Commun. Math. Phys. 195, 525 – 547 (1998)

Communications in
Mathematical Physics

Structure and Representations of the Quantum General Linear Supergroup

R. B. Zhang

Department of Pure Mathematics, University of Adelaide, Adelaide, Australia

Received: 27 November 1996 / Accepted: 10 December 1997

Abstract: The structure and representations of the quantum general linear supergroup $GL_q(m|n)$ are studied systematically by investigating the Hopf superalgebra G_q of its representative functions. G_q is factorized into $G_q^\pi \ G_q^{\bar\pi}$, and a Peter–Weyl basis is constructed for each factor. Parabolic induction for the quantum supergroup is developed. The underlying geometry of induced representations is discussed, and an analog of Frobenius reciprocity is obtained. A quantum Borel–Weil theorem is proven for the irreducible covariant and contravariant tensorial representations, and explicit realizations are given for classes of irreducible tensorial representations in terms of sections of quantum super vector bundles over quantum projective superspaces.

1. Introduction

Quantized universal enveloping superalgebras [1, 2] (which will be called quantum superalgebras for simplicity) represent the most important generalizations of the Drinfeld–Jimbo [3] quantized universal enveloping algebras. Their origin can be traced back to the Perk-Schultz solution of the Yang–Baxter equation and also the work of Bazhanov and Shadrikov [4]. However, systematical investigations of such algebraic structures only started about six years ago, but in an intensive manner. By now the subject has been developed quite extensively: the quasi-triangular Hopf superalgebraic structure of the quantum superalgebras was investigated [5]; the representation theory of large classes of quantum (affine)superalgebras and super Yangians was developed [6, 7]; applications of quantum superalgebras to two dimensional integrable models in statistical mechanics and quantum field theory were extensively explored [1, 8]. Quantum superalgebras have also been applied to the study of knot theory and 3-manifolds [9, 10], yielding many new topological invariants, notably, the multi-parameter generalizations of Alexander–Conway polynomials.

Closely related to the Drinfeld–Jimbo algebras are the quantum groups introduced by Woronowicz and Faddeev–Reshetikhin–Takhatajan [11], which are, in the spirit of

Tannaka–Krein duality theory, the "groups" associated with the quantized universal enveloping algebras. One very important aspect of quantum groups is their geometrical significance: they provide a concrete framework for developing noncommutative geometry [12], in particular, for investigating notions such as quantum flag varieties [13] and quantum fibre bundles.

Our aim here is to study the structure and representations of the quantum general linear supergroup $GL_q(m|n)$ in a systematical fashion by investigating the algebra of its representative functions. We start in Sect. 2 with a concise treatment of finite dimensional unitary representations of $U_q(gl(m|n))$. Results will be repeatedly used in the remainder of the paper. In Sect. 3 we define the quantum general linear supergroup $GL_q(m|n)$, or more exactly, the superalgebra G_q of functions on it. This is done by first defining the bi-superalgebras G_q^π and $G_q^{\bar{\pi}}$, which are respectively generated by the matrix elements of the vector representation and its dual irreducible representation. Peter–Weyl type of bases for these bi-superalgebras are constructed. The G_q is defined to be generated by G_q^π and $G_q^{\bar{\pi}}$ with some extra relations. It has the structures of a $*$-Hopf superalgebra, which separates points of $U_q(gl(m|n))$, and factorizes into $G_q^\pi G_q^{\bar{\pi}}$. Section 4 treats the representation theory of the quantum supergroup, and in particular, parabolic induction. The geometrical interpretation of induced representations is discussed, leading naturally to the concepts of quantum homogeneous spaces and quantum super vector bundles. A quantum analog of Frobenius reciprocity is obtained; and a quantum version of the Borel–Weil theorem is proven for the irreducible covariant and contravariant tensorial representations. Section 5 gives the explicit realizations of two infinite classes of irreducible tensorial representations in terms of sections of quantum super vector bundles over the quantum projective superspace. In doing this, we also treat the quantum projective superspace in some detail.

2. Unitary Representations of $U_q(gl(m|n))$

The finite dimensional unitary representations of $U_q(gl(m|n))$ were classified in [15]. Here we will reformulate the results on the covariant and contravariant tensor representations so that they can be readily used in the remainder of the paper. The material presented here also heavily relies on references [6] and [14].

2.1. Hopf $*$-superalgebras and unitary representations. Let A be a $\mathbb{Z}_2$-graded associative algebra over the complex field $\mathbb{C}$. Its underlying $\mathbb{Z}_2$-graded vector space is the direct sum $A = A_0 \oplus A_1$ of the even subspace A_0 and the odd subspace A_1. We introduce the grading index $[\ ] : A_0 \cup A_1 \to \mathbb{Z}_2$ such that $[a] = \theta$ if $a \in A_\theta$. We will call A a $\mathbb{Z}_2$-graded $*$-algebra, or $*$-superalgebra, if there exists an even anti-linear anti-automorphism $* : A \to A$ such that $* \circ * = id_A$. We will denote $*(a)$ by a^*. Needless to say, $*(ab) = b^* a^*$, $a, b \in A$.

An important new feature of the $\mathbb{Z}_2$-graded case is that for a given $*$-operation of A, there exists an associated $*'$ such that

$$*'(a) = (-1)^{[a]} a^*, \tag{1}$$

for a being homogeneous, and extends to the whole of A anti-linearly. There also exist the so-called graded $*$-operations, which, however, are not useful for this paper, and thus will not be discussed any further.

Let A and B be two $\mathbb{Z}_2$-graded $*$-algebras. Then $A \otimes_{\mathbb{C}} B$ has a natural $\mathbb{Z}_2$-graded $*$-algebra structure, with the $*$-operation defined for homogeneous elements by

$$*(a \otimes b) = (-1)^{[a][b]} a^* \otimes b^*,$$

and for all the elements by extending this anti-linearly.

Consider a $\mathbb{Z}_2$-graded Hopf algebra (also called Hopf superalgebra) H, with multiplication m, unit 1_H, co-multiplication Δ, co-unit ϵ and antipode S. We emphasize that the antipode is a *linear* anti- automorphism of the underlying algebra of H. In particular, for homogeneous $a, b \in A$, we have $S(ab) = (-1)^{[a][b]} S(b)S(a)$. H will be called a $\mathbb{Z}_2$-graded Hopf $*$-algebra, or Hopf $*$-superalgebra, if the underlying algebra of H is a $*$-superalgebra such that Δ and ϵ are $*$-homomorphisms, i.e.,

$$* \circ \Delta = \Delta \circ *, \; * \circ \epsilon = \epsilon \circ *.$$

These properties together with the defining relations of the antipode

$$m \circ (S \otimes id)\Delta = m \circ (id \otimes S)\Delta = 1_H \epsilon$$

imply that

$$S \circ * \circ S \circ * = id_H.$$

Let V be a left H-module. If there exists a non-degenerate sesquilinear form $(\ ,\ ) : V \times V \to \mathbb{C}$, such that

$$\begin{aligned} &(i).\ (av,\ u) = (v,\ a^* u),\ \forall u, v \in V,\ a \in H, \\ &(ii).\qquad (v,\ v) \geq 0, \qquad (v,\ v) = 0 \ \text{ iff } \ v = 0, \end{aligned}$$

we call V and the associated representation of H unitary.

Unitary representations have the following important properties:

i) *A unitary representation is completely reducible;*
ii) *The tensor product of two unitary (with respect to the same $*$-operation) representations is again unitary;*
iii) *If a representation is unitary with respect to $*$, then its dual is unitary with respect to $*'$.*

All the three assertions are well known, but there are some related matters worth discussing. One is concerned with the requirement that two representations must be unitary with respect to the same $*$-operation in order for their tensor product to be unitary as well. The tensor product $V \otimes_{\mathbb{C}} W$ of two H-modules has a natural H module structure

$$\begin{aligned} a\{v \otimes w\} &= \Delta(a)\{v \otimes w\} \\ &= \sum_{(a)} (-1)^{[a_{(2)}][v]} a_{(1)} v \otimes a_{(2)} w. \end{aligned}$$

If both V and W are equipped with sesquilinear forms $(\ ,\ ) : V \times V \to \mathbb{C}$, and $(\ ,\ ) : W \times W \to \mathbb{C}$, we can define a sesquilinear form $((\ ,\ )) : (V \otimes_{\mathbb{C}} W)^{\times 2} \to \mathbb{C}$ by

$$((v_1 \otimes w_1,\ v_2 \otimes w_2)) = (v_1,\ v_2)(w_1,\ w_2).$$

Now if both V and W are unitary with respect to the same $*$- operation, then $((\ ,\ ))$ is clearly positive definite and nondegenerate. Furthermore,

$$((v_1 \otimes w_1,\ a\{v_2 \otimes w_2\})) = \sum_{(a)} (-1)^{[a_{(2)}][v_2]}((a^*_{(1)}v_1 \otimes a^*_{(2)}w_1,\ v_2 \otimes w_2))$$
$$= ((a^*\{v_1 \otimes w_1\},\ v_2 \otimes w_2)).$$

Therefore, $V \otimes_{\mathbb{C}} W$ indeed furnishes a unitary H-module. On the other hand, if, say, V is $*$-unitary, while W is $*'$-unitary, then one can easily see that the above calculations will fail to go through.

The other concerns the third assertion, the validity of which actually requires some qualification, namely, the Hopf *-superalgebra H in question must admit an even group like element $K_{2\rho}$ satisfying

$$K^*_{2\rho} = K_{2\rho},\ S^2(a) = K_{2\rho} a K^{-1}_{2\rho}, \quad \forall a \in H. \tag{2}$$

Let V be a locally finite module over H, which is unitary with respect to the sesquilinear form $(\ ,\ ) : V \times V \to \mathbb{C}$. For every $v \in V$, we define $v^\dagger$ by $v^\dagger(w) = (v, w)$, $\forall w \in V$, and denote the linear span of all such $v^\dagger$ by $V^\dagger$, which is a subspace of the dual vector space of V. The $V^\dagger$ has a natural H module structure, with the action of H given by

$$(av^\dagger)(w) = (-1)^{[a][v^\dagger]} v^\dagger(S(a)w), \quad w \in V.$$

Unitarity of V leads to

$$av^\dagger = (-1)^{[a][v]} (*S(a)v)^\dagger .$$

We define a sesquilinear form $(\ ,\ )' : V^\dagger \times V^\dagger \to \mathbb{C}$ by

$$(v^\dagger,\ w^\dagger)' = (K_{2\rho} w,\ v).$$

It follows from the properties of the original form on V that $(\ ,\ )'$ is positive definite and nondegenerate. A straightforward calculation shows that

$$(av^\dagger,\ w^\dagger)' = (v^\dagger, *'(a) w^\dagger)',$$

where $*'$ is defined by (1).

2.2. $U_q(gl(m|n))$. Throughout the paper, we will denote by $\mathfrak{g}$ the complex Lie superalgebra $gl(m|n)$, and by $U(\mathfrak{g})$ its universal enveloping algebra. As is well known, there are the Drinfeld and Jimbo versions of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$, which, though, have very similar properties at generic q.

It is the Jimbo version of $U_q(\mathfrak{g})$ that will be used in this paper. Now $U_q(\mathfrak{g})$ is a $\mathbb{Z}_2$-graded unital associative algebra over $\mathbb{C}(q, q^{-1})$, q being an indeterminate, generated by $\{K_a,\ K_a^{-1},\ a \in \mathbf{I};\ E_{b\ b+1},\ E_{b+1,b},\ b \in \mathbf{I}'\}$, $\mathbf{I} = \{1, 2, ..., m + n\}$, $\mathbf{I}' = \{1, 2, ..., m + n - 1\}$, subject to the following relations:

$$\begin{aligned}
K_a K_a^{-1} &= 1, \quad K_a^{\pm 1} K_b^{\pm 1} = K_b^{\pm 1} K_a^{\pm 1},\\
K_a E_{b\ b\pm 1} K_a^{-1} &= q_a^{\delta_{ab} - \delta_{a\ b\pm 1}} E_{b\ b\pm 1},\\
[E_{a\,a+1},\ E_{b+1\,b}\} &= \delta_{ab}(K_a K_{a+1}^{-1} - K_a^{-1} K_{a+1})/(q_a - q_a^{-1}),\\
(E_{m\,m+1})^2 &= (E_{m+1\,m})^2 = 0,\\
E_{a\,a+1} E_{b\,b+1} &= E_{b\,b+1} E_{a\,a+1},\\
E_{a+1\,a} E_{b+1\,b} &= E_{b+1\,b} E_{a+1\,a}, \quad |a - b| \geq 2,\\
\mathcal{S}^{(+)}_{a\ a\pm 1} &= \mathcal{S}^{(-)}_{a\ a\pm 1} = 0, \quad a \neq m,\\
\{E_{m-1\,m+2},\ E_{m\,m+1}\} &= \{E_{m+2\,m-1},\ E_{m+1\,m}\} = 0,
\end{aligned} \tag{3}$$

where $q_a = q^{(-1)^{[a]}}$,

$$\begin{aligned}\mathcal{S}^{(+)}_{a\,a\pm1} &= (E_{a\,a+1})^2 E_{a\pm1\,a+1\pm1} - (q+q^{-1}) E_{a\,a+1}\, E_{a\pm1\,a+1\pm1}\, E_{a\,a+1}\\ &\quad + E_{a\pm1\,a+1\pm1}\,(E_{a\,a+1})^2,\\ \mathcal{S}^{(-)}_{a\,a\pm1} &= (E_{a+1\,a})^2\, E_{a+1\pm1\,a\pm1} - (q+q^{-1}) E_{a+1\,a}\, E_{a+1\pm1\,a\pm1}\, E_{a+1\,a}\\ &\quad + E_{a+1\pm1\,a\pm1}\,(E_{a+1\,a})^2,\end{aligned}$$

and $E_{m-1\,m+2}$ and $E_{m+2\,m-1}$ are the $a = m-1$, $b = m+1$, cases of the following elements [16, 6]:

$$\begin{aligned}E_{a\,b} &= E_{a\,c}E_{c\,b} - q_c^{-1} E_{c\,b}E_{a\,c},\\ E_{b\,a} &= E_{b\,c}E_{c\,a} - q_c E_{c\,a}E_{b\,c}, \quad a < c < b.\end{aligned}$$

The $\mathbb{Z}_2$ grading of the algebra is specified such that the elements $K_a^{\pm1}$, $\forall a \in \mathbf{I}$, and $E_{b\,b+1}$, $E_{b+1\,b}$, $b \neq m$, are even, while $E_{m\,m+1}$ and $E_{m+1\,m}$ are odd. Above, we have also used the notation $[a] = \begin{cases} 0, \text{ if } a \le m,\\ 1, \text{ if } a > m.\end{cases}$

On the other hand, the Drinfeld version of $U_q(\mathfrak{g})$ is defined over $\mathbb{C}[[\hbar]]$, $q = \exp(\hbar)$, and is completed with respect to the $\hbar$-adic topology of $\mathbb{C}[[\hbar]]$. It is generated by $\{E_{a\,a},\ a \in \mathbf{I};\ E_{b\ b+1},\ E_{b+1,b},\ b \in \mathbf{I}'\}$, subject to the same relations (3) with

$$K_a = q_a^{E_{a\,a}}.$$

It is well known that $U_q(\mathfrak{g})$ has the structure of a $\mathbb{Z}_2$ graded Hopf algebra, with a co-multiplication

$$\begin{aligned}\Delta(E_{a\,a+1}) &= E_{a\,a+1} \otimes K_a K_{a+1}^{-1} + 1 \otimes E_{a\,a+1},\\ \Delta(E_{a+1\,a}) &= E_{a+1\,a} \otimes 1 + K_a^{-1} K_{a+1} \otimes E_{a+1\,a},\\ \Delta(K_a^{\pm1}) &= K_a^{\pm1} \otimes K_a^{\pm1},\end{aligned}$$

co-unit

$$\begin{aligned}\epsilon(E_{a\,a+1}) &= \epsilon(E_{a+1\,a}) = 0, \quad \forall a \in \mathbf{I}',\\ \epsilon(K_b^{\pm1}) &= 1, \quad \forall b \in \mathbf{I},\end{aligned}$$

and antipode

$$\begin{aligned}S(E_{a\,a+1}) &= -E_{a\,a+1} K_a^{-1} K_{a+1},\\ S(E_{a+1\,a}) &= -K_a K_{a+1}^{-1} E_{a+1\,a},\\ S(K_a^{\pm1}) &= K_a^{\mp1} \otimes K_a^{\mp1}.\end{aligned}$$

At generic q, the Jimbo version of $U_q(\mathfrak{g})$ has more or less the same representation theory as that of the Drinfeld version [6]. Let $\{\epsilon_a | a \in \mathbf{I}\}$ be the basis of a vector space with a bilinear form $(\epsilon_a,\ \epsilon_b) = (-1)^{[a]}\delta_{ab}$. The roots of the classical Lie superalgebra $gl(m|n)$ can be expressed as

$$\epsilon_a - \epsilon_b,\ a \neq b,\ a,\ b \in \mathbf{I}.$$

For later use, we define

$$2\rho = \sum_{a\le b}(-1)^{[a]+[b]}(\epsilon_a - \epsilon_b).$$

From [6] we know that every finite dimensional irreducible $U_q(\mathfrak{g})$ module is of highest weight type and is essentially uniquely characterized by a highest weight. Let $W(\lambda)$ be an irreducible $U_q(\mathfrak{g})$ module with highest weight $\lambda = \sum_a \lambda_a \epsilon_a$, $\lambda_a \in \mathbb{C}$. There exists a unique (up to scalar multiples) vector $v_+^\lambda \neq 0$ in $W(\lambda)$, called the highest weight vector, such that

$$\begin{aligned} E_{a a+1} v_+^\lambda &= 0, \quad a \in \mathbf{I}', \\ K_b v_+^\lambda &= q_b^{\lambda_b} v_+^\lambda, \quad b \in \mathbf{I}. \end{aligned}$$

$W(\lambda)$ is finite dimensional if and only if λ satisfies $\lambda_a - \lambda_{a+1} \in \mathbb{Z}_+$, $a \neq m$, and in that case, it has the same weight space decomposition as that of the corresponding irreducible $gl(m|n)$ module with the same highest weight.

2.3. Unitarity of covariant and contravariant tensor representations. From this section on, we will assume that $U_q(\mathfrak{g})$ is obtained from the Jimbo algebra by specializing q to a *real positive parameter* different from 1. To construct a $*$-operation for $U_q(\mathfrak{g})$, we first consider the Hopf subalgebra generated by $e = E_{a\ a+1}$, $f = E_{a+1\ a}$, and $k = K_a K_{a+1}^{-1}$, for a fixed $a \neq m$. It is not difficult to show that $*(e) = fk$, $*(f) = k^{-1}e$, $*(k^{\pm 1}) = k^{\pm 1}$ defines a $*$-operation for this $U_q(sl(2))$ subalgebra. Possible generalizations of this to $U_q(\mathfrak{g})$ are

$$\begin{aligned} *(E_{a\, a+1}) &= (-1)^{(\theta+1)\delta_{ma}} E_{a+1\, a} K_a K_{a+1}^{-1}, \\ *(E_{a+1\, a}) &= (-1)^{(\theta+1)\delta_{ma}} K_a^{-1} K_{a+1} E_{a\, a+1}, \\ *(K_a^{\pm 1}) &= K_a^{\pm 1}, \end{aligned} \tag{4}$$

where $\theta = 1$ or 2. It is quite obvious that the “quadratic” relations of (3) are preserved by the $*$-operations, and we have also explicitly checked that the “Serre relations” are preserved as well. We will call the $*$-operations type 1 and type 2 respectively when $\theta = 1$ and 2.

It is also well known that

$$K_{2\rho} = \prod_{a<b} \left(K_a\, K_b^{-1}\right)^{(-1)^{[a]+[b]}}$$

satisfies Eq. (2).

Now we consider the irreducible covariant and contravariant tensor representations of $U_q(\mathfrak{g})$. The vector representation π of $U_q(\mathfrak{g})$ is of highest weight ϵ_1. The corresponding module $\mathbb{E}$ has the standard basis $\{v_a | a \in \mathbf{I}\}$, such that

$$\begin{aligned} K_a v_b &= q_a^{\delta_{ab}} v_b, \\ E_{a\ a\pm 1} v_b &= \delta_{b\ a\pm 1} v_a. \end{aligned}$$

Define a sesquilinear form on $\mathbb{E} \times \mathbb{E}$ by

$$(v_a,\ v_b) = \delta_{ab} \prod_{c=1}^{a-1} q_c^{-1}.$$

Then it is straightforward to show that with respect to the type 1 $*$-operation, we have

$$(E_{a\ a\pm 1}v_b,\ v_c) = (v_b,\ E^*_{a\ a\pm 1}v_c),$$
$$(K_a v_b,\ v_c) = (v_b,\ K_a v_c).$$

Therefore, the vector representation is unitary of type 1.

The $U_q(\mathfrak{g})$ modules $\mathbb{E}^{\otimes k}$, $k \in \mathbb{Z}_+$ ($\mathbb{E}^0 = \mathbb{C}$), obtained by repeated tensor products of the vector module with itself can be decomposed into direct sums of irreducible type 1 unitary modules, and we will call each direct summand an irreducible contravariant tensor module, and the corresponding irreducible representation an irreducible contravariant tensor representation.

The irreducible contravariant tensor representations can be characterized in the following way. Let $\mathbb{Z}_+$ be the set of nonnegative integers. Define a subset $\mathcal{P}$ of ${\mathbb{Z}_+}^{\otimes(m+n)}$ by

$$\mathcal{P} = \{p = (p_1, p_2, ..., p_{m+n}) \in {\mathbb{Z}_+}^{\otimes(m+n)} \mid\ p_{m+1} \leq n,\ p_a \geq p_{a+1},\ a \in \mathbf{I}'\}.$$

We associate with each $p \in \mathcal{P}$ a $\lambda^{(p)} = \sum_{a=1}^{m+n} \lambda_a \epsilon_a$ defined by

$$\lambda_a = p_a, \quad a \leq m,$$
$$\sum_{\mu=1}^{n} \lambda_{m+\mu}\epsilon_{m+\mu} = \sum_{\nu=1}^{n} \sum_{\mu=1}^{p_{m+\nu}} \epsilon_{m+\mu}.$$

Introduce the set

$$\Lambda^{(1)} = \{\lambda^{(p)} \mid p \in \mathcal{P}\}. \tag{5}$$

From results of [6, 14] we know that an irreducible representation of $U_q(\mathfrak{g})$ is a contravariant tensor if and only if its highest weight belongs to $\Lambda^{(1)}$. Needless to say, all such irreducible representations are type 1 unitary.

Let $W(\lambda)$ be an irreducible contravariant tensor $U_q(\mathfrak{g})$ module with highest weight $\lambda \in \Lambda^{(1)}$. We define $\overline{\lambda}$ to be its lowest weight, and set $\lambda^\dagger = -\overline{\lambda}$. An explicit formula for $\lambda^\dagger$ was given in [14] (Sect. III. B.), where a more compact characterization was also given for the sets $\Lambda^{(1)}$ and

$$\Lambda^{(2)} := \{\lambda^\dagger \mid \lambda \in \Lambda^{(1)}\}. \tag{6}$$

We refer to that paper for details. Now the dual module $W(\lambda)^\dagger$ of $W(\lambda)$, which we will call a covariant tensor module, has highest weight $\lambda^\dagger$. All irreducible covariant tensor modules are unitary of type 2. The most important example is the covariant vector module $\mathbb{E}^\dagger$, which is the dual of the vector module $\mathbb{E}$. Its highest weight is given by $-\epsilon_{m+n}$.

We summarize our discussions in the following

Proposition 1. 1. *Each $U_q(\mathfrak{g})$ module $\mathbb{E}^{\otimes k}$ (resp. $(\mathbb{E}^\dagger)^{\otimes k}$), $k \in \mathbb{Z}_+$, can be decomposed into a direct sum of irreducible modules with highest weights belonging to $\Lambda^{(1)}$ (resp. $\Lambda^{(2)}$).*

2. *Every irreducible $U_q(\mathfrak{g})$ module with highest weight belonging to $\Lambda^{(1)}$ (resp. $\Lambda^{(2)}$) is contained in some repeated tensor products of $\mathbb{E}$ (resp. $\mathbb{E}^\dagger$) as an irreducible component.*

More detailed structures of the irreducible covariant and contravariant tensor representations can be understood, e.g., their characters and super characters can be computed, the Clebsch-Gordan problem of irreducible representations within a given tensor type can also be resolved by using the supersymmetric Young diagram method. Here we elucidate some general aspects of the Clebsch-Gordan problem, which will play an important role in the remainder of the paper.

Denote by $[\lambda]$ the equivalence class of irreducible representations with highest weight λ. For λ and λ' both belonging to $\Lambda^{(1)}$, we interpret $[\lambda] + [\lambda']$ as the equivalence class of the direct sum representations, and $[\lambda] \cdot [\lambda']$ as that of the direct products. Let $\left[\Lambda^{(1)}\right]$ be the $\mathbb{Z}^+$ module with a basis $\{[\lambda] \mid \lambda \in \Lambda^{(1)}\}$. Then the "$\cdot$" operation defines a multiplication on $\left[\Lambda^{(1)}\right]$. Clearly $[\lambda] \cdot [0] = [0] \cdot [\lambda] = [\lambda]$. Furthermore, from Sect. V of [14] we can deduce that if $[\lambda] \cdot [\lambda'] = [\lambda^1] + [\lambda^2] + \ldots + [\lambda^k]$, then none of the λ^i is zero unless both λ and λ' are zero. This is in agreement with the fact that

$$\Lambda^{(1)} \bigcap \Lambda^{(2)} = \{0\}.$$

The discussions above can be repeated word by word for the irreducible representations with highest weights belonging to $\Lambda^{(2)}$.

3. Quantum General Linear Supergroup $GL_q(m|n)$

For compact Lie groups in the classical setting, there exists the celebrated Tannaka-Krein duality theory [18], which enables the reconstruction of a group from the Hopf algebra of its representative functions. The theory of quantum groups [11] makes essential use of a quantum analog of the duality [17], and is formulated entirely in terms of the algebra of functions. We will adopt the same philosophy here to formulate and study quantum supergroups. However, we should mention that Lie supergroups are much more complicated than ordinary compact Lie groups in structures; at the best, the Tannaka-Krein duality holds in a restricted sense for Lie supergroups even at the classical situation, though we have not come across any treatment of the problem in the literature.

3.1. Subalgebra of functions associated with the vector representation. As before, we denote by π the vector representation of $U_q(\mathfrak{g})$ relative to the standard basis $\{v_a \mid a \in \mathbf{I}\}$ of $\mathbb{E}$. Then

$$x v_a = \sum_b \pi(x)_{b\,a} v_b, \quad x \in U_q(\mathfrak{g}).$$

Let $(U_q(\mathfrak{g}))^0$ be the finite dual of $U_q(\mathfrak{g})$. Consider the elements $t_{a\,b}$, $a, b \in \mathbf{I}$ of $(U_q(\mathfrak{g}))^0$ satisfying

$$t_{a\,b}(x) = \pi(x)_{a\,b}, \quad \forall x \in U_q(\mathfrak{g}).$$

It is easy to show that the $t_{a\,b}$ indeed belong to $(U_q(\mathfrak{g}))^0$. Also note that $t_{a\,b}$ is even if $[a] + [b] \equiv 0 (mod\, 2)$, and odd otherwise.

Standard Hopf algebra theory asserts that $(U_q(\mathfrak{g}))^0$ is a $\mathbb{Z}_2$ - graded Hopf algebra with its structures dualizing those of $U_q(\mathfrak{g})$. Consider the subalgebra G_q^π of $(U_q(\mathfrak{g}))^0$ generated by $t_{a\,b}$, $a, b \in \mathbf{I}$. The multiplication which G_q^π inherits from $(U_q(\mathfrak{g}))^0$ is given by

$$\langle t\, t',\ x\rangle = \sum_{(x)} \langle t \otimes t',\ x_{(1)} \otimes x_{(2)}\rangle$$
$$= \sum_{(x)} (-1)^{[t'][x_{(1)}]} \langle t, x_{(1)}\rangle \langle t',\ x_{(2)}\rangle, \quad \forall t, t' \in G_q^\pi,\ x \in U_q(\mathfrak{g}). \tag{7}$$

To better understand the algebraic structure of G_q^π, we recall that the Drinfeld version of $U_q(\mathfrak{g})$ admits a universal R matrix, which in particular satisfies

$$R\Delta(x) = \Delta'(x)R, \quad \forall x \in U_q(\mathfrak{g}). \tag{8}$$

Applying $\pi \otimes \pi$ to both sides of the equation yields

$$R^{\pi\,\pi}(\pi \otimes \pi)\Delta(x) = (\pi \otimes \pi)\Delta'(x)R^{\pi\,\pi}, \tag{9}$$

where $R^{\pi\,\pi} := (\pi \otimes \pi)R$. The universal R-matrix of $U_q(\mathfrak{g})$ can be extracted from the Khoroshkin-Tolstoy paper of [5] by appropriately adjusting the conventions. We can then apply $\pi \otimes \pi$ to R to get $R^{\pi\,\pi}$. The matrices $R^{\bar\pi\,\pi}$ and $R^{\bar\pi\,\bar\pi}$ which will be used later, can also be obtained similarly. However, the explicit form of these matrices can be extracted more easily from the results of [16]. Here we copy $R^{\pi\,\pi}$ from that reference

$$R^{\pi\,\pi} = q^{\sum_{a\in\mathbf{I}} e_{a\,a}\otimes e_{a\,a}(-1)^{[a]}} + (q - q^{-1}) \sum_{a<b} e_{a\,b} \otimes e_{b\,a}(-1)^{[b]}.$$

We may also mention that $R^{\pi\,\pi}$ is the infinite spectral parameter limit of the Perk-Schultz R-matrix.

It is important to realize that Eq. (9) makes perfect sense within the Jimbo formulation of the quantized universal enveloping algebra $U_q(\mathfrak{g})$, even when q is specialized to a real parameter. We can re-interpret the equation in terms of the $t_{a\,b}$. Then by setting $t = \sum_{a,b} e_{a\,b} \otimes t_{a\,b}$, we have

$$R_{12}^{\pi\,\pi}\, t_1\, t_2 = t_2\, t_1\, R_{12}^{\pi\,\pi}. \tag{10}$$

The co-multiplication Δ of G_q^π is also defined in the standard way by

$$\langle \Delta(t_{a\,b}),\ x \otimes y\rangle = \langle t_{a\,b},\ xy\rangle = \pi(xy)_{a\,b}, \quad \forall x, y \in U_q(\mathfrak{g}).$$

We have

$$\Delta(t_{a\,b}) = \sum_{c\in\mathbf{I}} (-1)^{([a]+[c])([c]+[b])} t_{a\,c} \otimes t_{c\,b}. \tag{11}$$

G_q^π also has the unit ϵ, and the co-unit $1_{U_q(\mathfrak{g})}$. Therefore, G_q^π has the structures of a $\mathbb{Z}_2$-graded bi- algebra. However, it does not admit an antipode, as we will explain later.

Let $\pi^{(\lambda)}$ be an arbitrary irreducible contravariant tensor representation of $U_q(\mathfrak{g})$. We may also regard $\pi^{(\lambda)}$ as a representative of $[\lambda]$, where $\lambda \in \Lambda^{(1)}$. Define the elements $t_{i\,j}^{(\lambda)}$, $i,\ j = 1, 2, ..., dim_{\mathbb{C}}\pi^{(\lambda)}$, of $(U_q(\mathfrak{g}))^0$ by

$$t_{i\,j}^{(\lambda)}(x) = \pi^{(\lambda)}(x)_{i\,j}, \quad \forall x \in U_q(\mathfrak{g}).$$

It is an immediate consequence of Proposition 1 that $t_{i\,j}^{(\lambda)} \in G_q^\pi$, for all $i,\ j$ and $\lambda \in \Lambda^{(1)}$, and every $f \in G_q^\pi$ can be expressed as a linear sum of these elements. From the representation theory of $U_q(\mathfrak{g})$ we can deduce that these elements are also linearly independent. Introduce the vector spaces

$$T^{(\lambda)} = \bigoplus_{i,j=1}^{dim\pi^{(\lambda)}} \mathbb{C}t^{(\lambda)}_{i\,j}.$$

Then

Proposition 2. *As a vector space,*

$$G_q^\pi = \bigoplus_{\lambda\in\Lambda^{(1)}} T^{(\lambda)}.$$

To return to the question why G_q^π admits no antipode, we consider an arbitrary $t^{(\lambda)}_{i\,j} \in G_q^\pi$ with $\lambda \neq 0$. Denote also by S the antipode of $(U_q(\mathfrak{g}))^0$. Then

$$S(t^{(\lambda)}_{i\,j})(x) = t^{(\lambda)}_{i\,j}(S(x)), \quad \forall x \in U_q(\mathfrak{g}).$$

That is, $S(t^{(\lambda)}_{i\,j})$ are the matrix elements of the dual irreducible representation of $\pi^{(\lambda)}$, the highest weight of which is not contained in $\Lambda^{(1)}$ unless the irreducible representation $\pi^{(\lambda)}$ is trivial, i.e., $\lambda = 0$. Therefore, $S(t^{(\lambda)}_{i\,j}) \notin G_q^\pi$.

Let us now recapitulate that G_q^π is defined as the sub bi-superalgebra of $(U_q(\mathfrak{g}))^0$ generated by the matrix elements of the vector representation of $U_q(\mathfrak{g})$. Equation (10) is a set of relations satisfied by the t_{ab} as elements of $(U_q(\mathfrak{g}))^0$. However, we may also consider a bi-superalgebra $\mathcal{T}_q$ generated by $\tau_{ab},\ a, b \in \mathbf{I}$, subject to the same relations as (10) but with t replaced by $\sum_{a,b} e_{a\,b} \otimes \tau_{a\,b}$, and with a similar co-multiplication as (11). Clearly, the bi-superalgebra map

$$\psi : \mathcal{T}_q \to G_q^\pi, \qquad \tau_{ab} \mapsto t_{ab},$$

is surjective. Now a natural and important question is whether ψ is also injective. The answer to this question is affirmative, as can be shown by adapting the method of Takeuchi [19] to the present situation. The question is also closely related to the problem of "connectedness" of quantum supergroups (see [19] about the corresponding problem for ordinary $GL_q(n)$), which will be treated in detail on another occasion.

3.2. Subalgebra of functions associated with the dual vector representation. Let $\{\bar{v}_a \mid a \in \mathbf{I}\}$ be the basis of $\mathbb{E}^\dagger$ dual to the standard basis of $\mathbb{E}$, i.e.,

$$\bar{v}_a(v_b) = \delta_{a\,b}.$$

Denote by $\bar{\pi}$ the covariant vector representation relative to this basis. Let $\bar{t}_{a\,b},\ a, b \in \mathbf{I}$, be the elements of $(U_q(\mathfrak{g}))^0$ such that

$$\bar{t}_{a\,b}(x) = \bar{\pi}(x)_{a\,b}, \quad \forall x \in U_q(\mathfrak{g}).$$

Note that $\bar{t}_{a\,b}$ is even if $[a]+[b] \equiv 0(mod\,2)$, and odd otherwise. These elements generate a $\mathbb{Z}_2$-graded bi- subalgebra $G_q^{\bar{\pi}}$ of $(U_q(\mathfrak{g}))^0$ in the standard fashion. Here we merely point out that they obey the relation

$$R_{12}^{\bar{\pi}\,\bar{\pi}}\ \bar{t}_1\ \bar{t}_2 = \bar{t}_2\ \bar{t}_1 R_{12}^{\bar{\pi}\,\bar{\pi}}, \tag{12}$$

with

$$\bar{t} := \sum_{a,b} e_{a\,b} \otimes \bar{t}_{b\,a}, \qquad R^{\bar{\pi}\,\bar{\pi}} := (\bar{\pi} \otimes \bar{\pi})R.$$

The following explicit form of $R^{\bar{\pi}\,\bar{\pi}}$ is obtained from [16]

$$R^{\bar{\pi}\,\bar{\pi}} = q^{\sum_{a\in\mathbf{I}} e_{a\,a}\otimes e_{a\,a}(-1)^{[a]}} + (q - q^{-1})\sum_{a>b} e_{a\,b} \otimes e_{b\,a}(-1)^{[b]}.$$

Also, the co-multiplication is given by

$$\Delta(\bar{t}_{a\,b}) = \sum_{c\in\mathbf{I}}(-1)^{([a]+[c])([c]+[b])}\bar{t}_{a\,c} \otimes \bar{t}_{c\,b}.$$

Denote by $\bar{\pi}^{(-\bar{\lambda})}$ the irreducible representation dual to $\pi^{(\lambda)}$, $\lambda \in \Lambda^{(1)}$, in a given homogeneous basis. Introduce the elements $\bar{t}^{(-\bar{\lambda})}_{i\,j}$, $i,\ j = 1, 2, ..., dim_{\mathbb{C}}\pi^{(\lambda)}$, of $(U_q(\mathfrak{g}))^0$ such that

$$\bar{t}^{(-\bar{\lambda})}_{i\,j}(x) = \bar{\pi}^{(-\bar{\lambda})}_{i\,j}(x), \quad \forall x \in U_q(\mathfrak{g}).$$

Then it follows from Proposition 1 that these elements form a basis of $G_q^{\bar{\pi}}$. Set

$$\bar{T}^{(\mu)} = \oplus_{i,j}\mathbb{C}\bar{t}^{(\mu)}_{i\,j}.$$

We have

Proposition 3.

$$G_q^{\bar{\pi}} = \bigoplus_{\mu\in\Lambda^{(2)}} \bar{T}^{(\mu)}.$$

3.3. Algebra G_q of functions on $GL_q(m|n)$. We define the algebra G_q of functions on the quantum general linear supergroup $GL_q(m|n)$ to be the $\mathbb{Z}_2$-graded subalgebra of $(U_q(\mathfrak{g}))^0$ generated by $\{t_{a\,b},\ \bar{t}_{a\,b} \mid a, b \in \mathbf{I}\}$. The $t_{a\,b}$ and $\bar{t}_{a\,b}$, besides obeying the relations (10) and (12), also satisfy

$$R_{12}^{\bar{\pi}\,\pi}\ \bar{t}_1\ t_2 = t_2\ \bar{t}_1\ R_{12}^{\bar{\pi}\,\pi}, \tag{13}$$

where $R^{\bar{\pi}\,\pi} := (\bar{\pi} \otimes \pi)R$. Equation (13) arises by first applying $\bar{\pi} \otimes \pi$ to both sides of (8), then interpreting the resulting equation in terms of the $t_{a\,b}$ and $\bar{t}_{a\,b}$. The following explicit expression of $R^{\bar{\pi}\,\pi}$ is extracted from [16],

$$R^{\bar{\pi}\,\pi} = q^{-\sum_{a\in\mathbf{I}} e_{a\,a}\otimes e_{a\,a}(-1)^{[a]}} - (q - q^{-1})\sum_{a<b} e_{b\,a} \otimes e_{b\,a}(-1)^{[a]+[b]+[a][b]}.$$

Equation (13) enables us to factorize G_q into

$$G_q = G_q^{\pi}\, G_q^{\bar{\pi}}. \tag{14}$$

As both G_q^{π} and $G_q^{\bar{\pi}}$ are $\mathbb{Z}_2$-graded bi-algebras, G_q inherits a natural bi-algebra structure. It also admits an antipode. By considering

$$(x\bar{v}_a)(v_b) = (-1)^{[x][a]}\bar{v}_a(S(x)v_b), \quad x \in U_q(\mathfrak{g}),$$

where $\{v_a\}$ is the standard basis of the vector representation, and $\{\bar{v}_a\}$ is the basis of the covariant vector representation dual to $\{v_a\}$, we arrive at

Lemma 1. *The antipode $S : G_q \to G_q$ is a linear anti-automorphism given by*

$$S(t_{a\,b}) = (-1)^{[a][b]+[a]}\bar{t}_{b\,a},$$
$$S(\bar{t}_{a\,b}) = (-1)^{[a][b]+[b]}q^{(2\rho,\ \epsilon_a-\epsilon_b)}t_{b\,a}. \tag{15}$$

Therefore, G_q has the structures of a $\mathbb{Z}_2$-graded Hopf algebra.

Furthermore, $*$-operations can also be constructed for G_q, thus turning it into a Hopf $*$-superalgebra. We have

$$*(t_{a\,b}) = (-1)^{(\theta+[a])([a]+[b])}\bar{t}_{a\,b},$$
$$*(\bar{t}_{a\,b}) = (-1)^{(\theta+[a])([a]+[b])}t_{a\,b},$$

where $\theta \in \mathbb{Z}_2$.

An important property of G_q is that it separates points of $U_q(\mathfrak{g})$, that is, for any nonvanishing $x \in U_q(\mathfrak{g})$, there exists $f \in G_q$ such that $f(x) \neq 0$. As a matter of fact, G_q^π by itself separates points of $U_q(\mathfrak{g})$. Put differently, for any $u \in U_q(\mathfrak{g})$, if $u \neq 0$, then $\pi^{\otimes p}(u) \neq 0$ for some $p \in \mathbb{Z}_+$.

To verify our assertion, we first consider the corresponding proposition in the classical situation of $U(\mathfrak{g})$ in detail. Let $E^{(0)}_{a\,b}$, $a,\ b \in \mathbf{I}$, be the standard generators of $\mathfrak{g}$ embedded in its universal enveloping algebra. In the vector representation $\pi^{(0)}$, one has

$$\pi^{(0)}(E^{(0)}_{a\,b}) = e_{a\,b}.$$

We isolate the $u(1)$ subalgebra of $\mathfrak{g}$ with the generator

$$Z^{(0)} = \sum_{a\in\mathbf{I}} E^{(0)}_{a\,a},$$

and denote by $X^{(0)}_A$, $A = 1, ..., (m+n)^2 - 1$, the elements $E^{(0)}_{c\,c} - E^{(0)}_{c+1\ c+1}$, $c \in \mathbf{I}'$, and $E^{(0)}_{a\,b}$, $a \neq b$, in any fixed ordering. Then a Poincaré–Birkhoff–Witt basis for $U(\mathfrak{g})$ is given by,

$$\{B^{(0)}_{k,\ A_1\ldots A_l} = (Z^{(0)})^k X^{(0)}_{A_1}...X^{(0)}_{A_l} \mid k, l \in \mathbb{Z}_+,\ A_i \leq A_{i+1},\ A_i \neq A_{i+1} \text{ if } [X^{(0)}_{A_i}] = 1\}.$$

Set $\pi^{(0)}(X^{(0)}_A) = e_A$. Denote by $\mathcal{M}$ the vector space of $(m+n)\times(m+n)$ matrices, and define

$$\mathcal{R}_k = \sum_{i=0}^{k-1} \underbrace{\mathcal{M}\otimes ... \otimes \mathcal{M}}_{i} \otimes I \otimes \underbrace{\mathcal{M}\otimes ... \otimes \mathcal{M}}_{k-1-i}.$$

Let

$$b_{A_1\,...\,A_k} = \sum_{\sigma\in S_k} (-1)^{|\sigma_{\{A\}}|} e_{A_{\sigma(1)}} \otimes e_{A_{\sigma(2)}} \otimes ... \otimes e_{A_{\sigma(k)}},$$

where $|\sigma_{\{A\}}|$ is the number of permutations required amongst odd elements in order to change $X_{A_1} \otimes X_{A_2} \otimes ... \otimes X_{A_k}$ to $X_{A_{\sigma(1)}} \otimes X_{A_{\sigma(1)}} \otimes ... \otimes X_{A_{\sigma(k)}}$. Clearly, the elements

$$\{b_{A_1\,...\,A_k} \mid k \in \mathbb{Z}_+,\ A_i \leq A_{i+1},\ A_i \neq A_{i+1} \text{ if } [X_{A_i}] = 1\}$$

are linearly independent in $\mathcal{M}^{\otimes k}$, and we will denote by $\mathcal{L}_k$ their linear span. By considering the trace (not the supertrace!) on each factor of $\mathcal{M}^{\otimes k}$, we can easily see that $\mathcal{L}_k$ intersects $\mathcal{R}_k$ trivially. Therefore,

$$\begin{aligned}(\pi^{(0)})^{\otimes(k+p)}(B^{(0)}_{0,\ A_1\ldots A_k}) &= \left((\pi^{(0)})^{\otimes k} \otimes (\pi^{(0)})^{\otimes p}\right)(B^{(0)}_{0,\ A_1\ldots A_k}) \\ &= b_{A_1\ldots A_k} \otimes I^{\otimes p} + r_{k,\,p}, \quad r_{k,\,p} \in \mathcal{R}_k \otimes \mathcal{M}^{\otimes p},\end{aligned}$$

are linearly indepenedent as elements of $\mathcal{M}^{\otimes(k+p)}$.

Consider $u \in U(\mathfrak{g})$ given by

$$u = \sum_{k=0}^{K}\sum_{l=0}^{L}\sum_{\{A\}} C_{k,\ A_1\ldots A_l} B^{(0)}_{k,\ A_1\ldots A_l}, \quad C_{k,\ A_1\ldots A_l} \in \mathbb{C}.$$

Using

$$(\pi^{(0)})^{\otimes p}(Z^k) = p^k\ I^{\otimes p},$$

we immediately see that $(\pi^{(0)})^{\otimes p}(u) = 0$, $\forall p > L$, requires

$$\sum_{k=0}^{K} p^k C_{k,\ A_1\ldots A_l} = 0, \quad \forall p > L,$$

which forces all the $C_{k,\ A_1\ldots A_l}$ to vanish. This completes the proof for the classical case.

Remarks. There is something slightly unnatural about our proof, that is, the combination $E^{(0)}_{m\,m} - E^{(0)}_{m+1\ m+1}$ does not belong to $sl(m|n) \subset \mathfrak{g}$, and this in turn forced us to consider the ordinary trace instead of the supertrace in proving $\mathcal{L}_k \cap \mathcal{R}_k = \{0\}$. We can avoid this unnaturalness when $m \neq n$ by using $E^{(0)}_{m\,m} + E^{(0)}_{m+1\ m+1}$ instead, but not when $m = n$.

With the above preparations we can now readily prove our assertion for the quantum superalgebra. We first consider the Drinfeld version of $U_q(\mathfrak{g})$. Similar to the classical case, we set

$$Z = \sum_{a\in\mathbf{I}} E_{a\,a},$$

and denote by X_A, $A = 1, \ldots., (m+n)^2 - 1$, the elements $E_{c\,c} - E_{c+1\ c+1}$, $c \in \mathbf{I}'$ and $E_{a\,b}$, $a \neq b$, in a fixed ordering. Then

$$\{B_{k,\ A_1\ldots A_l} = Z^k X_{A_1}\ldots X_{A_l} \,|\, k, l \in \mathbb{Z}_+,\ A_i \leq A_{i+1},\ A_i \neq A_{i+1} \text{ if } [X_{A_i}] = 1\}$$

forms a Poincaré–Birkhoff–Witt basis for $U_q(\mathfrak{g})$ [6]. Given

$$u = \hbar^k(u_0 + \hbar u_1 + \hbar^2 u_2 + \ldots),$$

where each u_i is a finite $\mathbb{C}$-combination of some $B_{k,\ A_1\ldots A_l}$, and u_0 is assumed to be nonzero. Then it follows from the classical case that there exist infinitely many $p \in \mathbb{Z}_+$ such that

$$\pi^{\otimes p}(u) \not\equiv 0 (mod\, \hbar^{k+1}).$$

For the Jimbo algebra, we observe that ordered monomials in $E_{a\,b}$, $a \neq b$, and $K_a^{\pm 1}$ form a basis of $U_q(\mathfrak{g})$. Given $u \in U_q(\mathfrak{g})$, and a positive integer p, we consider the matrix elements of $\pi^{\otimes p}(u)|_{q=\exp(\hbar)}$ as a power series in $\hbar$. $\pi^{\otimes p}(u) \not\equiv 0$ if and only if some of these power series do not vanish identically. Now for the purpose of computing $\pi^{\otimes p}(u)|_{q=\exp(\hbar)}$, we can make the identification

$$\pi^{\otimes p}(K_a) = \sum_{k=0}^{\infty} \frac{(-1)^{k[a]}\hbar^k}{k!} e_{a\,a}(p)^k,$$
$$e_{a\,a}(p) = \sum_{i=0}^{p-1} \underbrace{I \otimes ... \otimes I}_{i} \otimes e_{a\,a} \otimes \underbrace{I \otimes ... \otimes I}_{p-i-1}.$$

This takes us back to the Drinfeld algebra situation, and we have already shown that in that situation the $\pi^{\otimes p}$, $p \in \mathbb{Z}_+$, separates points of $U_q(\mathfrak{g})$.

We summarize the discussions of this section into a proposition, points ii) and iii) of which may be considered as a partial generalization of the classical Peter-Weyl theorem to the quantum supergroup in an algebraic setting:

Proposition 4. (i) *G_q is a $*$-Hopf superalgebra;*
(ii) *G_q separates points of $U_q(\mathfrak{g})$;*
(iii) *The following elements span G_q:*

$$t^{(\lambda)}_{i\,j}\, \bar{t}^{(\mu)}_{i'\,j'}, \quad i,\ j = 1, 2, ..., dim\pi^{(\lambda)}, \ \ \lambda \in \Lambda^{(1)},$$
$$i',\ j' = 1, 2, ..., dim\bar{\pi}^{(\mu)}, \ \ \mu \in \Lambda^{(2)}.$$

However, we should point out that these elements are *not* linearly independent.

4. Induced Representations of G_q

We will develop parabolic induction for representations of $GL_q(m|n)$ in this section. Recall that corresponding to every locally finite right co-module $\omega : W \to W \otimes G_q$ over G_q, there exists a unique left $U_q(\mathfrak{g})$ module $U_q(\mathfrak{g}) \otimes W \to W$ with the module action defined by

$$x\, w = \omega(w)(x), \quad x \in U_q(\mathfrak{g}), \ \ w \in W.$$

A similar correspondence exists for left G_q co-modules and right U_q modules. Therefore, we can describe the representation theory of G_q in both the G_q co-module language and $U_q(\mathfrak{g})$ module language, depending on which one is more convenient in a given situation. We will largely use the latter here.

4.1. Parabolic subalgebras of $U_q(gl(m|n))$. Let Θ be a subset of $\mathbf{I}'$. Introduce the following sets of elements of $U_q(\mathfrak{g})$:

$$\mathcal{S}_l = \{K_a^{\pm 1}, a \in \mathbf{I};\ \ E_{c\ c+1},\ E_{c+1\ c},\ c \in \Theta\};$$
$$\mathcal{S}_{p_+} = \mathcal{S}_l \cup \{E_{c\ c+1},\ c \in \mathbf{I}'\backslash\Theta\};$$
$$\mathcal{S}_{p_-} = \mathcal{S}_l \cup \{E_{c+1\ c},\ c \in \mathbf{I}'\backslash\Theta\}.$$

The elements of each set generate a $\mathbb{Z}_2$ -graded Hopf subalgebra of $U_q(\mathfrak{g})$. We denote by $U_q(\mathfrak{l})$ the Hopf subalgebra generated by the elements of $\mathcal{S}_l$, and by $U_q(\mathfrak{p}_\pm)$ the Hopf subalgebras respectively generated by the elements of $\mathcal{S}_{p_\pm}$. In the classical limit, the Hopf subalgebras $U_q(\mathfrak{p}_\pm)$ coincide with the universal enveloping algebras of parabolic subalgebras of the Lie superalgebra $\mathfrak{g}$. Therefore, we will call $U_q(\mathfrak{p}_\pm)$ parabolic subalgebras of $U_q(\mathfrak{g})$.

Let V_μ be a finite dimensional irreducible $U_q(\mathfrak{l})$ module. Then V_μ is of highest weight type. Let μ be the highest weight and $\tilde{\mu}$ the lowest weight of V_μ respectively. We can extend V_μ in a unique fashion to a $U_q(\mathfrak{p}_+)$ module, which we still denote by V_μ, such that the elements of $\mathcal{S}_{p_+}\backslash\mathcal{S}_l$ act by zero. Similarly, V_μ also leads to a $U_q(\mathfrak{p}_-)$ module, on which the elements of $\mathcal{S}_{p_-}\backslash\mathcal{S}_l$ act by zero. It is not difficult to see that all finite dimensional irreducible $U_q(\mathfrak{p}_\pm)$ modules are of this kind.

Consider a finite dimensional irreducible $U_q(\mathfrak{g})$ module $W(\lambda)$ with highest weight λ and lowest weight $\overline{\lambda}$. $W(\lambda)$ can be restricted into a $U_q(\mathfrak{p}_+)$ or $U_q(\mathfrak{p}_-)$ module in a natural way, and the resultant module is always indecomposable, but not irreducible in general.

Consider first the case of $U_q(\mathfrak{p}_+)$. We wish to examine the $\mathbb{Z}_2$-graded vector space $Hom_{U_q(\mathfrak{p}_+)}(W(\lambda),\ V_\mu)$, which graded-commutes with $U_q(\mathfrak{p}_+)$, namely,

$$p\,\phi - (-1)^{[p][\phi]}\phi\, p = 0, \quad p \in U_q(\mathfrak{p}_+),\ \phi \in Hom_{U_q(\mathfrak{p}_+)}(W(\lambda),\ V_\mu).$$

Because of the irreducibility of V_μ, every non-zero $\phi \in Hom_{U_q(\mathfrak{p}_+)}(W(\lambda),\ V_\mu)$ must be surjective, and thus $V_\mu \cong W(\lambda)/Ker\phi$. As a $U_q(\mathfrak{p}_+)$ module, $W(\lambda)$ is indecomposable, and contains a unique maximal proper submodule M such that the lowest weight vector w_- of $W(\lambda)$ does not belong to M. Therefore, $Ker\phi = M$, and $V_\mu = \phi(U_q(\mathfrak{l})w_-)$. This forces $\overline{\lambda} = \tilde{\mu}$, and all elements of $Hom_{U_q(\mathfrak{p}_+)}(W(\lambda),\ V_\mu)$ are scalar multiples of one another. It is worth observing that the map ϕ may be odd. In fact its degree is given by $[\phi] \equiv [w_-] + [\phi(w_-)]\,(mod 2)$. The case of $U_q(\mathfrak{p}_-)$ can be studied in exactly the same way. To summarize, we have

Lemma 2.

$$\dim_{\mathbb{C}} Hom_{U_q(\mathfrak{p}_+)}(W(\lambda),\ V_\mu) = \begin{cases} 1, & \overline{\lambda} = \tilde{\mu}, \\ 0, & \overline{\lambda} \neq \tilde{\mu}. \end{cases}$$

$$\dim_{\mathbb{C}} Hom_{U_q(\mathfrak{p}_-)}(W(\lambda),\ V_\mu) = \begin{cases} 1, & \lambda = \mu, \\ 0, & \lambda \neq \mu. \end{cases}$$

4.2. Induced representations and quantum superbundles. Let us first introduce two types of left actions of $U_q(\mathfrak{g})$ on G_q, which correspond to the left and right translations in the classical situation.

Define a bilinear map $\cdot : U_q(\mathfrak{g}) \otimes G_q \to G_q$ by

$$\begin{aligned} x \otimes f \mapsto\ & x \cdot f \\ &= \sum_{(f)} \langle f_{(1)},\ S^{-1}(x)\rangle f_{(2)}, \end{aligned} \tag{16}$$

which can be easily shown to satisfy

$$\begin{aligned} &(x \cdot f)(y) = (-1)^{[x][y]} f(S^{-1}(x)y), \\ &x \cdot (y \cdot f) = (xy) \cdot f, \quad x,\ y \in U_q(\mathfrak{g}),\ f \in G_q. \end{aligned}$$

(We assume that the elements $x,\ y \in U_q(\mathfrak{g})$ and $g,\ f \in G_q$ are homogeneous for the sake of simplicity. All the statements below generalize to inhomogeneous elements in the obvious way.) Therefore, this defines a left action of $U_q(\mathfrak{g})$ on G_q, which corresponds to the left translation of Lie groups in the classical situation. It is worth observing that we may replace S^{-1} in the above definition, and arrive at a different left action.

Another left action "$\circ$" of $U_q(\mathfrak{g})$ on G_q can be defined by

$$x \circ f = \sum_{(f)} (-1)^{[x]([f]+[x])} f_{(1)} \langle f_{(2)}, x \rangle. \tag{17}$$

Straightforward calculations can show that

$$\begin{aligned} x \circ (y \circ f) &= (xy) \circ f; \\ (x \circ f)(y) &= f(yx), \\ (id_{G_q} \otimes x\circ)\Delta(f) &= \Delta(x \circ f). \end{aligned}$$

This corresponds to the right translation in the classical theory. It graded-commutes with the action "$\cdot$", namely,

$$x \circ (y \cdot f) = (-1)^{[x][y]} y \cdot (x \circ f).$$

Let $U_q(\mathfrak{p})$ denote either $U_q(\mathfrak{p}_+)$ or $U_q(\mathfrak{p}_-)$. Given any finite dimensional left $U_q(\mathfrak{p})$ module V, we form the tensor product $V \otimes_{\mathbb{C}} G_q$, which is a subspace of functions $U_q(\mathfrak{g}) \to V$:

$$\begin{aligned} \zeta &= \sum v_i \otimes f_i \in V \otimes G_q, \\ x &\in U_q(\mathfrak{g}), \\ \zeta(x) &= \sum f_i(x) v_i. \end{aligned}$$

The left actions "$\cdot$" and "$\circ$" of $U_q(\mathfrak{g})$ on G_q can be extended in an obvious way to actions on $V \otimes_{\mathbb{C}} G_q$,

$$\begin{aligned} x \cdot \zeta &= \sum (-1)^{[x][v_i]} v_i \otimes x \cdot f_i, \\ x \circ \zeta &= \sum (-1)^{[x][v_i]} v_i \otimes x \circ f_i, \quad x \in U_q(\mathfrak{g}). \end{aligned}$$

Furthermore, there also exists a co-action ω of G_q on $V \otimes_{\mathbb{C}} G_q$ defined by $\omega = id_V \otimes \Delta'$, where Δ' represents the opposite co-multiplication of G_q.

Consider the subspace of $V \otimes_{\mathbb{C}} G_q$ defined by

$$\mathcal{O}^V = \{\zeta \in V \otimes_{\mathbb{C}} G_q \mid p \circ \zeta = (S(p) \otimes id_{G_q})\zeta, \ \forall\, p \in U_q(\mathfrak{p})\}. \tag{18}$$

Lemma 3. *$\mathcal{O}^V$ furnishes a left $U_q(\mathfrak{g})$ module under "$\cdot$", and at the same time a right G_q co-module under ω.*

Proof. The lemma can be confirmed by direct calculations. For $x \in U_q(\mathfrak{g})$, $p \in U_q(\mathfrak{p})$, $\zeta \in \mathcal{O}^V$, we have

$$\begin{aligned} p \circ (x \cdot \zeta) &= (-1)^{[x][p]} x \cdot (p \circ \zeta) \\ &= (S(p) \otimes id_{G_q})(x \cdot \zeta); \\ (p \circ \otimes id_{G_q})\omega(\zeta) &= (p \circ \otimes id_{G_q})(id_V \otimes \Delta')\zeta \\ &= (id_V \otimes \tau)(id_V \otimes id_{G_q} \otimes p\circ)(id_V \otimes \Delta)\zeta \\ &= (id_V \otimes \tau)(id_V \otimes \Delta')(p \circ \zeta) \\ &= \omega(S(p) \otimes id_{G_q})\zeta, \end{aligned}$$

where τ is the flip mapping.

We call $\mathcal{O}^V$ the induced $U_q(\mathfrak{g})$ module, and also the induced G_q co-module, which gives rise to a co-representation of G_q.

A conceptual understanding of $\mathcal{O}^V$ can be gained by considering its classical analog, which was investigated by Manin [20] and Penkov [21]. Very briefly (precise and extensive treatments can be found in the references just given.), if P is a parabolic subgroup of the complex Lie supergroup $SL(m|n)$, and E a finite dimensional representation of P, then $SL(m|n) \times_P E$, the quotient space of $SL(m|n) \times E$ under the equivalence relation $(g, v) \sim (gp, p^{-1}v)$ for all $p \in P$, defines a super vector bundle over the supermanifold $SL(m|n)/P$. A function $f : SL(m|n) \to E$ satisfying $f(gp) = p^{-1}f(g)$, $\forall p \in P$ defines a section of the bundle $s_f : SL(m|n)/P \to SL(m|n) \times_P E$. Analogously, we may regard $\mathcal{O}^V$ as the vector space of sections of a quantum super vector bundle over the quantum counterpart of $SL(m|n)/P$.

It is of great importance to systematically develop the theory of quantum homogeneous super vector bundles, and we hope to return to the subject in the future. In this paper, we will restrict ourselves to issues directly related to representation theory, and will not further ponder noncommutative geometry, except for the last section, where we will discuss in some detail quantum projective superspaces when dealing with explicit realizations of the irreducible skew supersymmetric tensor representations and their duals.

We have the following quantum analog of Frobenius reciprocity.

Proposition 5. *Let W be a quotient $U_q(\mathfrak{g})$ module of $\bigoplus_{k,l=0}^{\infty} \mathbb{E}^{\otimes k} \otimes (\mathbb{E}^*)^{\otimes l}$ (the restriction of which furnishes a $U_q(\mathfrak{p})$ module in a natural way). Then there is a canonical isomorphism*

$$Hom_{U_q(\mathfrak{g})}(W,\ \mathcal{O}^V) \cong Hom_{U_q(\mathfrak{p})}(W,\ V). \tag{19}$$

Proof. We prove the proposition by explicitly constructing the isomorphism, which we claim to be the linear map

$$\begin{aligned} F : \mathrm{Hom}_{U_q(\mathfrak{g})}(W,\ \mathcal{O}^V) &\to \mathrm{Hom}_{U_q(\mathfrak{p})}(W,\ V), \\ \psi &\mapsto \psi(1_{U_q(\mathfrak{g})}), \end{aligned}$$

with the inverse map

$$\begin{aligned} \overline{F} : \mathrm{Hom}_{U_q(\mathfrak{p})}(W,\ V) &\to \mathrm{Hom}_{U_q(\mathfrak{g})}(W,\ \mathcal{O}^V), \\ \phi &\mapsto \overline{\phi}, \end{aligned}$$

where $\overline{\phi}$ is defined by

$$\overline{\phi}(w)(x) = (-1)^{[x]([w]+1)}\phi(S(x)w), \quad x \in U_q(\mathfrak{g}),\ w \in W.$$

As for F, we need to show that its image is contained in $\mathrm{Hom}_{U_q(\mathfrak{p})}(W,\ V)$. This is indeed the case, as

$$\begin{aligned} p(F\psi(w)) &= (p \cdot \psi(w))(1_{U_q(\mathfrak{g})}) \\ &= (-1)^{[\psi][p]}F\psi(pw), \quad p \in U_q(\mathfrak{p}),\ \ w \in W. \end{aligned}$$

In order to show that $\overline{F}$ is the inverse of F, we first need to demonstrate that the image $Im(\overline{F})$ of $\overline{F}$ is contained in $\mathrm{Hom}_{U_q(\mathfrak{g})}(W,\ \mathcal{O}^V)$. Note that $Im(\overline{F}) \subset \mathrm{Hom}_{\mathbb{C}}(W,\ V \otimes G_q)$, since W is a subquotient of $\bigoplus_{k,l=0}^{\infty} \mathbb{E}^{\otimes k} \otimes (\mathbb{E}^*)^{\otimes l}$. Some relatively simple manipulations lead to

$$\begin{aligned}(y\cdot\bar{\phi}(w))(x) &= (-1)^{[y][\bar{\phi}]+[x]([w]+[x]+[y])}\phi(S(x)yw)\\ &= (-1)^{[y][\bar{\phi}]}\bar{\phi}(yw)(x),\\ (p\circ\bar{\phi}(w))(x) &= (-1)^{[x]([w]+1)+[p][\phi]}\phi(S(p)S(x)w)\\ &= S(p)(\bar{\phi}(w)(x)),\quad x,\ y\in U_q(\mathfrak{g}),\ \ p\in U_q(\mathfrak{p}),\ \ w\in W.\end{aligned}$$

Therefore, $Im(\bar{F})\subset \mathrm{Hom}_{U_q(\mathfrak{g})}(W,\ \mathcal{O}^V)$.

Now we show that F and $\bar{F}$ are inverse to each other. For $\psi\in\mathrm{Hom}_{U_q(\mathfrak{g})}(W,\ \mathcal{O}^V)$, and $\phi\in\mathrm{Hom}_{U_q(\mathfrak{p})}(W,\ V)$, we have

$$\begin{aligned}(F\bar{F}\phi)(w) &= (\bar{F}\phi)(w)(1_{U_q(\mathfrak{g})})\\ &= \phi(w),\\ (\bar{F}F\psi)(w)(x) &= (-1)^{[x]([w]+1)}(F\psi)(S(x)w)\\ &= (-1)^{[x]([w]+1)}\psi(S(x)w)(1_{U_q(\mathfrak{g})})\\ &= (-1)^{[x]([\psi(w)]+1)}(S(x)\cdot\psi(w))(1_{U_q(\mathfrak{g})})\\ &= \psi(w)(x),\quad x\in U_q(\mathfrak{g}),\ \ w\in W.\end{aligned}$$

This completes the proof of the proposition.

4.3. Quantum Borel–Weil theorem for the irreducible covariant and contravariant tensor representations. In this subsection we study in detail the irreducible covariant and contravariant tensor representations of $U_q(\mathfrak{g})$ within the framework of parabolic induction. Our main result here will be a quantum version of the Borel–Weil theorem for these irreducible representations.

For the classical Lie supergroups, the program of developing a Bott–Borel–Weil theory was initiated and extensively investigated by Penkov and co-workers [21, 22], although much remains to be done on the subject. Their program has also revealed a very rich content and various interesting new phenomena. It appears that the Hopf algebraic approach to the Bott–Borel–Weil theory developed here is also worth exploring at the classical level, and is likely to provide a new method complementary to the geometric approach of [21].

Let V be a finite dimensional irreducible $U_q(\mathfrak{p})$ module, with the $U_q(\mathfrak{l})$ highest weight μ and $U_q(\mathfrak{l})$ lowest weight $\tilde{\mu}$. For the purpose of studying the tensor representations, we need to consider

$$\begin{aligned}\overline{\mathcal{O}}(\mu) &= \mathcal{O}^V\cap\left(V\otimes G_q^{\pi}\right),\\ \mathcal{O}(\mu) &= \mathcal{O}^V\cap\left(V\otimes G_q^{\bar{\pi}}\right).\end{aligned}\tag{20}$$

Let us study $\mathcal{O}(\mu)$ first. A typical element of $\mathcal{O}(\mu)$ is of the form

$$\zeta=\sum_{\lambda\in\Lambda^{(1)}}\sum_{\alpha,\beta,i}c^{\lambda}_{\alpha\beta,\ i}\ v_i\otimes\bar{t}^{(\lambda^{\dagger})}_{\alpha\beta},$$

where $\{v_i\}$ is a basis of V, and the $c^{\lambda}_{\alpha\beta,\ i}$ are complex numbers. The $\bar{t}^{(\lambda^{\dagger})}_{\alpha\beta}$ are elements of the Peter-Weyl basis for $G_q^{\bar{\pi}}$, which, needless to say, are polynomials in $\bar{t}_{ab}$, $a,\ b\in\mathbf{I}$. The property that $(p\circ\zeta)=(\bar{S}(p)\otimes id_{G_q})\zeta$, $\forall p\in U_q(\mathfrak{p})$ leads to

$$\sum_{\gamma,i}(-1)^{[p]([\gamma]+[v_i])}c^{\lambda}_{\alpha\gamma,\ i}\ t^{(\lambda)}_{\gamma\beta}(p)v_i=\sum_i c^{\lambda}_{\alpha\beta,\ i}\ p\,v_i,\quad \forall p\in U_q(\mathfrak{p}).\tag{21}$$

Let $W(\lambda)$ with the basis $\{w_\alpha\}$ be the irreducible $U_q(\mathfrak{g})$ module associated with the irreducible representation $t^{(\lambda)}$. We define the linear maps between $\mathbb{Z}_2$ graded *vector spaces*

$$\phi_\lambda^{(\alpha)}: \ W(\lambda) \to V, \\ w_\beta \mapsto \sum_i c^\lambda_{\alpha\,\beta,\,i}\, v_i.$$

There is no particular significance attached to the maps at this stage, apart from the mere fact that they can be employed to re-express Eq. (21) as

$$\sum_\gamma (-1)^{[p][\phi_\lambda^{(\alpha)}]} t^{(\lambda)}_{\gamma\,\beta}(p)\phi_\lambda^{(\alpha)}(w_\gamma) = p\,\phi_\lambda^{(\alpha)}(w_\beta).$$

We emphasize that this equation is entirely equivalent to (21). Now something of crucial importance appears: this equation requires that each $\phi_\lambda^{(\alpha)}$ be a $U_q(\mathfrak{p})$ module homomorphism of degree $[\phi_\lambda^{(\alpha)}]$. Lemma 2 forces

$$\phi_\lambda^{(\alpha)} = c_\alpha\,\phi_\lambda,\; c_\alpha \in \mathbb{C},$$

and ϕ_λ may be nonzero only when

$$i)\ \overline{\lambda} = \tilde{\mu}, \text{ if } U_q(\mathfrak{p}) = U_q(\mathfrak{p}_+), \\ ii)\ \lambda = \mu, \text{ if } U_q(\mathfrak{p}) = U_q(\mathfrak{p}_-).$$

In these cases, $\mathcal{O}(\mu)$ is spanned by

$$\zeta_\alpha = \sum_\beta \phi_\lambda(w_\beta) \otimes \overline{t}^{(\lambda^\dagger)}_{\alpha\,\beta},$$

which are obviously linearly independent. Furthermore,

$$x \cdot \zeta_\alpha = (-1)^{[x][\phi_\lambda]} \sum_\beta t^{(\lambda)}_{\beta\,\alpha}(x)\,\zeta_\beta, \qquad x \in U_q(\mathfrak{g}). \tag{22}$$

The case of $\overline{\mathcal{O}}(\mu)$ can be studied in exactly the same way. To summarize, we have the following quantum analog of the Borel–Weil theorem for the irreducible covariant and contravariant tensor representations

Proposition 6. *As $U_q(\mathfrak{g})$ modules,*

$$\mathcal{O}(\mu) \cong \begin{cases} W((-\tilde{\mu})^\dagger), & \textit{if } \tilde{\mu} \in -\Lambda^{(2)},\ U_q(\mathfrak{p}) = U_q(\mathfrak{p}_+), \\ W(\mu), & \textit{if } \mu \in \Lambda^{(1)},\ \ U_q(\mathfrak{p}) = U_q(\mathfrak{p}_-), \\ \{0\}, & \textit{otherwise}. \end{cases} \tag{23}$$

$$\overline{\mathcal{O}}(\mu) \cong \begin{cases} W((-\tilde{\mu})^\dagger), & \textit{if } \tilde{\mu} \in -\Lambda^{(1)},\ U_q(\mathfrak{p}) = U_q(\mathfrak{p}_+), \\ W(\mu), & \textit{if } \mu \in \Lambda^{(2)},\ \ U_q(\mathfrak{p}) = U_q(\mathfrak{p}_-), \\ \{0\}, & \textit{otherwise}. \end{cases} \tag{24}$$

In the proposition, the notation $W(\lambda)$ signifies the irreducible $U_q(\mathfrak{g})$ module with highest weight λ.

Remarks. $\mathcal{O}(\mu)$ and $\overline{\mathcal{O}}(\mu)$, which form irreducible $U_q(\mathfrak{g})$-modules, are proper subspaces of $\mathcal{O}^V$. Although $\mathcal{O}^V$ itself also furnishes a left $U_q(\mathfrak{g})$-module, it is not irreducible in general. This fact differs drastically from the ordinary quantum group case, where the counter part of $\mathcal{O}^V$, which may be regarded as the quantum analog of the sheaf of holomorphic sections of a homogeneous vector bundle, forms an irreducible module over the corresponding quantized universal enveloping algebra.

5. Quantum Projective Superspaces and Skew Supersymmetric Tensors

We will apply the general theory developed in the last section to study two infinite classes of irreducible representations, namely, the irreducible skew supersymmetric tensor representations and their duals. Explicit realizations of these irreducible representations will be given in terms of sections of quantum super vector bundles over quantum projective superspaces.

5.1. Quantum projective superspaces. Let $U_q(\mathfrak{g}')$, $\mathfrak{g}' = gl(m|n-1)$, be the subalgebra of $U_q(\mathfrak{g})$ generated by the following elements:

$$\{K_a,\ a \in \mathbf{I}';E_{c\,c+1},\ E_{c+1\ c},\ \ c \in \mathbf{I}'\backslash\{m+n-1\}\}.$$

Clearly $U_q(\mathfrak{g}')$ is a Hopf subalgebra. Define

$$\begin{aligned}\mathcal{A}_+ &= \{f \in G_q^{\pi} \mid f(xp) = \epsilon(p)f(x),\ \forall x \in U_q(\mathfrak{g}),\ p \in U_q(\mathfrak{g}')\},\\ \mathcal{A}_- &= \{f \in G_q^{\bar{\pi}} \mid f(xp) = \epsilon(p)f(x),\ \forall x \in U_q(\mathfrak{g}),\ p \in U_q(\mathfrak{g}')\}.\end{aligned}$$

The Hopf algebra structure of $U_q(\mathfrak{g}')$ implies that both $\mathcal{A}_+$ and $\mathcal{A}_-$ are subalgebras of G_q. Together they generate another subalgebra of G_q, which we will denote by $\mathbb{S}_q^{m|n-1}$. Set

$$z_a = t_{a\ m+n},\ \bar{z}_a = \bar{t}_{a\ m+n},\ a \in \mathbf{I}.$$

Then z_a and $\bar{z}_a$ are conjugate to each other under the $*$-operation with $\theta = 0$. More explicitly,

$$*(z_a) = \bar{z}_a, \quad \forall a \in \mathbf{I}.$$

Now $\mathbb{S}_q^{m|n-1}$ is generated by the z's and $\bar{z}$'s, which satisfy the following commutations relations:

$$\begin{aligned} z_a\, z_b &= (-1)^{[z_a][z_b]}\, q\, z_b\, z_a, \quad a < b,\\ (z_c)^2 &= 0, \quad c \leq m;\\ \bar{z}_a \bar{z}_b &= (-1)^{[\bar{z}_a][\bar{z}_b]} q^{-1}\, \bar{z}_b \bar{z}_a, \quad a < b,\\ (\bar{z}_c)^2 &= 0, \quad c \leq m;\\ \bar{z}_a\, z_b &= q(-1)^{[\bar{z}_a][z_b]} z_b\, \bar{z}_a + \delta_{a\,b} \Big\{(1 - q_a^{-1})\bar{z}_a\, z_a \\ &\quad - (-1)^{[\bar{z}_a]}(q - q^{-1}) \sum_{c<a} \bar{z}_c\, z_c \Big\}, \quad \forall a,\, b \in \mathbf{I},\\ \sum_{c\in\mathbf{I}} \bar{z}_c\, z_c &= 1.\end{aligned}$$

It can be shown that the last two equations imply that

$$\sum_{c\in\mathbf{I}} q^{(2\rho,\ \epsilon_c)}\, z_c\, \overline{z}_c = q^{(2\rho,\ \epsilon_{m+n})}.$$

$\mathbb{S}_q^{m|n-1}$ furnishes a right G_q co-module algebra, with the co-module action $\omega : \mathbb{S}_q^{m|n-1} \to \mathbb{S}_q^{m|n-1} \otimes G_q$ defined by

$$\omega(z_a) = \sum_{c\in\mathbf{I}} z_c \otimes t_{a\,c},$$
$$\omega(\overline{z}_a) = \sum_{c\in\mathbf{I}} \overline{z}_c \otimes \overline{t}_{a\,c}.$$

Also, $\mathbb{S}_q^{m|n-1}$ gives rise to a right $U_q(\mathfrak{g})$ module algebra with the module action "$\circ$" defined by (17). This module algebra structure restricts naturally to a module algebra structure over $U_q(\mathfrak{g}')\otimes U_q(gl(1))$, where $U_q(gl(1))$ is generated by $K_{m+n}^{\pm 1}$. The action of $U_q(\mathfrak{g}')$ on $\mathbb{S}_q^{m|n-1}$ is trivial following the definitions of $\mathcal{A}_\pm$; $U_q(gl(1))$ also acts in a very simple manner. To be explicit, we introduce the notations that for $L = (\theta_1, ..., \theta_m; l_1, ..., l_n) \in \{0,\ 1\}^{\otimes m} \otimes \mathbb{Z}_+^{\otimes n}$, $|L| = \sum_{i=1}^m \theta_i + \sum_{\mu=1}^n l_\mu$. Set

$$\begin{aligned} Z^L &= z_1^{\theta_1}...z_m^{\theta_m} z_{m+1}^{l_1}...z_{m+n}^{l_n}, \\ \overline{Z}^L &= \overline{z}_1^{\theta_1}...\overline{z}_m^{\theta_m} \overline{z}_{m+1}^{l_1}...\overline{z}_{m+n}^{l_n}. \end{aligned} \tag{25}$$

Then for any $k \in \mathbb{Z}$, and $p \in U_q(\mathfrak{g}')$, we have

$$(pK_{m+n}^k) \circ (Z^L\, \overline{Z}^{L'}) = \epsilon(p) q^{k(|L'|-|L|)} Z^L\, \overline{Z}^{L'}. \tag{26}$$

We will define the quantum projective superspace $\mathbb{C}P_q^{m|n-1}$ to be the $U_q(gl(1))$ invariant subalgebra of $\mathbb{S}_q^{m|n-1}$, namely,

$$\mathbb{C}P_q^{m|n-1} = \left(\mathbb{S}_q^{m|n-1}\right)^{U_q(gl(1))}. \tag{27}$$

5.2. Irreducible skew supersymmetric tensor representations and their duals. We specialize $U_q(\mathfrak{p}_+)$ and $U_q(\mathfrak{p}_-)$ to the case with $\Theta = \mathbf{I}'\backslash\{m + n - 1\}$. Consider a one-dimensional irreducible $U_q(\mathfrak{p}_+)$ module $V_+ = \mathbb{C}v$ such that

$$\begin{aligned} E_{b\,b+1} v &= E_{c+1\,c} v = 0, \\ K_b v &= v, \\ K_{m+n} v &= q^{-k} v, \\ k \in \mathbb{Z}_+,&\quad b,\ c \in \mathbf{I}',\ c < m + n - 1, \end{aligned}$$

and denote the associated representation by ϕ. Define

$$\overline{\mathcal{O}}_k = \left\{\zeta \in V_+ \otimes G_q^\pi \,|\, (p \circ \zeta)(x) = \phi(S(p))\zeta(x),\ \forall x \in U_q(\mathfrak{g}),\ p \in U_q(\mathfrak{p}_+)\right\}.$$

Direct calculations can show that

$$\overline{\mathcal{O}}_k = \bigoplus_{|L|=k} \mathbb{C}v \otimes Z^L, \tag{28}$$

where Z^L is defined by (25). Then $\overline{\mathcal{O}}_k$ gives rise to the rank k irreducible skew supersymmetric tensor representation of $U_q(\mathfrak{g})$, with the highest weight

$$\lambda = \begin{cases} \sum_{i=1}^{k} \epsilon_i, & k \leq m, \\ \sum_{i=1}^{m} \epsilon_i + (k-m)\epsilon_{m+1}, & k > m. \end{cases}$$

Now let $V_- = \mathbb{C}w$ be a one dimensional irreducible $U_q(\mathfrak{p}_-)$ module such that

$$\begin{aligned} E_{c\,c+1} v &= E_{b+1\,b} v = 0, \\ K_b v &= v, \\ K_{m+n} v &= q^k v, \\ k \in \mathbb{Z}_+, &\quad b,\ c \in \mathbf{I}',\ c < m+n-1, \end{aligned}$$

and denote the corresponding irreducible representation by ψ. Define

$$\mathcal{O}_k = \left\{ \zeta \in V_- \otimes G_q^{\bar{\pi}} \,|\, (p \circ \zeta)(x) = \psi(S(p))\zeta(x),\ \forall x \in U_q(\mathfrak{g}),\ p \in U_q(\mathfrak{p}_-) \right\}.$$

Then

$$\mathcal{O}_k = \bigoplus_{|L|=k} \mathbb{C}w \otimes \overline{Z}^L. \tag{29}$$

This time $\mathcal{O}_k$ yields an irreducible representationwith highest weight

$$\lambda = -k\epsilon_{m+n},$$

which is dual to the rank k irreducible skew supersymmetric tensor representation.

References

1. Bracken, A.J., Gould, M.D. and Zhang, R.B.: Quantum supergroups and solutions of the Yang–Baxter equation. Mod. Phys. Lett. **A5**, 831–840 (1990);
   Zhang, R.B., Bracken, A.J. and Gould, M.D.: Solution of the graded Yang–Baxter equation associated with the vector representation of $U_q(osp(M/2n))$. Phys. Lett. **B257**, 133–139 (1991)
2. Chaichian, M. and Kulish, P.P.: Phys. Lett. **B234**, 72 (1990);
   Floreanini, R., Spiridonov, V.P. and Vinet, L.: Commun. Math. Phys. **137**, 149 (1991);
   Scheunert, M.: Serre- type relations for special linear Lie superalgebras. Lett. Math. Phys. **34**, 320 (1993);
   Yamane, H.: A Serre type theorem for affine Lie superalgebras and their quantized universal enveloping superalgebras. Proc. Japan Acad. bf A70, 31 (1994)
3. Drinfeld, V.G.: Quantum groups. Proc. Internl. Cong. Math., Berkeley, **1**, 789 (1986);
   Jimbo, M.: A q-difference analog of $U(g)$ and the Yang–Baxter equation. Lett. Math. Phys., **10**, 63 (1985)
4. Perk, J.H.H. and Schultz, C.L.: Phys. Lett. **A84**, 407 (1981);
   Bazhanov, V.V. and Shadrikov, A.G.: Theor. Math. Phys. **73**, 1302 (1987)
5. Gould, M.D., Zhang, R.B. and Bracken, A.J.: Quantum double construction for graded Hopf algebras. Bull. Australian Math. Soc. **47**, 353–375 (1993);
   Yamane, H.: Universal R-matrices for quantum groups associated to Lie superalgebras. Proc. Japan. Acad. bf A67, 108 (1991);
   Khoroshkin, S.M. and Tolstoy, V.N.: Universal R-matrix for quantized (super)algebras. Commun. Math. Phys. **141**, 599 (1991)
6. Zhang, R.B.: Finite dimensional irreducible representations of the quantum supergroup $U_q(gl(m/n))$. J. Math. Phys., **34**, 1236–1254 (1993)

7. Zhang, R.B.: Finite dimensional representations of $U_q(C(n))$ at arbitrary q. J. Phys. **A26**, 7041–7059 (1993);
Palev, T.D., Stoilova, N.I. and Van der Jeugt, J.: Finite-dimensional representations of the quantum superalgebra $U_q[gl(m|n)]$ and related q-identities. Commun. Math. Phys. **166**, 367 (1994);
Zhang, R.B.: *The $gl(M|N)$ super Yangian and its finite dimensional representations*. Lett. Math. Phys. **37**, 419–434 (1996);
Zhang, R.B.: *Symmetrizable quantum affine superalgebras and their representations*. J. Math. Phys. **38**, 535 (1997)
8. Bracken, A.J., Gould, M.D., Zhang, Y.Z. and Delius, G.W.: Solutions to the quantum Yang-Baxter equation with extra non-additive parameters. J. Phys. **27 A**, 6551–6561 (1994);
Bracken, A.J., Gould, M.D., Links, J.R. and Zhang, Y.Z.: A new supersymmetric and exactly solvable model of correlated electrons Phys. Rev. Lett. **74**, 2768–2771 (1995)
9. Zhang, R.B.: Braid group representations arising from quantum supergroups with arbitrary q and link polynomials. J. Math. Phys. **33**, 3918–3930 (1992);
Gould, M.D., Tsohantjis, I. and Bracken, A.J.: *Quantum supergroups and link polynomials*. Rev. Math. Phys. **l5**, 533–549 (1993);
Links, J.R., Gould, and Zhang, R.B.: Quantum supergroups, link polynomials and representations of the braid generators. Rev. Math. Phys., **5**, 345–361 (1993)
10. Zhang, R.B.: Quantum supergroups and topological invariants of three-manifolds. Rev. Math. Phys. **7**, 809–831 (1995);
Zhang, R.B. and Lee, H.C.: Lickorish invariant and quantum OSP(1|2). Mod. Phys. Lett. **A 11**, 2397 (1996)
11. Woronowicz, S.L.: Compact matrix pseudo groups. Commun. Math. Phys. **111**, 613 (1987);
Faddeev, L.D., Reshetikhin, N.Yu. and Takhtajan, L.A.: Quantization of Lie groups and Lie algebras. Leningrad Math. J. **1**, 193 (1990)
12. Manin, Yu.I.: Quantum Groups and Noncommutative Geometry. Universite de Montreal, Centre de Recherches Mathematiques, Montreal, PQ (1988);
Woronowicz, S.L.: Differential calculus on compact matrix pseudo groups (quantum groups). Commun. Math. Phys. **122**, 125 (1989);
Chu, C.S.,Ho, P.M. and Zumino, B.: Some complex quantum manifolds and their geometry. Preprint
13. Lakshmibai, V. and Reshetikhin, N.Yu.: Quantum deformation of flag and Schubert schemes. C. R. Acad. Sci. Paris. Ser. I. Math. **313** No 3, 121–126(1991)
14. Gould, M.D. and Zhang, R.B.: Classification of all star irreps of $gl(m \mid n)$. J. Math. Phys., **31**, 2552–2559 (1990)
15. Gould, M.D. and Scheunert, M. Classification of finite dimensional unitary irreps for $U_q[gl(m \mid n)]$. J. Math. Phys. **36**, 435–452 (1995)
16. Zhang, R.B.: Universal L-operator and invariants of the quantum supergroup $U_q(gl(m|n))$. J. Math. Phys., **33**, 1970–1979 (1992)
17. Woronowicz, S.L.: Tannaha–Krein duality for compact matrix pseudo groups. Inv. Math., **93**, 35 (1988)
18. Hewitt, E. and Ross, K.: *Abstract harmonic analysis, vol.2.* . 2nd edition, New York: Springer-Verlag, 1979
19. Takeuchi, M.: Some topics on $GL_q(n)$. J. Algebra **147**, 379 (1992)
20. Manin, Y.I.: *Gauge field theory and complex geometry*. Berlin: Springer-Verlag, 1988
21. Penkov, I.B.. *Borel–Weil-Bott Theory for Classical Lie Supergroups*. Sovr. Probl. Math. **32**, Moscow: VINITI, 1988 pp. 71–124
22. Penkov, I.B. and Serganova, V. Cohomology of G/P for Classical Complex Lie Groups of G and Characters of Some Typical G-Modules. Annales de L'institut Fourier, **39** , 846–873 (1989)

Communicated by T. Miwa