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Quantum Symmetry Groups of Finite Spaces

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Dedicated to Marc A. Rieffel on the occasion of his sixtieth birthday

Abstract: We determine the quantum automorphism groups of finite spaces. These are compact matrix quantum groups in the sense of Woronowicz.

1. Introduction

At Les Houches Summer School on Quantum Symmetries in 1995, Alain Connes posed the following problem: *What is the quantum automorphism group of a space?* Here the notion of a space is taken in the sense of noncommutative geometry [4], hence it can be either commutative or noncommutative.

To put this problem in a proper context, let us recall that the notion of a group arises most naturally as symmetries of various kinds of spaces. As a matter of fact, this is how the notion of a group was discovered historically. However, the notion of a quantum group was discovered from several different points of view [10, 11, 8, 28, 29, 30, 31, 9], the most important of which is to view quantum groups as deformations of ordinary Lie groups or Lie algebras, instead of viewing them as quantum symmetry objects of noncommutative spaces. In [13], an important first step was made by Manin in this latter direction, where quantum groups are described as quantum symmetry objects of quadratic algebras.

In this paper, we solve the problem above for finite spaces (viz. finite dimensional C^*-algebras). That is, we explicitly determine the quantum automorphism groups of such spaces. These spaces do not carry the additional geometric (Riemannian) structures in the sense of [4, 5]. The quantum automorphism groups for the latter geometric finite spaces can be termed quantum isometry groups. At the end of his book [4], Connes poses the problem of finding a finite quantum symmetry group for the finite geometric space used in his formulation of the Standard Model in particle physics. This problem is clearly related to the problem above he posed at Les Houches Summer School. We expect that the results in our paper will be useful for this problem. As a matter of fact, the quantum symmetry group for the finite geometric space of [4] should be a quantum subgroup of an appropriate quantum automorphism group described in this paper. The main difficulty is to find the *natural* quantum finite subgroup of the latter that deserves to be called the quantum isometry group.

This paper can be viewed as a continuation of the work of Manin [13] in the sense that the quantum groups we consider here are also quantum symmetry objects. However, it differs from the work of Manin in three main aspects. First, the noncommutative spaces on which Manin considers symmetries are quadratic algebras and are infinite; while the spaces on which we consider symmetries are not quadratic and are finite. Second, Manin's quantum groups are generated by infinitely many multiplicative matrices and admits many actions on the spaces in question, one action for each multiplicative matrix (for the notion of multiplicative matrices, see Manin [13]); while our quantum groups are generated by a single multiplicative matrix and they act on the spaces in question in *one* natural manner. Finally, Manin's quantum groups do not give rise to natural structures of C^*-algebras in general (see [18]); while our quantum groups, besides having a purely algebraic formulation, are compact matrix quantum groups in the sense of Woronowicz [30]. Consequently we need to invoke some basic results of Woronowicz [30]. Loosely speaking, Manin's quantum groups are noncompact quantum groups. But to the best knowledge of the author, it is not known as to how one can make this precise in the strict sense of Woronowicz [32]. On the other hand, it is natural to expect that quantum automorphism groups of finite spaces are compact quantum groups without knowing their explicit descriptions in this paper.

The ideas in our earlier papers [19, 20, 18] on universal quantum groups play an important role in this paper. Note that finite spaces are just finite dimensional C^*-algebras, no deformation is involved. Moreover, as in [19, 20, 18], the quantum groups considered in this paper are *intrinsic* objects, not as deformations of groups, so they are different from the quantum groups obtained by the traditional method of deformations of Lie groups (cf. [8, 9, 29, 31, 12, 16, 23]).

We summarize the contents of this paper. In Sect. 2, we recall some basic notions concerning actions of quantum groups and define the notion of a quantum automorphism group of a space. The most natural way to define a quantum automorphism group is by categorical method, viz, to define it as a universal object in a certain category of quantum transformation groups. Sections. 3, 4, 5 are devoted to explicit determination of quantum automorphism groups for several categories of quantum transformation groups of the spaces X_n, $M_n(\mathbb{C})$, and $\oplus_{k=1}^m M_{n_k}(\mathbb{C})$, respectively. Though the main idea in the construction of quantum automorphism groups is the same for each of the spaces X_n, $M_n(\mathbb{C})$ and $\oplus_{k=1}^m M_{n_k}(\mathbb{C})$, the two special cases X_n and M_n offers interesting phenomena in their own right. Hence we deal with them separately and begin by considering the simplest case X_n. In Sect. 6, using the results of Sects. 3, 4, 5, we prove that a finite space has a quantum automorphism group in the category of *all* compact quantum transformation groups if and only if the finite space is X_n, and that a measured finite space (i.e. a finite space endowed with a positive functional) always has a quantum automorphism group.

A convention on terminology: In the following, we will use interchangeably both the term compact quantum groups and the term Woronowicz Hopf C^*-algebras. The meaning should be clear from the context (cf. [19, 20, 23, 18]).

Notation. For every natural number n, and every *-algebra A, $M_n(A)$ denotes the *-algebra of $n \times n$ matrix with entries in A. We also use M_n to denote $M_n(\mathbb{C})$, where $\mathbb{C}$ is the algebra of complex numbers. For every matrix $u = (a_{ij}) \in M_n(A)$, u^t denotes the transpose of u; $\overline{u} = (a_{ij}^*)$ denotes the conjugate matrix of u; $u^* = \overline{u}^t$ denotes the adjoint matrix of u (this defines the ordinary *-operation on $M_n(A)$). The symbol $X(A)$ denotes the set of all unital *-homomorphism from A to $\mathbb{C}$. Finally, $X_n = \{x_1, \cdots, x_n\}$ is the finite space with n letters.

2. The Notion of Quantum Automorphism Groups

Part of the problem of Connes mentioned in the introduction is to make precise the notion of a quantum automorphism group, which we address in this section. First recall that the usual automorphism group $Aut(X)$ of a space X consists of the set of all transformations on X that preserve the structure of X. A quantum group is not a set of transformations in general. Thus a naive imitation of the above definition of $Aut(X)$ for quantum automorphisms will not work. However, we recapture the definition of $Aut(X)$ from the following universal property of $Aut(X)$ in the category of transformation groups of X: If G is any group acting on X, then there is a unique morphism of transformation groups from G to $Aut(X)$. This motivates our Definition 2.3 of quantum automorphism groups below.

The automorphism groups of finite spaces are compact Lie groups (e.g. $Aut(X_n) = S_n$, the symmetric group on n letters, and $Aut(M_n) = SU(n)$). For this reason, it is natural to expect that the quantum automorphism groups of such spaces are compact quantum groups, viz., Woronowicz Hopf C^*-algebras. We will consider only such quantum groups in this paper. For basic notions on compact quantum groups, we refer the reader to [30, 19, 20]. Note that for every compact quantum group, there corresponds a full Woronowicz Hopf C^*-algebra and a reduced Woronowicz Hopf C^*-algebra [1, 22]. We will assume that all the Woronowicz Hopf C^*-algebras in this paper are full, as morphisms behave well only with such algebras (see the discussions in III.7 of [22]). Let A be a compact quantum group. Let ϵ be the unit of this quantum group (or counit of the full Woronowicz Hopf C^*-algebra). Let $\mathcal{A}$ denote the canonical dense Hopf *-subalgebra of A consisting of coefficients of finite dimensional representations of the quantum group A.

Definition 2.1. *[cf. [1, 3, 14]] A* **left action** *of a compact quantum group A on a C^*-algebra B is a unital *-homomorphism α from B to $B \otimes A$ such that*

(1) $(id_B \otimes \Phi)\alpha = (\alpha \otimes id_A)\alpha$, *where Φ is the coproduct on A;*
(2) $(id_B \otimes \epsilon)\alpha = id_B$*;*
(3) *There is a dense *-subalgebra $\mathcal{B}$ of B, such that α restricts to a right coaction of the Hopf *-algebra $\mathcal{A}$ on $\mathcal{B}$.*

We also call (A, α) a **left quantum transformation group** *of B. Let $(\tilde{A}, \tilde{\alpha})$ be another left quantum transformation group of B. We define a* **morphism** *from $(\tilde{A}, \tilde{\alpha})$ to (A, α) to be a morphism π of quantum groups from $\tilde{A}$ to A (which is the same thing as a morphism of Woronowicz Hopf C^*-algebras from A to $\tilde{A}$, see [20]), such that*

$$\tilde{\alpha} = (id_B \otimes \pi)\alpha.$$

It is easy to see that left quantum transformation groups of B form a category with the morphisms defined above. We call it the **category of left quantum transformation groups** *of B.*

Our definition of an action of a quantum group above appears to be different from the one in [14], but it is equivalent to the latter. More precisely, conditions (2) and (3) above are equivalent to the following density requirement, which is used in [1, 3, 14] for the definition of an action:

$$(I \otimes A)\alpha(B) \ \text{ is norm dense in } \ B \otimes A,$$

but they are more natural and convenient for our purposes. It is not clear whether the injectivity condition on α imposed in [1, 3] is implied by the three conditions in the definition above. Our definition coincides with the notion of actions of groups on spaces when the quantum group A is a group and B is an ordinary space (simply by reversing the arrows).

The above definition is commonly called the *right coaction* of a unital Hopf C^*-algebra. Note that for the Hopf C^*-algebra $A = C(G)$ of continuous functions over a compact group G, the notion of right coaction of A corresponds to the notion of left action of G on a C^*-algebra B. For this reason, when we are dealing with a compact quantum group A, we call a right coaction of the underlying Woronowicz Hopf C^*-algebra of A a **left action** of the quantum group A. In the following, we will omit the word **left** for actions of quantum transformation groups. This should not cause confusion.

Definition 2.2. *Let (A, α) be a quantum transformation group of B. An element b of B is said to be* **fixed under** α *(or* **invariant under** α*) if*

$$\alpha(b) = b \otimes 1_A.$$

The **fixed point algebra** A^α *of the action α is*

$$\{b \in B \mid \alpha(b) = b \otimes 1_A\}.$$

The quantum transformation group (A, α) is said to be **ergodic** *if $A^\alpha = \mathbb{C}I$. A (continuous) functional ϕ on B is said to be* **invariant under** α *if*

$$(\phi \otimes id_A)\alpha(b) = \phi(b)I_A$$

for all $b \in B$. For a given functional ϕ on B, we define the **category of quantum transformation groups of the pair** (B, ϕ) *to be the category with objects that leave invariant the functional ϕ. This is a subcategory of the category of all quantum transformation groups.*

Besides the two categories of quantum transformation groups mentioned above, we also have the category of quantum transformation groups of Kac type for B, which is a full subcategory of the category of quantum transformation groups of B.

Definition 2.3. *Let $\mathcal{C}$ be a category of quantum transformation groups of B. The* **quantum automorphism group** *of B in $\mathcal{C}$ is a universal final object in the category $\mathcal{C}$. That is, if $(\tilde{A}, \tilde{\alpha})$ is an object in this category, then there is a unique morphism π of quantum transformation groups from $(\tilde{A}, \tilde{\alpha})$ to (A, α).*

Let ϕ be a continuous functional on the algebra B. We define **quantum automorphism group of the pair** (B, ϕ) *to be the universal object in the category of quantum transformation groups of the pair (B, ϕ) (cf. Definition 2.1).*

From categorical abstract nonsense, the quantum automorphism group of B (in a given category) is unique (up to isomorphism) if it exists. We emphasize in particular that the notion of a quantum automorphism group depends on the category of quantum transformation groups of B, not only on B. As a matter of fact, for a finite space B other than X_n, we will show in Theorem 6.1 that the quantum automorphism group does not exist for the category of all quantum transformation groups. In the subcategory of quantum transformation groups of B with objects consisting of compact transformation groups, the universal object is precisely the ordinary automorphism group $Aut(B)$, as mentioned in the beginning of this section.

We will also use the following notion, which generalizes the usual notion of a faithful group action.

Definition 2.4. *Let (A, α) be a quantum transformation group of B. We say that the action α is* **faithful** *if there is no proper Woronowicz Hopf C^*-subalgebra A_1 of A such that α is an action of A_1 on B.*

If (A, α) is a quantum automorphism group in some category of quantum transformation groups on B, then the action α is faithful. We leave the verification of this to the reader as an exercise.

3. Quantum Automorphism Group of Finite Space X_n

By the Gelfand–Naimark theorem, we can identify $X_n = \{x_1, \cdots, x_n\}$ with the C^*-algebra $B = C(X_n)$ of continuous functions on X_n. The algebra B has the following presentation,

$$B = C^*\{e_i \mid e_i^2 = e_i = e_i^*, \ \ \sum_{r=1}^{n} e_r = 1, \ \ i = 1, \cdots, n\}.$$

The ordinary automorphism group $Aut(X_n) = Aut(B)$ of X_n is the symmetric group S_n on n symbols. We can put the group S_n in the framework of Woronowicz as follows. As a transformation group, S_n can be thought of as the collection of all permutation matrices

$$g = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

When g varies in S_n, the a_{ij}'s ($i, j = 1, \cdots, n$) are functions on the group S_n satisfying the following relations:

$$a_{ij}^2 = a_{ij} = a_{ij}^*, \quad i, j = 1, \cdots, n, \tag{3.1}$$

$$\sum_{j=1}^{n} a_{ij} = 1, \quad i = 1, \cdots, n, \tag{3.2}$$

$$\sum_{i=1}^{n} a_{ij} = 1, \quad i = 1, \cdots, n. \tag{3.3}$$

It is easy to see that the commutative C^*-algebra generated by the above commutation relations is the Woronowicz Hopf C^*-algebra $C(S_n)$. In other words, the group S_n is completely determined by these relations. The following theorem shows that we have obtained much more: If we remove the condition that the a_{ij}'s commute with each other, these relations define the quantum automorphism group of X_n.

Theorem 3.1. *Let A be the C^*-algebra with generators a_{ij} ($i, j = 1, \cdots, n$) and defining relations (3.1)–(3.3). Then*

(1) *A is a compact quantum group of Kac type;*
(2) *The formulas*

$$\alpha(e_j) = \sum_{i=1}^{n} e_i \otimes a_{ij}, \quad j = 1, \cdots, n$$

defines a quantum transformation group (A, α) of B. It is the quantum automorphism group of B in the category of all compact quantum transformation groups (hence also in the category of compact quantum groups of Kac type) of B, and it contains the ordinary automorphism group $Aut(X_n) = S_n$ (in fact, $\{(\chi(a_{ij})) \mid \chi \in X(A)\}$ is precisely the set of permutation matrices).

Because of (2) above, we will denote the quantum group above by $A_{aut}(X_n)$. We will call it the **quantum permutation group** on n symbols.

Proof. (1) It is easy to check that there is a well-defined homomorphism Φ from A to $A \otimes A$ with the property

$$\Phi(a_{ij}) = \sum_{k=1}^{n} a_{ik} \otimes a_{kj}, \quad i, j = 1, \cdots, n.$$

Using (3.1)–(3.3), it is also easy to check that $u = (a_{ij})$ is an orthogonal matrix. Hence (A, u) is a quantum subgroup of $A_o(n)$, so it is of Kac type (cf. [19, 20, 18]).

To prove (2), note that the generators $\{e_i\}_{i=1}^n$ form a basis of the vector space B, so an action $\tilde{\alpha}$ of any quantum group $\tilde{A}$ on B is uniquely determined by its effect on the e_i's:

$$\tilde{\alpha}(e_j) = \sum_{i=1}^{n} e_i \otimes \tilde{a}_{ij}, \quad j = 1, \cdots, n.$$

The condition that $\tilde{\alpha}$ is a *-homomorphism together with the equations

$$e_i^2 = e_i = e_i^*, \quad i = 1, \cdots, n$$

shows that the $\tilde{a}_{ij}$'s satisfy the relations (3.1). The condition that $\tilde{\alpha}$ is a unital homomorphism together with the equation

$$\sum_{i=1}^{n} e_i = 1$$

shows that the $\tilde{a}_{ij}$'s satisfy (3.2). Let $\tilde{u} = (\tilde{a}_{ij})$. Then we have

$$\tilde{u}\tilde{u}^* = I_n.$$

The condition in Definition 2.1 (2) means that

$$\epsilon(\tilde{a}_{ij}) = \delta_{ij}, \quad i, j = 1, \cdots, n.$$

By condition (3) of Definition 2.1, the $\tilde{a}_{ij}$'s are in $\tilde{A}$. Hence by Proposition 3.2 of [30], it follows that $\tilde{u} = (\tilde{a}_{ij})$ is a non-degenerate smooth representation of the quantum group $\tilde{A}$. In particular, $\tilde{u}$ is also left invertible,

$$\tilde{u}^* \tilde{u} = I_n.$$

This implies that the $\tilde{a}_{ij}$'s satisfy the relations (3.3). From these we see that (A, α) is a universal quantum transformation group of B: there is a unique morphism π of quantum transformation groups from $(\tilde{A}, \tilde{\alpha})$ to (A, α) such that

$$\pi(a_{ij}) = \tilde{a}_{ij}, \quad i, j = 1, \cdots, n.$$

It is clear that the maximal subgroup of the quantum group A is S_n, that is, the set $\{(\chi(a_{ij})) \mid \chi \in X(A)\}$ is precisely the set of permutation matrices. $\square$

Remarks. (1) For each pair i, j, let A_{ij} be the group C^*-algebra $C^*(\mathbb{Z}/2\mathbb{Z})$ with generator p_{ij}, $p_{ij}^2 = p_{ij} = p_{ij}^*$ $(i, j = 1, \cdots, n)$. Then the C^*-algebra A is isomorphic to the following quotient C^*-algebra of the free product of the A_{ij}'s:

$$(*_{i,j=1}^n A_{ij}) / < \sum_{r=1}^n p_{rj} = 1 = \sum_{s=1}^n p_{is}, \quad i, j = 1, \cdots, n > .$$

(2) Let ϕ be the unique S_n-invariant probability measure on X_n. Then it is easy to see that ϕ is a fixed functional under the action of the quantum group $A_{aut}(X_n)$ defined in Theorem 3.1. Hence $A_{aut}(X_n)$ is also the quantum automorphism group for the pair (X_n, ϕ).

(3) Let $Q > 0$ be a positive $n \times n$ matrix. Let $A_{aut}^Q(X_n)$ be the C^*-algebra with generators a_{ij} $(i, j = 1, \cdots, n)$ and the defining relations given by (3.1)–(3.2) along with the following set of relations:

$$u^t Q u Q^{-1} = I_n = Q u Q^{-1} u^t, \tag{3.4}$$

where $u = (a_{ij})$. Then it not hard to verify that $(A_{aut}^Q(X_n), \alpha)$ is a compact quantum transformation subgroup of the one defined in Theorem 3.1 (hence the a_{ij}'s also satisfy the relations (3.3)), here α is as in Theorem 3.1. Note also for $Q = I_n$, $A_{aut}^Q(X_n) = A_{aut}(X_n)$.

4. Quantum Automorphism Group of Finite Space $M_n(\mathbb{C})$

Notation. Let $u = (a_{ij}^{kl})_{i,j,k,l=1}^n$ and $v = (b_{ij}^{kl})_{i,j,k,l=1}^n$ with entries from a *-algebra. Define uv to be the matrix whose entries are given by

$$(uv)_{ij}^{kl} = \sum_{r,s=1}^n a_{rs}^{kl} b_{ij}^{rs}, \quad i, j, k, l = 1, \cdots, n.$$

Let $\psi = Tr$ be the trace functional on M_n (so $\phi = \frac{1}{n}\psi$ is the unique $Aut(M_n)$-invariant state on M_n). The C^*-algebra M_n has the following presentation:

$$B = C^*\{e_{ij} \mid e_{ij} e_{kl} = \delta_{jk} e_{il}, \ e_{ij}^* = e_{ji}, \ \sum_{r=1}^n e_{rr} = 1, \ i, j, k, l = 1, \cdots, n\}.$$

Theorem 4.1. *Let A be the C^*-algebra with generators a_{ij}^{kl} and the following defining relations (4.1)–(4.5):*

$$\sum_{v=1}^{n} a_{ij}^{kv} a_{rs}^{vl} = \delta_{jr} a_{is}^{kl}, \quad i,j,k,l,r,s = 1,\cdots,n, \tag{4.1}$$

$$\sum_{v=1}^{n} a_{lv}^{sr} a_{vk}^{ji} = \delta_{jr} a_{lk}^{si}, \quad i,j,k,l,r,s = 1,\cdots,n, \tag{4.2}$$

$$a_{ij}^{kl*} = a_{ji}^{lk}, \quad i,j,k,l = 1,\cdots,n, \tag{4.3}$$

$$\sum_{r=1}^{n} a_{rr}^{kl} = \delta_{kl}, \quad k,l = 1,\cdots,n, \tag{4.4}$$

$$\sum_{r=1}^{n} a_{kl}^{rr} = \delta_{kl}, \quad k,l = 1,\cdots,n. \tag{4.5}$$

Then

(1) *A is a compact quantum group of Kac type;*
(2) *The formulas*

$$\alpha(e_{ij}) = \sum_{k,l=1}^{n} e_{kl} \otimes a_{ij}^{kl}, \quad i,j = 1,\cdots,n$$

defines a quantum transformation group (A,α) of (M_n,ψ). It is the quantum automorphism group of (M_n,ψ) in the category of compact quantum transformation groups (hence also in the category of compact quantum groups of Kac type) of (M_n,ψ), and it contains the ordinary automorphism group $Aut(M_n) = SU(n)$.

We will denote the quantum group above by $A_{aut}(M_n)$.

Proof. (1) It is easy to check that the matrix $u = (a_{ij}^{kl})$ as well as its conjugate $\bar{u} = (a_{ij}^{kl*})$ are both unitary matrices, and that the formulas

$$\Phi(a_{ij}^{kl}) = \sum_{r,s=1}^{n} a_{rs}^{kl} \otimes a_{ij}^{rs}, \quad i,j,k,l = 1,\cdots,n$$

gives a well-defined map from A to $A \otimes A$ (this is the coproduct). Hence A is a quantum subgroup of $A_u(m)$ (with $m = n^2$), so it is of Kac type (cf. [19, 20, 18]).

(2) Let $(\tilde{A},\tilde{\alpha})$ be any quantum transformation group of M_n. Being a basis for the vector space M_n, the e_{ij}'s uniquely determine the action $\tilde{\alpha}$:

$$\tilde{\alpha}(e_{ij}) = \sum_{k,l=1}^{n} e_{kl} \otimes \tilde{a}_{ij}^{kl}, \quad i,j = 1,\cdots,n.$$

The condition that $\tilde{\alpha}$ is a homomorphism together with the equations

$$e_{ij} e_{kl} = \delta_{jk} e_{il}, \quad i,j,k,l = 1,\cdots,n$$

shows that the $\tilde{a}_{ij}^{kl}$'s satisfy (4.1). The condition that $\tilde{\alpha}$ preserves the *-operation together with the equations

$$e_{ij}^* = e_{ji}, \quad i, j = 1, \cdots, n$$

shows that the $\tilde{a}_{ij}^{kl}$'s satisfy (4.3). The condition that $\tilde{\alpha}$ preserves the units together with the identity

$$\sum_r e_{rr} = 1$$

shows that the $\tilde{a}_{ij}^{kl}$'s satisfy (4.4). The condition that $\tilde{\alpha}$ leaves the trace ψ invariant shows that the $\tilde{a}_{ij}^{kl}$'s satisfy (4.5).

To show that the $\tilde{a}_{ij}^{kl}$'s satisfy (4.2), first it is an easy check that

$$\tilde{u}^* \tilde{u} = I_n^{\otimes 2},$$

where $\tilde{u} = (\tilde{a}_{ij}^{kl})_{i,j,k,l=1}^n$. By condition (3) of Definition 2.1, the $\tilde{a}_{ij}^{kl}$'s are in $\tilde{\mathcal{A}}$. Hence by Proposition 3.2 of [30], we see that $\tilde{u}$ is a non-degenerate smooth representation of the quantum group $\tilde{A}$. In particular, $\tilde{u}$ is also right invertible,

$$\tilde{u}\tilde{u}^* = I_n^{\otimes 2},$$

which means that

$$\sum_{i,j=1}^n \tilde{a}_{ij}^{kl} \tilde{a}_{ji}^{sr} = \delta_{kr}\delta_{ls}, \quad k, l, r, s = 1, \cdots, n.$$

From these relations and the relations (4.1), (4.3)-(4.5), we deduce that both matrices $\tilde{u}$ and $\tilde{u}^t$ are unitary. This shows that the quantum group A_1 generated by the coefficients $\tilde{a}_{ij}^{kl}$ is a compact quantum group of Kac type. That is, the antipode $\tilde{\kappa}$ is a bounded *-antihomomorphism when restricted to A_1. Put

$$v = (b_{ij}^{kl}) = (\tilde{\kappa}(\tilde{a}_{ij}^{kl})) = (\tilde{a}_{lk}^{ji}).$$

Then in the opposite algebra ${A_1}^{op}$ (which has the same elements as A_1 with multiplication reserved), the b_{ij}^{kl}'s satisfy the relations (4.1), which means that the $\tilde{a}_{ij}^{kl}$'s satisfy the relations (4.2) in the algebra $\tilde{A}$.

From the above consideration we see that (A, α) is a quantum transformation group of M_n, and that there is a unique morphism π of quantum groups from $\tilde{A}$ to A such that

$$\pi(a_{ij}^{kl}) = \tilde{a}_{ij}^{kl}, \quad i, j, k, l = 1, \cdots, n.$$

It is routine to check that π is the unique morphism π of quantum transformation groups from $(\tilde{A}, \tilde{\alpha})$ to (A, α).

From the relations (4.1)–(4.5), one can show that each matrix $(\chi(a_{ij}^{kl}))$ $(\chi \in X(A_{aut}(M_n)))$ defines an automorphism of M_n by the formulas in Theorem 4.1 (2). This means that the maximal subgroup $X(A_{aut}(M_n))$ is naturally embedded in $Aut(M_n)$. Conversely, it is clear that $Aut(M_n)$ can be embedded as a subgroup of the maximal subgroup $X(A_{aut}(M_n))$ of $A_{aut}(M_n)$. $\Box$

Remark. Consider the quantum group $(A_u(n), (a_{ij}))$ (cf. [20, 18]). Put $\tilde{a}_{ij}^{kl} = a_{ki}a_{lj}^*$. Then the $\tilde{a}_{ij}^{kl}$'s satisfies the relations (4.1)–(4.5). From this we see that the $\tilde{a}_{ij}^{kl}$'s determines a quantum subgroup of $A_{aut}(M_n)$. Hence the Woronowicz Hopf C^*-algebra $A_{aut}(M_n)$ is noncommutative and noncocommutative. How big is the subalgebra of $A_u(n)$ generated by the $\tilde{a}_{ij}^{kl}$? An answer to this question will shed light on the structure of the C^*-algebra $A_{aut}(M_n)$.

Proposition 4.2. *Let $Q > 0$ be a positive matrix in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Let A be the C^*-algebra with generators a_{ij}^{kl} and defining relations given by (4.1), (4.3), (4.4), along with the following set of relations:*

$$u^* Q u Q^{-1} = I_n^{\otimes 2} = Q u Q^{-1} u^*, \tag{4.6}$$

where $u = (a_{ij}^{kl})$. Then A is a compact quantum group that acts faithfully on M_n in the following manner,

$$\alpha(e_{ij}) = \sum_{k,l=1}^{n} e_{kl} \otimes a_{ij}^{kl}, \quad i, j = 1, \cdots, n,$$

and its maximal subgroup is isomorphic to a subgroup of $Aut(M_n) \cong SU(n)$. Any faithful compact quantum transformation group of M_n is a quantum subgroup of (A, α) for some positive Q.

Proof. First we show that A is a compact quantum group. Let $v = Q^{1/2} u Q^{-1/2}$. Then (4.6) is equivalent to

$$v^* v = I_n^{\otimes 2} = v v^*.$$

Hence the C^*-algebra A is well defined. The set of relations in (4.6) shows that u is invertible. We claim that u^t is also invertible. For simplicity of notation in the following computation, let $\tilde{Q} = (\tilde{q}_{ij}^{kl}) = Q^{-1}$. Then (4.6) becomes

$$\sum_{k,l,r,s,x,y=1}^{n} a_{ij}^{lk} q_{rs}^{kl} a_{xy}^{rs} \tilde{q}_{ef}^{xy} = \delta_{ef}^{ij} = \sum_{k,l,r,s,x,y=1}^{n} q_{kl}^{ij} a_{rs}^{kl} \tilde{q}_{xy}^{rs} a_{fe}^{yx},$$

where $i, j, e, f = 1, \cdots, n$. Put $P = (p_{ij}^{kl})$ and $\tilde{P} = (\tilde{p}_{ij}^{kl})$, where

$$p_{ij}^{kl} = q_{ij}^{lk}, \quad \tilde{p}_{ij}^{kl} = q_{ji}^{kl}, \quad i, j, k, l = 1, \cdots, n.$$

Then $P^{-1} = \tilde{P}$, and the relations (4.6) becomes

$$u^t P u P^{-1} = I_n^{\otimes 2} = P u P^{-1} u^t.$$

This proves our claim.

Now it is easy to check that A is a compact matrix quantum group with coproduct Φ given by the same formulas as in the proof of Theorem 4.1(1).

Let $(\tilde{A}, \tilde{\alpha})$ be a faithful quantum transformation group of M_n. We saw in the proof of Theorem 4.1 that there are elements $\tilde{a}_{ij}^{kl}$ $(i, j, k, l = 1, \cdots, n)$ in the C^*-algebra $\tilde{A}$ that satisfy the relations (4.1), (4.3) and (4.4). The condition in Definition 2.1 (2) means that

$$\epsilon(\tilde{a}_{ij}^{kl}) = \delta_{ij}^{kl}, \quad i, j, k, l = 1, \cdots, n.$$

By condition (3) of Definition 2.1, the $\tilde{a}_{ij}$'s are in $\tilde{\mathcal{A}}$. Hence by Proposition 3.2 of [30], this implies that $\tilde{u} = (\tilde{a}_{ij}^{kl})$ is a non-degenerate smooth representation of the quantum group $\mathcal{A}$. From the proof of Theorem 5.2 of [30], with

$$Q = (id \otimes \tilde{h})(\tilde{u}^* \tilde{u}),$$

we have $Q > 0$ and $\tilde{u}$ satisfies (4.6). The assumption that $(\tilde{A}, \tilde{\alpha})$ is faithful implies that $\tilde{A}$ is generated by the elements $\tilde{a}_{ij}^{kl}$ $(i, j = 1, \cdots, n)$. This shows that (A, α) is a well defined faithful quantum transformation group of M_n and that the compact quantum transformation group $(\tilde{A}, \tilde{\alpha})$ is a quantum subgroup of (A, α).

Let $\chi \in X(A)$. From the defining relations for A, we see that $(\chi(a_{kl,ij}))$ defines an ordinary transformation for M_n via the formulas in Theorem 4.2. Hence the maximal subgroup $X(A)$ is embedded in $Aut(M_n)$. $\square$

Note. We will denote the quantum group above by $A^Q_{aut}(M_n)$. If $Q = I_n^{\otimes 2}$, then it is easy to see that the square of the coinverse (i.e. antipode) map is the identity map. From this one can show that this quantum group reduces to the quantum group $A_{aut}(M_n)$ in Theorem 4.1.

5. Quantum Automorphism Group of Finite Space $\bigoplus_{k=1}^m M_{n_k}(\mathbb{C})$

Notation. Let $u = (a^{kl}_{rs,xy})$ and $v = (b^{kl}_{rs,xy})$ be two matrices with entries from a *-algebra, where

$$k, l = 1, \cdots, n_x, \quad r, s = 1, \cdots, n_y, \quad x, y = 1, \cdots, m.$$

Define uv to be the matrix whose entries are given by

$$(uv)^{kl}_{rs,xy} = \sum_{p=1}^{m} \sum_{i,j=1}^{n_p} a^{kl}_{ij,xp} b^{ij}_{rs,py}.$$

Using the same method as above, we now study the quantum automorphism group of the finite space $B = \bigoplus_{k=1}^m M_{n_k}$, where n_k is a positive integer. The C^*-algebra B has the following presentation:

$$B = C^*\{e_{kl,i} \mid \ e_{kl,i}e_{rs,j} = \delta_{ij}\delta_{lr}e_{ks}, \ e^*_{kl,i} = e_{lk,i}, \ \sum_{q=1}^{m}\sum_{p=1}^{n_q} e_{pp,q} = 1,$$
$$k, l = 1, \cdots, n_i, \quad r, s = 1, \cdots, n_j, \quad i, j = 1, \cdots, m\}.$$

Let ψ be the positive functional on B defined by

$$\psi(e_{kl,i}) = Tr(e_{kl,i}) = \delta_{kl}, \quad k, l = 1, \cdots, n_i, \quad i = 1, \cdots, m.$$

The defining relations for the quantum group of (B, ψ) are obtained as a combination of the relations of the quantum automorphism groups $A_{aut}(X_n)$ and $A_{aut}(M_n)$.

Theorem 5.1. *Let A be the C^*-algebra with generators $a^{kl}_{rs,xy}$*

$$k, l = 1, \cdots, n_x, \quad r, s = 1, \cdots, n_y, \quad x, y = 1, \cdots, m,$$

and the following defining relations (5.1)–(5.5):

$$\sum_{v=1}^{n_x} a^{kv}_{ij,xy} a^{vl}_{rs,xz} = \delta_{jr}\delta_{yz} a^{kl}_{is,xy}, \tag{5.1}$$

$$i, j = 1, \cdots, n_y, \quad r, s = 1, \cdots, n_z, \quad k, l = 1, \cdots, n_x, \quad x, y, z = 1, \cdots, m,$$

$$\sum_{v=1}^{n_x} a^{sr}_{lv,yx} a^{ji}_{vk,zx} = \delta_{jr}\delta_{yz} a^{si}_{lk,yx}, \tag{5.2}$$

$$i, j = 1, \cdots, n_z, \quad r, s = 1, \cdots, n_y, \quad k, l = 1, \cdots, n_x, \quad x, y, z = 1, \cdots, m,$$

$${a^{kl}_{ij,yz}}^* = a^{lk}_{ji,yz}, \tag{5.3}$$

$$i, j = 1, \cdots, n_z, \quad k, l = 1, \cdots, n_y, \quad y, z = 1, \cdots, m,$$

$$\sum_{z=1}^{m} \sum_{r=1}^{n_z} a^{kl}_{rr,yz} = \delta_{kl}, \quad k, l = 1, \cdots, n_y, \quad y = 1, \cdots, m, \tag{5.4}$$

$$\sum_{y=1}^{m} \sum_{r=1}^{n_y} a^{rr}_{kl,yz} = \delta_{kl}, \quad k, l = 1, \cdots, n_z, \quad z = 1, \cdots, m. \tag{5.5}$$

Then

(1) *A is a compact quantum group of Kac type;*
(2) *The formulas*

$$\alpha(e_{rs,j}) = \sum_{i=1}^{m} \sum_{k,l}^{n_i} e_{kl,i} \otimes a^{kl}_{rs,ij}, \quad r, s = 1, \cdots, n_j, \quad j = 1, \cdots, m$$

define a quantum transformation group (A, α) of (B, ψ). This is the quantum automorphism group of (B, ψ) in the category of compact quantum transformation groups (hence also in the category of compact quantum groups of Kac type) of (B, ψ), and it contains the ordinary automorphism group $Aut(B)$.

We will denote the quantum group above by $A_{aut}(B)$.

Proof. The proof of this theorem follows the lines of the proof of Theorem 4.1. The coproduct is given by

$$\Phi(a^{kl}_{ij,xy}) = \sum_{p=1}^{m} \sum_{r,s=1}^{n_p} a^{kl}_{rs,xp} \otimes a^{rs}_{ij,py}, \quad k, l = 1, \cdots, n_x, \quad x, y = 1, \cdots, m. \qquad \square$$

Note that when $n_k = 1$ for all k, then the quantum group $A_{aut}(B)$ reduces to the quantum group $A_{aut}(X_n)$ in Theorem 3.1, and when $m = 1$, $A_{aut}(B)$ reduces to the quantum group $A_{aut}(M_n)$ in Theorem 4.1.

Let $Q = (q^{kl}_{rs,xy}) > 0$ $(k, l = 1, \cdots, n_x, \quad r, s = 1, \cdots, n_y, \quad x, y = 1, \cdots, m)$ be a positive matrix with complex entries. Define $\delta^{kl}_{rs,xy}$ to be 1 if $k = r, l = s, x = y$ and 0 otherwise, and let I be the matrix with entries $\delta^{kl}_{rs,xy}$, where

$$k, l = 1, \cdots, n_x, \quad r, s = 1, \cdots, n_y, \quad x, y = 1, \cdots, m.$$

Proposition 5.2. *Let Q and I be as above. Let A be the C^*-algebra with generators $a^{kl}_{rs,xy}$,*

$$k,l = 1,\cdots,n_x, \quad r,s = 1,\cdots,n_y, \quad x,y = 1,\cdots,m,$$

and defining relations (5.1), (5.3), (5.4), along with the following set of relations:

$$u^* Q u Q^{-1} = I = Q u Q^{-1} u^*, \tag{5.6}$$

where $u = (a^{kl}_{rs,xy})$. Then A is a compact quantum group that acts faithfully on B in the following manner,

$$\alpha(e_{rs,j}) = \sum_{i=1}^{m} \sum_{k,l}^{n_i} e_{kl,i} \otimes a^{kl}_{rs,ij}, \quad r,s = 1,\cdots,n_j, \quad j = 1,\cdots,m.$$

Any faithful compact quantum transformation group of B is a quantum subgroup of (A,α) for some positive Q.

Proof. The proof follows the lines of Theorem 4.2. □

We will denote the quantum group above by $A^Q_{aut}(B)$, or simply by A^Q_{aut}. When $Q = I_n^{\otimes 2}$, then $A^Q_{aut}(B)$ is just $A_{aut}(B)$. Note that for n_k's distinct, the automorphism group $Aut(\oplus_{k=1}^m M_{n_k})$ is isomorphic to the group $\times_{k=1}^m Aut(M_{n_k})$. A natural problem related to this is

Problem 5.3. For n_k's distinct, the quantum automorphism group $A_{aut}(\oplus_{k=1}^m M_{n_k})$ is isomorphic to the quantum group $\otimes_{k=1}^m A_{aut}(M_{n_k})$ (cf. [21]).

For each fixed $1 \leq k_0 \leq m$, $A_{aut}(M_{k_0})$ as defined in the last section is a quantum subgroup of $A_{aut}(B)$. (This is seen as follows. Let $\tilde{a}^{kl}_{rs,xy} = \delta_{xk_0}\delta_{yk_0}a^{kl}_{rs}$, where the a^{kl}_{rs}'s are generators of $A_{aut}(M_{n_{k_0}})$. Then the $\tilde{a}^{kl}_{rs,xy}$'s satisfy the defining relations for $A_{aut}(B)$.) Note also that if $n_k = n$ for all k, then $A_{aut}(X_m)$ is a quantum subgroup of $A_{aut}(B)$. (This is seen as follows. Let $\tilde{a}^{kl}_{rs,xy} = \delta_{kr}\delta_{ls}a_{xy}$, where the a_{xy}'s are generators of $A_{aut}(X_m)$. Then the $\tilde{a}^{kl}_{rs,xy}$'s satisfy the defining relations for $A_{aut}(B)$.) In view of the fact that the ordinary automorphism group $Aut(\oplus_1^m M_n)$ is isomorphic to the semi-direct product $SU(n) \rtimes S_m$, it would be interesting to solve the following problem.

Problem 5.4. Is it possible to express $A_{aut}(\oplus_1^m M_n)$ in terms of $A_{aut}(M_n)$ and $A_{aut}(X_m)$ as a certain semi-direct product that generalizes [21]?

6. The Main Result

Summarizing the previous sections, we can now state the main result of this paper.

Theorem 6.1. *Let B be a finite space of the form $\oplus_{k=1}^m M_{n_k}$.*

(1) *Quantum automorphism group of B exists in the category of (left) quantum transformation groups if and only if B is the finite space X_m.*
(2) *The quantum automorphism group for (B,ψ) exists and is defined as in Theorem 5.1 (see also Theorem 3.1, Theorem 4.1).*

Proof. (1) If B is X_m, we saw in Theorem 3.1 that $A_{aut}(X_m)$ is the quantum automorphism group of X_m in the category of all quantum transformation groups.

Now assume that $B \neq C(X_m)$, and assume that the quantum automorphism group of B exists in the category of all quantum transformation groups. Call it (A_0, α_0). As in Theorem 5.1 and Theorem 5.2, α_0 is determined by its effect on the basis $e_{rs,j}$ of B,

$$\alpha_0(e_{rs,j}) = \sum_{i=1}^{m} \sum_{k,l}^{n_i} e_{kl,i} \otimes \tilde{a}^{kl}_{rs,ij}, \quad r,s = 1, \cdots, n_j, \quad j = 1, \cdots, m.$$

Since (A_0, α_0) is the quantum automorphism group of B, the action α_0 is faithful (cf. Definition 2.4). This implies that the $\tilde{a}^{kl}_{rs,ij}$'s generates the C^*-algebra A_0. As in Theorem 5.2 (see also Theorem 4.2), there is a positive Q_0, such that the $\tilde{a}^{kl}_{rs,xy}$'s satisfy the relations (5.1), (5.3), (5.4), along with the following set of relations:

$$\tilde{u}^* Q_0 \tilde{u} Q_0^{-1} = I = Q_0 \tilde{u} Q_0^{-1} \tilde{u}^*, \tag{6.1}$$

where $\tilde{u} = (\tilde{a}^{kl}_{rs,xy})$. By the universal property of (A_0, α_0), we conclude that $A_0 = A^{Q_0}_{aut}$ (see also the last statement in Theorem 5.2). For every positive Q, the unique morphism from (A^Q_{aut}, α) to (A_0, α_0) sends the generators $\tilde{a}^{kl}_{rs,xy}$ of $A^{Q_0}_{aut}$ to the corresponding generators $a^{kl}_{rs,xy}$ of A^Q_{aut} (again because of faithfulness of the quantum transformation group A^Q_{aut} and the universality of $A^{Q_0}_{aut}$). Hence the generators $a^{kl}_{rs,xy}$ also satisfy the relations (6.1). This is impossible because we can choose Q so that A^Q_{aut} and $A^{Q_0}_{aut}$ have different *classical points* in the *vector space* with coordinates $a^{kl}_{rs,xy}$ $(k,l = 1, \cdots, n_x, \;\; r,s = 1, \cdots, n_y, \;\; x,y = 1, \cdots, m)$.

(2) This is proved in the previous sections. □

Concluding Remarks. (1) In this paper, we only described the quantum automorphism group of (B, ψ) for the special choice of functional ψ, because this quantum automorphism group is closest to the ordinary automorphism group $Aut(B)$ of B, and it contains the latter. One can also use the same method to describe quantum automorphism groups of B endowed with other functionals or a collection of functionals.

(2) For each $1 \leq k \leq n$, consider the delta measure χ_k on X_n corresponding to the point x_k. Then the quantum automorphism group of (X_n, χ_k) is isomorphic to the *quantum permutation group* of the space X_{n-1}, just as in the case of ordinary permutation groups.

(3) If we remove condition (3) in Definition 2.1, then we obtain the notion of an action of a quantum semi-group on a C^*-algebra. The relations (5.1), (5.3), (5.4) define *the universal quantum semi-group* $E(B)$ acting on B, even though B is not a quadratic algebra in the sense of Manin [13]. From the main theorem of this paper, the Hopf envelope $H(B)$ of this quantum semi-group in the sense of Manin cannot be a compact quantum group (see also the last section of [18]).

After this paper was submitted for publication, we received the papers [6, 7], where a finite quantum group symmetry $A(F)$ for M_3 is described, following the work of Connes [5]. The finite quantum group $A(F)$ in these papers is not a finite quantum group in the sense of [30] (because it does not have a compatible C^* norm), so it cannot be a quantum subgroup of the COMPACT quantum symmetry groups $A_{aut}(M_3)$ and $A^Q_{aut}(M_3)$ in our paper; but it is a quantum subgroup of the Hopf envelope $H(B)$ of the quantum semi-group $E(B)$ mentioned in the last paragraph.

Our paper gives solutions to the "intricate problem" mentioned in the end of Sect. 2 of the paper [7]: find the biggest quantum group acting on M_3. This "intricate problem" has two solutions: the first, Theorem 6.1, solves the problem in the category of compact quantum groups; the second, the remarks in the last two paragraphs, solves the problem in the category of all quantum groups–Hopf algebras that need not have C^*-norms.

(4) In [13], the quantum group $SU_q(2)$ is described as the quantum automorphism group of the quantum plane (i.e. the deformed plane). In view of the fact that the automorphism group $Aut(M_2)$ is $SU(2)$, one might be able to describe $SU_q(2)$ as a quantum automorphism group of the non-deformed space M_2 endowed with a collection of functionals.

Appendix

In [18], we introduced a compact matrix quantum group $A_o(Q)$ for each non-singular matrix Q. It has the following presentation:

$$\bar{u} = u,$$

$$uu^t = I_m = u^t u,$$

$$u^t Q u Q^{-1} = I_m = Q u Q^{-1} u^t,$$

where $u = (a_{ij})$.

As a matter of fact, it is more appropriate to use the notation $A_o(Q)$ (and we will do so from now on) for the compact matrix quantum group with the following sets of relations (where Q is positive):

$$\bar{u} = u,$$

$$u^t Q u Q^{-1} = I_m = Q u Q^{-1} u^t.$$

(Let $v = Q^{1/2} u Q^{-1/2}$. Then v is a unitary matrix. Hence the C^*-algebra A exists. From this it is easy to see that $A_o(Q)$ is a compact matrix quantum group.) This quantum group has all the properties listed in [18] for the old $A_o(Q)$. The old $A_o(Q)$ is the intersection of the quantum groups $A_o(n)$ and the new $A_o(Q)$ defined above. Moreover, if Q is a real matrix, the new $A_o(Q)$ is a compact quantum group of Kac type.

Finally, we note that the quantum group denoted by $A_o(F)$ in [2] is the same as the quantum group $B_u(Q)$ in [24, 26] with $Q = F^*$, so it is different from the quantum group $A_o(Q)$ above unless F is the trivial matrix I_n.

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