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Twisted Quantum Affine Algebras

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Abstract: We give a highest weight classification of the finite-dimensional irreducible representations of twisted quantum affine algebras. As in the untwisted case, such representations are in one-to-one correspondence with n-tuples of monic polynomials in one variable. But whereas in the untwisted case n is the rank of the underlying finite-dimensional complex simple Lie algebra $\mathfrak{g}$, in the twisted case n is the rank of the subalgebra of $\mathfrak{g}$ fixed by the diagram automorphism. The way in which such an n-tuple determines a representation is also more complicated than in the untwisted case.

Introduction

Quantum affine algebras are one of the most important classes of quantum groups. Their finite-dimensional representations lead to solutions of the quantum Yang–Baxter equation which are trigonometric functions of the spectral parameter (see [7], Sect. 12.5 B) and are thus related to various types of integrable models in statistical mechanics and field theory. Quantum affine algebras have also been shown to arise as "quantum symmetry groups" of certain integrable quantum field theories, such as affine Toda field theories (see [2] and [10]). More precisely, there is an affine Toda field theory associated to any affine Lie algebra $\mathfrak{k}$, and this theory admits as a quantum symmetry group the quantum affine algebra $U_q(\mathfrak{k}^*)$, where $\mathfrak{k}^*$ is the affine Lie algebra *dual* to $\mathfrak{k}$ (whose Dynkin diagram is obtained from that of $\mathfrak{k}$ by reversing the arrows). Since $\mathfrak{k}^*$ is often twisted even if $\mathfrak{k}$ is untwisted, this shows that the representation theory of twisted quantum affine algebras is, in this context at least, just as important as that of untwisted ones. However, there appear to be virtually no results in the literature on the twisted case. The only exceptions appear to be [12] and [14], which prove the existence of finite-dimensional irreducible representations of twisted quantum affine algebras $U_q(\mathfrak{k})$ which are irreducible under certain subalgebras of the form $U_q(\mathfrak{m})$, where $\mathfrak{m}$ is a finite-dimensional Lie subalgebra of $\mathfrak{k}$, and [15] and [17] which construct, by vertex operator methods, quantum analogues of the standard modules (which are, of course, infinite-dimensional).

In [6] and [8], we gave a classification of the finite-dimensional irreducible representations of untwisted quantum affine algebras in terms of their highest weights, which are in one-to-one correspondence with n-tuples of polynomials in one variable with constant coefficient one (n being the rank of the underlying finite-dimensional Lie algebra). The purpose of this paper is to extend this result to the twisted case. We find that the finite-dimensional irreducible representations of twisted quantum affine algebras are again parametrized by n-tuples of polynomials. But n is now the rank of the fixed point subalgebra of the diagram automorphism, and the way in which such an n-tuple determines a highest weight is more complicated than in the untwisted case.

In the analogous classical situation, we classified in [8] the finite-dimensional irreducible representations of the twisted affine Lie algebra $\hat{\mathfrak{g}}^\sigma$, associated to a diagram automorphism σ of a finite-dimensional complex simple Lie algebra $\mathfrak{g}$, by using the canonical embedding of $\hat{\mathfrak{g}}^\sigma$ in the untwisted affine Lie algebra $\hat{\mathfrak{g}}$. Namely, we showed that every finite-dimensional irreducible representation of $\hat{\mathfrak{g}}$ decomposes under $\hat{\mathfrak{g}}^\sigma$ into a finite direct sum of irreducibles, and that every finite-dimensional irreducible representation of $\hat{\mathfrak{g}}^\sigma$ arises in this way. Together with the results of [6] and [7], this gave the desired classification. In the quantum case, Jing [16] has shown how to embed $U_q(\hat{\mathfrak{g}}^\sigma)$ into $U_q(\hat{\mathfrak{g}})$, but this embedding is not as simple as in the classical case and we have preferred to use a direct approach, following the method used for untwisted quantum affine algebras in [6] and [8]. Since the proofs are similar to those for the untwisted case, we omit many of the details.

1. Twisted Quantum Affine Algebras

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra with Cartan matrix $A = (a_{ij})_{i,j\in I}$. Let $\sigma : I \to I$ be a bijection such that $a_{\sigma(i)\sigma(j)} = a_{ij}$ for all $i, j \in I$, and let m be the order of σ; we assume that $m > 1$ (thus, m = 2 or 3). We also denote by σ the corresponding Lie algebra automorphism of $\mathfrak{g}$.

Fix a primitive mth root of unity $\omega \in \mathbb{C}^\times$. For $r \in \mathbb{Z}/m\mathbb{Z}$, let $\mathfrak{g}_r$ be the eigenspace of σ on $\mathfrak{g}$ with eigenvalue ω^r. Then,

$$\mathfrak{g} = \bigoplus_{r\in\mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_r$$

is a $\mathbb{Z}/m\mathbb{Z}$-gradation of $\mathfrak{g}$ (see [18], Chapter 8).

The fixed point set $\mathfrak{g}_0$ of σ is a simple Lie algebra. The nodes of its Dynkin diagram are naturally indexed by I_σ, the set of σ-orbits on I. Moreover, $\mathfrak{g}_1$ is an irreducible representation of $\mathfrak{g}_0$. Let $\{\alpha_i\}_{i\in I_\sigma}$ be a set of simple roots of $\mathfrak{g}_0$, and let θ be the highest weight of $\mathfrak{g}_1$ as a representation of $\mathfrak{g}_0$. Let $\{n_i\}_{i\in I_\sigma}$ be the positive integers such that

$$\theta = \sum_{i\in I_\sigma} n_i\alpha_i.$$

The twisted affine Lie algebra $\hat{\mathfrak{g}}^\sigma$ is the universal central extension (with one-dimensional centre) of the twisted loop algebra

$$L(\mathfrak{g})^\sigma = \{f \in \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \mid f(\omega t) = \sigma(f(t))\},$$

where t is an indeterminate. It is well known (see [18]) that $\hat{\mathfrak{g}}^\sigma$ is a symmetrizable Kac–Moody algebra whose Dynkin diagram has nodes indexed by

$$\hat{I}_\sigma = I_\sigma \amalg \{0\}.$$

(Note: the node labelled 0 here is labelled ϵ in [18].) Let $A^\sigma = (a_{ij}^\sigma)_{i,j\in\hat{I}_\sigma}$ be the (generalized) Cartan matrix of $\hat{\mathfrak{g}}^\sigma$, and let $\{d_i\}_{i\in\hat{I}_\sigma}$ be the coprime positive integers such that the matrix $(d_i a_{ij}^\sigma)$ is symmetric. Setting $n_0 = 1$, we have

$$\sum_{i\in\hat{I}_\sigma} n_i d_i a_{ij}^\sigma = 0 \quad \text{for all } j \in \hat{I}_\sigma. \tag{1}$$

Since $\hat{\mathfrak{g}}^\sigma$ is a symmetrizable Kac–Moody algebra there is, according to Drinfel'd and Jimbo, a corresponding quantum group $U_q(\hat{\mathfrak{g}}^\sigma)$. Namely, let q be a non-zero complex number, *assumed throughout this paper not to be a root of unity*. Let $q_i = q^{d_i}$ for $i \in \hat{I}_\sigma$. If $n \in \mathbb{Z}$, set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}},$$

and for $n \geq r \geq 0$,

$$[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q,$$
$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}.$$

Proposition 1.1. *There is a Hopf algebra $U_q(\hat{\mathfrak{g}}^\sigma)$ over $\mathbb{C}$ which is generated as an algebra by elements $e_i^\pm$, $k_i^{\pm1}$ ($i \in \hat{I}_\sigma$), with the following defining relations:*

$$k_i k_i^{-1} = k_i^{-1} k_i = 1; \quad k_i k_j = k_j k_i;$$
$$k_i e_j^\pm k_i^{-1} = q_i^{\pm a_{ij}^\sigma} e_j^\pm;$$
$$[e_i^+, e_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}};$$
$$\sum_{r=0}^{1-a_{ij}^\sigma} (-1)^r \begin{bmatrix} 1 - a_{ij}^\sigma \\ r \end{bmatrix}_{q_i} (e_i^\pm)^r e_j^\pm (e_i^\pm)^{1-a_{ij}^\sigma - r} = 0 \quad \textit{if} \quad i \neq j.$$

The comultiplication Δ of $U_q(\hat{\mathfrak{g}}^\sigma)$ is given by

$$\Delta(e_i^+) = e_i^+ \otimes k_i + 1 \otimes e_i^+, \quad \Delta(e_i^-) = e_i^- \otimes 1 + k_i^{-1} \otimes e_i^-, \quad \Delta(k_i^{\pm1}) = k_i^{\pm1} \otimes k_i^{\pm1}.$$

It follows from (1) that

$$c = \prod_{i\in\hat{I}_\sigma} k_i^{n_i}$$

lies in the centre of $U_q(\hat{\mathfrak{g}}^\sigma)$. Let $U_q(L(\mathfrak{g})^\sigma)$ be the quotient of $U_q(\hat{\mathfrak{g}}^\sigma)$ by the ideal generated by $c - 1$. Note that, since c is group-like, $U_q(L(\mathfrak{g})^\sigma)$ inherits a natural Hopf algebra structure from $U_q(\hat{\mathfrak{g}}^\sigma)$.

The following theorem is an analogue of a result of Drinfel'd ([13], Theorem 4). To state it, we introduce a quantity $\lambda_{\mathfrak{g}}$, which is equal to 2 if $\mathfrak{g}$ is of type A_{2n} for some

$n \geq 1$, and equal to 1 otherwise. Further, let u_1 and u_2 be independent indeterminates and, for $i, j \in I$, define $d_{ij} \in \mathbb{Q}$, $P_{ij}^{\pm}, F_{ij}^{\pm}, G_{ij}^{\pm} \in \mathbb{C}[u_1, u_2]$ as follows:

if $\sigma(i) = i$, then $d_{ij} = \frac{1}{2}$, $P_{ij}^{\pm}(u_1, u_2) = 1$;

if $a_{i\sigma(i)} = 0$ and $\sigma(j) \neq j$, then $d_{ij} = \frac{1}{4m}$, $P_{ij}^{\pm}(u_1, u_2) = 1$;

if $a_{i\sigma(i)} = 0$ and $\sigma(j) = j$, then $d_{ij} = \frac{1}{2}$, $P_{ij}^{\pm}(u_1, u_2) = \frac{u_1^m q^{\pm 2m} - u_2^m}{u_1 q^{\pm 2} - u_2}$;

if $a_{i\sigma(i)} = -1$, then $d_{ij} = \frac{1}{8}$, $P_{ij}^{\pm}(u_1, u_2) = u_1 q^{\pm 1} + u_2$;

$$F_{ij}^{\pm}(u_1, u_2) = \prod_{r \in \mathbb{Z}/m\mathbb{Z}} (u_1 - \omega^r q^{\pm \lambda_{\mathfrak{g}} a_{i\sigma^r(j)}} u_2);$$

$$G_{ij}^{\pm}(u_1, u_2) = \prod_{r \in \mathbb{Z}/m\mathbb{Z}} (u_1 q^{\pm \lambda_{\mathfrak{g}} a_{i\sigma^r(j)}} - \omega^r u_2).$$

We note that $G_{ij}^{\pm}(u_1, u_2) = -F_{ji}^{\pm}(u_2, u_1)$.

Definition 1.2. *For $i \in I$, let $\bar{i} \in I_\sigma$ be the σ-orbit of i. Let $D_q(\mathfrak{g})^\sigma$ be the associative algebra over $\mathbb{C}$ with generators $X_{i,k}^{\pm}$ ($i \in I$, $k \in \mathbb{Z}$), $H_{i,k}$ ($i \in I$, $k \in \mathbb{Z}\backslash\{0\}$), $K_i^{\pm 1}$ ($i \in I$), and the following defining relations:*

$$X_{\sigma(i),k}^{\pm} = \omega^k X_{i,k}^{\pm}; \quad H_{\sigma(i),k} = \omega^k H_{i,k}; \quad K_{\sigma(i)}^{\pm 1} = K_i^{\pm 1};$$

$$K_i K_i^{-1} = K_i^{-1} K_i = 1; \quad K_i K_j = K_j K_i;$$

$$H_{i,k} H_{j,l} = H_{j,l} H_{i,k}; \quad K_i H_{j,l} = H_{j,l} K_i;$$

$$K_i X_{j,k}^{\pm} K_i^{-1} = q^{\pm \lambda_{\mathfrak{g}} \sum_{r \in \mathbb{Z}/m\mathbb{Z}} a_{i\sigma^r(j)}} X_{j,k}^{\pm};$$

$$[H_{i,k}, X_{j,l}^{\pm}] = \pm \frac{1}{k} \left(\sum_{r \in \mathbb{Z}/m\mathbb{Z}} [k a_{i\sigma^r(j)}/d_{\bar{i}}]_{q_{\bar{i}}} \omega^{kr} \right) X_{j,k+l}^{\pm};$$

$$[X_{i,k}^{+}, X_{j,l}^{-}] = \sum_{r \in \mathbb{Z}/m\mathbb{Z}} \delta_{\sigma^r(i),j} \omega^{rl} \left(\frac{\Psi_{i,k+l}^{+} - \Psi_{i,k+l}^{-}}{q_{\bar{i}} - q_{\bar{i}}^{-1}} \right),$$

where the $\Psi_{i,k}^{\pm}$ are defined by

$$\sum_{k=0}^{\infty} \Psi_{i,\pm k}^{\pm} u^k = K_i^{\pm 1} \exp \left(\pm (q_{\bar{i}} - q_{\bar{i}}^{-1}) \sum_{l=1}^{\infty} H_{i,\pm l} u^l \right),$$

u being an indeterminate, and $\Psi_{i,k}^{\pm} = 0$ if $\mp k > 0$;

$$F_{ij}^{\pm}(u_1, u_2) X_i^{\pm}(u_1) X_j^{\pm}(u_2) = G_{ij}^{\pm}(u_1, u_2) X_j^{\pm}(u_2) X_i^{\pm}(u_1),$$

where

$$X_i^{\pm}(u) = \sum_{k \in \mathbb{Z}} X_{i,k}^{\pm} u^{-k};$$

$$\mathrm{Sym}\{P_{ij}^{\pm}(u_1, u_2)(X_j^{\pm}(v) X_i^{\pm}(u_1) X_i^{\pm}(u_2) - (q^{2md_{ij}} + q^{-2md_{ij}}) X_i^{\pm}(u_1) X_j^{\pm}(v) X_i^{\pm}(u_2) + X_i^{\pm}(u_1) X_i^{\pm}(u_2) X_j^{\pm}(v))\} = 0$$

if $a_{ij} = -1$ and $\sigma(i) \neq j$, where u_1, u_2 and v are independent indeterminates and Sym denotes symmetrization over u_1, u_2;

$$\mathrm{Sym}\{(q^{3\lambda_{\mathfrak{g}}/2}u_1^{\mp 1} - (q^{\lambda_{\mathfrak{g}}/2} + q^{-\lambda_{\mathfrak{g}}/2})u_2^{\mp 1} + q^{-3\lambda_{\mathfrak{g}}/2}u_3^{\mp 1})X_i^{\pm}(u_1)X_i^{\pm}(u_2)X_i^{\pm}(u_3)\} = 0 \tag{$2^{\pm}$}$$

and

$$\mathrm{Sym}\{(q^{-3\lambda_{\mathfrak{g}}/2}u_1^{\pm 1} - (q^{\lambda_{\mathfrak{g}}/2} + q^{-\lambda_{\mathfrak{g}}/2})u_2^{\pm 1} + q^{3\lambda_{\mathfrak{g}}/2}u_3^{\pm 1})X_i^{\pm}(u_1)X_i^{\pm}(u_2)X_i^{\pm}(u_3)\} = 0 \tag{$3^{\pm}$}$$

if $a_{i\sigma(i)} = -1$, where Sym denotes symmetrization over the independent indeterminates u_1, u_2, u_3.

Theorem 1.3. *There exists an isomorphism of algebras between $U_q(L(\mathfrak{g})^\sigma)$ and $D_q(\mathfrak{g})^\sigma$ such that*

$$e_{\bar{i}}^{+} = X_{i,0}^{+}, \quad e_{\bar{i}}^{-} = \frac{1}{p_i}X_{i,0}^{-}, \quad k_{\bar{i}} = K_i,$$

where $i \in I$ belongs to the σ-orbit $\bar{i}$.

Remarks. 1. There is a similar realization of $U_q(\hat{\mathfrak{g}}^\sigma)$. Theorem 1.3 is, however, sufficient for our purposes since it can be shown (cf. [7], Proposition 12.2.3) that the central element c acts as one on every finite-dimensional representation of $U_q(\hat{\mathfrak{g}}^\sigma)$.

2. Relations (2) and (3) are present only when $\hat{\mathfrak{g}}^\sigma$ is of type $A_{2n}^{(2)}$. Drinfel'd ([13], Theorem 4) has analogues of only two of these four relations (namely (2^-) and (3^+)). The other two can be shown to be consequences of these together with the other defining relations of $D_q(\mathfrak{g})^\sigma$. We have included all four partly for reasons of symmetry, and partly because they are all needed in subsequent calculations.
3. The isomorphism in 1.3 depends on the choice of the section $\bar{i} \mapsto i$ of the canonical projection $I \to I_\sigma$, but any two such isomorphisms differ only by a rescaling on the generators $e_{\bar{i}}^{\pm}$ ($\bar{i} \in \hat{I}_\sigma$).

For a proof of this theorem, and an explicit description of the isomorphism, see [16], Theorem 3.1 and [17], Proposition 2.1. However, in the $A_{2n}^{(2)}$ case, the q in [16] and [17] must be replaced by q^2 to get the algebras denoted here by $U_q(L(\mathfrak{g})^\sigma)$ and $D_q(\mathfrak{g})^\sigma$. Compare also [1] and [11] for analogous results in the untwisted case.

For later use, we record here the defining relations, and the form of the isomorphism in 1.3, for the simplest twisted quantum affine algebra $U_q(L(sl_3)^\tau)$, where τ is the non-trivial diagram automorphism of $sl_3(\mathbb{C})$. In this case, we may drop the index i from the generators of $U_q(L(sl_3)^\tau)$ (since $|I_\tau| = 1$). The generalized Cartan matrix is

$$A^\tau = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix},$$

so that $d_0 = 4$ and $d_1 = 1$.

The defining relations are as follows:

$$KK^{-1} = K^{-1}K = 1, \quad KH_k = H_kK, \quad H_kH_l = H_lH_k,$$
$$KX_k^{\pm}K^{-1} = q^{\pm 2}X_k^{\pm}, [X_k^+, X_l^-] = \frac{\psi_{k+l}^+ - \psi_{k+l}^-}{q - q^{-1}},$$
$$\text{where } \sum_{k=0}^{\infty} \psi_{\pm k}^{\pm} u^k = K^{\pm 1} \exp\left(\pm(q - q^{-1}) \sum_{l=1}^{\infty} H_{\pm l} u^l\right),$$
$$[H_k, X_l^{\pm}] = \pm \frac{[2k]_q}{k}(q^{2k} + q^{-2k} + (-1)^{k+1})X_{k+l}^{\pm} \quad \text{if } k \neq 0,$$
$$X_{k+2}^{\pm}X_l^{\pm} + (q^{\mp 2} - q^{\pm 4})X_{k+1}^{\pm}X_{l+1}^{\pm} - q^{\pm 2}X_k^{\pm}X_{l+2}^{\pm}$$
$$= q^{\pm 2}X_l^{\pm}X_{k+2}^{\pm} + (q^{\pm 4} - q^{\mp 2})X_{l+1}^{\pm}X_{k+1}^{\pm} - X_{l+2}^{\pm}X_k^{\pm},$$
$$\mathrm{Sym}(q^3X_{k\mp 1}^{\pm}X_l^{\pm}X_m^{\pm} - (q + q^{-1})X_k^{\pm}X_{l\mp 1}^{\pm}X_m^{\pm} + q^{-3}X_k^{\pm}X_l^{\pm}X_{m\mp 1}^{\pm}) = 0,$$
$$\mathrm{Sym}(q^{-3}X_{k\pm 1}^{\pm}X_l^{\pm}X_m^{\pm} - (q + q^{-1})X_k^{\pm}X_{l\pm 1}^{\pm}X_m^{\pm} + q^3X_k^{\pm}X_l^{\pm}X_{m\pm 1}^{\pm}) = 0,$$

where Sym means the sum over all permutations of k, l, m.

The isomorphism in 1.3 is given by

$$e_0^+ = K^{-2}(X_0^-X_1^- - q^2X_1^-X_0^-), \; e_0^- = \frac{1}{[4]_q^2}(X_{-1}^+X_0^+ - q^{-2}X_0^+X_{-1}^+)K^2,$$
$$k_0 = K^{-2}, \; e_1^+ = X_0^+, \; e_1^- = X_0^-, \; k_1 = K.$$

Let U_+^σ (resp. U_-^σ, U_0^σ) be the subalgebras of U^σ generated by the $X_{i,k}^+$ (resp. by the $X_{i,k}^-$, by the $\Psi_{i,k}^{\pm}$) for $i \in I$, $k \in \mathbb{Z}$.

Proposition 1.4. $U^\sigma = U_-^\sigma.U_0^\sigma.U_+^\sigma$.

The proof is straightforward.

2. Some Subalgebras of $U_q(L(\mathfrak{g})^\sigma)$

The study of untwisted quantum affine algebras can be reduced, to some extent at least, to the case of quantum affine sl_2, by noting that any algebra of the former type can be generated by finitely many copies of the latter (see [1], Proposition 3.8). In the twisted case, one needs $U_q(L(sl_3)^\tau)$ in addition, where τ is the unique non-trivial diagram automorphism of $sl_3(\mathbb{C})$.

We recall the definition of quantum affine sl_2:

Definition 2.1. *$U_q(L(sl_2))$ is the associative algebra with generators $X_k^{\pm}$ ($k \in \mathbb{Z}$), H_k ($k \in \mathbb{Z}\backslash\{0\}$), $K^{\pm 1}$, and the following defining relations:*

$$KK^{-1} = K^{-1}K = 1; \; KH_k = H_kK; \; H_kH_l = H_lH_k;$$
$$KX_k^{\pm}K^{-1} = q^{\pm 2}X_k^{\pm};$$
$$[H_k, X_l^{\pm}] = \pm\frac{1}{k}[2k]_qX_{k+l}^{\pm};$$
$$X_{k+1}^{\pm}X_l^{\pm} - q^{\pm 2}X_l^{\pm}X_{k+1}^{\pm} = q^{\pm 2}X_k^{\pm}X_{l+1}^{\pm} - X_{l+1}^{\pm}X_k^{\pm};$$
$$[X_k^+, X_l^-] = \frac{\Psi_{k+l}^+ - \Psi_{k+l}^-}{q - q^{-1}},$$

where

$$\sum_{k=0}^{\infty} \Psi^{\pm}_{\pm k} u^k = K^{\pm 1} exp\left(\pm(q-q^{-1})\sum_{l=1}^{\infty} H_{\pm l} u^l\right),$$

and $\Psi^{\pm}_k = 0$ *if* $\mp k > 0$.

See [7], Theorem 12.2.1 – we have set the central element $\mathcal{C}^{1/2}$ equal to one.
The result we need is the following:

Proposition 2.2. *Let* $i \in I$.

(i) *If* $\sigma(i) \neq i$ *and* $a_{i\sigma(i)} \neq 0$, *there is a homomorphism of algebras* $\varphi_i : U_q(L(sl_3)^\tau) \to U_q(L(\mathfrak{g})^\sigma)$ *such that*

$$\varphi_i(X^{\pm}_k) = X^{\pm}_{i,k}, \quad \varphi_i(H_k) = H_{i,k}, \quad \varphi_i(\Psi^{\pm}_k) = \Psi^{\pm}_{i,k}, \quad \varphi_i(K) = K_i.$$

(ii) *If* $\sigma(i) \neq i$ *and* $a_{i\sigma(i)} = 0$, *there is a homomorphism of algebras* $\varphi_i : U_q(L(sl_2)) \to U_q(L(\mathfrak{g})^\sigma)$ *such that*

$$\varphi_i(X^{\pm}_k) = X^{\pm}_{i,k}, \quad \varphi_i(H_k) = H_{i,k}, \quad \varphi_i(\Psi^{\pm}_k) = \Psi^{\pm}_{i,k}, \quad \varphi_i(K) = K_i.$$

(iii) *If* $\sigma(i) = i$, *there is a homomorphism of algebras* $\varphi_i : U_{q^m}(L(sl_2)) \to U_q(L(\mathfrak{g})^\sigma)$ *such that*

$$\varphi_i(X^{+}_k) = \frac{1}{m} X^{+}_{i,mk}, \quad \varphi_i(X^{-}_k) = X^{-}_{i,mk},$$
$$\varphi_i(H_k) = H_{i,mk}, \quad \varphi_i(\Psi^{\pm}_k) = \Psi^{\pm}_{i,mk}, \quad \varphi_i(K) = K_i.$$

Proof. Straightforward verification, using 1.3 and 2.1. □

Remarks. 1. We have dropped the subscript i from the generators of $U_q(L(sl_3)^\tau)$ in (i), since $|I_\tau| = 1$.
2. In (iii), the generators $X^{\pm}_{i,k}$, $H_{i,k}$, $\Psi^{\pm}_{i,k}$ of $U_q(L(\mathfrak{g})^\sigma)$ vanish if k is not a multiple of m.
3. We expect that the homomorphisms φ_i are injective, but we shall not need this.

3. Finite-Dimensional Representations

A representation V of U^σ (i.e. a left U^σ-module) is said to be of type I if each k_i ($i \in \hat{I}_\sigma$) acts semisimply on V with eigenvalues which are integer powers of q_i. It is not difficult to show that every finite-dimensional irreducible representation of U^σ can be obtained from a type I representation by twisting with an automorphism of U^σ of the form $e^+_i \mapsto \epsilon_i e^+_i$, $e^-_i \mapsto e^-_i$, $k_i \mapsto \epsilon_i k_i$, where each $\epsilon_i = \pm 1$ (cf. [7], Proposition 12.2.3).

If V is a type I representation of U^σ, a vector $v \in V$ is said to be a highest weight vector if v is annihilated by the $X^+_{i,k}$ for all $i \in I$, $k \in \mathbb{Z}$, and is a simultaneous eigenvector for the elements of U^σ_0. If, in addition, $V = U^\sigma.v$, then V is said to be a highest weight representation. Moreover, if $\{\psi^{\pm}_{i,k}\}_{i \in I, k \in \mathbb{Z}}$ are the scalars such that

$$\Psi^{\pm}_{i,k}.v = \psi^{\pm}_{i,k} v,$$

the pair of $(I \times \mathbb{Z})$-tuples $\boldsymbol{\psi}^{\pm} = \{\psi^{\pm}_{i,k}\}_{i\in I, k\in\mathbb{Z}}$ is called the highest weight of V (or the weight of v). Note that we necessarily have

$$\begin{aligned} \psi^{\pm}_{i,k} &= 0 \quad \text{if } \mp k > 0, \quad \psi^{+}_{i,0}\psi^{-}_{i,0} = 1, \\ \psi^{\pm}_{\sigma(i),k} &= \omega^k \psi^{\pm}_{i,k} \quad \text{for all } i \in I, k \in \mathbb{Z}. \end{aligned} \tag{4}$$

Conversely, by the usual Verma module construction, it is easy to show that, for any $\boldsymbol{\psi}^{\pm} = \{\psi^{\pm}_{i,k}\}_{i\in I,k\in\mathbb{Z}}$ satisfying (4), there is, up to isomorphism, exactly one irreducible representation $V(\boldsymbol{\psi}^{\pm})$ with highest weight $\boldsymbol{\psi}^{\pm}$.

The following theorem is the main result of this paper:

Theorem 3.1. *(i) Every finite-dimensional irreducible type I representation of $U_q(L(\mathfrak{g})^\sigma)$ is highest weight.*

(ii) If $\boldsymbol{\psi}^{\pm} = \{\psi^{\pm}_{i,k}\}_{i\in I,k\in\mathbb{Z}}$, the highest weight representation $V(\boldsymbol{\psi}^{\pm})$ of $U_q(L(\mathfrak{g})^\sigma)$ is finite-dimensional if and only if there exist polynomials $P_i \in \mathbb{C}[u]$ $(i \in I)$ with constant coefficient one such that

$$\sum_{k=0}^{\infty} \psi^{+}_{i,k} u^k = \sum_{k=0}^{\infty} \psi^{-}_{i,-k} u^{-k} = \begin{cases} q^{m \deg P_i} \frac{P_i(q^{-2m}u)}{P_i(u)} & \text{if } \sigma(i) \neq i \text{ and } a_{i\sigma(i)} \neq 0, \\ q^{\deg P_i} \frac{P_i(q^{-2}u)}{P_i(u)} & \text{if } \sigma(i) \neq i \text{ and } a_{i\sigma(i)} = 0, \\ q^{m \deg P_i} \frac{P_i(q^{-2m}u^m)}{P_i(u^m)} & \text{if } \sigma(i) = i, \end{cases}$$

in the sense that the first two terms are the Laurent expansions of the third term about $u = 0$ and $u = \infty$, respectively.

The proof of (i) is straightforward (cf. [7], Proposition 12.2.3). The proof of (ii) will occupy the next two sections.

Remark. Since $\Psi^{\pm}_{\sigma(i),k} = \omega^k \Psi^{\pm}_{i,k}$, the polynomials P_i, if they exist, necessarily satisfy the condition

$$P_{\sigma(i)}(u) = P_i(\omega u). \tag{5}$$

Let Π be the set of I-tuples of polynomials $P_i \in \mathbb{C}[u]$ with constant coefficient one satisfying (5). If $\mathbf{P} = \{P_i\}_{i\in I} \in \Pi$, we denote by $V(\mathbf{P})$ the irreducible highest weight representation $V(\boldsymbol{\psi}^{\pm})$ of U^σ (abusing notation), the relation between $\mathbf{P}$ and $\boldsymbol{\psi}^{\pm}$ being as in 3.1.

Proposition 3.2. *Let $\mathbf{P} = \{P_i\}_{i\in I}, \mathbf{Q} = \{Q_i\}_{i\in I} \in \Pi$, and let $v_{\mathbf{P}} \in V(\mathbf{P})$, $v_{\mathbf{Q}} \in V(\mathbf{Q})$ be highest weight vectors. Then, $v_{\mathbf{P}} \otimes v_{\mathbf{Q}}$ is a highest weight vector in $V(\mathbf{P}) \otimes V(\mathbf{Q})$ of weight $\boldsymbol{\phi}^{\pm}$, where $\boldsymbol{\phi}^{\pm}$ is related to the I-tuple $\{P_iQ_i\}_{i\in I}$ in the same way as $\boldsymbol{\psi}^{\pm}$ is related to $\{P_i\}_{i\in I}$ in 3.1.*

This will be proved in Sects. 4 and 5. The following corollary is immediate:

Corollary 3.3. *Let the notation be as in 3.2, and denote the I-tuple $\{P_iQ_i\}_{i\in I}$ by $\mathbf{P} \otimes \mathbf{Q}$. Then, $V(\mathbf{P} \otimes \mathbf{Q})$ is isomorphic as a representation of $U_q(L(\mathfrak{g})^\sigma)$ to a subquotient of $V(\mathbf{P}) \otimes V(\mathbf{Q})$.*

4. The $U_q(L(sl_3)^\tau)$ Case

In this section, we prove 3.1 and 3.2 for $U_q(L(sl_3)^\tau)$, where τ is the non-trivial diagram automorphism of $sl_3(\mathbb{C})$, and we denote $U_q(L(sl_3)^\tau)$ by U^τ. The explicit form of the generators and relations of U^τ was given at the end of Sect. 1. It will be convenient to set

$$\tilde{e}_0 = X_0^- X_1^- - q^2 X_1^- X_0^-,$$

so that, in the isomorphism in 1.3, $e_0^+ = K^{-2}\tilde{e}_0$, and to write

$$(X_k^\pm)^{(r)} = \frac{(X_k^\pm)^r}{[r]_q!}, \quad (\tilde{e}_0)^{(r)} = \frac{(\tilde{e}_0)^r}{[r]_{q^4}!}.$$

The crucial result for the proof of 3.1 in this case is the next proposition.

Definition 4.1. *Define elements* $\{\mathcal{P}_r\}_{r\in\mathbb{N}}$ *in* U_0^τ *by* $\mathcal{P}_0 = 1$ *and*

$$\mathcal{P}_r = -\frac{1}{1-q^{-4r}} \sum_{j=0}^{r-1} \Psi_{j+1}^+ \mathcal{P}_{r-j-1} K^{-1}. \tag{6}$$

If we introduce the formal power series

$$\mathcal{P}(u) = \sum_{r=0}^{\infty} \mathcal{P}_r u^r, \quad \Psi^\pm(u) = \sum_{r=0}^{\infty} \Psi_{\pm r}^\pm u^{\pm r}$$

in an indeterminate u, then 4.1 is equivalent to saying that $\mathcal{P}(u)$ has constant coefficient one and that

$$\Psi^+(u) = K\frac{\mathcal{P}(q^{-4}u)}{\mathcal{P}(u)}.$$

Let X^+ be the linear subspace of U^τ spanned by the X_k^+ for $k \in \mathbb{Z}$.

Proposition 4.2. *For all* $r \in \mathbb{N}$*, we have the following congruences (* mod $U^\tau X^+$*):*

(i)$_r$ $(X_0^+)^{(2r+2)}(\tilde{e}_0)^{(r+1)} \equiv (-1)^{r+1} q^{-2(r+1)(2r+1)} [4]_q^{r+1} \mathcal{P}_{r+1} K^{2r+2}$;

(ii)$_r$ $(X_0^+)^{(2r+1)}(\tilde{e}_0)^{(r+1)} \equiv (-1)^r q^{-4r(r+1)} [4]_q^{r+1} \sum_{j=0}^{r} X_{j+1}^- \mathcal{P}_{r-j} K^{2r+1}$;

(iii)$_r$ $(X_0^+)^{(2r+1)}(\tilde{e}_0)^{(r+1)} \equiv q^{-6r+2}[4]_q K X_1^- (X_0^+)^{(2r)}(\tilde{e}_0)^{(r)}$

$$+q^{-8r+4}\frac{[4]_q}{[2]_q[3]_q} K^2 [H_1, (X_0^+)^{(2r-1)}(\tilde{e}_0)^{(r)}].$$

Proof. All three congruences are easily checked when $r = 0$.

Assuming (iii)$_r$, one deduces (ii)$_r$ from (i)$_{r-1}$, (ii)$_{r-1}$ and (6), and then (i)$_r$ follows from (ii)$_r$ by multiplying on the left by X_0^+ and using (6) again.

Thus, the main point is to prove (iii)$_r$. For this, one needs identities (7)–(16) below:

$$(X_0^+)^{(r)} X_1^+ = -q^{-3r}[r-1]_q X_1^+ (X_0^+)^{(r)} + q^{-3r+3} X_0^+ X_1^+ (X_0^+)^{(r-1)}; \tag{7}$$

$$[H_1, (X_0^+)^{(r)}] = [3]_q \left\{ \left(\frac{q^{-3r+3} + q^{-r+3} - q^{-r+1} - q^{-r-1}}{q - q^{-1}} \right) X_1^+ (X_0^+)^{(r-1)} \right.$$
$$\left. + q^{-2r+4} X_0^+ X_1^+ (X_0^+)^{(r-2)} \right\}; \tag{8}$$

$$[(X_0^+)^{(r)}, X_1^-] = q^{-r+1} K H_1 (X_0^+)^{(r-1)} + \left(\frac{q^{-2r+1} + q^{-2r-1} - q^{-2r+5} - q^{-4r+5}}{q - q^{-1}} \right) K X_1^+ (X_0^+)^{(r-2)} - q^{-3r+5} K X_0^+ X_1^+ (X_0^+)^{(r-3)}; \tag{9}$$

$$[(X_0^+)^{(r)}, \tilde{e}_0] = q^{-r+3}[4]_q K X_1^- (X_0^+)^{(r-1)} + q^{-2r+4}(q^2 + q^{-2}) K^2 H_1 (X_0^+)^{(r-2)} + q^{-3r+6} \left(\frac{q^{-5} + q^{-3} - q^3 - q^{-2r+3}}{q - q^{-1}} \right) K^2 X_1^+ (X_0^+)^{(r-3)} - q^{-4r+8} K^2 X_0^+ X_1^+ (X_0^+)^{(r-4)}; \tag{10}$$

$$\tilde{e}_0 X_1^- = q^4 X_1^- \tilde{e}_0 \; ; \tag{11}$$

$$[H_1, \tilde{e}_0^{(r)}] = -q^{-4r+5}(q - q^{-1})[3]_q[4]_q \tilde{e}_0^{(r-1)} (X_1^-)^2 \; ; \tag{12}$$

$$[X_0^+, \tilde{e}_0^{(r)}] = q^{-4r+4}[4]_q \tilde{e}_0^{(r-1)} X_1^- K \; ; \tag{13}$$

$$[(X_0^+)^{(r+1)}, \tilde{e}_0] = q^{-r+4}[2]_q K X_1^- (X_0^+)^{(r)} + q^{-r}[2]_q K (X_0^+)^{(r)} X_1^- + \frac{q^{-2r}}{[3]_q} K^2 [H_1, (X_0^+)^{(r-1)}] + q^{-2r+3}(q - q^{-1}) K^2 H_1 (X_0^+)^{(r-1)} \; ; \tag{14}$$

$$X_1^- \tilde{e}_0^{(r)} K \equiv \frac{1}{[4]_q} X_0^+ \tilde{e}_0^{(r+1)} \quad (\mathrm{mod}\ U^\tau X^+) \; ; \tag{15}$$

$$\tilde{e}_0^{(r-1)} (X_1^-)^2 \equiv \frac{q^{8r-2}}{[4]_q^2} (X_0^+)^2 \tilde{e}_0^{(r+1)} K^{-2} - \frac{q^{4r-2}}{[4]_q} \tilde{e}_0^{(r)} H_1 \quad (\mathrm{mod}\ U^\tau X^+) \, . \tag{16}$$

Identities (7)–(10) are proved successively by induction on r; (11) is a consequence of (3^-); (12) and (13) are proved by induction on r using (11); (14) follows from (8) and (9); congruence (15) follows from (11) and (13); and (16) follows by a double application of (13).

Finally, to prove (iii)$_r$, we compute

$$\begin{aligned} &[r+1]_{q^4} [(X_0^+)^{(2r+1)} \, , \, \tilde{e}_0^{(r+1)}] \\ &\quad = q^{-2r+4}[2]_q K X_1^- (X_0^+)^{(2r)} \tilde{e}_0^{(r)} + q^{-2r}[2]_q K (X_0^+)^{(2r)} X_1^- \tilde{e}_0^{(r)} \\ &\qquad + \frac{q^{-4r}}{[3]_q} K^2 [H_1, (X_0^+)^{(2r-1)}] \tilde{e}_0^{(r)} + q^{-4r+3}(q - q^{-1}) K^2 H_1 (X_0^+)^{(2r-1)} \tilde{e}_0^{(r)} \\ &\qquad\qquad \text{(by (14))} \end{aligned}$$

$$
\begin{aligned}
&= q^{-2r+4}[2]_q K X_1^-(X_0^+)^{(2r)}\tilde{e}_0^{(r)} + q^{-2r}[2]_q K (X_0^+)^{(2r)} X_1^- \tilde{e}_0^{(r)} \\
&\quad + \frac{q^{-4r}}{[3]_q} K^2 [H_1, (X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}] - \frac{q^{-4r}}{[3]_q} K^2 (X_0^+)^{(2r-1)}[H_1, \tilde{e}_0^{(r)}] \\
&\quad + q^{-4r+3}(q-q^{-1})K^2[H_1,(X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}] + q^{-4r+3}(q-q^{-1})K^2(X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}H_1 \\
&= q^{-2r+4}[2]_q K X_1^-(X_0^+)^{(2r)}\tilde{e}_0^{(r)} + q^{-2r}[2]_q K (X_0^+)^{(2r)} X_1^- \tilde{e}_0^{(r)} \\
&\quad + \frac{q^{-4r+6}}{[3]_q} K^2 [H_1, (X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}] - \frac{q^{-4r}}{[3]_q} K^2 (X_0^+)^{(2r-1)}[H_1, \tilde{e}_0^{(r)}] \\
&\quad + q^{-4r+3}(q-q^{-1})K^2(X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}H_1 \\
&\equiv q^{-2r+4}[2]_q K X_1^-(X_0^+)^{(2r)}\tilde{e}_0^{(r)} + q^{-2r-2}\frac{[2]_q[2r+1]_q}{[4]_q}(X_0^+)^{(2r+1)}\tilde{e}_0^{(r+1)} \\
&\quad + \frac{q^{-4r+6}}{[3]_q} K^2 [H_1, (X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}] - \frac{q^{-4r}}{[3]_q} K^2 (X_0^+)^{(2r-1)}[H_1, \tilde{e}_0^{(r)}] \\
&\quad + q^{-4r+3}(q-q^{-1})K^2(X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}H_1 \qquad (\text{mod } U^\tau X^+) \\
&\qquad\quad (\text{ by (15) applied to the second term}) \\
&\equiv q^{-2r+4}[2]_q K X_1^-(X_0^+)^{(2r)}\tilde{e}_0^{(r)} + q^{-2r-2}\frac{[2]_q[2r+1]_q}{[4]_q}(X_0^+)^{(2r+1)}\tilde{e}_0^{(r+1)} \\
&\quad + \frac{q^{-4r+6}}{[3]_q} K^2 [H_1, (X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}] + q^{-8r+5}(q-q^{-1})[4]_q K^2 (X_0^+)^{(2r-1)}\tilde{e}_0^{(r-1)}(X_1^-)^2 \\
&\quad + q^{-4r+3}(q-q^{-1})K^2(X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}H_1 \qquad (\text{mod } U^\tau X^+) \\
&\qquad\quad (\text{by (12) applied to the fourth term}) \\
&\equiv q^{-2r+4}[2]_q K X_1^-(X_0^+)^{(2r)}\tilde{e}_0^{(r)} + q^{-2r-2}\frac{[2]_q[2r+1]_q}{[4]_q}(X_0^+)^{(2r+1)}\tilde{e}_0^{(r+1)} \\
&\quad + \frac{q^{-4r+6}}{[3]_q} K^2 [H_1, (X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}] \\
&\quad + q^{-8r+5}(q-q^{-1})[4]_q K^2 (X_0^+)^{(2r-1)}\left(\frac{q^{8r-2}}{[4]_q^2}(X_0^+)^2\tilde{e}_0^{(r+1)}K^{-2} - \frac{q^{4r-2}}{[4]_q}\tilde{e}_0^{(r)}H_1\right) \\
&\quad + q^{-4r+3}(q-q^{-1})K^2(X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}H_1 \qquad (\text{mod } U^\tau X^+) \\
&\qquad\quad (\text{by (16) applied to the fourth term}) \\
&\equiv q^{-2r+4}[2]_q K X_1^-(X_0^+)^{(2r)}\tilde{e}_0^{(r)} + q^{-2r-2}\frac{[2]_q[2r+1]_q}{[4]_q}(X_0^+)^{(2r+1)}\tilde{e}_0^{(r+1)} \\
&\quad + \frac{q^{-4r+6}}{[3]_q} K^2 [H_1, (X_0^+)^{(2r-1)}\tilde{e}_0^{(r)}] + \frac{1-q^{-2}}{[4]_q}[2r+1]_q[2r]_q(X_0^+)^{(2r+1)}\tilde{e}_0^{(r+1)} \\
&\qquad\quad (\text{ mod } U^\tau X^+).
\end{aligned}
$$

Collecting the terms involving $(X_0^+)^{(2r+1)}\tilde{e}_0^{(r+1)}$ on the left-hand side and simplifying gives (iii)$_r$. □

Let V be the finite-dimensional irreducible type I representation of U^τ with highest weight given by the pair of $\mathbb{Z}$-tuples $\{\psi_k^\pm\}_{k\in\mathbb{Z}}$, and let v be a highest weight vector in V. We have

$$k_0.v = q^{4r_0}v, \quad k_1.v = q^{r_1}v$$

for some $r_0, r_1 \in \mathbb{Z}$. Note that $r_1 = -2r_0$; in particular, r_1 is even. Write $r_0 = -r$, $r_1 = 2r$ from now on.

Regarding V as a representation of the $U_q(sl_2)$ subalgebra of U^τ generated by $e_1^\pm$ and $k_1^{\pm 1}$, v is a highest weight vector (so that $r \geq 0$), and we have a direct sum decomposition

$$V \cong \bigoplus_{p \in \mathbb{N}} V_p^{n_p},$$

where V_p is the irreducible representation of $U_q(sl_2)$ of dimension $p+1$ (and on which k_1 acts with eigenvalues in $q^{\mathbb{Z}}$), and the $n_p \geq 0$ are certain multiplicities. By 1.4, $n_p = 0$ if p is odd or $> 2r$. Applying both sides of (i)$_s$ in 4.2 to v, it follows that $\mathcal{P}_s.v = 0$ if $s > r$. Hence,

$$\mathcal{P}(u).v = P(u)v,$$

where $P \in \mathbb{C}[u]$ is a polynomial with constant coefficient 1 and degree $\leq r$. By the remarks following 4.1,

$$\Psi^+(u).v = q^{2r}\frac{P(q^{-4}u)}{P(u)}v. \tag{17}$$

Multiplying both sides of (ii)$_r$ on the left by X^+_{-n-1}, where $n \in \mathbb{N}$, we see that

$$\sum_{j=n}^{r} \Psi^+_{j-n}\mathcal{P}_{r-j}.v = \sum_{j=0}^{n} \Psi^-_{j-n}\mathcal{P}_{r-j}.v \tag{18}$$

if $n \leq r$, and

$$\sum_{j=0}^{r} \Psi^-_{j-n}\mathcal{P}_{r-j}.v = 0 \tag{19}$$

if $n > r$. By 4.1, (18) is equivalent to

$$q^{2r}q^{-4(r-n)}\mathcal{P}_{r-n}.v = \sum_{j=0}^{n} \Psi^-_{j-n}\mathcal{P}_{r-j}.v. \tag{20}$$

Equations (19) and (20) are together equivalent to

$$\Psi^-(u).v = q^{2r}\frac{P(q^{-4}u)}{P(u)}v. \tag{21}$$

Finally, to compute $\deg P$, note that if $\deg P = s$, Eq. (21) implies that

$$\Psi^-(u).v = q^{2r-4s}\frac{(q^{-4}u)^{-s}P(q^{-4}u)}{u^{-s}P(u)}v,$$

and hence that $K^{-1}.v = q^{2r-4s}v$. But from Eq. (17) we have $K.v = q^{2r}v$, so $s = r$.

This completes the proof of the "only if" part of 3.1(ii) in the U^τ case. Before proving the "if" part, we prove 3.2 in the U^τ case. This depends on the following proposition.

Proposition 4.3. *Let* $k \geq 0$.

(i) $\Delta(X_k^+) \equiv \sum_{j=0}^{k} X_{k-j}^+ \otimes \Psi_j^+ + 1 \otimes X_k^+ \ (\bmod\ U^\tau (X^+)^2 \otimes U^\tau)$;

(ii) $\Delta(\Psi_k^+) \equiv \sum_{j=0}^{k} \Psi_j^+ \otimes \Psi_{k-j}^+ \ (\bmod\ U^\tau X^+ \otimes U^\tau + U^\tau \otimes U^\tau X^+)$.

Proof. Making use of 1.1 and the isomorphism in 1.3, one computes that

$$\Delta(H_1) = H_1 \otimes 1 + 1 \otimes H_1 - (q-q^{-1})[2]_q[3]_q X_0^+ \otimes X_1^- + q^{-1}(q-q^{-1})[3]_q (X_0^+)^2 \otimes K^2 e_0^+.$$

The formula in (i) now follows by an easy induction on k. Then (ii) follows from (i) by using

$$\Psi_k^+ = (q - q^{-1})[X_k^+, X_0^-]. \quad \square$$

Part (ii) of 4.3 implies that, when acting on a tensor product of two highest weight vectors, $\Psi^+(u)$ acts as a group-like element of the formal power series Hopf algebra $U^\tau[[u]]$. Proposition 3.2 follows.

To prove the "if" part of 3.1(ii) for U^τ, it suffices by 3.3 to show that $V(P)$ is finite-dimensional when $\deg P = 1$. This is accomplished in the next proposition.

Proposition 4.4. *The following is a representation of* U^τ*, for any* $a \in \mathbb{C}^\times$*:*

$$X_k^+ \mapsto a^k [2]_q \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & (-1)^k q^{2k} \\ 0 & 0 & 0 \end{pmatrix}, \quad X_k^- \mapsto a^k \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & (-1)^k q^{2k} & 0 \end{pmatrix},$$

$$H_k \mapsto a^k \frac{[2k]_q}{k} \begin{pmatrix} q^{-2k} & 0 & 0 \\ 0 & (-1)^k - q^{2k} & 0 \\ 0 & 0 & (-1)^{k+1} q^{4k} \end{pmatrix}, \quad K \mapsto \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & q^{-2} \end{pmatrix},$$

$$\Psi_k^+ \mapsto (q^2 - q^{-2}) a^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^k q^{2k} - 1 & 0 \\ 0 & 0 & (-1)^{k+1} q^{2k} \end{pmatrix} \quad \text{if } k > 0,$$

$$\Psi_k^- \mapsto -(q^2 - q^{-2}) a^k \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^k q^{2k} - 1 & 0 \\ 0 & 0 & (-1)^{k+1} q^{2k} \end{pmatrix} \quad \text{if } k < 0.$$

Proof. Straightforward verification. $\square$

The representation defined in 4.4, say V_a, is clearly irreducible and of type I. Moreover, if $\{\psi_k^\pm\}_{k\in\mathbb{Z}}$ is its highest weight, we have

$$\sum_{k=0}^{\infty} \psi_k^+ u^k = q^2 + \sum_{k=1}^{\infty} (q^2 - q^{-2}) a^k u^k = q^2 \frac{P(q^{-4}u)}{P(u)},$$

where $P(u) = 1 - au$. Thus, we have exhibited a finite-dimensional irreducible type I representation of U^τ with highest weight given (as in 3.1) by an arbitrary polynomial of degree one. This completes the proof of 3.1(ii) in the U^τ case. (Note that, if $P = 1$ is the constant polynomial, then $V(P)$ is the trivial representation.)

5. The general case

In this section, we outline the proofs of 3.1 and 3.2 for an arbitrary twisted quantum affine algebra $U_q(L(\mathfrak{g})^\sigma)$, which we denote by U^σ.

Suppose then that V is a finite-dimensional highest weight representation of U^σ with highest weight vector v and highest weight $\{\psi^\pm_{i,k}\}_{i\in I,k\in\mathbb{Z}}$. We consider three cases, as in Proposition 2.2:

Case (i). $\sigma(i) \neq i$ and $a_{i\sigma(i)} \neq 0$. Note that $m = 2$ in this case. Using the homomorphism φ_i described in 2.1(i), we can view V as a representation of $U_q(L(sl_3)^\tau)$, and as such v is still a highest weight vector in V. By the $U_q(L(sl_3)^\tau)$ case of 3.1, proved in the previous section, there exists $P_i \in \Pi$ such that

$$\sum_{k=0}^{\infty} \psi^+_{i,k} u^k = \sum_{k=0}^{\infty} \psi^-_{i,-k} u^{-k} = q^{2 \deg P_i} \frac{P_i(q^{-4}u)}{P_i(u)}.$$

Case (ii). $\sigma(i) \neq i$ and $a_{i\sigma(i)} = 0$. This time, we can view V as a representation of $U_q(L(sl_2))$. By [6], Theorem 3.4, there exists $P_i \in \Pi$ such that

$$\sum_{k=0}^{\infty} \psi^+_{i,k} u^k = \sum_{k=0}^{\infty} \psi^-_{i,-k} u^{-k} = q^{\deg P_i} \frac{P_i(q^{-2}u)}{P_i(u)}.$$

Case (iii). $\sigma(i) = i$. Viewing V as a representation of $U_{q^m}(L(sl_2))$, there exists $P_i \in \Pi$ such that

$$\sum_{k=0}^{\infty} \psi^+_{i,mk} u^k = \sum_{k=0}^{\infty} \psi^-_{i,-mk} u^{-k} = q^{m \deg P_i} \frac{P_i(q^{-2m}u)}{P_i(u)}.$$

Noting that $\Psi^\pm_{i,k} = 0$ unless k is divisible by m, we find that

$$\sum_{k=0}^{\infty} \psi^+_{i,k} u^k = \sum_{k=0}^{\infty} \psi^-_{i,-k} u^{-k} = q^{m \deg P_i} \frac{P_i(q^{-2m}u^m)}{P_i(u^m)}.$$

This proves the “only if” part of 3.1(ii). The “if” part is proved by an argument similar to that used in the untwisted case in [8], Sect. 5. In that case, the crucial point was to establish the result for $U_q(L(sl_2))$. In the present case, we also need the result for $U_q(L(sl_3)^\tau)$, which was proved at the end of Sect. 4. We omit further details.

Finally, to prove 3.3 in the general case, one uses the methods of [9], Sect. 2. Let U_i denote $U_q(L(sl_3)^\tau)$, $U_q(L(sl_2))$ or $U_{q^m}(L(sl_2))$ in cases (i), (ii) or (iii) of 2.2, respectively. If V is an irreducible highest weight representation of U^σ with highest weight vector v, denote the representation $\varphi_i(U_i).v$ of U_i by V_i.

Lemma 5.1. *With the above notation, V_i is an irreducible representation of U_i with highest weight vector v.*

The proof is similar to that of Lemma 2.3 in [9]. In particular, for any $\mathbf{P} = \{P_i\}_{i\in I} \in \Pi$, $V(\mathbf{P})_i \cong V(P_i)$, where $V(P_i)$ is the finite-dimensional irreducible representation of U_i associated to the polynomial P_i as in 3.1(ii) if $U_i = U_q(L(sl_3)^\tau)$, and as in Theorem 3.4 in [6] if $U_i = U_q(L(sl_2))$ or $U_{q^m}(L(sl_2))$.

If V and W are two irreducible highest weight representations of U^σ with highest weight vectors v and w, then, for any $i \in I$, $V_i \otimes W_i$ is a representation of U_i via $(\varphi_i \otimes \varphi_i)\circ$

Δ_i. On the other hand, it is not difficult to show that the subspace $V_i \otimes W_i$ of $V \otimes W$ is preserved by $(\Delta \circ \varphi_i)(U_i)$, giving a second way of viewing $V_i \otimes W_i$ as a representation of U_i. We denote these representations by $V_i \otimes_i W_i$ and $V_i \otimes W_i$, respectively.

Lemma 5.2. *With the above notation, the identity map $V_i \otimes_i W_i \to V_i \otimes W_i$ is an isomorphism of representations of U_i.*

The proof is similar to that of Proposition 2.2 in [9]. The necessary facts about the comultiplication of $U_q(L(\mathfrak{g})^\sigma)$ can be established by computations similar to those used to prove Theorem 2.2 in [17].

Now let $\mathbf{P} = \{P_i\}_{i \in I}$, $\mathbf{Q} = \{Q_i\}_{i \in I} \in \Pi$. Then, by the $U_q(L(sl_3)^\tau)$ case of 3.2, proved in the previous section, and the analogous result for $U_q(L(sl_2))$ (Proposition 4.3 in [6]), we have the following isomorphisms of representations of U_i:

$$V(\mathbf{P})_i \otimes_i V(\mathbf{Q})_i \cong V(P_i) \otimes_i V(Q_i) \cong V(P_i Q_i).$$

If $v_{\mathbf{P}}$ and $v_{\mathbf{Q}}$ are highest weight vectors in $V(\mathbf{P})$ and $V(\mathbf{Q})$, it follows from 5.2 that

$$\Psi^{\pm}_{i,k}.(v_{\mathbf{P}} \otimes v_{\mathbf{Q}}) = \psi^{\pm}_{i,k}(v_{\mathbf{P}} \otimes v_{\mathbf{Q}}),$$

the action on the left being given by Δ, where $\{\psi^{\pm}_{i,k}\}_{i \in I, k \in \mathbb{Z}}$ corresponds to the polynomial $P_i Q_i$ as in the $U_q(L(sl_3)^\tau)$ case of 3.1(ii) (proved in the previous section) or the analogous result for $U_q(L(sl_2))$ or $U_{q^m}(L(sl_2))$ (Theorem 3.4 in [6]). This proves 3.2 for U^σ.

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