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A Representation for Fermionic Correlation Functions

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Abstract: Let $d\mu_S(a)$ be a Gaussian measure on the finitely generated Grassmann algebra **A**. Given an even $W(a) \in$ **A**, we construct an operator R on **A** such that

$$\frac{1}{\mathcal{Z}} \int f(a)\, e^{\mathrm{W}(a)}\, d\mu_{S(a)} = \int (\mathbb{1} - \mathrm{R})^{-1} (f)\, d\mu_S$$

for all $f(a) \in$ **A**. This representation of the Schwinger functional iteratively builds up Feynman graphs by successively appending lines farther and farther from f. It allows the Pauli exclusion principle to be implemented quantitatively by a simple application of Gram's inequality.

1. Introduction

Let $\mathbf{A}(a_1, \cdots, a_n)$ be the finite dimensional, complex Grassmann algebra freely generated by $a_1, \cdots, a_n$. Let $\mathcal{M}_r = \Big\{ (i_1,\cdots,i_r) \,\Big|\, 1\le i_1,\cdots,i_r \le n \Big\}$ be the set of all multi indices of degree $r \ge 0$. For each multi index $\mathrm{I} = (i_1,\cdots,i_r)$, set $a_\mathrm{I} = a_{i_1} \cdots a_{i_r}$. By convention, $a_\emptyset = 1$. Let

$$\mathcal{I} = \bigcup_{0\le r\le n} \Big\{ (i_1,\cdots,i_r) \,\Big|\, 1\le i_1<\cdots<i_r\le n \Big\}$$

be the family of all strictly increasing multi indices. The set of monomials $\{ a_\mathrm{I} \,|\, \mathrm{I} \in \mathcal{I} \}$ is a basis for $\mathbf{A}(a_1, \cdots, a_n)$.

Let $S = \big(S_{ij}\big)$ be a skew symmetric matrix of even order n. Recall that the Grassmann, Gaussian integral with covariance S is the unique linear map

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$$f(a,b) \;\in\; \mathbf{A}(a_1,\cdots,a_n,b_1,\cdots,b_n) \quad\longrightarrow\quad \int f(a,b)\; d\mu_{S(a)} \;\in\; \mathbf{A}(b_1,\cdots,b_n)$$

satisfying

$$\int e^{\Sigma\, a_i b_i}\; d\mu_{S(a)} = e^{-\frac{1}{2}\Sigma\, b_i S_{ij} b_j}.$$

To manipulate Grassmann, Gaussian integrals, we can "integrate by parts" with respect to the generator a_k , $k = 1,\cdots,n$,

$$\int a_k f(a_1,\cdots,a_n)\; d\mu_S = \sum_{\ell=1}^{n} S_{k\ell} \int \left(\frac{\partial}{\partial a_\ell} f(a_1,\cdots,a_n)\right)\; d\mu_S .$$

The left partial derivative

$$\frac{\partial}{\partial a_\ell} f(a) = \sum_{\mathrm{I}\in\mathcal{I}} f_{\mathrm{I}} \frac{\partial}{\partial a_\ell} a_{\mathrm{I}}$$

is determined by

$$\frac{\partial}{\partial a_\ell} a_{\mathrm{I}} = \begin{cases} 0\,, & \ell \notin \mathrm{I} \\ (-1)^{|\mathrm{J}|}\, a_{\mathrm{J}}\, a_{\mathrm{K}}\,, & a_{\mathrm{I}} = a_{\mathrm{J}}\, a_\ell\, a_{\mathrm{K}}. \end{cases}$$

Here, $|\mathrm{J}|$ is the degree of J. Integrating by parts with respect to a_{i_1} and then arguing by induction on r we find

$$\int a_{\mathrm{I}}\; d\mu_{S(\mathbf{a})} = \mathrm{Pf}\big(S_{\mathrm{I}}\big)$$

where, for any multi index $\mathrm{I} = (i_1,\cdots,i_r)$ in $\mathcal{M}_r$, $S_{\mathrm{I}} = \Big(S_{i_k i_\ell}\Big)$ is the skew symmetric matrix with elements $S_{i_k i_\ell}$, $k,\ell = 1,\cdots,r$, and where

$$\mathrm{Pf}\big(S_{\mathrm{I}}\big) = \frac{1}{2^s s!} \sum_{k_1,\cdots,k_r=1}^{r} \varepsilon^{k_1\cdots k_r}\, S_{i_{k_1} i_{k_2}} \cdots S_{i_{k_{r-1}} i_{k_r}}$$

is its Pfaffian when $r = 2s$ is even. By convention, the Pfaffian of a skew symmetric matrix of odd order is zero. As usual,

$$\varepsilon^{i_1\cdots i_r} = \begin{cases} 1\,, & i_1,\cdots,i_r \text{ is an even permutation of } 1,\cdots,r \\ -1\,, & i_1,\cdots,i_r \text{ is an odd permutation of } 1,\cdots,r \\ 0\,, & i_1,\cdots,i_r \text{ are not all distinct} \end{cases}.$$

Let $\mathrm{W}(a)$ be an element of the commutative subalgebra $\mathbf{A}^0(a_1,\cdots,a_n)$ of all "even" Grassmann polynomials in $\mathbf{A}(a_1,\cdots,a_n)$. That is,

$$\mathrm{W}(a) = \sum_{r\ge 0} \sum_{j_1,\cdots,j_r} \mathrm{w}_r(j_1,\cdots,j_r)\, a_{j_1}\cdots a_{j_r},$$

where, $\mathrm{w}_r(j_1,\cdots,j_r)$, is an antisymmetric function of its arguments $1\le j_1,\cdots,j_r\le n$ that vanishes identically when r is odd. By definition, the "Schwinger functional" $\mathcal{S}(f)$ on $\mathbf{A}(a_1,\cdots,a_n)$ corresponding to the "interaction" $\mathrm{W}(a)$ and the "propagator" $S = \big(S_{ij}\big)$ is

$$\mathcal{S}(f) = \frac{1}{\mathcal{Z}} \int f(a)\, e^{\mathrm{W}(a)}\; d\mu_{S(a)},$$

where $\mathcal{Z} = \int e^{\mathrm{W}(a)}\, d\mu_S$. The associated “correlation functions” $\mathcal{S}_{m(j_1,\cdots,j_m)}$, $m \geq 0$, are given by

$$\mathcal{S}_{m(j_1,\cdots,j_m)} = \frac{1}{\mathcal{Z}} \int a_{j_1} \cdots a_{j_m}\, e^{\mathrm{W}(a)}\, d\mu_{S(a)}.$$

In this paper we introduce an operator R on $\mathbf{A}(a_1, \cdots, a_n)$ such that

$$\mathcal{S}(f) = \int (\mathbb{1} - \mathrm{R})^{-1}(f)\, d\mu_S$$

holds for all f in $\mathbf{A}^0(a_1, \cdots, a_n)$. The utility of this representation of the Schwinger functional is demonstrated in Sect. III, where an elementary, but archetypical, bound on the correlation functions $\mathcal{S}_{m(j_1,\cdots,j_r)}$, $m \geq 1$ is obtained by bounding the operator norm of R in terms of a “naive power counting” norm on $\mathrm{W}(a)$. The tools developed here will be used to simplify the rigorous construction of a class of two dimensional Fermi liquids outlined in [FKLT].

The representation of the Schwinger functional derived in this paper grew out of the “integration by parts expansion” of [FMRT, FKLT] which, in turn, was developed as a replacement for the traditional “cluster/Mayer expansion”. In fact, the integration by parts expansion can be obtained by first expanding the inverse $(\mathbb{1} - \mathrm{R})^{-1}$ in a Neumann series and then selectively expanding, by repeated partial integrations, the Grassmann, Gaussian integrals appearing in the definition of R. Apart from its conciseness, the advantage of the representation of the Schwinger functional given in this paper lies in the fact that the Pauli exclusion principle can be implemented quantitatively by a simple application of Gram’s inequality. This is in contrast to the “integration by parts expansion”, where the Pauli exclusion principle is implemented by a more physical, but more complicated, approach that involves carefully counting the number of fields in position space cubes whose dimensions are matched to the decay of the free propagator. There is a large literature devoted to the mathematical structure of Fermionic field theories, for example [C, FMRS, GK, BW].

One more notion is required for the detailed formulation of our results. For this purpose, let $S^\sharp$ be the complex, skew symmetric matrix of order $2n$ given by

$$S^\sharp = \begin{pmatrix} \mathbf{0} & S \\ S & S \end{pmatrix}.$$

Also, for all multi indices I and J in $\mathcal{M}$, let $S_{:\mathrm{I}:\mathrm{J}}$ be the skew symmetric matrix of order $|\mathrm{I}| + |\mathrm{J}|$ defined by

$$S_{:\mathrm{I}:\mathrm{J}} = \begin{pmatrix} \mathbf{0} & S_{\mathrm{I},\mathrm{J}} \\ S_{\mathrm{J},\mathrm{I}} & S_{\mathrm{J}} \end{pmatrix} = S_{\mathrm{IJ}} - \begin{pmatrix} S_{\mathrm{I}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Here, IJ is the juxtaposition of I and J and

$$S_{\mathrm{I},\mathrm{J}} = \left(S_{i_k j_\ell} \right)$$

is the matrix with elements $S_{i_k j_\ell}$, $k = 1, \cdots, r$ and $\ell = 1, \cdots, s$. Now, suppose $a^\sharp_{\mathrm{I}}\, a_{\mathrm{J}}$ is the monomial in the Grassmann algebra $\mathbf{A}(a^\sharp_1, \cdots, a^\sharp_n, a_1, \cdots, a_n)$ corresponding to I and J in $\mathcal{M}$. Then, by construction,

$$\int a^\sharp_{\mathrm{I}}\, a_{\mathrm{J}}\; d\mu_{S^\sharp(\mathbf{a}^\sharp,\mathbf{a})} = \mathrm{Pf}\left(S_{:\mathrm{I}:\mathrm{J}}\right)$$

In other words, the Grassmann, Gaussian integral with covariance $S^\sharp$ excludes “contractions” between any pair of the generators $a_1^\sharp, \cdots, a_n^\sharp$ in $\mathbf{A}(a_1^\sharp, \cdots, a_n^\sharp, a_1, \cdots, a_n)$.

For any $f(\mathbf{a})$ in $\mathbf{A}(a_1, \cdots, a_n)$, let $f(\mathbf{a})^\sharp$ be the Grassmann polynomial belonging to $\mathbf{A}(a_1^\sharp, \cdots, a_n^\sharp)$ defined by $f(\mathbf{a})^\sharp = f(\mathbf{a}^\sharp)$. If S is an invertible skew symmetric matrix, then there is a unique linear map, called the *Wick map with respect to* S, from $f(\mathbf{a})$ in the Grassmann algebra $\mathbf{A}(a_1, \cdots, a_n)$ to $: f(\mathbf{a}) :_S$ in $\mathbf{A}(a_1, \cdots, a_n)$ such that

$$\int \, : f(\mathbf{a}) :_S \; g(\mathbf{a}) \; d\mu_S(\mathbf{a}) = \int f(\mathbf{a})^\sharp \; g(\mathbf{a}) \; d\mu_{S^\sharp}(\mathbf{a}^\sharp, \mathbf{a})$$

for all $f(\mathbf{a})$, $g(\mathbf{a})$ in $\mathbf{A}(a_1, \cdots, a_n)$.

The Wick map has a unique extension from $\mathbf{A}(a_1, \cdots, a_n)$ to the Grassmann algebra $\mathbf{A}(a_1, \cdots, a_n, b_1, \cdots, b_n)$ that is left linear over the subalgebra $\mathbf{A}(b_1, \cdots, b_n)$. For example, we can apply the Wick map to the exponential $e^{\Sigma a_i b_i}$ as a Grassmann polynomial in the generators $a_1, \cdots, a_n$ with coefficients in $\mathbf{A}(b_1, \cdots, b_n)$. By definition,

$$\int \, : e^{\Sigma a_i b_i} :_S \; e^{\Sigma a_i b_i^\sharp} \; d\mu_S(\mathbf{a}) = \int \; e^{\Sigma a_i^\sharp b_i} \, e^{\Sigma a_i b_i^\sharp} \; d\mu_{S^\sharp}(\mathbf{a}^\sharp, \mathbf{a}) = e^{-\Sigma b_i S_{ij} b_j^\sharp} \, e^{-\frac{1}{2}\Sigma b_i^\sharp S_{ij} b_j^\sharp}.$$

On the other hand,

$$\int \, \left(e^{\Sigma a_i b_i} \;\; e^{\frac{1}{2}\Sigma b_i S_{ij} b_j}\right) e^{\Sigma a_i b_i^\sharp} \, d\mu_S(\mathbf{a}) = e^{-\Sigma b_i S_{ij} b_j^\sharp} \, e^{-\frac{1}{2}\Sigma b_i^\sharp S_{ij} b_j^\sharp}$$

and consequently,

$$: e^{\Sigma a_i b_i} :_S \; = \; e^{\frac{1}{2}\Sigma b_i S_{ij} b_j} \; e^{\Sigma a_i b_i}.$$

It follows from the last identity that the Wick map $: \cdot :_S$ depends continuously on S and has a unique continuous extension to the vector space of all skew symmetric matrices.

Definition 1.1. *For any Grassmann polynomial*

$$h(a, b) = \sum_{\mathrm{I},\mathrm{J} \in \mathcal{I}} g_{\mathrm{I}\,\mathrm{J}} \; a_{\mathrm{I}} \, b_{\mathrm{J}}$$

in $\mathbf{A}(a_1, \cdots, a_n, b_1, \cdots, b_n)$,

$$: h(a, b) :_{S,a} = \sum_{\mathrm{I},\mathrm{J} \in \mathcal{I}} g_{\mathrm{I}\,\mathrm{J}} \;\; : a_{\mathrm{I}} :_S \;\; b_{\mathrm{J}},$$

$$: h(a, b) :_{S,b} = \sum_{\mathrm{I},\mathrm{J} \in \mathcal{I}} g_{\mathrm{I}\,\mathrm{J}} \; a_{\mathrm{I}} \;\; : b_{\mathrm{J}} :_S \; .$$

In other words, the Wick map with respect to S is applied to $h(a, b)$ *as a Grassmann polynomial in the generators* $a_1, \cdots, a_n$ *with coefficients in* $\mathbf{A}(b_1, \cdots, b_n)$ to obtain $: h(a, b) :_{S,a}$. Similarly, it is applied to $h(a, b)$ *as a Grassmann polynomial in the generators* $b_1, \cdots, b_n$ *with coefficients in* $\mathbf{A}(a_1, \cdots, a_n)$ to obtain $: h(a, b) :_{S,b}$.

Definition 1.2. *The linear map* R *from* $\mathbf{A}(a_1, \cdots, a_n)$ *to itself is (consciously suppressing the dependence on* $\mathrm{W}(a)$ *and* S*) given by*

$$\big(\mathrm{R}(f)\big)(a) = \int \, : e^{\mathrm{W}(a+b)-\mathrm{W}(a)} - 1 :_{S,b} \;\; f(b) \, d\mu_{S}(b).$$

Our first result is

Theorem 1.3. *Suppose* 1 *is not in the spectrum of* R. *Then,*

$$\mathcal{S}(f) = \int (\mathbb{1} - \mathrm{R})^{-1}(f)\, d\mu_S$$

for every f *in* $\mathbf{A}^0(a_1, \cdots, a_n)$.

To exploit the representation of Theorem 1.3 we decompose R by expanding the exponential in $:e^{\mathrm{W}(a+b)-\mathrm{W}(a)} - 1 :_{S,b}$.

Definition 1.4. *For each pair* $\mathbf{r}, \mathbf{s} \in \mathbb{N}^\ell$, $\ell \geq 1$, *and every polynomial* f *in* $\mathbf{A}(a_1, \cdots, a_n)$, *the complex valued kernel* $\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)(\mathrm{K}_1, \cdots, \mathrm{K}_\ell)$ *on* $\mathcal{M}_{r_1 - s_1} \times \cdots \times \mathcal{M}_{r_\ell - s_\ell}$ *is (consciously suppressing the dependence on* $\mathrm{W}(a)$ *and* S*) given by*

$$\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)(\mathrm{K}_1, \cdots, \mathrm{K}_\ell) = \pm(\mathbf{s})\, \frac{1}{\ell!} \sum_{\mathrm{J}_1 \in \mathcal{M}_{s_1}} \cdots \sum_{\mathrm{J}_\ell \in \mathcal{M}_{s_\ell}} \int :\prod_{i=1}^{\ell} \tbinom{r_i}{s_i} \mathrm{w}_{r_i}(\mathrm{J}_i, \mathrm{K}_i)\, b_{\mathrm{J}_i} :_S\ f(b)\ d\mu_S(b)$$

when $r_i \geq s_i \geq 1$, $i = 1, \cdots, \ell$. *Otherwise,* $\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)(\mathrm{K}_1, \cdots, \mathrm{K}_\ell) = 0$. *Here,*

$$\pm(\mathbf{s}) = \prod_{i=1}^{\ell} (-1)^{s_i(s_{i+1} + \cdots + s_\ell)}.$$

The corresponding linear map $\mathrm{R}^{\mathbf{r}\,\mathbf{s}}$ *from* $\mathbf{A}(a_1, \cdots, a_n)$ *to itself is*

$$\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f) = \sum_{\mathrm{K}_1 \in \mathcal{M}_{r_1 - s_1}} \cdots \sum_{\mathrm{K}_\ell \in \mathcal{M}_{r_\ell - s_\ell}} \mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)(\mathrm{K}_1, \cdots, \mathrm{K}_\ell) \prod_{i=1}^{\ell} a_{\mathrm{K}_i}.$$

Theorem 1.5. *For every* f *in* $\mathbf{A}^0(a_1, \cdots, a_n)$,

$$\mathrm{R}(f) = \sum_{\ell \geq 1} \sum_{\mathbf{r}, \mathbf{s} \in \mathbb{N}^\ell} \mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f).$$

Remark 1.6. At the end of the introduction, we use Theorem 1.5 to interpret Definition 1.4 and Theorem 1.3 in terms of Feynman graphs.

We can combine Theorem 1.3 and Theorem 1.5 to obtain analytic control over the Schwinger functional $\mathcal{S}(f)$. To do this, choose a nondecreasing function Φ on $\mathbb{N}$ and a $\Lambda > 0$ such that

$$\left| \int a_{\mathrm{I}}\ :a_{\mathrm{J}}:_S\ d\mu_S(a) \right| \leq \begin{cases} \Phi(|\mathrm{I}|)\, \Lambda^{\frac{1}{2}(|\mathrm{I}| + |\mathrm{J}|)}, & |\mathrm{J}| \leq |\mathrm{I}| \\ 0\,, & |\mathrm{J}| > |\mathrm{I}| \end{cases}$$

for all multi indices I and J. The number Λ is morally the supremum $\|S\|_\infty = \sup_{i,j \in \{1, \cdots, n\}} |S_{ij}|$ of the covariance S. The number $\Phi(|\mathrm{I}|)$ is intuitively the degree of cancellation between the at most $|\mathrm{I}|!$ nonzero terms contributing to the Pfaffian

$$\mathrm{Pf}\begin{pmatrix} S_{\mathrm{I}} & S_{\mathrm{I},\mathrm{J}} \\ S_{\mathrm{J},\mathrm{I}} & \mathbf{0} \end{pmatrix} = \int a_{\mathrm{I}}\ :a_{\mathrm{J}}:_S\ d\mu_S(a).$$

At one extreme (see, Example 3.4), we can always choose $\Phi(|\mathrm{I}|) = |\mathrm{I}|!$ for all multi indices I and $\Lambda = \|S\|_\infty$. At the other extreme (see, Example 3.5), suppose that

$$S = \begin{pmatrix} \mathbf{0} & \Sigma \\ -\Sigma^{\mathrm{T}} & \mathbf{0} \end{pmatrix}$$

for some matrix $\Sigma = \left(\Sigma_{ij}\right)$ of order $\frac{n}{2}$, and further that there is a complex Hilbert space $\mathcal{H}$, elements $v_i, w_i \in \mathcal{H}$, $i = 1, \cdots, \frac{n}{2}$, and a constant $\Lambda > 0$ with

$$\begin{aligned} \langle v_i, w_j \rangle_{\mathcal{H}} &= \Sigma_{ij}, \\ \|v_i\|_{\mathcal{H}}\, , \|w_j\|_{\mathcal{H}} &\le L^{\frac{1}{2}} \end{aligned}$$

for all $i, j = 1, \cdots, \frac{n}{2}$. Then, by a variant of Gram's inequality, we can choose $\Phi(|\mathrm{I}|) = 1$ for all multi indices I and $\Lambda = 2L$.

For example, the Grassmann algebra associated to a many fermion system has an equal number of "annihilation" $a_1, \cdots, a_m$ and "creation" $\bar{a}_1, \cdots, \bar{a}_m$ generators. Furthermore, the physical covariance C cannot pair two annihilation or two creation generators. That is, $C = \begin{pmatrix} \mathbf{0} & C_{i\bar{j}} \\ C_{\bar{i}j} & \mathbf{0} \end{pmatrix}$. It is also often possible to write $C_{i\bar{j}}$ as an inner product between vectors in an appropriate Hilbert space with "naturally" bounded norms so that $\Phi(|\mathrm{I}|) = 1$, $\mathrm{I} \in \mathcal{M}$, can be achieved in models of physical interest. See, [FMRT, p.682].

Now, let

$$f(a) = \sum_{m \ge 0} f^{(m)}(a)$$

be a Grassmann polynomial in $\mathbf{A}(a_1, \cdots, a_n)$ where, for each $m \ge 0$,

$$f^{(m)}(a) = \sum_{j_1, \cdots, j_m} f_m(j_1, \cdots, j_m)\, a_{j_1} \cdots a_{j_m}$$

and the kernel $f_m(j_1, \cdots, j_m)$ is an antisymmetric function of its arguments.

Definition 1.7. *For all $\alpha \ge 2$, the "external" and "internal" naive power counting norms $\|f\|_\alpha$ and $|||f|||_\alpha$ of the Grassmann polynomial $f(a)$ are*

$$\|f\|_\alpha = \sum_{m \ge 0} \|f^{(m)}\|_\alpha = \sum_{m \ge 0} \alpha^m\, \Lambda^{\frac{1}{2}m}\, \|f_m\|_1$$

and

$$|||f|||_\alpha = \sum_{m \ge 0} |||f^{(m)}|||_\alpha = \sum_{m \ge 0} \alpha^m\, \|S\|_{1,\infty}\, \Lambda^{\frac{1}{2}(m-2)}\, \|f_m\|_{1,\infty},$$

where

$$\begin{aligned} \|f_m\|_1 &= \sum_{j_1, \cdots, j_m} |f_m(j_1, \cdots, j_m)|, \\ \|f_m\|_{1,\infty} &= \sup_{j_1 \in \{1, \cdots, n\}} \sum_{j_2, \cdots, j_m} |f_m(j_1, j_2, \cdots, j_m)| \end{aligned}$$

are the L^1 and " mixed $\mathrm{L}^1, \mathrm{L}^\infty$ " norms of the antisymmetric kernels $f_m(j_1, \cdots, j_m)$.

In Sect. 3 we prove

Theorem 1.8. *Suppose* $2\,|||\mathrm{W}|||_{\alpha+1} \le 1$. *Then, for all polynomials f in the Grassmann algebra* $\mathbf{A}(a_1, \cdots, a_n)$,

$$\|\mathrm{R}(f)\|_\alpha \le 2\Phi(n)\ \ |||\mathrm{W}|||_{\alpha+1}\ \ \|f\|_\alpha .$$

In particular, the spectrum of R *is bounded away from* 1 *uniformly in the number of degrees of freedom n when Φ is constant and $|||\mathrm{W}|||_{\alpha+1}$ is small enough.*

A simple consequence of Theorem 1.8 is the archetypical bound on correlation functions

Theorem 1.9. *Suppose* $2\,(1+\Phi(n))\,|||\mathrm{W}|||_{\alpha+1} < 1$. *Then, for each $m \ge 0$ and all sequences of indices $1 \le j_1, \cdots, j_m \le n$,*

$$\left|\mathcal{S}_m(j_1, \cdots, j_m)\right| \le \frac{\Phi(n)}{1 - 2\,\Phi(n)\,|||\mathrm{W}|||_{\alpha+1}}\ \alpha^m\ \Lambda^{\frac{1}{2}m} .$$

In particular, the correlation functions are bounded uniformly in the number of degrees of freedom n when Φ is constant and $\|\mathrm{W}\|_{\alpha+1}$ is small enough.

1.1. Decomposition of Feynman graphs into annuli. In the rest of the introduction we motivate and interpret Definition 1.4 and Theorem 1.3 “graphically”. However, we emphasize that the purely algebraic proof of Theorem 1.3 given in the next section is completely independent of this discussion and, in particular, does not refer to graphs.

Recall that, for $\mathrm{I} = (i_1, \cdots, i_r)$ in $\mathcal{M}_r$ with r even,

$$\int a_{\mathrm{I}}\ d\mu_S(a) = \mathrm{Pf}\left(S_{\mathrm{I}}\right) = \sum_{\substack{k_1, \cdots, k_r = 1 \\ k_{2i-1} < k_{2i} \text{ for } i \le r/2 \\ k_1 < k_3 < \cdots < k_{r-1}}}^{r} \varepsilon^{k_1 \cdots k_r} S_{i_{k_1} i_{k_2}} \cdots S_{i_{k_{r-1}} i_{k_r}}$$

can be thought of as the sum of the amplitudes of all graphs having r vertices labelled $i_1, \cdots, i_r$ and having precisely one line attached to each vertex. The amplitude of the graph having lines $\{i_{k_1}, i_{k_2}\}$, $\cdots$, $\{i_{k_{r-1}}, i_{k_r}\}$ is $\varepsilon^{k_1 \cdots k_r} S_{i_{k_1} i_{k_2}} \cdots S_{i_{k_{r-1}} i_{k_r}}$.

This graphical representation has an immediate extension to

$$\int a_{\mathrm{H}}\ e^{\mathrm{W}(a)}\ \ d\mu_S(a) = \sum_{\ell \ge 0} \sum_{\mathbf{r} \in \mathbb{N}^\ell} \frac{1}{\ell!} \sum_{\mathrm{I}_1 \in \mathcal{M}_{r_1}} \cdots \sum_{\mathrm{I}_\ell \in \mathcal{M}_{r_\ell}} \int a_{\mathrm{H}} \prod_{i=1}^{\ell} \mathrm{w}_{r_i}(\mathrm{I}_i)\, a_{\mathrm{I}_i}\ d\mu_S(a)$$

with $\mathrm{H} \in \mathcal{M}_m$. One merely has to substitute $\mathrm{H}\mathrm{I}_1 \cdots \mathrm{I}_\ell$ for I. In general, a “Feynman graph Γ” with m external legs and ℓ internal vertices $\mathrm{w}_{r_1}, \cdots, \mathrm{w}_{r_\ell}$ (the vertex w_{r_i} having r_i legs) is a partition of the $m + r_1 + \cdots + r_\ell$ legs into disjoint pairs that are represented as lines. The amplitude of the graph Γ is defined to be

$$\mathrm{Am}(\Gamma)(\mathrm{H}) = \frac{\varepsilon}{\ell!} \sum_{\mathrm{I}_1 \in \mathcal{M}_{r_1}} \cdots \sum_{\mathrm{I}_\ell \in \mathcal{M}_{r_\ell}} \prod_{i=1}^{\ell} \mathrm{w}_{r_i}(\mathrm{I}_i) \prod_{\text{lines c in } \Gamma} S_{i_{\mathrm{c}}\, j_{\mathrm{c}}} ,$$

where we choose any ordering $(i_{\mathrm{c}}\,, j_{\mathrm{c}})$ of the pairs in the partition determined by the lines c of Γ and where ε is the signature of the permutation that brings the juxtaposition of these pairs to the juxtaposition $\mathrm{H}\mathrm{I}_1 \cdots \mathrm{I}_\ell$ of the multi indices $\mathrm{H}\,, \mathrm{I}_1\,, \cdots, \mathrm{I}_\ell$. Then, for $\mathrm{H} \in \mathcal{M}_m$,

$$\int a_{\mathrm{H}}\, e^{\mathrm{W}(a)}\ d\mu_{S(a)} = \sum_{\text{Feynman graphs }\Gamma} \mathrm{Am}(\Gamma)(\mathrm{H}).$$

A Feynman graph Γ is "externally connected" when each connected component contains a line with an external leg. In other words, there are no "vacuum components". It is well known (and can also be derived from Theorem 1.3 and Theorem 1.5) that the correlation function $\mathcal{S}(a_{\mathrm{H}}) = \frac{1}{\mathcal{Z}} \int a_{\mathrm{H}}\, e^{\mathrm{W}(a)}\, d\mu_{S(a)}$ is the sum of the amplitudes of all externally connected graphs. Roughly speaking, the representation $\mathcal{S}(f) = \sum_{n=0}^{\infty} \int \mathrm{R}^n(f) d\mu_S$ generates these graphs a bit at a time with the m^{th} application of R adding those lines that are of distance $m-1$ from f and those vertices that are of distance m from f.

To make this more precise, we return to Definition 1.4 and choose $\mathbf{r}, \mathbf{s} \in \mathbb{N}^\ell$ satisfying $r_i \geq s_i \geq 1\,,\ i=1,\cdots,\ell$. We first explain how to visualize the action of the operator $\mathrm{R}^{\mathbf{r}\,\mathbf{s}}$ on a homogeneous Grassmann polynomial

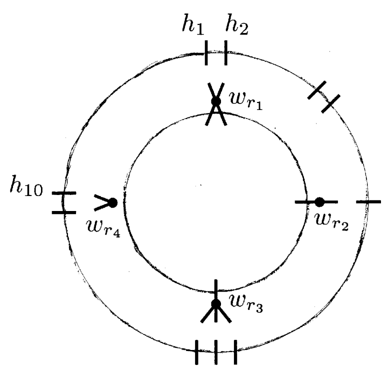
$$f^{(m)}(a) = \sum_{j_1,\cdots,j_m} f_m(j_1,\cdots,j_m)\, a_{j_1} \cdots a_{j_m} = \sum_{\mathrm{H}\in\mathcal{M}_m} f_m(\mathrm{H})\, a_{\mathrm{H}}$$

of degree m.

Suppose, $f(a) = a_{\mathrm{H}}$ is a monomial where, $\mathrm{H} = (h_1,\cdots,h_m)$, and select a multi index $\mathrm{K}_i \in \mathcal{M}_{r_i - s_i}$, $i = 1,\cdots,\ell$. To graphically interpret the coefficient

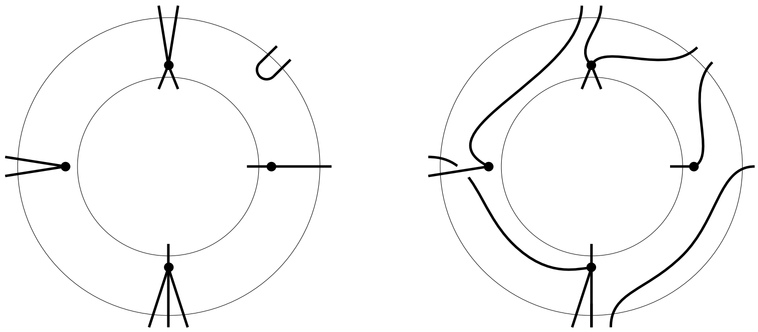
$$\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(a_{\mathrm{H}})(\mathrm{K}_1,\cdots,\mathrm{K}_\ell) = \pm(\mathbf{s})\, \frac{1}{\ell!} \sum_{\mathrm{J}_1\in\mathcal{M}_{s_1}} \cdots \sum_{\mathrm{J}_\ell\in\mathcal{M}_{s_\ell}} \int :\prod_{i=1}^{\ell} \tbinom{r_i}{s_i}\, \mathrm{w}_{r_i}(\mathrm{J}_i,\mathrm{K}_i)\, b_{\mathrm{J}_i} :_S\ b_{\mathrm{H}}\, d\mu_{S(b)}$$

of $a_{\mathrm{K}_1} \cdots a_{\mathrm{K}_\ell}$ in $\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(a_{\mathrm{H}})$, we imagine an annulus that has, for each generator in the incoming monomial b_{H}, an "external leg" entering some point on its outer boundary and that has, for each generator in the outgoing product $\prod_{i=1}^{\ell} a_{\mathrm{K}_i}$ of $\mathrm{R}^{\mathbf{r}\,\mathbf{s}}$ an "external leg" leaving some point on its inner boundary, and that furthermore contains the ℓ "vertices" $w_{r_1},\cdots,w_{r_\ell}$ in its interior. There are no other external legs or vertices. The vertex w_{r_i} , $i = 1,\cdots,\ell$, has r_i "legs". One leg, for each generator in the monomial $\mathrm{w}_{r_i}(\mathrm{J}_i,\mathrm{K}_i)\, b_{\mathrm{J}_i}\, a_{\mathrm{K}_i}$. The leg attached to w_{r_i} for a given generator in a_{K_i} is joined by a "passive" line to the corresponding external leg leaving the inner boundary of the annulus.



$\ell = 4 \quad m = 10 \quad \vec{r} = (4,2,4,2) \quad \vec{s} = (2,1,3,2)$

We construct "annular graphs of type $m, \mathbf{r}, \mathbf{s}$" out of our annulus, when $m \geq s_1 + \cdots + s_\ell$ and $m + s_1 + \cdots + s_\ell$ is even, by connecting each leg of w_{r_i}, $i = 1, \cdots, \ell$, representing a generator in b_{J_i}, $|\mathrm{J}_i| = s_i$, by an "active" line with an external leg representing a generator in b_{H} entering the outer boundary or connecting, again by an "active" line, two external legs entering the outer boundary corresponding to a pair of generators in b_{H}. Each vertex is connected to at least one external leg entering the outer boundary because $s_i \geq 1$, $1 \leq i \leq \ell$. Note that there is a bijection between annular graphs and partitions P of the disjoint union $\mathrm{H} \,\dot\cup \bigcup\limits^{\cdot}_{1 \leq i \leq \ell} \mathrm{J}_i$ into disjoint unordered pairs such that each element of $\bigcup\limits^{\cdot}_{1 \leq i \leq \ell} \mathrm{J}_i$ is paired with an element of H.



Two annular graphs of type 10, $(4, 2, 4, 2)$, $(2, 1, 3, 2)$

For each sequence $(\mathrm{H}, \mathrm{K}_1, \cdots, \mathrm{K}_\ell)$ of multi indices in $\mathcal{M}_m \times \mathcal{M}_{r_1 - s_1} \times \cdots \times \mathcal{M}_{r_\ell - s_\ell}$, the amplitude $\mathrm{Am(A)}(\mathrm{H}, \mathrm{K}_1, \cdots, \mathrm{K}_\ell)$ of an annular graph A of type $m, \mathbf{r}, \mathbf{s}$ is

$$\mathrm{Am(A)}(\mathrm{H}, \mathrm{K}_1, \cdots, \mathrm{K}_\ell) = \pm(\mathbf{s}) \, \frac{1}{\ell!} \, \varepsilon \sum_{\mathrm{J}_1 \in \mathcal{M}_{s_1}} \cdots \sum_{\mathrm{J}_\ell \in \mathcal{M}_{s_\ell}} \prod_{i=1}^{\ell} \binom{r_i}{s_i} \mathrm{w}_{r_i}(\mathrm{J}_i, \mathrm{K}_i) \prod_{\substack{\text{active lines} \\ \text{c in A}}} S_{i_{\mathrm{c}} \, j_{\mathrm{c}}},$$

where we choose any ordering $(i_{\mathrm{c}}, j_{\mathrm{c}})$ of the pairs in the partition P_{A} determined by the active lines c of A and where ε is the signature of the permutation that brings the juxtaposition of these pairs to the juxtaposition $\mathrm{H}\mathrm{J}_1 \cdots \mathrm{J}_\ell$ of the multi indices $\mathrm{H}, \mathrm{J}_1, \cdots, \mathrm{J}_\ell$. The amplitude $\mathrm{Am(A)}(\mathrm{H}, \mathrm{K}_1, \cdots, \mathrm{K}_\ell)$ is a function of the external legs of the annular graph A.

Recall that the Grassmann, Gaussian integral

$$\int : \prod_{i=1}^{\ell} b_{\mathrm{J}_i} :_S \; b_{\mathrm{H}} \, d\mu_{S}(b) = \int \Big(\prod_{i=1}^{\ell} b^{\sharp}_{\mathrm{J}_i} \Big) \, b_{\mathrm{H}} \, d\mu_{S^\sharp}(b^\sharp, b)$$

is equal to a Pfaffian that is the sum over all the partitions of the product $\Big(\prod\limits_{i=1}^{\ell} b^{\sharp}_{\mathrm{J}_i} \Big) \, b_{\mathrm{H}}$ into disjoint pairs such that each generator in $\prod\limits_{i=1}^{\ell} b^{\sharp}_{\mathrm{J}_i}$ contracts, via a matrix element of S, to a generator in b_{H}. Therefore, by construction,

$$\pm(\mathbf{s})\ \frac{1}{\ell!}\sum_{\mathrm{J}_1\in\mathcal{M}_{s_1}}\cdots\sum_{\mathrm{J}_\ell\in\mathcal{M}_{s_\ell}}\int :\prod_{i=1}^{\ell}\tbinom{r_i}{s_i}\,\mathrm{w}_{r_i}(\mathrm{J}_i,\mathrm{K}_i)\,b_{\mathrm{J}_i}\ :_S\ b_{\mathrm{H}}\ d\mu_{S}(b)$$
$$=\sum_{\substack{\text{annular graphs A}\\ \text{of type }|\mathrm{H}|,\mathbf{r},\mathbf{s}}}\mathrm{Am}(\mathrm{A})(\mathrm{H},\mathrm{K}_1,\cdots,\mathrm{K}_\ell).$$

That is,

$$\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})(\mathrm{K}_1,\cdots,\mathrm{K}_\ell)=\sum_{\mathrm{H}\in\mathcal{M}_m}f_m(\mathrm{H})\sum_{\substack{\text{annular graphs A}\\ \text{of type }m,\mathbf{r},\mathbf{s}}}\mathrm{Am}(\mathrm{A})(\mathrm{H},\mathrm{K}_1,\cdots,\mathrm{K}_\ell).$$

Suppose, $|\lambda|\ll 1$. Then, the partition function $\mathcal{Z}(\lambda)=\int e^{\lambda\mathrm{W}(a)}\,d\mu_S\neq 0$ and all the eigenvalues of $\mathrm{R}=O(\lambda)$ lie strictly inside the unit disc. In this case, the Neumann series $(\mathbb{1}-\mathrm{R})^{-1}=\sum_{p\geq 0}\mathrm{R}^p$ converges. Writing out all the terms,

$$\int(\mathbb{1}-\mathrm{R})^{-1}(f^{(m)})\,d\mu_S=\sum_{\substack{p\geq 0,\\ \ell_1,\cdots,\ell_p\geq 1}}\ \sum_{\mathbf{r}_1,\mathbf{s}_1\in\mathbb{N}^{\ell_1}}\cdots\sum_{\mathbf{r}_p,\mathbf{s}_p\in\mathbb{N}^{\ell_p}}\int\mathrm{R}^{\mathbf{r}_p\,\mathbf{s}_p}\cdots\mathrm{R}^{\mathbf{r}_1\,\mathbf{s}_1}(f^{(m)})\,d\mu_S.$$

To convey the intuition that leads to the statement of Theorem 1.3, we examine both the action of a product $\mathrm{R}^{\mathbf{r}_p\,\mathbf{s}_p}\cdots\mathrm{R}^{\mathbf{r}_1\,\mathbf{s}_1}$ contributing to R^p on $f^{(m)}$ and the corresponding Grassmann Gaussian integrals

$$\int\mathrm{R}^{\mathbf{r}_p\,\mathbf{s}_p}\cdots\mathrm{R}^{\mathbf{r}_1\,\mathbf{s}_1}(f^{(m)})\,d\mu_S.$$

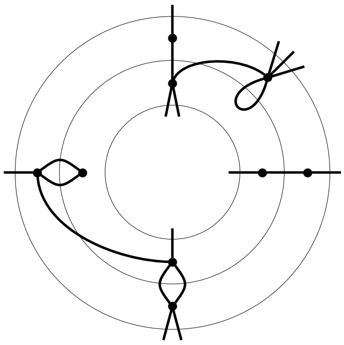
For $p=2$,

$$\mathrm{R}_{\mathbf{r}_2\,\mathbf{s}_2}\Big(\mathrm{R}^{\mathbf{r}_1\,\mathbf{s}_1}(f^{(m)})\Big)(\mathrm{L}_1,\cdots,\mathrm{L}_{\ell_2})$$
$$=\sum_{\mathrm{H}\in\mathcal{M}_{m_1}}f_{m_1}(\mathrm{H})\sum_{\mathrm{K}_1,\cdots,\mathrm{K}_{\ell_1}}\ \sum_{\substack{\text{annular graphs A}_i\\ \text{of type }m_i,\mathbf{r}_i,\mathbf{s}_i\\ \text{for }i=1,2}}\mathrm{Am}(\mathrm{A}_1)(\mathrm{H},\mathrm{K}_1,\cdots,\mathrm{K}_{\ell_1})\ \mathrm{Am}(\mathrm{A}_2)(\mathrm{K}_1\cdots\mathrm{K}_{\ell_1},\mathrm{L}_1,\cdots,\mathrm{L}_{\ell_2}).$$

The degree $m_1=m$ and the degree $m_2=r_{1,1}-s_{1,1}+\cdots+r_{1,\ell_1}-s_{1,\ell_1}$, the second sum is over all sequences of multi indices $(\mathrm{K}_1,\cdots,\mathrm{K}_{\ell_1})$ in $\mathcal{M}_{r_{1,1}-s_{1,1}}\times\cdots\times\mathcal{M}_{r_{1,\ell_1}-s_{1,\ell_1}}$, and the multi index $\mathrm{K}_1\cdots\mathrm{K}_{\ell_1}$ occurring in the amplitude $\mathrm{Am}(\mathrm{A}_2)$ is the juxtaposition of the multi indices $\mathrm{K}_1,\cdots,\mathrm{K}_{\ell_1}$. Now,

$$\mathrm{Am}(\mathrm{A}_1\mathrm{A}_2)(\mathrm{H},\mathrm{L}_1,\cdots,\mathrm{L}_{\ell_2})=\sum_{\mathrm{K}_1,\cdots,\mathrm{K}_{\ell_1}}\mathrm{Am}(\mathrm{A}_1)(\mathrm{H},\mathrm{K}_1,\cdots,\mathrm{K}_{\ell_1})\ \mathrm{Am}(\mathrm{A}_2)(\mathrm{K}_1\cdots\mathrm{K}_{\ell_1},\mathrm{L}_1,\cdots,\mathrm{L}_{\ell_2})$$

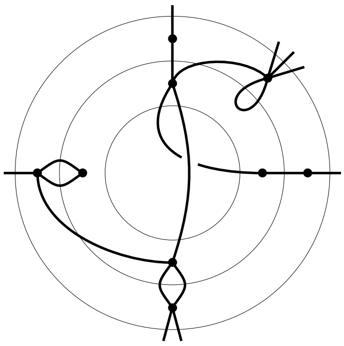
is the "amplitude of the double annular graph $\mathrm{A}_1\mathrm{A}_2$ of type $m,\mathbf{r}_1,\mathbf{s}_1,\mathbf{r}_2,\mathbf{s}_2$" obtained by inserting A_2 just inside the inner boundary of A_1 and then, for each generator in $\prod_{i=1}^{\ell_1}a_{\mathrm{K}_i}$, joining the associated external leg (at the end of a passive line) leaving the inner boundary of A_1 to its mate (at the beginning of an active line) entering the outer boundary of A_2. Notice that, by construction, each vertex in A_2 is connected by a line to at least one vertex in A_1 that, in turn, is connected to at least one external leg entering the outer boundary of A_1.



A double annular graph A_1A_2 with A_1 of type $8, (2,6,2,4,4), (1,3,1,2,1)$ and A_2 of type $10, (4,2,4,2), (2,1,2,2)$

For $p \geq 3$, one obtains the amplitude of the completely analogous p-annular graph $\mathrm{A}_1 \cdots \mathrm{A}_p$ of type $m, \mathbf{r}_1, \mathbf{s}_1, \cdots, \mathbf{r}_p, \mathbf{s}_p$ in which each vertex in the annular graph A_i , $i = 2, \cdots, p$ of type $m_i = \sum_{j=1}^{\ell_{i-1}} r_{i-1,j} - s_{i-1,j}$, $\mathbf{r}_i, \mathbf{s}_i$, is connected by a line to at least one vertex in A_{i-1} and ultimately to at least one external leg entering the outer boundary of A_1.

An "externally connected Feynman graph Γ of type $m, \mathbf{r}_1, \mathbf{s}_1, \cdots, \mathbf{r}_p, \mathbf{s}_p$" is a p-annular graph $\mathrm{A}_1 \cdots \mathrm{A}_p$ of type $m, \mathbf{r}_1, \mathbf{s}_1, \cdots, \mathbf{r}_p, \mathbf{s}_p$, as in the last paragraph, together with a partition P of the legs emanating from the inner boundary of A_p into disjoint pairs that are joined to form lines.



Suppose Γ is an externally connected Feynman graph with m external legs and the internal vertices $\mathrm{w}_{r_1}, \cdots, \mathrm{w}_{r_n}$. Set

$$\mathrm{A}_1(\Gamma) = \left\{ \mathrm{w}_{r_j} \,\middle|\, \text{at least one leg of } \mathrm{w}_{r_j} \text{ is connected by a line in } \Gamma \text{ to an external leg} \right\}$$

and then define $\mathrm{A}_i(\Gamma)$, $i \geq 2$, inductively by

$$\mathrm{A}_i(\Gamma) = \left\{ \mathrm{w}_{r_j} \notin \bigcup_{h=1}^{i-1} \mathrm{A}_h(\Gamma) \,\middle|\, \text{at least one leg of } \mathrm{w}_{r_j} \text{ is connected by a line in } \Gamma \text{ to a vertex in } \mathrm{A}_{i-1} \right\}.$$

Also, set $\ell_i = |\mathrm{A}_i(\Gamma)|$ for $i \geq 1$. There is a $1 \leq p \leq n$ such that $\mathrm{A}_p(\Gamma) \neq \emptyset$, while $\mathrm{A}_{p+1}(\Gamma) = \emptyset$, and

$$\{\mathrm{w}_{r_1}, \cdots, \mathrm{w}_{r_n}\} = \bigcup_{i=1}^{p} \mathrm{A}_i(\Gamma).$$

Let $\mathrm{w}_{r_{j_1}}, \cdots, \mathrm{w}_{r_{j_{\ell_i}}}$ be the vertices in $\mathrm{A}_i(\Gamma)$. For each $k=1,\cdots,\ell_i$, let s_{j_k} be the number of legs of the vertex $\mathrm{w}_{r_{j_k}}$ that are attached by lines to a vertex in $\mathrm{A}_{i-1}(\Gamma)$. Also, let m_i^Γ be the total number of legs emanating from the vertices in $\mathrm{A}_{i-1}(\Gamma)$ that are not attached by lines to vertices in $\mathrm{A}_{i-2}(\Gamma)$. For each $i = 1, \cdots, p$, set $\mathbf{r}_i^\Gamma = (r_{j_1}, \cdots, r_{j_{\ell_i}})$ and $\mathbf{s}_i^\Gamma = (s_{j_1}, \cdots, s_{j_{\ell_i}})$. Observe that the graph Γ induces a unique annular graph structure on $\mathrm{A}_i(\Gamma)$ of type $m_i^\Gamma, \mathbf{r}_i^\Gamma, \mathbf{s}_i^\Gamma$ for each $i = 1, \cdots, p$, and a partition P_Γ of the legs leaving the inner boundary of $\mathrm{A}_p(\Gamma)$ into pairs. Thus, every externally connected Feynman graph Γ with m external legs corresponds to a unique externally connected Feynman graph of type $m, \mathbf{r}_1^\Gamma, \mathbf{s}_1^\Gamma, \cdots, \mathbf{r}_p^\Gamma, \mathbf{s}_p^\Gamma$.

For each multi index H in $\mathcal{M}_m$, the amplitude $\mathrm{Am}(\Gamma)(\mathrm{H})$ of the externally connected Feynman graph Γ of type $m, \mathbf{r}_1, \mathbf{s}_1, \cdots, \mathbf{r}_p, \mathbf{s}_p$ is defined by

$$\mathrm{Am}(\Gamma)(\mathrm{H}) = \varepsilon(\mathrm{P}_\Gamma) \sum_{\mathrm{M}_1, \cdots, \mathrm{M}_{\ell_p}} \mathrm{Am}(\mathrm{A}_1 \cdots \mathrm{A}_p)(\mathrm{H}, \mathrm{M}_1, \cdots, \mathrm{M}_{\ell_p}) \prod_{(i,j)\in \mathrm{P}_\Gamma} S_{i\,j},$$

where $\varepsilon(\mathrm{P}_\Gamma)$ is the signature of the permutation that brings the juxtaposition of the pairs in the partition P_Γ to the juxtaposition $\mathrm{M}_1 \cdots \mathrm{M}_{\ell_p}$. The sum is over all sequences of multi indices $(\mathrm{M}_1, \cdots, \mathrm{M}_{\ell_p})$ in $\mathcal{M}_{r_{p,1}-s_{p,1}} \times \cdots \times \mathcal{M}_{r_{p,\ell_p}-s_{p,\ell_p}}$. By definition, the amplitude $\mathrm{Am}(\Gamma)(\mathrm{H})$ of an externally connected Feynman graph Γ is the amplitude of the corresponding externally connected Feynman graph of type $m, \mathbf{r}_1^\Gamma, \mathbf{s}_1^\Gamma, \cdots, \mathbf{r}_p^\Gamma, \mathbf{s}_p^\Gamma$.

We can now write

$$\begin{aligned}
&\int \mathrm{R}^{\mathbf{r}_2\,\mathbf{s}_2}\mathrm{R}^{\mathbf{r}_1\,\mathbf{s}_1}(f^{(m)})\, d\mu_S \\
&= \sum_{\mathrm{H}\in\mathcal{M}_m} f_m(\mathrm{H}) \int \mathrm{R}^{\mathbf{r}_2\,\mathbf{s}_2}\mathrm{R}^{\mathbf{r}_1\,\mathbf{s}_1}(a_\mathrm{H})\, d\mu_S \\
&= \sum_{\mathrm{H}\in\mathcal{M}_m} f_m(\mathrm{H}) \sum_{\mathrm{L}_1, \cdots, \mathrm{L}_{\ell_2}} \sum_{\substack{\text{annular graphs } \mathrm{A}_i \\ \text{of type } m_i, \mathbf{r}_i, \mathbf{s}_i \\ \text{for } i=1,2}} \mathrm{Am}(\mathrm{A}_1\mathrm{A}_2)(\mathrm{H}, \mathrm{L}_1, \cdots, \mathrm{L}_{\ell_2}) \int \prod_{i=1}^{\ell_2} a_{\mathrm{L}_i}\, d\mu_S \\
&= \sum_{\mathrm{H}\in\mathcal{M}_m} f_m(\mathrm{H}) \sum_{\substack{\text{externally connected} \\ \text{Feynman graphs } \Gamma \text{ of} \\ \text{type } m_1, \mathbf{r}_1, \mathbf{s}_1, m_2, \mathbf{r}_2, \mathbf{s}_2}} \mathrm{Am}(\Gamma)(\mathrm{H})
\end{aligned}$$

since, the integral $\int \prod_{i=1}^{\ell_2} a_{\mathrm{L}_i}\, d\mu_S$ is equal to a Pfaffian that is the sum over all partitions of the product $\prod_{i=1}^{\ell_2} a_{\mathrm{L}_i}$ into disjoint pairs that are contracted via matrix elements of S. Similarly,

$$\int \mathrm{R}^{\mathbf{r}_p\,\mathbf{s}_p} \cdots \mathrm{R}^{\mathbf{r}_1\,\mathbf{s}_1}(f^{(m)})\, d\mu_S = \sum_{\mathrm{H}\in\mathcal{M}_m} f_m(\mathrm{H}) \sum_{\substack{\text{externally connected} \\ \text{Feynman graphs } \Gamma \text{ of} \\ \text{type } m_1, \mathbf{r}_1, \mathbf{s}_1, \cdots, m_p, \mathbf{r}_p, \mathbf{s}_p}} \mathrm{Am}(\Gamma)(\mathrm{H})$$

for all $p \geq 3$. It follows that

$$\int (\mathbb{1} - \mathrm{R})^{-1} (f^{(m)})\, d\mu_S$$
$$= \sum_{\mathrm{H}\in\mathcal{M}_m} f_m(\mathrm{H}) \sum_{\substack{p\geq 0,\\ \ell_1,\cdots,\ell_p\geq 1}} \sum_{\mathbf{r}_1,\mathbf{s}_1\in\mathbb{N}^{\ell_1}} \cdots \sum_{\mathbf{r}_p,\mathbf{s}_p\in\mathbb{N}^{\ell_p}} \sum_{\substack{\text{externally connected}\\ \text{Feynman graphs }\Gamma\text{ of}\\ \text{type }m_1,\mathbf{r}_1,\mathbf{s}_1,\cdots,m_p,\mathbf{r}_p,\mathbf{s}_p}} \mathrm{Am}(\Gamma)(\mathrm{H})$$

or, equivalently,

$$\int (\mathbb{1} - \mathrm{R})^{-1} (f^{(m)})\, d\mu_S = \sum_{\mathrm{H}\in\mathcal{M}_m} f_m(\mathrm{H}) \sum_{\substack{\text{externally connected}\\ \text{Feynman graphs }\Gamma}} \mathrm{Am}(\Gamma)(\mathrm{H}),$$

where the last sum is over all externally connected Feymann graphs with m external legs and any finite set $\mathrm{w}_{r_1}, \cdots, \mathrm{w}_{r_n}$ of vertices chosen from w_r, $r \geq 1$. As mentioned before, it is "a well known fact" that

$$\mathcal{S}(f^{(m)}) = \sum_{\mathrm{H}\in\mathcal{M}_m} f_m(\mathrm{H}) \sum_{\substack{\text{externally connected}\\ \text{Feynman graphs }\Gamma}} \mathrm{Am}(\Gamma)(\mathrm{H}).$$

Therefore,

$$\mathcal{S}(f^{(m)}) = \int (\mathbb{1} - \mathrm{R})^{-1} (f^{(m)}).\ d\mu_S$$

This completes the graphical interpretation of Theorem 1.3.

2. The Proofs of Theorem 1.3 and Theorem 1.5

Again, let $S = \big(S_{ij}\big)$ be a skew symmetric matrix of even order n.

Lemma 2.1. *For each $h(b, c)$ in $\mathbf{A}(b_1, \cdots, b_n, c_1, \cdots, c_n)$,*

$$\int\int h(b, a{+}b^\sharp)\ d\mu_{S^\sharp}(b^\sharp,b)\, d\mu_S(a) = \int h(b, c)\ d\mu_{\left(\begin{smallmatrix} S & S\\ S & S\end{smallmatrix}\right)}(b,c).$$

Proof. Let Γ be the linear functional on the Grassmann algebra $\mathbf{A}(b_1, \cdots, b_n, c_1, \cdots, c_n)$ given by

$$\Gamma(h) = \int\int h(b, a{+}b^\sharp)\ d\mu_{S^\sharp}(b^\sharp,b)\, d\mu_S(a),$$

Then,

$$\begin{aligned}
\Gamma\big(e^{\Sigma\, b_i d_i + c_i e_i}\big) &= \int\int e^{\Sigma\, b_i d_i + (a+b^\sharp)_i e_i}\ d\mu_{S^\sharp}(b^\sharp,b)\, d\mu_S(a)\\
&= \int e^{\Sigma\, a_i e_i} \int e^{\Sigma\, b_i^\sharp e_i + b_i d_i}\ d\mu_{S^\sharp}(b^\sharp,b)\, d\mu_S(a)\\
&= e^{-\Sigma\, e_i S_{ij} d_j}\ e^{-\frac12 \Sigma\, d_i S_{ij} d_j} \int e^{\Sigma\, a_i e_i}\ d\mu_S(a)\\
&= e^{-\frac12 \Sigma\, e_i S_{ij} e_j}\ e^{-\Sigma\, e_i S_{ij} d_j}\ e^{-\frac12 \Sigma\, d_i S_{ij} d_j}.
\end{aligned}$$

By uniqueness,

$$\Gamma(h) = \int h(b, c)\ d\mu_{\left(\begin{smallmatrix} S & S\\ S & S\end{smallmatrix}\right)}(b,c).$$

□

The main ingredient required for the proof of Theorem 1.3 is

Proposition 2.2. *For all f and g in $\mathbf{A}(a_1,\cdots,a_n)$,*

$$\int\int f(b)\ :g(a+b):_{S,b}\ d\mu_S(b)\,d\mu_S(a)=\int f(a)\,g(a)\ d\mu_S(a).$$

Proof. By Lemma 2.1,

$$\int\int f(b)\ :g(a+b):_{S,b}\ d\mu_S(b)\,d\mu_S(a)=\int\int f(b)\,g(a+b^\sharp)\ d\mu_{S^\sharp}(b^\sharp,b)\,d\mu_S(a)$$
$$=\int f(b)\,g(c)\ d\mu_{\left(\begin{smallmatrix}S&S\\S&S\end{smallmatrix}\right)}(b,c).$$

Now, observe that for all multi indices $\mathrm{I}=(i_1,\cdots,i_r)$ and $\mathrm{J}=(j_1,\cdots,j_s)$ in $\mathcal{M}$, the juxtaposition

$$\mathrm{I}(\mathrm{J}+n)=(i_1,\cdots,i_r\,,j_1+n,\cdots,j_s+n)$$

is a multi index in $\{1,\cdots,2n\}^{r+s}$ and by construction,

$$\int b_{\mathrm{I}}\,c_{\mathrm{J}}\ d\mu_{\left(\begin{smallmatrix}S&S\\S&S\end{smallmatrix}\right)}(b,c)=\mathrm{Pf}\Big(\left(\begin{smallmatrix}S&S\\S&S\end{smallmatrix}\right)_{\mathrm{I}(\mathrm{J}+n)}\Big)=\mathrm{Pf}\big(S_{\mathrm{IJ}}\big)=\int a_{\mathrm{I}}\,a_{\mathrm{J}}\ d\mu_S(a).$$

It follows that

$$\int\int f(b)\ :g(a+b):_{S,b}\ d\mu_S(b)\,d\mu_S(a)=\int f(a)\,g(a)\ d\mu_S(a)$$

□

Again, let

$$\mathrm{W}(a)=\sum_{r\ge0}\ \sum_{j_1,\cdots,j_r}\mathrm{w}_r(j_1,\cdots,j_r)\ a_{j_1}\cdots a_{j_r}$$

be an even Grassmann polynomial where, $\mathrm{w}_r(j_1,\cdots,j_r)$, is an antisymmetric function of its arguments $1\le j_1,\cdots,j_r\le n$ that vanishes identically when r is odd. Let R be the linear map on $\mathbf{A}(a_1,\cdots,a_n)$, corresponding to $\mathrm{W}(a)$ and S, that was introduced in Definition I.2.

Theorem 2.3. *For all f in $\mathbf{A}(a_1,\cdots,a_n)$,*

$$\int f(a)\,e^{\mathrm{W}(a)}\ d\mu_S(a)=\int(\mathrm{R}_0+\mathrm{R})(f)(a)\ e^{\mathrm{W}(a)}\ d\mu_S(a)$$

where,

$$\mathrm{R}_0(f)(a)=\int f(b)\ d\mu_S(b)\ a_\emptyset.$$

Proof. By Proposition 2.2,

$$\int(\mathrm{R}_0+\mathrm{R})(f)(a)\ e^{\mathrm{W}(a)}\ d\mu_S(a)=\int\int\ :e^{\mathrm{W}(a+b)-\mathrm{W}(a)}:_{S,b}\ f(b)\ d\mu_S(b)\ e^{\mathrm{W}(a)}\ d\mu_S(a)$$
$$=\int\int\ :e^{\mathrm{W}(a+b)}:_{S,b}\ f(b)\ d\mu_S(b)\ d\mu_S(a)$$
$$=\int f(a)\,e^{\mathrm{W}(a)}\ d\mu_S(a),$$

□

It is now easy to give the

Proof of Theorem 1.3. If $|\lambda| \le r \ll 1$, then

$$\mathcal{Z}(\lambda) = \int e^{\lambda \mathrm{W}(a)}\, d\mu_S \neq 0,$$

and all the eigenvalues of $\mathrm{R} = O(\lambda)$ lie strictly inside the unit disc. In this case,

$$\frac{1}{\mathcal{Z}} \int a_{\mathrm{H}}\, e^{\lambda \mathrm{W}(a)}\, d\mu_S = \int a_{\mathrm{H}}\, d\mu_S + \frac{1}{\mathcal{Z}} \int \mathrm{R}(a_{\mathrm{H}})\, e^{\lambda \mathrm{W}(a)}\, d\mu_S$$

and

$$\sum_{s \ge 0} \mathrm{R}^s(a_{\mathrm{H}}) = (\mathbb{1} - \mathrm{R})^{-1}\, a_{\mathrm{H}}.$$

Iterating,

$$\begin{aligned} \frac{1}{\mathcal{Z}} \int a_{\mathrm{H}}\, e^{\lambda \mathrm{W}(a)}\, d\mu_S &= \int a_{\mathrm{H}}\, d\mu_S + \int \mathrm{R}(a_{\mathrm{H}})\, d\mu_S + \frac{1}{\mathcal{Z}} \int \mathrm{R}^2(a_{\mathrm{H}})\, e^{\lambda \mathrm{W}(a)}\, d\mu_S \\ &= \int \sum_{s=0}^{t} \mathrm{R}^s(a_{\mathrm{H}})\, d\mu_S + \frac{1}{\mathcal{Z}} \int \mathrm{R}^{t+1}(a_{\mathrm{H}})\, e^{\lambda \mathrm{W}(a)}\, d\mu_S \end{aligned}$$

for all $t \ge 0$. In the limit,

$$\frac{1}{\mathcal{Z}} \int a_{\mathrm{H}}\, e^{\lambda \mathrm{W}(a)}\, d\mu_S = \int (\mathbb{1} - \mathrm{R})^{-1}\, a_{\mathrm{H}}\, d\mu_S$$

when $|\lambda| \le r \ll 1$. To complete the proof, observe that both sides of the last identity are rational functions of $\lambda \in \mathbb{C}$. $\square$

To prove Theorem I.5 we make

Convention 2.4. *Let* $\mathrm{I} = (i_1, \cdots, i_r)$ *be any multi index. A "sub multi index"* $\mathrm{J} \subset \mathrm{I}$ *is a multi index* $\mathrm{J} = (j_1, \cdots, j_s)$ *together with a strictly increasing map* ν_{J} *from* $\{1, \cdots, s\}$ *to* $\{1, \cdots, r\}$ *such that* $j_k = i_{\nu_{\mathrm{J}}(k)}$, $k = 1, \cdots, s$. *If the multi indices* I *and* J *belong to* $\mathcal{I}$ *and, in addition,* $\mathrm{J} \subset \mathrm{I}$ *as sets, then* J *is uniquely determined as a sub multi index by the inclusion map of* $\{j_1, \cdots, j_s\}$ *into* $\{i_1, \cdots, i_r\}$. *For every sub multi index* $\mathrm{J} \subset \mathrm{I}$, *there is a unique complementary sub multi index* $\mathrm{I} \setminus \mathrm{J} \subset \mathrm{I}$ *such that the image of* $\nu_{\mathrm{I} \setminus \mathrm{J}}$ *is the complement of* $\nu_{\mathrm{J}}(\{1, \cdots, s\})$ *in* $\{1, \cdots, r\}$. *The "relative sign"* $\rho(\mathrm{J}, \mathrm{I})$ *of the pair* $\mathrm{J} \subset \mathrm{I}$ *is the signature of the permutation that brings the sequence* $(1, \cdots, r-s, r-s+1, \cdots, r)$ *to* $(\nu_{\mathrm{J}}(1), \cdots, \nu_{\mathrm{J}}(s), \nu_{\mathrm{I} \setminus \mathrm{J}}(1), \cdots, \nu_{\mathrm{I} \setminus \mathrm{J}}(r-s))$. *By construction,* $a_{\mathrm{I}} = \rho(\mathrm{J}, \mathrm{I})\, a_{\mathrm{J}}\, a_{\mathrm{I} \setminus \mathrm{J}}$. *The relative sign is defined on all of* $\mathcal{I} \times \mathcal{I}$ *by*

$$\rho(\mathrm{J}, \mathrm{I}) = \begin{cases} \rho(\mathrm{J}, \mathrm{I}), & \mathrm{J} \subset \mathrm{I} \\ 0, & \mathrm{J} \not\subset \mathrm{I}. \end{cases}$$

Proof of Theorem 1.5. Observe that for each $r \ge 1$,

$$\begin{aligned} \sum_{\mathrm{I} \in \mathcal{M}_r} \mathrm{w}_r(\mathrm{I}) \big((a+b)_{\mathrm{I}} - a_{\mathrm{I}} \big) &= \sum_{\mathrm{I} \in \mathcal{M}_r} \mathrm{w}_r(\mathrm{I}) \sum_{1 \le s \le r} \sum_{\substack{\mathrm{J}\ \text{a subindex} \\ \text{of I in } \mathcal{M}_s}} \rho(\mathrm{J}, \mathrm{I})\, b_{\mathrm{J}}\, a_{\mathrm{I} \setminus \mathrm{J}} \\ &= \sum_{1 \le s \le r} \sum_{\mathrm{I} \in \mathcal{M}_r} \sum_{\substack{\mathrm{J}\ \text{a subindex} \\ \text{of I in } \mathcal{M}_s}} \rho(\mathrm{J}, \mathrm{I})\, \rho(\mathrm{J}, \mathrm{I})\, \mathrm{w}_r(\mathrm{J}, \mathrm{I} \setminus \mathrm{J})\, b_{\mathrm{J}}\, a_{\mathrm{I} \setminus \mathrm{J}} \\ &= \sum_{1 \le s \le r} \sum_{\mathrm{J} \in \mathcal{M}_s} \sum_{\mathrm{K} \in \mathcal{M}_{r-s}} \binom{r}{s}\, \mathrm{w}_r(\mathrm{J}, \mathrm{K})\, b_{\mathrm{J}}\, a_{\mathrm{K}} \end{aligned}$$

and consequently, for each $\mathbf{r} \in \mathbb{N}^\ell$,

$$\begin{aligned}
&\prod_{i=1}^{\ell} \Big(\sum_{\mathrm{I}_i \in \mathcal{M}_{r_i}} \mathrm{w}_{r_i}(\mathrm{I}_i)\big((a+b)_{\mathrm{I}_i} - a_{\mathrm{I}_i}\big)\Big) \\
&= \prod_{i=1}^{\ell} \Big(\sum_{1\le s_i \le r_i} \sum_{\mathrm{J}_i \in \mathcal{M}_{s_i}} \sum_{\mathrm{K}_i \in \mathcal{M}_{r_i - s_i}} \tbinom{r_i}{s_i} \mathrm{w}_{r_i}(\mathrm{J}_i, \mathrm{K}_i)\, b_{\mathrm{J}_i}\, a_{\mathrm{K}_i}\Big) \\
&= \sum_{\substack{\mathbf{s}\in\mathbb{N}^\ell \\ \mathbf{r}\ge\mathbf{s}\ge\mathbf{1}}} \sum_{\substack{\mathrm{J}_1\in\mathcal{M}_{s_1} \\ \mathrm{K}_1 \in \mathcal{M}_{r_1-s_1}}} \cdots \sum_{\substack{\mathrm{J}_\ell\in\mathcal{M}_{s_\ell} \\ \mathrm{K}_\ell \in \mathcal{M}_{r_\ell-s_\ell}}} \Big(\prod_{i=1}^{\ell} \tbinom{r_i}{s_i} \mathrm{w}_{r_i}(\mathrm{J}_i, \mathrm{K}_i)\, b_{\mathrm{J}_i}\, a_{\mathrm{K}_i}\Big) \\
&= \sum_{\substack{\mathbf{s}\in\mathbb{N}^\ell \\ \mathbf{r}\ge\mathbf{s}\ge\mathbf{1}}} \sum_{\substack{\mathrm{J}_1\in\mathcal{M}_{s_1} \\ \mathrm{K}_1 \in \mathcal{M}_{r_1-s_1}}} \cdots \sum_{\substack{\mathrm{J}_\ell\in\mathcal{M}_{s_\ell} \\ \mathrm{K}_\ell \in \mathcal{M}_{r_\ell-s_\ell}}} \pm(\mathbf{s}) \prod_{i=1}^{\ell} \tbinom{r_i}{s_i} \mathrm{w}_{r_i}(\mathrm{J}_i, \mathrm{K}_i)\, b_{\mathrm{J}_i} \prod_{i=1}^{\ell} a_{\mathrm{K}_i}.
\end{aligned}$$

Now, we can expand the exponential to obtain

$$\begin{aligned}
&: e^{\mathrm{W}(a+b)-\mathrm{W}(a)} - 1 :_{S,b} \\
&= \sum_{\ell\ge 1} \frac{1}{\ell!} \sum_{\mathbf{r}\in\mathbb{N}^\ell} : \prod_{i=1}^{\ell} \Big(\sum_{\mathrm{I}_i \in \mathcal{M}_{r_i}} \mathrm{w}_{r_i}(\mathrm{I}_i)\big((a+b)_{\mathrm{I}_i} - a_{\mathrm{I}_i}\big)\Big) :_{S,b} \\
&= \sum_{\ell\ge 1} \frac{1}{\ell!} \sum_{\substack{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell \\ \mathbf{r}\ge\mathbf{s}\ge\mathbf{1}}} \sum_{\substack{\mathrm{J}_1\in\mathcal{M}_{s_1} \\ \mathrm{K}_1 \in \mathcal{M}_{r_1-s_1}}} \cdots \sum_{\substack{\mathrm{J}_\ell\in\mathcal{M}_{s_\ell} \\ \mathrm{K}_\ell \in \mathcal{M}_{r_\ell-s_\ell}}} \pm(\mathbf{s}) : \prod_{i=1}^{\ell} \tbinom{r_i}{s_i} \mathrm{w}_{r_i}(\mathrm{J}_i, \mathrm{K}_i)\, b_{\mathrm{J}_i} :_S \prod_{i=1}^{\ell} a_{\mathrm{K}_i} \\
&= \sum_{\ell\ge 1} \sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \sum_{\mathrm{K}_1 \in \mathcal{M}_{r_1-s_1}} \cdots \sum_{\mathrm{K}_\ell \in \mathcal{M}_{r_\ell-s_\ell}} \mathrm{Q}_{\mathbf{r}\,\mathbf{s}}(\mathrm{K}_1,\cdots,\mathrm{K}_\ell,b) \prod_{i=1}^{\ell} a_{\mathrm{K}_i},
\end{aligned}$$

where

$$\mathrm{Q}_{\mathbf{r}\,\mathbf{s}}(\mathrm{K}_1,\cdots,\mathrm{K}_\ell,b) = \begin{cases} \pm(\mathbf{s}) \frac{1}{\ell!} \sum\limits_{\mathrm{J}_1\in\mathcal{M}_{s_1}} \cdots \sum\limits_{\mathrm{J}_\ell\in\mathcal{M}_{s_\ell}} : \prod\limits_{i=1}^{\ell} \binom{r_i}{s_i} \mathrm{w}_{r_i}(\mathrm{J}_i, \mathrm{K}_i)\, b_{\mathrm{J}_i} :_S , & r_i \ge s_i \ge 1,\ i=1,\cdots,\ell \\ 0\,, & \text{otherwise} \end{cases}.$$

Integrating,

$$\begin{aligned}
&\int : e^{\mathrm{W}(a+b)-\mathrm{W}(a)} - 1 :_{S,b}\ f(b)\ d\mu_S(b) \\
&= \sum_{\ell\ge 1} \sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \sum_{\mathrm{K}_1 \in \mathcal{M}_{r_1-s_1}} \cdots \sum_{\mathrm{K}_\ell \in \mathcal{M}_{r_\ell-s_\ell}} \int \mathrm{Q}_{\mathbf{r}\,\mathbf{s}}(\mathrm{K}_1,\cdots,\mathrm{K}_\ell,b)\, f(b)\ d\mu_S(b) \prod_{i=1}^{\ell} a_{\mathrm{K}_i} \\
&= \sum_{\ell\ge 1} \sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \sum_{\mathrm{K}_1 \in \mathcal{M}_{r_1-s_1}} \cdots \sum_{\mathrm{K}_\ell \in \mathcal{M}_{r_\ell-s_\ell}} \mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)(\mathrm{K}_1,\cdots,\mathrm{K}_\ell) \prod_{i=1}^{\ell} a_{\mathrm{K}_i} \\
&= \sum_{\ell\ge 1} \sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f)
\end{aligned}$$

That is,

$$\mathrm{R}(f) = \sum_{\ell \geq 1} \sum_{\mathbf{r},\mathbf{s} \in \mathbb{N}^\ell} \mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f).$$

□

3. An Archetypical Bound and "Naive Power Counting"

Fix a complex, skew symmetric matrix $S = \left(S_{ij}\right)$ of order n and an even Grassmann polynomial

$$\mathrm{W}(a) = \sum_{r \geq 0} \sum_{j_1,\cdots,j_r} \mathrm{w}_r(j_1,\cdots,j_r)\, a_{j_1} \cdots a_{j_r},$$

where $\mathrm{w}_r(j_1,\cdots,j_r)$ is an antisymmetric function of its arguments $1 \leq j_1,\cdots,j_r \leq n$ that vanishes identically when r is odd. In this section we introduce a family of norms on $\mathbf{A}(a_1,\cdots,a_n)$ and then derive an archetypical bound on

$$\mathrm{R}(f) = \sum_{\ell \geq 1} \sum_{\mathbf{r},\mathbf{s} \in \mathbb{N}^\ell} \mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f)$$

for every f in $\mathbf{A}(a_1,\cdots,a_n)$. Recall that

$$\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f) = \sum_{\mathrm{K}_1 \in \mathcal{M}_{t_1}} \cdots \sum_{\mathrm{K}_\ell \in \mathcal{M}_{t_\ell}} \mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)(\mathrm{K}_1,\cdots,\mathrm{K}_\ell) \prod_{i=1}^{\ell} a_{\mathrm{K}_i}$$

for all $\mathbf{r},\mathbf{s} \in \mathbb{N}^\ell$ with the convention $\mathbf{t} = \mathbf{r} - \mathbf{s}$, where

$$\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)(\mathrm{K}_1,\cdots,\mathrm{K}_\ell) = \pm \frac{1}{\ell!} \sum_{\mathrm{J}_1 \in \mathcal{M}_{s_1}} \cdots \sum_{\mathrm{J}_\ell \in \mathcal{M}_{s_\ell}} \int :\prod_{i=1}^{\ell} \tbinom{r_i}{s_i} \mathrm{w}_{r_i}(\mathrm{J}_i,\mathrm{K}_i)\, b_{\mathrm{J}_i} :_S \; f(b)\, d\mu_S$$

when $r_i \geq s_i \geq 1$, $i = 1,\cdots,\ell$, and $\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)(\mathrm{K}_1,\cdots,\mathrm{K}_\ell) = 0$ otherwise. The sign $\pm = \pm(\mathbf{s})$ is given by $\pm(\mathbf{s}) = \prod_{i=1}^{\ell} (-1)^{s_i(s_{i+1}+\cdots+s_\ell)}$.

A first prerequisite for introducing an appropriate family of norms on $\mathbf{A}(a_1,\cdots,a_n)$ is to define the " L^1 norm" $\|\mathrm{u}\|_1$ and the " mixed $\mathrm{L}^1, \mathrm{L}^\infty$ norm" $\|\mathrm{u}\|_{1,\infty}$ of a function $\mathrm{u}(j_1,\cdots,j_r)$ on $\{1,\cdots,n\}^r$ by

$$\|\mathrm{u}\|_1 = \sum_{j_1,\cdots,j_r} |\mathrm{u}(j_1,\cdots,j_r)|$$

and

$$\|\mathrm{u}\|_{1,\infty} = \sup_{i=1,\cdots,r} \sup_{j_i \in \{1,\cdots,n\}} \sum_{j_1,\cdots,j_{i-1}} \sum_{j_{i+1},\cdots,j_r} |\mathrm{u}(j_1,\cdots,j_{i-1},j_i,j_{i+1},\cdots,j_r)|.$$

If $\mathrm{u}(j_1,\cdots,j_r)$ is an antisymmetric function of its arguments, then

$$\|\mathrm{u}\|_{1,\infty} = \sup_{j_1 \in \{1,\cdots,n\}} \sum_{j_2,\cdots,j_r} |\mathrm{u}(j_1,j_2,\cdots,j_r)|.$$

For example,

$$\|S\|_{1,\infty} = \sup_{i \in \{1,\cdots,n\}} \sum_{j} |S_{ij}|.$$

Remark 3.1. Let $\mathrm{u}(j_1,\cdots,j_r)$ be a function on $\{1,\cdots,n\}^r$ and set

$$\mathrm{Alt}\,\mathrm{u}(j_1,\cdots,j_r) = \frac{1}{r!}\sum_{\pi\in S_r} \mathrm{sgn}(\pi)\ \pi\cdot\mathrm{u}(j_1,\cdots,j_r),$$

where $\pi\cdot\mathrm{u}(j_1,\cdots,j_r) = \mathrm{u}(j_{\pi(1)},\cdots,j_{\pi(r)})$. Observe that $\|\pi\cdot\mathrm{u}\|_1 = \|\mathrm{u}\|_1$ for all $\pi\in S_r$ and consequently,

$$\|\mathrm{Alt}\,\mathrm{u}\|_1 \le \frac{1}{r!}\sum_{\pi\in S_r} \|\pi\cdot\mathrm{u}\|_1 = \|\mathrm{u}\|_1.$$

That is, $\|\mathrm{Alt}\,\mathrm{u}\|_1 \le \|\mathrm{u}\|_1$. Similarly, $\|\mathrm{Alt}\,\mathrm{u}\|_{1,\infty} \le \|\mathrm{u}\|_{1,\infty}$.

Proposition 3.2 (Tree Bound). *Let* $f(h_1,\cdots,h_m)$ *and* $\mathrm{u}_i(j,\mathrm{K}_i) = \mathrm{u}_i(j,k_{i1},\cdots,k_{it_i})$ $i=1,\cdots,\ell$*, be antisymmetric functions of their arguments with* $m\ge\ell$*. Let*

$$\mathrm{T}(\mathrm{K}_1,\cdots,\mathrm{K}_\ell) = \sum_{h_1,\cdots,h_m} |f(h_1,\cdots,h_m)| \ \prod_{i=1}^{\ell}\Big(\sum_{j=1}^{n} |S_{h_ij}|\ |\mathrm{u}_i(j,\mathrm{K}_i)|\Big).$$

Then,

$$\|\mathrm{T}\|_1 \le \|f\|_1 \ \prod_{i=1}^{\ell} \|S\|_{1,\infty}\ \|\mathrm{u}_i\|_{1,\infty}.$$

Proof. We have

$$\begin{aligned}
\|\mathrm{T}\|_1 &= \sum_{\mathrm{K}_1,\cdots,\mathrm{K}_\ell} |\mathrm{T}(\mathrm{K}_1,\cdots,\mathrm{K}_\ell)| \\
&= \sum_{\mathrm{K}_1,\cdots,\mathrm{K}_\ell}\ \sum_{h_1,\cdots,h_m} |f(h_1,\cdots,h_m)| \ \prod_{i=1}^{\ell}\Big(\sum_{j=1}^{n} |S_{h_ij}|\ |\mathrm{u}_i(j,\mathrm{K}_i)|\Big) \\
&\le \sum_{h_1,\cdots,h_m} |f(h_1,\cdots,h_m)| \ \prod_{i=1}^{\ell}\Big(\sum_{j=1}^{n} |S_{h_ij}|\Big)\ \prod_{i=1}^{\ell} \|\mathrm{u}_i\|_{1,\infty} \\
&\le \sum_{h_1,\cdots,h_m} |f(h_1,\cdots,h_m)| \ \prod_{i=1}^{\ell} \|S\|_{1,\infty}\ \prod_{i=1}^{\ell} \|\mathrm{u}_i\|_{1,\infty} \\
&= \|f\|_1 \ \prod_{i=1}^{\ell} \|S\|_{1,\infty}\ \|\mathrm{u}_i\|_{1,\infty}.
\end{aligned}$$

□

A second prerequisite for introducing a family of norms on $\mathbf{A}(a_1,\cdots,a_n)$ that "correctly measures" the size of $\mathrm{R}(f)$ is to choose a nondecreasing function Φ on $\mathbb{N}$ and a $\Lambda>0$ satisfying the

Hypothesis 3.3. *For all multi indices* I *and* J*,*

$$\Big|\int a_{\mathrm{I}}\ :a_{\mathrm{J}}:_S\ d\mu_S\Big| \le \begin{cases} \Phi(|\mathrm{I}|)\ \Lambda^{\frac12(|\mathrm{I}|+|\mathrm{J}|)}, & |\mathrm{J}|\le|\mathrm{I}| \\ 0, & |\mathrm{J}|>|\mathrm{I}|. \end{cases}$$

The form of this hypothesis is motivated by two examples.

Example 3.4 (Global Factorial). For any complex, skew symmetric matrix $S = \big(S_{ij}\big)$, the bound

$$\Big| \int a_{\mathrm{I}} \ :a_{\mathrm{J}}:_S \ \ d\mu_S \Big| \le \begin{cases} |\mathrm{I}|! \ \|S\|_\infty^{\frac12(|\mathrm{I}|+|\mathrm{J}|)}, & |\mathrm{J}| \le |\mathrm{I}| \\ 0, & |\mathrm{J}| > |\mathrm{I}| \end{cases}$$

holds for all multi indices I and J. The proof of this crude inequality is by induction on $|\mathrm{J}|$. Suppose $|\mathrm{J}| = 0$. If $\mathrm{I} = \{i_1,\cdots,i_r\}$, then

$$\int a_{\mathrm{I}} \ d\mu_S = \begin{cases} \mathrm{Pf}\Big(S_{i_k i_\ell}\Big), & r \text{ is even} \\ 0, & r \text{ is odd,} \end{cases}$$

where $\mathrm{Pf}\Big(S_{i_k i_\ell}\Big)$ is the Pfaffian of the matrix with elements $S_{i_k i_\ell}$, $k, \ell = 1, \cdots, r$. We have

$$\begin{aligned} \Big| \int a_{\mathrm{I}} \ d\mu_S \Big| &\le \sum_{k_1,\cdots,k_r=1}^{r} |\varepsilon^{k_1\cdots k_r}| \, |S_{i_{k_1} i_{k_2}}| \cdots |S_{i_{k_{r-1}} i_{k_r}}| \\ &\le \|S\|_\infty^{\frac12 r} \sum_{k_1,\cdots,k_r=1}^{r} |\varepsilon^{k_1\cdots k_r}| \\ &= \|S\|_\infty^{\frac12 r} \ r!. \end{aligned}$$

Suppose $|\mathrm{J}| > 0$. Integration by parts with respect to $a^\sharp_{j_1}$ gives

$$\begin{aligned} \int a_{\mathrm{I}} \ :a_{\mathrm{J}}:_S \ \ d\mu_S &= \int a_{\mathrm{I}}\, a^\sharp_{\mathrm{J}} \ d\mu_{S^\sharp} \\ &= (-1)^{|\mathrm{I}|} \int a^\sharp_{j_1} a_{\mathrm{I}}\, a^\sharp_{\mathrm{J}\setminus\{j_1\}} \ d\mu_{S^\sharp} \\ &= (-1)^{|\mathrm{I}|} \sum_{\ell=1}^{|\mathrm{I}|} (-1)^{\ell-1} S_{j_1 i_\ell} \int a_{\mathrm{I}\setminus\{i_\ell\}}\, a^\sharp_{\mathrm{J}\setminus\{j_1\}} \ d\mu_{S^\sharp} \\ &= (-1)^{|\mathrm{I}|} \sum_{\ell=1}^{|\mathrm{I}|} (-1)^{\ell-1} S_{j_1 i_\ell} \int a_{\mathrm{I}\setminus\{i_\ell\}} \ :a_{\mathrm{J}\setminus\{j_1\}}:_S \ \ d\mu_S. \end{aligned}$$

Our induction hypothesis implies that

$$\Big| \int a_{\mathrm{I}\setminus\{i_\ell\}} \ :a_{\mathrm{J}\setminus\{j_1\}}:_S \ \ d\mu_S \Big| \le \|S\|_\infty^{\frac12(|\mathrm{I}|+|\mathrm{J}|-2)} \, (|\mathrm{I}|-1)!$$

for each $\ell = 1, \cdots, |\mathrm{I}|$. Now,

$$\begin{aligned} \Big| \int a_{\mathrm{I}} \ :a_{\mathrm{J}}:_S \ \ d\mu_S \Big| &\le \sum_{\ell=1}^{|\mathrm{I}|} |S_{j_1 i_\ell}| \, \Big| \int a_{\mathrm{I}\setminus\{i_\ell\}} \ :a_{\mathrm{J}\setminus\{j_1\}}:_S \ \ d\mu_S \Big| \\ &\le \|S\|_\infty^{\frac12(|\mathrm{I}|+|\mathrm{J}|-2)} \, (|\mathrm{I}|-1)! \sum_{\ell=1}^{|\mathrm{I}|} |S_{j_1 i_\ell}| \\ &\le \|S\|_\infty^{\frac12(|\mathrm{I}|+|\mathrm{J}|)} \, (|\mathrm{I}|-1)! \ \big|\mathrm{I}\big|. \end{aligned}$$

.

This "perturbative bound" is obtained by ignoring all potential cancellations between the at most $|\mathrm{I}|!$ nonzero terms appearing in Pfaffian equal to $\int a_{\mathrm{I}}\, a^{\sharp}_{\mathrm{J}}\ d\mu_{S^\sharp}$.

Example 3.5 (Gram's Inequality). Suppose that $S = \big(S_{ij}\big)$ is a complex, skew symmetric matrix of the form

$$S = \begin{pmatrix} \mathbf{0} & \Sigma \\ -\Sigma^t & \mathbf{0} \end{pmatrix},$$

where $\Sigma = \big(\Sigma_{ij}\big)$ is a matrix of order $\frac{n}{2}$. Suppose, in addition, that there is a complex Hilbert space $\mathcal{H}$, elements $v_i, w_i \in \mathcal{H}$, $i = 1, \cdots, \frac{n}{2}$, and a constant $\Lambda > 0$ with

$$\Sigma_{ij} = \langle v_i, w_j\rangle_{\mathcal{H}}$$

and

$$\|v_i\|_{\mathcal{H}}\,,\ \|w_j\|_{\mathcal{H}} \le \Big(\frac{\Lambda}{2}\Big)^{\frac{1}{2}}$$

for all $i, j = 1, \cdots, \frac{n}{2}$. Then, the "nonperturbative bound"

$$\Big|\int a_{\mathrm{I}}\ :a_{\mathrm{J}}:_S\ \ d\mu_S\Big| \le \begin{cases} \Lambda^{\frac{1}{2}(|\mathrm{I}|+|\mathrm{J}|)}\,, & |\mathrm{J}| \le |\mathrm{I}| \\ 0\,, & |\mathrm{J}| > |\mathrm{I}| \end{cases}$$

holds for all multi indices I and J. The proof is presented in the Appendix.

Now, let

$$f(a) = \sum_{m\ge 0} f^{(m)}(a)$$

be a Grassmann polynomial in $\mathbf{A}(a_1, \cdots, a_n)$ where, for each $m \ge 0$,

$$f^{(m)}(a) = \sum_{j_1,\cdots,j_m} f_m(j_1,\cdots,j_m)\ a_{j_1}\cdots a_{j_m}$$

and the kernel $f_m(j_1,\cdots,j_m)$ is an antisymmetric function of its arguments. Fix a complex, skew symmetric matrix $S = \big(S_{ij}\big)$ of order n satisfying Hypothesis 3.3. We recall

Definition 1.4. *For all $\alpha \ge 2$, the "external" and "internal" naive power counting norms $\|f\|_\alpha$ and $|||f|||_\alpha$ of the Grassmann polynomial $f(a)$ are*

$$\|f\|_\alpha = \sum_{m\ge 0} \|f^{(m)}\|_\alpha = \sum_{m\ge 0} \alpha^m\ \Lambda^{\frac{1}{2}m}\ \|f_m\|_1$$

and

$$|||f|||_\alpha = \sum_{m\ge 0} |||f^{(m)}|||_\alpha = \sum_{m\ge 0} \alpha^m\ \|S\|_{1,\infty}\ \Lambda^{\frac{1}{2}(m-2)}\ \|f_m\|_{1,\infty}.$$

By the triangle inequality,

$$\|\mathrm{R}(f)\|_\alpha \le \sum_{\ell\ge 1}\ \sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \|\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f)\|_\alpha \le \sum_{m\ge 0}\ \sum_{\ell\ge 1}\ \sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \|\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_\alpha$$

and consequently,

$$\|\mathrm{R}(f)\|_\alpha \le \sum_{m\ge 1}\sum_{\ell=1}^{m}\sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \|\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_\alpha$$

since, $\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)}) = 0$ for all $\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell$ when $\ell > m$. Furthermore,

$$\|\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_\alpha = \alpha^{\left(\sum_{i=1}^{\ell}(r_i-s_i)\right)}\,\Lambda^{\frac{1}{2}\sum_{i=1}^{\ell}(r_i-s_i)}\,\|\mathrm{Alt}\,\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_1$$

since,

$$\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)}) = \sum_{j_1,\cdots,j_{\mathrm{M}}} \mathrm{Alt}\,\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})(j_1,\cdots,j_{\mathrm{M}})\; a_{j_1}\cdots a_{j_{\mathrm{M}}}$$

with $\mathrm{M} = (r_1-s_1)+\cdots+(r_\ell-s_\ell)$. Altogether,

$$\|\mathrm{R}(f)\|_\alpha \le \sum_{m\ge 1}\sum_{\ell=1}^{m}\sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \alpha^{\left(\sum_{i=1}^{\ell}(r_i-s_i)\right)}\,\Lambda^{\frac{1}{2}\sum_{i=1}^{\ell}(r_i-s_i)}\,\|\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_1.$$

Proposition 3.2 will now be used to obtain a bound on the norm $\|\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_1$ of the kernel $\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})(\mathrm{K}_1,\cdots,\mathrm{K}_\ell)$.

Lemma 3.6. *Let* $\mathrm{H},\mathrm{J}_1,\cdots,\mathrm{J}_\ell$ *be multi indices with* $|\mathrm{H}| = m \ge \ell$. *Then,*

$$\left|\int :\prod_{i=1}^{\ell} a_{\mathrm{J}_i} :_S \; a_{\mathrm{H}}\, d\mu_S\right| \le \mathrm{M}(\mathrm{H},\mathrm{J}_1,\cdots,\mathrm{J}_\ell) \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}} \prod_{i=1}^{\ell} |S_{h_{\mu_i} j_{i1}}|,$$

where

$$\mathrm{M}(\mathrm{H},\mathrm{J}_1,\cdots,\mathrm{J}_\ell) = \sup_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}} \left|\int a_{\mathrm{H}\setminus\{h_{\mu_1},\cdots,h_{\mu_\ell}\}} \;:\prod_{i=1}^{\ell} a_{\mathrm{J}_i\setminus\{j_{i1}\}} :_S \; d\mu_S\right|.$$

Proof. For convenience, set $k_i = j_{i1}$, $i = 1,\cdots,\ell$. By antisymmetry, the integrand can be rewritten so that

$$\int a_{\mathrm{H}} \;:\prod_{i=1}^{\ell} a_{\mathrm{J}_i} :_S \; d\mu_S = \pm \int a^\sharp_{k_\ell}\cdots a^\sharp_{k_1}\; a_{\mathrm{H}} \prod_{i=1}^{\ell} a^\sharp_{\mathrm{J}_i\setminus\{k_i\}}\, d\mu_{S^\sharp}.$$

Now, integrate by parts successively with respect to $a^\sharp_{k_\ell},\cdots,a^\sharp_{k_1}$, and then apply Leibniz's rule to obtain

$$\begin{aligned}\int a_{\mathrm{H}} \;:\prod_{i=1}^{\ell} a_{\mathrm{J}_i} :_S \; d\mu_S &= \pm \int \left[\prod_{i=1}^{\ell}\left(\sum_{m=1}^{n} S_{k_i m}\frac{\partial}{\partial a_m}\right) a_{\mathrm{H}}\right] \prod_{i=1}^{\ell} a^\sharp_{\mathrm{J}_i\setminus\{k_i\}}\, d\mu_{S^\sharp}\\ &= \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}} \pm \prod_{i=1}^{\ell} S_{k_i h_{\mu_i}} \int a_{\mathrm{H}\setminus\{h_{\mu_1},\cdots,h_{\mu_\ell}\}} \prod_{i=1}^{\ell} a^\sharp_{\mathrm{J}_i\setminus\{k_i\}}\, d\mu_{S^\sharp}\end{aligned}$$

since

$$\prod_{i=1}^{\ell}\Big(\sum_{m=1}^{n} S_{k_i m}\,\frac{\partial}{\partial a_m}\Big)\,a_{\rm H} = \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}} \Big(\prod_{i=1}^{\ell} S_{k_i h_{\mu_i}}\,\frac{\partial}{\partial a_{h_{\mu_i}}}\Big)\,a_{\rm H}$$
$$= \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}} \pm\prod_{i=1}^{\ell} S_{k_i h_{\mu_i}}\ \ a_{{\rm H}\setminus\{h_{\mu_1},\cdots,h_{\mu_\ell}\}}.$$

It follows immediately that

$$\Big|\int a_{\rm H}\ :\prod_{i=1}^{\ell} a_{{\rm J}_i}:_S\ d\mu_S\Big|$$
$$\le \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}}\ \prod_{i=1}^{\ell}|S_{h_{\mu_i}k_i}|\ \Big|\int a_{{\rm H}\setminus\{h_{\mu_1},\cdots,h_{\mu_\ell}\}}\ :\prod_{i=1}^{\ell} a_{{\rm J}_i\setminus\{k_i\}}:_S\ d\mu_S\Big|$$
$$\le {\rm M}({\rm H},{\rm J}_1,\cdots,{\rm J}_\ell) \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}}\ \prod_{i=1}^{\ell}|S_{h_{\mu_i}k_i}|.$$

□

Proposition 3.7. *Let*

$$f^{(m)}(a) = \sum_{h_1,\cdots,h_m} f_m(h_1,\cdots,h_m)\ a_{h_1}\cdots a_{h_m}$$

be a homogeneous Grassmann polynomial of degree m, where $f_m(h_1,\cdots,h_m)$ is an anti-symmetric function of its arguments. Let ${\rm J}_1,\cdots,{\rm J}_\ell$ be multi indices with $m\ge\ell$. Then,

$$\Big|\int :\prod_{i=1}^{\ell} b_{{\rm J}_i}:_S\ \ f^{(m)}(b)\ d\mu_S\Big|$$
$$\le \ell!\binom{m}{\ell}\,{\rm M}(m,{\rm J}_1,\cdots,{\rm J}_\ell)\sum_{h_1,\cdots,h_m}|f_m(h_1,\cdots,h_m)|\ \prod_{i=1}^{\ell}|S_{h_i j_{i1}}|,$$

where ${\rm M}(m,{\rm J}_1,\cdots,{\rm J}_\ell) = \sup_{|{\rm H}|=m}\ {\rm M}({\rm H},{\rm J}_1,\cdots,{\rm J}_\ell)$.

Proof. For convenience, set $k_i = j_{i1}$, $i = 1,\cdots,\ell$. By the preceding lemma,

$$\Big|\int :\prod_{i=1}^{\ell} b_{{\rm J}_i}:_S\ \ f^{(m)}(b)\ d\mu_S\Big| \le \sum_{|{\rm H}|=m}|f_m({\rm H})|\ \ {\rm M}({\rm H},{\rm J}_1,\cdots,{\rm J}_\ell) \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}}\ \prod_{i=1}^{\ell}|S_{h_{\mu_i}k_i}|$$
$$\le {\rm M}(m,{\rm J}_1,\cdots,{\rm J}_\ell)\sum_{|{\rm H}|=m}|f_m({\rm H})| \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}}\ \prod_{i=1}^{\ell}|S_{h_{\mu_i}k_i}|.$$

Observe that, by the antisymmetry of f_m,

$$\begin{aligned}
&\sum_{|\mathrm{H}|=m} |f_m(\mathrm{H})| \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}} \prod_{i=1}^{\ell} |S_{h_{\mu_i}k_i}| \\
&= \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}} \sum_{h_1,\cdots,h_m} |f_m(h_1,\cdots,h_m)| \prod_{i=1}^{\ell} |S_{h_{\mu_i}k_i}| \\
&= \sum_{\substack{1\le\mu_1,\cdots,\mu_\ell\le m\\ \text{pairwise}\\ \text{different}}} \sum_{h_1,\cdots,h_m} |f_m(h_1,\cdots,h_m)| \prod_{i=1}^{\ell} |S_{h_ik_i}| \\
&= \ell! \binom{m}{\ell} \sum_{h_1,\cdots,h_m} |f_m(h_1,\cdots,h_m)| \prod_{i=1}^{\ell} |S_{h_ik_i}|.
\end{aligned}$$

□

Proposition 3.8. *Let*

$$f^{(m)}(a) = \sum_{h_1,\cdots,h_m} f_m(h_1,\cdots,h_m)\, a_{h_1}\cdots a_{h_m}$$

be as above. Let $\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell$ *with* $m\ge\ell$*. Then, the* L^1 *norm* $\|\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_1$ *of the kernel* $\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})(\mathrm{K}_1,\cdots,\mathrm{K}_\ell)$ *is bounded by*

(a)

$$\|\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_1 \le \binom{m}{\ell} \mathrm{M}(m,\mathbf{s})\, \|f_m\|_1 \Big(\prod_{i=1}^{\ell} \tbinom{r_i}{s_i}\, \|S\|_{1,\infty}\, \|\mathrm{w}_{r_i}\|_{1,\infty}\Big),$$

where $\mathrm{M}(m,\mathbf{s}) = \sup_{\substack{|\mathrm{J}_i|=s_i\\ i=1,\cdots,\ell}} \mathrm{M}(m,\mathrm{J}_1,\cdots,\mathrm{J}_\ell)$.

(b)

$$\begin{aligned}
&\|\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_1 \\
&\le \Phi(m-\ell) \binom{m}{\ell} \Lambda^{\frac12\left(m-\sum_{i=1}^{\ell}(r_i-s_i)\right)} \|f_m\|_1 \Big(\prod_{i=1}^{\ell} \tbinom{r_i}{s_i}\, \|S\|_{1,\infty}\, \Lambda^{\frac12(r_i-2)}\, \|\mathrm{w}_{r_i}\|_{1,\infty}\Big)
\end{aligned}$$

when, in addition, Hypothesis 3.3 is satisfied.

Proof. To verify (a), set

$$\mathrm{u}_i(j,\mathrm{K}_i) = \sum_{\substack{|\mathrm{J}'_i|=s_i-1\\ i=1,\cdots,\ell}} \big|\mathrm{w}_{r_i}(j,\mathrm{J}'_i,\mathrm{K}_i)\big|.$$

for each $i=1,\cdots,\ell$. By construction, $\|\mathrm{u}_i\|_{1,\infty} = \|\mathrm{w}_{r_i}\|_{1,\infty}$, $i=1,\cdots,\ell$. Also, set

$$\mathrm{T}(\mathrm{K}_1,\cdots,\mathrm{K}_\ell) = \sum_{h_1,\cdots,h_m} |f_m(h_1,\cdots,h_m)| \prod_{i=1}^{\ell} \Big(\sum_{j=1}^{n} |S_{h_ij}|\, |\mathrm{u}_i(j,\mathrm{K}_i)|\Big)$$

By Proposition 3.7,

$$\left| \mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)_{(\mathrm{K}_1,\cdots,\mathrm{K}_\ell)} \right| \le \binom{m}{\ell} \mathrm{M}(m,\mathbf{s}) \left(\prod_{i=1}^{\ell} \tbinom{r_i}{s_i} \right) \ \mathrm{T}_{(\mathrm{K}_1,\cdots,\mathrm{K}_\ell)}$$

since,

$$\begin{aligned}
&\sum_{\substack{|\mathrm{J}_i|=s_i \\ i=1,\cdots,\ell}} \mathrm{M}(m,\mathrm{J}_1,\cdots,\mathrm{J}_\ell) \sum_{h_1,\cdots,h_m} |f_m(h_1,\cdots,h_m)| \ \prod_{i=1}^{\ell} |S_{h_i j_{i1}}| \ |\mathrm{w}_{r_i}(\mathrm{J}_i,\mathrm{K}_i)| \\
&\le \mathrm{M}(m,\mathbf{s}) \sum_{\substack{|\mathrm{J}'_i|=s_i-1 \\ i=1,\cdots,\ell}} \ \sum_{h_1,\cdots,h_m} |f_m(h_1,\cdots,h_m)| \ \prod_{i=1}^{\ell} \Big(\sum_{j=1}^{n} |S_{h_i j}| \ |\mathrm{w}_{r_i}(j,\mathrm{J}'_i,\mathrm{K}_i)| \Big) \\
&= \mathrm{M}(m,\mathbf{s}) \sum_{h_1,\cdots,h_m} |f_m(h_1,\cdots,h_m)| \ \prod_{i=1}^{\ell} \Big(\sum_{j=1}^{n} |S_{h_i j}| \ |\mathrm{u}_i(j,\mathrm{K}_i)| \Big).
\end{aligned}$$

It follows from Proposition 3.2 that

$$\begin{aligned}
\|\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f)\|_1 &\le \binom{m}{\ell} \mathrm{M}(m,\mathbf{s}) \left(\prod_{i=1}^{\ell} \tbinom{r_i}{s_i} \right) \ \|\mathrm{T}\|_1 \\
&\le \binom{m}{\ell} \mathrm{M}(m,\mathbf{s}) \left(\prod_{i=1}^{\ell} \tbinom{r_i}{s_i} \right) \ \|f_m\|_1 \ \prod_{i=1}^{\ell} \|S\|_{1,\infty} \, \|\mathrm{w}_{r_i}\|_{1,\infty}
\end{aligned}$$

For (b), simply observe that, by Hypothesis 3.3,

$$\left| \int a_{\mathrm{H}\setminus\{h_{\mu_1},\cdots,h_{\mu_\ell}\}} \ : \prod_{i=1}^{\ell} a_{\mathrm{J}_i\setminus\{j_{i1}\}} :_S \ d\mu_S \right| \le \Phi(m-\ell) \ \Lambda^{\frac{1}{2}\left(m+\sum_{i=1}^{\ell}(|\mathrm{J}_i|-2)\right)}$$

for any multi indices $\mathrm{H}, \mathrm{J}_1, \cdots, \mathrm{J}_\ell$ with $|\mathrm{H}| = m \ge \ell$ and any pairwise different sequence of indices $1 \le \mu_1, \cdots, \mu_\ell \le m$. Consequently,

$$\mathrm{M}(m,\mathbf{s}) = \sup_{\substack{|\mathrm{J}_i|=s_i \\ i=1,\cdots,\ell}} \ \sup_{|\mathrm{H}_i|=m} \ \mathrm{M}(\mathrm{H},\mathrm{J}_1,\cdots,\mathrm{J}_\ell) \le \Phi(m-\ell) \ \Lambda^{\frac{1}{2}\left(m+\sum_{i=1}^{\ell}(s_i-2)\right)}.$$

□

We have developed all the material required for a useful bound on the operator R. For the rest of this section we assume

Lemma 3.9. *Let*

$$f^{(m)}(a) = \sum_{h_1,\cdots,h_m} f_m(h_1,\cdots,h_m) \, a_{h_1} \cdots a_{h_m}$$

be as above and let $m \ge \ell$*. Then, for all* $\alpha \ge 2$*,*

$$\sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \|\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_\alpha \le \Phi(m-\ell) \ \|f^{(m)}\|_\alpha \ |||\mathrm{W}|||_{\alpha+1}^{\ell}.$$

Proof. By Proposition 3.8 (b),

$$\begin{aligned}&\|\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_\alpha\\ &\le \alpha^{\left(\sum_{i=1}^{\ell}(r_i-s_i)\right)}\ \Lambda^{\frac12\sum_{i=1}^{\ell}(r_i-s_i)}\ \|\mathrm{R}_{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_1 \le \Phi(m-\ell)\binom{m}{\ell}\ \Lambda^{\frac12 m}\ \|f_m\|_1\ \mathcal{P}_{\mathbf{r}\,\mathbf{s}}\end{aligned}$$

where, for convenience, $\mathcal{P}_{\mathbf{r}\,\mathbf{s}} = \prod_{i=1}^{\ell}\binom{r_i}{s_i}\ \alpha^{r_i-s_i}\ \|S\|_{1,\infty}\ \Lambda^{\frac12(r_i-2)}\ \|\mathrm{w}_{r_i}\|_{1,\infty}$. However,

$$\binom{m}{\ell}\ \Lambda^{\frac12 m}\ \|f_m\|_1 = \frac{1}{\alpha^m}\binom{m}{\ell}\ \alpha^m\ \ \Lambda^{\frac12 m}\ \|f_m\|_1 \le \|f^{(m)}\|_\alpha$$

when $\alpha \ge 2$, and consequently,

$$\sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell}\|\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_\alpha \le \Phi(m-\ell)\ \|f^{(m)}\|_\alpha \sum_{\substack{r_i\ge s_i\ge 1\\ i=1,\cdots,\ell}}\mathcal{P}_{\mathbf{r}\,\mathbf{s}}.$$

Observe that

$$\sum_{\substack{r_i\ge s_i\ge 1\\ i=1,\cdots,\ell}}\mathcal{P}_{\mathbf{r}\,\mathbf{s}} = \left(\sum_{r\ge s\ge 1}\binom{r}{s}\ \alpha^{r-s}\ \|S\|_{1,\infty}\ \Lambda^{\frac12(r-2)}\ \|\mathrm{w}_r\|_{1,\infty}\right)^{\ell} \le |||\mathrm{W}|||_{\alpha+1}^{\ell}$$

since

$$\begin{aligned}\sum_{r\ge s\ge 1}\binom{r}{s}\ \alpha^{r-s}\ \|S\|_{1,\infty}\ \Lambda^{\frac12(r-2)}\ \|\mathrm{w}_r\|_{1,\infty} &\le \sum_{r\ge s\ge 0}\binom{r}{s}\ \alpha^{r-s}\ \|S\|_{1,\infty}\ \Lambda^{\frac12(r-2)}\ \|\mathrm{w}_r\|_{1,\infty}\\ &= \sum_{r\ge 0}(\alpha+1)^r\ \|S\|_{1,\infty}\ \Lambda^{\frac12(r-2)}\ \|\mathrm{w}_r\|_{1,\infty}\\ &= |||\mathrm{W}|||_{\alpha+1}.\end{aligned}$$

Therefore,

$$\sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell}\|\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_\alpha \le \Phi(m-\ell)\ \|f^{(m)}\|_\alpha\ \ |||\mathrm{W}|||_{\alpha+1}^{\ell}.$$

□

We can now prove

Theorem 1.8. *Suppose* $2\,|||\mathrm{W}|||_{\alpha+1} \le 1$. *Then, for all polynomials* f *in the Grassmann algebra* $\mathbf{A}(a_1,\cdots,a_n)$,

$$\|\mathrm{R}(f)\|_\alpha \le 2\ \Phi(n)\ \ |||\mathrm{W}|||_{\alpha+1}\ \ \|f\|_\alpha.$$

Proof. By Lemma 3.9,

$$\begin{aligned}
\|\mathrm{R}(f)\|_\alpha &\le \sum_{m\ge 1} \sum_{\ell=1}^{m} \sum_{\mathbf{r},\mathbf{s}\in\mathbb{N}^\ell} \|\mathrm{R}^{\mathbf{r}\,\mathbf{s}}(f^{(m)})\|_\alpha \\
&\le \Phi(n) \sum_{m\ge 1} \|f^{(m)}\|_\alpha \sum_{\ell=1}^{m} |||\mathrm{W}|||_{\alpha+1}^{\ell} \\
&\le \Phi(n)\ \|f\|_\alpha\ \frac{1}{1-|||\mathrm{W}|||_{\alpha+1}}\ |||\mathrm{W}|||_{\alpha+1} \\
&\le 2\ \Phi(n)\ \|f\|_\alpha\ |||\mathrm{W}|||_{\alpha+1}
\end{aligned}$$

□

Corollary 3.10. *Suppose* $2\big(1+\Phi(n)\big)\,|||\mathrm{W}|||_{\alpha+1} < 1$. *Then, for all polynomials* f *in the Grassmann algebra* $\mathbf{A}(a_1,\cdots,a_n)$,

$$\|(\mathbb{1}-\mathrm{R})^{-1}(f)\|_\alpha \le \frac{1}{1-2\,\Phi(n)\,|||\mathrm{W}|||_{\alpha+1}}\ \|f\|_\alpha.$$

Lemma 3.11. *For all Grassmann polynomials* f *in* $\mathbf{A}(a_1,\cdots,a_n)$,

$$\Big|\int f(a)\ d\mu_S\Big| \le \Phi(n)\ \|f\|_\alpha$$

Proof. As usual, write

$$f(a) = \sum_{m\ge 0} f^{(m)}(a),$$

where, for each $m \ge 0$, $f^{(m)}(a) = \sum_{j_1,\cdots,j_m} f_{m(j_1,\cdots,j_m)}\ a_{j_1}\cdots a_{j_m}$ and the kernel $f_{m(j_1,\cdots,j_m)}$ is an antisymmetric function of its arguments. Then, by Hypothesis 3.3,

$$\begin{aligned}
\Big|\int f(a)\ d\mu_S\Big| &\le \sum_{m\ge 0} \Big|\int f^{(m)}(a)\ d\mu_S\Big| \\
&\le \sum_{m\ge 0}\ \sum_{j_1,\cdots,j_m} |f_{m(j_1,\cdots,j_m)}|\ \Big|\int a_{j_1}\cdots a_{j_m}\ d\mu_S\Big| \\
&\le \sum_{m\ge 0}\ \sum_{j_1,\cdots,j_m} |f_{m(j_1,\cdots,j_m)}|\ \Phi(m)\ \Lambda^{\frac{1}{2}m} \\
&\le \sum_{m\ge 0} \|f_m\|_1\ \Phi(m)\ \alpha^m\ \Lambda^{\frac{1}{2}m} \\
&\le \Phi(n)\,\|f\|_\alpha.
\end{aligned}$$

□

Recall that the correlation functions $\mathcal{S}_{m(j_1,\cdots,j_m)}$, $m \ge 0$, corresponding to the interaction $\mathrm{W}(a)$ and the propagator $S = \big(S_{ij}\big)$ are given by

$$\mathcal{S}_{m(j_1,\cdots,j_m)} = \frac{1}{\mathcal{Z}} \int a_{j_1}\cdots a_{j_m}\ e^{\mathrm{W}(a)}\ d\mu_{S(a)}.$$

Theorem 1.9. *Suppose* $2\big(1+\Phi(n)\big)\,|||\mathrm{W}|||_{\alpha+1} < 1$. *Then, for each* $m \geq 0$ *and all sequences of indices* $1 \leq j_1, \cdots, j_m \leq n$,

$$|\mathcal{S}_m(j_1,\cdots,j_m)| \leq \frac{\Phi(n)}{1 - 2\,\Phi(n)\,|||\mathrm{W}|||_{\alpha+1}}\ \alpha^m\ \Lambda^{\frac{1}{2}m}.$$

Proof. Fix $1 \leq j_1, \cdots, j_m \leq n$ and rewrite the monomial $a_{\mathrm{J}} = a_{j_1} \cdots a_{j_m}$ as

$$a_{\mathrm{J}} = \sum_{k_1,\cdots,k_m} \mathrm{Alt}(\delta_{k_1,j_1} \cdots \delta_{k_m,j_m})(k_1,\cdots,k_m)\ a_{k_1} \cdots a_{k_m}.$$

Then,

$$\|a_{\mathrm{J}}\|_\alpha = \alpha^m\ \Lambda^{\frac{1}{2}m}\ \|\mathrm{Alt}(\delta_{\cdot,j_1} \cdots \delta_{\cdot,j_m})\|_1 \leq \alpha^m\ \Lambda^{\frac{1}{2}m}.$$

By Theorem I.3, Lemma 3.11 and Corollary 3.10,

$$\begin{aligned}
|\mathcal{S}_m(j_1,\cdots,j_m)| &= \Big|\int (\mathbb{1} - \mathrm{R})^{-1}\,(a_{\mathrm{J}})\ d\mu_S\Big| \\
&\leq \Phi(n)\ \|(\mathbb{1} - \mathrm{R})^{-1}\,(a_{\mathrm{J}})\|_\alpha \\
&\leq \frac{\Phi(n)}{1 - 2\,\Phi(n)\,|||\mathrm{W}|||_{\alpha+1}}\ \|a_{\mathrm{J}}\|_\alpha,
\end{aligned}$$

so that

$$|\mathcal{S}_m(j_1,\cdots,j_m)| \leq \frac{\Phi(n)}{1 - 2\,\Phi(n)\,|||\mathrm{W}|||_{\alpha+1}}\ \alpha^m\ \Lambda^{\frac{1}{2}m}.$$

□

Appendix: Gram's Inequality for Pfaffians

Proposition. *Suppose that* $S = \big(S_{ij}\big)$ *is a complex, skew symmetric matrix of the form*

$$S = \begin{pmatrix} \mathbf{0} & \Sigma \\ -\Sigma^t & \mathbf{0} \end{pmatrix},$$

where $\Sigma = \big(\Sigma_{ij}\big)$ *is a matrix of order* $\frac{n}{2}$. *Suppose, in addition, that there is a complex Hilbert space* $\mathcal{H}$, *elements* $v_i, w_i \in \mathcal{H}$, $i = 1, \cdots, \frac{n}{2}$, *and a constant* $\Lambda > 0$ *with*

$$\Sigma_{ij} = \langle v_i, w_j \rangle_{\mathcal{H}}$$

and

$$\|v_i\|_{\mathcal{H}}\,,\ \|w_j\|_{\mathcal{H}} \leq \Big(\frac{\Lambda}{2}\Big)^{\frac{1}{2}}$$

for all $i, j = 1, \cdots, \frac{n}{2}$. *Then, for all multi indices* I *and* J,

$$\Big|\int a_{\mathrm{I}}\ d\mu_S\Big| \leq \Big(\frac{\Lambda}{2}\Big)^{\frac{1}{2}|\mathrm{I}|}$$

and

$$\Big|\int a_{\mathrm{I}}\ :a_{\mathrm{J}}:_S\ \ d\mu_S\Big| \leq \begin{cases} \Lambda^{\frac{1}{2}(|\mathrm{I}|+|\mathrm{J}|)}\,, & |\mathrm{J}| \leq |\mathrm{I}| \\ 0\,, & |\mathrm{J}| > |\mathrm{I}|. \end{cases}$$

Proof. To prove the first inequality, suppose $i_1<\cdots<i_r$ and observe that

$$\int a_{i_1}\cdots a_{i_r}\ d\mu_S = \mathrm{Pf}\Big(S_{i_k i_\ell}\Big),$$

where $\mathrm{Pf}\Big(S_{i_k i_\ell}\Big)$ is the Pfaffian of the matrix with elements $S_{i_k i_\ell}$, $k,\ell = 1,\cdots,r$, given by

$$S_{i_k i_\ell} = \begin{cases} 0\ , & 1\le i_k, i_\ell\le \frac{n}{2} \\ \Sigma_{i_k\ i_\ell-\frac{n}{2}}\ , & 1\le i_k\le\frac{n}{2} \text{ and } \frac{n}{2}<i_\ell\le n \\ -\Sigma_{i_\ell\ i_k-\frac{n}{2}}\ , & \frac{n}{2}<i_k\le n \text{ and } 1\le i_\ell\le\frac{n}{2} \\ 0\ , & \frac{n}{2}<i_k\le n \text{ and } \frac{n}{2}<i_\ell\le n. \end{cases}$$

More concisely,

$$\int a_{i_1}\cdots a_{i_r}\ d\mu_S = \mathrm{Pf}\begin{pmatrix} \mathbf{0} & U \\ -U^t & \mathbf{0}\end{pmatrix},$$

where $U = \big(U_{k\ell}\big)$ is the $\rho = \max\big\{k\big| i_k\le\frac{n}{2}\big\}$ by $r-\rho$ matrix with elements

$$U_{k\ell} = \Sigma_{i_k\ i_{\ell+\rho}-\frac{n}{2}} = \big\langle v_{i_k}\ ,\ w_{i_{\ell+\rho}-\frac{n}{2}}\big\rangle_{\mathcal{H}}\ .$$

By direct inspection,

$$\mathrm{Pf}\begin{pmatrix} \mathbf{0} & U \\ -U^t & \mathbf{0}\end{pmatrix} = \begin{cases} 0\ , & \rho\ne r-\rho \\ (-1)^{\frac{1}{2}\rho(\rho-1)}\ \det(U)\ , & \rho=r-\rho. \end{cases}$$

If $r = 2\rho$, then by Gram's inequality for determinants

$$\begin{aligned} &\Big|\int a_{i_1}\cdots a_{i_r}\ d\mu_S\Big| \\ &= \Big|\det\Big(\big\langle v_{i_k}\ ,\ w_{i_{\ell+\rho}-\frac{n}{2}}\big\rangle_{\mathcal{H}}\Big)\Big| \le \prod_{k=1}^{\rho}\|v_{i_k}\|_{\mathcal{H}}\ \|w_{i_{k+\rho}-\frac{n}{2}}\|_{\mathcal{H}} \le \big(\frac{\Lambda}{2}\big)^{\rho}. \end{aligned}$$

Finally, by antisymmetry,

$$\Big|\int a_{\mathrm{I}}\ d\mu_S\Big| \le \big(\frac{\Lambda}{2}\big)^{\frac{1}{2}|\mathrm{I}|}$$

for any multi index I.

To prove the second inequality, set

$$\Sigma^\sharp = \begin{pmatrix} \mathbf{0} & \Sigma \\ \Sigma & \Sigma\end{pmatrix}.$$

The matrix

$$S^\sharp = \begin{pmatrix} \mathbf{0} & S \\ S & S\end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \Sigma \\ \mathbf{0} & \mathbf{0} & -\Sigma^t & \mathbf{0} \\ \mathbf{0} & \Sigma & \mathbf{0} & \Sigma \\ -\Sigma^t & \mathbf{0} & -\Sigma^t & \mathbf{0}\end{pmatrix}$$

is conjugated by the permutation matrix

$$\begin{pmatrix} \mathbb{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1} & \mathbf{0} \\ \mathbf{0} & \mathbb{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{1}\end{pmatrix}$$

to

$$\begin{pmatrix} \mathbf{0} & \Sigma^\sharp \\ -\Sigma^{\sharp\, t} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \Sigma \\ \mathbf{0} & \mathbf{0} & \Sigma & \Sigma \\ \mathbf{0} & -\Sigma^t & \mathbf{0} & \mathbf{0} \\ -\Sigma^t & -\Sigma^t & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Also, define the vectors $v_i^\sharp, w_i^\sharp$, $i = 1, \cdots, n$, in the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ by

$$v_i^\sharp = \begin{cases} (0, v_i) & 1 \le i \le \frac{n}{2} \\ (v_i, v_i)\ , & \frac{n}{2} < i \le n. \end{cases},$$

$$w_i^\sharp = \begin{cases} (w_i, 0)\ , & 1 \le i \le \frac{n}{2} \\ (0, w_i)\ , & \frac{n}{2} < i \le n \end{cases}.$$

Then,

$$\Sigma_{ij}^\sharp = \left\langle v_i^\sharp\, ,\, w_j^\sharp \right\rangle_{\mathcal{H}\oplus\mathcal{H}}$$

and

$$\|v_i^\sharp\|_{\mathcal{H}\oplus\mathcal{H}}\, ,\ \|w_j^\sharp\|_{\mathcal{H}\oplus\mathcal{H}} \le \Lambda^{\frac{1}{2}}$$

for all $i, j = 1, \cdots, n$. The second inequality has now been reduced to the first for the matrix

$$\begin{pmatrix} \mathbf{0} & \Sigma^\sharp \\ -\Sigma^{\sharp\, t} & \mathbf{0} \end{pmatrix},$$

the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ and the vectors $v_i^\sharp, w_i^\sharp$, $i = 1, \cdots, n$. $\square$

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