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Solutions of the Quantum Dynamical Yang–Baxter Equation and Dynamical Quantum Groups*

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Abstract: The quantum dynamical Yang–Baxter (QDYB) equation is a useful generalization of the quantum Yang–Baxter (QYB) equation. This generalization was introduced by Gervais, Neveu, and Felder. Unlike the QYB equation, the QDYB equation is not an algebraic but a difference equation, with respect to a matrix function rather than a matrix. The QDYB equation and its quasiclassical analogue (the classical dynamical Yang–Baxter equation) arise in several areas of mathematics and mathematical physics (conformal field theory, integrable systems, representation theory). The most interesting solution of the QDYB equation is the elliptic solution, discovered by Felder.

In this paper, we prove the first classification results for solutions of the QDYB equation. These results are parallel to the classification of solutions of the classical dynamical Yang–Baxter equation, obtained in our previous paper. All solutions we found can be obtained from Felder's elliptic solution by a limiting process and gauge transformations.

Fifteen years ago the quantum Yang–Baxter equation gave rise to the theory of quantum groups. Namely, it turned out that the language of quantum groups (Hopf algebras) is the adequate algebraic language to talk about solutions of the quantum Yang–Baxter equation.

In this paper we propose a similar language, originating from Felder's ideas, which we found to be adequate for the dynamical Yang–Baxter equation. This is the language of dynamical quantum groups (or $\mathfrak{h}$-Hopf algebroids), which is the quantum counterpart of the language of dynamical Poisson groupoids, introduced in our previous paper.

Introduction

This paper is devoted to the quantum dynamical Yang–Baxter equation, its solutions, and the related algebraic structures (quantum groupoids, Hopf algebroids); abusing language, we will call these structures by the collective name **"dynamical quantum groups"**.

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Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra over $\mathbb{C}$, V a semisimple finite dimensional $\mathfrak{h}$-module, and γ a complex number. The quantum dynamical Yang–Baxter (QDYB) equation is the equation

$$\begin{aligned} R^{12}(\lambda-\gamma h^{(3)})\, R^{13}(\lambda)\, R^{23}(\lambda-\gamma h^{(1)}) \\ = R^{23}(\lambda)\, R^{13}(\lambda-\gamma h^{(2)})\, R^{12}(\lambda) \end{aligned} \tag{1}$$

with respect to a meromorphic function $R:\mathfrak{h}^*\to \mathrm{End}(V\otimes V)$, where by definition $R^{12}(\lambda-\gamma h^{(3)})(v_1\otimes v_2\otimes v_3) := (R^{12}(\lambda-\gamma\mu)(v_1\otimes v_2))\otimes v_3$ if v_3 has weight μ, and $R^{13}(\lambda-\gamma h^{(2)})$, $R^{23}(\lambda-\gamma h^{(1)})$ are defined analogously.

It is also useful to consider the quantum dynamical Yang–Baxter equation with spectral parameter, with respect to a meromorphic function $R:\mathbb{C}\times\mathfrak{h}^*\to\mathrm{End}(V\otimes V)$. By definition, the QDYB equation with spectral parameter is just Eq. (1), with $R^{ij}(*)$ replaced by $R^{ij}(z_i-z_j,*)$, where $z_1,z_2,z_3\in\mathbb{C}$.

Solutions of the QDYB equation which are invariant under $\mathfrak{h}$ are called quantum dynamical R-matrices.

A brief history of the QDYB equation is as follows. The QDYB equation was proposed by Felder [F2] as a quantization of the classical dynamical Yang–Baxter equation [F1], but it also appeared earlier in physical literature [GN]. Examples of dynamical R-matrices appeared in [Fad1, AF]). As Felder showed [F2], the QDYB equation is equivalent to the star-triangle relation in statistical mechanics. The most interesting known solution of the QDYB equation with spectral parameter is the elliptic solution given in [F1, F2]. As was shown in [TV], this solution arises when one studies monodromies of the quantum KZ equation introduced in [FR], see also [FTV1-2]. The algebraic structure corresponding to this solution was described in [F1,F2, FV1-3] and called "the elliptic quantum group". Although the elliptic quantum group is not a Hopf algebra, it is very similar to a Hopf algebra in many respects. For example, its category of representations, with a suitable definition of the tensor product, is a tensor category, which was studied in [FV1, FV2].

This paper has two goals.

1. To classify quantum dynamical R-matrices in the case when $\mathfrak{h}\subset\mathrm{End}(V)$ is the algebra of all diagonal operators in some basis.
2. To describe the axiomatics of the algebraic structure corresponding to a quantum dynamical R-matrix.

The first goal is partially attained in Chapters 1 and 2.

In Chapter 1, we study dynamical R-matrices without spectral parameter. We define the notion of a dynamical R-matrix of Hecke type which is a dynamical R-matrix satisfying a generalized unitarity condition. Then we define gauge transformations, which map the set of such dynamical R-matrices to itself. After this, we classify dynamical R-matrices of Hecke type, with $\mathfrak{h}$ as above. The answer turns out to be completely parallel to the classical case ([EV], Chapter 3). In particular, any classical dynamical r-matrix from [EV] without spectral parameter (for the Lie algebra $\mathfrak{gl}_N$) can be quantized.

In Chapter 2, we study dynamical R-matrices with spectral parameter, satisfying the unitarity condition. As in Chapter 1, we define gauge transformations, which map the set of such dynamical R-matrices to itself. After this, we list all known examples, and give a partial classification result (for R-matrices given by a power series in γ, which are quantizations of elliptic r-matrices from [EV], Chapter 4). As before, the results are parallel to the classical case. In particular, any classical dynamical r-matrix from [EV] with spectral parameter (for the Lie algebra $\mathfrak{gl}_N$) can be quantized.

Remark. We were not able to obtain a nice classification result for dynamical R-matrices with spectral parameter and numerical γ, since we do not understand what is the correct analogue of the residue condition in [EV]. However, we expect that such a result can be obtained along the same lines as in Chapter 4 of [EV], and Chapter 1 of this paper.

The second goal is attained in Chapters 3–6.

In Chapter 3, we explain the connection between dynamical R-matrices and monoidal categories. We introduce the tensor category of $\mathfrak{h}$-vector spaces, and show that a tensor functor from a braided monoidal category to the category of $\mathfrak{h}$-vector spaces gives a dynamical R-matrix, in the same way as a tensor functor from a braided monoidal category to the category of vector spaces gives a usual R-matrix. We also attach to every dynamical R-matrix a tensor category of its representations, following the ideas of [F1, F2, FV1, FV2]. This category is nontrivial (for example, it contains the basic representation), has natural notions of the left and right dual objects, and is equipped with a canonical tensor functor to $\mathfrak{h}$-vector spaces.

In Chapter 4 we introduce the notions of an $\mathfrak{h}$-algebra, $\mathfrak{h}$-bialgebroid, and $\mathfrak{h}$-Hopf algebroid, which are generalizations of the notions of an algebra, bialgebra, and Hopf algebra. We define the notion of a dynamical representation of an $\mathfrak{h}$-algebra, and show that the category of dynamical representations Rep(A) of an $\mathfrak{h}$-bialgebroid A is a tensor category with a natural tensor functor to $\mathfrak{h}$-vector spaces. If A is an $\mathfrak{h}$-Hopf algebroid, this category in addition has natural notions of the left and right dual representation.

Using a generalization of the Faddeev–Reshetikhin–Sklyanin–Takhtajan formalism [FRT, FT] which assigns a Hopf algebra to any R-matrix, we assign an $\mathfrak{h}$-bialgebroid A_R to any dynamical R-matrix R. If R has an additional rigidity property, then A_R is an $\mathfrak{h}$-Hopf algebroid. We call the bialgebroid A_R the dynamical quantum group associated to R. We show that the category of representations of R is equivalent to the category Rep(A_R) as a tensor category with duality and with a functor to $\mathfrak{h}$-vector spaces.

In Chapter 5, we define quantum counterparts of the quasiclassical objects defined in [EV] (in the setting of perturbation theory). More specifically, we define the notions of a biequivariant algebra (biequivariant quantum space), a biequivariant Hopf algebroid (biequivariant quantum groupoid), a dynamical Hopf algebroid (dynamical quantum groupoid), which are the quantum analogues of the notions of a biequivariant Poisson algebra (biequivariant Poisson manifold), a biequivariant Poisson–Hopf algebroid (biequivariant Poisson groupoid), a dynamical Poisson–Hopf algebroid (dynamical Poisson groupoid), introduced in [EV]. We introduce the notion of quantization for biequivariant and dynamical objects, and conjecture that any dynamical Poisson groupoid can be quantized.

This material is a generalization of the material of Chapter 4, because, as we explain in Sect. 5.5, the notion of an $\mathfrak{h}$-algebra ($\mathfrak{h}$-bialgebroid, $\mathfrak{h}$-Hopf algebroid) is essentially a special case of the notion of a biequivariant algebra (bialgebroid, Hopf algebroid).

Remark. The general notion of a Hopf algebroid was introduced by J. H. Lu [Lu]. It is easy to check that biequivariant and dynamical Hopf algebroids as defined in Chapter 5 of our paper are Hopf algebroids in the sense of Lu. However, the notion considered in [Lu] is more general than the one considered in this paper.

In Chapter 6, we study $\mathfrak{h}$-bialgebroids associated to dynamical R-matrices of strong Hecke type. Using the semisimplicity of the Hecke algebra for a generic value of the parameter, we prove a Poincare–Birkhoff–Witt theorem for such bialgebroids. This result explains the meaning of the Hecke type condition, which was artificially introduced in Chapter 1. Using the same method, we show that the $\mathfrak{h}$-Hopf algebroid associated to a dynamical R-matrix of Hecke type of the form $R = 1 - \gamma r + \ldots$ is a flat deformation (quantization) of the Poisson–Hopf algebroid corresponding to r.

In the next papers, we plan to develop the theory of dynamical quantum groups. We plan to describe the infinite-dimensional dynamical quantum groups associated to dynamical R-matrices with spectral parameter, and dynamical quantum groups (both finite and infinite dimensional) associated to Lie groups other than GL_N. We plan to develop the representation theory of dynamical quantum groups, and explain its connection with exchange (Zamolodchikov) algebras, Kazhdan–Lusztig functors, KZ and quantum KZ equations.

1. Classification of Quantum Dynamical R-matrices without Spectral Parameter

1.1. Quantum dynamical R-matrix. Let $\mathfrak{h}$ be an abelian finite dimensional Lie algebra. A finite dimensional diagonalizable $\mathfrak{h}$-module is a complex finite dimensional vector space V with a weight decomposition $V = \oplus_{\mu\in\mathfrak{h}^*} V[\mu]$, such that $\mathfrak{h}$ acts on $V[\mu]$ by $xv = \mu(x)v$, where $x \in \mathfrak{h}$, $v \in V[\mu]$.

Let $V_i,\ i = 1, 2, 3,$ be finite dimensional diagonalizable $\mathfrak{h}$-modules,

$$R_{V_iV_j} : \mathfrak{h}^* \to \mathrm{End}(V_i \otimes V_j), \qquad 1 \le i < j \le 3,$$

meromorphic functions, γ a nonzero complex number. The equation in $\mathrm{End}(V_1\otimes V_2\otimes V_3)$,

$$\begin{aligned} R^{12}_{V_1V_2}(\lambda - \gamma h^{(3)})\, R^{13}_{V_1V_3}(\lambda)\, R^{23}_{V_2V_3}(\lambda - \gamma h^{(1)}) \\ = R^{23}_{V_2V_3}(\lambda)\, R^{13}_{V_1V_3}(\lambda - \gamma h^{(2)})\, R^{12}_{V_1V_2}(\lambda) \end{aligned} \tag{1.1.1}$$

is called *the quantum dynamical Yang–Baxter equation with step* γ (QDYB equation).

Here we use the following notation. If $X \in \mathrm{End}(V_i)$, then we denote by $X^{(i)} \in \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ the operator $\cdots \otimes \mathrm{Id} \otimes X \otimes \mathrm{Id} \otimes \cdots$, acting non-trivially on the i^{th} factor of a tensor product of vector spaces, and if $X = \sum X_k \otimes Y_k \in \mathrm{End}(V_i \otimes V_j)$, then we set $X^{ij} = \sum X_k^{(i)} Y_k^{(j)}$. The shift of λ by $\gamma h^{(i)}$ is defined in the standard way. For instance, $R^{12}_{V_1V_2}(\lambda - \gamma h^{(3)})$ acts on a tensor $v_1 \otimes v_2 \otimes v_3$ as $R^{12}_{V_1V_2}(\lambda - \gamma\mu_3) \otimes \mathrm{Id}$ if v_3 has weight μ_3.

A function $R_{V_iV_j} : \mathfrak{h}^* \to \mathrm{End}(V_i \otimes V_j)$ is called *a function of zero weight* if

$$[R_{V_iV_j}(\lambda), h \otimes 1 + 1 \otimes h] = 0 \tag{1.1.2}$$

for all $h \in \mathfrak{h}$, $\lambda \in \mathfrak{h}^*$. A solution $\{R_{V_iV_j}\}_{1\le i<j\le 3}$ of the QDYB equation is called a solution of zero weight if each of the functions is of zero weight.

If all the spaces V_i are equal to a space V, then consider the QDYB equation on one function $R : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$,

$$\begin{aligned} R^{12}(\lambda - \gamma h^{(3)})\, R^{13}(\lambda)\, R^{23}(\lambda - \gamma h^{(1)}) \\ = R^{23}(\lambda)\, R^{13}(\lambda - \gamma h^{(2)})\, R^{12}(\lambda). \end{aligned} \tag{1.1.3}$$

An invertible function R of zero weight satisfying the QDYB Eq. (1.1.3) is called *a quantum dynamical R-matrix*.

1.2. Quantization and quasiclassical limit. Let $x_1, \ldots, x_N$ be a basis in $\mathfrak{h}$. The basis defines a linear system of coordinates on $\mathfrak{h}^*$. For any $\lambda \in \mathfrak{h}^*$, set $\lambda_i = x_i(\lambda)$, $i = 1, \ldots, N$.

Let $R_\gamma : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ be a smooth family of solutions to the QDYB equation with step γ such that

$$R_\gamma(\lambda) = 1 - \gamma\, r(\lambda) + O(\gamma^2). \tag{1.2.1}$$

Then the function $r : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ satisfies the classical dynamical Yang–Baxter Eq. (CDYB),

$$\begin{gathered}\sum_{i=1}^{N} x_i^{(1)} \frac{\partial r^{23}}{\partial x_i} + \sum_{i=1}^{N} x_i^{(2)} \frac{\partial r^{31}}{\partial x_i} + \sum_{i=1}^{N} x_i^{(3)} \frac{\partial r^{12}}{\partial x_i} + \\ [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0\,.\end{gathered} \tag{1.2.2}$$

A function r of zero weight satisfying the CDYB equation is called *a classical dynamical r-matrix*. The function r in (1.2.1) is called *the quasiclassical limit of* R, and the function R is called *a quantization of* r.

Let $U \subset \mathfrak{h}^*$ be an open set, and let $R : U \to \mathrm{End}(V \otimes V)$ be a zero weight meromorphic function on U. We will say that R is a quantum dynamical R-matrix on U if the QDYB equation is satisfied for R whenever it makes sense.

Remark. If U is a bounded set, this notion is only interesting for small γ, so that the QDYB equation makes sense on a nonempty open set $U' \subset U$.

A classical dynamical r-matrix $r(\lambda)$ on U is called *quantizable* if there exists a power series in γ,

$$R_\gamma(\lambda) = 1 - \gamma\, r(\lambda) + \sum_{n=2}^{\infty} \gamma^n r_n(\lambda), \tag{1.2.3}$$

convergent for small $|\gamma|$ for any fixed $\lambda \in U$ and such that $R_\gamma(\lambda)$ is a quantum dynamical R-matrix on U with step γ.

1.3. Quantum dynamical R-matrices of Hecke type. Let $\mathfrak{h}$ be an abelian Lie algebra of dimension N. Let V be a diagonalizable $\mathfrak{h}$-module of the same dimension N such that its weights $\omega_1, ..., \omega_N$ form a basis in $\mathfrak{h}^*$. Let $x_1, ..., x_N$ be the dual basis of $\mathfrak{h}$. Let $v_1, ..., v_N$ be an eigenbasis for $\mathfrak{h}$ in V such that $x_i v_j = \delta_{ij} v_j$. Then the $\mathfrak{h}$-module $V \otimes V$ has the weight decomposition,

$$V \otimes V = \oplus_{a=1}^{N} V_{aa} \,\oplus \oplus_{a<b} V_{ab}, \tag{1.3.1}$$

where $V_{aa} = \mathbb{C}\, v_a \otimes v_a$ and $V_{ab} = \mathbb{C}\, v_a \otimes v_b \oplus \mathbb{C}\, v_b \otimes v_a$.

Introduce a basis E_{ij} in $\mathrm{End}(V)$ by $E_{ij} v_k = \delta_{jk} v_i$.

A quantum dynamical R-matrix $R : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ for these $\mathfrak{h}$ and V will be called an R-matrix of gl_N type.

The zero weight condition implies that the R-matrix preserves the weight decomposition (1.3.1) and has the form

$$R(\lambda) = \sum_{a,b=1}^{N} \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}(\lambda)\, E_{ba} \otimes E_{ab}, \tag{1.3.2}$$

where $\alpha_{ab}, \beta_{ab} : \mathfrak{h}^* \to \mathbb{C}$ are suitable meromorphic functions.

Let $P \in \mathrm{End}(V \otimes V)$ be the permutation of factors. Set $R^\vee = PR$.

Let p, q be nonzero complex numbers, $p \neq -q$. A function $R : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ will be called *a function of Hecke type with parameters* p, q if

(1.3.3) The function preserves the weight decomposition (1.3.1).
(1.3.4) For any $a = 1, ..., N$ and $\lambda \in \mathfrak{h}^*$, we have $R^\vee(\lambda) v_a \otimes v_a \;=\; p\, v_a \otimes v_a$.
(1.3.5) For any $a \neq b$ and $\lambda \in \mathfrak{h}^*$, the operator $R^\vee(\lambda)$ restricted to the two dimensional space V_{ab} has eigenvalues p and $-q$.

A function $R : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ will be called *a function of weak Hecke type with parameters* p, q if it preserves the weight decomposition (1.3.1) and for any $\lambda \in \mathfrak{h}^*$ satisfies the equation

$$(R^\vee(\lambda) - p)\,(R^\vee(\lambda) + q) = 0. \tag{1.3.6}$$

A relation between Hecke types is given by the following simple observation. Let $R_t : \mathfrak{h}^* \to \mathrm{End}(V \otimes V), t \in [0, 1]$, be a continuous family of meromorphic functions, which is analytic when $t \in (0, 1)$. Assume that for any t the function R_t is of weak Hecke type and $R_{t=0} = \mathrm{Id}$. Then R_t is of Hecke type for any t. In fact, the matrix $R^\vee_{t=0} = P$ satisfies (1.3.4–5) and hence $R^\vee_t$ satisfies (1.3.4–5) for any t.

In the following sections we classify quantum dynamical R-matrices of gl_N Hecke type.

1.4. Gauge transformations and multiplicative closed 2-forms. In this subsection we introduce gauge transformations of quantum dynamical R-matrices of Hecke type. We shall use the notion of a multiplicative form.

A multiplicative k-form on a vector space with a linear coordinate system $\lambda_1, ..., \lambda_N$ is a collection,

$$\varphi = \{\varphi_{a_1,...,a_k}(\lambda_1, ..., \lambda_N)\},$$

of meromorphic functions , where $a_1, ..., a_k$ run through all ordered k element subsets of $\{1, ..., N\}$, such that for any subset $a_1, ..., a_k$ and any $i,\ 1 \le i < k$, we have

$$\varphi_{a_1,...,a_{i+1},a_i,...,a_k}(\lambda_1, ..., \lambda_N)\,\varphi_{a_1,...,a_k}(\lambda_1, ..., \lambda_N) \;=\; 1.$$

Let Ω^k be the set of all multiplicative k-forms.

If φ and ψ are multiplicative k-forms, then $\{\varphi_{a_1,...,a_k}(\lambda_1, ..., \lambda_N) \cdot \psi_{a_1,...,a_k}(\lambda_1, ..., \lambda_N)\}$ and $\{\varphi_{a_1,...,a_k}(\lambda_1, ..., \lambda_N) / \psi_{a_1,...,a_k}(\lambda_1, ..., \lambda_N)\}$ are multiplicative k-forms. This gives an abelian group structure on Ω^k. The zero element in Ω^k is the form $\{\varphi_{a_1,...,a_k}(\lambda_1, ..., \lambda_N) \equiv 1\}$.

Fix a nonzero complex number γ. For any $a = 1, ..., N$, introduce an operator δ_a on the space of meromorphic functions $f(\lambda_1, ..., \lambda_N)$ by

$$\delta_a \;:\; f(\lambda_1, ..., \lambda_N) \;\mapsto\; f(\lambda_1, ..., \lambda_N) \,/\, f(\lambda_1, ..., \lambda_a - \gamma, ..., \lambda_N)$$

and an operator $d_\gamma : \Omega^k \to \Omega^{k+1},\ \varphi \mapsto d_\gamma\varphi$, by

$$(d_\gamma\varphi)_{a_1,...,a_{k+1}}(\lambda_1, ..., \lambda_N) = \prod_{i=1}^{k+1} (\delta_{a_i}\varphi_{a_1,...,a_{i-1},a_{i+1},...,a_{k+1}}(\lambda_1, ..., \lambda_N))^{(-1)^{i+1}}.$$

We have $d^2_\gamma = 0$. A form φ will be called *γ-closed* if $d_\gamma\varphi = 0$.

Let $\varphi(\gamma) = \{\varphi_{a_1,...,a_k}(\lambda_1, ..., \lambda_N, \gamma)\}$ be a smooth family of multiplicative k-forms such that for all $a_1, ..., a_k$,

$$\varphi_{a_1,...,a_k}(\lambda, \gamma) = 1 - \gamma\, C_{a_1,...,a_k}(\lambda) + O(\gamma^2)$$

for suitable functions $C_{a_1,\dots,a_k}(\lambda)$. Then the functions $\{C_{a_1,\dots,a_k}(\lambda)\}$ are skew-symmetric with respect to permutation of the indices, so it is natural to consider a differential form $C = \sum_{a_1<\dots<a_k} C_{a_1,\dots,a_k}(\lambda)\, dx_{a_1} \wedge \dots \wedge dx_{a_k}$. The differential form C is called the *quasiclassical limit* of the multiplicative form $\varphi(\gamma)$ and the multiplicative form $\varphi(\gamma)$ is called a *quantization* of the differential form C. It is easy to see that if $\varphi(\gamma)$ is γ-closed, then C is closed.

Let $U \subset \mathbb{C}^N$ be an open set, and let φ be a multiplicative meromorphic k-form on U. We will say that φ is γ-closed if the equation $d_\gamma \varphi = 0$ is satisfied whenever it makes sense.

A closed differential form $\{C_{a_1,\dots,a_k}(\lambda)\}$ is called *quantizable* if there exists a power series in γ,

$$\varphi_{a_1,\dots,a_k}(\lambda,\gamma) = 1 - \gamma\, C_{a_1,\dots,a_k}(\lambda) + \sum_{n=2}^{\infty} \gamma^n C_{n;\, a_1,\dots,a_k}(\lambda),$$

convergent for small $|\gamma|$ for a fixed $\lambda \in U$ and such that $\{\varphi_{a_1,\dots,a_k}(\lambda,\gamma)\}$ is a γ-closed multiplicative k-form.

Lemma 1.1. *Every closed holomorphic differential k-form C defined on an open polydisc is quantizable to a holomorphic multiplicative closed k-form $\varphi(\gamma)$.*

Proof. Since U is a polydisc, we can find a holomorphic $(k-1)$-form E on U such that $dE = C$. Define a multiplicative $(k-1)$-form θ on U by $\theta_{a_1\dots a_{k-1}} = e^{-E_{a_1\dots a_{k-1}}}$. Set $\varphi(\gamma) = d_\gamma \theta$. Since $d_\gamma^2 = 0$, the form $\varphi(\gamma)$ is a desired multiplicative closed k-form. □

Remark. The Taylor expansion of $\varphi(\gamma)$ in powers of γ is well defined in U, but for each particular (even very small) nonzero γ, the form $\varphi(\gamma)$ is defined in a smaller open subset $U'(\gamma) \subset U$ which tends to U as $\gamma \to 0$.

Now we introduce gauge transformations of quantum dynamical R-matrices, $R : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$, of form (1.3.2) with step γ.

(1.4.1) Let $\{\varphi_{ab}\}$ be a meromorphic γ-closed multiplicative 2-form on $\mathfrak{h}^*$. Set

$$R(\lambda) \mapsto \sum_{a=1}^{N} \alpha_{aa}(\lambda)\, E_{aa} \otimes E_{aa} + \sum_{a\neq b} \varphi_{ab}(\lambda)\, \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a\neq b} \beta_{ab}(\lambda)\, E_{ba} \otimes E_{ab}.$$

(1.4.2) Let the symmetric group S_N , the Weyl group of gl_N, act on $\mathfrak{h}^*$ and V by permutation of coordinates. For any permutation $\sigma \in S_N$, set

$$R(\lambda) \;\mapsto\; (\sigma \otimes \sigma)\, R(\sigma^{-1} \cdot \lambda)\, (\sigma^{-1} \otimes \sigma^{-1}).$$

(1.4.3) For a nonzero complex number c, set

$$R(\lambda) \;\mapsto\; c\, R(\lambda).$$

(1.4.4) For a nonzero complex number c and an element $\mu \in \mathfrak{h}^*$, set

$$R(\lambda) \;\mapsto\; R(c\,\lambda + \mu).$$

It is clear that any gauge transformation of types (1.4.2)–(1.4.3) transforms a quantum dynamical R-matrix with step γ to a quantum dynamical R-matrix with step γ. Any gauge transformation of type (1.4.4) transforms a quantum dynamical R-matrix with step γ to a quantum dynamical R-matrix with step γ/c. In all cases, if the R-matrix is of Hecke type, then the transformed matrix is of Hecke type. If the transformation is of type (1.4.3) and the Hecke parameters of the R-matrix are p and q, then the Hecke parameters of the transformed matrix are cp and cq.

Theorem 1.1. *Any gauge transformation of type (1.4.1) transforms a quantum dynamical R-matrix with step γ to a quantum dynamical R-matrix with step γ. If the R-matrix is of Hecke type, then the transformed matrix is of Hecke type with the same parameters.*

Theorem 1.1 is proved in Sect. 1.9.

Two R-matrices $R : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ and $R' : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ will be called *equivalent* if one of them can be transformed into another by a sequence of gauge transformations.

1.5. Classification of quantum dynamical R-matrices of Hecke type with parameters p, q such that $q = p$. If Hecke parameters satisfy $p = q$, then the Hecke Eq. (1.3.6) can be written as

$$R^{21}(\lambda)\, R(\lambda) = q^2\, \mathrm{Id}.$$

Let $X \subset \{1, ..., N\}$ be a subset. Say that X *is decomposed into disjoint intervals,* $X = X_1 \cup ... \cup X_n$, if every X_k has the form $\{a_k, a_k + 1, ..., b_k\}$ and $a_{k+1} > b_k$ for $k = 1, ..., n - 1$.

A meromorphic function $\mu(\lambda)$ will be called γ*-quasiconstant* if $\delta_a \mu = 0$ for all a. Fix a γ-quasiconstant $\mu : \mathfrak{h}^* \to \mathfrak{h}^*$ with $\gamma = 1$. Define scalar meromorphic γ-quasiconstant functions $\mu_{ab} : \mathfrak{h}^* \to \mathbb{C}$ by $\mu_{ab}(\lambda) = x_a(\mu(\lambda)) - x_b(\mu(\lambda))$. Let λ_{ab} denote $\lambda_a - \lambda_b$.

Define $R_{\cup X_k} : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ by

$$R_{\cup X_k}(\lambda) = \sum_{a,b=1}^{N} E_{aa} \otimes E_{bb} + \sum_{k=1}^{n} \sum_{a,b \in X_k\ a \neq b} \frac{1}{\lambda_{ab} - \mu_{ab}(\lambda)} \,(\, E_{aa} \otimes E_{bb} + E_{ba} \otimes E_{ab}\,). \tag{1.5.1}$$

Theorem 1.2. *1. For every $X \subset \{1, ..., N\}$, the R-matrix $R_{\cup X_k}$ defined by (1.5.1) is a quantum dynamical R-matrix of Hecke type with parameters $p = 1,\ q = 1$ and step $\gamma = 1$.*

2. Every quantum dynamical R-matrix of Hecke type with parameters p, q, such that $p = q$, is equivalent to one of the matrices (1.5.1).

Theorem 1.2 is proved in Sect. 1.11.

1.6. Classification of quantum dynamical R-matrices of Hecke type with parameters p, q such that $q \neq p$. Assume that for any $a, b,\ a \neq b$, a γ-quasiconstant $\mu_{ab} : \mathfrak{h}^* \to \mathbb{C}$ is given. We say that this collection of quasiconstants is *multiplicative* if

(1.6.1) For any a, b, we have

$$\mu_{ab}(\lambda)\, \mu_{ba}(\lambda) = 1.$$

(1.6.2) For any a, b, c, we have

$$\mu_{ac}(\lambda) = \mu_{ab}(\lambda)\, \mu_{bc}(\lambda).$$

Fix a multiplicative family of γ-quasiconstants with $\gamma = 1$.

Fix a complex number ϵ such that $e^\epsilon \neq 1$. Let $X \subset \{1, ..., N\}$ be a subset, $X = X_1 \cup ... \cup X_n$ its decomposition into disjoint intervals.

For any $a, b \in \{1, ..., N\}$, $a \neq b$, we shall introduce functions $\alpha_{ab}, \beta_{ab} : \mathfrak{h}^* \to \mathbb{C}$. We shall introduce functions β_{ab} and then set $\alpha_{ab} = e^\epsilon + \beta_{ab}$.

If $a, b \in X_k$ for some k, then we set

$$\beta_{ab}(\lambda) = \frac{e^\epsilon - 1}{\mu_{ab}(\lambda) e^{\epsilon \lambda_{ab}} - 1}. \tag{1.6.3}$$

Otherwise we set $\beta_{ab}(\lambda) = 0$, if $a < b$, and $\beta_{ab}(\lambda) = 1 - e^\epsilon$, if $a > b$.

Define $R_{\cup X_k} : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ by

$$R_{\cup X_k,\epsilon}(\lambda) = \sum_{a=1}^N E_{aa} \otimes E_{aa} + \sum_{a \neq b} \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}(\lambda)\, E_{ba} \otimes E_{ab}. \tag{1.6.4}$$

Theorem 1.3. *1. For every $X \subset \{1, ..., N\}$, the R-matrix $R_{\cup X_k,\epsilon}$ defined by (1.6.4) is a quantum dynamical R-matrix of Hecke type with parameters $p = 1$, $q = e^\epsilon$ and step $\gamma = 1$.*

2. Every quantum dynamical R-matrix of Hecke type with parameters p, q such that $q \neq p$ is equivalent to one of the matrices (1.6.4).

Theorem 1.3 is proved in Sect. 1.12.

1.7. Quantization of classical dynamical r-matrices of gl_N type. Let V be the N dimensional $\mathfrak{h}$-module considered in Sect. 1.3. Let $r : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ be a zero weight meromorphic function satisfying CDYB (1.2.2). Assume that r satisfies *the unitarity condition*,

$$r(\lambda) + r^{21}(\lambda) = \epsilon P + \delta\, \mathrm{Id} \tag{1.7.1}$$

for some constants $\epsilon, \delta \in \mathbb{C}$ and all λ. The constant ϵ is called *the coupling constant*, the constant δ is called *the secondary coupling constant.* The zero weight condition implies that r has the form

$$r(\lambda) = \sum_{a,b=1}^N \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}(\lambda)\, E_{ab} \otimes E_{ba}. \tag{1.7.2}$$

We recall a classification of such r-matrices. First we introduce gauge transformations of classical dynamical r-matrices.

(1.7.3) Let $\psi = \sum_{a,b} \psi_{ab}(\lambda) dx_a \wedge dx_b$ be a closed meromorphic differential 2-form on $\mathfrak{h}^*$
(and the notion of a closed differential form has the standard meaning). Set

$$r(\lambda) \mapsto r(\lambda) + \sum_{a \neq b}^N \psi_{ab}(\lambda)\, E_{aa} \otimes E_{bb}.$$

(1.7.4) For $\mu \in \mathfrak{h}^*$, set

$$r(\lambda) \mapsto r(\lambda + \mu).$$

(1.7.5) Let the symmetric group S_N act on $\mathfrak{h}^*$ and V by permutation of coordinates. For any permutation $\sigma \in S_N$, set

$$r(\lambda) \mapsto (\sigma \otimes \sigma)\, r(\sigma^{-1} \cdot \lambda)\, (\sigma^{-1} \otimes \sigma^{-1}).$$

(1.7.6) For a nonzero complex number c, set

$$r(\lambda) \mapsto c\, r(c\lambda).$$

(1.7.7) For a nonzero complex number c, set

$$r(\lambda) \mapsto r(\lambda) + c\,\mathrm{Id}.$$

Any gauge transformation transforms a classical dynamical r-matrix to a classical dynamical r-matrix [EV]. Two classical dynamical r-matrices $r(\lambda)$ and $r'(\lambda)$ will be called *equivalent* if one of them can be transformed into another by a sequence of gauge transformations.

The gauge transformations of quantum dynamical R-matrices described in Sect. 1.4 are analogs of gauge transformations of classical dynamical r-matrices.

Classification of r-matrices with zero coupling constant, $\epsilon = 0$. Let $X \subset \{1, ..., N\}$ be a subset, $X = X_1 \cup ... \cup X_n$ its decomposition into disjoint intervals.

Define a map $r : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ by

$$r_{\cup X_k}(\lambda) = \sum_{k=1}^{n} \sum_{a,b \in X_k\ a \neq b} \frac{1}{\lambda_{ba}} E_{ba} \otimes E_{ab}. \tag{1.7.8}$$

Theorem 1.4. *1. For any X and its decomposition $X = X_1 \cup ... \cup X_n$ into disjoint intervals, the function $r_{\cup X_k}$ defined by (1.7.8) is a classical dynamical r-matrix with zero coupling constant.*

2. Any classical dynamical r-matrix $r : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ with zero coupling constant is equivalent to one of the matrices (1.7.8).

Theorem 1.4 follows from [EV].

Classification of r-matrices with nonzero coupling constant, $\epsilon \neq 0$. Let $X \subset \{1, ..., N\}$ be a subset, $X = X_1 \cup ... \cup X_n$ its decomposition into disjoint intervals.

For any $a, b \in \{1, ..., N\}$, $a \neq b$, we introduce functions $\beta_{ab} : \mathfrak{h}^* \to \mathbb{C}$. If $a, b \in X_k$ for some k, then we set

$$\beta_{ab}(\lambda) = \mathrm{cotanh}\,(\lambda_{ba})\,.$$

Otherwise we set $\beta_{ab}(\lambda) = -1$, if $a < b$, and $\beta_{ab}(\lambda) = 1$, if $a > b$.

Define $r_{\cup X_k} : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ by

$$r_{\cup X_k}(\lambda) = P + \sum_{a \neq b} \beta_{ab}(\lambda)\, E_{ba} \otimes E_{ab}. \tag{1.7.9}$$

Theorem 1.5. *1. For every $X \subset \{1, ..., N\}$ and its decomposition $X = X_1 \cup ... \cup X_n$ into disjoint intervals, the function $r_{\cup X_k}$ defined by (1.7.9) is a classical dynamical r-matrix with nonzero coupling constant $\epsilon = 2$ and the secondary coupling constant $\delta = 0$.*

2. Every classical dynamical r-matrix $r : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ with nonzero coupling constant is equivalent to one of the matrices (1.7.9).

Theorem 1.5 follows from [EV].

Theorem 1.6. *1. Every classical dynamical r-matrix r with zero coupling constant, holomorphic on an open polydisc $U \subset \mathfrak{h}^*$, can be quantized to a quantum dynamical R-matrix R_γ on U, of Hecke type with parameters p, q such that $p = q$.*

2. Every classical dynamical r-matrix r with nonzero coupling constant, holomorphic on an open polydisc $U \subset \mathfrak{h}^*$, can be quantized to a quantum dynamical R-matrix R_γ on U, of Hecke type with parameters p, q such that $p \neq q$.

Proof. The R-matrix

$$R_{\cup X_k}(\lambda, \gamma) = \sum_{a,b=1}^{N} E_{aa} \otimes E_{bb} + \sum_{k=1}^{n} \sum_{a,b \in X_k\ a \neq b} \frac{\gamma}{\lambda_{ab}} (E_{aa} \otimes E_{bb} + E_{ba} \otimes E_{ab})$$

is a quantum dynamical R-matrix of Hecke type with parameters $p = q = 1$ and step γ. Its quasiclassical limit is

$$r'(\lambda) = \sum_{k=1}^{n} \sum_{a,b \in X_k\ a \neq b} \frac{-1}{\lambda_{ab}} (E_{aa} \otimes E_{bb} + E_{ba} \otimes E_{ab}).$$

Making the gauge transformation (1.7.3) corresponding to the closed form $\sum_k \sum_{a,b \in X_k, a<b} \lambda_{ab}^{-1}\, dx_a \wedge dx_b$, we get the r-matrix $r_{\cup X_k}$ defined by (1.7.8). This remark and Lemma 1.1 easily imply the first statement of the theorem. The second statement is proved analogously. □

1.8. Quantum dynamical Yang–Baxter equation in coordinates. Consider a quantum dynamical R-matrix $R(\lambda)$ of form (1.3.2). Assume that the matrix is of Hecke type, with step $\gamma = 1$ and Hecke parameters $p = 1$ and q. Any R-matrix of Hecke type can be reduced to such an R-matrix by gauge transformations of types (1.4.3) and (1.4.4).

The Hecke property implies that $\alpha_{aa} = 1$ and hence the matrix has the form

$$R(\lambda) = \sum_{a=1}^{N} E_{aa} \otimes E_{aa} + \sum_{a \neq b} \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}(\lambda)\, E_{ba} \otimes E_{ab}. \quad (1.8.1)$$

The Hecke property also implies that for every $a, c \in \{1, \ldots, N\},\ a \neq c$, we have

$$\beta_{ac}(\lambda) + \beta_{ca}(\lambda) = 1 - q, \quad (1.8.2)$$

$$\beta_{ac}(\lambda)\, \beta_{ca}(\lambda) - \alpha_{ac}(\lambda)\, \alpha_{ca}(\lambda) = -q, \quad (1.8.3)$$

this is the trace and the determinant of $R^\vee$ restricted to V_{ac}.

Applying both sides of the QDYB Eq. (1.1.3) to a basis vector $v_a \otimes v_a \otimes v_c \in V^{\otimes 3}$, $a \neq c$, we get equations

$$\alpha_{ca}(\lambda - \omega_a)\, \beta_{ac}(\lambda)\, \alpha_{ac}(\lambda - \omega_a) + \beta_{ac}(\lambda - \omega_a)^2 = \beta_{ac}(\lambda - \omega_a), \quad (1.8.4)$$

$$\begin{aligned} \beta_{ca}(\lambda - \omega_a)\, \beta_{ac}(\lambda)\, \alpha_{ac}(\lambda - \omega_a) + \alpha_{ac}(\lambda - \omega_a)\, \beta_{ac}(\lambda - \omega_a) = \\ \beta_{ac}(\lambda)\, \alpha_{ac}(\lambda - \omega_a). \end{aligned} \quad (1.8.5)$$

Applying both sides of the QDYB Eq. (1.1.3) to a basis vector $v_a \otimes v_b \otimes v_c \in V^{\otimes 3}$ with pairwise distinct a, b, c we get equations

$$\alpha_{ab}(\lambda - \omega_c)\,\alpha_{ac}(\lambda)\,\alpha_{bc}(\lambda - \omega_a) = \alpha_{bc}(\lambda)\,\alpha_{ac}(\lambda - \omega_b)\,\alpha_{ab}(\lambda), \tag{1.8.6}$$

$$\alpha_{ac}(\lambda - \omega_b)\,\alpha_{ab}(\lambda)\,\beta_{bc}(\lambda - \omega_a) = \beta_{bc}(\lambda)\,\alpha_{ac}(\lambda - \omega_b)\,\alpha_{ab}(\lambda), \tag{1.8.7}$$

$$\beta_{ab}(\lambda - \omega_c)\,\alpha_{ac}(\lambda)\,\alpha_{bc}(\lambda - \omega_a) = \alpha_{ac}(\lambda)\,\alpha_{bc}(\lambda - \omega_a)\,\beta_{ab}(\lambda), \tag{1.8.8}$$

$$\beta_{cb}(\lambda - \omega_a)\,\beta_{ac}(\lambda)\,\alpha_{bc}(\lambda - \omega_a) + \alpha_{bc}(\lambda - \omega_a)\,\beta_{ab}(\lambda)\,\beta_{bc}(\lambda - \omega_a) = \beta_{ac}(\lambda)\,\alpha_{bc}(\lambda - \omega_a)\,\beta_{ab}(\lambda), \tag{1.8.9}$$

$$\alpha_{cb}(\lambda - \omega_a)\,\beta_{ac}(\lambda)\,\alpha_{bc}(\lambda - \omega_a) + \beta_{bc}(\lambda - \omega_a)\,\beta_{ab}(\lambda)\,\beta_{bc}(\lambda - \omega_a) = \alpha_{ba}(\lambda)\,\beta_{ac}(\lambda - \omega_b)\,\alpha_{ab}(\lambda) + \beta_{ab}(\lambda)\,\beta_{bc}(\lambda - \omega_a)\,\beta_{ab}(\lambda), \tag{1.8.10}$$

$$\beta_{ac}(\lambda - \omega_b)\,\alpha_{ab}(\lambda)\,\beta_{bc}(\lambda - \omega_a) = \beta_{ba}(\lambda)\,\beta_{ac}(\lambda - \omega_b)\,\alpha_{ab}(\lambda) + \alpha_{ab}(\lambda)\,\beta_{bc}(\lambda - \omega_a)\,\beta_{ab}(\lambda). \tag{1.8.11}$$

Lemma 1.2. *For any $a, c,\ a \neq c$, the functions $\alpha_{ac}(\lambda)$ and $q + \beta_{ac}(\lambda)$ are not identically equal to zero.*

Proof. If $\alpha_{ac} \equiv 0$, then Eqs. $(1.8.2)_{ac}$, $(1.8.3)_{ac}$, $(1.8.4)_{ac}$, and $(1.8.4)_{ca}$ give a contradiction. Thus, α_{ac} and α_{ca} are not identically equal to zero. Equations $(1.8.2)_{ac}$, $(1.8.3)_{ac}$ imply

$$\alpha_{ac}(\lambda)\,\alpha_{ca}(\lambda) = (q + \beta_{ac}(\lambda))\,(q + \beta_{ca}(\lambda)). \tag{1.8.12}$$

The lemma is proved. □

1.9. Proof of Theorem 1.1. Let $\{\varphi_{ab}\}$ be a γ-closed multiplicative 2-form on $\mathfrak{h}^*$. It is easy to see that Eqs. (1.8.2)–(1.8.11) are invariant with respect to the gauge transformation (1.4.1). This proves Theorem 1.1. □

1.10. Relation $\alpha_{ac} = q + \beta_{ac}$. Consider a quantum dynamical R-matrix $R(\lambda)$ of form (1.3.2). Assume that the matrix is of Hecke type with step $\gamma = 1$ and Hecke parameters $p = 1$ and q. For any $a, c,\ a \neq c$, set

$$\varphi_{ac}(\lambda) = \frac{q + \beta_{ac}(\lambda)}{\alpha_{ac}(\lambda)}. \tag{1.10.1}$$

Lemma 1.3. *The collection of functions $\varphi = \{\varphi_{ac}\}$ is a γ-closed multiplicative 2-form with $\gamma = 1$.*

Corollary 1.1. *Apply to the R-matrix $R(\lambda)$ the gauge transformation (1.4.1) corresponding to the multiplicative 2-form φ^{-1}. Then the coefficients of the transformed matrix satisfy the equation*

$$\alpha_{ac} = q + \beta_{ac} \tag{1.10.2}$$

for all a, c.

Proof of Lemma 1.3. Equation $\varphi_{ac}\varphi_{ca} = 1$ follows from (1.8.12). Equation $d_\gamma \varphi = 0$ is a direct corollary of (1.8.6) and (1.8.7). □

1.11. Proof of Theorem 1.2. Let $R(\lambda)$ be a quantum dynamical R-matrix of Hecke type with parameters p, q such that $p = q$. Using gauge transformations (1.4.3) and (1.4.4) we can make step $\gamma = 1$ and $p = q = 1$. By Lemma 1.3 we may assume that $\alpha_{ac}(\lambda) = 1 + \beta_{ac}(\lambda)$ for all $a \neq c$. By (1.8.2) we have $\beta_{ac}(\lambda) = -\beta_{ca}(\lambda)$ for all $a \neq c$.

Fix $a, c,\ a \neq c$, and solve Eqs. $(1.8.4)_{ac}$, $(1.8.5)_{ac}$, $(1.8.4)_{ca}$, $(1.8.5)_{ca}$.

Lemma 1.4. *Any solution* $\beta_{ac}(\lambda)$, $\beta_{ca}(\lambda)$ *of Eqs.* $(1.8.4)_{ac}, (1.8.5)_{ac}$, $(1.8.4)_{ca}$, $(1.8.5)_{ca}$ *has one of the following two forms.*

1. $\beta_{ac} = \beta_{ca} = 0$.
2.

$$\beta_{ac}(\lambda) = \frac{1}{\lambda_{ac} - \mu_{ac}}, \qquad \beta_{ca}(\lambda) = \frac{1}{\lambda_{ca} - \mu_{ca}}, \tag{1.11.1}$$

where $\mu_{ac} = -\mu_{ca}$ *and* $\mu_{ac}(\lambda)$ *is a meromorphic function periodic with respect to shifts of* λ *by* ω_a *and* ω_c, $\mu_{ac}(\lambda - \omega_a) = \mu_{ac}(\lambda - \omega_c) = \mu_{ac}(\lambda)$.

Proof. It is easy to see that $\beta_{ac}(\lambda) = \beta_{ca}(\lambda) \equiv 0$ is a solution. Now assume that $\beta_{ac} = -\beta_{ca} \not\equiv 0$. Then $(1.8.5)_{ac}$ gives

$$\frac{1}{\beta_{ac}(\lambda)} + \frac{1}{\beta_{ac}(\lambda - \omega_a)} = 1,$$

and $(1.8.5)_{ca}$ gives

$$\frac{1}{\beta_{ac}(\lambda)} + \frac{1}{\beta_{ac}(\lambda - \omega_c)} = -1.$$

Let $\mu_{ac}(\lambda) = \lambda_{ac} - 1/\beta_{ac}(\lambda)$. Then $\mu_{ac}(\lambda - \omega_a) = \mu_{ac}(\lambda)$ and $\mu_{ac}(\lambda - \omega_c) = \mu_{ac}(\lambda)$. Hence

$$\beta_{ac}(\lambda) = \frac{1}{\lambda_{ac} - \mu_{ac}},$$

where $\mu_{ac}(\lambda)$ is a meromorphic function periodic in ω_a and ω_c. Similarly,

$$\beta_{ca}(\lambda) = \frac{1}{\lambda_{ca} - \mu_{ca}},$$

where $\mu_{ca}(\lambda)$ is a function periodic in ω_a and ω_c. We have $\mu_{ac} = -\mu_{ca}$ since $\beta_{ac} = -\beta_{ca}$. It is easy to see that these functions β_{ac} and β_{ca} solve Eqs. $(1.8.4)_{ac}$ and $(1.8.4)_{ca}$. The lemma is proved. □

Equation (1.8.7) shows that the function $\beta_{ac}(\lambda)$ and hence the function $\mu_{ac}(\lambda)$ is periodic with respects to shifts of λ by ω_b for any b different from a and c.

Consider Eq. $(1.8.9)_{abc}$ on functions $\beta_{ab}(\lambda)$, $\beta_{bc}(\lambda)$, $\beta_{ac}(\lambda)$. It is easy to see that if one of these three functions is identically equal to zero, then there is another function in this triple which is identically equal to zero.

Introduce a relation on the set $\{1, ..., N\}$. For any $a \in \{1, ..., N\}$, let a be related to a. For any $a, b \in \{1, ..., N\}, a \not\equiv b$, let a be related to b if the function $\beta_{ab}(\lambda)$ is not identically equal to zero. It is easy to see that this is an equivalence relation.

Let $Y \subset \{1, ..., N\}$ be the union of all the equivalence classes containing more than one element. Let $Y = Y_1 \cup ... \cup Y_n$ be its decomposition into equivalence classes.

If pairwise distinct $a, b, c \in \{1, ..., N\}$ do not belong to the same equivalence class, then at least two of the three functions $\beta_{ab}(\lambda)$, $\beta_{bc}(\lambda)$, $\beta_{ac}(\lambda)$ are identically equal to zero. Hence this triple of functions satisfies Eq. $(1.8.9)_{abc}$. If all three elements a, b, c belong to the same equivalence class, then equation $(1.8.9)_{abc}$ takes the form

$$\frac{1}{\lambda_{cb} - \mu_{cb}} \frac{1}{\lambda_{ac} - \mu_{ac}} + \frac{1}{\lambda_{ab} - \mu_{ab}} \frac{1}{\lambda_{bc} - \mu_{bc}} = \frac{1}{\lambda_{ac} - \mu_{ac}} \frac{1}{\lambda_{ab} - \mu_{ab}}.$$

This implies that $\mu_{ac}(\lambda) = \mu_{ab}(\lambda) + \mu_{bc}(\lambda)$. Therefore there exists a 1-quasiconstant meromorphic map $\mu : \mathfrak{h}^* \to \mathfrak{h}^*$ such that $\mu_{ac}(\lambda) = x_a(\mu(\lambda)) - x_c(\mu(\lambda))$ for all a, c such that $\mu_{ac}(\lambda)$ is not identically equal to zero. It is easy to see that if the functions $\mu_{ab}(\lambda)$ have this property then Eqs. (1.8.8) and (1.8.10) are also satisfied.

Let σ be a permutation of $\{1, ..., N\}$ which transforms the set Y and the decomposition $Y = Y_1 \cup ... \cup Y_n$ into a set $X \subset \{1, ..., N\}$ and its decomposition into disjoint intervals $X = X_1 \cup ... \cup X_n$. Apply to the R-matrix $R(\lambda)$ the gauge transformation (1.4.2) corresponding to the permutation σ. Then the transformed R-matrix will have form (1.5.1) corresponding to the constructed decomposition $X = X_1 \cup ... \cup X_n$. Theorem 1.2 is proved. $\square$

1.12. Proof of Theorem 1.3. Let $R(\lambda)$ be a quantum dynamical R-matrix of Hecke type with parameters p, q such that $p \neq q$. Using gauge transformations (1.4.3) and (1.4.4) we can make step $\gamma = 1$ and $p = 1$. Fix a number ϵ such that $q = e^{\epsilon}$.

By Lemma 1.3 we may assume that $\alpha_{ac}(\lambda) = q + \beta_{ac}(\lambda)$ for all $a \neq c$. By (1.8.2) we have $\beta_{ca}(\lambda) = 1 - q - \beta_{ac}(\lambda)$ for all $a \neq c$.

Fix $a, c,\ a \neq c$, and solve Eqs. $(1.8.4)_{ac}, (1.8.5)_{ac}, (1.8.4)_{ca}, (1.8.5)_{ca}$.

Lemma 1.5. *Any solution $\beta_{ac}(\lambda), \beta_{ca}(\lambda)$ of Eqs. $(1.8.4)_{ac}, (1.8.5)_{ac}, (1.8.4)_{ca}, (1.8.5)_{ca}$ has one of the following two forms.*

1. $\beta_{ac} = 0, \beta_{ca} = 1 - q$ *or* $\beta_{ca} = 0, \beta_{ac} = 1 - q$.
2.
$$\beta_{ac}(\lambda) = \frac{e^{\epsilon} - 1}{\mu_{ac}(\lambda)e^{\epsilon\lambda_{ac}} - 1}, \qquad \beta_{ca}(\lambda) = \frac{e^{\epsilon} - 1}{\mu_{ca}(\lambda)e^{\epsilon\lambda_{ca}} - 1}, \tag{1.12.1}$$

where $\mu_{ac}(\lambda)\mu_{ca}(\lambda) = 1$ and $\mu_{ac}(\lambda)$ is a meromorphic function periodic with respect to shifts of λ by ω_a and ω_c, $\mu_{ac}(\lambda - \omega_a) = \mu_{ac}(\lambda - \omega_c) = \mu_{ac}(\lambda)$.

Proof. Equation $(1.8.4)_{ac}$ can be written in the form

$$(q + \beta_{ac}(\lambda - \omega_a))\,(1 - \beta_{ac}(\lambda - \omega_a))\,\beta_{ac}(\lambda) = (1 - \beta_{ac}(\lambda - \omega_a))\,\beta_{ac}(\lambda - \omega_a).$$

Hence $\beta_{ac}(\lambda) \equiv 1$ or

$$(q + \beta_{ac}(\lambda - \omega_a))\,\beta_{ac}(\lambda) = \beta_{ac}(\lambda - \omega_a). \tag{1.12.2}$$

The function $\beta_{ac}(\lambda)$ cannot be identically equal to 1. In fact, if $\beta_{ac}(\lambda) \equiv 1$, then Eq. $(1.8.4)_{ca}$ gives $0 = -q(1 + q)$ which is impossible since we always assume that $-q \neq p$.

Equation $(1.12.2)_{ac}$ has constant solutions $\beta_{ac}(\lambda) = 0$ or $\beta_{ac}(\lambda) = 1 - q$ which correspond to the first statement of the lemma. Now assume that $\beta_{ac}(\lambda)$ is not constant. Introduce a new meromorphic function $y_{ac}(\lambda) = (\beta_{ac}(\lambda) + q - 1)/\beta_{ac}(\lambda)$. It is easy to see that $y_{ac}(\lambda)\, y_{ca}(\lambda) = 1$. Now Eqs. $(1.12.2)_{ac}$, $(1.12.2)_{ca}$ can be written as

$$y_{ac}(\lambda) = q\, y_{ac}(\lambda - \omega_a), \qquad y_{ac}(\lambda) = q^{-1}\, y_{ac}(\lambda - \omega_c). \tag{1.12.3}$$

Set $\mu_{ac}(\lambda) = y_{ac}(\lambda)\, e^{-\epsilon\lambda_{ac}}$. Then the function $\mu_{ac}(\lambda)$ is periodic with respect to shifts of λ by ω_a and ω_c. We have $\mu_{ac}(\lambda)\,\mu_{ca}(\lambda) = 1$. Returning to functions $\beta_{ac}(\lambda)$ and $\beta_{ca}(\lambda)$ we get the second type of solutions. The lemma is proved. $\square$

Equation (1.8.7) shows that the function $\beta_{ac}(\lambda)$ and hence the function $\mu_{ac}(\lambda)$ is periodic with respect to shifts of λ by ω_b for any b different from a and c.

If the function $\beta_{ac}(\lambda)$ has form (1.12.1), then we say that the function $\mu_{ac}(\lambda)$ is *finite*. If $\beta_{ac}(\lambda) = 1 - q$, then we say that $\mu_{ac}(\lambda) = 0$. If $\beta_{ac}(\lambda) = 0$, then we say that $\mu_{ac}(\lambda) = \infty$. If $\mu_{ac}(\lambda) = 0$, then $\mu_{ca}(\lambda) = \infty$. If $\mu_{ac}(\lambda) = \infty$, then $\mu_{ca}(\lambda) = 0$.

For pairwise distinct a, b, c, we shall say that the equation

$$\mu_{ab}(\lambda)\,\mu_{bc}(\lambda) = \mu_{ac}(\lambda) \tag{1.12.4}$$

holds if one of the following four conditions is satisfied.

(1.12.5) All three functions $\mu_{ab}(\lambda)$, $\mu_{bc}(\lambda)$, $\mu_{ac}(\lambda)$, are finite, and satisfy (1.12.4).
(1.12.6) $\mu_{ac}(\lambda) = \infty$ and at least one of the functions $\mu_{ab}(\lambda)$, $\mu_{bc}(\lambda)$ is equal to ∞.
(1.12.7) $\mu_{ac}(\lambda) = 0$ and at least one of the functions $\mu_{ab}(\lambda)$, $\mu_{bc}(\lambda)$ is equal to 0.
(1.12.8) $\mu_{ac}(\lambda)$ is finite, one of the functions $\mu_{ab}(\lambda)$, $\mu_{bc}(\lambda)$ is equal to zero and the other is equal to infinity.

Lemma 1.6. *For any pairwise distinct a, b, c, Eq. (1.12.4) holds.*

The lemma easily follows from Eq. (1.8.9).
Introduce

$$Y = \{(a, b) \,|\, (a, b) \in \{1, ..., N\}, a \neq b,\ \mu_{ab} = \infty\}. \tag{1.12.9}$$

Then

(1.12.10) If $(a, b) \in Y$ and $(b, c) \in Y$, then $(a, c) \in Y$.
(1.12.11) If (a, b) belongs to Y, then (b, a) does not belong to Y.

By Theorem 3.11 in [EV], there exists a permutation σ of numbers $\{1, ..., N\}$ such that for the new order on $\{1, ..., N\}$, if $(a, b) \in Y$, then $a < b$. Apply to the R-matrix $R(\lambda)$ the gauge transformation (1.4.2) corresponding to the permutation σ. Then the set Y defined by (1.12.9) for the transformed R-matrix is such that if $(a, b) \in Y$, then $a < b$. From now on we denote by $R(\lambda)$ the transformed matrix.

Let $Z = \{(a, b) \,|\, a < b\} - Y$.

Lemma 1.7. *1. If (a, b) belongs to Z, then all pairs $(c, c+1)$, $c = a, a+1, ..., b-1$, belong to Z.*
2. If for some $a, b,\ a < b$, all pairs $(c, c+1)$ for $c = a, a+1, ..., b-1$ belong to Z, then (a, b) belongs to Z.

Lemma 1.7 is a special case of Lemma 3.13 in [EV].

Consider the subset $X \subset \{1, ..., N\}$ of all a such that there exists b with the property that (a, b) or (b, a) belongs to Z.

Introduce a relation on the set X. For any $a \in X$, let a be related to a. For any $a, b \in X, a < b$, let a be related to b if $(a, b) \in Z$. Lemma 1.7 implies that this relation is an equivalence relation. Let $X = X_1 \cup ... \cup X_n$ be the decomposition of X into equivalence classes. Lemma 1.7 implies that $X = X_1 \cup ... \cup X_n$ is a decomposition into a union of disjoint intervals. It is easy to see that the R-matrix $R(\lambda)$ has form (1.6.4) for the constructed decomposition $X = X_1 \cup ... \cup X_n$. Theorem 1.3 is proved. □

1.13. Quantum dynamical R-matrices as an extrapolation of constant quantum R-matrices. Consider the vector representation V of the quantum group $U_q(gl_N)$. Then its R-matrix $\mathcal{R} \in \mathrm{End}(V \otimes V)$ has the form,

$$\mathcal{R} = \sum_{a=1}^{N} E_{aa} \otimes E_{aa} + \sum_{a \neq b} \alpha_{ab}\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}\, E_{ba} \otimes E_{ab}, \tag{1.13.1}$$

where the numbers α_{ab}, β_{ab} are defined as follows: $\alpha_{ab} = q,\ \beta_{ab} = 0$ if $a < b$ and $\alpha_{ab} = 1,\ \beta_{ab} = 1 - q$ if $a > b$. The matrix $\mathcal{R}$ is a constant solution of the quantum dynamical Yang–Baxter equation (1.1.3).

For any permutation σ of numbers $\{1, ..., N\}$ we construct a new constant solution, $\mathcal{R}_\sigma$, of the quantum Yang–Baxter equation. $\mathcal{R}_\sigma$ has form (1.13.1) where the numbers α_{ab}, β_{ab} are defined by the rule: $\alpha_{ab} = q,\ \beta_{ab} = 0$ if $\sigma(a) < \sigma(b)$ and $\alpha_{ab} = 1,\ \beta_{ab} = 1-q$ if $\sigma(a) > \sigma(b)$.

Fix a complex number ϵ such that $e^\epsilon = q$. Consider the matrix

$$R(\lambda) = \sum_{a=1}^{N} E_{aa} \otimes E_{aa} + \sum_{a\neq b} \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a\neq b} \beta_{ab}(\lambda)\, E_{ba} \otimes E_{ab}, \quad (1.13.2)$$

where the functions $\alpha_{ac}(\lambda)$ and $\beta_{ac}(\lambda)$ are defined by

$$\beta_{ab}(\lambda) = \frac{e^\epsilon - 1}{e^{\epsilon\lambda_{ab}} - 1}, \qquad \alpha_{ab} = e^\epsilon + \beta_{ab}.$$

The matrix $R(\lambda)$ is the R-matrix of form (1.6.4) corresponding to data $X = X_1 = \{1, ..., N\}$.

The R-matrix $R(\lambda)$ extrapolates the constant R-matrices $\{\mathcal{R}_\sigma\}$ in the following sense. Let $\rho = (\frac{N-1}{2}, \frac{N-3}{2}, ..., \frac{1-N}{2}) \in \mathfrak{h}^*$. Let $\sigma(\rho)$ be the vector obtained from ρ by permutation of coordinates by σ. Then

$$\lim_{t\to+\infty} R(\frac{t}{\epsilon}\,\sigma(\rho)) = \mathcal{R}_\sigma. \quad (1.13.3)$$

2. Quantum Dynamical R-matrices with Spectral Parameter

2.1. Definition. Let $\mathfrak{h}$ be an abelian finite dimensional Lie algebra. Let $V_i,\ i = 1, 2, 3$, be finite dimensional diagonalizable $\mathfrak{h}$-modules,

$$R_{V_iV_j} : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(V_i \otimes V_j), \qquad 1 \le i < j \le 3,$$

meromorphic functions, γ a nonzero complex number. The equation in $\mathrm{End}(V_1\otimes V_2\otimes V_3)$,

$$\begin{aligned} &R^{12}_{V_1V_2}(z_1-z_2, \lambda - \gamma h^{(3)})\, R^{13}_{V_1V_3}(z_1 - z_3, \lambda)\, R^{23}_{V_2V_3}(z_2 - z_3, \lambda - \gamma h^{(1)}) \\ &\quad = R^{23}_{V_2V_3}(z_2 - z_3, \lambda)\, R^{13}_{V_1V_3}(z_1 - z_3, \lambda - \gamma h^{(2)})\, R^{12}_{V_1V_2}(z_1, -z_2, \lambda) \end{aligned} \quad (2.1.1)$$

is called *the quantum dynamical Yang–Baxter equation with spectral parameter and step* γ (QDYB equation). In what follows we will use a notation $z_{ij} = z_i - z_j$.

A function $R_{V_iV_j} : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(V_i \otimes V_j)$ is called *a function of zero weight* if

$$[R_{V_iV_j}(z, \lambda), h \otimes 1 + 1 \otimes h] = 0 \quad (2.1.2)$$

for all $h \in \mathfrak{h}$, $z \in \mathbb{C}$, $\lambda \in \mathfrak{h}^*$. A solution $\{R_{V_iV_j}\}_{1\le i<j\le 3}$ of the QDYB Eq. (2.1.1) is called a solution of zero weight if each of the functions is of zero weight.

If all the spaces V_i are equal to a space V, then we consider the QDYB equation on one function $R : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$,

$$\begin{aligned} &R^{12}(z_{12},\ \lambda - \gamma h^{(3)})\, R^{13}(z_{13},\ \lambda)\, R^{23}(z_{23},\ \lambda - \gamma h^{(1)}) \\ &\quad = R^{23}(z_{23},\ \lambda)\, R^{13}(z_{13},\ \lambda - \gamma h^{(2)})\, R^{12}(z_{12},\ \lambda). \end{aligned} \quad (2.1.3)$$

A zero weight function R satisfying the QDYB Eq. (2.1.3) is called *a quantum dynamical R-matrix with spectral parameter*. An R-matrix is called *unitary*, if it satisfies the *unitarity condition*

$$R(z,\lambda)\,R^{21}(-z,\lambda) = 1. \tag{2.1.4}$$

2.2. Quantization and quasiclassical limit. Let $x_1,\dots,x_N$ be a basis in $\mathfrak{h}$. The basis defines a linear system of coordinates on $\mathfrak{h}^*$. For any $\lambda \in \mathfrak{h}^*$, set $\lambda_i = x_i(\lambda)$, $i = 1,\dots,N$.

Let $R_\gamma : \mathbb{C}\times\mathfrak{h}^* \to \mathrm{End}(V\otimes V)$ be a smooth family of solutions to Eqs. (2.1.3) and (2.1.4) with step γ such that

$$R_\gamma(z,\lambda) = 1 - \gamma\, r(\lambda) + O(\gamma^2). \tag{2.2.1}$$

Then the function $r : \mathbb{C}\times\mathfrak{h}^* \to \mathrm{End}(V\otimes V)$ satisfies the zero weight condition

$$[r(z,\lambda), h\otimes 1 + 1\otimes h] \;=\; 0 \tag{2.2.2}$$

for all $h\in\mathfrak{h}$, $z\in\mathbb{C}$, $\lambda\in\mathfrak{h}^*$, the unitarity condition

$$r(z,\lambda) + r^{21}(-z,\lambda) = 0 \tag{2.2.3}$$

and the classical dynamical Yang–Baxter equation with spectral parameter (CDYB),

$$\sum_{i=1}^{N} x_i^{(1)}\frac{\partial r^{23}}{\partial x_i}(z_{23},\lambda) \;+\; \sum_{i=1}^{N} x_i^{(2)}\frac{\partial r^{31}}{\partial x_i}(z_{31},\lambda) \;+\; \sum_{i=1}^{N} x_i^{(3)}\frac{\partial r^{12}}{\partial x_i}(z_{12},\lambda) \;+$$
$$[r^{12}(z_{12},\lambda), r^{13}(z_{13},\lambda)] + [r^{12}(z_{12},\lambda), r^{23}(z_{23},\lambda)] \;+\; [r^{13}(z_{13},\lambda), r^{23}(z_{23},\lambda)] \;=\; 0\,. \tag{2.2.4}$$

A function $r(z,\lambda)$ with properties (2.2.2)–(2.2.4) is called a *classical dynamical r-matrix with spectral parameter.*

The function r in (2.2.1) is called *the quasiclassical limit of* R, and the function R is called *a quantization of* r.

Let $U\subset\mathfrak{h}^*$ be an open set, and let $R:\mathbb{C}\times U\to \mathrm{End}(V\otimes V)$ be a zero weight meromorphic function on $\mathbb{C}\times U$. We will say that R is a quantum dynamical R-matrix with spectral parameter on $\mathbb{C}\times U$ if the QDYB equation with spectral parameter is satisfied for R whenever it makes sense.

A classical dynamical r-matrix $r(z,\lambda)$ with spectral parameter on $\mathbb{C}\times U$ is called *quantizable* if there exists a power series in γ,

$$R_\gamma(z,\lambda) = 1 - \gamma\, r(z,\lambda) + \sum_{n=2}^{\infty}\gamma^n r_n(z,\lambda) \tag{2.2.5}$$

convergent for small $|\gamma|$ for any fixed $(z,\lambda)\in\mathbb{C}\times U$, such that $R_\gamma(z,\lambda)$ is a quantum dynamical R-matrix on $\mathbb{C}\times U$ with spectral parameter and step γ.

2.3. R-matrices of gl_N type. Let $\mathfrak{h}$ be an abelian Lie algebra of dimension N. Let V be a diagonalizable $\mathfrak{h}$-module of the same dimension such that its weights $\omega_1,\dots,\omega_N$ form a basis in $\mathfrak{h}^*$. Let $x_1,\dots,x_N$ be the dual basis of $\mathfrak{h}$. Let $v_1,\dots,v_N$ be an eigenbasis for $\mathfrak{h}$ in V such that $x_i v_j = \delta_{ij} v_j$. Then the $\mathfrak{h}$-module $V\otimes V$ has the weight decomposition,

$$V\otimes V = \oplus_{a=1}^{N} V_{aa} \;\oplus \oplus_{a<b} V_{ab}, \tag{2.3.1}$$

where $V_{aa} \;=\; \mathbb{C}\, v_a\otimes v_a$ and $V_{ab} \;=\; \mathbb{C}\, v_a\otimes v_b \oplus \mathbb{C}\, v_b\otimes v_a$.

A quantum dynamical R-matrix with spectral parameter, $R : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$, for these $\mathfrak{h}$ and V will be called an R-matrix of gl_N type.

The zero weight condition implies that the R-matrix preserves the weight decomposition (2.3.1) and has the form

$$R(z,\lambda) = \sum_{a,b=1}^{N} \alpha_{ab}(z,\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}(z,\lambda)\, E_{ba} \otimes E_{ab}, \tag{2.3.2}$$

where $\alpha_{ab}, \beta_{ab} : \mathbb{C} \times \mathfrak{h}^* \to \mathbb{C}$ are suitable meromorphic functions.

2.4. Gauge transformations. Fix a nonzero complex number γ. Let $\psi : \mathfrak{h}^* \to \mathbb{C}$ be a function. For any $a, b = 1, \dots, N$, set

$$\begin{aligned} \partial_a \psi(\lambda) &= \psi(\lambda) - \psi(\lambda - \omega_a), \\ L_{ab}\psi(\lambda) &= \partial_a \psi(\lambda) - \partial_b \psi(\lambda - \omega_a) = \psi(\lambda) - 2\psi(\lambda - \omega_a) + \psi(\lambda - \omega_a - \omega_b). \end{aligned}$$

Introduce gauge transformations of quantum dynamical R-matrices, $R : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$, of type (2.3.2) with step γ.

(2.4.1) Let ψ be a meromorphic function on $\mathfrak{h}^*$. Set

$$\begin{aligned} &R(z,\lambda) \mapsto \\ &\sum_{a,b=1}^{N} e^{z\partial_a\partial_b\psi(\lambda)}\, \alpha_{ab}(z,\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} e^{zL_{ab}\psi(\lambda)}\, \beta_{ab}(z,\lambda)\, E_{ba} \otimes E_{ab}. \end{aligned}$$

(2.4.2) Let $\{\varphi_{ab}\}$ be a meromorphic γ-closed multiplicative 2-form on $\mathfrak{h}^*$. Set

$$\begin{aligned} R(z,\lambda) \mapsto &\sum_{a=1}^{N} \alpha_{aa}(z,\lambda)\, E_{aa} \otimes E_{aa} + \\ &\sum_{a \neq b} \varphi_{ab}(\lambda)\, \alpha_{ab}(z,\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}(z,\lambda)\, E_{ba} \otimes E_{ab}. \end{aligned}$$

(2.4.3) Let the symmetric group S_N act on $\mathfrak{h}^*$ and V by permutation of coordinates. For any permutation $\sigma \in S_N$, set

$$R(z,\lambda) \mapsto (\sigma \otimes \sigma)\, R(z, \sigma^{-1} \cdot \lambda)\, (\sigma^{-1} \otimes \sigma^{-1}).$$

(2.4.4) For a nonzero holomorphic scalar function $c(z)$, set

$$R(z,\lambda) \mapsto c(z)\, R(z,\lambda).$$

(2.4.5) For nonzero complex number b, c and an element $\mu \in \mathfrak{h}^*$, set

$$R(z,\lambda) \mapsto R(bz,\, c\lambda + \mu).$$

It is clear that any gauge transformation of type (2.4.3) transforms a (unitary) quantum dynamical R-matrix with spectral parameter and step γ to a (unitary) quantum dynamical R-matrix with spectral parameter and step γ. Any gauge transformation of type (2.4.4) transforms a quantum dynamical R-matrix with spectral parameter and step γ to a quantum dynamical R-matrix with spectral parameter and step γ. If in addition we have $c(z)c(z^{-1}) = 1$, then the gauge transformation of type (2.4.4) transforms a unitary quantum dynamical R-matrix with spectral parameter and step γ to a unitary quantum dynamical R-matrix with spectral parameter and step γ. Any gauge transformation of type (2.4.5) transforms a (unitary) quantum dynamical R-matrix with spectral parameter and step γ to a (unitary) quantum dynamical R-matrix with spectral parameter and step γ/c.

Theorem 2.1. *Any gauge transformation of type (2.4.1) or (2.4.2) transforms a quantum dynamical R-matrix with spectral parameter and step γ to a quantum dynamical R-matrix with spectral parameter and step γ. Moreover, if the initial quantum dynamical R-matrix is unitary, then the transformed R-matrix is unitary.*

Theorem 2.1 is analogous to Theorem 1.1 and is also proved by direct verification. Namely, in order to prove Theorem 2.1 it is enough to write the QDYB Eq. (2.1.3) in coordinates, as it was done for Eq. (1.1.3) in Sect. 1.8, and then check that if functions $\alpha_{ab}(z,\alpha)$ and $\beta_{ab}(z,\alpha)$ form a solution of the coordinate equations, then the transformed functions also form a solution.

Two R-matrices $R : \mathbb{C}\times\mathfrak{h}^* \to \mathrm{End}(V\otimes V)$ and $R' : \mathbb{C}\times\mathfrak{h}^* \to \mathrm{End}(V\otimes V)$ will be called *equivalent* if one of them can be transformed into another by a sequence of gauge transformations.

2.5. Examples.

The elliptic R-matrix. Fix a point τ in the upper half plane and a complex number γ. Let

$$\theta(z,\tau) = -\sum_{j\in\mathbb{Z}+\frac{1}{2}} e^{\pi i j^2\tau + 2\pi i j(z+\frac{1}{2})}$$

be Jacobi's first theta function.

Let $\mathfrak{h}$ be the Cartan subalgebra of gl_N. It is the abelian Lie algebra of diagonal complex $N\times N$ matrices with the standard basis $x_i = \mathrm{diag}(0,\dots,0,1_i,0,\dots,0)$, $i = 1,\dots,N$. Its dual space $\mathfrak{h}^*$ has the dual basis ω_i.

The vector representation of gl_N is $V = \mathbb{C}^N$ with the standard basis $v_1,\dots,v_N$, $x_i v_j = \delta_{ij} v_j$.

Let $R^{ell}_{\gamma,\tau}(z,\lambda) \in \mathrm{End}(V\otimes V)$ be the R-matrix of the elliptic quantum group $E_{\tau,\gamma/2}(sl_N)$, [F1-2, FV2]. It is a function of the spectral parameter $z\in\mathbb{C}$ and an additional variable $\lambda = (\lambda_1,\dots,\lambda_N)\in\mathfrak{h}^*$. It is a solution of the CDYB Eq. (2.1.3) and satisfies the unitarity condition (2.1.4) [F1-2]. The formula for $R^{ell}_{\gamma,\tau}$ is

$$R^{ell}_{\gamma,\tau}(z,\lambda) = \sum_{a=1}^{N} E_{aa}\otimes E_{aa} + \sum_{a\neq b}\alpha(z,\lambda_{ab})E_{aa}\otimes E_{bb} + \sum_{a\neq b}\beta(z,\lambda_{ab})E_{ba}\otimes E_{ab}, \quad (2.5.1)$$

where $\lambda_{ab} = \lambda_a - \lambda_b$ and the functions α,β are ratios of theta functions:

$$\alpha(z,\lambda) = \frac{\theta(\lambda+\gamma,\tau)\theta(z,\tau)}{\theta(\lambda,\tau)\theta(z-\gamma,\tau)}, \qquad \beta(z,\lambda) = \frac{\theta(z-\lambda,\tau)\theta(\gamma,\tau)}{\theta(z-\gamma,\tau)\theta(\lambda,\tau)}. \quad (2.5.2)$$

Trigonometric R-matrices. Let $X \subset \{1, ..., N\}$ be a subset, $X = X_1 \cup ... \cup X_n$ its decomposition into disjoint intervals.

For any $a, b \in \{1, ..., N\}$, $a \neq b$, we introduce functions $\alpha_{ab}, \beta_{ab} : \mathbb{C} \times \mathfrak{h}^* \to \mathbb{C}$.

If $a, b \in X_k$ for some k, then we set

$$\alpha_{ab}(z,\lambda) = \frac{\sin(\lambda_{ab}+\gamma)\,\sin(z)}{\sin(\lambda_{ab})\,\sin(z-\gamma)}, \qquad \beta_{ab}(z,\lambda) = \frac{\sin(z-\lambda_{ab})\,\sin(\gamma)}{\sin(\lambda_{ab})\,\sin(z-\gamma)}. \tag{2.5.3}$$

Otherwise we set

$$\alpha_{ab}(z,\lambda) = e^{-i\gamma}\,\frac{\sin(z)}{\sin(z-\gamma)}, \qquad \beta_{ab}(z,\lambda) = -\,e^{iz}\,\frac{\sin(\gamma)}{\sin(z-\gamma)} \tag{2.5.4}$$

if $a < b$, and

$$\alpha_{ab}(z,\lambda) = e^{i\gamma}\,\frac{\sin(z)}{\sin(z-\gamma)}, \qquad \beta_{ab}(z,\lambda) = -\,e^{-iz}\,\frac{\sin(\gamma)}{\sin(z-\gamma)} \tag{2.5.5}$$

if $a > b$.

Define a function $R^{trig}_{\cup X_k,\gamma} : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ by

$$R^{trig}_{\cup X_k,\gamma}(z,\lambda) \;=\; \sum_{a=1}^{N} E_{aa} \otimes E_{aa} + \sum_{a \neq b} \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}(\lambda)\, E_{ba} \otimes E_{ab}, \tag{2.5.6}$$

where α_{ab} and β_{ab} are defined by (2.5.3) - (2.5.5).

Rational R-matrices. Let $X \subset \{1, ..., N\}$ be a subset, $X = X_1 \cup ... \cup X_n$ its decomposition into disjoint intervals.

For any $a, b \in \{1, ..., N\}$, $a \neq b$, we shall introduce functions $\alpha_{ab}, \beta_{ab} : \mathbb{C} \times \mathfrak{h}^* \to \mathbb{C}$.

If $a, b \in X_k$ for some k, then we set

$$\alpha_{ab}(z,\lambda) = \frac{(\lambda_{ab}+\gamma)\,z}{\lambda_{ab}\,(z-\gamma)}, \qquad \beta_{ab}(z,\lambda) = \frac{(z-\lambda_{ab})\,\gamma}{\lambda_{ab}\,(z-\gamma)}. \tag{2.5.7}$$

Otherwise we set

$$\alpha_{ab}(z,\lambda) = \frac{z}{z-\gamma}, \qquad \beta_{ab}(z,\lambda) = -\,\frac{\gamma}{z-\gamma}. \tag{2.5.8}$$

Define a function $R^{rat}_{\cup X_k,\gamma} : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ by

$$R^{rat}_{\cup X_k,\gamma}(z,\lambda) \;=\; \sum_{a=1}^{N} E_{aa} \otimes E_{aa} + \sum_{a \neq b} \alpha_{ab}(\lambda)\, E_{aa} \otimes E_{bb} + \sum_{a \neq b} \beta_{ab}(\lambda)\, E_{ba} \otimes E_{ab}, \tag{2.5.9}$$

where α_{ab} and β_{ab} are defined by (2.5.7) - (2.5.8).

Theorem 2.2. *For any subset $X \subset \{1, ..., N\}$ and its decomposition $X = X_1 \cup ... \cup X_n$ into disjoint intervals, the functions $R^{trig}_{\cup X_k,\gamma}$ and $R^{rat}_{\cup X_k,\gamma}$ are zero weight solutions of the QDYB Eq. (2.1.3) satisfying the unitarity condition (2.1.4).*

Proof. According to [F1-2] the elliptic R-matrix $R^{ell}_{\gamma,\tau}$ is a zero weight solution of the QDYB Eq. (2.1.3) satisfying the unitarity condition (2.1.4).

If $q = e^{2\pi i\tau} \to 0$, then $\theta(z) \sim 2q^{1/8}\sin(\pi z)$.

These two facts show that the R-matrix $R^0(z,\lambda)$ of the form (2.3.2), with

$$\alpha_{ab}(z,\lambda) = \frac{\sin(\lambda_{ab}+\gamma)\,\sin(z)}{\sin(\lambda_{ab})\,\sin(z-\gamma)}, \qquad \beta_{ab}(z,\lambda) = \frac{\sin(z-\lambda_{ab})\,\sin(\gamma)}{\sin(\lambda_{ab})\,\sin(z-\gamma)}$$

for all $a \neq b$ and $\alpha_{aa} \equiv 1$ for all a, is a zero weight solution of the QDYB Eq. (2.1.3) satisfying the unitarity condition (2.1.4).

For any fixed $d \in \mathfrak{h}^*$, the R-matrix $R^0(z,\lambda+d)$ is also a zero weight solution of the QDYB Eq. (2.1.3) satisfying the unitarity condition (2.1.4).

Fix a subset $X \subset \{1,\dots,N\}$ and its decomposition $X = X_1 \cup \dots \cup X_n$ into disjoint intervals. It is easy to see that there exists a sequence of elements $d_i \in \mathfrak{h}^*$, $i = 1,2,\dots$ such that the R-matrix $R^0(z,\lambda+d_i)$ has a limit when i tends to infinity, and this limit is equal to $R^{trig}_{\cup X_k,\gamma}(z,\lambda)$. This observation shows that $R^{trig}_{\cup X_k,\gamma}(z,\lambda)$ is a zero weight solution of the QDYB Eq. (2.1.3) satisfying the unitarity condition (2.1.4).
Rescale the R-matrix $R^{trig}_{\cup X_k,\gamma}(z,\lambda)$ and consider a matrix $R_\epsilon(z,\lambda) = R^{trig}_{\cup X_k,\epsilon\gamma}(\epsilon z,\epsilon\lambda)$, where ϵ is a new parameter. Let γ, z, λ be fixed and let ϵ tends to 0. Then the limit of $R_\epsilon(z,\lambda)$ is equal to $R^{rat}_{\cup X_k,\gamma}(z,\lambda)$. Hence, $R^{rat}_{\cup X_k,\gamma}(z,\lambda)$ is a zero weight solution of the QDYB Eq. (2.1.3) satisfying the unitarity condition (2.1.4). Theorem 2.2 is proved. □

2.6. Quantization of classical dynamical r-matrices of gl_N type with spectral parameter. Let V be the N dimensional $\mathfrak{h}$-module considered in Sect. 2.3. Let $r : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ be a zero weight meromorphic function satisfying CDYB (2.2.4) and the unitarity condition (2.2.3).

The zero weight condition implies that r has the form

$$r(z,\lambda) = \sum_{a,b=1}^{N} \alpha_{ab}(z,\lambda)\,E_{aa}\otimes E_{bb} + \sum_{a\neq b} \beta_{ab}(z,\lambda)\,E_{ab}\otimes E_{ba}. \tag{2.6.1}$$

Assume that the function r satisfies also the *residue condition*

$$\mathrm{Res}_{z=0}\, r(\lambda,z) \;=\; \epsilon\, P \;+\delta\,\mathrm{Id}.$$

Here $P \in \mathrm{End}(V\otimes V)$ is the permutation of factors and $\mathrm{Id} \in \mathrm{End}(V\otimes V)$ is the identity operator. The complex numbers ϵ and δ are called *the coupling constant* and *the secondary coupling constant*, respectively. We always assume that the coupling constant ϵ is not equal to zero.

We recall a classification of such r-matrices. First we introduce gauge transformations of classical dynamical r-matrices with spectral parameter.

(2.6.2) Let $\psi = \sum_{a,b}\psi_{ab}(\lambda)dx_a \wedge dx_b$ be a closed meromorphic differential 2-form on $\mathfrak{h}^*$. Set

$$r(z,\lambda) \;\mapsto\; r(z,\lambda) + \sum_{a\neq b}\psi_{ab}(\lambda)\,E_{aa}\otimes E_{bb}.$$

(2.6.3) For a holomorphic function $\psi : \mathfrak{h}^* \to \mathbb{C}$, set

$$r(z,\lambda) \mapsto \sum_{a,b=1}^{N} (\alpha_{ab}(z,\lambda) + z\,\frac{\partial^2\psi}{\partial x_a\,\partial x_b}(\lambda))\,E_{aa}\otimes E_{bb} + \sum_{a\neq b} \beta_{ab}(z,\lambda)\,e^{z(\frac{\partial\psi}{\partial x_a}(\lambda)-\frac{\partial\psi}{\partial x_b}(\lambda))}\,E_{ab}\otimes E_{ba}.$$

(2.6.4) For $\mu\in\mathfrak{h}^*$, set

$$r(z,\lambda) \mapsto r(z,\lambda+\mu).$$

(2.6.5) Let the symmetric group S_N act on $\mathfrak{h}^*$ and V by permutation of coordinates. For any permutation $\sigma\in S_N$, set

$$r(z,\lambda) \mapsto (\sigma\otimes\sigma)\,r(z,\sigma^{-1}\cdot\lambda)\,(\sigma^{-1}\otimes\sigma^{-1}).$$

(2.6.6) For a nonzero complex number c, set

$$r(z,\lambda) \mapsto c\,r(z,c\lambda).$$

(2.6.7) For an odd scalar meromorphic function $f(z)$, $f(z)+f(-z)=0$, set

$$r(z,\lambda) \mapsto r(z,\lambda) + f(z)\,\mathrm{Id}.$$

Any gauge transformation transforms a classical dynamical r-matrix with spectral parameter to a classical dynamical r-matrix with spectral parameter [EV]. Two classical dynamical r-matrices $r(z,\lambda)$ and $r'(z,\lambda)$ will be called *equivalent* if one of them can be transformed into another by a sequence of gauge transformations.

The gauge transformations of quantum dynamical R-matrices with spectral parameter described in Sect. 2.4 are analogs of the gauge transformations of classical dynamical r-matrices with spectral parameter.

Classification of the classical dynamical r-matrices with spectral parameter.

The elliptic r-matrix. Fix a point τ in the upper half plane. Introduce the functions

$$\sigma_w(z) = \frac{\theta(w-z,\tau)\,\theta'(0,\tau)}{\theta(w,\tau)\,\theta(z,\tau)}, \qquad \rho(z) = \frac{\theta'(z,\tau)}{\theta(z,\tau)},$$

where $\theta'(z,\tau) = \frac{\partial\theta(z,\tau)}{\partial z}$. Set

$$r_\tau^{ell}(z,\lambda) = \rho(z)\sum_{a=1}^{N} E_{aa}\otimes E_{aa} + \sum_{a\neq b}\sigma_{\lambda_{ba}}(z)E_{ab}\otimes E_{ba}. \tag{2.6.8}$$

For every $\tau\in\mathbb{C}$, Im $\tau>0$, the function $r_\tau^{ell}(z,\lambda)$ is a classical dynamical r-matrix with spectral parameter z, coupling constant $\epsilon=1$ and secondary constant $\delta=0$, [FW].

Trigonometric r-matrices. Let $X\subset\{1,...,N\}$ be a subset, $X = X_1\cup...\cup X_n$ its decomposition into disjoint intervals.

For any $a,b\in\{1,...,N\}$, $a\neq b$, we introduce a function $\beta_{ab} : \mathbb{C}\oplus\mathfrak{h}^*\to\mathbb{C}$.

If $a,b\in X_k$ for some k, then we set

$$\beta_{ab}(z,\lambda) = -\,\frac{\sin(\lambda_{ab}+z)}{\sin(\lambda_{ab})\,\sin(z)}.$$

Otherwise we set

$$\beta_{ab}(z,\lambda) = \frac{e^{-iz}}{\sin(z)}, \qquad \text{for } a < b, \qquad \beta_{ab}(z,\lambda) = \frac{e^{iz}}{\sin(z)} \qquad \text{for } a > b.$$

We introduce a trigonometric r-matrix $r^{trig}_{\cup X_k,\gamma} : \mathbb{C} \oplus \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ by

$$r^{trig}_{\cup X_k}(z,\lambda) = \mathrm{cotan}\,(z) \sum_{a=1}^{N} E_{aa} \otimes E_{aa} + \sum_{a \neq b} \beta_{ab}(z,\lambda)\; E_{ab} \otimes E_{ba}, \tag{2.6.9}$$

where $\mathrm{cotan}\,(z) = \cos(z)\,/\sin(z)$.

Rational r-matrices. Let $X \subset \{1, ..., N\}$ be a subset, $X = X_1 \cup ... \cup X_n$ its decomposition into disjoint intervals. Set

$$r^{rat}_{\cup X_k}(z,\lambda) = \frac{P}{z} + \sum_{k=1}^{n} \sum_{a,b \in X_k,\, a \neq b} \frac{1}{\lambda_{ab}} E_{ab} \otimes E_{ba}\,. \tag{2.6.10}$$

Theorem 2.3. *1. For every subset $X \subset \{1, ..., N\}$ and its decomposition $X = X_1 \cup ... \cup X_n$ into disjoint intervals, the matrices $r^{trig}_{\cup X_k}$ and $r^{rat}_{\cup X_k}$ are classical dynamical r-matrices with spectral parameter.*
2. Every classical dynamical r-matrix $r : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ with nonzero coupling constant is equivalent to one of the matrices (2.6.8)–(2.6.10).

Theorem 2.3 follows from [EV].

Theorem 2.4. *Let $r(z,\lambda)$ be a unitary classical dynamical r-matrix with spectral parameter and nonzero coupling constant, meromorphic on $\mathbb{C} \times U$, where U is an open polydisc. Assume that for any $\lambda \in U$ there exists $z \in \mathbb{C}$ such that r is holomorphic at (λ, z). Then r can be quantized to a unitary quantum dynamical R-matrix R_γ on $\mathbb{C} \times U$ of gl_N type. Moreover, if a classical dynamical r-matrix with spectral parameter and nonzero coupling constant is equivalent to the elliptic r-matrix (2.6.8) (resp., a trigonometric r-matrix (2.6.9) or a rational r-matrix (2.6.10)), then it has a quantization equivalent to the elliptic R-matrix (2.5.1) (resp., a trigonometric R-matrix (2.5.6) or a rational R-matrix (2.5.9)).*

Proof. We shall prove that if a classical dynamical r-matrix is equivalent to the elliptic r-matrix (2.6.8), then it is quantizable to a quantum dynamical R-matrix equivalent to the elliptic R-matrix (2.5.1). The other statements of the theorem are proved similarly.

Compute the quasiclassical limit of $R^{ell}_{\gamma,\tau}(z,\lambda)$. For the functions $\alpha(z,\lambda,\gamma)$ and $\beta(z,\lambda,\gamma)$ defined in (2.5.2), we have

$$\lim_{\gamma\to 0} \frac{\alpha(z,\lambda,\gamma) - 1}{\gamma} = \frac{\theta'(\lambda)}{\theta(\lambda)} + \frac{\theta'(z)}{\theta(z)}, \qquad \lim_{\gamma\to 0} \frac{\beta(z,\lambda,\gamma)}{\gamma} = \frac{\theta'(0)\theta(z-\lambda)}{\theta(\lambda)\theta(z)}.$$

Hence

$$R^{ell}_{\gamma,\tau}(z,\lambda) = 1 - \gamma\, r(z,\lambda) + O(\gamma^2),$$

where

$$r(z,\lambda) = -\sum_{a\neq b}\left(\frac{\theta'(\lambda_{ab})}{\theta(\lambda_{ab})} + \frac{\theta'(z)}{\theta(z)}\right)E_{aa}\otimes E_{bb} - \sum_{a\neq b}\frac{\theta'(0)\theta(z-\lambda_{ab})}{\theta(\lambda_{ab})\theta(z)}E_{ba}\otimes E_{ab}$$
$$= -\sum_{a\neq b}\left(\frac{\theta'(\lambda_{ab})}{\theta(\lambda_{ab})} + \frac{\theta'(z)}{\theta(z)}\right)E_{aa}\otimes E_{bb} + \sum_{a\neq b}\sigma_{\lambda_{ba}}(z)\,E_{ab}\otimes E_{ba}.$$

Now applying to the r-matrix $r(z,\lambda)$ the transformation (2.6.2) corresponding to the closed differential 2-form

$$\sum_{a\neq b}\frac{\theta'(\lambda_{ab})}{\theta(\lambda_{ab})}\,dx_a\wedge dx_b,$$

and then applying to the result the transformation (2.6.7) corresponding to the function $f(z) = \theta'(z)/\theta(z)$ we get the matrix $r^{ell}_\tau(z,\lambda)$ defined by (2.6.8). This remark and Lemma 1.1 easily imply the statement of the Theorem concerning the elliptic r-matrix. Theorem 2.4 is proved. □

Remark. The elliptic quantum dynamical R-matrix (2.5.1) was invented by G. Felder [F1-2] as a quantization of the classical dynamical r-matrix (2.6.8).

2.7. Formal dynamical R-matrices and gauge fixing conditions. Let $R_\gamma(z,\lambda) = 1-\gamma r(z,\lambda)+\sum_{n\geq 2}\gamma^n r_n(z,\lambda)$ be a power series in λ and γ, whose coefficients are meromorphic functions of z, taking values in End($V\otimes V$). The series R_γ is called a formal quantum dynamical R-matrix of gl_N type with spectral parameter and step γ if it is of zero weight and satisfies the quantum dynamical Yang–Baxter equation. In addition, R_γ is called unitary if it satisfies the unitarity condition (2.1.4). In this section for brevity we will refer to formal quantum dynamical R-matrices of gl_N type with spectral parameter and step γ as "formal dynamical R-matrices". As we know, any such R-matrix has form (2.3.2).

The theory of formal dynamical R-matrices is completely analogous to the theory of analytic dynamical R-matrices. In particular, one can define formal classical dynamical r-matrices and formal gauge transformations in an obvious way. If $R_\gamma = 1-\gamma r + \ldots$ is a (unitary) formal dynamical R-matrix, then r is a (unitary) formal dynamical r-matrix.

An example of a formal dynamical R-matrix is the Taylor expansion of an analytic dynamical R-matrix $R_\gamma(z,\lambda)$ at a point $\gamma = 0, \lambda = \lambda_0$, such that R is regular at this point for generic values of z.

Proposition 2.1. *Let $R_\gamma = 1-\gamma r+\ldots$ be a unitary formal dynamical R-matrix, and $z_0\in\mathbb{C}$ a point where R_γ is regular. Let α_{ab}, β_{ab} be the matrix coefficients of R_γ, see (2.3.2). Then R_γ can be transformed, by a sequence of formal gauge transformations, to a unitary formal dynamical R-matrix satisfying the following conditions:*

1) for every a,b,c, the ratio $\frac{\alpha_{ab}(z,\lambda-\gamma\omega_c)}{\alpha_{ab}(z,\lambda)}$ is independent of z;
2) for every $a<b$, $\alpha_{ab}(z_0,\lambda) = 1$;
3) the coefficient $\alpha_{11}(z,\lambda)$ is independent of z.

Proof. The QDYB equation with spectral parameter implies the equation

$$\alpha_{ab}(u,\lambda-\gamma\omega_c)\,\alpha_{ac}(u+v,\lambda)\,\alpha_{bc}(v,\lambda-\gamma\omega_a) = \alpha_{bc}(v,\lambda)\,\alpha_{ac}(u+v,\lambda-\gamma\omega_b)\,\alpha_{ab}(u,\lambda) \tag{2.7.1}$$

for any a,b,c. Therefore, we have

$$\frac{\alpha_{ab}(u,\lambda-\gamma\omega_c)}{\alpha_{ab}(u,\lambda)}=H_{abc}(\lambda)e^{D_{abc}(\lambda)u}, \tag{2.7.2}$$

for suitable power series $H_{abc}(\lambda),\ D_{abc}(\lambda)$.

Lemma 2.1. *There exists a formal power series $\psi(\lambda)$ such that $D_{abc}=\partial_a\partial_b\partial_c\psi$.*

Proof. From (2.7.1) it follows that D_{abc} is symmetric. From (2.7.2) it follows that $\partial_d D_{abc}$ is symmetric. The rest of the proof of the lemma follows from the basic theory of difference equations with infinitesimal shift. □

Corollary 2.1. *Performing a gauge transformation (2.4.1), we can arrange $D=0$, i.e. condition 1.*

From now on we assume that $D=0$, i.e.

$$\frac{\alpha_{ab}(u,\lambda-\gamma\omega_c)}{\alpha_{ab}(u,\lambda)}=H_{abc}(\lambda). \tag{2.7.3}$$

This implies that $\alpha_{ab}(u,\lambda)=\alpha^1_{ab}(u)\alpha^2_{ab}(\lambda)$, where α^i_{ab} are new functions.

Consider the multiplicative 2-form φ defined by $\varphi_{ab}(\lambda)=\alpha_{ab}(z_0,\lambda)$, $a<b$. It follows from (2.7.1) that $d_\gamma\varphi=0$. Therefore, by a gauge transformation of type (2.4.2) we can arrange $\varphi=1$, i.e. condition 2.

It remains to arrange condition 3. By (2.7.3), $\alpha_{11}(z,\lambda)=f(z)g(\lambda)$ for a suitable formal power series $g(\lambda)$ and a meromorphic function $f(z)$ such that $f(z)f(-z)=1$. Applying transformation (2.4.4) with $c(z)=1/f(z)$, we get condition 3. The proposition is proved. □

We will call conditions 1–3 the gauge fixing conditions.

2.8. Classification of unitary formal dynamical R-matrices with elliptic quasiclassical limit. We will say that a formal classical dynamical r-matrix r is of elliptic, trigonometric, or rational type if it is gauge equivalent (by formal gauge transformations) to an r-matrix of the form (2.6.8), (2.6.9),(2.6.10), respectively, expanded near a point $\lambda_0\in\mathfrak{h}^*$. It follows from [EV] that any formal classical dynamical r-matrix satisfying the residue condition with coupling constant $\epsilon\neq 0$ is of elliptic, trigonometric, or rational type.

Theorem 2.5. *Let $R_\gamma=1-\gamma r+O(\gamma^2)$ be a unitary formal dynamical R-matrix whose quasiclassical limit r is of the elliptic type. Then there exist a point $\lambda_0\in\mathfrak{h}^*$ and a power series $\tau(\gamma)=\tau_0+O(\gamma)\in\mathbb{C}[[\gamma]]$, $Im(\tau_0)>0$ such that the R-matrix R_γ can be transformed, by a sequence of formal gauge transformations, into the Taylor series of $R^{ell}_{\gamma,\tau(\gamma)}(z,\lambda-\lambda_0)$, where $R^{ell}_{\gamma,\tau}(z,\lambda)$ is the elliptic R-matrix (2.5.1).*

The proof of this Theorem occupies the next section.

2.9. Proof of Theorem 2.5 . Let X^0 be the space of unitary formal classical dynamical r-matrices with spectral parameter and a nonzero coupling constant. Let X^0_* be the subset of elements of X^0 which satisfy the following gauge fixing conditions:

1c) $\frac{\partial}{\partial\lambda_c}\alpha_{ab}(z,\lambda)$ is independent of z;
2c) $\alpha_{ab}(z_0,\lambda)=0$, $a<b$;
3c) $\alpha_{11}(z,\lambda)$ is independent of z (these conditions are quasiclassical analogues of conditions 1–3 above).

According to the results of [EV], the space X_*^0 is a connected, finite-dimensional complex manifold (with singularities), and any element of X^0 is gauge equivalent to an element of X_*^0. (i.e. X_*^0 is a "cross-section"). Moreover, since $r \in X_*^0$ is of elliptic type, the manifold X_*^0 is smooth at r.

Let X be the space of unitary formal quantum dynamical R-matrices with spectral parameter, and X_* the subset of elements of X satisfying the gauge fixing conditions 1-3.

As we have shown in Sect. 2.7, we can assume that our family R_γ is in X_*. In this case, $r \in X_*^0$.

Now let us prove the statement of the theorem modulo γ^{m+1} by induction in m.

For $m = 1$, the theorem is a tautology. Suppose we know the theorem for $m = k \geq 1$, and want to prove it for $m = k + 1$.

We have a polynomial $R_k = 1 - \gamma r + ... + \gamma^k r_k$ which satisfies the condition $R_k \in X_*$ modulo γ^{k+1}. We know that R_k satisfies the conclusion of Theorem 2.5 modulo γ^{k+1}, i.e. is of the form (2.5.1) modulo γ^{k+1}.

Consider any extension of this polynomial to order $k+1$: $R_{k+1} = R_k + \gamma^{k+1} r_{k+1}$. The condition that $R_{k+1} \in X_*$ modulo γ^{k+2} can be expressed as a nonhomogeneous linear equation with respect to r_{k+1} having the form $A\, r_{k+1} = s_{k+1}(r_k, ..., r_2, r)$, where A is a linear operator.

The obvious, but crucial observation now is the following.

Lemma 2.2. *Ker $A = T_r X_*^0$, where $T_r X_*^0$ denotes the tangent space at the point r.*

Proof. Indeed, it is easy to see by an explicit calculation that the linear homogeneous equation $A\rho = 0$ is nothing else but the equation for a tangent vector to X_*^0 at the point r. □

Corollary 2.2. *The dimension of the space of solutions of $Ar_{k+1} = s_{k+1}$ is less than or equal to K = dim(X_*^0).*

However, by Theorem 2.4, we already have a family of elements of X_* with K parameters – the quantizations of elements of X_*^0. Therefore, using dimension arguments, we obtain that if r_{k+1} satisfies $Ar_{k+1} = s_{k+1}$, then $R_{k+1}(\gamma)$ has to be in this K-parametric family, which completes the induction step.

The theorem is proved. □

Remark. If r is not elliptic but rational or trigonometric, the result of Theorem 2.5 can be generalized, in the sense that formal dynamical R-matrices $R_\gamma = 1 - \gamma r + ...$ with rational or trigonometric r can be explicitly classified up to gauge transformations by the same method as above. However, both the statement and the proof in this case are more delicate, as the manifold X_0^* may now be singular at r, and it is necessary to describe carefully these singularities. For simplicity one should first consider the case dim $V = 2$, and then generalize to an arbitrary dimension. We are not giving this argument here.

3. Quantum Dynamical R-matrices and Monoidal Categories

Let us briefly recall some standard notions of the category theory [Mac, Kass].

Recall that a morphism $a : F \to G$ of two functors from a category $\mathcal{C}$ to a category $\mathcal{C}'$ is a choice of a morphism $a_X : F(X) \to G(X)$ for any object X in $\mathcal{C}$, such that for any two objects $X, Y \in \mathcal{C}$ and any morphism $g : X \to Y$ we have $a_Y \circ F(g) = G(g) \circ a_X$. An endomorphism of a functor is just a morphism of this functor into itself.

Recall that a *monoidal category* is a category $\mathcal{C}$ with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (i.e. a functor with respect to each factor), called the tensor product, and an isomorphism of functors $\Phi : (* \otimes *) \otimes * \to * \otimes (* \otimes *)$, called the associativity isomorphism, such that Φ satisfies the pentagon relation, and there exists a unit object $\mathbf{1} \in \mathbb{C}$ with certain properties. A *braided monoidal category* is a monoidal category with a functorial isomorphism $\beta : \otimes \to \otimes^{op}$ called the commutativity isomorphism, which satisfies the hexagon relations. A braided category is called *symmetric* if $\beta^2 = 1$. A monoidal category will be called a *tensor category* if it has an additive structure $\oplus$, such that $\otimes$ is distributive with respect to $\oplus$.

3.1. The category of $\mathfrak{h}$*-vector spaces.* Let $\mathfrak{h}$ be a finite-dimensional commutative Lie algebra over $\mathbb{C}$. Let $M_{\mathfrak{h}^*}$ denote the field of meromorphic functions on $\mathfrak{h}^*$. Fix a complex number γ.

Let $\mathcal{V}_{\mathfrak{h}}$ denote the category whose objects are diagonalizable $\mathfrak{h}$-modules, and morphisms are defined by $\mathrm{Hom}_{\mathcal{V}_{\mathfrak{h}}}(X, Y) = \mathrm{Hom}_{\mathfrak{h}}(X, Y \otimes_{\mathbb{C}} M_{\mathfrak{h}^*})$.

Let $W \otimes *$ be the functor of multiplication by W. For any $W \in \mathcal{V}_{\mathfrak{h}}$ and $f \in \mathrm{End}_{\mathcal{V}_{\mathfrak{h}}}(W)$, define $f(* - \gamma h^{(2)}) \in \mathrm{End}(W \otimes *)$ by the formula

$$f_V(\lambda - \gamma h^{(2)})(w \otimes v) = f_V(\lambda - \gamma\mu) w \otimes v, \tag{3.1.1}$$

for any $v \in V$ of weight μ (cf. Sect. 1.1).

Define a bifunctor $\bar{\otimes} : \mathcal{V}_{\mathfrak{h}} \times \mathcal{V}_{\mathfrak{h}} \to \mathcal{V}_{\mathfrak{h}}$ as follows. For any $X, Y \in \mathcal{V}_{\mathfrak{h}}$, define $X \bar{\otimes} Y$ to be the usual tensor product $X \otimes Y$. For any two morphisms $f : X \to X'$, $g : Y \to Y'$ define the morphism $f \bar{\otimes} g : X \otimes Y \to X' \otimes Y'$ by the formula

$$f \bar{\otimes} g(\lambda) = f^{(1)}(\lambda - \gamma h^{(2)})(1 \otimes g(\lambda)). \tag{3.1.2}$$

It is easy to see that the category $\mathcal{V}_{\mathfrak{h}}$ equipped with the bifunctor $\bar{\otimes}$ is a tensor category (cf. [Mac]). Indeed, the functors $* \bar{\otimes} (* \bar{\otimes} *)$ and $(* \bar{\otimes} *) \bar{\otimes} *$ are equal, so $\bar{\otimes}$ is associative. Moreover, the object $\mathbf{1} = \mathbb{C}$ (the trivial $\mathfrak{h}$-module), satisfies the condition $\mathbf{1} = \mathbf{1} \bar{\otimes} \mathbf{1}$, and the functors $X \to \mathbf{1} \bar{\otimes} X$, $X \to X \bar{\otimes} \mathbf{1}$ are autoequivalences of $\mathcal{V}_{\mathfrak{h}}$, so $\mathbf{1}$ is an identity object in $\mathcal{V}_{\mathfrak{h}}$.

We will call this monoidal category the category of $\mathfrak{h}$-vector spaces. If $\mathfrak{h} = 0$, the category $\mathcal{V}_{\mathfrak{h}}$ coincides with the category of complex vector spaces.

If $\gamma = 0$, the category $\mathcal{V}_{\mathfrak{h}}$ is equivalent, as a tensor category, to the category of diagonalizable $\mathfrak{h}$-modules, with scalars extended from $\mathbb{C}$ to $M_{\mathfrak{h}^*}$. This case is not very interesting, so from now on we will assume that $\gamma \neq 0$.

The category $\mathcal{V}_{\mathfrak{h}}$ depends on γ, but the categories with different nonzero γ are obviously equivalent. We will suppress the dependence of $\mathcal{V}_{\mathfrak{h}}$ on γ in the notation.

Remark. It is clear that for any two objects $X, Y \in \mathcal{V}_{\mathfrak{h}}$ the permutation operator $\sigma_{XY} : X \bar{\otimes} Y \to Y \bar{\otimes} X$ is an isomorphism in $\mathcal{V}_{\mathfrak{h}}$. However, if $\mathfrak{h} \neq 0$, then this isomorphism is not functorial in X and Y. In fact, it is quite easy to see that if $\mathfrak{h} \neq 0$, there is no functorial isomorphism between $X \bar{\otimes} Y$ and $Y \bar{\otimes} X$: such an isomorphism would have to conjugate $f^{(1)}(\lambda - \gamma h^{(2)})(1 \otimes g(\lambda))$ into $g^{(1)}(\lambda - \gamma h^{(2)})(1 \otimes f(\lambda))$ for any f, g, which is impossible, since there is no relation between $f(\lambda)$ and $f(\lambda - \gamma\mu)$ for a generic function f. Thus, the category $\mathcal{V}_{\mathfrak{h}}$ is a tensor category which in general does not admit a braided structure.

3.2. Dynamical quantum R-matrices and tensor functors. It is known from the theory of quantum groups that if we are given a braided monoidal category $\mathcal{B}$, a symmetric tensor category $\mathcal{V}$, and a tensor functor $F : \mathcal{B} \to \mathcal{V}$, then for any object $X \in \mathcal{B}$ we can construct an element $R(\mathcal{B}, F, X) \in \mathrm{Aut}_{\mathcal{V}}(F(X) \otimes F(X))$ which satisfies the quantum Yang–Baxter equation, by the formula

$$R(\mathcal{B}, F, X) = \sigma F(\beta_{XX}), \tag{3.2.1}$$

where

$$\beta_{XY} : X \otimes Y \to Y \otimes X$$

is the braiding in $\mathcal{B}$, and σ is the permutation. For brevity we will write $R(\mathcal{B}, F, X)$ as R_X.

Suppose now that we are given a braided monoidal category $\mathcal{B}$ and a tensor functor $F : \mathcal{B} \to \mathcal{V}_{\mathfrak{h}}$. Observe that formula (3.2.1) makes sense in this situation. However, since σ_{XY} is not a functorial isomorphism, we should not expect R_X to be a solution to the quantum Yang–Baxter equation. Still, it turns out that R_X satisfies a modified version of the quantum Yang–Baxter equation, namely, the quantum dynamical Yang–Baxter Eq. (1.1.3).

Theorem 3.1. *The element R_X satisfies the quantum dynamical Yang–Baxter Eq. (1.1.3) in* $\mathrm{End}_{\mathcal{V}_{\mathfrak{h}}}(F(X)^{\bar{\otimes} 3})$.

Proof. We start with the braid relation

$$(\beta \otimes 1)(1 \otimes \beta)(\beta \otimes 1) = (1 \otimes \beta)(\beta \otimes 1)(1 \otimes \beta). \tag{3.2.2}$$

Applying the functor F to (3.2.2), and using the definition of the tensor product of morphisms in $\mathcal{V}_{\mathfrak{h}}$, we get (1.1.3). □

3.3. Representations of a quantum dynamical R-matrix. The notions discussed in this section were introduced in [F1, F2, FV1].

Let $R : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ be a quantum dynamical R-matrix (see Chapter 1).

Definition. *A representation of R is an object $W \in \mathcal{V}_{\mathfrak{h}}$ endowed with an invertible morphism $L \in \mathrm{End}_{\mathcal{V}_{\mathfrak{h}}}(V \bar{\otimes} W)$, called the L-operator, such that*

$$\begin{aligned} R^{12}(\lambda - \gamma h^{(3)})\, L^{13}(\lambda)\, L^{23}(\lambda - \gamma h^{(1)}) \\ = L^{23}(\lambda)\, L^{13}(\lambda - \gamma h^{(2)})\, R^{12}(\lambda), \end{aligned} \tag{3.3.1}$$

in $\mathrm{End}_{\mathcal{V}_{\mathfrak{h}}}(V \bar{\otimes} V \bar{\otimes} W)$.

Examples. 1. The trivial representation: $W = \mathbb{C}$, $L = \mathrm{Id}$.
2. The basic representation: $W = V$, $L = R$.

Let (W, L) be a representation of R. Let $A \in \mathrm{Aut}_{\mathcal{V}_{\mathfrak{h}}}(W)$. Let $L^A(\lambda) := (1 \otimes A(\lambda)^{-1}) L(\lambda)(1 \otimes A(\lambda - \gamma h^{(1)}))$.

Lemma 3.1. (W, L^A) *is a representation of R.*

Proof. Straightforward. □

Let (W, L_W) and (U, L_U) be representations of R.

Definition. *A morphism $A \in Hom_{\mathcal{V}_{\mathfrak{h}}}(W, U)$ is called an R-morphism if*

$$(1 \otimes A(\lambda))L_W(\lambda) = L_U(\lambda)(1 \otimes A(\lambda - \gamma h^{(1)})), \tag{3.3.2}$$

Denote the space of R-morphisms from W to U by $\mathrm{Hom}_R(W,U)$.

It is clear that the composition of two R-morphisms is again an R-morphism. Thus, representations of R form a category, which we denote by $Rep(R)$. This category is additive, with the obvious notion of direct sum.

Definition. *The tensor product of W and U is the pair $(W\otimes U, L_{W\otimes U})$, where*

$$L_{W\otimes U}(\lambda) = L_W^{12}(\lambda - \gamma h^{(3)})L_U^{13}(\lambda). \tag{3.3.3}$$

Lemma 3.2. *$(W\otimes U, L_{W\otimes U})$ is a representation of R.*

Proof. Straightforward. □

It is clear that $(W\otimes U)\otimes X = W\otimes(U\otimes X)$.

Lemma 3.3. *If W, W', U, U' are representations of R and f, g are R-morphisms then $f\bar{\otimes} g$ is an R-morphism.*

Proof. Straightforward. □

Thus, we have equipped the category $Rep(R)$ with a structure of a tensor category. Moreover, the forgetful functor $F: Rep(R)\to \mathcal{V}_{\mathfrak{h}}$ is naturally a tensor functor.

Theorem 3.1 shows that any pair $(\mathcal{B},\ F:\mathcal{B}\to\mathcal{V}_{\mathfrak{h}})$ defines a system of quantum dynamical R-matrices. It turns out that conversely, any quantum dynamical R-matrix R defines $\mathcal{B}$, F, and X, such that $R = R(\mathcal{B}, F, X)$. The construction of $\mathcal{B}, F, X$ is parallel to the case of usual R-matrices ($\mathfrak{h} = 0$), where it is well known.

Namely, let $\mathcal{B}$ be the subcategory of $Rep(R)$ whose objects are tensor powers of V, and morphisms are the same as in Rep(R). It is clearly a monoidal category. Define a braiding β on $\mathcal{B}$ by $\beta_{VV} = \sigma R$. It is easy to check using the hexagon axioms for the braiding that there exists a unique braiding on $\mathcal{B}$ with such β_{VV}.

Let $F:\mathcal{B}\to\mathcal{V}_{\mathfrak{h}}$ be the forgetful functor. We assign the pair $(\mathcal{B}, F)$ to R. It is clear that $R = R(\mathcal{B}, F, X)$ if we take $X = V$.

3.4. Dual representations. It is useful to define the notion of the left and right dual representations.

Definition. *Let (W, L_W) be a representation of R. The right dual representation to W is the pair (W^*, L_{W^*}), where W^* denotes the $\mathfrak{h}$-graded dual of W, and*

$$L_{W^*}(\lambda) = L_W^{-1}(\lambda+\gamma h^{(2)})^{t_2}, \tag{3.4.1}$$

provided that the r.h.s. of (3.4.1) is invertible (here t_2 denotes dualization in the second component). The left dual representation to W is the pair $({}^*W, L_{{}^*W})$, where ${}^*W = W^*$, and

$$L_{{}^*W}(\lambda) = L_W^{t_2}(\lambda-\gamma h^{(2)})^{-1}, \tag{3.4.2}$$

provided that the r.h.s. of (3.4.2) is well defined.

Remark 1. Here $L_W^{-1}(\lambda+\gamma h^{(2)})^{t_2}$ denotes the result of three operations applied successively to L_W: inversion, shifting of the argument, and dualization in the second component. Similarly, $L_W^{t_2}(\lambda-\gamma h^{(2)})^{-1}$ denotes the result of three operations applied successively to L_W: dualization in the second component, shifting of the argument, and inversion.

Remark 2. We do not define the representation W^* if L_{W^*} is not invertible, and do not define the representation ${}^*W$ if $L_W^{t_2}$ is not invertible.

Lemma 3.4. *The right dual representation* (W^*, L_{W^*}) *and the left dual representation* $({}^*W, L_{{}^*W})$ *are representations of* R*, and if* W *has finite dimensional weight subspaces then* ${}^*(W^*) = ({}^*W)^* = W$.

Proof. The lemma can be checked by a direct calculation. It also follows from Propositions 4.1 and 4.4 below. $\square$

Lemma 3.5. *If* $A : W_1 \to W_2$ *is a homomorphism of representations of* R*, then the linear map* $A^*(\lambda) := A(\lambda+\gamma h^{(1)})^t = A^t(\lambda-\gamma h^{(1)})$ *is a homomorphism of representations* $W_2^* \to W_1^*$*, and is a homomorphism of representations* ${}^*W_2 \to {}^*W_1$*, when these representations are defined.*

Proof. The lemma can be checked by a direct calculation. It also follows from Propositions 4.1 and 4.4. $\square$

Remark. It is easy to show that for two finite dimensional representations W_1, W_2 of R, the representation $(W_1 \otimes W_2)^*$ is naturally isomorphic to $W_2^* \otimes W_1^*$, and similarly for the left dual, if the corresponding dual representations are defined.

4. $\mathfrak{h}$-Hopf Algebroids and Their Dynamical Representations

In this chapter we will define the notion of an $\mathfrak{h}$-bialgebroid, and give the simplest nontrivial examples – dynamical quantum groups associated to quantum dynamical R-matrices from Chapter 1. We will generalize this material in the next chapter.

4.1. $\mathfrak{h}$*-bialgebroids.* Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra over $\mathbb{C}$, and γ a nonzero complex number. Recall that $M_{\mathfrak{h}^*}$ denotes the field of meromorphic functions on $\mathfrak{h}^*$.

Definition. *An* $\mathfrak{h}$*-algebra with step* γ *is an associative algebra* A *over* $\mathbb{C}$ *with* 1*, endowed with an* $\mathfrak{h}^*$*-bigrading* $A = \oplus_{\alpha,\beta\in\mathfrak{h}^*} A_{\alpha\beta}$ *(called the weight decomposition), and two algebra embeddings* $\mu_l, \mu_r : M_{\mathfrak{h}^*} \to A_{00}$ *(the left and the right moment maps), such that for any* $a \in A_{\alpha\beta}$ *and* $f \in M_{\mathfrak{h}^*}$*, we have*

$$\mu_l(f(\lambda))a = a\mu_l(f(\lambda + \gamma\alpha)), \quad \mu_r(f(\lambda))a = a\mu_r(f(\lambda + \gamma\beta)). \tag{4.1.1}$$

A morphism $\varphi : A \to B$ of two $\mathfrak{h}$-algebras is an algebra homomorphism, preserving the moment maps. By (4.1.1), such a homomorphism also preserves the weight decomposition.

Let A, B be two $\mathfrak{h}$-algebras with step γ, and $\mu_l^A, \mu_r^A, \mu_l^B, \mu_r^B$ their moment maps. Define their "matrix tensor product", $A\widetilde{\otimes}B$, which is also an $\mathfrak{h}$-algebra.

Definition. *Let*

$$(A\widetilde{\otimes}B)_{\alpha\delta} := \oplus_\beta A_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} B_{\beta\delta}, \tag{4.1.2}$$

where $\otimes_{M_{\mathfrak{h}^*}}$ *means the usual tensor product modulo the relation* $\mu_r^A(f)a \otimes b = a \otimes \mu_l^B(f)b$*, for any* $a \in A, b \in B, f \in M_{\mathfrak{h}^*}$.

Introduce a multiplication in $A\widetilde{\otimes}B$ by the rule $(a\otimes b)(a'\otimes b') = aa'\otimes bb'$. It is easy to show that this product is well defined (cf. Proposition 5.1). Define the moment maps for $A\widetilde{\otimes}B$ by $\mu_l^{A\widetilde{\otimes}B}(f) = \mu_l^A(f)\otimes 1$, $\mu_r^{A\widetilde{\otimes}B}(f) = 1\otimes\mu_r^B(f)$. It is easy to check that this makes $A\widetilde{\otimes}B$ into an $\mathfrak{h}$-algebra. It is clear that $\widetilde{\otimes}$ is functorial with respect to both factors, and $(A\widetilde{\otimes}B)\widetilde{\otimes}C = A\widetilde{\otimes}(B\widetilde{\otimes}C)$. However, $A\widetilde{\otimes}B$ is not, in general, isomorphic to $B\widetilde{\otimes}A$.

Remark. The name "matrix tensor product" is used because formula (4.1.2) reminds of the matrix multiplication.

Definition. *A coproduct on an $\mathfrak{h}$-algebra A is a homomorphism of $\mathfrak{h}$-algebras $\Delta : A \to A\widetilde{\otimes}A$.*

Let $D_{\mathfrak{h}}$ be the algebra of difference operators $M_{\mathfrak{h}^*}\to M_{\mathfrak{h}^*}$, i.e. operators of the form $\sum_{i=1}^n f_i(\lambda)T_{\beta_i}$, where $f_i\in M_{\mathfrak{h}^*}$, and for $\beta\in\mathfrak{h}^*$ we denote by T_β the field automorphism of $M_{\mathfrak{h}^*}$ given by $(T_\beta f)(\lambda) = f(\lambda+\gamma\beta)$.

The algebra $D_{\mathfrak{h}}$ is the simplest nontrivial example of an $\mathfrak{h}$-algebra. Indeed if we define the weight decomposition by $D_{\mathfrak{h}} = \oplus(D_{\mathfrak{h}})_{\alpha\beta}$, where $(D_{\mathfrak{h}})_{\alpha\beta} = 0$ if $\alpha\neq\beta$, and $(D_{\mathfrak{h}})_{\alpha\alpha} = \{f(\lambda)T_\alpha^{-1} : f\in M_{\mathfrak{h}^*}\}$, and the moment maps $\mu_l = \mu_r : M_{\mathfrak{h}^*}\to (D_{\mathfrak{h}})_{00}$ to be the tautological isomorphism, then $D_{\mathfrak{h}}$ becomes an $\mathfrak{h}$-algebra.

Lemma 4.1. *For any $\mathfrak{h}$-algebra A, the algebras $A\widetilde{\otimes}D_{\mathfrak{h}}$ and $D_{\mathfrak{h}}\widetilde{\otimes}A$ are canonically isomorphic to A.*

Proof. Straightforward. □

Lemma 4.1 shows that the category of $\mathfrak{h}$-algebras equipped with the product $\widetilde{\otimes}$ is a monoidal category, where the unit object is $D_{\mathfrak{h}}$.

Definition. *A counit on an $\mathfrak{h}$-algebra A is a homomorphism of $\mathfrak{h}$-algebras $\varepsilon : A\to D_{\mathfrak{h}}$.*

Definition. *An $\mathfrak{h}$-bialgebroid is an $\mathfrak{h}$-algebra A equipped with a coassociative coproduct Δ (i.e. such that $(\Delta\otimes\mathrm{Id}_A)\circ\Delta = (\mathrm{Id}_A\otimes\Delta)\circ\Delta$), and a counit ε such that $(\varepsilon\otimes\mathrm{Id}_A)\circ\Delta = (\mathrm{Id}_A\otimes\varepsilon)\circ\Delta = \mathrm{Id}_A$.*

The property of the counit in the definition makes sense because of Lemma 4.1.

4.2. Dynamical representations of $\mathfrak{h}$-bialgebroids. Let W be a diagonalizable $\mathfrak{h}$-module, and let $D^\alpha_{\mathfrak{h},W}\subset\mathrm{Hom}_{\mathbb{C}}(W, W\otimes D_{\mathfrak{h}})$ be the space of all difference operators on $\mathfrak{h}^*$ with coefficients in $\mathrm{End}_{\mathbb{C}}(W)$, which have weight α with respect to the action of $\mathfrak{h}$ in W.

Consider the algebra $D_{\mathfrak{h},W} = \oplus_\alpha D^\alpha_{\mathfrak{h},W}$. This algebra has a weight decomposition $D_{\mathfrak{h},W} = \oplus_{\alpha,\beta}(D_{\mathfrak{h},W})_{\alpha\beta}$ defined as follows: if $g\in\mathrm{Hom}_{\mathbb{C}}(W, W\otimes M_{\mathfrak{h}^*})$ is an operator of weight $\beta-\alpha$ then $gT_\beta^{-1}\in(D_{\mathfrak{h},W})_{\alpha\beta}$.

Define the moment maps $\mu_l,\mu_r : M_{\mathfrak{h}^*}\to(D_{\mathfrak{h},W})_{00}$ by the formulas $\mu_r(f(\lambda)) = f(\lambda)$, $\mu_l(f(\lambda)) = f(\lambda-\gamma h)$.

Lemma 4.2. *The algebra $D_{\mathfrak{h},W}$ equipped with this weight decomposition and these moment maps is an $\mathfrak{h}$-algebra.*

Proof. Straightforward. □

Lemma 4.3. *There is a natural embedding of* $\mathfrak{h}$*-algebras* $\theta_{WU}: D_{\mathfrak{h},W}\tilde{\otimes} D_{\mathfrak{h},U} \to D_{\mathfrak{h},W\otimes U}$, *given by the formula* $fT_\beta \otimes gT_\delta \to (f\bar{\otimes} g)T_\delta$, *where* $\bar{\otimes}$ *is defined in Chapter 3, and* $f \in Hom(W, W \otimes M_{\mathfrak{h}^*})$. *This embedding is an isomorphism if* W, U *are finite-dimensional.*

Proof. We have to show that the map θ_{WU} is well defined, and is an embedding. We also have to show that θ_{WU} is a homomorphism of $\mathfrak{h}$-algebras, which is an isomorphism in the finite-dimensional case.

The fact that θ_{WU} is well defined follows from the identity $\varphi(\lambda)f\bar{\otimes}g = f\bar{\otimes}\varphi(\lambda - \gamma h)g$, for any function $\varphi \in M_{\mathfrak{h}^*}$. The injectivity of θ_{WU}, and its surjectivity in the finite dimensional case are straightforward.

It remains to show that θ_{WU} is a homomorphism of $\mathfrak{h}$-algebras. It is obvious that θ_{WU} preserves the moment maps, so it remains to show that it is multiplicative. We have

$$\begin{aligned}
\theta_{WU}((f(\lambda)T_\beta^{-1} \otimes g(\lambda)T_\delta^{-1})(f'(\lambda)T_{\beta'}^{-1} \otimes g'(\lambda)T_{\delta'}^{-1})) &= \\
\theta_{WU}(f(\lambda)f'(\lambda-\gamma\beta)T_{\beta+\beta'}^{-1} \otimes g(\lambda)g'(\lambda-\gamma\delta)T_{\delta+\delta'}^{-1}) &= \\
f^{(1)}(\lambda-\gamma h^{(2)})f'^{(1)}(\lambda-\gamma h^{(2)}-\gamma\beta)(1\otimes g(\lambda)g'(\lambda-\gamma\delta))T_{\delta+\delta'}^{-1} &= \\
f^{(1)}(\lambda-\gamma h^{(2)})(1\otimes g(\lambda))f'^{(1)}(\lambda-\gamma h^{(2)}-\gamma\delta)(1\otimes g'(\lambda-\gamma\delta))T_{\delta+\delta'}^{-1} &= \\
f^{(1)}(\lambda-\gamma h^{(2)})(1\otimes g(\lambda))T_\delta^{-1} f'^{(1)}(\lambda-\gamma h^{(2)})(1\otimes g'(\lambda))T_{\delta'}^{-1} &= \\
\theta_{WU}(f(\lambda)T_\beta^{-1} \otimes g(\lambda)T_\delta^{-1})\theta_{WU}(f'(\lambda)T_{\beta'}^{-1} \otimes g'(\lambda)T_{\delta'}^{-1}).&
\end{aligned} \tag{4.2.1}$$

The lemma is proved. □

Definition. *A dynamical representation of an* $\mathfrak{h}$*-algebra* A *is a diagonalizable* $\mathfrak{h}$*-module* W *endowed with a homomorphism of* $\mathfrak{h}$*-algebras* $\pi_W : A \to D_{\mathfrak{h},W}$. *A homomorphism of dynamical representations* $\varphi : W_1 \to W_2$ *is an element of* $Hom_{\mathbb{C}}(W_1, W_2 \otimes M_{\mathfrak{h}^*})$ *such that* $\varphi \circ \pi_{W_1}(x) = \pi_{W_2}(x) \circ \varphi$ *for all* $x \in A$.

Example. If A has a counit, then it has the trivial representation: $W = \mathbb{C}$, $\pi = \varepsilon$.

Suppose now that A is an $\mathfrak{h}$-bialgebroid. Then, if W and U are two dynamical representations of A, the $\mathfrak{h}$-module $W \otimes U$ also has a natural structure of a dynamical representation, defined by $\pi_{W\otimes U}(x) = \theta_{WU} \circ (\pi_W \otimes \pi_U) \circ \Delta(x)$.

It is easy to show that if $f : W_1 \to W_2$ and $g : U_1 \to U_2$ are homomorphisms of dynamical representations, then $f\bar{\otimes}g$ is a homomorphism $W_1 \otimes U_1 \to W_2 \otimes U_2$ (where $\bar{\otimes}$ is defined in Chapter 3). This gives a rule of tensoring morphisms. Thus, dynamical representations of A form a monoidal category Rep(A), whose identity object is the trivial representation.

Moreover, the category Rep(A) is equipped with a natural tensor functor Rep(A) $\to \mathcal{V}_{\mathfrak{h}}$ to the category of $\mathfrak{h}$-vector spaces – the forgetful functor.

4.3. $\mathfrak{h}$*-Hopf algebroids and dual representations.* Let us introduce the notion of an antipode on an $\mathfrak{h}$-bialgebroid.

Let A be an $\mathfrak{h}$-algebra. A linear map $S : A \to A$ is called an antiautomorphism of $\mathfrak{h}$-algebras if it is an antiautomorphism of algebras and $\mu_r \circ S = \mu_l$, $\mu_l \circ S = \mu_r$. From these conditions it follows that $S(A_{\alpha\beta}) = A_{-\beta,-\alpha}$.

Let A be an $\mathfrak{h}$-bialgebroid, and let Δ, ε be the coproduct and counit of A. For $a \in A$, let

$$\Delta(a) = \sum_i a_i^1 \otimes a_i^2. \tag{4.3.1}$$

Definition. *An antipode on the $\mathfrak{h}$-bialgebroid A is an antiautomorphism of $\mathfrak{h}$-algebras $S : A \to A$ such that for any $a \in A$ and any presentation (4.3.1) of $\Delta(a)$, one has*

$$\sum_i a_i^1 S(a_i^2) = \mu_l(\varepsilon(a)1), \ \sum_i S(a_i^1)a_i^2 = \mu_r(\varepsilon(a)1), \tag{4.3.2}$$

where $\varepsilon(a)1 \in M_{\mathfrak{h}^*}$ is the result of application of the difference operator $\varepsilon(a)$ to the constant function 1.

Remark. It is easy to see that $\sum_i a_i^1 S(a_i^2)$ and $\sum_i S(a_i^1)a_i^2$ depends only on a and not on the choice of the presentation (4.3.1).

Definition. *An $\mathfrak{h}$-bialgebroid with an antipode is called an $\mathfrak{h}$-Hopf algebroid.*

Remark. If $\mathfrak{h} = 0$, the notions of an $\mathfrak{h}$-algebra, $\mathfrak{h}$-bialgebroid, $\mathfrak{h}$-Hopf algebroid coincide with the notions of an algebra, bialgebra, and Hopf algebra, respectively.

For any $\mathfrak{h}$-Hopf algebroid A, the category Rep(A) has the following natural notion of the left and right dual representation.

If (W, π_W) is a dynamical representation of an $\mathfrak{h}$-algebra A, we denote by $\pi_W^0 : A \to \mathrm{Hom}(W, W \otimes M_{\mathfrak{h}^*})$ the map defined by $\pi_W^0(x)w = \pi_W(x)w$, $w \in W$ (the difference operator $\pi_W(x)$ restricted to the constant functions). It is clear that π_W is completely determined by π_W^0.

Definition. *Let (W, π_W) be a dynamical representation of A. Then the right dual representation to W is (W^*, π_{W^*}), where W^* is the $\mathfrak{h}$-graded dual to W, and*

$$\pi_{W^*}^0(x)(\lambda) = \pi_W^0(S(x))(\lambda + \gamma h - \gamma\alpha)^t \tag{4.3.3}$$

for $x \in A_{\alpha\beta}$, where t denotes dualization. The left dual representation to W is $({}^*W, \pi_{{}^*W})$, where ${}^*W = W^*$, and

$$\pi_{{}^*W}^0(x)(\lambda) = \pi_W^0(S^{-1}(x))(\lambda + \gamma h - \gamma\alpha)^t \tag{4.3.4}$$

for $x \in A_{\alpha\beta}$.

Proposition 4.1. *Formulas (4.3.3) and (4.3.4) define dynamical representations of A. Moreover, if $A(\lambda) : W_1 \to W_2$ is a morphism of dynamical representations, then $A^*(\lambda) := A(\lambda + \gamma h)^t$ defines a morphism $W_2^* \to W_1^*$ and ${}^*W_2 \to {}^*W_1$.*

Proof. Let $x \in A_{\alpha_x\beta_x}$, $y \in A_{\alpha_y\beta_y}$. Then $\pi_W^0(xy)(\lambda) = \pi_W^0(x)(\lambda)\pi_W^0(y)(\lambda - \gamma\beta_x)$ by the definition of a dynamical representation. Therefore, we have

$$\begin{aligned}\pi_{W^*}^0(xy)(\lambda)^t = \pi_W^0(S(xy))(\lambda + \gamma h - \gamma\alpha_x - \gamma\alpha_y) = \\ \pi_W^0(S(y)S(x))(\lambda + \gamma h - \gamma\alpha_x - \gamma\alpha_y) = \\ \pi_W^0(S(y))(\lambda + \gamma h - \gamma\alpha_x - \gamma\alpha_y + \gamma\alpha_{S(x)} \\ -\gamma\beta_{S(x)})\pi_W^0(S(x))(\lambda + \gamma h - \gamma\alpha_x - \gamma\alpha_y - \beta_{S(y)}) = \\ \pi_W^0(S(y))(\lambda + \gamma h - \gamma\alpha_y - \gamma\beta_x)\pi_W^0(S(x))(\lambda + \gamma h - \gamma\alpha_x).\end{aligned} \tag{4.3.5}$$

Dualizing (4.3.5), we get

$$\pi^0_{W^*}(xy)(\lambda) = \pi^0_W(S(x))(\lambda+\gamma h-\gamma\alpha_x)^t\pi^0_W(S(y))(\lambda+\gamma h-\gamma\alpha_y-\gamma\beta_x)^t = \pi^0_{W^*}(x)(\lambda)\pi^0_{W^*}(y)(\lambda-\gamma\beta_x), \tag{4.3.6}$$

which implies the first statement of the proposition for W^*. The proof for ${}^*W$ is obtained by replacing S by S^{-1}.

Let us prove the second statement. The intertwining property of $A(\lambda)$ can be written as

$$A(\lambda)\pi^0_W(x)(\lambda) = \pi^0_W(x)(\lambda)A(\lambda-\gamma\beta_x). \tag{4.3.7}$$

Replacing x with $S(x)$ and shifting the arguments, we get

$$\begin{aligned} A(\lambda+\gamma h-\gamma\beta_x)\pi^0_W(S(x))(\lambda+\gamma h-\gamma\alpha_x) = \\ \pi^0_W(S(x))(\lambda+\gamma h-\gamma\alpha_x)A(\lambda+\gamma h-\gamma\alpha_x-\gamma\beta_S(x)). \end{aligned} \tag{4.3.8}$$

Dualizing (4.3.8) and using the identity $\beta_{S(x)}+\alpha_x = 0$, we get the second statement of the proposition. The proposition is proved. □

4.4. $\mathfrak{h}$-bialgebroids associated to a function $R : \mathfrak{h}^* \to \mathrm{End}(V\otimes V)$. Let $\mathfrak{h}$ be a finite dimensional commutative Lie algebra, and $V = \oplus_{\alpha\in\mathfrak{h}^*}V_\alpha$ a finite dimensional diagonalizable $\mathfrak{h}$-module. Let $R(\lambda)$ be a meromorphic function $\mathfrak{h}^*\to\mathrm{End}(V\otimes V)$ of zero weight, such that $R(\lambda)$ is invertible for a generic λ. Using R, we will now define an $\mathfrak{h}$-bialgebroid A_R which we call the *dynamical quantum group* corresponding to R. This construction is analogous to the Faddeev–Reshetikhin–Sklyanin–Takhtajan construction of the quantum function algebra on GL_N.

As an algebra, A_R by definition is generated by two copies of $M_{\mathfrak{h}^*}$ (embedded as subalgebras) and certain new generators, which are matrix elements of the operators $L^{\pm1}\in\mathrm{End}(V)\otimes A_R$. We denote the elements of the first copy of $M_{\mathfrak{h}^*}$ as $f(\lambda^1)$ and of the second copy as $f(\lambda^2)$, where $f\in M_{\mathfrak{h}^*}$. We denote by $(L^{\pm1})_{\alpha\beta}$ the weight components of $L^{\pm1}$ with respect to the natural $\mathfrak{h}$-bigrading on End(V), so that $L^{\pm1} = (L^{\pm1}_{\alpha\beta})$, where $L^{\pm1}_{\alpha\beta}\in\mathrm{Hom}_{\mathbb{C}}(V_\beta,V_\alpha)\otimes A_R$.

Then the defining relations for A_R are:

$$f(\lambda^1)L_{\alpha\beta} = L_{\alpha\beta}f(\lambda^1+\gamma\alpha);\ f(\lambda^2)L_{\alpha\beta} = L_{\alpha\beta}f(\lambda^2+\gamma\beta);\ [f(\lambda^1),g(\lambda^2)] = 0; \tag{4.4.1}$$

$$LL^{-1} = L^{-1}L = 1; \tag{4.4.2}$$

and the dynamical Yang–Baxter relation

$$R^{12}(\lambda^1)L^{13}L^{23} =: L^{23}L^{13}R^{12}(\lambda^2) : . \tag{4.4.3}$$

Here the :: sign (“normal ordering”) means that the matrix elements of L should be put on the right of the matrix elements of R. Thus, if $\{v_a\}$ is a homogeneous basis of V, and $L = \sum E_{ab}\otimes L_{ab}$, $R(\lambda)(v_a\otimes v_b) = \sum R^{ab}_{cd}(\lambda)v_c\otimes v_d$, then (4.4.3) has the form

$$\sum R^{xy}_{ac}(\lambda^1)L_{xb}L_{yd} = \sum R^{bd}_{xy}(\lambda^2)L_{cy}L_{ax}, \tag{4.4.4}$$

where we sum over repeated indices.

More precisely, the algebra A_R is, by definition, the quotient of the algebra $\tilde{A}$ freely generated by $M_{\mathfrak{h}^*}\otimes M_{\mathfrak{h}^*}$ and elements $L_{ab}, (L^{-1})_{ab}$, $a,b = 1,\ldots,\mathrm{dim}V$, by the ideal defined by relations (4.4.1)–(4.4.3).

Introduce the moment maps for A_R by $\mu_l(f) = f(\lambda^1)$, $\mu_r(f) = f(\lambda^2)$, and the weight decomposition by $f(\lambda^1), f(\lambda^2) \in (A_R)_{00}$, $L_{\alpha\beta} \in \mathrm{Hom}_{\mathbb{C}}(V_\beta, V_\alpha) \otimes (A_R)_{\alpha\beta}$. It is clear that A_R equipped with such structures is an $\mathfrak{h}$-algebra.

Now define the coproduct on A_R, $\Delta : A_R \to A_R \widetilde{\otimes} A_R$, by the usual Lie-theoretic formulas

$$\Delta(L) = L^{12}L^{13}, \Delta(L^{-1}) = (L^{-1})^{13}(L^{-1})^{12} \tag{4.4.5}$$

(here Δ is applied to the second component of L, L^{-1}).

Proposition 4.2. *Δ extends to a well defined homomorphism $A \to A\widetilde{\otimes}A$.*

Proof. From (4.4.5) we get

$$\Delta(L_{\alpha\beta}) = \sum_\theta L^{12}_{\alpha\theta} L^{13}_{\theta\beta}. \tag{4.4.6}$$

So it remains to show that the defining relations of A_R are invariant under Δ. The invariance of relations (4.4.1) follows directly from (4.4.6). Relation (4.4.2) is obviously invariant. To check the invariance of relation (4.4.3), we have to show that

$$R^{12}(\lambda_1^1)L^{13}L^{14}L^{23}L^{24} =: L^{23}L^{24}L^{13}L^{14}R^{12}(\lambda_2^2) : \tag{4.4.7}$$

(the subscripts $1, 2$ under λ indicate that the corresponding functions are taken from the first and the second components of A_R in the product $A_R\widetilde{\otimes}A_R$; and, as before, the :: sign indicates that the functions of λ^i are written on the left from the L-operators).

We have

$$\begin{aligned} &R^{12}(\lambda_1^1)L^{13}L^{14}L^{23}L^{24} = R^{12}(\lambda_1^1)L^{13}L^{23}L^{14}L^{24} =: L^{23}L^{13}R^{12}(\lambda_1^2) : L^{14}L^{24} = \\ &L^{23}L^{13}R^{12}(\lambda_2^1)L^{14}L^{24} = L^{23}L^{13} : L^{24}L^{14}R^{12}(\lambda_2^2) :=: L^{23}L^{24}L^{13}L^{14}R^{12}(\lambda_2^2) : . \end{aligned} \tag{4.4.8}$$

(We replaced λ_1^2 by λ_2^1 in the middle of (4.4.8) since $A_R\widetilde{\otimes}A_R$ is by definition inside of the tensor product $A_R \otimes_{M_{\mathfrak{h}^*}} A_R$, where $M_{\mathfrak{h}^*}$ is mapped into the first component of A_R by μ_r and into the second by μ_l, acting from the left). The proposition is proved. □

Now define the counit on the algebra A_R. Recall that the counit has to be an algebra homomorphism $\varepsilon : A_R \to D_{\mathfrak{h}}$.

Define the counit by the formula

$$\varepsilon(L_{\alpha\beta}) = \delta_{\alpha\beta}\mathrm{Id}_{V_\alpha} \otimes T_\alpha^{-1}, \varepsilon((L^{-1})_{\alpha\beta}) = \delta_{\alpha\beta}\mathrm{Id}_{V_\alpha} \otimes T_\alpha, \tag{4.4.9}$$

where $\mathrm{Id}_{V_\alpha} : V_\alpha \to V_\alpha$ is the identity operator.

We need to check that the counit is well defined, i.e. that the defining relations are annihilated by it. For relations (4.4.1),(4.4.2) it is obvious. Relation (4.4.3) reduces to checking that

$$(\sum_{\alpha,\beta} R^{12}(\lambda)(\mathrm{Id}_{V_\alpha} \otimes \mathrm{Id}_{V_\beta})) \otimes T^{-1}_{\alpha+\beta} = (\sum_{\alpha,\beta}(\mathrm{Id}_{V_\alpha} \otimes \mathrm{Id}_{V_\beta})R^{12}(\lambda)) \otimes T^{-1}_{\alpha+\beta}, \tag{4.4.10}$$

which holds because R has zero weight.

Proposition 4.3. *The counit axiom* $(\mathrm{Id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id}$ *is satisfied for A_R.*

Proof. We need to check the relations on L. These relations follow from the fact that the elements $T_\alpha^{-1} \otimes L_{\alpha\beta}$, $L_{\alpha\beta} \otimes T_\beta^{-1}$ are mapped to $L_{\alpha\beta}$ under the natural isomorphisms $D_{\mathfrak{h}} \widetilde{\otimes} A_R \to A_R$, $A_R \widetilde{\otimes} D_{\mathfrak{h}} \to A_R$. $\square$

Thus, A_R is an $\mathfrak{h}$-bialgebroid. We will call it the *dynamical quantum group* corresponding to the function R.

It is also possible to consider the algebra generated by $f(\lambda_1), f(\lambda_2)$, L (without L^{-1}). Denote this algebra by $\bar{A}_R$. The algebra $\bar{A}_R$ is an $\mathfrak{h}$-bialgebroid, which is naturally mapped to A_R.

Remark. The algebra $\bar{A}_R$ was introduced in [FV1] under the name of "the operator algebra".

4.5. The antipode on A_R. Let A, B be algebras with 1. For $X \in B \otimes A$, define $i(X)$ to be the inverse of X, and $i_*(X)$ to be the inverse of X in the algebra $B \otimes A_{op}$, where A_{op} is A with the reversed order of multiplication. Clearly, $i^2 = i_*^2 = \mathrm{Id}$.

Let I be the group freely generated by i, i_* with relations $i^2 = i_*^2 = \mathrm{Id}$. We will say that an element X is *strongly invertible* if for any $g \in I$ the element $g(X)$ is well defined.

Definition. *An invertible, weight zero matrix function R is said to be rigid if the element $L \in \mathrm{End}(V) \otimes A_R$ is strongly invertible.*

Proposition 4.4. *R is rigid if and only if A_R admits an antipode S such that $S(L) = L^{-1}$. In this case, $S^{2n}(L) = (i_*i)^n(L), S^{2n+1}(L) = i(i_*i)^n(L)$. In particular, $S(L^{-1}) = i_*i(L)$.*

Proof. Suppose that R is rigid. Extend the definition of the antipode by $S(L^{-1}) = i_*(L^{-1}) = i_*i(L)$. It is easy to see that the relations of A_R are preserved, so this indeed defines an antihomomorphism $S : A \to A$. Moreover, S is an isomorphism: the inverse is given by $S^{-1}(L^{-1}) = L$, $S^{-1}(L) = i_*(L)$.

Now suppose that S is defined. Then it is easy to check that $(i_*i)^n(L) = S^{2n}(L)$, $i(i_*i)^n(L) = S^{2n+1}(L)$, $n \in \mathbb{Z}$. This defines $g(L)$ for all $g \in I$. The proposition is proved. $\square$

Remark 1. The proposition shows that for rigidity of R, it is sufficient that $i_*(L)$ and $i_*(L^{-1})$ be defined.

Remark 2. Observe that in general $S^2 \neq 1$.

Thus, if R is rigid then A_R is an $\mathfrak{h}$-Hopf algebroid.

4.6. Representation theory of A_R. Now consider the representation theory of A_R. As was pointed out in [FV1], the category $\mathrm{Rep}(A_R)$ of dynamical representations of A_R is tautologically isomorphic to the category $\mathrm{Rep}(R)$ of representations of R.

Proposition 4.5. *The tensor categories $Rep(A_R)$ and $Rep(R)$ are equivalent.*

Proof. Define the functor $\Gamma : Rep(A_R) \to Rep(R)$ to be the identity at the level of vector spaces, and set

$$L_{\Gamma(W)} = \pi_W^0(L). \tag{4.6.1}$$

Define the functor $\Gamma^{-1} : Rep(R) \to Rep(A_R)$ by

$$\pi^0_{\Gamma^{-1}(W)}(L) = L_W. \tag{4.6.2}$$

These functors preserve tensor structure, and are obviously inverse to each other. The proposition is proved. $\square$

It is easy to see that the functor Γ commutes with the duality functors. Therefore, if R is rigid, then the representations $W^*, {}^*W$ of R are well defined for any W, and the category $Rep_f(R)$ of finite-dimensional representations of R (= the category $Rep_f(A_R)$ of finite dimensional dynamical representations of A_R) is a rigid tensor category[DM]. This explains our use of the word "rigid".

Although A_R is an $\mathfrak{h}$-Hopf algebroid for any rigid zero weight function R, it does not always have nice properties. For a generic R, this algebra will be very small and will not have interesting dynamical representations. However if R is a dynamical quantum R-matrix, then the category Rep(R) is nontrivial (it contains the basic representation defined in Chapter 3), so by Proposition 4.4 the category Rep(A_R) is also nontrivial. Thus, algebras A_R with R being a dynamical quantum R-matrix form a good class of $\mathfrak{h}$-Hopf algebroids. From now on we will only consider A_R for R being a dynamical quantum R-matrix.

4.7. Sufficient conditions for rigidity. Unfortunately, the definition of rigidity cannot be effectively checked, since it depends on the properties of the algebra A_R, about whose structure we do not know very much. Therefore, we would like to find some effective sufficient conditions of rigidity.

For any function $X : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$, define the function $\tilde{X} : \mathfrak{h}^* \to \mathrm{End}(V \otimes V)$ as follows. Suppose that for $v, w \in V$ one has $X(\lambda)(v \otimes w) = \sum_i f_i(\lambda) v_i \otimes w_i$, where $f_i \in M_{\mathfrak{h}^*}$ and w_i are homogeneous. Then set $\tilde{X}(\lambda)(v \otimes w) = \sum_i f_i(\lambda + \gamma\, wt(w_i)) v_i \otimes w_i$, where $wt(w_i)$ denotes the weight of w_i.

Let R be a dynamical quantum R-matrix with step γ. Assume that $i_*(\tilde{R})$ is defined.

Let us write $\tilde{R}$ in the form $\tilde{R} = \sum a_i \otimes b_i$, and $i_*(\tilde{R})$ in the form $i_*(\tilde{R}) = \sum c_i \otimes d_i$.

Define the operators $Q = \sum d_i c_i, Q' = \sum c_i d_i : \mathfrak{h}^* \to \mathrm{End}(V)$. These operators are of weight zero with respect to $\mathfrak{h}$, since R is of weight zero.

Proposition 4.6. *Suppose R is such that $i_*(\tilde{R})$ is defined, and R satisfies the following conditions:*

(i) The operator Q is invertible for a generic λ.
(ii) The operator Q' is invertible for a generic λ.

Then R is rigid, and

$$\begin{aligned} i_*(L^{-1}) = S^2(L) &=: (Q(\lambda^1) \otimes 1) L (Q^{-1}(\lambda^2) \otimes 1) : \\ &=: (Q'(\lambda^1 + \gamma h)^{-1} \otimes 1) L (Q'(\lambda^2 + \gamma h) \otimes 1) : . \end{aligned} \tag{4.7.1}$$

Remark. It is clear that (i) and (ii) are satisfied for $R = 1$ and are open conditions. Therefore, Proposition 4.5 shows that if R_γ is a continuous family of quantum dynamical R-matrices with step γ such that $R_0 = 1$, then R_γ is rigid for small γ.

Proof. First of all, let us deduce a commutation relation between L and L^{-1}.

Multiplying the dynamical Yang–Baxter equation by $(L^{-1})^{23}$ on the right, we get

$$R^{12}(\lambda^1)L^{13} =: L^{23}L^{13}R^{12}(\lambda^2)(L^{23})^{-1} : . \tag{4.7.2}$$

Let $\{v_a\}$ be an $\mathfrak{h}$-homogeneous basis of V, and $L = \sum E_{ab} \otimes L_{ab}$. Denote by ω_a the weight of v_a. Then we have

$$\begin{aligned} : L^{23}L^{13}R^{12}(\lambda^2) : (L^{23})^{-1} = \sum E_{ab}^{(2)} : L_{ab}^{(3)}L^{13}R^{12}(\lambda^2)(L^{23})^{-1} :=& \\ \sum E_{ab}^{(2)}L_{ab}^{(3)} : L^{13}R^{12}((\lambda + \gamma\omega_b)^2)(L^{23})^{-1} :=& \\ \sum E_{ab}^{(2)}L_{ab}^{(3)} : L^{13}\tilde{R}^{12}(\lambda^2)(L^{23})^{-1} := L^{23} : L^{13}\tilde{R}^{12}(\lambda^2) : (L^{23})^{-1}.& \end{aligned} \tag{4.7.3}$$

Therefore, multiplying (4.7.2) on the left by $(L^{23})^{-1}$ we get

$$(L^{23})^{-1} : R^{12}(\lambda^1)L^{13} :=: L^{13}\tilde{R}^{12}(\lambda^2)(L^{23})^{-1} : . \tag{4.7.4}$$

Transforming the left hand side of this equation similarly to (4.7.3), we arrive at the equation

$$: (L^{23})^{-1}\tilde{R}^{12}(\lambda^1)L^{13} :=: L^{13}\tilde{R}^{12}(\lambda^2)(L^{23})^{-1} :, \tag{4.7.5}$$

which is the desired commutation relation.

Now, using property (i), define

$$T =: (Q(\lambda^1) \otimes 1)L(Q^{-1}(\lambda^2) \otimes 1) :\in \mathrm{End}(V) \otimes A_R. \tag{4.7.6}$$

Let * denote the product in the algebra $\mathrm{End}(V) \otimes (A_R)_{op}$. Let us compute the product $L^{-1} * T$.

Set $L^{-1} = \sum E_{ab} \otimes (L^{-1})_{ab}$. Then we get

$$L^{-1} * T = \sum (E_{pq}Q(\lambda^2)E_{rs}Q^{-1}(\lambda^1) \otimes 1)(1 \otimes L_{rs}(L^{-1})_{pq}). \tag{4.7.7}$$

Using (4.7.5), we can rewrite (4.7.7) in the form

$$L^{-1} * T = \sum (d_i(\lambda^2)E_{rs}b_j(\lambda^1)Q(\lambda^1)a_j(\lambda^1)E_{pq}c_i(\lambda^2)Q^{-1}(\lambda^2) \otimes 1)(1 \otimes L_{rs}(L^{-1})_{pq}). \tag{4.7.8}$$

Using the definition of Q, we have

$$\sum b_iQa_i = 1. \tag{4.7.9}$$

Substituting (4.7.9) into (4.7.8), we get $L^{-1} * T = 1$.

Now, using property (ii), define

$$T' =: (Q'(\lambda^1 + \gamma h)^{-1} \otimes 1)L(Q'(\lambda^2 + \gamma h) \otimes 1) : . \tag{4.7.10}$$

Then, analogously to the above, we get $T' * L^{-1} = 1$. Thus, $T = T' = i_*(L^{-1})$.

It is easy to see that

$$i_*(L) =: (Q^{-1}(\lambda^2) \otimes 1)L(Q(\lambda^1) \otimes 1) : . \tag{4.7.11}$$

Thus, R is rigid. $\square$

Now we will show that any rigid quantum dynamical R-matrix satisfies a certain crossing symmetry condition.

For an invertible zero weight function $X(\lambda) \in \mathrm{End}(V \otimes V)$, set

$$\tau(X)(\lambda) = X^{-1}(\lambda + \gamma h^{(2)})^{t_2}. \tag{4.7.12}$$

Corollary 4.1. *Let R be a rigid quantum dynamical R-matrix on V. Then $\tau(R)$ is invertible, and R satisfies the crossing symmetry condition*

$$\tau^2(R) = (Q(\lambda - \gamma h^{(2)}) \otimes 1)R(\lambda)(Q^{-1}(\lambda) \otimes 1). \tag{4.7.13}$$

Proof. It is clear that $\tau^2(R) = L_{V^{**}}$, where V is the basic representation of R. Therefore, using (4.7.1) in the basic representation, we get (4.7.12). □

4.8. Dynamical quantum groups associated to dynamical R-matrices of gl_N type. Now suppose that R is a dynamical R-matrix of gl_N-type. Then it has form (1.3.2), and we can write the defining relations for A_R more explicitly. Since all weight subspaces of V are 1-dimensional, we have $(L^{\pm 1})_{\alpha\beta} \in A$. For brevity we write $(L^{\pm 1})_{ab}$ for $(L^{\pm 1})_{\omega_a \omega_b}$. Thus, we have $L^{\pm 1} = \sum E_{ab} \otimes (L^{\pm 1})_{ab}$.

In this notation, the defining relations for A_R look like

$$\begin{gathered}
LL^{-1} = L^{-1}L = 1, \\
f(\lambda^1)L_{bc} = L_{bc}f(\lambda^1 + \gamma\omega_b),\ f(\lambda^2)L_{bc} = L_{bc}f(\lambda^2 + \gamma\omega_c), \\
L_{as}L_{at} = \frac{\alpha_{st}(\lambda^2)}{1 - \beta_{ts}(\lambda^2)} L_{at}L_{as}, s \neq t, \\
L_{bs}L_{as} = \frac{\alpha_{ab}(\lambda^1)}{1 - \beta_{ab}(\lambda^1)} L_{as}L_{bs}, a \neq b, \\
\alpha_{ab}(\lambda_1)L_{as}L_{bt} - \alpha_{st}(\lambda_2)L_{bt}L_{as} = (\beta_{ts}(\lambda_2) - \beta_{ab}(\lambda_1))L_{bs}L_{at}, a \neq b, s \neq t,
\end{gathered} \tag{4.8.1}$$

where α_{ab}, β_{ab} are the functions from (1.3.2).

Remark. It is also possible to define dynamical quantum groups associated with dynamical R-matrices with spectral parameter. It is done analogously to the above, and we will do it in detail in a forthcoming paper. For example, if $R(z, \lambda)$ is a quantum dynamical R-matrix with spectral parameter of elliptic type (i.e. of the form (2.5.1)), we will get the elliptic quantum group defined in [F1, F2, FV1, FV2]. Relations (4.8.1) (for dynamical R-matrices of gl_N Hecke type) can be obtained as a limiting case of the defining relations for the elliptic quantum group.

4.9. Rigidity of the rational and the trigonometric dynamical R-matrix. Consider the trigonometric dynamical R-matrix $R(\lambda)$ defined by (1.6.4), with $X = \{1, ..., N\}$, and $\mu_{ab} = 1$.

Proposition 4.7. *$R(\lambda)$ is rigid, and the matrices Q, Q' are given by the formulas*

$$\begin{gathered}
Q = \mathit{diag}(Q_1, ..., Q_N),\ Q' = \mathit{diag}(Q'_1, ..., Q'_N), \\
Q_a(\lambda) = \prod_{i \neq a} \frac{q^{1+\lambda_i} - q^{\lambda_a}}{q^{\lambda_i} - q^{\lambda_a}}, \\
Q'_a(\lambda) = qQ_a^{-1}(\lambda),
\end{gathered} \tag{4.9.1}$$

where $q = e^{\epsilon}$.

Proof. First of all, it is not hard to show by a direct computation that the matrix $i_*(\tilde{R})$ is defined. So it remains to show that the elements Q, Q' are invertible.

Let $P(\lambda) = Q'(\lambda + \gamma h)$. Let P_i, Q_i be the diagonal entries of P, Q. As we know, these entries are defined by the following systems of linear equations:

$$\begin{aligned} Q_a + \sum_{b \neq a} \beta_{ab}(\lambda + \gamma\omega_a) Q_b &= 1, \\ P_a + \sum_{b \neq a} \beta_{ba}(\lambda + \gamma\omega_b) P_b &= 1. \end{aligned} \tag{4.9.2}$$

The explicit form of the systems (4.9.2) is

$$Q_a + \sum_{b \neq a} \frac{q-1}{q^{1+\lambda_a-\lambda_b}-1} Q_b = 1,$$

$$P_a + \sum_{b \neq a} \frac{q-1}{q^{1+\lambda_b-\lambda_a}-1} P_b = 1.$$

Thus, if one of these systems is nondegenerate (which we show below) then $Q(\lambda) = P(-\lambda)$.

From now on we consider only the first system. Note that it can be conveniently written as

$$\sum_b \frac{q-1}{q^{1+\lambda_a-\lambda_b}-1} Q_b = 1. \tag{4.9.3}$$

Define $X_b = q^{\lambda_b} Q_b$. Then (4.9.3) can be written as

$$\sum_b \frac{1}{[1+\lambda_a]-[\lambda_b]} X_b = 1, \tag{4.9.4}$$

where $[x] = \frac{q^x-1}{q-1}$. Thus, the vector X is defined by $X = C^{-1}\mathbf{1}$, where $C_{ab} = \frac{1}{[1+\lambda_a]-[\lambda_b]}$, and $\mathbf{1}$ is the vector whose components are all equal to 1.

To invert the matrix C, we use the well known combinatorial identity (which is called the “Bose-Fermi correspondence” in physics):

$$\det\left(\frac{1}{x_i - y_j}\right) = \frac{\prod_{i<j}(x_i - x_j)\prod_{i<j}(y_i - y_j)}{\prod_{i,j}(x_i - y_j)}. \tag{4.9.5}$$

Applying this identity to $x_i = [1+\lambda_i], y_i = [\lambda_i]$, and using the usual rule of inverting matrices, we get

$$(C^{-1})_{ab} = \frac{\prod_{(i,j): i=b \text{ or } j=a}(x_i - y_j)}{\prod_{j \neq b}(x_b - x_j)\prod_{i \neq a}(y_i - y_a)}. \tag{4.9.6}$$

In particular,

$$X_a = \sum_b (C^{-1})_{ab} = \frac{\prod_i (x_i - y_a)}{\prod_{i \neq a}(y_i - y_a)} \sum_b \frac{\prod_{j \neq a}(x_b - y_j)}{\prod_{j \neq b}(x_b - x_j)}. \tag{4.9.7}$$

Claim 1.

$$\sum_b \frac{\prod_{j \neq a}(x_b - y_j)}{\prod_{j \neq b}(x_b - x_j)} = 1. \tag{4.9.8}$$

Proof of the claim. Consider the expression on the l.h.s. of (4.9.8) as a rational function of $z = x_a$ for fixed x_b, $b \neq a$. This function has no more than simple poles at x_b, $b \neq a$, and no other singularities; it equals 1 at infinity. Thus, it suffices to show that its residues vanish, which is obvious: only two terms contribute to each residue, ant these two terms cancel each other.

Thus, we get:

$$Q_a(\lambda) = q^{-\lambda_a} \frac{\prod_i([1+\lambda_i] - [\lambda_a])}{\prod_{i\neq a}([\lambda_i] - [\lambda_a])}, \tag{4.9.9}$$

i.e.

$$\begin{aligned} Q_a(\lambda) &= \prod_{i\neq a} \frac{q^{1+\lambda_i} - q^{\lambda_a}}{q^{\lambda_i} - q^{\lambda_a}}, \\ P_a(\lambda) &= \prod_{i\neq a} \frac{q^{1-\lambda_i} - q^{-\lambda_a}}{q^{-\lambda_i} - q^{-\lambda_a}}. \end{aligned} \tag{4.9.10}$$

Therefore,

$$Q'_a(\lambda) = P_a(\lambda - \omega_a) = \prod_{i\neq a} \frac{q^{1-\lambda_i} - q^{1-\lambda_a}}{q^{-\lambda_i} - q^{1-\lambda_a}} = \prod_{i\neq a} \frac{q^{1+\lambda_i} - q^{1+\lambda_a}}{q^{1+\lambda_i} - q^{\lambda_a}} = qQ_a^{-1}(\lambda). \tag{4.9.11}$$

Thus, R is rigid, and Q, Q' are given by formula (4.9.1). The proposition is proved. □

An analogous theorem holds for the rational dynamical R-matrix (1.5.1) (with $X = \{1, ..., N\}$ and $\mu_{ab} = 0$). The formulas for Q, Q' for such R are obtained from (4.9.1) as $q \to 1$.

It is easy to show that the property of rigidity is preserved by gauge transformations, so we get

Corollary 4.2. *Any quantum dynamical R-matrix R of gl_N Hecke type is rigid.*

Clearly, the elements Q, Q' for any such R can be easily computed from (4.9.1).

5. *H*-Biequivariant Hopf Algebroids

In this chapter we generalize the notions of an $\mathfrak{h}$-algebra, $\mathfrak{h}$-bialgebroid, $\mathfrak{h}$-Hopf algebroid to the case when the Lie algebra $\mathfrak{h}$ is not necessarily commutative, and define quantum counterparts of the quasiclassical notions introduced in Chapters 1-2 of [EV].

We will define the notions of an H-biequivariant Hopf algebroid and quantum groupoid. The notion of an H-biequivariant quantum groupoid is a quantum analogue of the notion of an H-biequivariant Poisson groupoid, introduced in [EV]. We will also introduce less general notions of a dynamical quantum groupoid and Hopf algebroid, which are quantum analogues of the notions of a dynamical Poisson groupoid and Hopf algebroid.

In this chapter we will work mostly in the setting of perturbation theory. That is, quantum objects will be defined over $k[[\hbar]]$, where k is some field, and give classical objects modulo $\hbar$ and quasiclassical ones modulo $\hbar^2$. We discuss the relationship between the quasiclassical and quantum objects, and questions regarding quantization.

5.1. Quantization of Poisson algebras. In this section we will recall some well known facts from the theory of deformation quantization.

Let k be a field of characteristic zero. Let $K = k[[\hbar]]$. By a topologically free K-module we mean a K-module of the form $V[[\hbar]]$, where V is a k-vector space. All K-modules we will use will be topologically free. By tensor product of two such modules we will always mean completed tensor product over K.

Let A_0 be a commutative algebra over k with 1. Recall that according to Grothendieck, a linear operator $D : A_0 \to A_0$ is a differential operator of order $\leq N, N \geq 1$ if for any $a \in A_0$ the operator $f \to D(af) - aDf$ is a differential operator of order $\leq N - 1$, and a differential operator of order 0 is the operator of multiplication by an element of A_0. If A_0 is the algebra of regular functions on a manifold (smooth, analytic, algebraic, formal) then "differential operator of order N" means what it usually means in geometry.

Let A_0 be a Poisson algebra over k with 1, with Poisson bracket $\{,\}$. Recall that by a quantization of A_0 is meant a K-module $A = A_0[[\hbar]]$ equipped with a K-linear binary operation $* : A \otimes A \to A$, which defines an associative algebra structure on A, such that $A/\hbar A = A_0$ as an algebra, and $\frac{1}{\hbar}(f * g - g * f)$ mod $\hbar = \{f, g\}$, $f, g \in A_0 \subset A$. In this case A_0 is called the quasiclassical limit of A.

Let $f, g \in A_0$. Then

$$f * g = fg + \hbar c_1(f, g) + \hbar^2 c_2(f, g) + ..., \tag{5.1.1}$$

where $c_i : A_0 \otimes A_0 \to A_0$ are linear maps. A quantization defined by (5.1.1) is called local if $c_i(f, g)$ is a differential operator in f and g for any i. If A_0 is the algebra $\mathcal{O}(X)$ of regular functions on a smooth manifold X, and A is a local quantization of A_0, then A defines (by formula (5.1.1)) a quantization A_U of the algebra $(A_U)_0 = \mathcal{O}(U)$ of regular functions on any open subset U of X. In other words, it defines a quantization of the sheaf of regular functions. This holds also in the holomorphic and algebraic situations, if X is affine.

Let X be a manifold, and let T^*X be its cotangent bundle. Let $A_0 = \mathcal{O}(T^*X)_p$ be the Poisson algebra of regular functions on T^*X which are fiberwise polynomial of a uniformly bounded degree. This Poisson algebra has a distinguished quantization $A = \mathcal{O}_q(T^*X)_p$ called the canonical quantization (q is not a parameter here but the first letter of the word "quantum"). Namely, A is the algebra of formal series of the form $\sum_{n \geq 0} \hbar^n D_n$, where D_i are differential operators on X, such that $n \geq$ order(D_n), and $n -$ order(D_n) $\to +\infty$, as $n \to \infty$. It is easy to check that this quantization is local, so it defines a quantization $A_U = \mathcal{O}_q(U)$ of the Poisson algebra $(A_U)_0 = \mathcal{O}(U)$ of regular functions on an open subset $U \in T^*X$.

Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{g}^*$ be its dual space, with the usual Poisson structure. Consider the Poisson algebra $\mathcal{O}(\mathfrak{g}^*)_p$ of polynomial functions on $\mathfrak{g}^*$. This algebra has a distinguished quantization $A = \mathcal{O}_q(\mathfrak{g}^*)_p$, called the geometric quantization. Namely, A is the algebra of formal series of the form $\sum_{n \geq 0} \hbar^n D_n$, where $D_i \in U(\mathfrak{g})$, $n \geq$ order(D_n), and $n -$ order(D_n) $\to +\infty$, $n \to \infty$. It is easy to check that this quantization is local, so it defines a quantization $A_U = \mathcal{O}_q(U)$ of the Poisson algebra $(A_U)_0 = \mathcal{O}(U)$ of regular functions on an open subset $U \in \mathfrak{g}^*$.

5.2. H-biequivariant associative algebras. In this section we will introduce the notion of an H-biequivariant associative algebra. This notion is a quantum analogue of the notion of an H-biequivariant Poisson algebra, introduced in a previous paper [EV].

Let A be an associative algebra over K with 1, which is commutative mod $\hbar$, H a connected affine algebraic group over k, and $\psi : A \times H \to A$ be a right algebraic action of H on A by automorphisms, defined over k. This means that A, as a representation of H, has the form $A_0[[\hbar]]$, where A_0 is a sum of finite dimensional representations of H over k.

Let $\mathfrak{h}$ be the Lie algebra of H. Let $U \subset \mathfrak{h}^*$ be an H-invariant open set. A homomorphism $\mu : \mathcal{O}_q(U) \to A$ is called a quantum moment map for ψ if for any linear function on U given by $a \in \mathfrak{h}$ and any $f \in A$ we have

$$[\mu(a), f] = \hbar d\psi|_{h=1}(a, f). \tag{5.2.1}$$

Here $d\psi|_{h=1} : \mathfrak{h} \times A \to A$ is the differential of ψ at $h = 1 \in H$. Using the Leibnitz identity for the operator $g \to [\mu(g), f]$, from (5.2.1) one can compute $[\mu(g), f]$ for any rational function g.

For a left action of H a quantum moment map is defined in the same way, with the only difference that it is an anti-homomorphism rather than a homomorphism.

Definition. *An H-biequivariant associative algebra over U is a 5-tuple (A, l, r, μ_l, μ_r), where A is an associative algebra with 1 over K, which is commutative mod $\hbar$, l, r is a pair of commuting algebraic actions of H on A (a left action and a right action) by algebra automorphisms, defined over k, and $\mu_l, \mu_r : \mathcal{O}_q(U) \to A$ are quantum moment maps for l, r, such that*

(i) μ_l, μ_r are embeddings, and their images commute;
(ii) There exists an $l(H) \times r(H)$-invariant k-subspace A_0^l of A such that the multiplication map $\mu_r(\mathcal{O}_q(U)) \otimes A_0^l \to A$ is a linear isomorphism; there exists an $l(H) \times r(H)$-invariant k-subspace A_0^r of A such that the multiplication map $\mu_l(\mathcal{O}_q(U)) \otimes A_0^r \to A$ is a linear isomorphism.

A morphism of H-biequivariant associative algebras over U is a morphism of algebras which preserves l, r and μ_l, μ_r.

Remark 1. From $[l, r] = 0$ it follows that $[\mu_l \circ x, \mu_r \circ y]$ is a central element for $x, y \in \mathfrak{h}$, but it does not follow that this commutator equals 0. So we require that it is zero by condition (i).

Remark 2. Condition (ii) is of a technical nature and is not very important in the discussion below.

Denote the category of H-biequivariant associative algebras over U by $\mathcal{A}_U^q$ (q stands for “quantum”).

For convenience we will write $l(h)a$ as ha and $r(h)a$ as ah.

Let us now describe the monoidal structure on $\mathcal{A}_U^q$.

Let $A, B \in \mathcal{A}_U^q$. Then the group H acts in $A \otimes B$ by $\Delta(h)(a \otimes b) = ah^{-1} \otimes hb$. We will construct a new H-biequivariant associative algebra $A\widetilde{\otimes}B$, which is obtained by quantum Hamiltonian reduction of $A \otimes B$ by the action of H.

Denote by $A * B$ the space $A \otimes_{\mathcal{O}_q(U)} B$, where $\mathcal{O}_q(U)$ is mapped to A via μ_r^A, and to B via μ_l^B, acting in both algebras from the left. Then $A * B$ is the quotient of $A \otimes B$ by the linear span I of elements of the form $\mu_r^A(f)a \otimes b - a \otimes \mu_l^B(f)b$, $f \in \mathcal{O}_q(U)$, $a \in A, b \in B$. The space $A * B$ has two commuting actions of H ($l_A \otimes 1$ and $1 \otimes r_B$). But we cannot claim that $A * B \in \mathcal{A}_U^q$, since the algebra structure on $A \otimes B$ does not, in general, descend to $A * B$ (I is only a right ideal and not necessarily a left ideal).

However, the action Δ of H on $A \otimes B$ descends to one on $A * B$, so we can define $A\widetilde{\otimes}B := (A * B)^H$, where H acts by Δ.

Proposition 5.1. *The algebra structure on $A \otimes B$ descends to one on $A\widetilde{\otimes}B$.*

Proof. Let $x, y \in A\widetilde{\otimes}B$. We can regard x, y as elements of $A * B$. Choose their liftings $X = \sum a_i \otimes b_i$, $Y = \sum c_i \otimes d_i$ into $A \otimes B$. By definition, xy is the image of XY in $A * B$.

We have to check two things.

1. That xy is H-invariant.

2. That xy does not depend on the choice of liftings X, Y.

First we check property 1. Since x, y are H-invariant, we have

$$\begin{gathered}\sum[\mu_r^A(z), a_i] \otimes b_i + \sum a_i \otimes [\mu_l^B(z), b_i] \in I,\\ \sum[\mu_r^A(z), c_i] \otimes d_i + \sum c_i \otimes [\mu_l^B(z), d_i] \in I, z \in \mathfrak{h}.\end{gathered} \tag{5.2.2}$$

Therefore, since I is a right ideal,

$$\sum[\mu_r^A(z), a_i c_j] \otimes b_i d_j + \sum a_i c_j \otimes [\mu_l^B(z), b_i d_j] \in XI + I. \tag{5.2.3}$$

Lemma 5.1. *If X is H-invariant modulo I, then $XI \subset I$.*

Proof of the Lemma. Since $\sum c_j \otimes d_j$ is H-invariant modulo I, for any $z \in \mathfrak{h}$ we have

$$\sum c_j \mu_r^A(z) \otimes d_j - \sum c_j \otimes d_j \mu_l^B(z) \in I. \tag{5.2.4}$$

Therefore, the same equality holds any rational function $g \in \mathcal{O}_q(U)$ instead of z. This proves the lemma.

The Lemma shows that the RHS of (5.2.3) is in I, i.e. xy is H-invariant.

Now we check property 2. If X', Y' are any other liftings of x and y, then $X - X' \in I$, and $Y - Y' \in I$. So it remains to show that $X(Y - Y') \in I$. But this follows from the lemma. □

Thus, we have shown that the product descends to $A\widetilde{\otimes}B$. The two commuting actions of H on $A \otimes B$ by $(h_1, h_2)(a \otimes b) = h_1 a \otimes b h_2$, and the corresponding quantum moment maps descend to $A\widetilde{\otimes}B$. So, in order to check that $A\widetilde{\otimes}B \in \mathcal{A}_U^q$, it suffices to check properties (i) and (ii).

Using properties (i) and (ii) of the quantum moment maps $\mu_l^A, \mu_r^A, \mu_l^B, \mu_r^B$, it is easy to see that $A * B$ is naturally identified with $\mu_l^A(\mathcal{O}_q(U)) \otimes A_0^r \otimes B_0^r$, and $A\widetilde{\otimes}B$ is identified with $\mu_l^A(\mathcal{O}_q(U)) \otimes (A_0^r \otimes B_0^r)^H$, where H acts by $a \otimes b \to ah^{-1} \otimes hb$. This implies properties (i) and (ii) for the quantum moment map $\mu_l^A \otimes 1 : \mathcal{O}_q(U) \to A\widetilde{\otimes}B$, corresponding to the left action of H on $A\widetilde{\otimes}B$ (with $(A\widetilde{\otimes}B)_0^r = (A_0^r \otimes B_0^r)^H$). For the quantum moment map $1 \otimes \mu_r^B : \mathcal{O}_q(U) \to A\widetilde{\otimes}B$ corresponding to the right action, these properties are proved analogously.

Thus, $A\widetilde{\otimes}B \in \mathcal{A}_U^q$. It is clear that the assignment $A, B \to A\widetilde{\otimes}B$ is a bifunctor $\mathcal{A}_U^q \times \mathcal{A}_U^q \to \mathcal{A}_U^q$.

Recall [EV] that $(T^*H)_U$ denotes the variety of points $(h, p) \in T^*H$ such that $h^{-1}p \in U$. Consider the algebra $\mathcal{O}_q((T^*H)_U)$, which is the canonical quantization of the standard symplectic structure on $(T^*H)_U$. It is equipped with the standard actions l, r of H on left and right given by $(x, p) \to (h_1 x h_2, h_1 p h_2)$ (these actions obviously respect the quantization).

Let $\mu_{l,r} : \mathcal{O}_q(U) \to \mathcal{O}_q((T^*H)_U)$ be the embeddings, which assign to an element of $U(\mathfrak{h})$ the corresponding right-, respectively left-invariant differential operator on H. It is easy to check that $\mu_{l,r}$ are quantum moment maps for l, r.

Let $\mathbf{1} = (\mathcal{O}_q((T^*H)_U), l, r, \mu_l, \mu_r)$. It is easy to check that we have natural isomorphisms $A\widetilde{\otimes}\mathbf{1} \equiv A \equiv \mathbf{1}\widetilde{\otimes}A$.

Proposition 5.2. *(i)* $(A\widetilde{\otimes}B)\widetilde{\otimes}C = A\widetilde{\otimes}(B\widetilde{\otimes}C)$.
(ii) $\mathbf{1}$ *is a unit object in* $\mathcal{A}_U^q$ *with respect to* $\widetilde{\otimes}$, *and* $(\mathcal{A}_U^q, \widetilde{\otimes}, \mathbf{1})$ *is a monoidal category.*

Proof. Easy. □

Let $A \in \mathcal{A}_U^q$. Denote by $\bar{A}$ the new object of $\mathcal{A}_U^q$ obtained as follows: $\bar{A}$ is A^{op} (the opposite algebra), with the left and the right actions of H permuted (i.e. the left, respectively right, action of h on $\bar{A}$ is the right, respectively left, action of h^{-1} on A), and the quantum moment maps also permuted. We will call $\bar{A}$ the dual object to A. By a *quasireflection* on A we will mean a morphism $i : \bar{A} \to A$. Note that unlike [EV], here we do not require that $i^2 = 1$.

Let $A \in \mathcal{A}_U^q$ and $i : \bar{A} \to A$ be a quasireflection. Let $\varphi_+^i, \varphi_-^i : A \otimes A \to A$ be given by the formulas $\varphi_+^i(a \otimes b) = ai(b)$, $\varphi_-^i(a \otimes b) = i(a)b$. It is easy to see that these maps descend to linear maps $\psi_\pm^i : A\widetilde{\otimes}A \to A$.

5.3. H-biequivariant Hopf algebroids. Now let us define the quantum version of the notion of an H-biequivariant Poisson–Hopf algebroid.

Definition. *Let A be an H-biequivariant associative algebra. Then A is called an H-biequivariant Hopf algebroid over U if it is equipped with a coassociative $\mathcal{A}_U^q$-morphism $\Delta : A \to A\widetilde{\otimes}A$ called the coproduct, a $\mathcal{A}_U^q$-morphism $\varepsilon : A \to \mathbf{1}$ called the counit, and a quasireflection $S : \bar{A} \to A$ called the antipode, such that*

(i) $(id \bullet \varepsilon) \circ \Delta = (\varepsilon \bullet id) \circ \Delta = id$, *and*
(ii) $\psi_+^S \circ \Delta = \mu_l \circ P \circ \varepsilon$, $\psi_-^S \circ \Delta = \mu_r \circ P \circ \varepsilon$, *where $P : \mathbf{1} \to \mathcal{O}_q(U)$ is the map which assigns to a differential operator on H its value at the identity element (which is in $U(\mathfrak{h})$).*

The same structure without the antipode will be called an H-biequivariant bialgebroid.

If $H = 1$, then these notions coincide with notions of a Hopf algebra and a bialgebra over K.

Remark 1. In the above discussion, U is a Zariski open set. If $k = \mathbb{R}$ or $\mathbb{C}$, then we can take U to be an open set in the usual sense, and define $\mathcal{O}(U)$ to be the algebra of smooth, respectively analytic, functions on U. Then we can repeat Sect. 5.2, 5.3, and thus define the notions of an H-biequivariant associative algebra and Hopf algebroid over U. Similarly, one can take U to be the infinitesimal neighborhood of zero in $\mathfrak{h}^*$ (i.e. $\mathcal{O}(U) = k[[\mathfrak{h}]]$). The material of Sects. 5.2 and 5.3 can be generalized to this case as well.

Remark 2. In the smooth, analytic, and formal case one has to drop the condition that A is the sum of finite dimensional representations of H (because $\mathcal{O}_q(U)$ does not satisfy this condition). One should instead require that A is a representation of $\mathfrak{h}$. One should also impose the locality condition for a quantum moment map μ: for any $f \in A$ the operation $g \to [\mu(g), f]$ is local in g, in the sense that $[\mu(g), f] = \sum \mu(D_i g) f_i$, where $f_i \in A$, and D_i are h-adically convergent series of differential operators on U. Using (5.2.1) and the locality property, one can compute $[\mu(g), f]$ not only for rational functions g but for arbitrary smooth, holomorphic, or formal functions.

5.4. Quantization of H-biequivariant Poisson–Hopf algebroids and Poisson groupoids. In this section we will heavily use notations and definitions from [EV], Chapters 1 and 2. We advise the reader to look through these chapters before reading this section.

Consider the following two settings.

1. Let A_0 be an H-biequivariant Poisson algebra (see Sect. 2.3 of [EV]). Let $A = A_0[[\hbar]]$. Suppose that A is equipped with an associative product $*$ in such a way that A is a local quantization of A_0 as a Poisson algebra, and the 5-tuple (A, l, r, μ_l, μ_r) is an H-biequivariant associative algebra (where l, r, μ_r, μ_r are the K-linear extensions of the structure maps of A_0 to A).

2. Assume that in addition A_0 is an H-biequivariant Poisson–Hopf algebroid, i.e. it is equipped with maps $\Delta_0, \varepsilon_0, S_0$ satisfying certain axioms (see Sect. 2.4 of [EV]). Suppose that A is as above, and in addition that A is equipped with maps Δ, ε, S, which make A an H-biequivariant Hopf algebroid, and equal $\Delta_0, \varepsilon_0, S_0$ modulo $\hbar$.

Definition. *In these cases, A_0 is called the quasiclassical limit of A, and A is called a quantization of A_0.*

If $H = 1$, then this definition is the usual definition of a quantization of a Poisson and Poisson–Hopf algebra.

Now consider the geometric version of this definition. Let X be an H-biequivariant Poisson manifold over U. Let $A_0 = \mathcal{O}(X)$. Then A_0 satisfies the axioms of an H-biequivariant Poisson algebra, except for maybe property (ii). The notion of quantization of A_0 is defined as above. A quantization A of A_0 will be called an *H-biequivariant quantum space*.

If X is in addition an H-biequivariant Poisson groupoid, then A_0 satisfies the axioms of an H-biequivariant Poisson–Hopf algebroid, except for property (ii) and the fact that the coproduct Δ maps A_0 to $A_0^2 := \mathcal{O}(X \bullet X)[[\hbar]]$, which is a completion of $A_0 \widetilde{\otimes} A_0$, but not to $A_0 \widetilde{\otimes} A_0$ itself (here $X \bullet Y$ is the product of the X-biequivariant Poisson manifolds, defined in [EV]). (This problem already exists for Lie groups, where the coproduct maps $\mathcal{O}(G)$ to $\mathcal{O}(G \times G)$ and not to $\mathcal{O}(G) \otimes \mathcal{O}(G)$.) The notion of quantization of A_0 is defined as above. The quantization is called local if $f * g$ is a bidifferential operator of f, g modulo any power of $\hbar$, and $\Delta(f) = D\Delta_0(f)$, where D is a differential operator modulo any power of $\hbar$. A local quantization A of A_0 will be called an H-biequivariant quantum groupoid.

Suppose that $X = X(G, H, U)$ is a dynamical Poisson groupoid (see Chapter 1 of [EV]), and $A_0 = \mathcal{O}(X)$ is as above. In this case a local quantization A of A_0 will be called a *dynamical quantum groupoid*. If the subspace $\mathcal{O}(U) \otimes \mathcal{O}(G) \otimes \mathcal{O}(U)[[\hbar]] \subset A$ is closed under the product, then it is an H-biequivariant Hopf algebroid. Such Hopf algebroid is called a *dynamical Hopf algebroid*.

Recall that by a preferred quantization of a Poisson Lie group is meant to be a quantization in which the coproduct is undeformed. The notion of a preferred quantization of an H-biequivariant Poisson groupoid or Poisson–Hopf algebroid is defined in the same way.

Conjecture. *(i) Any dynamical Poisson groupoid admits a quantization.*
(ii) Any quasitriangular dynamical Poisson groupoid admits a preferred quantization.

In the case $H = 1$ (Poisson–Lie groups), this conjecture goes back to Drinfeld and is proved in [EK1, EK2].

5.5. The case $H = (\mathbb{C}^*)^N$. In this section we will consider the special case when $H = (\mathbb{C}^*)^N$, and establish the connection between the constructions of this chapter and Chapter 4.

Let $H = (\mathbb{C}^*)^N$. In this case, the main notions of Chapter 5 are simplified:

1. Since H is commutative, the algebra $\mathcal{O}_q(U)$ is just $\mathcal{O}(U)[[\hbar]]$.

2. Denote by $P \subset \mathfrak{h}^*$ the lattice of characters of H ($P = \mathbb{Z}^n$). Let A be an H-biequivariant associative algebra. Then the algebra A can be written as $A = \oplus_{\alpha,\beta\in P} A_{\alpha\beta}$, where $A_{\alpha\beta}$ is the set of elements $a \in A$ such that $h_1 a h_2 = \alpha(h_1)\beta(h_2)a$ (the direct sum is understood in the $\hbar$-adically complete sense). The images of the maps μ_l, μ_r are in A_{00}. The product $A\widetilde{\otimes}B$ can be written in the form $(A\widetilde{\otimes}B)_{\alpha\delta} = \oplus_{\beta\in P} A_{\alpha\beta} \otimes_{\mathcal{O}(U)} B_{\beta\delta}$, where $\mathcal{O}(U)$ is embedded in A via μ_r^A and in B via μ_l^B, and acts from the left (thus this product is similar to the matrix product).

3. The algebra $\mathcal{O}_q((T^*H)_U) = \mathbf{1}$ can be written in form $\mathcal{O}(U) \otimes \mathcal{O}(H)[[\hbar]] = \mathcal{O}(U) \otimes \mathbb{C}[P][[\hbar]]$, where the commutation relations between P and $\mathcal{O}(U)$ are given by $f\chi = \chi f^\chi$, $f \in \mathcal{O}(U)$, $\chi \in P$, where $f^\chi(u) = f(u + \hbar\chi)$.

In particular, in this case we can replace the algebra $\mathcal{O}(U)$ with the field $M_{\mathfrak{h}^*}$ of meromorphic functions on $\mathfrak{h}^*$, imposing the locality condition (see Remark 2, Sect. 5.3). Then Eq. (5.2.1) together with the locality condition implies identities (4.1.1).

Now nothing prevents us from setting $\hbar$ to be no longer a formal parameter, but a nonzero complex number γ. In this situation, it is easy to see that an H-biequivariant algebra (bialgebroid, Hopf algebroid) is the same as an $\mathfrak{h}$-algebra ($\mathfrak{h}$-bialgebroid, $\mathfrak{h}$-Hopf algebroid) with weights belonging to $P \subset \mathfrak{h}^*$. This gives a connection between Chapters 4 and 5.

6. $\mathfrak{h}$-Bialgebroids Associated to Quantum Dynamical R-Matrices of Hecke Type

6.1. The Hecke condition. Let $R : \mathfrak{h}^* \to \underline{\mathrm{E}}\mathrm{nd}(V \otimes V)$ be a quantum dynamical R-matrix with step γ. Consider the $\mathfrak{h}$-bialgebroid A_R introduced in Chapter 4.

It is clear that if $R = 1$ and $\gamma = 0$ then $A_R = M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes \mathcal{O}(\mathrm{End}(V))$. Therefore, for $R \neq 1$ we want the algebra A_R to look like a quantum deformation of $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes \mathcal{O}(\mathrm{End}(V))$.

A natural formalization of this wish is the PBW property, defined below.

The algebra $\bar{A}_R$ has a natural $\mathbb{Z}_+$-grading, given by $deg(f(\lambda^i)) = 0, deg(L_{ab}) = 1$. Denote by $\bar{A}_R^n$ the degree n component of $\bar{A}_R$. It is clear that $\bar{A}_R^n$ are $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$-modules, where the two components of $M_{\mathfrak{h}^*}$ act by left multiplication by $f(\lambda^1)$ and $f(\lambda^2)$.

Definition. *The algebra $\bar{A}_R$ is said to satisfy the Poincare–Birkhoff–Witt (PBW) property if the $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$-module $\bar{A}_R^n$ is isomorphic to the free module $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes S^n End(V)$.*

For a general dynamical R-matrix, the PBW property is not the case. However, the property holds if one imposes an additional "Hecke type" condition on R.

Definition. *R is said to be of strong Hecke type if*

(i) R satisfies Eq. (1.3.6) for some nonzero parameters $p, q \in \mathbb{C}$, $p \neq -q$, such that q/p is not a root of unity, and

(ii) There exists a continuous family $R(t)$, $t \in [0,1]$, of quantum dynamical R-matrices with step $\gamma(t)$ continuously depending on t, satisfying (i) with parameters $p(t)$, $q(t)$, such that $R(0) = 1, p(0) = q(0) = 1, \gamma(0) = 0$, $R(1) = R, p(1) = p, q(1) = q, \gamma(1) = \gamma$.

Example. It is easy to see from the classification that all dynamical R-matrices of gl_N Hecke type are of strong Hecke type. Thus, for dynamical R-matrices of gl_N-type, strong Hecke type is the same as the Hecke type.

Theorem 6.1. *If R is of strong Hecke type then $\bar{A}_R$ satisfies the PBW property.*

This theorem explains the meaning of the Hecke type conditions introduced in Chapter 1. If $\mathfrak{h} = 0$, this theorem is well known (see [FRT]).

6.2. Proof of Theorem 6.1. Let $\tilde{A}$ be the algebra with the same generators as $\bar{A}_R$ and the same relations except the Yang–Baxter relation. Then, as a vector space, the algebra $\tilde{A}$ has the form $\oplus_{n\geq 0}\tilde{A}^n$, $\tilde{A}^n = M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes (\mathrm{End}(V))^{\otimes n}$, and $\bar{A}_R$ is the quotient of $\tilde{A}$ by the Yang–Baxter relation.

Let $H_n(v)$ be the Hecke algebra of type A_n with parameter v. It is the algebra generated by elements T_i, $1 \leq i \leq n-1$, with relations

$$[T_i, T_j] = 0, |i-j| \geq 2;\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1};\ (T_i - 1)(T_i + v) = 0. \tag{6.2.1}$$

If v is not a root of unity of degree n, this algebra is isomorphic to $\mathbb{C}[S_n]$ and therefore semisimple.

Denote by $R^{ii+1}(\lambda)$ the operator $1^{i-1}\bar{\otimes}R(\lambda)\bar{\otimes}1^{n-i-1} : V^{\otimes n} \to M_{\mathfrak{h}^*} \otimes V^{\otimes n}$, where $\bar{\otimes}$ has the meaning defined by (3.1.2).

If R satisfies condition (i), then we have an action of $H_n(v)$, $v = q/p$, on the $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$-module $\tilde{A}^n$, defined by the formula

$$T_iX = P_{ii+1} : R^{ii+1}(\lambda^1)XR^{ii+1}(\lambda^2)^{-1} : P_{ii+1}, \tag{6.2.2}$$

where P_{ii+1} is the permutation of the i^{th} and the $i+1^{\text{st}}$ components in the tensor product $V^{\otimes n}$. This construction explains the origin of the term "Hecke type".

The Yang–Baxter relation in A_R implies that the degree n component $\bar{A}_R^n$ of $\bar{A}_R$ is isomorphic to the space of coinvariants of $T_1, ..., T_{n-1}$ in $\tilde{A}^n$. By semisimplicity of $H_n(v)$, this space is isomorphic to the space of vectors in $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes (\mathrm{End}(V))^{\otimes n}$, which are invariant under T_i.

Now recall that R satisfies condition (ii). Let $R(t)$ be the corresponding family. Consider the corresponding modules $\bar{A}_{R(t)}^n$. Since they can be defined both as coinvariants and invariants, their dimensions cannot jump, which implies that $\bar{A}_{R(0)}^n$ is isomorphic to $\bar{A}_{R(1)}^n$ as a $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*}$-module. However, by our assumptions, $\bar{A}_{R(0)}^n = M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes S^n\mathrm{End}(V)$, while $\bar{A}_{R(1)}^n = \bar{A}_R^n$. This proves the theorem. □

6.3. Hecke condition and quantization. Theorem 6.1 has the following generalization to the case when the step γ is a formal parameter.

Let $R_\gamma = 1 - \gamma r + \sum \gamma^n r_n$ be a formal series whose coefficients are meromorphic functions $\mathfrak{h}^* \to \mathrm{End}(V \otimes V)$. Suppose that R is a quantum dynamical R-matrix with step γ. Let A_{R_γ}, $\bar{A}_{R_\gamma}$ denote the algebras over $K := \mathbb{C}[[\gamma]]$ defined as in Chapter 4.

It is clear that $\bar{A}_{R_\gamma}/\gamma\bar{A}_{R_\gamma} = M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes \mathcal{O}(\mathrm{End}(V))$. Thus the analogue of the PBW property for $\bar{A}_{R_\gamma}$ in this case is the property that the K-module $\bar{A}_{R_\gamma}$ is a topologically free module, i.e. provides a flat deformation of $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes \mathcal{O}(\mathrm{End}(V))$.

Theorem 6.2. *If R_γ satisfies the Hecke equation (1.3.6) for some $p(\gamma) = 1 + O(\gamma)$, $q(\gamma) = 1 + O(\gamma)$, then $\bar{A}_{R_\gamma}$ is a flat deformation of $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes \mathcal{O}(End(V))$.*

Proof. Analogous to the proof of Theorem 6.1 □

Corollary 6.1. *Under the assumption of Theorem 6.2, A_{R_γ} is a flat deformation of $M_{\mathfrak{h}^*} \otimes M_{\mathfrak{h}^*} \otimes \mathcal{O}(GL(V))$.*

If R_γ is holomorphic in an open set $U \subset \mathfrak{h}^*$ then we can define algebras $\bar{A}^U_{R_\gamma}$, $A^U_{R_\gamma}$ in the same way as $\bar{A}_{R_\gamma}$, A_{R_γ}, except that $M_{\mathfrak{h}^*}$ is replaced with the algebra of holomorphic functions $\mathcal{O}(U)$ on U. It is clear that Theorem 6.2 and Corollary 6.1 are valid for these algebras:

Proposition 6.1. *Under the assumptions of Theorem 6.2, the algebras $\bar{A}^U_{R_\gamma}$, $A^U_{R_\gamma}$ are topologically free over K.*

Now let $R_\gamma : U \to \mathrm{End}(V \otimes V)[[\gamma]]$ be a quantum dynamical R-matrix holomorphic on U which satisfies the condition of Theorem 6.2. Let $p(\gamma) = 1 + a\gamma + O(\gamma^2)$, $q(\gamma) = 1 + b\gamma + O(\gamma^2)$, $\gamma \to 0$. Then from the quadratic equation for $R^\vee$ we get the unitarity condition

$$r^{21} + r = (b - a)P - (b + a), \tag{6.3.1}$$

and from the quantum dynamical Yang–Baxter equation for R we get the classical dynamical Yang–Baxter equation for r. Thus, according to Chapter 1 of [EV], r defines a structure of a quasitriangular dynamical Poisson groupoid on $U \times GL(V) \times U$. In particular, we have the corresponding dynamical Poisson–Hopf algebroid $A^{0U}_r = \mathcal{O}(U) \otimes \mathcal{O}(GL(V)) \otimes \mathcal{O}(U)$ (here $\mathcal{O}(G)$ denotes the algebra of polynomial functions on G).

Theorem 6.3. *The dynamical Hopf algebroid $A^U_{R_\gamma}$ is a quantization of the dynamical Poisson–Hopf algebroid A^{0U}_r.*

Proof. Since we know that $A^U_{R_\gamma}$ is topologically free, the proof is the direct computation of the quasiclassical limit and then comparison with Chapter 1 of [EV]. □

Let $G = GL(V)$, H be a maximal torus in G, and $U \subset \mathfrak{h}^*$ a polydisc. Let $X(G, H, U)$ be the Lie groupoid $U \times G \times U$ with two actions of H, defined in Chapter 1 of [EV].

Theorem 6.4. *Any structure of a quasitriangular dynamical Poisson groupoid on $X(G, H, U)$ admits a preferred quantization.*

Proof. The statement follows from Theorem 1.6 and Theorem 6.3. □

Remark. Notice that if R_γ fails to satisfy the Hecke condition modulo γ^2, then the algebra A_{R_γ} is not topologically free. Indeed, in this case r does not satisfy the unitarity condition, so according to Chapter 1 of [EV] the bracket defined by r on $U \times GL(V) \times U$ is not Poisson (i.e. does not satisfy the Jacobi identity). This means that the corresponding deformation is not flat, since a flat deformation of a commutative algebra induces a Poisson structure on this algebra. Thus, the Hecke condition seems to be intrinsic for good properties of the algebra A_R.

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