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Self-Duality of the SL_2 Hitchin Integrable System at Genus 2

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Abstract: We revisit the Hitchin integrable system [11, 21] whose phase space is the bundle cotangent to the moduli space $\mathcal{N}$ of holomorphic SL_2-bundles over a smooth complex curve of genus 2. As shown in [18], $\mathcal{N}$ may be identified with the 3-dimensional projective space of theta functions of the 2$^{\text{nd}}$ order, i.e. $\mathcal{N} \cong \mathbb{P}^3$. We prove that the Hitchin system on $T^*\mathcal{N} \cong T^*\mathbb{P}^3$ possesses a remarkable symmetry: it is invariant under the interchange of positions and momenta. This property allows to complete the work of van Geemen–Previato [21] which, basing on the classical results on geometry of the Kummer quartic surfaces, specified the explicit form of the Hamiltonians of the Hitchin system. The resulting integrable system resembles the classic Neumann systems which are also self-dual. Its quantization produces a commuting family of differential operators of the 2$^{\text{nd}}$ order acting on homogeneous polynomials in four complex variables. As recently shown by van Geemen–deJong [22], these operators realize the Knizhnik–Zamolodchikov–Bernard–Hitchin connection for group $SU(2)$ and genus 2 curves.

1. Introduction

In [11], Nigel Hitchin has discovered an interesting family of classical integrable models related to modular geometry of holomorphic vector bundles or to 2-dimensional gauge fields. The input data for Hitchin's construction are a complex Lie group G and a complex curve Σ of genus γ. The configuration space of the integrable system is the moduli space $\mathcal{N}$ of (semi)stable holomorphic G-bundles over Σ. This is a finite-dimensional complex variety and Hitchin's construction is done in the holomorphic category. It exhibits a complete family of Poisson-commuting Hamiltonians on the (complex) phase space $T^*\mathcal{N}$. The Hitchin Hamiltonians have open subsets of abelian varieties as generic level sets on which they induce additive flows [11]. More recently, Hitchin's construction was extended to the case of singular or punctured curves [16, 19, 7] providing a unified construction of a vast family of classical integrable systems. For $\Sigma = \mathbb{C}P^1$ with punctures, one obtains this way the so called Gaudin chains and for $G = SL_N$ and Σ of genus 1 with one puncture, the elliptic Calogero-Sutherland models which found an unexpected application in the supersymmetric 4-dimensional gauge theories [6].

In Sect. 2 of the present paper we briefly recall the basic idea of Hitchin's construction. The main aim of this contribution is to treat in detail the case of $G = SL_2$ and Σ of genus 2 (no punctures). The genus 2 curves are hyperelliptic, i.e., given by the equation

$$\zeta^2 = \prod_{s=1}^{6}(\lambda - \lambda_s), \tag{1.1}$$

where λ_s are 6 different complex numbers. The semistable moduli space $\mathcal{N}$ has a particularly simple form for genus 2, [18]: it is the projectivized space of theta functions of the 2$^{\text{nd}}$ order:

$$\mathcal{N} = \mathbb{P}H^0(L^2_\Theta), \tag{1.2}$$

where L_Θ is the theta-bundle over the Jacobian J^1 of (the isomorphism classes of) degree $\gamma - 1 = 1$ line bundles[1] l over Σ. $dim_{\mathbf{C}}(H^0(L^2_\Theta)) = 4$ so that $\mathcal{N} \cong \mathbb{P}^3$. This picture of $\mathcal{N}$ is related to the realization of SL_2-bundles as extensions of degree 1 line bundles. We review some of the results in this direction in Sect. 3 using a less sophisticated language than that of the original work [18]. The relation between the extensions and the theta functions is lifted to the level of the cotangent bundle $T^*\mathcal{N}$ in Sect. 4. The language of extensions proves suitable for a direct description of the Hitchin Hamiltonians on $T^*\mathcal{N}$. The main aim is, however, to present the Hitchin system as an explicit 3-dimensional family of integrable systems on $T^*\mathbb{P}^3$, parametrized by the moduli of the curve. This was first attempted, and almost achieved, in reference [21].

Let us recall that the Hitchin Hamiltonians are components of the map

$$\mathcal{H}:\ T^*\mathcal{N}\ \longrightarrow\ H^0(K^2) \tag{1.3}$$

with values in the (holomorphic) quadratic differentials (K denotes the canonical bundle of Σ). Due to relation (1.2), the map $\mathcal{H}$ may be viewed as a $H^0(K^2)$-valued function of pairs (θ, ϕ), where $\theta \in H^0(L^2_\Theta)$ and ϕ from the dual space $H^0(L^2_\Theta)^*$ are s.t. $\langle\theta, \phi\rangle = 0$. Fix a holomorphic trivialization of L_Θ around $l \in J^1$ and denote by ϕ_l the linear form that computes the value of the theta function at l. As was observed in [21],

$$\mathcal{H}(\theta, \phi_l) = -\tfrac{1}{16\pi^2}(d\theta(l))^2 \tag{1.4}$$

(with appropriate normalizations). In the above formula, θ is viewed as a function on J^1 and $d\theta(l)$ as an element of $H^0(K)$. Since $\theta(l) = 0$, the equation is consistent with changes of the trivialization of L_Θ.

The map $J^1 \ni l \mapsto \phi_l$ induces an embedding of the Kummer surface $J^1/\mathbb{Z}_2$ with l and $l^{-1}K$ identified into a quartic $\mathcal{K}^*$ in $\mathbb{P}H^0(L^2_\Theta)^*$. The Kummer quartic is a carrier of a rich but classical structure, a subject of an intensive study of the nineteenth century geometers, see [13] and also the last chapter of [10]. The reference [21] used the relation (1.4) and a mixture of the classical results and of more modern algebraic geometry to recover an explicit form of the components of the Hitchin map $\mathcal{H}$ up to a multiplication by a function on the configuration space. The authors of [21] checked that the simplest

[1] We use the multiplicative notation for the tensor product of line bundles.

way to fix this ambiguity leads to Poisson-commuting functions but they fell short of showing that the latter coincide with the ones of the Hitchin construction.

Among the aims of the present paper is to fill the gap left in [21]. We observe that the proposal of [21] has a remarkable **self-duality** property: it is invariant under the interchange of the positions and momenta in $T^*\mathbb{P}^3$. We show that the Hitchin construction leads to a system with the same symmetry. This limits the ambiguity left by the analysis of [21] to a multiplication of the components of $\mathcal{H}$ by constants. A direct check based on Eq. (1.4) fixes the normalizations and results in a formula for the Hitchin map which uses the hyperelliptic description (1.1) of the curve. Namely,

$$\mathcal{H} = -\tfrac{1}{128\pi^2}\sum_{1\leq s\neq t\leq 6}\frac{r_{st}}{(\lambda-\lambda_s)(\lambda-\lambda_t)}\,(d\lambda)^2, \tag{1.5}$$

where r_{st} are explicit polynomials in (θ,ϕ) given, upon representation of (θ,ϕ) by pairs $(q,p)\in\mathbb{C}^4\times\mathbb{C}^4$, by Eqs. (7.7) below. The above expression for $\mathcal{H}$ has a similar form as that for the Hitchin map on the Riemann sphere with 6 insertion points λ_s, see e.g. Sect. 4 of [9], except for the structure of the terms r_{st}. This is not an accident but is connected to the reduction of conformal field theory on genus 2 surfaces to an orbifold theory in genus 0 [14, 23]. We plan to return to this relation in a future publication.

Let us discuss in more detail how we establish the self-duality of the Hitchin Hamiltonians. The main tool here is an explicit expression for the values of the Hitchin map off the Kummer quartic $\mathcal{K}^*$ which we obtain in Sect. 5. Our formula for $\mathcal{H}(\theta,\phi)$ requires a choice of a pair of perpendicular 2-dimensional subspaces $(\Pi,\Pi^\perp)$, where $\theta\in\Pi\subset H^0(L_\Theta^2)$ and $\phi\in\Pi^\perp\subset H^0(L_\Theta^2)^*$ (there is a complex line of such choices). The plane $\Pi^\perp$ corresponds to a line $\mathbb{P}\Pi^\perp$ in $\mathbb{P}H^0(L_\Theta^2)^*$ which intersects the Kummer quartic $\mathcal{K}^*$ in four points $\mathbb{C}^*\phi_{l_j}$, $j=1,2,3,4$, (counting with multiplicity). Whereas the analysis of [21] was mainly concerned with the geometry of bitangents to $\mathcal{K}^*$ with two pairs of coincident ϕ_{l_j}'s, we concentrate on the generic situation with ϕ_{l_j}'s different. Then any two of them, say $\mathbb{C}^*\phi_{l_1}$ and $\mathbb{C}^*\phi_{l_2}$, span $\Pi^\perp$. Π is composed of the 2$^{\text{nd}}$ order theta functions vanishing at l_1 and l_2. In particular,

$$\phi = a_1\phi_{l_1} + a_2\phi_{l_2} \quad\text{and}\quad \theta(l_1) = 0 = \theta(l_2). \tag{1.6}$$

Let x_1+x_2 and x_3+x_4 be the divisors of l_1l_2 and of $l_1l_2^{-1}K$, respectively, where x_i are four points[2] in Σ. If $l_1^2\neq K$, which holds in a general situation, then the quadratic differential $\mathcal{H}(\theta,\phi)$ is determined by its values at x_i which, as we show in Sect. 5, are

$$\mathcal{H}(\theta,\phi)(x_i) \;=\; -\tfrac{1}{16\pi^2}\left(a_1d\theta(l_1)\pm a_2d\theta(l_2)\right)^2(x_i). \tag{1.7}$$

Sign plus is taken for x_1 and x_2 and sign minus for x_3 and x_4. Note that for $\phi=\phi_l$ with $\theta(l)=0$ the above equation reproduces the result (1.4).

As we recall at the end of Sect. 3, there exists an almost natural linear isomorphism ι between $H^0(L_\Theta^2)^*$ and $H^0(L_\Theta^2)$. What follows is independent of the remaining ambiguity in the choice of ι. The identity $\langle\theta,\phi\rangle = \langle\iota(\phi),\iota^{-1}(\theta)\rangle$ implies that if (θ,ϕ) is a perpendicular pair then so is (θ',ϕ') where $\theta'=\iota(\phi)$ and $\phi'=\iota^{-1}(\theta)$. Thus ι interchanges the positions and momenta in $T^*\mathcal{N}$. We may take $(\Pi',\Pi'^\perp)=(\iota(\Pi^\perp),\iota^{-1}(\Pi))$ as a pair of perpendicular subspaces containing (θ',ϕ'). The line $\mathbb{P}\Pi'^\perp$ meets $\mathcal{K}^*$ in four points

[2] The other two lines of intersection of $\mathbb{P}\Pi^\perp$ with $\mathcal{K}^*$ correspond to l_3 and l_4 with $l_1l_3=\mathcal{O}(x_1+x_3)$, $l_1l_3^{-1}K=\mathcal{O}(x_2+x_4)$, $l_1l_4=\mathcal{O}(x_1+x_4)$, $l_1l_4^{-1}K=\mathcal{O}(x_2+x_3)$.

$\mathbb{C}^*\phi_{l'_j}$. Equivalently, $\mathbb{C}^*\iota(\phi_{l'_j})$ are the points of intersection of $\mathbb{P}\Pi$ with the Kummer quartic $\mathcal{K} = \iota(\mathcal{K}^*) \subset \mathbb{P}H^0(L^2_\Theta)$. In a general situation, $\Pi'^{\perp}$ is spanned by any pair of $\phi_{l'_j}$'s so that

$$\phi' = a'_1\phi_{l'_1} + a'_2\phi_{l'_2} \quad \text{and} \quad \theta'(l'_1) = 0 = \theta'(l'_2) \tag{1.8}$$

which is the dual version of relations (1.6). Equivalently,

$$\theta = a'_1\iota(\phi_{l'_1}) + a'_2\iota(\phi_{l'_2}) \quad \text{and} \quad \langle\iota(\phi_{l'_1}),\phi\rangle = 0 = \langle\iota(\phi_{l'_2}),\phi\rangle. \tag{1.9}$$

Let y_i be the points associated to l'_j the same way as the points x_i were associated to l_j. l'_j may be chosen so that y_i and x_i coincide modulo the natural involution of Σ fixing the six Weierstrass points. Formula (1.7) implies then that

$$\mathcal{H}(\theta',\phi')(y_i) \;=\; -\tfrac{1}{16\pi^2}\left(a'_1 d\theta'(l'_1) \pm a'_2 d\theta'(l'_2)\right)^2(y_i). \tag{1.10}$$

Points y_i in Eq. (1.10) may be replaced by x_i since the quadratic differentials are equal at point x if and only if they are equal at the image of x by the involution of Σ. A direct calculation of the coefficients a_1, a_2 and a'_1, a'_2 appearing on the right-hand sides of Eqs. (1.7) and (1.10) shows then that both expressions coincide, establishing the self-duality of $\mathcal{H}$. The verification of this equality is the subject of Sect. 6.

In Sect. 7, we recall the main result of reference [21] and show how the self-duality may be used to complete the analysis performed there and to obtain the explicit form (1.5) of the Hitchin map. We briefly discuss the relation of that form to the classical Yang-Baxter equation.

An appropriate quantization of Hitchin Hamiltonians leads to operators acting on holomorphic sections of powers of the determinant line bundle over $\mathcal{N}$ and defining the Knizhnik–Zamolodchikov–Bernard–Hitchin [15, 4, 5, 12] connection. In our case, the sections of the powers of the determinant bundle are simply homogeneous polynomials on $H^0(L^2_\Theta)$. It is easy to quantize the Hamiltonians corresponding to the components of the Hitchin map (1.5) in such a way that one obtains an explicit family of commuting 2^{nd} order differential operators acting on such polynomials. The corresponding connection coincides with the explicit form of the (projective) KZBH connection worked out recently[3] in [22].

The quantization of the genus 2 Hitchin system is briefly discussed in the Conclusions, where we also mention other possible directions for further research. Four appendices which close the paper contain some more technical material.

We would like to end the presentation of our paper by expressing some regrets. We apologize to Ernst Eduard Kummer and other nineteenth century giants for our insufficient knowledge of their classic work. The apologies are also due to a few contemporary algebraic geometers who could be interested in the present work for an analytic character of our arguments. To the specialist in integrability we apologize for the yet incomplete analysis of the integrable system studied here and, finally, we apologize to ourselves for not having finished this work 2 years ago.

[3] We thank B. van Geemen for attracting our attention to ref. [22] and for pointing out that this work may be used to fix indirectly the precise form of the Hitchin map.

2. Hitchin's Construction

Let us assume, for simplicity, that the complex Lie group G is simple, connected and simply connected. We shall denote by **g** its Lie algebra. The complex curve Σ will be assumed smooth, compact and connected. Topologically, all G-bundles on Σ are trivial and the complex structures in the trivial bundle may be described by giving operators $\bar\partial + A$, where A are smooth **g**-valued 0,1-forms on Σ [1]. Let $\mathcal{A}$ denote the space of such forms (i.e. of chiral gauge fields). The group $\mathcal{G}$ of local (chiral) gauge transformations composed of smooth maps h from Σ to G acts on operators $\bar\partial + A$ by conjugation and on the gauge fields A by

$$A \;\longmapsto\; {}^{h}\!A \equiv hAh^{-1} + h\bar\partial h^{-1}.$$

Two holomorphic G-bundles are equivalent iff the corresponding gauge fields are in the same orbit of $\mathcal{G}$. Hence the space of orbits $\mathcal{A}/\mathcal{G}$ coincides with the (moduli) space of inequivalent holomorphic G-bundles. It may be supplied with a structure of a variety provided one gets rid of bad orbits. This may be achieved by limiting the considerations to (semi)stable bundles, i.e. such that the vector bundle associated with the adjoint representations of G contains only holomorphic subbundles with negative (non-positive) first Chern number. For genus $\gamma > 1$, the moduli space $\mathcal{N}_s \equiv \mathcal{A}_s/\mathcal{G}$ of stable G-bundles is a smooth complex variety with a natural compactification to a variety $\mathcal{N}_{ss}$, the (Seshadri-) moduli space of semistable bundles [18].

The complex cotangent bundle $T^*\mathcal{N}_s$ may be obtained from the infinite-dimensional bundle $T^*\mathcal{A}_s$ by the symplectic reduction. $T^*\mathcal{A}_s$ may be realized as the space of pairs (A, Φ), where Φ is a (possibly distributional) **g**-valued 1,0-form on Σ, $A \in \mathcal{A}_s$ and the duality with the vectors δA tangent to $\mathcal{A}$ is given by

$$\int_\Sigma tr\ \Phi \wedge \delta A$$

with tr standing for the Killing form. The action of the local gauge group $\mathcal{G}$ on $\mathcal{A}_s$ lifts to a symplectic action on $T^*\mathcal{A}_s$ by

$$\Phi \;\longmapsto\; {}^{h}\Phi \equiv h\Phi h^{-1}\ .$$

The moment map μ for the action of $\mathcal{G}$ on $T^*\mathcal{N}_s$ is

$$\mu(A, \Phi) = \bar\partial\Phi + A \wedge \Phi + \Phi \wedge A \;\equiv\; \bar\partial_A \Phi\ .$$

Note that it takes values in **g**-valued 2-forms on Σ. These may be naturally viewed as elements of the space dual to the Lie algebra of $\mathcal{G}$. The symplectic reduction of $T^*\mathcal{A}_s$ realizes $T^*\mathcal{N}_s$ as the space of $\mathcal{G}$-orbits in the zero level of μ:

$$T^*\mathcal{N}_s \;\cong\; \mu^{-1}(\{0\})/\mathcal{G}\ .$$

For a homogeneous G-invariant polynomial P on **g** of degree d_P, the gauge invariant expression $P(\Phi)$ defines a section of the bundle K^{d_P} of d_P-differentials on Σ. If Φ is in the zero level of μ then $P(\Phi)$ is also holomorphic. Hence the map $\Phi \mapsto P(\Phi)$ induces a map

$$\mathcal{H}_P:\ T^*\mathcal{N}_s \;\longrightarrow\; H^0(K^{d_P})$$

into the finite dimensional vector space of holomorphic differentials of degree d_P on Σ. The components of such vector-valued Hamiltonians clearly Poisson-commute since upstairs (on $T^*\mathcal{A}_s$) they depend only on the momentum variables Φ. By a beautiful argument, Hitchin showed [11] that taking all polynomials P one obtains a complete system of Hamiltonians in involution and that the collection of maps $\mathcal{H}_P$ defines in generic points a foliation of $T^*\mathcal{N}_s$ into (open subsets of) abelian varieties.

Let us briefly sketch Hitchin's argument for $G = SL_2$. There is only one (up to normalization) non-trivial invariant polynomial P_2 on sl_2 given by, say, half of the Killing form. $\mathcal{H} \equiv \mathcal{H}_{P_2}$ maps into the space of quadratic differentials. A non-trivial holomorphic quadratic differential ρ determines a (spectral) curve $\Sigma' \subset K$ given by the equation

$$\xi^2 = \rho(\pi(\xi)), \tag{2.1}$$

where $\xi \in K$ and π is the projection of K on Σ. The map $\xi \mapsto -\xi$ gives an involution σ of Σ'. Restriction of π to Σ' is a 2-fold covering of Σ ramified over $4(\gamma-1)$ points fixed by σ, the zeros of ρ. Σ' has genus $\gamma' = 4\gamma - 3$. If $\rho = \frac{1}{2}tr\,(\Phi)^2$ then relation (2.1) coincides with the eigen-value equation

$$det(\Phi - \xi \cdot I) = 0$$

for the Lax matrix Φ. Let for each $0 \not\equiv \xi \in \Sigma'$, l_ξ denote the corresponding eigen-subspace of Φ. By continuity, l_ξ extend to vanishing ξ in Σ' and $\cup_\xi l_\xi$ forms a line subbundle l of $\Sigma' \times \mathbb{C}^2$. In fact, l is a holomorphic subbundle with respect to the complex structure defined on $\Sigma' \times \mathbb{C}^2$ by $\bar\partial + A \circ \pi$. The degree of l is $-2(\gamma-1)$. Besides,

$$l(\sigma^* l) = \pi^* K^{-1}. \tag{2.2}$$

Conversely, given Σ' and a holomorphic line bundle l of degree $-2(\gamma-1)$ on it satisfying (2.2), we may recover a rank 2 holomorphic bundle E of trivial determinant over Σ as a pushdown of l to Σ. Thus for $0 \not\equiv \xi \in \Sigma'$, $E_{\pi(\xi)} = l_\xi \oplus l_{-\xi}$. E corresponds to a unique holomorphic SL_2-bundle which, if stable (what happens on an open subset of l's) defines a point in the moduli space $\mathcal{N}_s$. A holomorphic 1,0-form with values in the traceless endomorphisms of E acting as multiplication by $\pm\xi$ on $l_{\pm\xi} \subset E_{\pi(\xi)}$ defines then a unique covector of $T^*\mathcal{N}_s$. Thus Σ' encodes the values of the quadratic Hitchin Hamiltonian $\mathcal{H}$ (i.e., of the action variables) whereas the line bundles l satisfying relation (2.2) form the abelian (Prym) variety (of the angle variables) describing the level set of $\mathcal{H}$.

3. SL_2 Moduli Space at Genus 2

We shall present briefly the description of the moduli space $\mathcal{N}_s$ for $G = SL_2$ and $\gamma = 2$ which was worked out in [18].

Let us start by recalling some basic facts about theta functions. We shall use a coordinate rather than an abstract language. The space of degree $\gamma - 1$ holomorphic line bundles forms a Jacobian torus $J^{\gamma-1}$ of complex dimension γ. Fixing a marking (a symplectic homology basis $(A_a, B_b),\ a, b = 1, \dots, \gamma$), we may identify $J^{\gamma-1}$ with $\mathbb{C}^\gamma/(\mathbb{Z}^\gamma + \tau\mathbb{Z}^\gamma)$. $\tau \equiv (\tau^{ab})$ is the period matrix, i.e. $\tau^{ab} = \int_{B_b}\omega^a$, where ω^a are the basic holomorphic forms on Σ normalized so that $\int_{A_a}\omega^b = \delta^{ab}$. The point $0 \in \mathbb{C}^\gamma$ corresponds in $J^{\gamma-1}$ to a (marking dependent) spin structure S_0, i.e. a degree 1 bundle such that $S_0^2 = K$. $u \in \mathbb{C}^\gamma$ describes the line bundle $V(u)S_0$, where $V(u)$ is the flat line bundle with the twists $e^{2\pi i u^b}$ along the B_b cycles. The set of degree 1 bundles l with non-trivial holomorphic sections forms a divisor Θ of a holomorphic line bundle L_Θ over $J^{\gamma-1}$. Holomorphic sections of the k^{th} power ($k > 0$) of L_Θ are called theta function of order k. With the use of a marking, they may be represented by holomorphic functions $u \mapsto \theta(u)$ on $\mathbb{C}^2$ satisfying

$$\theta(u + p + \tau q) = e^{-\pi i k q\cdot\tau q - 2\pi i k q\cdot u}\ \theta(u) \tag{3.1}$$

for $p, q \in \mathbb{Z}^\gamma$. The functions

$$\theta_{k,e}(u) = \sum_{n\in\mathbb{Z}^\gamma} e^{\pi i k(n+e/k)\cdot\tau(n+e/k)+2\pi i k(n+e/k)\cdot u}, \tag{3.2}$$

where $e \in \mathbb{Z}^\gamma/k\mathbb{Z}^\gamma$ form a basis of the theta functions of order k. Hence $dim H^0(L^k_\Theta) = k^\gamma$. In particular, the Riemann theta function $\theta_{1,0}(u) \equiv \vartheta(u)$ represents the unique (up to normalization) non-trivial holomorphic section of L_Θ. It vanishes on the set

$$\{\sum_{i=1}^{\gamma-1}\int_{x_0}^{x_i}\omega - \Delta \mid x_1 \in \Sigma, \ldots, x_{\gamma-1} \in \Sigma\}$$

representing the divisor Θ. Here $\Delta \in \mathbb{C}^\gamma$ denotes the (x_0-dependent) vector of Riemann constants. All theta functions of order 1 and 2 are even functions of u.

For $\gamma = 2$, the divisor Θ is formed by the bundles $\mathcal{O}(x)$ with divisors $x \in \Sigma$. $\mathcal{O}(x) = V(\int_{x_0}^{x}\omega - \Delta)S_0$. The pullback of the theta bundle L_Θ by means of the map $x \mapsto \mathcal{O}(x)$ is equivalent to the canonical bundle K. The equivalence assigns 1,0-forms to functions representing sections of the pullback of L_Θ:

$$\epsilon^{ab}\partial_b\vartheta(\int_{x_0}^{x}\omega - \Delta) \quad\mapsto\quad \omega^a(x). \tag{3.3}$$

This is consistent since vanishing of $\vartheta(\int_{x_0}^{x}\omega - \Delta)$ implies that

$$\partial_a\vartheta(\int_{x_0}^{x}\omega - \Delta)\,\omega^a(x) = 0\ .$$

Hence any multivalued function on Σ picking up a factor $e^{-\pi i\tau^{aa} - 2\pi i(\int_{x_0}^{x}\omega^a - \Delta^a)}$ when x goes around the B_a cycle and univalued around the A_a cycles may be identified with a 1,0-form on Σ.

As already suggested by the discussion at the end of Sect.2, for the SL_2 group it is more convenient to use the language of holomorphic vector bundles (of rank 2 and trivial determinant) than to work with principal SL_2-bundles. Of course the first ones are just associated to the second ones by the fundamental representation of SL_2. Any stable rank 2 bundle E with trivial determinant is an extension of a degree 1 line bundle l ([18], Lemmas 5.5 and 5.8), i.e. it appears in an exact sequence of holomorphic vector bundles

$$0 \longrightarrow l^{-1} \overset{\sigma}{\longrightarrow} E \overset{\varpi}{\longrightarrow} l \longrightarrow 0\ . \tag{3.4}$$

The inequivalent extensions (3.4) are classified by the cohomology classes in $H^1(l^{-2})$. This may be seen as follows. Taking a section of ϖ, i.e., a smooth bundle homomorphism

$s : l \to E$ such that $\varpi \circ s = id_l$, we infer that $\varpi \bar{\partial} s = 0$ and hence that $\bar{\partial} s = \sigma b$ for b a 0,1-form with values in $Hom(l, l^{-1}) = l^{-2}$, i.e. $b \in \wedge^{01}(l^{-2})$. b is determined up to $\bar{\partial}\varphi$, where φ is a smooth section of l^{-2}, i.e. $\varphi \in \Gamma(l^{-2})$. The class $[b]$ in $\wedge^{01}(l^{-2})/\Gamma(l^{-2}) \cong H^1(l^{-2})$ determines the extension (3.4) up to equivalence. Each b corresponds to an extension: one may simply take E equal to $l^{-1} \oplus l$ with the $\bar{\partial}$-operator given by $\bar{\partial}_{l^{-1}\oplus l} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Proportional $[b]$ correspond to equivalent bundles E. If E is a stable bundle then the extension (3.4) is necessarily nontrivial, i.e. $[b] \not= 0$.

Let C_E denote the set of degree 1 line bundles l s.t. $H^0(l \otimes E) \not= 0$ (equivalently, s.t. E is an extension of l). This is a complex 1-dimensional variety. It was shown in [18] that C_E characterizes the bundle E up to isomorphism and that there exists a theta function θ of the 2nd order which vanishes exactly on C_E. The assignment $E \mapsto \mathbb{C}^*\theta$ gives an injective map

$$m : \mathcal{N}_s \longrightarrow \mathbb{P}H^0(L^2_\Theta) \,. \tag{3.5}$$

Let $V(u_1)S_0 \equiv l_{u_1} \in C_E$. E may be realized as an extension of l_{u_1} which is characterized by $[b] \in H^1(l_{u_1}^{-2})$. Then one may take

$$\theta(u) \;=\; \int_\Sigma K(x; u_1, u) \wedge b(x), \tag{3.6}$$

where

$$K(x; u_1, u) = \vartheta(\int_{x_0}^{x} \omega - u_1 - u - \Delta)\, \vartheta(\int_{x_0}^{x} \omega - u_1 + u - \Delta) \cdot \left(\epsilon^{ab}\partial_b\vartheta(\int_{x_0}^{x} \omega - \Delta)\right)^{-1} \omega^a(x) \tag{3.7}$$

(it does not depend on the choice of $a = 1, 2$). Let us explain the above formulae. $K(x; u_1, u)$, in its dependence on x, is a multivalued holomorphic 1,0-form. More exactly, the function

$$x \;\mapsto\; \vartheta(\int_{x_0}^{x} \omega - u_1 - u - \Delta) \tag{3.8}$$

is multivalued around the B_a-cycles picking up the factor

$$\mathrm{e}^{-\pi i \tau^{aa} - 2\pi i(\int_{x_0}^{x} \omega^a - u_1^a - u^a - \Delta^a)}$$

when x goes around B_a so that it describes an element $s_2 \in H^0(l_{u_1} l_u)$ (non-vanishing if $u_1 + u \notin \mathbb{Z}^2 + \tau\mathbb{Z}^2$). Similarly,

$$x \;\mapsto\; \vartheta(\int_{x_0}^{x} \omega - u_1 + u - \Delta)\; \left(\epsilon^{ab}\partial_b\vartheta(\int_{x_0}^{x} \omega - \Delta)\right)^{-1} \omega^a(x)$$

picks up the factor

$$\mathrm{e}^{2\pi i(u_1^a - u^a)}$$

when x goes around B_a and describes a holomorphic 1,0-form χ with values in $l_{u_1} l_u^{-1}$ (non-vanishing if $u_1 - u \notin \mathbb{Z}^2 + \tau\mathbb{Z}^2$). The product $s_2\chi = K(\cdot; u_1, u)$ is a holomorphic 1,0-form with values in $l_{u_1}^2$ and it may be paired with $b \in \wedge^{01}(l_{u_1}^{-2})$ via the integral over

x on the r.h.s. of Eq. (3.6). The integral is independent of the choice of the representative b of the cohomology class $[b]$. In its dependence on u, $K(x;u_1,u)$ is a theta function of the 2nd order and so is $\theta(u)$. In Appendix 1 we check explicitly that θ given by Eq. (3.6) possesses the required property.

The product of the two shifted Riemann theta functions $\vartheta(u'-u)\vartheta(u'+u)$ is a theta function of the 2nd order both in u' and in u (and it is invariant under the interchange $u'\leftrightarrow u$). Let ι denote the (marking dependent) linear isomorphism between the spaces $H^0(L_\Theta^2)^*$ and $H^0(L_\Theta^2)$ defined by

$$\iota(\phi)(u) = \langle \vartheta(\cdot - u)\vartheta(\cdot + u),\ \phi\rangle. \tag{3.9}$$

An easy calculation shows that

$$\vartheta(u'-u)\,\vartheta(u'+u) = \sum_e \theta_{2,e}(u')\,\theta_{2,e}(u). \tag{3.10}$$

Hence ι interchanges the basis $(\theta_{2,e})$ of $H^0(L_\Theta^2)$ with the dual basis $(\theta_{2,e}^*)$ of $H^0(L_\Theta^2)^*$. Denote by ϕ_u the linear form on $H^0(L_\Theta^2)$ that computes the value of the theta function at point $u\in\mathbb{C}^2$. The Kummer quartic $\mathcal{K}^*\subset H^0(L_\Theta^2)^*$, $\mathcal{K}^* = \{\mathbb{C}^*\phi_{u'}\,|\,u'\in\mathbb{C}^2\}$ is mapped by the isomorphism ι into a quartic $\mathcal{K}\subset H^0(L_\Theta^2)$ of theta functions proportional to

$$u\ \mapsto\ \vartheta(u'-u)\,\vartheta(u'+u)$$

for some $u'\in\mathbb{C}^2$.

One may define a projective action of $(\mathbb{Z}/2\mathbb{Z})^4$ on $H^0(L_\Theta^2)$ by assigning to an element $(e,e')\in(\mathbb{Z}/2\mathbb{Z})^4$, with $e,e' = (0,0),(1,0),(0,1)$ or $(1,1)$, a linear transformation $U_{e,e'}$ s.t.

$$(U_{e,e'}\theta)(u) = \mathrm{e}^{\frac{1}{2}\pi i e'\cdot\tau e' + 2\pi i e'\cdot u}\theta(u + \tfrac{1}{2}(e+\tau e')). \tag{3.11}$$

The relation $U_{e_1,e_1'}U_{e_2,e_2'} = (-1)^{e_1\cdot e_2'}U_{e_1+e_2,e_1'+e_2'}$ holds so that U lifts to the Heisenberg group. In the action on the basic theta functions,

$$U_{e_1,e_1'}\theta_{2,e} = (-1)^{e_1\cdot e}\,\theta_{2,e+e_1'}. \tag{3.12}$$

The marking-dependence of the isomorphism ι of Eq. (3.9) is given by the action of $(\mathbb{Z}/2\mathbb{Z})^4$. It is easy to check that this action preserves $\mathcal{K}$ and that the transposed action of $(\mathbb{Z}/2\mathbb{Z})^4$ preserves $\mathcal{K}^*$. The $(\mathbb{Z}/2\mathbb{Z})^4$ symmetry of the Kummer quartics allows to find easily their defining equation, see Appendix 3.

It was shown in [18] that the image of $\mathcal{N}_s$ under the map (3.5) contains all non-zero theta functions of the 2nd order except the ones in the Kummer quartic $\mathcal{K}$. The latter correspond, however, to the (Seshadri equivalence classes of) semistable but not stable bundles so that the map m extends to an isomorphism between $\mathcal{N}_{ss}$ and $\mathbb{P}H^0(L_\Theta^2)$ showing that $\mathcal{N}_{ss}$ is a smooth projective variety.

4. Cotangent Bundle

Let us describe the cotangent space of $\mathcal{N}_s$ at point E. The covectors tangent to $\mathcal{N}_s$ at E may be identified with holomorphic 1,0-forms Ψ with values in the bundle of traceless endomorphisms of E. We may assume that E is an extension of a line bundle l of degree 1 realized as $l^{-1}\oplus l$ with $\bar\partial_E = \bar\partial_{l^{-1}\oplus l} + B$, where $B = (\begin{smallmatrix}0 & b\\ 0 & 0\end{smallmatrix})$. Then

$$\Psi = \begin{pmatrix} -\mu & \nu \\ \eta & \mu \end{pmatrix}, \tag{4.1}$$

where $\mu \in \wedge^{10}$, $\nu \in \wedge^{10}(l^{-2})$, $\eta \in \wedge^{10}(l^2)$ and

$$\bar\partial_{l^2}\eta = 0, \qquad \bar\partial\mu = -\eta\wedge b, \qquad \bar\partial_{l^{-2}}\nu = 2\mu\wedge b\ . \tag{4.2}$$

It is easy to relate the above description of covectors tangent to $\mathcal{N}_s$ to the one of Sect. 2. Let $\mathcal{U} : l^{-1}\oplus l \to \Sigma\times\mathbb{C}^2$ be a smooth isomorphism of rank 2 bundles with trivial determinant. Then $\mathcal{U}\bar\partial_E\mathcal{U}^{-1} = \bar\partial + A$ for a certain sl_2-valued 0,1-form A and $\Phi = \mathcal{U}\Psi\mathcal{U}^{-1}$ satisfies $\bar\partial_A\Phi = 0$. The $\mathcal{G}$ orbit of (A,Φ) is independent of the choice of $\mathcal{U}$ and the quadratic Hitchin Hamiltonian takes value $\frac{1}{2}tr(\Phi)^2$ on it. The latter expression is clearly equal to $\frac{1}{2}tr(\Psi)^2 = \mu^2+\eta\nu$ which, as easily follows from relations (4.2), defines a holomorphic quadratic differential. Hence

$$\mathcal{H}(E,\Psi) = \mu^2 + \eta\nu\ . \tag{4.3}$$

We would like to express the latter using the theta function description of $T^*\mathcal{N}_{ss} = T^*\mathbb{P}H^0(L^2_\Theta)$; where the covectors tangent to $\mathcal{N}_{ss}$ at $\mathbb{C}^*\theta$ are represented by linear forms ϕ on $H^0(L^2_\Theta)$ s.t. $\langle\theta,\phi\rangle = 0$.

Let $l = l_{u_1}\in C_E$, i.e. $\theta(u_1) = 0$ for the theta function corresponding to E. We shall assume that $l^2 \not\cong K$ i.e. that $2u_1\notin\mathbb{Z}^2+\tau\mathbb{Z}^2$. An infinitesimal variation δE of the bundle E in $\mathcal{N}_s$ may be achieved by changing $\bar\partial_E = \bar\partial_{l^{-1}\oplus l} + B$ with $B = (\begin{smallmatrix}0 & b\\ 0 & 0\end{smallmatrix})$ to

$$\bar\partial_{l^{-1}\oplus l} + \begin{pmatrix} \pi\delta u_1(\mathrm{Im}\tau)^{-1}\bar\omega & b+\delta b \\ 0 & -\pi\delta u_1(\mathrm{Im}\tau)^{-1}\bar\omega \end{pmatrix} \equiv \bar\partial_E + \delta B \tag{4.4}$$

(all other variations of $\bar\partial_E$ may be obtained from (4.4) by infinitesimal gauge transformations). Clearly

$$\langle\delta E,\Psi\rangle = \int_\Sigma tr\ \Psi\wedge\delta B = -2\pi\delta u_1(\mathrm{Im}\tau)^{-1}\int_\Sigma \mu\wedge\bar\omega + \int_\Sigma \eta\wedge\delta b\ . \tag{4.5}$$

Note that the line bundle l_{u_1} with the $\bar\partial$-operator changed to $\bar\partial_{l_{u_1}} - \pi\delta u_1(\mathrm{Im}\tau)^{-1}\bar\omega$ is equivalent to $l_{u_1+\delta u_1}\equiv l'$ and the equivalence is established by multiplication by the multivalued function $x\mapsto \mathrm{e}^{2\pi i\delta u_1(\mathrm{Im}\tau)^{-1}\int_{x_0}^x \mathrm{Im}\omega}$. Hence $l^{-1}\oplus l$ with the $\bar\partial$-operator given by Eq. (4.4) is equivalent to $l'^{-1}\oplus l'$ with the $\bar\partial$-operator $\bar\partial_{l'^{-1}\oplus l'} + (\begin{smallmatrix}0 & b+\delta' b\\ 0 & 0\end{smallmatrix})$; where $\delta' b(x) = \delta b - 4\pi i\delta u_1(\mathrm{Im}\tau)^{-1}(\int_{x_0}^{x}\mathrm{Im}\omega) b(x)$. The last bundle corresponds by the relation (3.6) to the theta function

$$(\theta+\delta\theta)(u) = \int_\Sigma K(x; u_1+\delta u_1, u)\wedge(b(x)+\delta' b(x))\ .$$

Hence δE is represented by the variation

$$\delta\theta(u) = -\, 2\pi\delta u_1^a(\mathrm{Im}\tau)^{-1}_{ab}\int_\Sigma L^b(x;u_1,u)\wedge b(x) + \int_\Sigma K(x;u_1,u)\wedge\delta b(x) \quad (4.6)$$

of the theta function, where

$$L^a(x;u_1,u) = K(x;u_1,u)\int_{x_0}^{x}(\omega^a-\overline{\omega}^a) - \tfrac{1}{2\pi}\mathrm{Im}\tau^{ab}\partial_{u_1^b}K(x;u_1,u)\,. \quad (4.7)$$

Note that as functions of x, $L^a(x;u_1,u)$ are 1,0-forms with values in $l^2_{u_1}$ (as are $K(x;u_1,u)$). They are not holomorphic:

$$\bar\partial_x L^a(x;u_1,u) \;=\; K(x;u_1,u)\wedge\overline{\omega}^a(x).$$

As functions of u, $L^a(x;u_1,u)$ are theta functions of the 2nd order.

We would like to find an explicit form of the Lax matrix Ψ representing the linear form ϕ on $H^0(L^2_\Theta)$ s.t. $\langle\theta,\phi\rangle = 0$. We shall achieve this goal partially, finding the entries η and μ of the matrix (4.1). The correspondence between Ψ and ϕ is determined by the equality

$$\langle\delta E,\Psi\rangle \;=\; \langle\delta\theta,\phi\rangle.$$

Since the left-hand side is given by Eq. (4.5) and $\delta\theta$ by Eq. (4.6), we obtain

$$\begin{aligned}&-2\pi\delta u_1(\mathrm{Im}\tau)^{-1}\int_\Sigma \mu\wedge\overline{\omega} + \int_\Sigma \eta\wedge\delta b\\ &=\; -2\pi\delta u_1^a(\mathrm{Im}\tau)^{-1}_{ab}\int_\Sigma\langle L^b(x;u_1,\cdot),\phi\rangle\wedge b(x) + \int_\Sigma\langle K(x;u_1,\cdot),\phi\rangle\wedge\delta b(x). \quad (4.8)\end{aligned}$$

Taking $\delta u_1 = 0$ we infer that

$$\eta(x) = \langle K(x;u_1,\cdot),\phi\rangle \quad (4.9)$$

is the lower left entry of the matrix Ψ corresponding to the linear form ϕ.

It is easy to find the entry μ of Ψ representing the linear form ϕ_{u_1} (recall that ϕ_{u_1} computes the value of a theta function in $H^0(L^2_\Theta)$ at point u_1). Since $K(x;u_1,u_1) = 0$, it follows from Eq. (4.9) that $\eta = 0$ in this case. Equation (4.8) reduces then to

$$\begin{aligned}&-2\pi\delta u_1(\mathrm{Im}\tau)^{-1}\int_\Sigma\mu\wedge\overline{\omega} = \delta u_1^a\int_\Sigma\partial_{u_1^a}K(x;u_1,u_1)\wedge b(x)\\ &= -\delta u_1^a\int_\Sigma\partial_{u^a}K(x;u_1,u_1)\wedge b(x) = -\delta u_1^a\;\partial_a\theta(u_1).\end{aligned}$$

This fixes μ uniquely:

$$\mu = \tfrac{i}{4\pi}\partial_a\theta(u_1)\omega^a\,. \quad (4.10)$$

Let us check that there exists $\nu\in\wedge^{10}(l^{-2}_{u_1})$ such that the last equation of (4.2) holds. For this it is necessary and sufficient that

$$\int_\Sigma\kappa\mu\wedge b = 0 \quad (4.11)$$

for a non-zero holomorphic section κ of $l_{u_1}^2 = V(2u_1)K$ ($dim H^0(l_{u_1}^2) = 1$ if $2u_1 \notin \mathbb{Z}^2 + \tau\mathbb{Z}^2$). But such a section may be represented by the function

$$x \;\mapsto\; \vartheta(\int_{x_0}^{x} \omega - 2u_1 - \Delta)$$

so that, recalling the definition (3.7), we obtain

$$\int_\Sigma \kappa\omega^a \wedge b = \int_\Sigma \epsilon^{ab}\partial_{u^b} K(x; u_1, u_1) \wedge b(x) = \epsilon^{ab}\, \partial_b \theta(u_1)\,. \tag{4.12}$$

Hence the relation (4.11) follows for μ given by Eq. (4.10). The 1,0-form ν satisfying the last relation of (4.2) is now unique since $H^0(l_{u_1}^{-2}K) = \{0\}$.

We would like to find the entry μ of Ψ corresponding to more general linear forms ϕ s.t. $\langle\theta, \phi\rangle = 0$. Recall that θ with $\theta(u_1) = 0$ may be given by formula (3.6) with $b \in \wedge^{0,1}(l_{u_1}^{-2})$. Note that any 2nd-order theta function $\delta\theta$ vanishing at u_1 and not in the Kummer quartic $\mathcal{K}$ may be written as

$$\delta\theta(u) = \int_\Sigma K(x; u_1, u) \wedge \delta b(x) \tag{4.13}$$

with $\delta b \in \wedge^{01}(l_{u_1}^{-2})$ since it corresponds to an extension of l_{u_1}. The space of $\delta\theta$ vanishing at u_1 is 3-dimensional, as well as the space $H^1(l_{u_1}^{-2})$ of classes $[\delta b]$ and the assumption that $\delta\theta \notin \mathcal{K}$ is obviously superfluous. Set for a linear form ψ on $H^0(L_\Theta^2)$,

$$\eta_\psi(x) = \langle K(x; u_1, \cdot), \psi\rangle\,. \tag{4.14}$$

η_ψ defines a holomorphic 1,0-form with values in $l_{u_1}^2$. We have

$$\langle\delta\theta, \psi\rangle = \int_\Sigma \eta_\psi \wedge \delta b \tag{4.15}$$

for $\delta\theta$ given by Eq. (4.13). By dimensional count, the map $\psi \mapsto \eta_\psi$ is onto $H^0(l_{u_1}^2 K)$ with the 1-dimensional kernel spanned by ϕ_{u_1}. Specifying Eq. (4.15) to $\delta\theta \propto \theta$, we obtain the relation

$$\langle\theta, \psi\rangle = \int_\Sigma \eta_\psi \wedge b \tag{4.16}$$

which determines the class $[b] \in H^1(l_{u_1}^{-2})$ in terms of θ. On the other hand, taking $\psi = \phi$ in Eq. (4.14), we infer that $\eta = 0$ if and only if ϕ is proportional to ϕ_{u_1}, the case studied before.

If $\eta_\phi \not\equiv 0$ then μ depends on the choice of the representative b in the class $[b] \in H^1(l_{u_1}^{-2})$ characterizing E as the extension of l_{u_1}. Under the transformation $b \mapsto b + \bar\partial\varphi$, where φ is a section of $l_{u_1}^{-2}$,

$$\eta \mapsto \eta, \qquad \mu \mapsto \mu + \varphi\eta, \qquad \nu \mapsto \nu - 2\varphi\mu - \varphi^2\eta.$$

The pairing of the theta functions $L^a(x; u_1, \cdot)$ of Eq. (4.7) with the linear form ϕ gives two 1,0-forms with values in $l_{u_1}^2$:

$$\chi^a(x) = \langle L^a(x; \cdot, u_1), \phi\rangle \qquad \text{s.t.} \qquad \bar\partial\chi^a = \eta \wedge \bar\omega^a. \tag{4.17}$$

Specifying the equality (4.8) to the case with $\delta b = 0$, we infer the relation

$$\int_\Sigma \mu \wedge \bar{\omega}^a = \int_\Sigma \chi^a \wedge b \tag{4.18}$$

which, together with the equation

$$\bar{\partial}\mu = -\eta \wedge b \tag{4.19}$$

determines μ completely. In Appendix 2, we show that μ fixed this way satisfies the relation $\int_\Sigma \kappa\mu \wedge b = 0$ and hence defines a unique 1,0-form ν with values in $l_{u_1}^{-2}$ s.t. $\bar{\partial}\nu = 2\mu \wedge b$.

5. Hitchin Hamiltonians

From the relation (4.3) and the explicit form of Ψ corresponding to ϕ_{u_1} (η vanishing, μ given by Eq. (4.10)), one obtains

$$\mathcal{H}(\theta, a_1\phi_{u_1}) = -\frac{1}{16\pi^2} a_1^2 (\partial_a \theta(u_1)\, \omega^a)^2. \tag{5.1}$$

The right -and side is a quadratic differential. Equation (5.1), whose projective version was first obtained in [21], is consistent with the rescaling $\theta \mapsto t\theta$ and $\phi \mapsto t^{-1}\phi$ for $t \in \mathbb{C}^*$. It describes the value of the Hitchin map $\mathcal{H}$ on the special covectors, namely those represented by the pairs (θ, ϕ) s.t. $\mathbb{C}^*\phi$ is in the intersection $\mathcal{K}_E^*$ of the Kummer quartic $\mathcal{K}^*$ with the plane $\langle\theta, \phi\rangle = 0$. The linear span of $\mathcal{K}_E^*$ gives the whole cotangent space $T_E^*\mathcal{N}_{ss}$. Indeed, any theta function of the 2nd order $\delta\theta$ which vanishes on C_E has to be proportional to θ and defines a zero vector in $T_E\mathcal{N}_{ss}$. $\mathcal{K}_E^*$ is itself a quartic. Hence the restriction of the quadratic polynomial $\mathcal{H}$ to six lines in $\mathcal{K}_E^*$ in a general position determines $\mathcal{H}$ completely.

It is possible to find a more explicit description of the values of $\mathcal{H}$ away from $\mathcal{K}_E^*$ and this is the main aim of the rest of the present section. Suppose then that the entry η in Ψ does not vanish. Let x_i, $i = 1, \ldots, 4$, be its four zeros. We shall assume that η cannot be written as $\kappa\omega$ for $\kappa \in H^0(l_{u_1}^2)$ and $\omega \in H^0(K)$. This is true for generic ϕ. In this case, $\eta = a_2\eta_{\phi_{u_2}}$ for some $a_2 \in \mathbb{C}^*$ and for u_2 satisfying

$$u_1 + u_2 = \int_{x_0}^{x_1} \omega + \int_{x_0}^{x_2} \omega - 2\Delta \quad \text{and} \quad u_1 - u_2 = \int_{x_0}^{x_3} \omega + \int_{x_0}^{x_4} \omega - 2\Delta, \tag{5.2}$$

$u_1 \pm u_2 \notin \mathbb{Z} + \tau\mathbb{Z}$. Indeed, $\eta_{\phi_{u_2}}(x)$ is a holomorphic section of $l_{u_1}^2 K$ represented by the multivalued function $\vartheta(\int_{x_0}^x \omega - u_1 - u_2 - \Delta)\vartheta(\int_{x_0}^x \omega - u_1 + u_2 - \Delta)$ vanishing exactly at x_i and such a section is unique up to normalization. We infer that in the action on the theta functions of Eq. (4.13), the linear forms ϕ and $a_2\phi_{u_2}$ coincide. Since Eq. (4.13) gives all theta functions vanishing at u_1, it follows that

$$\phi = a_1\phi_{u_1} + a_2\phi_{u_2} \tag{5.3}$$

for some $a_1 \in \mathbb{C}$. Let us stress that, to fix normalizations, u_1 and u_2 should be viewed as elements of $\mathbb{C}^2$ with x_i in relations (5.2) belonging to the covering space $\widetilde{\Sigma}$ of Σ. The relation $\langle\theta, \phi\rangle = 0$ implies that $\theta(u_2) = 0$.

Summarizing, we have shown that a generic pair (θ,ϕ) s.t. $\langle\theta,\phi\rangle = 0$ may be obtained by first choosing u_1 and u_2 s.t. $2u_1, 2u_2, u_1 \pm u_2 \notin \mathbb{Z} + \tau\mathbb{Z}$ and then taking θ from the 2-dimensional space of theta functions vanishing at u_1 and u_2 and ϕ from the orthogonal subspace. The zeros x_i of η are determined from Eqs. (5.2) (as the zeros of $\vartheta(\int_{x_0}^{x}\omega - u_1 \pm u_2 - \Delta)$). For simplicity, we shall assume that they are distinct (this is true for generic ϕ). Then the differentials $\partial\eta(x_i) \in (l_{u_1}^2 K^2)_{x_i}$ do not vanish.

A quadratic differential $\rho \in H^0(K^2)$ is determined by its values at four points x_i which form a divisor of $l_{u_1}^2 K \not\simeq K^2$. Since $dim H^0(K^2) = 3$, there is one linear relation satisfied by all $\rho(x_i)$:

$$\sum_{i=1}^{4} \rho(x_i)\kappa(x_i)\partial\eta(x_i)^{-1} = 0$$

for $0 \neq \kappa \in H^0(l_{u_1}^2)$. It expresses the fact that the sum of residues of the meromorphic 1,0-form $\rho\kappa\eta^{-1}$ has to vanish. For $\rho = \mathcal{H}(\theta,\phi) = \mu^2 + \eta\nu$,

$$\rho(x_i) = \mu(x_i)^2$$

so that it is enough to know $\mu(x_i)$ in order to determine $\mathcal{H}(\theta,\phi)$. Note that although the 1,0-form μ depends on the choice of the representative b of the class $[b] \in H^1(l_{u_1}^{-2})$ defined by Eq. (4.16), the values $\mu(x_i)$ are invariant since under $b \mapsto b + \bar\partial\varphi$ the 1,0-form μ changes to $\mu + \varphi\eta$.

It remains to find $\mu(x_i)$. Consider the meromorphic function $\eta_\psi\eta^{-1}$. Viewed as a distribution, $\bar\partial(\eta_\psi\eta^{-1})$ is supported at the poles of $\eta_\psi\eta^{-1}$ and

$$\int_\Sigma \mu \wedge \bar\partial(\eta_\psi\eta^{-1}) = -2\pi i \sum_{i=1}^{4} \mu(x_i)\eta_\psi(x_i)\partial\eta(x_i)^{-1}$$

for any (smooth) 1,0-form μ. In particular, for μ satisfying Eq. (4.19) we obtain

$$\sum_{i=1}^{4} \mu(x_i)\eta_\psi(x_i)\partial\eta(x_i)^{-1} = \frac{1}{2\pi i}\int_\Sigma \eta_\psi \wedge b = \frac{1}{2\pi i}\langle\theta,\psi\rangle. \tag{5.4}$$

Recall that η_ψ run through the three-dimensional space $H^0(l_{u_1}^2 K)$. If $\eta_\psi(x_i) = 0$ for all i then η_ψ has to be proportional to $\eta = a_2\eta_{\phi_{u_2}}$. Hence vectors $(\eta_\psi(x_i))$ form a 2-dimensional subspace in $\bigoplus_i (l_{u_1}^2 K)_{x_i}$ and Eqs. (5.4) determine vector $(\mu(x_i)) \in \bigoplus_i K_{x_i}$ up to a 2-dimensional ambiguity spanned by $(\omega^a(x_i))$ (indeed, as the residues of the meromorphic 1,0-form $\eta_\psi\eta^{-1}\omega^a$, the numbers $\omega^a(x_i)\eta_\psi(x_i)\partial\eta(x_i)^{-1}$ sum to zero). It is clearly enough to take for ψ in Eq. (5.4) any two linear forms independent of ϕ_{u_1} and ϕ_{u_2}. In the generic situation, we may choose the forms $\partial_a\phi_{u_1}$ defined by

$$\langle\theta, \partial_a\phi_{u_1}\rangle = \partial_a\theta(u_1).$$

Denoting the corresponding 1,0-forms η_ψ by η'_a, we obtain 2 relations for $\mu(x_i)$:

$$\sum_{i=1}^{4} \mu(x_i)\eta'_a(x_i)\partial\eta(x_i)^{-1} = \frac{1}{2\pi i}\partial_a\theta(u_1). \tag{5.5}$$

Alternatively, we may choose for ψ the linear forms $\partial_a \phi_{u_2}$ corresponding to 1,0-forms η''_a. This gives the relations

$$\sum_{i=1}^{4} \mu(x_i)\eta''_a(x_i)\partial\eta(x_i)^{-1} = \frac{1}{2\pi i}\partial_a\theta(u_2). \tag{5.6}$$

η''_a must be linearly dependent from η'_a and η (in the generic situation):

$$\eta''_a = D_a^b \eta'_b + \eta \tag{5.7}$$

leading via Eqs. (5.5) and (5.6) to the relation

$$\partial_a\theta(u_2) = D_a^b\ \partial_b\theta(u_1).$$

We need 2 more equations to determine $\mu(x_i)$. They may be obtained from Eqs. (4.18) fixing the holomorphic contributions to μ. Indeed, using the 2[nd] equation in (4.17), and Eq. (4.19) we infer that

$$\begin{aligned}\int_\Sigma \mu \wedge \bar{\omega}^a = \int_\Sigma (\mu\eta^{-1})\eta \wedge \bar{\omega}^a = \int_\Sigma (\mu\eta^{-1})\bar{\partial}\chi^a = \int_\Sigma \chi^a \wedge \bar{\partial}(\mu\eta^{-1})\\ = \int_\Sigma \chi^a \wedge b - 2\pi i \sum_{i=1}^{4} \mu(x_i)\chi^a(x_i)\partial\eta(x_i)^{-1}\end{aligned} \tag{5.8}$$

so that Eq. (4.18) implies that

$$\sum_{i=1}^{4} \mu(x_i)\chi^a(x_i)\partial\eta(x_i)^{-1} = 0. \tag{5.9}$$

These are the two missing equations. To see this, repeat the calculation (5.8) for μ replaced by ω^b. This gives the relation

$$\frac{1}{\pi}\mathrm{Im}\tau^{ab} = \sum_{i=1}^{4} \omega^b(x_i)\chi^a(x_i)\partial\eta(x_i)^{-1}.$$

Suppose now that $d_a\chi^a(x_i) + e\eta_\psi(x_i) = 0$ for $i = 1, \ldots, 4$. It follows that

$$0 = \sum_{i=1}^{4} \omega^b(x_i)\left(d_a\chi^a(x_i) + e\eta_\psi(x_i)\right)\partial\eta(x_i)^{-1} = \frac{1}{\pi}\mathrm{Im}\tau^{ab}d_a$$

so that $d_a = 0$. Hence the vectors $(\chi^a(x_i))$ span a 2-dimensional subspace of $\bigoplus_i K_{x_i}$ transversal to the 2-dimensional subspace spanned by the vectors $(\eta_\psi(x_i))$ and the linear equations (5.4) and (5.9) determine $\mu(x_i)$ completely.

It is enough to consider the case $\phi = \phi_{u_2}$. Indeed, the shift $\phi \mapsto \phi + a_1\phi_{u_1}$ results in the change

$$\mu \quad \mapsto \quad \mu + \frac{i}{4\pi}a_1\partial_a\theta(u_1)\omega^a,$$

see Eq. (4.10). Identifying 1,0-forms with multivalued functions by the relation (3.3) and setting $\chi_a = 2\pi(\mathrm{Im}\tau)^{-1}_{ab}\chi^b$, $w_i = \int_{x_0}^{x_i}\omega - \Delta$, $G_1 = G_{12} = -G_2$ and $G_3 = G_{34} = -G_4$ where

$$G_{ij} = det \begin{pmatrix} \partial_1\vartheta(w_i) & \partial_1\vartheta(w_j) \\ \partial_2\vartheta(w_i) & \partial_2\vartheta(w_j) \end{pmatrix},$$

we obtain

$$\begin{array}{ll} \partial\eta(x_1) = G_1\,\vartheta(w_1 - w_3 - w_4), & \chi_a(x_1) = -\partial_a\vartheta(w_2)\,\vartheta(w_1 - w_3 - w_4), \\ \partial\eta(x_2) = G_2\,\vartheta(w_2 - w_3 - w_4), & \chi_a(x_2) = -\partial_a\vartheta(w_1)\,\vartheta(w_2 - w_3 - w_4), \\ \partial\eta(x_3) = G_3\,\vartheta(w_3 - w_1 - w_2), & \chi_a(x_3) = -\partial_a\vartheta(w_4)\,\vartheta(w_3 - w_1 - w_2), \\ \partial\eta(x_4) = G_4\,\vartheta(w_4 - w_1 - w_2), & \chi_a(x_4) = -\partial_a\vartheta(w_3)\,\vartheta(w_4 - w_1 - w_2), \end{array}$$

$$\begin{array}{ll} \eta'_a(x_1) = \partial_a\vartheta(w_1)\,\vartheta(w_2 + w_3 + w_4), & \eta''_a(x_1) = \partial_a\vartheta(w_2)\,\vartheta(w_1 - w_3 - w_4), \\ \eta'_a(x_2) = \partial_a\vartheta(w_2)\,\vartheta(w_1 + w_3 + w_4), & \eta''_a(x_2) = \partial_a\vartheta(w_1)\,\vartheta(w_2 - w_3 - w_4), \\ \eta'_a(x_3) = \partial_a\vartheta(w_3)\,\vartheta(w_1 + w_2 + w_4), & \eta''_a(x_3) = -\partial_a\vartheta(w_4)\,\vartheta(w_3 - w_1 - w_2), \\ \eta'_a(x_4) = \partial_a\vartheta(w_4)\,\vartheta(w_1 + w_2 + w_3), & \eta''_a(x_4) = -\partial_a\vartheta(w_3)\,\vartheta(w_4 - w_1 - w_2). \end{array}$$

Given these values, it is easy to find the explicit form of the matrix (D_a^b) appearing in the relation between the derivatives of $\partial_a\theta$ at u_1 and u_2 by specifying Eq. (5.7) to two of the points x_i. One form of these relations is

$$\begin{aligned} &\partial_2\vartheta(w_3)\partial_1\theta(u_2) - \partial_1\vartheta(w_3)\partial_2\theta(u_2) \\ &\qquad = -\tfrac{\vartheta(w_3-w_1-w_2)}{\vartheta(w_1+w_2+w_4)}\,(\partial_2\vartheta(w_4)\partial_1\theta(u_1) - \partial_1\vartheta(w_4)\partial_2\theta(u_1)), \\ &\partial_2\vartheta(w_4)\partial_1\theta(u_2) - \partial_1\vartheta(w_4)\partial_2\theta(u_2) \\ &\qquad = -\tfrac{\vartheta(w_4-w_1-w_2)}{\vartheta(w_1+w_2+w_3)}\,(\partial_2\vartheta(w_3)\partial_1\theta(u_1) - \partial_1\vartheta(w_3)\partial_2\theta(u_1)). \end{aligned}$$

Let us denote $\widetilde{\mu}(x_i) = \mu(x_i)/G_i$. Equations (5.9) have the general solution

$$(\widetilde{\mu}(x_1), \ldots, \widetilde{\mu}(x_4)) = g_1(G_{34}, 0, G_{23}, -G_{24}) + g_2(0, G_{34}, G_{13}, -G_{14})$$

and Eqs. (5.6) fix the values of g_1 and g_2 to

$$\begin{aligned} g_1 &= -\frac{\partial_2\vartheta(w_1)\partial_1\theta(u_2) - \partial_1\vartheta(w_1)\partial_2\theta(u_2)}{4\pi i G_{12}G_{34}}, \\ g_2 &= \frac{\partial_2\vartheta(w_2)\partial_1\theta(u_2) - \partial_1\vartheta(w_2)\partial_2\theta(u_2)}{4\pi i G_{12}G_{34}}. \end{aligned}$$

This leads to the following simple result:

$$\mu(x_i) = \pm\tfrac{i}{4\pi}(\partial_2\vartheta(w_i)\partial_1\theta(u_2) - \partial_1\vartheta(w_i)\partial_2\theta(u_2)) \tag{5.10}$$

or, in a more abstract notation from the introduction,

$$\mu(x_i) = \pm\tfrac{i}{4\pi}d\theta(l_{u_2})$$

with the plus sign for $i = 1, 2$ and the minus one for $i = 3, 4$.

Since the Hitchin Hamiltonian is quadratic in ϕ and its values on ϕ_{u_1} and ϕ_{u_2} are given by Eq. (5.1), it follows that

$$\begin{aligned} &\mathcal{H}(\theta, a_1\phi_{u_1} + a_2\phi_{u_2}) \\ &\quad = a_1^2\mathcal{H}(\theta, \phi_{u_1}) + a_2^2\mathcal{H}(\theta, \phi_{u_2}) + 2a_1a_2(c_1(\omega^1)^2 + c_2\omega^1\omega^2 + c_3(\omega^2)^2). \end{aligned}$$

The mixed term may be found from the linear equations

$$\frac{i}{4\pi}(\partial_2\vartheta(w_i)\partial_1\theta(u_1) - \partial_1\vartheta(w_i)\partial_2\theta(u_1))\,\widetilde{\mu}(x_i)G_i$$
$$= c_1\partial_2\vartheta(w_i)\partial_2\vartheta(w_i) - c_2\partial_2\vartheta(w_i)\partial_1\vartheta(w_i) + c_3\partial_1\vartheta(w_i)\partial_1\vartheta(w_i).$$

Their explicit solution leads to the expression

$$\begin{aligned}\mathcal{H}(\theta, a_1\phi_{u_1} + a_2\phi_{u_2}) = &-\tfrac{1}{16\pi^2}\,(a_1\partial_a\theta(u_1)\omega^a + a_2\partial_a\theta(u_2)\omega^a)^2\\ &+ \tfrac{a_1a_2}{4\pi^2G_{13}G_{23}}\,(\partial_2\vartheta(w_3)\partial_1\theta(u_1) - \partial_1\vartheta(w_3)\partial_2\theta(u_1))\\ &\cdot(\partial_2\vartheta(w_3)\partial_1\theta(u_2) - \partial_1\vartheta(w_3)\partial_2\theta(u_2))\,\partial_a\vartheta(w_1)\partial_b\vartheta(w_2)\omega^a\omega^b.\end{aligned} \tag{5.11}$$

The second term on the right-hand side is a quadratic differential that vanishes at x_1 and x_2 and is equal to $\frac{a_1a_2}{4\pi^2}\partial_a\theta(u_1)\partial_b\theta(u_2)\omega^a\omega^b$ at x_3 and x_4 so that

$$\mathcal{H}(\theta,\phi)(x_i) = -\frac{1}{16\pi^2}\,(a_1\partial_a\theta(u_1)\omega^a(x_i) \pm a_2\partial_a\theta(u_2)\omega^a(x_i))^2, \tag{5.12}$$

where sign plus should be taken for x_1 and x_2 and sign minus for x_3 and x_4. This is the result (1.7) described in Introduction.

6. Self-Duality

We would like to compare the values of the Hitchin Hamiltonians on the dual pairs (θ,ϕ) and (θ',ϕ'), where $\theta' = \iota(\phi)$ and $\phi' = \iota^{-1}(\theta)$ with ι defined by Eq. (3.9). Recall that, given u_1 s.t. $\theta(u_1) = 0$, we associated to the linear form ϕ a 1,0-form η by Eq. (4.9). Viewed as a holomorphic section of $l^2_{u_1}K$,

$$\eta(x) = \langle\vartheta(\int_{x_0}^{x}\omega - u_1 - \cdot - \Delta)\,\vartheta(\int_{x_0}^{x}\omega - u_1 + \cdot - \Delta)\,,\ \phi\rangle.$$

Let us denote

$$u_i' = \int_{x_0}^{x_i}\omega - u_1 - \Delta. \tag{6.1}$$

The vanishing of $\eta(x_i)$ implies then that the linear form ϕ annihilates the theta functions

$$u \ \mapsto\ \vartheta(u_i' - u)\vartheta(u_i' + u) = \iota(\phi_{u_i'})(u) \tag{6.2}$$

and also, if we rewrite $\eta(x_i)$ as $\iota(\phi)(u_i')$, that $\theta'(u_i') = 0$. Since $\phi = a_1\phi_{u_1} + a_2\phi_{u_2}$ and ϕ_{u_1} annihilates the theta functions (6.2) as well, it follows that they belong to Π. Hence $\mathbb{C}^*\iota(\phi_{u_i'})$ are the 4 points of intersection of the line $\mathbb{P}\Pi$ with the Kummer quartic $\mathcal{K}$. Equivalently, $\mathbb{C}^*\phi_{u_i'}$ are the points of intersection of $\mathbb{P}\Pi'^{\perp}$ with $\mathcal{K}^*$. In the generic situation, any pair of theta functions $\phi_{u_i'}$ spans $\Pi'^{\perp}$ and since $\phi' \in \Pi'^{\perp}$, we may write

$$\phi' = a_1'\phi_{v_1} + a_2'\phi_{v_2} \tag{6.3}$$

or, equivalently,

$$\theta = a_1'\iota(\phi_{v_1}) + a_2'\iota(\phi_{v_2}). \tag{6.4}$$

The involution $l \mapsto l^{-1}K$ of the Jacobian J^1 lifts to $\mathbb{C}^2$ to the flip of sign of u. By restriction to the bundles $\mathcal{O}(x)$, it induces the involution $x \mapsto x'$ of Σ which leaves 6 Weierstrass points invariant. The latter involution lifts to an involution (without fixed points) of the covering space $\widetilde{\Sigma}$ determined by the equation

$$\int_{x_0}^{x} \omega - \Delta \;=\; -\int_{x_0}^{x'} \omega + \Delta. \tag{6.5}$$

Definitions (6.1) together with Eqs. (5.2) give the relations

$$u_1' - u_2' = \int_{x_0}^{x_1} \omega - \int_{x_0}^{x_2} \omega \quad \text{and} \quad u_1' + u_2' = -\int_{x_0}^{x_3} \omega - \int_{x_0}^{x_4} \omega + 2\Delta$$

holding in $\mathbb{C}^2$, with $x_i \in \widetilde{\Sigma}$. They may be rewritten as

$$u_1' - u_2' = \int_{x_0}^{x_1} \omega + \int_{x_0}^{x_2'} \omega - 2\Delta \quad \text{and} \quad u_1' + u_2' = \int_{x_0}^{x_3'} \omega + \int_{x_0}^{x_4'} \omega - 2\Delta, \tag{6.6}$$

which, upon the flip of the sign of u_2' leaving $\phi_{u_2'}$ unchanged, provides the dual version of relations (5.2) corresponding to points $x_1, x_2', x_3', x_4' \in \widetilde{\Sigma}$. Applying the previous result (5.12) and using the possibility to exchange a point with its image under the involution of Σ in the argument of a quadratic differential, we infer that

$$\mathcal{H}(\theta', \phi')(x_i) = -\tfrac{1}{16\pi^2}\, \big(a_1'\partial_a\theta'(u_1')\omega^a(x_i) \mp a_2\partial_a\theta'(u_2')\omega^a(x_i)\big)^2. \tag{6.7}$$

The sign minus should be taken for x_1 and x_2 and sign plus for x_3 and x_4. The exchange of signs in comparison with Eq. (5.12) is due to the flip $u_2' \mapsto -u_2'$.

In order to compare expressions (5.12) and (6.7) we shall calculate the coefficients $a_{1,2}$ and $a_{1,2}'$ of the linear combinations (5.3) and (6.3). Note that the definition $\theta' = \iota(\phi)$ implies that

$$\theta'\Big(\int_{x_0}^{x} \omega - u_1 - \Delta\Big) = a_2\, \vartheta\Big(\int_{x_0}^{x} \omega - u_1 - u_2 - \Delta\Big)\, \vartheta\Big(\int_{x_0}^{x} \omega - u_1 + u_2 - \Delta\Big).$$

Taking the derivative over x at x_1, we obtain

$$\partial_a\theta'(u_1')\omega^a(x_1) \;=\; -a_2\, \vartheta(w_1 - w_3 - w_4)\, \partial_a\vartheta(w_2)\omega^a(x_1),$$

where we employed Eqs. (5.2) and the abbreviated notations $w_i = \int_{x_0}^{x_i} - \Delta$. Hence

$$a_2 \;=\; -\frac{\partial_a\theta'(u_1')\omega^a(x_1)}{\vartheta(w_1 - w_3 - w_4)\ \partial_a\vartheta(w_2)\omega^a(x_1)}. \tag{6.8}$$

Similarly,

$$\theta'\Big(\int_{x_0}^{x} \omega - u_2 - \Delta\Big) = a_1\, \vartheta\Big(\int_{x_0}^{x} \omega - u_1 - u_2 - \Delta\Big)\, \vartheta\Big(\int_{x_0}^{x} \omega + u_1 - u_2 - \Delta\Big).$$

Taking the derivative at $x = x_1$ and noting that $w_1 - u_2 = -u_2'$, we infer that

$$a_1 \;=\; \frac{\partial_a\theta'(u_2')\omega^a(x_1)}{\vartheta(w_1 + w_3 + w_4)\ \partial_a\vartheta(w_2)\omega^a(x_1)}. \tag{6.9}$$

To calculate $a'_{1,2}$, we note that Eq. (6.4) implies that

$$\theta(\int_{x_0}^{x} \omega - v_1 - \Delta) = a'_2 \, \vartheta(\int_{x_0}^{x} \omega - u'_1 - u'_2 - \Delta)\vartheta(\int_{x_0}^{x} \omega - u'_1 + u'_2 - \Delta).$$

Upon derivation at $x = x_1$ and with the use of relations (6.6) and (6.5), this gives

$$a'_2 = -\frac{\partial_a \theta(u_1)\omega^a(x_1)}{\vartheta(w_1+w_3+w_4)\ \partial_a \vartheta(w_2)\omega^a(x_1)}. \tag{6.10}$$

Finally, since

$$\theta(\int_{x_0}^{x} \omega + v_2 - \Delta) = a'_1 \, \vartheta(\int_{x_0}^{x} \omega - u'_1 + u'_2 - \Delta)\vartheta(\int_{x_0}^{x} \omega + u'_1 + u'_2 - \Delta),$$

and $w_1 + u'_2 = u_2$ we infer that

$$a'_1 = -\frac{\partial_a \theta(u_2)\omega^a(x_1)}{\vartheta(w_1-w_3-w_4)\ \partial_a \vartheta(w_2)\omega^a(x_1)}. \tag{6.11}$$

Substitution of expressions (6.9),(6.8),(6.11) and (6.10) shows equality of the right-hand sides of Eqs. (5.12) and (6.7) for $x_i = x_1$. Since there is a full symmetry between points x_i (hidden in our arbitrary choices of the order and the signs of u_j's and u'_j's), the self-duality

$$\mathcal{H}(\theta, \phi) = \mathcal{H}(\theta', \phi') \tag{6.12}$$

follows.

7. van Geemen–Previato's Result and Beyond

The genus 2 curves are hyperelliptic. The map $H^0(K) \ni \omega \mapsto \omega(x)$ defines an element of $\mathbb{P}H^0(K)^*$ and varying $x \in \Sigma$ one obtains a realization of Σ as a ramified double cover $\mathbb{P}H^0(K)^* \cong \mathbb{P}^1$. One may use the 1,0-forms $\omega^a \in H^0(K)$ to define the homogeneous coordinates on $\mathbb{P}H^0(K)^*$. Then

$$\lambda(x) = \frac{\omega^2(x)}{\omega^1(x)} = -\frac{\partial_1 \vartheta(\int_{x_0}^{x} \omega - \Delta)}{\partial_2 \vartheta(\int_{x_0}^{x} \omega - \Delta)} \tag{7.1}$$

becomes the inhomogeneous coordinate of the image in $\mathbb{P}^1$ of the point $x \in \Sigma$. If x' is the image of x under the involution $\mathcal{O}(x) \mapsto \mathcal{O}(-x)K = \mathcal{O}(x')$, i.e. if

$$\int_{x_0}^{x} \omega + \int_{x_0}^{x'} \omega - 2\Delta \ \in\ \mathbb{Z} + \tau\mathbb{Z} \qquad \text{then} \qquad \lambda(x) = \lambda(x').$$

Hence the involution $x \mapsto x'$ permutes the sheets of the covering $\Sigma \mapsto \mathbb{P}^1$ ramified over the 6 Weierstrass points x_s, $s = 1, \ldots, 6$, fixed by the involution. $\mathcal{O}(x_s)$ is an odd spin structure. i.e.

$$\int_{x_0}^{x_s} \omega - \Delta = E_s \mod(\mathbb{Z}^2 + \tau\mathbb{Z}^2)$$

and

$$\lambda_s \equiv \lambda(x_s) = -\tfrac{\partial_1\vartheta(E_s)}{\partial_2\vartheta(E_s)}, \tag{7.2}$$

where $E_s = \frac{1}{2}(e_s + \tau e'_s)$ with $e_s, e'_s = (1,0), (0,1)$ or $(1,1)$ such that $e_s \cdot e'_s$ is odd. The possibilities are:

$$\begin{aligned} &e_1 = (1,0),\ e'_1 = (1,0); \quad e_2 = (1,1),\ e'_2 = (1,0); \quad e_3 = (0,1),\ e'_3 = (0,1);\\ &e_4 = (1,1),\ e'_4 = (0,1); \quad e_5 = (0,1),\ e'_5 = (1,1); \quad e_6 = (1,0),\ e'_6 = (1,1), \end{aligned} \tag{7.3}$$

and we shall number the Weierstrass points (in a marking-dependent way) in agreement with this list. Σ may be identified with the hyperelliptic curve given by the equation

$$\zeta^2 = \prod_{s=1}^{6}(\lambda - \lambda_s) \tag{7.4}$$

with the involution mapping (λ, ζ) to $(\lambda, -\zeta)$. The expressions

$$\omega^1 = C\frac{d\lambda}{\zeta} \quad \text{and} \quad \omega^2 = C\frac{\lambda d\lambda}{\zeta}, \tag{7.5}$$

where C is a constant, give the basis of holomorphic 1,0-forms of Σ (the right-hand sides vanish exactly where the left-hand sides do).

Let us recall the main result of [21] based on the analysis of the formula (5.1) for the Hitchin Hamiltonians on the Kummer quartic $\mathcal{K}^*$. It will be convenient to identify the pairs (θ, ϕ) s.t. $\langle\theta, \phi\rangle = 0$ with pairs $(q,p) \in \mathbb{C}^4 \times \mathbb{C}^4$ s.t. $q \cdot p = 0$ by the relations

$$\begin{aligned} \theta &= q_1\theta_{2,(0,0)} + q_2\theta_{2,(1,0)} + q_3\theta_{2,(0,1)} + q_4\theta_{2,(1,1)},\\ \phi &= p_1\theta^*_{2,(0,0)} + p_2\theta^*_{2,(1,0)} + p_3\theta^*_{2,(0,1)} + p_4\theta^*_{2,(1,1)}. \end{aligned}$$

The symplectic form of $T^*\mathbb{P}^3$ is the standard $dp \wedge dq$ and the isomorphism ι interchanges p and q. By examining the values of the quadratic differentials given by $\mathcal{H}$ at the Weierstrass points x_s, van Geemen and Previato showed that

$$\mathcal{Z}_s(q) = \{p \mid q \cdot p = 0,\ \mathcal{H}(q,p)(x_s) = 0\}$$

is a union of a pair of bitangents to $\mathcal{K}^*$. Then classical results giving the equations for bitangents to the Kummer surface permitted the authors of [21] to write an almost explicit formula for $\mathcal{H}(x_s)$ in the form

$$\mathcal{H}(q,p)(x_s) = h_s \sum_{t \neq s} \frac{r_{st}(q,p)}{\lambda_s - \lambda_t}, \tag{7.6}$$

where $r_{st} = r_{ts}$ are homogeneous polynomials,

$$\begin{aligned} r_{12}(q,p) &= \ \ (q_1p_1 + q_2p_2 - q_3p_3 - q_4p_4)^2,\\ r_{13}(q,p) &= \ \ (q_1p_4 - q_2p_3 - q_3p_2 + q_4p_1)^2,\\ r_{14}(q,p) &= -(q_1p_4 + q_2p_3 - q_3p_2 - q_4p_1)^2,\\ r_{15}(q,p) &= -(q_1p_3 - q_2p_4 - q_3p_1 + q_4p_2)^2,\\ r_{16}(q,p) &= \ \ (q_1p_3 + q_2p_4 + q_3p_1 + q_4p_2)^2,\\ r_{23}(q,p) &= -(q_1p_4 - q_2p_3 + q_3p_2 - q_4p_1)^2,\\ r_{24}(q,p) &= \ \ (q_1p_4 + q_2p_3 + q_3p_2 + q_4p_1)^2,\\ r_{25}(q,p) &= \ \ (q_1p_3 - q_2p_4 + q_3p_1 - q_4p_2)^2, \end{aligned} \tag{7.7}$$

$$
\begin{aligned}
r_{26}(q,p) &= -(q_1p_3 + q_2p_4 - q_3p_1 - q_4p_2)^2,\\
r_{34}(q,p) &= \ \ (q_1p_1 - q_2p_2 + q_3p_3 - q_4p_4)^2,\\
r_{35}(q,p) &= \ \ (q_1p_2 + q_2p_1 + q_3p_4 + q_4p_3)^2,\\
r_{36}(q,p) &= -(q_1p_2 - q_2p_1 - q_3p_4 + q_4p_3)^2,\\
r_{45}(q,p) &= -(q_1p_2 - q_2p_1 + q_3p_4 - q_4p_3)^2,\\
r_{46}(q,p) &= \ \ (q_1p_2 + q_2p_1 - q_3p_4 - q_4p_3)^2,\\
r_{56}(q,p) &= \ \ (q_1p_1 - q_2p_2 - q_3p_3 + q_4p_4)^2,
\end{aligned}
$$

and $h_s \in K^2_{x_s}$ could still depend on q. In the original language of pairs (θ,ϕ), and of the $(\mathbb{Z}/2\mathbb{Z})^4$-action (3.12) on $H^0(L^2_\Theta)$ one has

$$r_{st}(\theta,\phi) = \langle U_{e_s,e'_s}U_{e_t,e'_t}\theta,\phi\rangle\langle U_{e_t,e'_t}U_{e_s,e'_s}\theta,\phi\rangle$$

with e_s, e'_s from the list (7.3). The polynomials r_{st} are self-dual:

$$r_{st}(q,p) = r_{st}(p,q) \tag{7.8}$$

and the self-duality of $\mathcal{H}$ proven in the present paper forces coefficients h_s in Eq. (7.6) to be q-independent filling partially the gap left in [21]. An easy but important identity is

$$\sum_{t\neq s} r_{st}(q,p) = (q\cdot p)^2 = 0 \tag{7.9}$$

for any fixed s. It implies that the Hamiltonians (7.6) are preserved up to normalization by the isomorphisms of the hyperelliptic surfaces induced by the fractional action $\lambda \mapsto \lambda' = \frac{a\lambda+b}{c\lambda+d}$ of $SL(2,\mathbb{C})$ on $\mathbb{P}^1$.

We would still like to fix the values of the constants h_s in Eqs. (7.6). We claim that they are such that the Hitchin map is given by Eq. (1.5), i.e. that

$$\mathcal{H}(q,p) \ = \ -\tfrac{1}{128\pi^2} \sum_{\substack{s,t=1,\dots,6,\\ s\neq t}} \frac{r_{st}(q,p)}{(\lambda-\lambda_s)(\lambda-\lambda_t)}(d\lambda)^2. \tag{7.10}$$

First note that the above formula is consistent with the $SL(2,\mathbb{C})$ transformations. Indeed, relations (7.9) imply that

$$\sum_{s\neq t} \frac{r_{st}}{(\lambda'-\lambda'_s)(\lambda'-\lambda'_t)}(d\lambda')^2 = \sum_{s\neq t} \frac{r_{st}}{(\lambda-\lambda_s)(\lambda-\lambda_t)}(d\lambda)^2$$

for $\lambda' = \frac{a\lambda+b}{c\lambda+d}$. Taking, in particular, $\lambda' = \lambda^{-1}$ one verifies that the quadratic differentials (7.10) are regular at infinity. They are also regular at the branching points since $\frac{d\lambda}{\sqrt{\lambda-\lambda_s}}$ is a local holomorphic differential around x_s. Hence the r.h.s. of Eq. (7.10) is indeed a (holomorphic) quadratic differential. Thus Eq. (7.10) is equivalent to relations (7.6) with $h_s = \frac{(d\lambda)^2}{(\lambda-\lambda_s)}|_{x_s}$, modulo an overall normalization. To prove Eq. (7.10) we shall verify it at a point of the phase space for which $\mathcal{H}(q,p)(x_s) \neq 0$ for $s \neq 1$. This will fix h_s for $s \neq 1$ and hence all of them (two quadratic differentials equal at points x_s with $s \neq 1$ have to coincide).

Consider a pair (θ,ϕ_{u_1}) lying in the product $\mathcal{K}\times\mathcal{K}^*$ of the Kummer quartics with

$$\theta(u) = \mathrm{e}^{\frac{1}{2}\pi i e_1'\cdot\tau e_1' + 2\pi i e_1'\cdot u_1}\,\vartheta(u_1+E_1-u)\,\vartheta(u_1+E_1+u) = \sum_e (U_{e_1,e_1'}\theta_{2,e})(u_1)\,\theta_{2,e}(u) \qquad (7.11)$$

for $e_1 = e_1' = (1,0)$. Note that $\langle\theta,\phi_{u_1}\rangle = 0$. Equation (5.1) together with the relations (7.5) and the equation

$$\partial_a\theta(u_1) = -\mathrm{e}^{\frac{1}{2}\pi i e_1'\cdot\tau e_1' + 2\pi i e_1'\cdot u_1}\,\partial_a\vartheta(E_1)\,\vartheta(2u_1+E_1)$$

results in the identity

$$\mathcal{H}(\theta,\phi_{u_1}) = -\tfrac{C^2}{16\pi^2}\,\mathrm{e}^{\pi i e_1'\cdot\tau e_1' + 4\pi i e_1'\cdot u_1}\,(\partial_2\vartheta(E_1))^2\vartheta(2u_1+E_1)^2\,(\lambda-\lambda_1)^2\,\tfrac{(d\lambda)^2}{\zeta^2}, \qquad (7.12)$$

where C is the constant appearing in Eq. (7.5). Note that $\mathcal{H}(\theta,\phi_{u_1}) \neq 0$ as long as $\vartheta(2u_1+E_1) \neq 0$. It follows that $\mathcal{H}(\theta,\phi_{u_1})$ is a quadratic differential proportional to $(\lambda-\lambda_1)^2\frac{(d\lambda)^2}{\zeta^2}$ which has the 4$^{\text{th}}$ order zero at x_1. The latter property characterizes it uniquely up to normalization.

It is not difficult to check that Eq. (7.10) gives a quadratic differential with the same property. Indeed, in the language of q's and p's, the linear form ϕ_{u_1} corresponds to a vector $p\in\mathbb{C}^4$ and θ to $q = (p_2,-p_1,p_4,-p_3)$. A straightforward verification shows that $r_{1t}(q,p) = 0$ for all $t\neq 1$. This implies that the quadratic differential given by Eq. (7.10) vanishes to the second order at x_1. The condition that it vanishes to the fourth order is

$$\sum_{\substack{s\neq t,\\ s,t\neq 1}} r_{st}((p_2,-p_1,p_4,-p_3),p)\prod_{v\neq 1,s,t}(\lambda_1-\lambda_v) = 0.$$

A direct calculation shows that this is exactly Eq. (A3.2) of the Kummer quartic with the coefficients (A3.4) so that it holds for p corresponding to ϕ_{u_1}. This establishes proportionality between the Hitchin map and the right-hand side of Eq. (7.10) with a coefficient that may be still curve-dependent.

Fixing the overall normalization of the Hitchin map is more involved. We shall calculate the value of the quadratic differential on the right-hand side of Eq. (7.12) at $\lambda=\lambda_2$ and compare it to the value given by Eq. (7.10). Since this is somewhat technical, we defer the argument to Appendix 4.

The system with Hamiltonians (7.6) bears some similarity to the classic Neumann systems[4], also anchored in modular geometry [17, 2]. The Hamiltonians of a Neumann system have the form

$$\mathcal{H}_s = \sum_{1\leq t\neq s\leq n}\frac{J_{st}^2}{\lambda_s-\lambda_t}, \qquad (7.13)$$

where $J_{st} = q_sp_t - q_tp_s$ are the functions on $T^*\mathbb{C}^n$ generating the infinitesimal action of the complex group SO_n:

[4] We thank M. Olshanetsky for attracting our attention to this fact.

$$\begin{aligned} \{J_{st}, J_{tv}\} &= -J_{sv} && \text{for } s,t,v \text{ different}, \\ \{J_{st}, J_{vw}\} &= 0 && \text{for } s,t,v,w \text{ different}. \end{aligned} \tag{7.14}$$

The fact that the Hamiltonians (7.6) (with constant h_s) Poisson commute reduces, as is well known, to the identities

$$\begin{aligned} \{r_{st} + r_{sv}, r_{tv}\} &= 0 && \text{and cyclic permutations thereof}, \\ \{r_{st}, r_{vw}\} &= 0 && \text{for } \quad \{s,t\} \cap \{v,w\} = \emptyset. \end{aligned} \tag{7.15}$$

If we set $r_{st} = J_{st}^2$ for the Neumann system, then Eqs. (7.15) follow from the relations (7.14). It appears that the same algebra stands behind the fact[5] that r_{st} given by Eq. (7.7) verify (7.15). The phase space $T^*\mathcal{N}_{ss} \cong \{(q,p)|q{\cdot}p = 0\}/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by $(q,p) \mapsto (tq, t^{-1}p)$, may be identified with the coadjoint orbit of the group SL_4 composed of the traceless complex $4{\times}4$ matrices $|p\rangle\langle q|$ of rank 1. Using the isomorphism of the complex Lie algebras $sl_4 \cong so_6$, we obtain the functions $J_{st} = -J_{ts}$ on this SL_4-orbit which generate the action of so_6 and have the Poisson brackets given by (7.14). A straightforward check shows that, for r_{st} of Eq. (7.7),

$$r_{st} = -4J_{st}^2 \tag{7.16}$$

so that Eq. (7.15) follows from the so_6-algebra (7.14).

Upon the introduction of the rational functions $\frac{r_{st}}{\lambda}$, Eqs. (7.15) take the form

$$\begin{aligned} &\left\{\tfrac{r_{st}}{\lambda_s-\lambda_t}, \tfrac{r_{sv}}{\lambda_s-\lambda_v}\right\} + \left\{\tfrac{r_{st}}{\lambda_s-\lambda_t}, \tfrac{r_{tv}}{\lambda_t-\lambda_v}\right\} + \left\{\tfrac{r_{sv}}{\lambda_s-\lambda_v}, \tfrac{r_{tv}}{\lambda_t-\lambda_v}\right\} = 0, \\ &\left\{\tfrac{r_{st}}{\lambda_s-\lambda_t}, \tfrac{r_{vw}}{\lambda_v-\lambda_w}\right\} = 0 \qquad \text{for} \quad \{s,t\} \cap \{v,w\} = \emptyset. \end{aligned} \tag{7.17}$$

The first of these identities is, essentially, the classical Yang-Baxter equation. Note, however, that r_{st}, unlike in the Gaudin and Neumann systems, is not an element of a product of two copies of a Poisson algebra of functions: there is no sign of an explicit product structure, or of a reduction thereof, in our phase space. The important question is whether r_{st} come from a rational solution of the CYBE. The conformal field theory work [14, 23] suggests that the answer may be positive, at least in some sense.

The knowledge of the explicit form of the quadratic differentials $\mathcal{H}(q,p)$ allows to write the explicit equations for the genus 5 spectral curve of the SL_2 Hitchin system at genus 2, see Eq. (2.1). They take the form

$$\zeta^2 = \prod_{s=1}^{6}(\lambda - \lambda_s), \qquad \xi^2 = \sum_{s\neq t} r_{st}(q,p) \prod_{v\neq s,t} (\lambda - \lambda_v). \tag{7.18}$$

The involution of the spectral curve flips the sign of ξ. To extract explicit formulae for the angle variables describing the point on the Prym variety of the spectral curve, we would need, however, a more explicit knowledge of the entire Lax matrix Ψ.

[5] This is the classical version of the observation of [22].

8. Conclusions

The main result of the present paper is the proof of self-duality of the Hitchin Hamiltonians on the cotangent bundle to the moduli space of the holomorphic SL_2 bundles on a genus 2 complex curve. The result was based on an expression for the Hitchin Hamiltonians off the Kummer quartic on which the values of the Hamiltonians were determined in [21]. Using the self-duality, we were able to complete the analysis of [21] and to obtain the explicit formula (1.5) for the Hitchin map (1.3) giving the action variables of the integrable system. The explicit formula for the angle variables remains still to be found. An interesting open problem is an extension of the present work to the case with insertion points.

Another important problem related to Hitchin's construction is the quantization of the corresponding integrable systems. For the SL_2 case such a quantization is essentially provided by the Knizhnik–Zamolodchikov–Bernard–Hitchin connection [15, 4, 5] which describes the variation of conformal blocks of the SU_2 WZW conformal field theory under the change of the complex structure of the curve. The (partition function) conformal blocks are holomorphic sections of the k^{th}-power of the determinant line bundle over the moduli space $\mathcal{N}_{ss}$ (k is the level of the WZW theory). In our case, they are simply k^{th}-order homogeneous polynomials on $H^0(L_\theta^2)$. It is easy to quantize the Hitchin Hamiltonians

$$H_s = \sum_{t\neq s} \frac{r_{st}}{\lambda_s - \lambda_t}.$$

If one keeps the original formulae (7.7) for r_{st} in which p_i stands now for $\frac{1}{i}\partial_{q_i}$, the relations (7.15) or (7.17) still hold after the replacement of the Poisson brackets by the commutators. One obtains this way the commuting operators H_s mapping the space of homogeneous, degree k polynomials in variables q into itself. Note, however, that now

$$\sum_{t\neq s} r_{st} = -k(k+4)$$

for each fixed s so that the quantization changes the conformal properties of the Hamiltonians. A direct construction of the projective version of the KZBH connection for group SU_2 and genus 2 has been recently given in ref. [22] by following Hitchin's approach [12]. It is consistent with the above *ad hoc* quantization of the classical Hitchin Hamiltonians.

The integral formulae for the conformal blocks [3, 20, 8] or, equivalently, the integral formulae for the scalar product of the conformal blocks [9] have been used at genus 0 and 1 to extract the Bethe Ansatz eigen-vectors and eigen-values of the quantized version of the quadratic Hitchin Hamiltonians. The Bethe-Ansatz type diagonalization of the quantization of the genus 2 Hitchin Hamiltonians is among the issues that will have to be examined.

Finally, as we stressed in the text, the relations between the conformal WZW field theory on a genus 2 surface and an orbifold theory in genus 0 requires further study.

Appendix 1

Let us check that θ given by Eq. (3.6) vanishes if and only if

$$H^0(l_u \otimes E) = \{(s_1, s_2) \mid s_2 \in H^0(l_u l_{u_1}),\ \bar{\partial}_{l_u^{-1} l_{u_1}} s_1 + s_2 b = 0\} \neq 0.$$

For $u - u_1 \in \mathbb{Z}^2 + \tau\mathbb{Z}^2$ the 1st theta function on the r.h.s. of Eq. (3.7) vanishes but $l_u = l_{u_1}$ and $l_{u_1} \in C_E$. Assume now that $u - u_1 \notin \mathbb{Z}^2 + \tau\mathbb{Z}^2$. Then $dim\ H^0(l_u^{-1} l_{u_1} K) = 1$ with a non-zero $\chi \in H^0(l_u^{-1} l_{u_1} K)$. The necessary and sufficient condition for the solvability of the equation $\partial_{l_u l_{u_1}^{-1}} s_1 + s_2 b = 0$ for a given $s_2 \in H^0(l_u l_{u_1})$ is

$$\int_\Sigma \chi s_2 b = 0. \tag{A1.1}$$

If $u + u_1 \in \mathbb{Z}^2 + \tau\mathbb{Z}^2$ then $l_u l_{u_1} = K$ and $dim H^0(l_u l_{u_1}) = 2$ so that there always is a non-zero solution but also $\theta(u) = 0$ in this case due to the vanishing of the 2nd theta function on the r.h.s. of Eq. (3.7). Finally, if $u \pm u_1 \notin \mathbb{Z}^2 + \tau\mathbb{Z}^2$ then $s_2 \in H^0(l_u l_{u_1})$ has to be proportional to the element defined by (3.8) and the condition (A1.1) coincides with the equation $\theta(u) = 0$.

Appendix 2

Let us show that the 1,0-form μ satisfying relations (4.18) and (4.19) automatically fulfills the condition

$$\int_\Sigma \kappa\mu \wedge b = 0. \tag{A2.1}$$

Among the infinitesimal gauge field variations δB given by Eq. (4.4) there are ones which are equivalent to infinitesimal gauge transformations:

$$\delta B = \bar{\partial}\Lambda + [B, \Lambda].$$

Explicitly, for $\Lambda = \begin{pmatrix} -\sigma & \varphi \\ \kappa & \sigma \end{pmatrix}$ with σ a function, φ a section of $l_{u_1}^{-2}$ and κ a section of $l_{u_1}^2$, this requires that

$$\bar{\partial}\kappa = 0, \qquad \pi\delta u_1 (\mathrm{Im}\tau)^{-1}\bar{\omega} = -\bar{\partial}\sigma + \kappa b, \qquad \delta b = \bar{\partial}\varphi + 2\sigma b. \tag{A2.2}$$

Such variations may only change the normalization of the theta function θ. Integrating the second of the above relations against forms ω^a and using Eq. (4.12) we find that

$$\delta u_1^a = -\tfrac{1}{2\pi i}\epsilon^{ab}\partial_b\theta(u_1) \tag{A2.3}$$

for the proper normalization of κ. For such δu_1 the first term on the right-hand side of Eq. (4.6) gives a theta function vanishing at $u = u_1$ and may be compensated by the second term. The 3rd equation of (A2.2) gives the compensating $\delta b \in \wedge^{01}(l_{u_1}^{-2})$. Pairing Eq. (4.6) with the above δu_1 and δb with the linear form ϕ, we obtain the identity

$$\tfrac{1}{i}\epsilon^{ab}\partial_b\theta(u_1)(\mathrm{Im}\tau)^{-1}_{ac}\int_\Sigma \chi^c \wedge b + 2\int_\Sigma \sigma\eta \wedge b = 0\ . \tag{A2.4}$$

On the other hand,

$$\int_\Sigma \kappa\mu\wedge b = \int_\Sigma \mu\wedge\bar\partial\sigma - \frac{1}{2i}\epsilon^{ab}\partial_b\theta(u_1)(\mathrm{Im})^{-1}_{ac}\int_\Sigma \mu\wedge\bar\omega^c$$
$$= -\int_\Sigma \sigma\eta\wedge b - \frac{1}{2i}\epsilon^{ab}\partial_b\theta(u_1)(\mathrm{Im})^{-1}_{ac}\int_\Sigma \chi^c\wedge b = 0,$$

where we have subsequently used the 2nd equation in (A2.2) with δu_1 given by Eq. (A2.3), the relation $\bar\partial\mu = -\eta\wedge b$ and Eq. (4.18) fixing μ and, finally, the identity (A2.4).

Appendix 3

It is not difficult to see that there exist a non-zero element $P\in S^4H^0(L^2_\Theta)$, a homogeneous polynomial of degree 4 on $H^0(L^2_\Theta)^*$, s.t.

$$P(\phi_{u'}) = 0$$

for all $u'\in\mathbb{C}^2$. Indeed, $dim S^4H^0(L^2_\Theta) = \binom{7}{3} = 35$ but the map $u'\mapsto P(\phi_{u'})$ defines an even theta function of order 8 and $dim H^0_{\mathrm{even}}(L^8_\Theta) = 34$. P is a quartic expression in $\theta_{2,e}(u')$ which vanishes for all u'. It has to be preserved by the $(\mathbb{Z}/2\mathbb{Z})^4$-action (3.12) and hence it must be of the form

$$\begin{aligned}P = {}& c_1(\theta^4_{2,(0,0)} + \theta^4_{2,(1,0)} + \theta^4_{2,(0,1)} + \theta^4_{2,(1,1)})\\ &+ c_2(\theta^2_{2,(0,0)}\theta^2_{2,(1,0)} + \theta^2_{2,(0,1)}\theta^2_{2,(1,1)})\\ &+ c_3(\theta^2_{2,(0,0)}\theta^2_{2,(0,1)} + \theta^2_{2,(1,0)}\theta^2_{2,(1,1)})\\ &+ c_4(\theta^2_{2,(0,0)}\theta^2_{2,(1,1)} + \theta^2_{2,(1,0)}\theta^2_{2,(0,1)})\\ &+ c_5\theta_{2,(0,0)}\theta_{2,(1,0)}\theta_{2,(0,1)}\theta_{2,(1,1)}.\end{aligned}$$

It is not difficult to calculate the values of coefficients c_i. Denoting $\alpha\equiv\theta_{2,(0,0)}(0)$, $\beta\equiv\theta_{2,(1,0)}(0)$, $\gamma\equiv\theta_{2,(0,1)}(0)$ and $\delta\equiv\theta_{2,(1,1)}(0)$, one has

$$\begin{aligned}c_1 &= \ (\alpha^2\beta^2-\gamma^2\delta^2)(\alpha^2\gamma^2-\beta^2\delta^2)(\alpha^2\delta^2-\beta^2\gamma^2),\\ c_2 &= -(\alpha^4+\beta^4-\gamma^4-\delta^4)(\alpha^2\gamma^2-\beta^2\delta^2)(\alpha^2\delta^2-\beta^2\gamma^2),\\ c_3 &= -(\alpha^4-\beta^4+\gamma^4-\delta^4)(\alpha^2\beta^2-\gamma^2\delta^2)(\alpha^2\delta^2-\beta^2\gamma^2),\\ c_4 &= -(\alpha^4-\beta^4-\gamma^4+\delta^4)(\alpha^2\beta^2-\gamma^2\delta^2)(\alpha^2\gamma^2-\beta^2\delta^2),\\ c_5 &= \ 2\alpha\beta\gamma\delta[(\alpha^4-\beta^4+\gamma^4-\delta^4)^2-4(\alpha^2\gamma^2-\beta^2\delta^2)^2].\end{aligned}\tag{A3.1}$$

If we use the basis dual to $(\theta_{2,e})$ to identify $\phi\in H^0(L^2_\Theta)^*$ with a vector $p = (p_1,p_2,p_3,p_4)\in\mathbb{C}^4$, the equation of the Kummer quartic $\mathcal{K}^*$ becomes

$$\begin{aligned}&c_1(p_1^4+p_2^4+p_3^4+p_4^4) + c_2(p_1^2p_2^2+p_3^2p_4^2) + c_3(p_1^2p_3^2+p_2^2p_4^2)\\ &\qquad + c_4(p_1^2p_4^2+p_2^2p_3^2) + c_5p_1p_2p_3p_4 = 0.\end{aligned}\tag{A3.2}$$

Similarly, identifying $\theta \in H^0(L_\Theta^2)$ with $q = (q_1, q_2, q_3, q_4) \in \mathbb{C}^4$ with the help of the basis $(\theta_{2,e})$, the same equation with p replaced by q defines the Kummer quartic $\mathcal{K}$, compare [13], p. 81.

We shall also need another well known presentation of the above equation using the inhomogeneous coordinates of the Weierstrass points λ_s given by Eq. (7.2). It is usually obtained by beautiful geometric considerations about quadratic line complexes, see [10]. It may be also obtained analytically by observing that the multivalued functions

$$x \;\mapsto\; \theta_{2,e}\Big(\int\limits_{x_0}^{x} \omega - \Delta\Big)$$

transform like bilinears in $\partial_a \vartheta(\int_{x_0}^{x} \omega - \Delta)$, i.e., that they represent quadratic differentials. It follows that

$$\begin{aligned}\sum_e \theta_{2,e}(E_s)\,\theta_{2,e}\Big(\int\limits_{x_0}^{x}\omega - \Delta\Big) &= \vartheta\Big(E_s + \int\limits_{x_0}^{x}\omega - \Delta\Big)\,\vartheta\Big(E_s - \int\limits_{x_0}^{x}\omega + \Delta\Big) \\ &= D_s\,\left(\partial_1\vartheta(E_s')\partial_2\vartheta\Big(\int\limits_{x_0}^{x}\omega - \Delta\Big) - \partial_2\vartheta(E_s')\partial_1\vartheta\Big(\int\limits_{x_0}^{x}\omega - \Delta\Big)\right) \\ &\quad\cdot\left(\partial_1\vartheta(E_s'')\partial_2\vartheta\Big(\int\limits_{x_0}^{x}\omega - \Delta\Big) - \partial_2\vartheta(E_s'')\partial_1\vartheta\Big(\int\limits_{x_0}^{x}\omega - \Delta\Big)\right),\end{aligned} \tag{A3.3}$$

where $E_s = \frac{1}{2}(e_s + \tau e_s')$ is an odd characteristics from the list (7.3) and E_s', E_s'' are the two other ones s.t. $E_s + E_s' = E_s''\mathrm{mod}(\mathbb{Z}^2 + \tau\mathbb{Z}^2)$. The odd characteristics E_s, E_s', E_s'' are either a permutation of E_1, E_4, E_5 or a permutation of E_2, E_3, E_6. The relations (A3.3) hold since both sides represent a quadratic differential with double zeros at the Weierstrass points corresponding to E_s' and E_s''. One may obtain expressions for the coefficients D_s by the de l'Hospital rule applied twice at those points. Specifying then $\int_{x_0}^{x}\omega - \Delta$ to E_s or to 3 remaining odd characteristics one obtains relations for quadratic combinations of $\theta_{2,e}(0)$ of the form $\pm\alpha^2 \pm \beta^2 \pm \gamma^2 \pm \delta^2$ with 2 plus and 2 minus signs as well as for $\alpha\beta \pm \gamma\delta$, $\alpha\gamma \pm \beta\delta$ and $\alpha\delta \pm \beta\gamma$. These relations may be used to compute the ratios of the coefficients c_i (A3.1) which become functions of λ_s only. One obtains this way an alternative expression for the coefficients c_i

$$\begin{aligned}
c_1 &= (\lambda_1-\lambda_2)(\lambda_3-\lambda_4)(\lambda_5-\lambda_6),\\
c_2 &= 2(\lambda_1-\lambda_2)((\lambda_3-\lambda_5)(\lambda_4-\lambda_6)+(\lambda_3-\lambda_6)(\lambda_4-\lambda_5)),\\
c_3 &= -2(\lambda_3-\lambda_4)((\lambda_1-\lambda_5)(\lambda_2-\lambda_6)+(\lambda_1-\lambda_6)(\lambda_2-\lambda_5)),\\
c_4 &= 2(\lambda_5-\lambda_6)((\lambda_1-\lambda_3)(\lambda_2-\lambda_4)+(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)),\\
c_5 &= -2(\lambda_1-\lambda_3)((\lambda_4-\lambda_5)(\lambda_2-\lambda_6)+(\lambda_4-\lambda_6)(\lambda_2-\lambda_5))\\
&\quad -2(\lambda_1-\lambda_4)((\lambda_3-\lambda_5)(\lambda_2-\lambda_6)+(\lambda_3-\lambda_6)(\lambda_2-\lambda_5))\\
&\quad -2(\lambda_1-\lambda_5)((\lambda_2-\lambda_4)(\lambda_3-\lambda_6)+(\lambda_2-\lambda_3)(\lambda_4-\lambda_6))\\
&\quad -2(\lambda_1-\lambda_6)((\lambda_2-\lambda_4)(\lambda_3-\lambda_5)+(\lambda_2-\lambda_3)(\lambda_4-\lambda_5))
\end{aligned} \tag{A3.4}$$

equivalent to the previous one up to normalization. Note that the $SL(2,\mathbb{C})$ transformations $\lambda_s \mapsto \frac{a\lambda_s+b}{c\lambda_s+d}$ preserve the form of the quartic equation. The virtue of the analytic approach is that it also provides useful expressions for the non-homogeneous ratios like e.g.

$$\frac{\alpha\beta + \gamma\delta}{\alpha^2\gamma^2 - \beta^2\delta^2} = -\frac{e^{-\frac{1}{2}\pi i(1,0)\cdot\tau(1,0)}}{2C^2\,(\partial_2\vartheta(E_1))^2}\;\frac{(\lambda_2-\lambda_5)(\lambda_2-\lambda_6)(\lambda_3-\lambda_4)}{\lambda_1-\lambda_2}. \tag{A3.5}$$

C^2 is given by the equations

$$C^2 = \frac{1}{2}\,\frac{(\partial_1\vartheta)^3\partial_2^3\vartheta - 3(\partial_1\vartheta)^2\partial_2\vartheta\,\partial_1\partial_2^2\vartheta + 3\partial_1\vartheta(\partial_2\vartheta)^2\partial_1^2\partial_2\vartheta - (\partial_2\vartheta)^3\partial_1^3\vartheta}{(\partial_2\vartheta)^4}\Bigg|_{E_s}\prod_{t\neq s}(\lambda_s-\lambda_t)$$

holding for any fixed s. It is not difficult to see by differentiating twice Eq. (7.1) at $x = x_s$ that C is the same constant that appears in Eq. (7.5). The expression (A3.5) is used below to fix the normalization of the Hitchin map.

Appendix 4

We shall show here that the overall normalization of the Hitchin map is as in Eq. (7.10). Since

$$\begin{aligned}
&e^{\pi i e_1'\cdot\tau e_1' + 4\pi i e_1'\cdot u_1}\,\vartheta(2u_1+E_1)^2\\
&= -e^{\pi i e_1'\cdot\tau e_1'}\,\vartheta(2u_1+E_1)\,\vartheta(2u_1-E_1) = -e^{\pi i e_1'\cdot\tau e_1'}\sum_e \theta_{2,e}(E_1)\,\theta_{2,e}(2u_1)\\
&= -e^{\frac{1}{2}\pi i(1,0)\cdot\tau(1,0)}\sum_e(-1)^{(1,0)\cdot e}\,\theta_{2,e+(1,0)}(0)\,\theta_{2,e}(2u_1),
\end{aligned}$$

the coefficient of $\frac{(d\lambda)^2}{\zeta^2}$ on the right-hand side of Eq. (7.12) takes at $\lambda = \lambda_2$ the value

$$\begin{aligned}
\frac{C^2}{16\pi^2}\,e^{\frac{1}{2}\pi i(1,0)\cdot\tau(1,0)}\,(\partial_2\vartheta(E_1))^2\,(\lambda_1-\lambda_2)^2\,&(\beta\theta_{2,(0,0)}(2u_1) - \alpha\theta_{2,(1,0)}(2u_1)\\
&+\delta\theta_{2,(0,1)}(2u_1) - \gamma\theta_{2,(1,1)}(2u_1))
\end{aligned} \tag{A4.1}$$

in the notations of Appendix 3. This coefficient should coincide with the one obtained from the right-hand side of Eq. (7.10) which is equal to

$$-\frac{1}{64\pi^2}\sum_{t\neq 2} r_{2t}(q,p)\prod_{v\neq 2,t}(\lambda_2-\lambda_v) \tag{A4.2}$$

calculated at (q,p) corresponding to (θ, ϕ_{u_1}) with θ given by Eq. (7.11). The respective values of r_{st} are:

$$\begin{aligned}
r_{1t} &= 0,\\
r_{23} &= 2(-\alpha\gamma^2\theta_{2,(0,0)}(2u_1) - \beta\delta^2\theta_{2,(1,0)}(2u_1) - \gamma\alpha^2\theta_{2,(0,1)}(2u_1)\\
&\quad -\delta\beta^2\theta_{2,(1,1)}(2u_1) - \beta\gamma\delta\theta_{2,(0,0)}(2u_1) - \alpha\gamma\delta\theta_{2,(1,0)}(2u_1)\\
&\quad -\alpha\beta\delta\theta_{2,(0,1)}(2u_1) - \alpha\beta\gamma\theta_{2,(1,1)}(2u_1)),\\
r_{24} &= 2(\alpha\gamma^2\theta_{2,(0,0)}(2u_1) + \beta\delta^2\theta_{2,(1,0)}(2u_1) + \gamma\alpha^2\theta_{2,(0,1)}(2u_1)\\
&\quad +\delta\beta^2\theta_{2,(1,1)}(2u_1) - \beta\gamma\delta\theta_{2,(0,0)}(2u_1) - \alpha\gamma\delta\theta_{2,(1,0)}(2u_1)\\
&\quad -\alpha\beta\delta\theta_{2,(0,1)}(2u_1) - \alpha\beta\gamma\theta_{2,(1,1)}(2u_1)),
\end{aligned} \tag{A4.3}$$

$$
\begin{aligned}
r_{25} = 2(&\alpha\delta^2\theta_{2,(0,0)}(2u_1) + \beta\gamma^2\theta_{2,(1,0)}(2u_1) + \gamma\beta^2\theta_{2,(0,1)}(2u_1) \\
&+\delta\alpha^2\theta_{2,(1,1)}(2u_1) + \beta\gamma\delta\theta_{2,(0,0)}(2u_1) + \alpha\gamma\delta\theta_{2,(1,0)}(2u_1) \\
&+\alpha\beta\delta\theta_{2,(0,1)}(2u_1) + \alpha\beta\gamma\theta_{2,(1,1)}(2u_1)),
\end{aligned}
$$

$$
\begin{aligned}
r_{26} = 2(&-\alpha\delta^2\theta_{2,(0,0)}(2u_1) - \beta\gamma^2\theta_{2,(1,0)}(2u_1) - \gamma\beta^2\theta_{2,(0,1)}(2u_1) \\
&-\delta\alpha^2\theta_{2,(1,1)}(2u_1) + \beta\gamma\delta\theta_{2,(0,0)}(2u_1) + \alpha\gamma\delta\theta_{2,(1,0)}(2u_1) \\
&+\alpha\beta\delta\theta_{2,(0,1)}(2u_1) + \alpha\beta\gamma\theta_{2,(1,1)}(2u_1)).
\end{aligned}
$$

Multiplying the coefficients at subsequent $\theta_{2,e}(2u_1)$ in expression (A4.1) by $\alpha, -\beta, \gamma$ and $-\delta$, respectively, and summing them up we obtain

$$\frac{C^2}{8\pi^2}\ \mathrm{e}^{\frac{1}{2}\pi i(1,0)\cdot\tau(1,0)}\ (\partial_2\vartheta(E_1))^2\ (\lambda_1 - \lambda_2)^2\ (\alpha\beta + \gamma\delta).$$

A similar operation on expression (A4.2) gives

$$-\frac{1}{16\pi^2}\ (\lambda_1 - \lambda_2)(\lambda_2 - \lambda_5)(\lambda_2 - \lambda_6)(\lambda_3 - \lambda_4)(\alpha^2\gamma^2 - \beta^2\delta^2).$$

The equality of the two expressions follows from Eq. (A3.5). This verifies the correctness of the overall normalization of the Hitchin map in Eq. (7.10).

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