-2r)} (1+|\xi|^p)^r (1+|\xi|^p)^{-r} \mathrm{d}^n\xi \leq$$

$$\|\!|f|\!\|_\alpha^{2-2r} \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^{2r} |\xi|^{2k+(2+\alpha)(2-2r)} (1+|\xi|^p)^r (1+|\xi|^p)^{-r} \mathrm{d}^n\xi \leq$$

$$\|\!|f|\!\|_\alpha^{2-2r} \left(\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\xi|^{(2k+(2+\alpha)(2-2r))/r} (1+|\xi|^p) \mathrm{d}^n\xi \right)^r \times$$

$$\left(\int_{\mathbb{R}^n} (1+|\xi|^p)^{-r/(1-r)} \mathrm{d}^n\xi \right)^{1-r}.$$

For the last integral to converge, we require that $p > r(1-r)n$, so let $\beta > 0$, and put $p = r(1-r)(n+\beta)$. Then the integral is

$$\int_{\mathbb{R}^n} (1 + |\xi|^{r(1-r)(n+\beta)})^{-1/r(1-r)} \mathrm{d}^n\xi \leq$$

$$|B^n| + n|B^n| \int_0^\infty r^{-(1+\beta)} \mathrm{d}r = |B^n|(1 + n/\beta)\,. \qquad \square$$

The next inequality shows that control of the sufficiently many moments and control on the L^2 norm together control the L^1 norm.

Theorem 4.2. *Let f be an integrable function on $\mathbb{R}^n$. Then for all $k > 0$,*

$$\int_{\mathbb{R}^n} |f(v)| \mathrm{d}^n v \leq C(n,k) \left(\int_{\mathbb{R}^n} |f(v)|^2 \mathrm{d}^n v \right)^{\frac{2k}{(n+4k)}} \left(\int_{\mathbb{R}^n} |v|^{2k} f(v) \mathrm{d}^n v \right)^{\frac{n}{(n+4k)}},$$

where

$$C(n,k) = \left[\left(\frac{n}{4k} \right)^{4k/(n+4k)} + \left(\frac{4k}{n} \right)^{n/(n+4k)} \right] |B^n|^{2k/(n+4k)}$$

and again, $|B^n|$ denotes the volume of the unit ball in $\mathbb{R}^n$.

Proof. We may assume that f is non-negative. Also, by translation invariance, we may assume that the infimum above is achieved at $v = 0$.

Let $R > 0$ be chosen. Then

$$\int_{\mathbb{R}^n} f(v) \mathrm{d}^n v = \int_{|v| \leq R} f(v) \mathrm{d}^n v + \int_{|v| \geq R} f(v) \mathrm{d}^n v \leq$$
$$\left(|B^n| R^n \right)^{1/2} \|f\|_{L^2} + R^{-2k} \int_{\mathbb{R}^n} |v|^{2k} f(v) \mathrm{d}^n v.$$

Optimizing in R now yields the result. $\square$

5. Optimal Exponential Convergence in the Strong L^2 Norm and Sobolev Norms for the Kac Equation

One now easily sees that the assertion in (1.16) is a consequence of Theorems 4.1 and 4.2, with a readily computed constant C_ϵ. Since Theorem 2.3 gives us a uniform bound on $\|f(\cdot,t)\|_{\mathrm{H}^m}$ provided it is finite initially, and provided the Fisher information $I(f)$ is initially finite, combining this with Theorem 3.1, we have:

Theorem 5.1. *Let f_0 be any probability density on the real line with unit variance, finite fourth moment m_4 and finite Fisher information $I(f_0)$. Then for any $\epsilon > 0$ there is a fixed constant m, depending only on ϵ, so that if*

$$\int_{\mathbb{R}} f(\mathbf{v}) |\mathbf{v}|^{2m} \mathrm{d}v + \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 |\xi|^{2m} \mathrm{d}\xi \leq K$$

there is a universal, computable constant C depending only on ϵ, m_4, $I(f_0)$ and K so that the solution of the Kac equation $f(\cdot,t)$ with initial data $f_0(\cdot)$ satisfies

$$\|f(\cdot,t) - M_f(\cdot)\|_{L^1} \leq C e^{-(1-\epsilon)\lambda_1 t},$$

where λ_1 is the spectral gap in the linearized collision operator L for the Kac equation; i.e., $\lambda_1 = 1/4$.

Moreover, increasing m we obtain the same result if the L^1 norm is replaced by any H^k *norm.*

This result proves a conjecture of McKean [Mk66] concerning the optimal rate of convergence for the Kac equation, though the result requires moments and Sobolev space regularity of all orders to reach the optimal rate of convergence. In contrast, McKean's proof used only a bound on the third moment and on the H^1 norm of the square root of the density f, but yielded a bound on the rate that was smaller by more than an order of magnitude, and that did not improve in the presence of greater regularity.

This concludes our treatment of the Kac model. We now turn to the Maxwellian case.

6. Geometric Lemmas for the Maxwellian Gain Term

Fix a unit vector $\mathbf{n}$ in $\mathbb{R}^3$, and consider the two maps $\Omega_\pm : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\Omega_\pm(\xi) = \xi_\pm = \frac{\xi \pm |\xi|\mathbf{n}}{2}. \tag{6.1}$$

Each of these maps "opens up" $\mathbb{R}^3$ into a half space with $\mathbf{n}$ as the bounding normal. More precisely, notice that

$$2\xi_+ \cdot \mathbf{n} = \xi \cdot \mathbf{n} + |\xi| \tag{6.2}$$

which is always positive unless ξ is a negative multiple of $\mathbf{n}$, in which case $\xi_+ = 0$. Thus, if we delete the ray antiparallel to $\mathbf{n}$ from the domain of Ω_+, its range is

$$H(\mathbf{n}) = \{\eta \mid \eta \cdot \mathbf{n} > \mathbf{0}\}. \tag{6.3}$$

Moreover, this restricted map is one-to-one on this range, and one easily works out, using

$$|\xi_+|^2 = |\xi|(\xi \cdot \mathbf{n}), \tag{6.4}$$

that the inverse map is

$$\Omega_+^{-1}(\eta) = 2\eta - \frac{|\eta|^2}{(\eta \cdot \mathbf{n})}\mathbf{n}. \tag{6.5}$$

Moreover, let θ and $\Theta_\pm$ be defined by

$$\cos(\theta) = \frac{\xi \cdot \mathbf{n}}{|\xi|} \quad \text{and} \quad \cos(\Theta_\pm)\frac{\xi_\pm \cdot \mathbf{n}}{|\xi_\pm|}. \tag{6.6}$$

Then one easily deduces from the above that

$$\Theta_+ = \theta/2 \qquad and \qquad |\xi_+||\xi| \cos(\theta/2). \tag{6.7}$$

It is now an easy matter to compute the Jacobian of the coordinate transformation given by Ω_+:

$$J\left(\frac{\partial \xi}{\partial \xi_+}\right) = \frac{4}{\cos^2(\Theta_+)} = \frac{4}{\cos^2(\theta/2)}. \tag{6.8}$$

An exactly analogous analysis of Ω_- leads to:

$$J\left(\frac{\partial \xi}{\partial \xi_-}\right)\frac{4}{\cos^2(\Theta_-)} = \frac{4}{\sin^2(\theta/2)}. \tag{6.9}$$

The first Jacobian is singular only near $\theta = \pi$, and the second only near $\theta = 0$. In particular, away from the origin, at least one of the two is bounded by 8.

7. Propagation of Smoothness for Maxwellian Molecules

Once again, we use the Fourier transform. Since the Maxwellian collision kernel is less directly related to convolution than the Kac collision kernel, the computation is more involved. But it has been carried out by Bobylev [Bo88] who found that

$$\widehat{\mathcal{Q}}^{+}(f,f)(\xi) = \int_{S^2} \widehat{f}(\xi_+)\widehat{f}(\xi_-)B(\mathbf{n}\cdot\xi/|\xi|)\mathrm{d}\mathbf{n}. \tag{7.1}$$

The notation is that introduced in the previous section. The next result contains Theorem 1.2, and provides some explicit information that we did not bring into the introduction. It is possible to present it in a form analogous to that of Theorem 2.1 for the Kac model, but this would require more work, and the present version suffices.

Moreover, to keep the formulas readable, we explicitly treat the case $B = 1$. It will be clear from the proof that all we really use is the fact that $\int |B(\xi\cdot\mathbf{n}/|\xi|)\mathrm{d}\mathbf{n} < \infty$. Thus Theorem 1.2 will be established in the stated generality.

Theorem 7.1 (*Smoothness bound on the gain term for Maxwellian molecules***).** *Let f be any probability density with finite second moments on $\mathbb{R}^3$ such that $\|f\|_{\mathrm{H}^m}$ is finite. Then there is a constant $C(m,T)$ so that*

$$\|\mathcal{Q}_+(f)\|^2_{\mathrm{H}^m} \le (1/2)\|f\|^2_{\mathrm{H}^m} + C(m,T) \tag{7.2}$$

whenever

$$\|f - M_f\|^2_{L^1} \le 2^{-(m+5)}. \tag{7.3}$$

Proof. Fix any $\epsilon > 0$. Then for $j = 1,2$, define the sets $A_j^{(\epsilon)}$ by

$$A_j^{(\epsilon)} = \{(\xi,\mathbf{n}) \mid ((1+(-1)^j\cos\theta)^2 \le (2\epsilon)^2\}. \tag{7.4}$$

Then

$$\int_{\mathcal{S}^2}\int_{\mathbb{R}^3} |\widehat{f}(\xi_+)|^2|\widehat{f}(\xi_-)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\mathbf{n} =$$
$$\sum_{j=1,2}\int_{\mathcal{S}^2}\int_{\mathbb{R}^3} 1_{A_j^{(\epsilon)}}(\xi,\mathbf{n})|\widehat{f}(\xi_+)|^2|\widehat{f}(\xi_-)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\mathbf{n} +$$
$$\int_{\mathcal{S}^2}\int_{\mathbb{R}^3}(1-\sum_{j=1,2}1_{A_j^{(\epsilon)}}(\xi,\mathbf{n}))|\widehat{f}(\xi_+)|^2|\widehat{f}(\xi_-)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\mathbf{n}.$$

Consider first the integral over $A_2^{(\epsilon)}$. On the set, ξ_+ and ξ are close to one another, but we have little control over ξ_-. Thus we use the Riemann-Lebesgue theorem to estimate:

$$|\widehat{f}(\xi_-)|^2 \le 1.$$

Next we observe that

$$1-\cos\theta = 1-\cos(2\Theta_+) = 2\sin^2(\Theta_+)$$

and that

$$\sin^2(\Theta_+) \ge (1-\cos\Theta_+).$$

Hence,

$$A_2^{(\epsilon)}(\xi,\mathbf{n}) \subset A_2^{(\epsilon)}(\xi_-,\mathbf{n}). \tag{7.5}$$

Furthermore, on $A_2^{(\epsilon)}$, $\cos\theta \geq 1-2\epsilon$. Hence $\cos^2(\theta/2) \geq 1-\epsilon$, and the Jacobian $J(\partial\xi/\partial\xi_\pm)$ is bounded above $4/(1-\epsilon)$. Finally, $|\xi| \leq |\xi_-|/\sqrt{1-\epsilon}$. Hence,

$$\begin{aligned}\int_{\mathcal{S}^2}\int_{\mathbb{R}^3} 1_{A_2^{(\epsilon)}}(\xi,\mathbf{n})|\widehat{f}(\xi_+)|^2|\widehat{f}(\xi_-)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\mathbf{n} \leq \\ \frac{4}{(1-\epsilon)^{(m+1)}}\int_{\mathcal{S}^2}\int_{H(\mathbf{n})} 1_{A_2^{(\epsilon)}}(\eta,\mathbf{n})|\widehat{f}(\eta)|^2||\eta|^{2m}\mathrm{d}\eta\mathrm{d}\mathbf{n} \leq \\ \frac{4}{(1-\epsilon)^{(m+1)}}\int_{\mathcal{S}^2}\int_{\mathbb{R}^3} 1_{A_2^{(\epsilon)}}(\eta,\mathbf{n})|\widehat{f}(\eta)|^2||\eta|^{2m}\mathrm{d}\eta\mathrm{d}\mathbf{n} = \\ \frac{4}{(1-\epsilon)^{(m+1)}}\int_{\mathbb{R}^3}\left(\int_{\mathcal{S}^2} 1_{A_2^{(\epsilon)}}(\xi,\mathbf{n})\mathrm{d}\mathbf{n}\right)|\widehat{f}(\eta)|^2||\eta|^{2m}\mathrm{d}\eta = \\ \frac{4}{(1-\epsilon)^{(m+1)}}\left(4\pi\epsilon\right)\|f\|^2_{\mathrm{H}^m},\end{aligned} \tag{7.6}$$

where $H(\mathbf{n})$ is defined in (6.3). Clearly, we get the same bound for the integral over $A_1^{(\epsilon)}$. Now fix some $R > 0$ to be chosen later, and split the remaining integration into the two parts where $|\xi| > R$ and $|\xi| \leq R$. On the latter region, we again use the Riemann–Lebesgue estimate to obtain

$$\begin{aligned}\int_{\mathcal{S}^2}\int_{|\xi|\leq R}(1-\sum_{j=1,2} 1_{A_j^{(\epsilon)}}(\xi,\mathbf{n}))|\widehat{f}(\xi_+)|^2|\widehat{f}(\xi_-)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\mathbf{n} \leq \\ \int_{\mathcal{S}^2}\int_{|\xi|\leq R}|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\mathbf{n} \leq \\ \frac{R^{2m+1}\pi}{2m+1}.\end{aligned} \tag{7.7}$$

To handle the integration over the remaining region, we first simplify notation by writing $c = \|f - M_f\|_{L^1}$. Then by one more use of the Riemann–Lebesgue lemma,

$$|\widehat{f}(\xi_\pm)| \leq c + |\widehat{M}_f(\xi_\pm)| = c + e^{-T|\xi_\pm|^2/2}. \tag{7.8}$$

Again with $\mathbf{n}$ fixed, consider those ξ in the final region of integration with $\xi\cdot\mathbf{n} \geq 0$. Recall that $|\xi_+| = |\xi|\cos\Theta_+$ and $|\xi_-| = |\xi|\sin\Theta_+$. This is the region where $\cos\Theta \leq \pi/4$, but $\sin^2\Theta \geq \epsilon$. Hence in this region

$$|\widehat{f}(\xi_-)| \leq c + e^{-\epsilon T R^2/2} \leq 2c\rho$$

when we choose $R = \sqrt{-2\ln c/T\epsilon}$. We now fix this choice of R. Making the same sort of estimates for $\xi\cdot\mathbf{n} \leq 0$, we get

$$\begin{aligned}\int_{\mathcal{S}^2}\int_{|\xi|>R}(1-\sum_{j=1,2} 1_{A_j^{(\epsilon)}}(\xi,\mathbf{n}))|\widehat{f}(\xi_+)|^2|\widehat{f}(\xi_-)|^2|\xi|^{2m}\mathrm{d}\xi\mathrm{d}\mathbf{n} \leq \\ 2c^2 2^m\frac{4}{1-\epsilon}\int_{\mathcal{S}^2}\int_{\mathbb{R}^3}|\widehat{f}(\eta)|^2|\eta|^{2m}\mathrm{d}\eta\mathrm{d}\mathbf{n} \leq \\ \frac{2^{m+2}c^2}{1-\epsilon}\|f\|^2_{\mathrm{H}^m}.\end{aligned} \tag{7.9}$$

Putting together all of the pieces yields

$$\|\mathcal{Q}_+(f)\|^2_{\mathrm{H}^m} \le \Big(\frac{2^{m+2}c^2}{1-\epsilon} + \frac{8\epsilon}{(1-\epsilon)^{(m+1)}}\Big)\|f\|^2_{\mathrm{H}^m} + K(c,T,\epsilon),$$

where

$$K(c,T,\epsilon) = \frac{1}{2m+1}(-2\ln c/T\epsilon)^{(2m+1)/2}.$$

Finally, choose $\epsilon = c^2/2$ to obtain the required multiple of $\|f\|^2_{\mathrm{H}^m}$. □

Several useful variants of the inequality are easily established. First, the Kullback–Cszilar inequality says that

$$\|f - M_f\|^2_{L^1} \le \frac{1}{2}\big(H(M_f) - H(f)\big)$$

and thus we immediately have

Theorem 7.2. *(Smoothness bound on the gain term for Maxwellian molecules, entropic version). Let f be any probability density on $\mathbb{R}^3$ with finite second moments such that $\|f\|_{\mathrm{H}^m}$ is finite. Then there is a constant $C(m,T)$ so that*

$$\|\mathcal{Q}_+(f)\|^2_{\mathrm{H}^m} \le (1/2)\|f\|^2_{\mathrm{H}^m} + C(m,T) \tag{7.10}$$

whenever

$$\big(H(M_f) - H(f)\big) \le 2^{-(m+4)}. \tag{7.11}$$

The utility of this simple variant lies in the monotonicity of the entropy: once we arrive at a time for which the condition (7.11) is satisfied, it remains satisfied forever.

The next version is given in terms of the Fisher information, and is different in that it does not require any "smallness" condition to apply.

Theorem 7.3. *(Smoothness bound on the gain term for Maxwellian molecules, Fisher information version). Let f be any probability density on $\mathbb{R}^3$ with finite second moments and such that $\|f\|_{\mathrm{H}^m}$ is finite. Then there is a constant $C(m,T,I(f))$ so that*

$$\|\mathcal{Q}_+(f)\|^2_{\mathrm{H}^m} \le \frac{1}{2}\|f\|^2_{\mathrm{H}^m} + C(m,T,I(f)). \tag{7.12}$$

Proof. We proceed as in the proof of the main theorem, except that we use $I(f)$, through Lemma 2.3, to control the size of $|\widehat{f}(\xi)|$ for large ξ. □

8. Optimal Exponential Convergence in the $\|\!|\cdot|\!\|$ Norm for Maxwellian Molecules

Here we prove Theorem 1.3, which is for Maxwellian molecules the analog of Theorem 3.1. In the course of the proof we obtain an interesting, somewhat indirect, explicit evaluation of λ_1.

Proof of Theorem 3.1. Once again, we put $\phi(\xi,t) = \widehat{f}(\xi,t) - \widehat{M}(\xi)$, and we let $P(\xi,t)$ denote the third degree Taylor polynomial of $\phi(\xi,t)$, and let $R(\xi)$ denote the third degree Taylor polynomial of $\widehat{M}(\xi)$. This is a bit more complicated than in the one dimensional Kac case, in that now there can be second degree differences between $P(\xi,t)$ and the third degree Taylor polynomial for $\widehat{M}(\xi)$.

However, we still proceed in the same way. First, let $K(\xi)$, a cut-off function, be any smooth approximation to $1_{\{|\xi|\leq 1\}}$ agreeing with it outside a thin neighborhood of the unit sphere. Then define

$$\widehat{S}(\xi,t) = K(\xi)(P(\xi,t) - R(\xi)). \tag{8.1}$$

Clearly, since the fourth moment of f is finite, so is $|||\phi(\cdot,t) - S(\cdot,t)|||$.

Moreover,[IT56], the coefficients of P are simply multiples of certain corresponding moments of f. And, as proved in this reference, these moments tend to their equilibrium values exponentially fast *with a rate that is independent of the initial data*, as in the Kac model case, provided only that the moment exists initially. Thus $S(\cdot,t)$ has the form $K(\xi)$ times a polynomial of degree three whose coefficients are decaying to zero exponentially fast.

Now any monomial in ξ multiplied by $K(\xi)$ is the Fourier transform of a function that has a finite norm in both L^1 and H^m for any n. Hence we have that

$$\left(\|S(\cdot,t)\|_{L^1} + \|S(\cdot,t)\|_{\mathrm{H}^m}\right) \leq Ce^{-t\lambda} \tag{8.2}$$

for some C depending on f and n, and some *universal* λ.

In our treatment of the Kac model, we computed the corresponding λ. Here we avoid this, and simply make the observation that we require: whatever the value of λ, it is the case that $\lambda \geq \lambda_1$.

Because of the universality of λ, this follows from the results of Cercignani, Lampis and Sgarra [CLS88] concerning the behavior of small perturbations of equilibrium. Specifically, consider the coefficient $c_{1,2}(t)$ of $K(\xi)\xi_1\xi_2$ in $S(\cdot,t)$. Pick δ small enough that $\delta v_1 v_2 + |v|^4 \geq 0$, and then pick ϵ small enough that

$$f_0(v) = C(1 + \epsilon(\delta v_1 v_2 + |v|^4))M(v) \tag{8.3}$$

in which C is the normalizing constant, is close enough to equilibrium for the theorem of [CGS88] to apply. Then the corresponding solution of the Boltzmann equation approches equilibrium like $e^{-t\lambda_1}$. But clearly it appraches equilibrium no faster than $c_{1,2}(t)$ tends to zero. Hence $c_{1,2}(t)$ must tend to zero at least as fast as $e^{-t\lambda_1}$. The argument for the other coefficients is the same. Thus, (8.3) holds with $\lambda = \lambda_1$.

Not only does $S(\cdot,t)$ decay at this rate, but since its coefficients in the Fourier transform respresentation satisfy the ordinary differential equations derived in [IT56], we have that the time derivative of $S(\cdot,t)$ does as well.

At this point we have stepped around most of the direct computation in the proof of Theorem 3.1 – which was only possible there because we made the simplifying assumption of constant rate function – and can easily conclude the proof.

Let $\phi_1 = \phi - S$ as before. Then

$$\frac{\partial \phi_1}{\partial t} + \phi_1 = \left(\frac{\partial S}{\partial t} + S\right) + \widehat{\mathcal{Q}}^+(\widehat{f},\widehat{S}) + \widehat{\mathcal{Q}}^+(\widehat{S},\widehat{M}) + \widehat{\mathcal{Q}}^+(\widehat{f},\phi_1) + \widehat{\mathcal{Q}}^+(\phi_1,\widehat{M}).$$

Then, all the terms involving S are estimated as in the proof of Theorem 3.1, and we obtain that

$$\left|\frac{\partial \phi_1}{\partial t} + \phi_1 - \left(\widehat{\mathcal{Q}}^+(\widehat{f},\phi_1) + \widehat{\mathcal{Q}}^+(\phi_1,\widehat{M})\right)\right| \leq C|\xi|^4 e^{-t\lambda_1}$$

for some constant C. Again as before, define $\phi_2(\xi,t) = \phi_1(\xi,t)/|\xi|^4$. Then the above estimate becomes

$$|\frac{\partial \phi_2}{\partial t} + \phi_2 - \big(\widehat{\mathcal{Q}}^+(\widehat{f},\phi_1) + \widehat{\mathcal{Q}}^+(\phi_1,\widehat{M})\big)/|\xi|^4| \le Ce^{-t\lambda_1}.$$

It remains to estimate

$$|\widehat{\mathcal{Q}}^+(\widehat{f},\phi_1) + \widehat{\mathcal{Q}}^+(\phi_1,\widehat{M})|/|\xi|^4. \tag{8.4}$$

Consider first

$$|\widehat{\mathcal{Q}}^+(\phi_1,\widehat{M})|/|\xi|^4 \le \int_{S^2} |\widehat{M}(\xi_-)||\phi_2(\xi_+)| \cos^4(\theta/2) B(\cos(\theta))\mathrm{d}\mathbf{n},$$

where we have used the formulas (6.6) and (6.7). This in turn is dominated by

$$\|\widehat{M}\|_{L^\infty}\|\phi_2\|_{L^\infty} \int_{S^2} \cos^4(\theta/2) B(\cos(\theta))\mathrm{d}\mathbf{n} = |\!|\!|\phi_1|\!|\!| \int_{S^2} \cos^4(\theta/2) B(\cos(\theta))\mathrm{d}\mathbf{n}.$$

Naturally we obtain the same estimate for $|\widehat{\mathcal{Q}}^+(\widehat{f},\phi_1)|/|\xi|^4$. Putting it all together, we then have

$$\frac{\partial |\!|\!|\phi_1|\!|\!|}{\partial t} + \big(1 - 2\int_{S^2} \cos^4(\theta/2) B(\cos(\theta))\mathrm{d}\mathbf{n}\big) |\!|\!|\phi_1|\!|\!| \le Ce^{-t\lambda_1}.$$

Solving this differential inequality gives us exponential decay at rate λ with

$$\lambda = \min\{\lambda_1, (1 - 2\int_{S^2} \cos^4(\theta/2) B(\cos(\theta))\mathrm{d}\mathbf{n}\big)\}.$$

We now claim that the terms in the minimum are actually equal.

To see this, let $h(v) = |v|^4 - b|v|^2 - c$, where b and c are chosen to make

$$\int_{\mathbb{R}^3} h(v)|v|^2 M(v)\mathrm{d}^3 v = \int_{\mathbb{R}^3} h(v) M(v)\mathrm{d}^3 v = 0.$$

Of course, h is just the spherically symmetric fourth order Hermite polynomial. Now pick a small enough that $1 + ah(v)$ is positive, and consider $f(v) = M(v)(1 + ah(v))$, then $\phi_1(\xi) = aM(\xi)|\xi|^4$, so that $\phi_2(\xi) = aM(\xi)$. In this case, we lose nothing in the estimate above:

$$|\widehat{\mathcal{Q}}^+(\phi_1,\widehat{M})|/\|\xi|^4 = aM(\xi)\int_{S^2} \cos^4(\theta/2) B(\cos(\theta))\mathrm{d}\mathbf{n}.$$

Again, as we have observed in Sect. 3,

$$\widehat{Q}^+(\phi_1,\widehat{M}) + \widehat{Q}^+(\widehat{M},\phi_1) - \phi_1 = \widehat{M\mathcal{L}h}, \tag{8.5}$$

and hence

$$\widehat{M\mathcal{L}h} = -\big(1 - 2\int_{S^2} \cos^4(\theta/2) B(\cos(\theta))\mathrm{d}\mathbf{n}\big)\big) h.$$

Thus, as is well known, $h(v)$ is an eigenvector of $\mathcal{L}$, and the corresponding eigenvalue is given by the above formula. This means that

$$\lambda_1 \le (1 - 2\int_{S^2} \cos^4(\theta/2)B(\cos(\theta))\mathrm{d}\mathbf{n})$$

and while at first sight one might suppose that another Hermite polynomial of order higher than four produces the gap, the estimate that was applied to (8.4) can be equally well applied to (8.5) for these higher eigenfunctions to see that this is not the case. Thus we have computed λ_1, and proved that

$$\frac{\partial \| \phi_2 \|}{\partial t} + \lambda_1 \| \phi_2 \| \le C e^{-t\lambda_1}.$$

This together with our bounds on S yield the theorem. □

9. Optimal Exponential Convergence in the Strong L^1 Norm and Sobolev Norms for Maxwellian Molecules

This section is very short; we only need collect results to provide a proof of Theorem 1.1, and to explain how to compute the constants involved in it.

Proof of Theorem 1.1. Since we have established Theorem 1.3 in Sect. 8, and have established Theorem 1.2 in Sect. 7, Theorem 1.1 follows from the interpolation inequalities, Theorems 4.1 and 4.2 , just as in the case of the Kac model. □

References

[Ar88] Arkeryd, L.: Stability in L^1 for the spatially homogeneous Boltzmann equation. Arch. Rat. Mech. Anal. **103**, 151–167 (1988)

[Bo88] Bobylev, A.V.: The theory of the nonlinear spatially uniform Boltzmann equation for Maxwellian molecules. Sov. Sci. Rev. c **7**, 111–233 (1988)

[BT92] Bobylev, A.V., Toscani, G.: On the generalization of the Boltzmann H-theorem for a spatially homogeneous Maxwell gas. J. Math. Phys. **33**, 2578–2586 (1992)

[Ca57] Carleman, T.: *Problèmes Matematiques dans la Théorie Cinétique des Gaz*. Uppsala: Almqvist-Wiksells, 1957

[Ce88] Cercignani, C.: *The Theory and Application of the Boltzmann Equation*. New York: Springer, 1988

[CLS88] Cercignani, C., Lampis, M., Sgarra, C.: L^2-stability near equilibrium of the solution of the homogeneous Boltzmann equation in the case of Maxwellian molecules. Meccanica **23**, 15–18 (1988)

[CC92] Carlen, E.A., Carvalho, M.C.: Strict entropy production bounds and stability of the rate of convergence to equilibrium for the Boltzmann equation. J. Stat. Phys. **67**, 575–608 (1992)

[CC94] Carlen, E.A., Carvalho, M.C.: Entropy production estimates for Boltzmann equations with physically realistic collision kernels.

[CELMR96] Carlen, E.A., Esposito, R., Lebowitz, J., Marra, R., Rokhlenko, A.: Kinetics of a Model Weakly Ionized Plasma in the Presence of Multiple Equilibria. Arch. Rat. Mech. Anal. **142**, 193–218 (1998

[El83] Elmroth, T.: Global boundedness of moments of solutions of the Boltzmann equation for forces of infinite range. Arch. Rat. Mech. Anal. **82**, 1–12 (1983)

[De94] Desvillettes, L.: On the regularizing properties of the non cut-off Kac equation. Commun. Math. Phys. **168**, 417–440 (1995)

[Ga95] Gabetta, E.: On a conjecture of McKean with application to Kac's model. Trans. Theo. Stat. Phys. **24**, 305–317 (1995)

[GP94] Gabetta, E., Pareschi, L.: About the non cut-off Kac equation: Uniqueness and asymptotic behaviour. Pubbl. Istituto Analisi Numerica CNR n. **911**, Pavia, 1994, Comm. Nonlinear Appl. Math. in press

[GP96] Gabetta, E., Pareschi, L.: Boundedness of moments and trend to equilibrium for the non "cut-off" Kac equation. Preprint n. **227**, Dip. Matematica Università di Ferrara, 1995, Rend. Sem. Math. Palermo, in press

[GTW95] Gabetta, E., Toscani, G., Wennberg, B.: Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation. J. Stat. Phys. **81**, 901–934 (1995)

[IT56] Ikenberry, E., Truesdell, C.: On the pressure and the flux of energy according to Maxwell's kinetic energy I. J. Rat. Mech. Anal. **5**, 1–54 (1956)

[Ka57] Kac, M.: *Probability and Related Topics in Physical Sciences*. London, New York: Interscience Publ. LTD., 1957

[Mk66] McKean Jr., H.P.: Speed of approach to equilibrium for Kac's caricature of a Maxwellian gas. Arch. Rational Mech. Anal. **21**, 343–367 (1966)

[Li95] Lions, P.L.: Compactness in Boltzmann's equation via Fourier integral operatorsand applications III. J. Math. Kyoto Univ. **34**, 539–584 (1994)

[LT95] Lions, P.L., Toscani, G.: A sthrenghtened central limit theorem for smooth densities. J. Funct. Anal. **128**, 148–167 (1995)

[To92] Toscani, G.: New a priori estimates for the spatially homogeneous Boltzmann equation. Cont. Mech. Termodyn. **4**, 81–93 (1992)

[To95] Toscani, G.: On regularity and asymptotic behaviour of a spatially homogeneous Maxwell gas. Preprint (1995); Rend. Seminario Matem. Palermo, in press

[WU70] Wang Chang, C.S., Uhlenbeck, G.E.: The kinetic Theory of Gases. In: *Studies in Statistical mechanics V*, eds. J. de Boer and G.E. Uhlenbeck, Amsterdam: North Holland Publishing Co., 1970 (This is a collection of Michigan Research Institute Reports written by the authors during the period 1948 to 1956)

[We93] Wennberg, B.: Stability and exponential convergence for the Boltzmann equation. Thesis, Chalmers University of Technology (1993)

[We94] Wennberg, B.: Regularity in the Boltzmann equation and the Radon transform. Commun. Part. Diff. Eq. **19**, 2057–2074 (1994)

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