Commun. Math. Phys. 194, 675 – 705 (1998)

Communications in
Mathematical Physics

D-Brane Bound States Redux*

Savdeep Sethi[1], **Mark Stern**[2]

[1] School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA.
E-mail: sethi@sns.ias.edu
[2] Department of Mathematics, Duke University, Durham, NC 27706, USA. E-mail: stern@math.duke.edu

Received: 23 May 1997 / Accepted: 5 November 1997

Abstract: We study the existence of D-brane bound states at threshold in Type II string theories. In a number of situations, we can reduce the question of existence to quadrature, and the study of a particular limit of the propagator for the system of D-branes. This involves a derivation of an index theorem for a family of non-Fredholm operators. In support of the conjectured relation between compactified eleven-dimensional supergravity and Type IIA string theory, we show that a bound state exists for two coincident zero-branes. This result also provides support for the conjectured description of M-theory as a matrix model. In addition, we provide further evidence that there are no BPS bound states for two and three-branes twice wrapped on Calabi–Yau vanishing cycles.

1. Introduction

Remarkable progress by Polchinski in describing the solitons of Type II string theory has provided the means by which many conjectured dualities involving string theories and M-theory can be stringently tested [1]. The low-energy dynamics of coincident D-branes has been described by Witten [2], who reduced the question of finding BPS bound states to one of studying the vacuum structure of various supersymmetric Yang-Mills theories. In simple cases, the BPS mass formula forbids the decay of a charged particle saturating the mass bound; hence, ensuring stability. However, there are a number of situations in which a particle is required that is only marginally stable against decay. Showing the existence of such particles, with energies at the decay threshold, is the goal of this paper. A similiar problem arose for finite $SU(2)$ N=2 Yang-Mills theory, studied in [3] and [4], where certain dyon bound states at threshold were shown to exist. The situations we shall presently study are significantly more difficult because the Hamiltonians are not as well behaved, and gauge invariance provides an added complexity.

* A preliminary version of this paper was circulated informally in November, 1996.

Let us briefly recall the low-energy dynamics of coincident Dirichlet p-branes, described in [2]. The world volume theory is the dimensional reduction of ten-dimensional N=1 Yang-Mills to the $p+1$-dimensional world volume of the brane. For a single brane, the gauge theory is abelian, and the dynamics therefore trivial in the infrared. For N coincident branes, the gauge symmetry is enhanced to $U(N)$ rather than $U(1)^N$. After factoring out a $U(1)$ corresponding to the center of mass motion, the existence of a bound state requires that the remaining $SU(N)$ $p+1$-dimensional Yang-Mills theory possess a normalizable supersymmetric vacuum. The bosonic potential for this model generally has flat directions, and so we encounter the problem of bound states at threshold. If a bound state is required by a conjectured duality, there is a consistency check, described by Sen [5], that can sometimes be performed. In favorable cases, one might be able to further compactify one direction of the superstring theory. If a bound state exists prior to compactification, it should give rise to BPS states in the further compactified theory which, for appropriate choices of momentum along the circle, are no longer marginally bound. The existence of these states can then be analyzed with more conventional techniques. Of course, for this consistency check, there have to be enough remaining uncompactified directions so that problems with infra-red divergences do not arise. More generally, however, the question of bound states at threshold must be addressed. Note that a normalizable state for a theory in a compact space generally does not remain normalizable when the volume is taken to infinity. The spectrum can and often does change discontinously, and showing the existence of the bound state in the non-compact situation requires a separate analysis.

In a similar spirit, we can arrive at descriptions of the effective dynamics of p-branes multiply-wrapped on supersymmetric cycles of a compactification space. In the case of p-branes wrapped on p-cycles, the resulting description of the low-energy dynamics is some flavor of quantum mechanics, although not generally just a supersymmetric gauge theory. Our aim in this paper is to address the fundamental issue – the existence of flat directions in the potential – which arises in studying binding in these situations. This analysis generalizes the discussion in [6], where we argued for the existence of a marginal bound state of a zero-brane and a four-brane, to the case where the gauge group is non-abelian. We shall see that there are very subtle issues that arise as a result of this complication.

The most exciting reason for studying this question is, however, the remarkable conjecture that M-theory may be described in terms of zero-brane dynamics in the limit where the number of branes goes to infinity [7]. This conjecture is, in part, founded on previous work studying the relation between supermembranes, and the $N \to \infty$ limit of type IIA zero-brane quantum mechanics [8,9]. In order for the M(atrix) model to have a chance at describing M theory, we need to be able to find states in the quantum mechanics which correspond to the gravitons of eleven-dimensional supergravity. The bound state that we shall find is precisely one of these particles. In the process of showing that such a bound state exists, we will provide a detailed study of the behavior of the propagator for the two zero-brane system when the zero-branes are far apart. There are a number of complications that make this analysis quite subtle. During the lengthy course of our investigation (which pre-dates the M(atrix) model), a number of germane papers have appeared. Among these papers have been interesting discussions of zero-brane scattering in various approximations [10,11], and more recently, an exciting extension of the original matrix model conjecture to the case of finite N [12]. There has also been an explicit argument showing that there are no normalizable ground states in a particular simplified matrix model [13], a heuristic attempt to argue for the existence of zero-brane bound states [14], and a recent paper which has some overlap with our results [15].

In the following section, we consider the case of $p = 0$. We describe a seven parameter family of theories, which are the primary focus of this discussion. This family of theories is derived fundamentally from the quantum mechanics describing the zero-brane in type IIA string theory by adding mass deformations. These parameters allow us to 'flow down' from ten dimensions to models that correspond to the reduction of N=1 Yang-Mills in lower dimensions by taking various mass terms to infinity. We discuss general features of these models, including the various physical scenarios in which they arise. Our approach to the question of counting bound states is described in section three. There, we argue that the L^2 index for this class of supersymmetric quantum mechanical Yang-Mills theories is actually computable. This involves a discussion of L^2 index theory for non-Fredholm operators, which is an area of mathematics that is relatively unexplored. In section four, we study the question of two-particle binding in these models, and we derive a formula for the principal contribution to the index. The final section is a study of the two-particle propagator in the limit where the two particles are far apart. With this analysis, we can compute a subtle additional contribution to the index. The way in which this contribution arises involves some rather surprising cancellations. In the class of models that we investigate, we find that only the case which corresponds to the reduction of supersymmetric Yang-Mills from ten dimensions can have a unique bound state. This answers, in large part, the question of why the large N limit of the reduced ten-dimensional Yang-Mills theory should be distinguished from the large N limit of reductions of lower-dimensional Yang-Mills theories.

2. Quantum Mechanical Gauge Theory

2.1. General Comments. Let us begin by considering models that arise from reducing supersymmetric d+1-dimensional $SU(N)$ Yang-Mills to quantum mechanics; see, for instance, [16] for the first discussion of quantum mechanical gauge theories, or perhaps [17]. Whether the Yang-Mills theory contains additional matter multiplets does not significantly change the following discussion; so, for simplicity, we shall assume no additional matter. On reducing the connection A_μ, we obtain scalar coordinates x^i where $i = 1, ..., \mathrm{d}$ which take values in the adjoint representation of the gauge group. We introduce canonical momenta obeying,

$$[x^i_A, p^j_B] = i\delta_{AB}\delta^{ij},$$

where the subscript A is a gauge index. With the generators T^A for the adjoint representation normalized so that $\mathrm{Tr}(T^A T^B) = N\delta^{AB}$, the Hamiltonian for the system takes the general form,

$$H = \frac{1}{2N}\mathrm{Tr}(p^i p^i) + V(x) + H_F. \tag{2.1}$$

The bosonic potential $V(x)$ is polynomial in x, and generally has flat directions. The term H_F is quadratic in the fermions and linear in x. Specific examples will be studied in the following subsection. The A_0 equation of motion gives a set of constraints, C_A, which must vanish on physical states by Gauss' law. The constraints obey the algebra,

$$[C_A, C_B] = if_{ABC}C_C, \tag{2.2}$$

where f_{ABC} are the structure constants. The constraints further obey the commutation relations $[C_A, H] = [C_A, Q] = 0$, where Q is a supersymmetry generator. The

supersymmetry algebra closes on the Hamiltonian if the constraints are set to zero. An N-particle BPS bound state corresponds to a normalizable, gauge-invariant ground state for this supersymmetric system.

Without detailed computation, what might we infer about the structure of the ground state? Away from the flat points, the wave function for the ground state will decay exponentially. The only interesting asymptotic behavior is expected near points where the potential is small. A preliminary comment about the structure of the flat directions is in order: for gauge group $SU(N)$, there are $d_c = (d-1)(N-1) + (N^2-1)$ commuting directions around a flat point, and $d_a = (d-1)(N^2-N)$ non-commuting directions. Let us consider the structure of the potential in the neighborhood of a flat point. As we shall subsequently describe in detail, the potential can be approximated by $V \sim -\frac{1}{2}r^2|v|^2$, where v parametrizes the transverse directions, and r is a radial coordinate for the flat directions. The Hamiltonian is then essentially a set of bosonic and fermionic harmonic oscillators for the transverse directions, and a free Laplacian along the flat directions. The frequency for oscillation along the massive directions depends on r. This observation provides one way of seeing that there are no scattering states in the spectrum of the bosonic Hamiltonian for these models, as discussed in [18]. Somewhat surprisingly, the spectrum of the bosonic models only contains discrete states. To construct a scattering state along the flat direction, one would want to put the transverse harmonic oscillators into their ground states; however, the zero point energy of the oscillators increases with r, essentially forbidding finite energy scattering states. The same argument does not apply to the supersymmetric case, since the ground state energy for the additional fermions now cancels the zero point energy from the bosons, as required by supersymmetry. If this were not the case, the subtleties in counting zero-brane bound states would not exist!

In a first approximation for large r, any zero-energy wavefunction, $\psi(x)$, roughly takes a product form corresponding to placing the transverse oscillators into the ground state, $\psi(x) \sim g(r,\theta)e^{-r|v|^2/2}$, where θ are angular variables for the flat directions. The leading dependence of $g(r,\theta)$ on r is believed to be power law decay for large r. Acting with the Hamiltonian for the massive directions on this wavefunction yields zero, since the zero point energies of the bosons and fermions cancel. We can now explain the key difficulty in studying the approximate asymptotic wavefunction: can the decay exponent be accurately estimated?

We note that this issue is critical, and cannot be resolved by simple approximations of the asymptotic behavior. For instance, even in this approximation, the function $g(r,\theta)$ is not simply the solution of a free Laplacian for the d_c-dimensional space of flat directions since the Laplacian, which for the radial coordinate is given by,

$$\Delta_r = -\frac{1}{r^{d_c-1}}\frac{\partial}{\partial r}r^{d_c-1}\frac{\partial}{\partial r}, \tag{2.3}$$

also acts on the harmonic oscillator component of the wavefunction. Actually, it is unlikely that the decay exponent can be accurately estimated without at least including the first excited mode for the massive direction into the approximation. We should also note that showing that the decay is fast enough to ensure normalizability is only a first step toward showing that a bound state exists. The structure of the wavefunction would need to be studied at small r where the non-abelian degrees of freedom are important. Currently, the only practical approach is to develop an appropriate index theory for the problem. As a final comment, note that the power law behavior of the asymptotic ground state wavefunction is a consequence of the lack of a mass gap in the spectrum. The supersymmetric theory contains a continuum of states which descend to zero energy, thanks to the existence of the flat direction [8].

2.2. A family of models. We now turn to the models of primary interest to us. Let us recall that strongly coupled Type IIA string theory in ten dimensions has a conjectured dual description as weakly coupled eleven-dimensional supergravity compactified on an S^1 [19,20]. To match the Kaluza-Klein spectrum of the compactified supergravity theory, Type IIA string theory requires electrically charged particle states. The Dirichlet zero-branes, which carry RR charge, seem to be the only candidates. Since there is a single Kaluza-Klein mode for each choice of momentum along the circle direction, we desire a single D-brane bound state for each N. Proving this conjecture was our original motivation for studying these theories.

Actually, Sen has argued in [5] that if a unique bound state exists in the quantum mechanics describing N zero-branes, then the spectrum of ultra-short multiplets in the toroidally compactified type II string agrees with the spectrum predicted by U-duality. The world-volume theory for the D-particle is given by the dimensional reduction of N=1 9 + 1-dimensional Yang-Mills to quantum mechanics. A Majorana-Weyl spinor in 9 + 1 dimensions has 16 real components, which means that the resulting quantum mechanical theory has N=16 supersymmetry. Let $\gamma^i_{\alpha\beta}$ be a real representation of the $SO(9)$ Clifford algebra with $i = 1, ..., 9$ and $\alpha = 1, ..., 16$. These Clifford matrices satisfy

$$\left\{\gamma^i, \gamma^j\right\} = 2\delta^{ij}.$$

After reduction, the Hamiltonian for this system takes the form,

$$H = \frac{1}{2N}\mathrm{Tr}(p^i p^i) - \frac{1}{4N}\sum_{ij}\mathrm{Tr}([x^i, x^j]^2) - \frac{1}{2N}\mathrm{Tr}(\psi\gamma^i[x^i, \psi]), \tag{2.4}$$

where the real fermions $\psi_{A\alpha}$ obey:

$$\{\psi_{A\alpha}, \psi_{B\beta}\} = \delta_{AB}\delta_{\alpha\beta}. \tag{2.5}$$

The Hilbert space is then composed of spinors on which the quantized fermions act as elements of a Clifford algebra. The spinor wavefunctions contain an extremely large number of components, even for small N, which makes an explicit construction of the zero energy bound state wavefunction at best difficult.[1] The supersymmetry algebra takes the form,

$$\{Q_\alpha, Q_\beta\} = 2\delta^{\alpha\beta}H + 2\gamma^i_{\alpha\beta}x^i_A C_A, \tag{2.6}$$

where,

$$Q_\alpha = \frac{1}{N}\gamma^i_{\alpha\beta}\mathrm{Tr}(\psi_\beta p^i) - \frac{i}{4N}\mathrm{Tr}([\gamma^i, \gamma^j]\psi[x^i, x^j])_\alpha,$$

while the constraint,

$$C = -i[x^i, p^i] - \frac{1}{2}[\psi_\alpha, \psi_\alpha],$$

or explicitly,

$$C_A = f_{ABC}(x^i_B p^i_C - \frac{i}{2}\psi_{B\alpha}\psi_{C\alpha}). \tag{2.7}$$

The constraint takes exactly the form assumed in the previous discussion. It is natural to call this a nine-dimensional model, although it is quantum mechanics, since there are nine bosonic variables in the adjoint of $SU(N)$, and the model is the reduction of a

[1] However, the existence of the nice $Spin(9)$ flavor symmetry might bring an explicit construction of the ground state wavefunction within the realm of possibility. We leave the attempt to construct the explicit solution to braver souls.

ten-dimensional theory. The flavor symmetry is clearly $Spin(9)$. Note that there is a nice correlation between fermion number and flavor representation that is worth mentioning at this point. The correlation essentially follows from spin-statistics in ten dimensions: fermionic states in the Hilbert space transform under spinor representations of the flavor group, while bosonic states appear in representations of $SO(9)$. If the ground state is unique, it must therefore be bosonic. There are similar relations for the other models that we shall soon discuss.

Let us consider what sort of deformations are possible in this theory. We would like to add mass terms to compactify some of the bosonic variables and effectively reduce the dimension, but we will also require that the supersymmetry algebra maintain its nice structure. In particular, we shall not consider deformations which introduce additional terms into the right hand side of the supersymmetry algebra (2.6), which are linear in momenta. The mass deformations that we shall describe correspond, in special cases, to breaking N=4 Yang-Mills in four-dimensions to N=2 or N=1 by giving masses to various chiral fields in the adjoint representation, and reducing the corresponding model to quantum mechanics.

To describe the allowed deformations, choose a real supersymmetry generator, $Q = Q_\alpha$. The generator can be split into terms involving momenta, and terms independent of momenta. Those depending on momenta can be expressed, schematically, as $\lambda^i_A p^i_A$, where λ^i is a real fermion, and i runs from 1 to 9. This leaves us with seven real fermions, ω^j_A in the adjoint of the gauge group, unpaired with a momentum operator, but each appearing in Q paired with an operator, f^j_A, quadratic in the coordinates. The supersymmetry generator is then roughly,

$$Q \sim \lambda^i_A p^i_A + \omega^j_A f^j_A + \dots .$$

The seven fermions, ω^j, then represent our deformation degrees of freedom. We can add any reasonable operator to f^j, independent of the momenta, and not generate a new term linear in momenta in the expression for $\{Q, Q\}$. There are many interesting possible deformations that preserve at least one supersymmetry. Some deformations can give quite exotic classical minima of the resulting bosonic potential. This is a topic that merits further investigation. As a special prosaic case, we could add the perturbation mx^i to one of the f^j, which would lift some of the flat directions. This is the family of deformations to which we shall restrict our discussion. More explicitly, consider a term f^j which squares to give the term in the potential, $|f^j|^2$. Adding the term mx^i to f^j changes the potential to $|f^j + mx^i|^2$. Taking $m \to \infty$ then effectively decouples x^i from the model. In this way, we generate a seven parameter family of models which depend on the values of the allowed masses for seven of the coordinates. Note that taking all masses to infinity leaves us with a two-dimensional model, and further compactification is not possible without introducing additional terms linear in the momenta into the supersymmetry algebra.

There are two cases of particular interest: the three and five-dimensional models. These models correspond to the reduction of N=1 Yang-Mills from four and six dimensions, respectively. For completeness and to fix annoying normalizations, we shall describe the Hamiltonian and supersymmetry algebra for both models explicitly.

In the three-dimensional case, the Hamiltonian is given by,

$$H = \frac{1}{2N}\mathrm{Tr}(p^i p^i) - \frac{1}{4N}\sum_{ij}\mathrm{Tr}([x^i, x^j]^2) + \frac{1}{N}\mathrm{Tr}(\overline{\psi}\sigma^i[x^i, \psi]), \tag{2.8}$$

where the index $i = 1, 2, 3$. The σ^i are the Pauli matrices, and the complex fermions ψ obey the anti-commutation relations:

$$\left\{\psi_{A\alpha}, \overline{\psi}_{B\beta}\right\} = \delta_{AB}\delta_{\alpha\beta},$$

where $\alpha = 1, 2$. The supersymmetry generators are now complex, but still take a form similar to the previous example,

$$Q_\alpha = \sigma^i_{\alpha\beta}\psi_{A\beta}p^i_A - \frac{1}{4}f_{ABC}[\sigma^i, \sigma^j]_{\alpha\beta}\psi_{A\beta}x^i_B x^j_C,$$

while the constraints are given by,

$$C_A = f_{ABC}(x^i_B p^i_C - i\overline{\psi}_{B\alpha}\psi_{C\alpha}). \tag{2.9}$$

The supersymmetry algebra is now,

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 0, \\ \left\{\overline{Q}_\alpha, \overline{Q}_\beta\right\} &= 0, \\ \left\{\overline{Q}_\alpha, Q_\beta\right\} &= 2\delta_{\alpha\beta}H - 2\sigma^i_{\alpha\beta}x^i_A C_A. \end{aligned} \tag{2.10}$$

The most glaring difference between this model and the nine-dimensional zero-brane case is that the Hilbert space is now a Fock space with a canonical vacuum. This model is quite special because the Pauli matrices form a Lie algebra, and so the complex supercharge can be expressed as [16],

$$Q_a = \sigma^i_{\alpha\beta}\psi_\beta(p^i_A - \frac{i}{2}f_{ABC}\epsilon^{ijk}x^j_B x^k_C).$$

After introducing a potential, $W = \frac{1}{6}f_{ABC}\epsilon^{ijk}x^i_A x^j_B x^k_C$, we can conjugate the supercharge in the following way:

$$e^W Q_a e^{-W} = \sigma^i_{\alpha\beta}\psi_\beta p^i_A.$$

The study of the ground state wavefunctions then takes on a cohomological flavor since the supercharge acts roughly as the operator,

$$Q \sim d + dW\wedge,$$

on the wavefunctions, which we can view as differential forms. We expect that, in this case, there should then be an explicit proof from studying the spectrum directly that shows there are no zero-energy L^2 wavefunctions for this model.

The five-dimensional case is governed by the Hamiltonian:

$$H = \frac{1}{2N}\mathrm{Tr}(p^i p^i) - \frac{1}{4N}\sum_{ij}\mathrm{Tr}([x^i, x^j]^2) + \frac{1}{N}\mathrm{Tr}(\overline{\psi}\gamma^i[x^i, \psi]). \tag{2.11}$$

The index i now runs from 1 to 5, and the matrices γ are elements of the $SO(5)$ Clifford algebra. Again, the fermions are complex, and obey the relations,

$$\left\{\psi_{A\alpha}, \overline{\psi}_{B\beta}\right\} = \delta_{AB}\delta_{\alpha\beta},$$

where $\alpha = 1, \ldots, 4$. The constraint has a form identical to the previous case (2.9), and the supersymmetry algebra is given by:

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= 0, \\ \{\overline{Q}_\alpha, \overline{Q}_\beta\} &= 0, \\ \{\overline{Q}_\alpha, Q_\beta\} &= 2\delta_{\alpha\beta} H - 2\gamma^i_{\alpha\beta} x^i_A C_A. \end{aligned} \tag{2.12}$$

2.3. Wrapped D-branes. Some of the models described in the previous section have already been realized from wrapped D-brane configurations. Let us begin by considering type IIB string theory, and the case of three-branes wrapped on a collapsing three-cycle. In his study of singularities near conifold points of Calabi–Yau manifolds, Strominger required only a single massless BPS state wrapped on the vanishing cycle [21]. That there should be no bound states has been argued from a somewhat different approach in [22]. The geometry of interest is $R \times S^3$, where the S^3 shrinks to zero size. Clearly, the effective theory on $R \times S^3$ is not N=4 Yang–Mills; such a theory would make little sense. Rather the world-volume theory of a D-brane on a curved space should be described by a topologically twisted theory [23,22]. As the size of the sphere shrinks, only the light degrees of freedom are relevant. The question, in this situation, then concerns the existence of a ground state in the theory obtained from the dimensional reduction of four-dimensional N=1 Yang–Mills, as first mentioned in [2].

We can also check the situation for type IIA, where we perform the same analysis for the case of two-branes wrapped on a vanishing two-cycle. The situation is exactly analogous to the case described above. The geometry is now $R \times S^2$, where the S^2 shrinks to zero size. The only difference involves the number of supersymmetries. The effective theory is now the reduction of N=1 Yang-Mills from six dimensions. Both models were explicitly described in the previous subsection. It seems plausible that other D-brane configurations will realize many, if not all, of the remaining models which we have discussed.

2.4. Gauge invariance. There is a rather nice feature of some of the computations that we shall describe that deserves a separate comment. Whether it provides a hint at how to formulate covariantly M-theory as a matrix model we leave to the judgement of the reader.[2] The gauge-fields in a quantum mechanical gauge theory are non-dynamical. They serve only to enforce the constraint that all states in the Hilbert space be gauge-invariant. How do we enforce such a constraint in the operator formulation? For very high temperatures, the partition function,

$$\mathrm{Z}(\beta) = \int dx \, \mathrm{tr}\, e^{-\beta H}(x, x) \tag{2.13}$$

can be well-approximated by perturbation theory. The notation that we will use throughout the paper may be unfamiliar, and so deserves a comment: we will often consider traces of some operator, say $\mathcal{O}$, which we will denote as,

$$\mathrm{tr}\, \mathcal{O}(x, y),$$

[2] When this section was originally written, the preceding comment seemed most appropriate. Subsequently, there has been an interesting proposal for a non-perturbative definition of the type IIB string, given in [24]. The high temperature limit of the partition function that we describe in this section reduces precisely to the model in that proposal. The relation between M(atrix) theory and the proposal in [24] seems to be in the spirit of a "T-duality" in the time direction.

where by (x, y) we mean the usual propagation of a particle from point y to point x. In an explicit basis of eigenfunctions, $\psi_n(x)$, with eigenvalue λ_n for $\mathcal{O}$, this expression takes the familiar form,

$$\sum_n \lambda_n \psi_n(x)\psi_n(y),$$

where n may index a continuous parameter.

However, in computing the partition function (2.13), it is inconvenient to try to trace over the gauge invariant spectrum of the Hamiltonian, i.e. states $|\psi(x) >$ satisfying $C_A|\psi(x) >= 0$. Our first task is then to implement the projection onto gauge invariant states explicitly, so we can trace over the full, unconstrained spectrum. The gauge constraints, C_A, split into two sets of $SU(N)$ generators: one generates rotations of the x^i, which we shall denote C^b, while the other, C^f, generates rotations of the fermions. Let us denote the operator generating a finite gauge transformation $g(t)$ on the fermions by $\Pi(g(t))$ where we shall drop the explicit dependence on t. To project onto gauge invariant states, we insert:

$$\begin{aligned} \mathrm{Z}(\beta) &= \int_{SU(N)} dt \int dx \,\mathrm{tr}\, e^{it_A C_A}\, e^{-\beta H}(x, x), \\ &= \int_{SU(N)} dt \int dx \,\mathrm{tr}\, \Pi(g)\, e^{-\beta H}(gx, x), \end{aligned} \tag{2.14}$$

where the measure for the $SU(N)$ integration is chosen so that $\int_{SU(N)} dt = 1$. The trace is now over the full Hilbert space, including gauge-variant states.

For small β, we can now construct a reasonable approximation for the propagator,

$$e^{-\beta H}(x, y) = \frac{1}{(2\pi\beta)^{l/2}} e^{-\frac{|x-y|^2}{2\beta}} e^{-\beta V} e^{-\beta H_F} + \ldots, \tag{2.15}$$

where $l = d(N^2 - 1)$ is the dimension of the space of scalars. We shall describe this approximation in somewhat more detail in the following section. The fermion projection operator can be expressed as $\Pi(g(t)) = e^{it_A C^f_A}$, which yields the expression,

$$\mathrm{Z}(\beta) = \int_{SU(N)} dt \int dx \,\mathrm{tr}\, \frac{1}{(2\pi\beta)^{l/2}} e^{-\frac{|x-gx|^2}{2\beta}} e^{-\beta V} e^{-\beta H_F} e^{it_A C^f_A} + \ldots.$$

As $\beta \to 0$, we see that the contribution from group elements away from the identity element is strongly suppressed. Indeed, we can then replace g by $I + i\vec{t}\cdot\vec{C}^b$, and the exponential term involving g becomes,

$$e^{\frac{-|i\vec{t}\cdot\vec{C}^b x|^2}{2\beta}}.$$

The term $i\vec{t}\cdot\vec{C}^b x$ is more transparent when written as $\frac{1}{N}\mathrm{Tr}[t, x^i]^2$, but this is precisely the form of a term in the potential energy, V. Indeed, in this limit, the gauge parameters combine exactly with the remaining coordinates to give a trace which is $SO(d+1)$ symmetric, rather than $SO(d)$ symmetric. Even the fermion projection operator combines naturally with H_F to give a complete symmetry between x^i and t, in the computation of this trace. We shall put this symmetry to good use in subsequent computations. Note that for the case of zero-branes in type IIA, the partition function appears to arise from a manifestly $SO(10)$ invariant Hamiltonian, without any hint of gauge constraints.

3. Counting Ground States

3.1. Defining the index. Ideally, to count the number of normalizable ground states for these models, we would like to compute the low temperature limit of the partition function,

$$\int dx \lim_{\beta\to\infty} \mathrm{tr}\, e^{-\beta H}(x,x).$$

Except for very simple systems, that computation is beyond reach. As usual, we are then interested in counting the number of L^2 ground states weighted by $(-1)^F$, where F is the fermion number. Therefore, we wish to compute the index,

$$\begin{aligned} \mathrm{Ind} &= \int dx \lim_{\beta\to\infty} \mathrm{tr}\,(-1)^F e^{-\beta H}(x,x), \\ &= n_B - n_F, \end{aligned} \tag{3.1}$$

where the trace is over the gauge invariant spectrum of the Hamiltonian. Let us first note that the index is perfectly well-defined. The only way that the index (3.1) could not be counting the net number of ground states is if the Hamiltonian had an extremely pathological low-energy spectrum, i.e. if the density of states diverged badly as $E \to 0$. That is certainly not the case for the models we are studying.

Whether the index is computable is another question entirely. The purpose of this section is to argue that our approach to computing the index actually counts the number of ground states. Before discussing the issues that arise in the non-Fredholm cases, let us discuss in some detail the situation where there is a gap in the spectrum. First, when the spectrum is actually discrete, the twisted partition function is β-independent. In these cases, we can compute the index in the $\beta \to 0$ limit, which reduces to a perturbative computation. The $\beta \to 0$ limit is what we will call the principal contribution to the index. Even in the case where the spectrum is discrete, the principal contribution, which is often computed as an integral over the coordinates, x, can be shifted to a boundary term. To see this, note that we should first perform all our analysis on a ball B_R, where $|x| < R$, and then take a limit $R \to \infty$. We can then write,

$$\begin{aligned} \mathrm{Ind} &= \lim_{R\to\infty} \int_{|x|<R} dx \lim_{\beta\to\infty} \mathrm{tr}\,(-1)^F e^{-\beta H}(x,x), \\ &= \lim_{R\to\infty} \lim_{\beta_o\to 0} \int_{|x|<R} dx \{\mathrm{tr}\,(-1)^F e^{-\beta_o H}(x,x) + \int_{\beta_o}^{\infty} d\beta \frac{\partial}{\partial\beta} \mathrm{tr}\,(-1)^F e^{-\beta H}(x,x)\}. \end{aligned}$$

Now, computing ∂_β of $\mathrm{tr}\,(-1)^F e^{-\beta H}$ brings down H, which we can replace by Q^2. When we try to run Q around the trace,

$$\begin{aligned} \mathrm{tr}\,(-1)^F Q^2 e^{-\beta H} &= -\mathrm{tr}\, Q(-1)^F Q e^{-\beta H} \\ &= -\mathrm{tr}\,(-1)^F Q^2 e^{-\beta H} - \frac{\partial}{\partial x^i} \mathrm{tr}\, e_i (-1)^F Q e^{-\beta H}, \end{aligned}$$

we find that the β variation vanishes up to a total divergence. In this expression, $e_i \frac{\partial}{\partial x^i}$, is the derivative term in the supercharge, Q. If we define e_n to be the fermion in the normal direction to the boundary, then the index can be written as a sum of two terms,

$$\mathrm{Ind} = \lim_{R\to\infty} \lim_{\beta_o\to 0} \{\int_{|x|<R} \mathrm{tr}\,(-1)^F e^{-\beta_o H} + \frac{1}{2} \int_{|x|=R} \int_{\beta_o}^{\infty} d\beta\, \mathrm{tr}\, e_n (-1)^F Q e^{-\beta H}\}. \tag{3.2}$$

At first sight, keeping track of these various limits may seem like a technicality; however, that is not the case. So, rather than continue a general discussion, let us revisit an old friend to see how these manipulations work concretely. A number of the diffculties that arise in the D-particle cases will become clearer.

3.2. The harmonic oscillator revisited. Let us consider a single supersymmetric harmonic oscillator, which has a unique ground state, and a discrete spectrum. The supercharge is given by,

$$Q = \psi_1 p + \psi_2 x,$$

where $\psi_1^2 = \psi_2^2 = 1$ and $\{\psi_1, \psi_2\} = 0$. The Hamiltonian is one-half the square of the supercharge,

$$\begin{aligned} H &= \frac{1}{2}Q^2 \\ &= \frac{1}{2}(p^2 + x^2 - i\psi_1\psi_2). \end{aligned}$$

Now to evaluate the principal term, we can consider the first term in (3.2), we which can write as,

$$\int_{-R}^{R} dx \frac{1}{\sqrt{2\pi\beta_o}} \,\mathrm{tr}\, i\psi_1\psi_2 e^{-\frac{\beta_o}{2}(x^2 - i\psi_1\psi_2)} + \dots,$$

where $(-1)^F = i\psi_1\psi_2$ in this case, and squares to the identity. The omitted terms are suppressed by powers of β. Evaluating the trace on the fermions, or in the equivalent path-integral language, integrating out the fermion zero modes, gives a leading contribution in β_o,

$$\int_{-R}^{R} dx \frac{1}{\sqrt{2\pi\beta_o}} \beta_o e^{-\beta_o x^2/2},$$

which gives,

$$\int_{-R\sqrt{\beta_o/2}}^{R\sqrt{\beta_o/2}} dx \frac{1}{\sqrt{\pi}} e^{-x^2}.$$

If we take β_o to zero faster than R^{-2}, this term vanishes, while if we take β_o to zero more slowly, we obtain the expected answer of one. Whether or not we get a contribution from this term depends on how we choose to take β_o to zero. When this term does not contribute, the second term in (3.2) contributes, and the principal contribution is shifted to a boundary term as we shall see. In this model, the principal contribution is the only contribution to the index.

The boundary term gets two equal contributions from R and $-R$ in this case, and so can be written,

$$\frac{1}{2}\int_{\beta_o}^{\infty} d\beta \,\mathrm{tr}\,(-i\psi_1)(i\psi_1\psi_2) Q e^{-\beta H}\Big|_{x=R}.$$

As R becomes large, the potential term $e^{-\beta V}$ damps the kernel, $e^{-\beta H}$, for large β. We therefore do not need non-perturbative information about the kernel to evaluate this contribution – a small β approximation suffices. Whenever there is a mass gap, we have this nice damping, which is the reason that the index is usually computable.

In this case, evaluating the trace on fermions gives,

$$\int_{\beta_o}^{\infty} d\beta \, x \frac{1}{\sqrt{2\pi\beta}} e^{-\beta x^2/2}\Big|_{x=R},$$

and on rescaling, we obtain:

$$\int_{\beta_o R^2/2}^{\infty} d\beta \, \frac{1}{\sqrt{\pi\beta}} e^{-\beta}.$$

Now we see that this term can contribute if β_o is taken to zero sufficiently quickly with R. Of course, the L^2 index is one regardless of how fast or slowly we choose to take β_o to zero. In the case of a model with potential V homogeneous in x with degree k, a similar argument can be applied. In that case, if we take β_o to zero slower than R^{-k} then the principal contribution is localized to the first term of (3.2), while if take β_o to zero faster than R^{-k}, the second term contributes. In the following section, we shall evaluate the principal contribution for the two D-particle case by letting β_o go to zero more slowly than R^{-4}. This seems computationally simpler than trying to localize the contribution to the boundary.

3.3. Reducing the principal term to quadrature. To evaluate the principal contribution, we have to construct a reasonable approximation to $e^{-\beta H}$. We will not need to alter the usual perturbative construction of the partition function because of the flat points of the potential, V. We start by writing,

$$e^{-\beta H} = \frac{1}{2\pi i} \int_\gamma e^{-\beta z} \frac{1}{H - z} dz,$$

where γ is a contour enclosing the spectrum of H. Let us consider the generic situation away from the flat points. We can approximate $(H - z)^{-1}$ by a perturbation series,

$$\frac{1}{H - z}(x, y) = \int \frac{e^{ik\cdot(x-y)}}{(k^2/2 + V - z)} (1 - \frac{H_F}{(k^2/2 + V - z)} + \dots), \tag{3.3}$$

where the first correction, proportional to H_F, is shown, and subsequent terms are constructed iteratively in powers of $(k^2/2 + V - z)^{-1}$. The corresponding propagator takes the form,

$$e^{-\beta H}(x, y) = \frac{1}{(2\pi\beta)^{l/2}} e^{-\frac{|x-y|^2}{2\beta}} e^{-\beta V} e^{-\beta H_F} + \dots, \tag{3.4}$$

where $l = d(N^2 - 1)$ is the dimension of the space of scalars. This approximation is reasonable for small β. The omitted terms which correct this approximation appear with a higher power of $(k^2/2 + V - z)^{-1}$, in (3.3), and consequently give rise to terms suppressed by powers of β in (3.4). This approximation then suffices for evaluating the first term in (3.2), where we choose to take β to zero more slowly than R^{-4}. Substituting the leading approximation for the propagator gives,

$$\begin{aligned} &\lim_{R\to\infty} \lim_{\beta\to 0} \int_{|x|<R} \int_{SU(N)} dt \, \mathrm{tr}\, (-1)^F e^{-\beta H} \Pi(g) (gx, x) \\ &= \lim_{R\to\infty} \lim_{\beta\to 0} \int_{|x|<R} \int_{SU(N)} dt \, \mathrm{tr}\, (-1)^F \frac{1}{(2\pi\beta)^{l/2}} e^{\frac{-|i\vec{t}\cdot\vec{C}^b x|^2}{2\beta}} e^{-\beta V} e^{-\beta H_F} e^{it_A C^f_A} + \dots, \end{aligned}$$

where we have approximated the group element by $I + i\vec{t}\cdot\vec{C}^b$, for reasons explained in the previous section. This amounts to localizing the integral over the gauge group to a small neighborhood of the identity – a slight twist on the usual localization to the minima of

the potential. As usual, the inclusion of $(-1)^F$ forces us to absorb fermion zero modes. In our trace, $(-1)^F$ is realized as the volume form for the Clifford algebra (2.5). There are two sources for fermions: the first is from $e^{-\beta H_F}$, while the second source is the fermion projection operator, $e^{it_A C^f_A}$, inserted into the trace. After writing the fermion term, $it_A C^f_A - \beta H_F$, as $\psi M \psi$ for some matrix M, the trace over the Clifford factors gives the Pfaffian of M, which is a polynomial in x and t. On rescaling the integral, we obtain,

$$\int dt \int dx \, \frac{1}{(2\pi)^{l/2}} e^{-|i\vec{t}\cdot\vec{C}^b x|^2/2} e^{-V(x)} \, \mathrm{Pf}(M), \tag{3.5}$$

where the integration region for t is now $\mathbf{R}^{N^2-1}$, while for x, the region is $\mathbf{R}^{d(N^2-1)}$. As will be clear from the subsequent explicit computation, the Pfaffian is of definite sign when d is odd, and the integral is thus non-vanishing. However, it is far from clear that this term yields an integer, and indeed, it generally is not integral. Therefore, there had better be a non-vanishing correction term.

We stress again that it is very natural to consider t on equal footing with the coordinates x^i. Let us denote t by x^0, and define γ^0 to be iI, where I is the identity matrix. The coordinates x^i_A now form an $(N^2-1)\times(d+1)$ matrix. In this notation, the matrix takes the form $M = -(i/2) f_{ABC} x^i_B \gamma^i$, and the integral admits an $SO(d+1)$ symmetry which we shall use in section four to compute explicit values for this term in the two-particle case.

3.4. The non-Fredholm case. When the Hamiltonian under consideration has continuous spectrum, the twisted partition function is generally β-dependent. The heuristic reason for the β-dependence is that the density of states for the bosonic and fermionic scattering states can differ. Supersymmetry pairs bosonic and fermionic modes, but does not necessarily preserve the spectral density. In these cases, the principal contribution to the index is not necessarily integer, and there must be an additional contribution from the second term in (3.2). In the case where there is a mass gap, this contribution can be perturbatively evaluated. What happens in the case where there is no mass gap?

Let us choose a real supercharge, Q, which squares to the Hamiltonian up to a gauge transformation. For this discussion, we will set the gauge constraints to zero. Q is then a self-adjoint elliptic first-order operator, which anti-commutes with our $\mathbf{Z}_2$ involution, $(-1)^F$. Let us define Q_+ as the restriction of Q to the +1 eigenspace of $(-1)^F$, i.e. the bosonic states. It may be helpful to think of Q as a matrix,

$$\begin{pmatrix} 0 & Q_+^* \\ Q_+ & 0 \end{pmatrix},$$

where $Q_+^* = Q_-$ is the restriction of Q to fermionic states. The computation that we need to perform is the calculation of the L^2 index of Q_+. This operator, though elliptic, is not Fredholm. Recall that a Fredholm operator, by definition, has a finite-dimensional kernel and cokernel. The fact that the continuous spectrum of $H = Q^2$ contains scattering states with arbitrarily small energies implies that the image of Q_+ is not closed, and the cokernel is infinite-dimensional, and distinct from the kernel of Q_-. Hence, we take the L^2 index of Q_+ to be the dimension of $\mathrm{Ker}(Q_+) \cap L^2$ minus the dimension of $\mathrm{Ker}(Q_+^*) \cap L^2$, which is not, in this case, the dimension of the kernel minus the dimension of the cokernel of Q_+.

Let us consider how to compute this index. Suppose that there exists a Green's function, G, for Q^2; i.e. a self-adjoint singular integral operator G which annihilates

the kernel of Q^2 and acts as $(Q^2)^{-1}$ on the orthogonal complement of the kernel. By definition, G obeys,

$$Q^2G = I - P,$$

where P denotes the orthogonal projection onto the kernel of Q^2. So, P then annihilates all states which are not zero-energy. We recall that by singular integral operator we mean an operator which is obtained by integrating against a matrix-valued kernel, $g(x, y)$, which has a well-understood singularity along the diagonal $x = y$. For example, the inverse of the Laplacian in three dimensions is the familiar kernel, $\sim |x - y|^{-1}$. Let us denote the restriction of G to the +1 eigenspace of $(-1)^F$ by G_+, where:

$$G_+ = G(I + (-1)^F)/2.$$

Let $P_\pm$ denote the orthogonal projection onto the L^2 kernel of $Q_\pm$. Then $QQG_+ = (I - P)(I + (-1)^F)/2 = (I + (-1)^F)/2 - P_+$, and similarly, using the fact that Q anti-commutes with $(-1)^F$ and commutes with G, we have $QG_+Q = (I - (-1)^F)/2 - P_-$. Therefore, the index of Q_+ can be expressed as,

$$\begin{aligned}
\mathrm{Ind}Q_+ &= \mathrm{tr}P_+ - \mathrm{tr}P_- \\
&= \mathrm{tr}[(I + (-1)^F)/2 - QQG_+] - \mathrm{tr}[(I - (-1)^F)/2 - QG_+Q] \\
&= \mathrm{tr}((-1)^F - [Q, QG_+]) \\
&= \mathrm{tr}((-1)^F + [Q, (-1)^F QG/2]).
\end{aligned}$$

Of course, the difficulty is that we cannot construct G explicitly – even the claim that G is represented by a singular integral operator requires some justification, which we will give shortly. This difficulty can be summarized in the following way: given a set of eigenstates, $\psi_{E\alpha}$, with eigenvalue E under Q^2, the inverse is formally,

$$(Q^2)^{-1} = \sum_{E\alpha, E>0} E^{-1}\psi_{E\alpha}(x)\psi_{E\alpha}(y),$$

where the sum may be over continuous indices. When the continuous spectrum is not bounded away from zero, it is not clear that the resulting sum converges to a function in any reasonable sense, since G is an unbounded operator on L^2 wavefunctions in this case. However, if the scattering states do not pile up at low-energies, then it should be intuitively reasonable that G is still nice, as, for example, in the free-particle case. We will see this later by realizing G as a limit of bounded singular integral operators G_w. Physically, this limiting procedure is equivalent to adding a mass term to the propagator, and taking the limit where the mass vanishes. The problem we are decribing is a common one in any theory with massless particles. So, let us proceed along the usual path by constructing an explicit approximation W to G for which we can compute the trace,

$$\mathrm{tr}((-1)^F + [Q, (-1)^F QW/2]).$$

We must then verify that for a carefully constructed W this trace is the same as the one computing the index of Q_+.

Our approximation will have the property that,

$$Q^2W = I - E,$$

for some compact error term E given by integrating against a matrix-valued kernel, $e(x, y)$. We also want $e(x, y)$ to decay polynomially in $(|x| + |y|)$ to a sufficiently high power which we need to determine, and study in greater detail later.

Let us describe what data is needed in order to insure that

$$\mathrm{tr}((-1)^F + [Q, (-1)^F QW/2])$$

computes the index. This is equivalent to the vanishing of $\mathrm{tr}[Q, (-1)^F Q(G-W)/2]$, which is the difference between $\mathrm{tr}((-1)^F + [Q, (-1)^F QG/2])$, which computes the index, and $\mathrm{tr}((-1)^F + [Q, (-1)^F QW/2])$, which we hope computes the index.

For any integer m, the operator W can be constructed so that the kernel for $G - W$ has m continuous derivatives. Let χ_R be the characteristic function of a ball B_R of radius R. The characteristic function is defined to be one on the ball and zero elsewhere. We will throw χ_R into the various traces to serve as a cut-off on the infra-red physics. By a similar argument to the one used in Sect. 3.1, we may use the divergence theorem to transform $\mathrm{tr}\chi_R[Q, (-1)^F Q(G-W)/2]$ into an integral over the boundary of B_R. Therefore, if we can show that $(-1)^F Q(G-W)(x,x)$ is decaying sufficiently rapidly at large $|x| = R$, we deduce that $\mathrm{tr}\chi_R[Q, (-1)^F Q(G-W)/2]$ converges to zero, and the index can be computed by replacing G by our approximation, W.

To prove that $Q(G-W)$ decays sufficiently quickly, we will need to use an argument that may be unfamiliar to the reader which establishes a correspondence between asymptotic estimates for Q^2 and decay rates for solutions ψ to $Q^2\psi = F$, where F satisfies some growth constraint. In order to orient the reader, let us first examine what the argument says in a much simpler case. Consider the differential equation in one variable r, on $(0, \infty)$,

$$(-\frac{d^2}{dr^2} + w^2)f = g,$$

where w is some constant. For this equation, a weak form of our general argument below says that if $e^{awr} g \in L^2$, for some $a \in (-1, 1)$, then we may conclude that $e^{awr} f$ is normalizable, if f satisfies the growth constraint $e^{-bwr} f \in L^2$, for some $b < 1$. The condition on the growth of f is clearly necessary to rule out the addition of the non-normalizable e^{rw} to any solution. In this simple case, we can prove this result by a direct integration. We will use the following analogous result, which is established in much greater generality in [25].

Suppose that for some positive constant c and some compact set K, we have an estimate of the following form:

$$\|Qf\|^2 \geq \|cf/r\|^2,$$

for all wavefunctions, f, which vanish outside K. With these assumptions, if $Q^2 F = e$, with $r^c e \in L^2$, and if F satisfies the growth constraint that F/r^{c-s} be normalizable for some positive s, then

$$\frac{r^{c-1}}{\ln(r)^{1+\epsilon}} F \in L^2,$$

for all positive ϵ. Also,

$$\frac{r^c}{\ln(r)^{\epsilon}} QF \in L^2,$$

for all positive ϵ. In the familiar case where we have c instead of c/r, we would obtain exponential decay as in the one-dimensional example. The reason we have a weaker decay rate in this example is that the decay is roughly no worse than $e^{-\psi}$, where ψ is a function with $|d\psi|^2$ less than the asymptotic lower bound for Q^2, which is c^2/r^2 in this case, and w^2 in the one-dimensional example.

For purposes of illustration, let us sketch a proof of these growth estimates. For any function u supported in the complement of K, integration by parts yields

$$0 = (Q^2F, u^2F) = \|QuF\|^2 - \|[Q,u]F\|^2.$$

The assumed estimate implies that,

$$\|cuF/r\|^2 \geq \|[Q,u]F\|^2.$$

From this formal inequality, we can deduce the normalizability of cuF/r when $cu/r > |[Q,u]|$. More accurately, we can deduce the integrability of $(c^2u^2/r^2 - |[Q,u]|^2)F^2$. Let us define a cutoff function ρ which is identically one outside a large ball K' containing K in its interior, and vanishing in K. Then taking u to be $r^c/\ln(r)^\epsilon$ times the cutoff function ρ gives formally,

$$\|cr^{c-1}F/\ln(r)^\epsilon\rho\|^2 \leq \|(r^{c-1}(c-\epsilon/\ln(r))\rho + r^c\rho')/\ln(r)^\epsilon F\|^2.$$

Collecting terms, we obtain:

$$\int r^{c-2}(2c\epsilon - \epsilon^2/\ln(r))|F|^2/\ln(r)^{1+\epsilon}\rho\|^2 \leq c_{K'}\int_{K'}|F|^2,$$

for some constant $c_{K'}$, which depends on K' and c. A limiting argument, approximating r by a sequence of bounded functions can be used to obtain from this formal inequality the boundedness of the left side, when $|F|$ is bounded on compact sets.

We will need the following variant of this inequality. Suppose that

$$\tilde{Q}^2 = -\frac{\partial^2}{\partial r^2} - \frac{(2d-1)}{r}\frac{\partial}{\partial r} + w,$$

where w is now a positive operator with $w \geq c^2/r^2$. Suppose that $\tilde{Q}^2F = 0$ in the complement of a compact set K. For some k to be determined, consider the point where r^kF is maximum. By the maximum principle, we have at the maximium that, $F' = -kF/r$ and $(r^kF)'' \leq 0$, where:

$$\begin{aligned}(r^kF)'' =& k(k-1)r^{k-2}F + 2kr^{k-1}F' + r^kF''\\ \geq& k(k-1)r^{k-2}F - 2k^2r^{k-2}F - (2d-1)r^{k-1}F' + r^{k-2}c^2F\\ =& (2d-k-2+c^2/k)kr^{k-2}F.\end{aligned}$$

If we choose $0 < k < 2(d-1)$, we may deduce that if the maximum exists it must occur in K. If Fr^a is bounded for any positive a, we may deduce that Fr^k is in fact bounded by its values on K for all k with $k < 2d + c^2/k - 2$. In our applications, c^2 will usually be the first or second eigenvalue of the standard spherical Laplacian, which is 0 or $(d-1)$. We recall that the standard Laplace operator on S^{d-1} has eigenvalues, $k(d+k-2)$, with $k = 0, 1, 2, \ldots$, where the multiplicity of each eigenvalue is, $\frac{(2k+d-2)(k+d-3)!}{k!(d-k)!}$. In the first case, we see that we get r^{d-2} decay; in the second we see that for d at least 3, we get faster than r^{2-2d} decay. These estimates extend easily to the case when $\tilde{Q}^2F = e$, where e satisfies the condition that $r^{2d}e$ is bounded. We will apply this to the case when $\tilde{Q}^2 = Q_x^2 + Q_y^2$, the hamiltonian in the first and second variables and F will be a kernel constructed from the difference between the Green's function and W. This argument formalizes the observation that the product of two elements of the kernel of H should

decay twice as fast as a single element of the kernel, and $G - W$ looks like such a product.

We will show that,

$$Q^2 = \Delta_r + u/r^2,$$

acting on wavefunctions supported outside a large compact set; here r is the distance along the flat direction, Δ_r is the radial part of the Euclidean Laplacian in the flat directions, and the operator u is semi-positive with first eigenvalue greater than the first or second eigenvalue of the spherical Laplacian. We can then apply the above argument to deduce that $G - W$ decays like r^{2-2d}. We can improve this estimate by observing that we get the second eigenvalue when the operator is restricted to wavefunctions with odd parity, and the first eigenvalue for the restriction to even parity wavefunctions. Split $(G - W)$ into its even and odd components. The odd component of $(G - W)$ decays faster than r^{1-d} by the preceding discussion, and it is easy to see that applying Q only improves the decay rate. To see this, use the inequality for $Q^2F = 0$,

$$\|wQF\| \leq 2\|[Q, w]F\|,$$

and choose w appropriately. For the even component of $G - W$, we have that its image under Q is odd; therefore, we again have the improved decay rate determined by the second eigenvalue of the sphere. This estimate requires that $(G - W)r^a$ be bounded for some positive a and that G be given as a singular integral operator. The last condition is needed to ensure that $(G - W)$ is a smooth function. We will not prove these results in detail but will merely sketch how they follow from the same sequence of ideas we have introduced. One considers first instead of G and W corresponding operators G_w and W_w, where G_w is $(Q^2 + w)^{-1}$ restricted to the orthogonal complement of the kernel of Q^2, etc. It is easy to get the desired initial r^a boundedness for $G_w - Q_w$ for each $w > 0$. One then can get the desired growth bound; i.e. we show that the max is controlled by an estimate on the compact set. We then allow w to tend to zero to get the desired result for $G - W = \lim_{w\to 0}(G_w - W_w)$. This type of argument can also be used to show that G is a singular integral operator.

This then establishes the desired decay estimate given one: a construction of W which leads to sufficiently small E; that is $|e(x, y)| \leq (|x| + |y|)^{-d-1}$, and two: a demonstration of the claimed asymptotic lower bound for Q^2. Under these two conditions, $\mathrm{tr}((-1)^F + [Q, (-1)^F QW/2])$ computes the index. We now turn to the evaluation of $[Q, (-1)^F QW/2]$, which we need to boil the problem down to a concrete computation.

It will be convenient to arrange the construction of W so that on a large compact set, say a ball B_R, its contribution to the index can be computed by the standard principal term computation described in the previous subsection. On the complement of this set we will need to use special coordinates to find a nice expression for W. This requires us to define an approximation A to QG rather than the approximation W to G, but since, off a compact set, A will clearly be of the form QW, this will not affect our preceding discussion. The use of two separate constructions for W in different regions may seem, perhaps, a bit unnatural when dealing with Euclidean space. It is forced on us, in part, by the need to obtain very good control of the error term E as $|x|$ tends to ∞. This rules out the use of the local computation used in B_R and described previously. Moreover, the special coordinates we use in the complement of the compact set, like polar coordinates, become singular at the origin. Therefore, we will need to use two sets of cutoff functions to patch the two approximate inverses together.

Let $\rho_{n,j}(x)$ be a sequence of cutoff functions which approach $\chi_{jR}(x)$, and set $\rho_n = \rho_{n,1}$. Let W' be our approximate Green's function near ∞, which we will construct in section five. The operator $\int_0^{\beta_0} d\beta\, Qe^{-\beta Q^2}(x,y)$ is the standard kernel that we will use in B_R. We can create a global approximation to QG by defining:

$$A(x,y) = \rho_n(x)\int_0^{\beta_0} d\beta\, Qe^{-\beta Q^2}(x,y)\rho_{n,2}(y) + (1-\rho_n(x))QW'(x,y)(1-\rho_{n,1/2}(y)).$$

The cutoff functions on the left of each operator are inserted to average the two operators. The cutoff functions on the right, however, are inserted so that the operators are localized to the domains where they are well-defined, and satisfy the desired estimates. The right cutoffs are of course chosen to be identically one on the support of the left cutoffs; otherwise, they would destroy the averaging effected by the left cutoffs. This is the reason for the second index on ρ.

Then to evaluate $[(-1)^F A, Q]$, we write:

$$\begin{aligned}
[(-1)^F A, Q] =& [Q,\rho_n](-1)^F \int_0^{\beta_0} d\beta\, Qe^{-\beta Q^2}\rho_{n,2} - [Q,\rho_n](-1)^F QW'(1-\rho_{n,1/2}) \\
&- \rho_n(-1)^F(I - e^{-\beta_0 Q^2})\rho_{n,2} + (1-\rho_n)(-1)^F(I-E)(1-\rho_{n,1/2}) \\
&+ (1-\rho_n)(-1)^F Q[Q,W'](1-\rho_{n,1/2}) + \dots, \\
=& [Q,\rho_n]\int_0^{\beta_0} d\beta\, Qe^{-\beta Q^2} - [Q,\rho_n](-1)^F QW' \\
&- (-1)^F(I - \rho_n e^{-\beta_0 Q^2}) + (1-\rho_n)(-1)^F(-E) \\
&+ (1-\rho_n)(-1)^F Q[Q,W'] + \dots,
\end{aligned}$$

where the omitted terms are terms that trace to zero. We will construct W' in section five so that the trace of $(1-\rho_n)(-1)^F(-E) + (1-\rho_n)(-1)^F Q[Q,W']$ will tend to zero as n tends to ∞. Thus subtracting off the $(-1)^F I$ term we are left to compute,

$$\mathrm{tr}[Q,\rho_n]\int_0^{\beta_0} d\beta\, Qe^{-\beta Q^2} - [Q,\rho_n](-1)^F QW' + (-1)^F \rho_n e^{-\beta_0 Q^2}.$$

The last term is the principal term which is given by evaluating the integral (3.5). Taking the limit as n tends to ∞, the two commutator terms converge to the boundary traces,

$$\int_{|x|=R}\left(\mathrm{tr}\, e_n \int_0^{\beta_0} d\beta\, Qe^{-\beta Q^2} - \mathrm{tr}\, e_n(-1)^F QW'\right).$$

Choosing β_0 to go to zero more slowly than R^{-4}, the integral,

$$\int_{|x|=R} \mathrm{tr}\, e_n \int_0^{\beta_0} d\beta\, Qe^{-\beta Q^2},$$

decomposes into two pieces. One is associated with a small neighborhood of the flat regions, which consist, say, of all points of distance at most one from a flat point. The other contribution is associated with the complementary region, i.e. almost all of the sphere of radius R. It is not difficult to show that the contribution from the flat region is squeezed to zero as R tends to ∞. The contribution from the complementary region is not

vanishing. Using similar arguments to those presented earlier in this section, it is possible to show without extensive computation that this term exactly cancels the principal term. Standard constructions give a W' whose contribution from the complementary region of tr $e_n(-1)^F QW'$ also exactly cancels the contribution of $\int_{|x|=R}$ tr $e_n \int_0^{\beta_0} d\beta\, Qe^{-\beta Q^2}$. Therefore, the total contribution of this boundary integral to the index comes from,

$$-\frac{1}{2}\int_{N_F(R)} \text{tr}\, e_n(-1)^F QW', \tag{3.6}$$

where $N_F(R)$ is a small neighborhood of the flat points on the boundary of the space, which is a sphere of radius R. We will show in section five that this integral converges to $-1/4$ in the two-particle case. In summary, the additional contribution to the index from the boundary is computed by evaluating (3.6), which is localized to the flat directions. The sum of (3.6) and the usual principal term from (3.5) must then be integer.

4. Two-Particle Binding

4.1. Symmetries and pfaffians. The simplest case to consider is $N = 2$; already, however, the integral (3.5) is thirty-dimensional for type IIA zero-branes! We shall have to use the various symmetries available to us to simplify the computation of the principal term. Recall that the coordinates x_A^i now form a $3 \times (d + 1)$ matrix. The integral (3.5) is invariant under the symmetry, $x \to gxh$, where g is an element of $SO(3)$ acting on the left while h is an element of $SO(d + 1)$ acting from the right on x. Note that the left action is a gauge transformation.

By using these symmetries, we shall rotate x into a special form,

$$\begin{pmatrix} b_1 & 0 & 0 \cdots & & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & 0 & \cdots 0 \end{pmatrix}, \tag{4.1}$$

and reduce our integral to one over only three variables.

In these special coordinates, the potential (including the gauge parameters) takes the special form,

$$\widetilde{V} = -\frac{1}{2}(b_1^2 b_2^2 + b_1^2 b_3^2 + b_2^2 b_3^2).$$

We shall now evaluate Pf(M) by evaluating the determinant of M. For the moment, let us return to general coordinates where $M = -(i/2) f_{ABC} x_B^i \gamma^i$. For convenience, let us denote $x_A^i \gamma^i$ by x_A. The matrix then takes the form,

$$M = \frac{i}{2}\begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}.$$

By row manipulations, or equivalently, by studying the eigenvalue equation, we find that the determinant can be expressed as,

$$\det(M) = \frac{1}{2^{3(d-1)}} \det(x_1 x_2 x_3) \det(1 - x_1 x_2^{-1} x_3 x_1^{-1} x_2 x_3^{-1}),$$

where $x_A^{-1} = (x_A - 2ix_A^0 I)/|x_A|^2$. After rotating x into our convenient set of coordinates (4.1), we can compute this determinant, and on taking the square root obtain,

$$\mathrm{Pf}(M) = \frac{1}{2^{2(d-1)}}(b_1 b_2 b_3)^{d-1}.$$

We can immediately see that when d is odd, the Pfaffian is an even function of the variables in both special and general coordinates, and the corresponding integral (3.5) is non-vanishing.

The last ingredient that we require to compute the integral (3.5) is the measure for our simplified coordinates (4.1). We shall obtain the measure by gauge-fixing the integral (3.5) using the Faddeev-Popov approach. Let us take x' to be,

$$\begin{pmatrix} b'_1 & 0 & 0 \cdots & & 0 \\ 0 & b'_2 & 0 & \cdots & 0 \\ 0 & 0 & b'_3 & 0 & \cdots 0 \end{pmatrix}.$$

We shall insert one into the integral (3.5) in the form,

$$\int db' dg dh\, \delta(x' - gxh) f(b),$$

where we have to determine $f(b)$. For some g_0 and h_0, $x_0 = g_0 x h_0$ takes the form (4.1). The integrals over g and h then reduce to integrals in a small neighborhood of g_o and h_o, with the exception of the $SO(d-2)$ subgroup of $SO(d+1)$ that leaves the form (4.1) invariant. If T is a generator for the left $SO(3)$ action, and R a generator of the right $SO(d+1)$ action which does not leave (4.1) invariant, then we can replace integration over g, h by,

$$\eta(d)\,\mathrm{vol}(SO(d-2)) \int db' dT dR\, \delta(x' - x_o - Tx_o - Rx_o) f(b).$$

The remaining integrals are straightforward, and we find that,

$$f(b) = \frac{1}{\eta(d)\,\mathrm{vol}(SO(d-2))}(b_1 b_2 b_3)^{d-2}|(b_1^2 - b_2^2)(b_1^2 - b_3^2)(b_2^2 - b_3^2)|.$$

The integrals over b are constrained such that $b_1 > b_2 > b_3$. The symmetry factor $\eta(d)$ is 4 for $d > 2$, but is 2 for $d = 2$ because the left and right symmetry groups are then both $SO(3)$. The value of the symmetry factor can also be checked by computing a Gaussian integral in the d-dimensional model, and comparing the result to the answer obtained using the measure for these special coordinates.

Finally, inserting one in this form into the integral (3.5), and integrating over x, g, h gives for $d > 2$,

$$\frac{1}{\mathrm{vol}(SO(3))}\frac{1}{(2\pi)^{3d/2}} \int db\, \frac{\mathrm{vol}(SO(d+1))\mathrm{vol}(SO(3))}{4\,\mathrm{vol}(SO(d-2))}(b_1 b_2 b_3)^{d-2}|(b_1^2 - b_2^2)(b_1^2 - b_3^2)\times$$

$$(b_2^2 - b_3^2)|\frac{1}{2^{2(d-1)}}(b_1 b_2 b_3)^{d-1} e^{\widetilde{V}}.$$

The first factor of $1/\mathrm{vol}(SO(3))$ comes from the normalization of the integration over the gauge group, where we recall that we chose the normalization so that $\int_{SU(2)} dt = 1$ prior to rescaling.

4.2. Computing the principal contribution. Now there is a nice change of variables that will allow us to evaluate this integral. Set: $y_1 = b_2 b_3/\sqrt{2}, y_2 = b_1 b_3/\sqrt{2}, y_3 = b_1 b_3/\sqrt{2}$, and the integral becomes,

$$\begin{aligned}
&\int db\,(b_1 b_2 b_3)^{d-2}|(b_1^2 - b_2^2)(b_1^2 - b_3^2)(b_2^2 - b_3^2)|\frac{1}{(2\pi)^{3d/2}}\frac{1}{2^{2(d-1)}}(b_1 b_2 b_3)^{d-1}e^{\widetilde{V}} \\
&= \int dy(y_1 y_2 y_3)^{d-3}|(y_1^2 - y_2^2)(y_1^2 - y_3^2)(y_2^2 - y_3^2)|\frac{1}{2}\frac{1}{2^{2(d-1)}}\frac{1}{\pi^{3d/2}}e^{-y_1^2+y_2^2+y_3^2} \\
&= \int_{\mathbf{R}^{3d}} dx e^{-|x|^2}\frac{1}{2}\frac{1}{2^{2(d-1)}}\frac{1}{\pi^{3d/2}}\frac{\eta(d-1)\,\mathrm{vol}(SO(d-3))}{\mathrm{vol}(SO(d))\mathrm{vol}(SO(3))} \\
&= \frac{1}{2}\frac{1}{2^{2(d-1)}}\frac{\eta(d-1)\,\mathrm{vol}(SO(d-3))}{\mathrm{vol}(SO(d))\mathrm{vol}(SO(3))}.
\end{aligned}$$

Lastly, we must multiply the result by the value of $\mathrm{tr}(I)$ from the trace over the fermions, which gives an extra factor $2^{3(d-1)}$. The net result is the formula:

$$P = 2^{d-2}\frac{\eta(d-1)\,\mathrm{vol}(SO(d+1))\mathrm{vol}(SO(d-3))}{\eta(d)\,\mathrm{vol}(SO(d-2))\mathrm{vol}(SO(d))\mathrm{vol}(SO(3))}, \tag{4.2}$$

for the principal contribution, P, for d odd, where we recall that $\mathrm{vol}(SO(n)) = \mathrm{vol}(S^{n-1})\cdots\mathrm{vol}(S^1)$, and that $\mathrm{vol}(S^n) = 2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2})$. Let us conclude this discussion by listing the explicit values in the following table:

Table 1. The principal contribution to the index

Dimension	Principal contribution
3	1/4
5	1/4
9	5/4

5. The Propagator for Well-Separated Branes

5.1. Some general comments. We have determined in the previous section that the principal contribution to the index is fractional. Since the index must be integer, there is a missing contribution. The manner in which this contribution arises is quite surprising, and involves a bizarre conspiracy of cancellations. Let us outline the procedure we will follow before presenting a detailed discussion. We will construct an approximation to the propagator for the two zero-branes when they are far apart. The approximation will be sufficiently good in the sense that any corrections will not contribute to the boundary term (3.6). At long distances, the only states that make a sizable contribution to the propagator are those localized along the flat directions of the potential. The simplest approximate description of the physics governing the light degrees of freedom is in terms of free particle propagation along the flat directions. This is immediately modified when we try to "integrate" out the massive modes. Let us label coordinates for the d_a massive directions by y. Then for example, the action of ∂_r^2, which is part of the Laplacian

(2.3), on the wavefunction for the flat direction is modified because of its action on the harmonic oscillator ground state,

$$\partial_r^2 \left(\frac{r}{\pi}\right)^{d_a/4} e^{-r|y|^2/2} = \left(\frac{r}{\pi}\right)^{d_a/4} e^{-r|y|^2/2} \left(\partial_r^2 + \left[\frac{d_a}{2r} - |y|^2\right]\partial_r + \frac{d_a(d_a-4)}{16r^2} - \frac{d_a}{4r}|y|^2 + \frac{|y|^4}{4}\right).$$

Each of the terms appearing on the right is of order $1/r^2$, and so none can a priori be neglected. In a similar way, the rest of the terms in the Hamiltonian modify the long distance behavior. This includes the $O(y^4)$ terms in the potential which we recall is of the form, $V \sim r^2|y|^2/2 + O(y^4)$. Since we need to include the $O(y^4)$ terms, this approximation is not one-loop in the usual sense. We will need to sum up all the corrections to free propagation, which are of order $1/r^2$, and surprisingly, they all cancel. The remaining index computation then involves free particle propagation on the moduli space which is $\mathbf{R}^{(d_c-2)}/\mathbf{Z}_2$. From this computation, we will recover the needed corrections to the index.

The construction that we shall describe is to be contrasted with the kind of effective action for the light modes that has been obtained using large-distance low-velocity expansions [7,11]. After integrating out the massive modes in a one-loop approximation, the leading correction to the effective Lagrangian at large r is a term of order $\sim v^4/r^7$ for the case of the nine-dimensional model. The connection between that approach, and the computations that we shall describe does not seem transparent. It seems possible that exploring the connection in detail will give insight, and dare we hope a proof, of the desired non-renormalization theorem for the F^4 term. Computing the leading corrections to the three and five-dimensional models also seems an interesting question. Since the amount of supersymmetry is reduced, we might suspect that there is a correction to the metric on the moduli space. However, in constructing the propagator using this approach, we do not find any fundamental difference between the three cases.

Let us start by discussing the form of the various operators that we need to study in special coordinates. Without rotating to a convenient set of coordinates, it will be very difficult to say anything about the structure of the partition function. We can rotate our coordinates, x, into a convenient basis by using a combination of gauge and flavor symmetries. So, we can choose a basis,

$$x = k\Lambda q,$$

with $k \in SO(3)$, $q \in SO(d)$ and Λ the following $3 \times d$ matrix,

$$\Lambda = \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & y_2^2 & \cdots & y_2^d \\ 0 & y_3^2 & \cdots & y_3^d \end{pmatrix}. \tag{5.1}$$

We have set $\Lambda_1^1 = r$, and let y denote the remaining $2 \times (d-1)$ matrix, with $\Lambda_A^i = y_A^i$, for $i, A > 1$. The reason for this choice is that the flat directions are now at the locus, $y = 0$. Note that the choice of k and q is not unique here. So the mapping,

$$m(k, \Lambda, q) \to k\Lambda q,$$

projects $SO(3) \times \mathbf{R}^{2d-1} \times SO(d)$ onto our space of matrices, x, but is clearly not one-to-one. The fibers of the map, m, are non-trivial. Any function that depends on

x can then be lifted to a function in the product space, which is constant under those transformations of k, Λ, q that leave x invariant. Most of our computations will focus on the neighborhood, N_F, of a flat point given by $|y|^2 < 1$, and r tending to ∞. This is the region that contributes to (3.6).

We need to write the Hamiltonian in terms of these new coordinates, which include the $d+1$ angular variables parametrizing the $d+2$-dimensional flat directions. The kinetic terms in the Hamiltonian can be determined by computing the Laplacian for the metric associated to this coordinate choice. Let us recall that given a metric, g, the Laplacian is given by:

$$\Delta = -\frac{1}{\sqrt{|g|}} T_i \left(\sqrt{|g|} g^{ij} \right) T_j,$$

where $|g|$ is det(g), and the T_i are a basis of vector-fields for some coordinate system. First, we need to make a choice of basis of vector-fields. For the coordinates, (5.1), it is natural to have $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial y^i_B}\}_{2 \le i \le d, B>1}$ as part of the basis. We need $d+1$ additional vector-fields. From the left $SO(3)$, we can choose the two vector-fields, $\{X_2, X_3\}$ which are associated to the two $SO(3)$ generators,

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

respectively. Similarly, we can add the $d-1$ vector-fields $\{V_j\}_{j>1}$, associated to the right $SO(d)$ generators, $z(j)$. The matrix $z(j)$ is a $d \times d$ anti-symmetric matrix with only one positive entry, $z(j)_{1j} = 1$. Our total basis is then composed of the subset of tangent vectors to the product space, $SO(3) \times \mathbf{R}^{2d-1} \times SO(d)$, given by $\{X_2, X_3\} \cup \{\frac{\partial}{\partial r}, \frac{\partial}{\partial y^i_B}\}_{2 \le i \le d, B>1} \cup \{V_j\}_{j>1}$.

We need to determine the metric, g, for this coordinate choice. The set of vector-fields, $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial y^i_B}\}$ are orthonormal and orthogonal to the rest of the basis. The rest of the basis have inner products,

$$(X_j, X_k) = r^2 \delta_{jk} + (yy^t)_{jk},$$
$$(V_j, V_k) = r^2 \delta_{jk} + (y^t y)_{jk},$$

and,

$$(X_j, V_k) = 2r y_{jk}.$$

These inner products can be determined by pushing forward the vector-fields under m, and computing the resulting norms. Now the metric can be written as a direct sum of two metrics, $g = g' \oplus g''$, where g'' is the identity matrix for the coordinates corresponding to (r, y). The interesting part of the metric is the part for the angular variables. So, let us write, $g' = r^2 I + K$, where K is determined from the above inner products. Then $(g')^{-1} = I/r^2 - K/r^4 + \ldots$, where the omitted terms are suppressed by more powers of r. To compute the Laplacian, we need:

$$\begin{aligned} \text{logdet}(g) = \text{logdet}(g') &= \text{trlog}(r^2 I) + \text{trlog}(I + K/r^2) \\ &= 2(d+1)\log(r) + \text{tr}(K/r^2) - \text{tr}(K^2/2r^4) + \ldots \\ &= 2(d+1)\log(r) - 2|y|^2/r^2 + \ldots, \end{aligned}$$

where omitted terms are again of lower order. Finally, this allows us to write down an expression for the Hamiltonian in these special coordinates at (k, Λ, q),

$$2H = -\frac{\partial^2}{\partial r^2} - \frac{(d+1)}{r}\frac{\partial}{\partial r} + \Delta_y + \frac{2y_B^j}{r^2}\frac{\partial}{\partial y_B^j} - \frac{1}{r^2}(\sum_{j>1} X_j^2 + \sum_{j>1} V_j^2) + r^2|y|^2 + \sum_{i>j>1}(y_2^i y_3^j - y_3^i y_2^j)^2 + k_{AM}\Lambda_M^s q_{sj} Y_A^j + \dots , \tag{5.2}$$

where Δ_y is the Laplacian in the y variables. We have written H_F as $k_{AM}\Lambda_M^s q_{sj} Y_A^j$, where $Y_A^j = i\gamma_{\alpha\beta}^j f_{ABC}\psi_{B\alpha}\psi_{C\beta}$, in the nine-dimensional model, and analogous expressions for the other cases. The omitted terms are all of order $O(1/r^3)$ or smaller.

5.2. Inverting the Hamiltonian. To invert the Hamiltonian, let us focus first on the harmonic oscillator term, $H_m = \Delta_y + r^2|y|^2$, with eigenvalues $2(d-1+n)r$, where n a non-negative integer. If there were no cancelling fermion term, this oscillator term would immediately guarantee a potential linearly increasing with r, and therefore, a discrete spectrum. In order to see the cancelling fermion term, we write $H_F = rY_r + H'_F$, where $Y_r = k_{A1}q_{1j}Y_A^j$, and H'_F consists simply of all those terms in H_F without a $\Lambda_1^1 = r$ factor. It is an easy task to show that the eigenvalues of Y_r range from $-2(d-1)$ to $2(d-1)$. In particular, when we consider wavefunctions for which the transverse bosons and fermions are simultaneously in their ground states, all terms of order r cancel, and we are left with only terms of order $1/r^2$ to worry about in the Hamiltonian. On any state with excited oscillators, we see that H has lowest eigenvalue of order r.

Our construction of W will use the nice factorization of the wavefunctions in terms of their behavior in the massive directions, and their behavior along the flat directions. We can think of H as a block 2×2 matrix, with respect to this decomposition, where the H_{11} piece corresponds to the terms in H which take the ground state for the oscillators back to itself. The piece H_{22} contains terms in H which send the state with one excited massive boson to itself, and the off-diagonal terms act in a similar way. As a first guess, one might try (and we did) to construct W as a perturbation of some nice approximation, W_1, to $H_{11}^{-1}\oplus H_{22}^{-1}$. Because of the large eigenvalue of the first excited state of $H_m + rY_r$, almost any construction should give a good H_{22}^{-1}. Inverting H_{11} is more problematic but not excessively so.

In a standard perturbative approach, we consider $HW_1 = I - E_1$. Here, E_1 will include, for example, such terms as $H_{12}W_1$. As a next step, set $W_2 = W_1 + W_1E_1$, with error $E_2 = -E_1^2$. One could iterate this construction to construct $W = W_n$ for some large n, if the errors were getting significantly smaller each time. For example if each E_k had a kernel $e_k(x,x')$ which was bounded by $(r(x)+r(x'))^{-k}$, this would lead us to the desired W after some number of iterations. With this in mind, it becomes clear what terms must be included in our initial approximation, W_1. Because any approximate H_{11}^{-1} will have upper bound of size r^2, we see that we cannot discard any term of size bigger than or equal to $O(1/r^2)$ in H which either acts on the ground state, or maps an excited state to the ground state. For these terms lead to errors which are not decreasing under iteration. Actually, this is an overstatement. If a term B maps us, for example, from the lowest state into a higher state we see that the error will enter as $H_{22}^{-1}BH_{11}^{-1}$. Because H_{22}^{-1} is bounded above by $(r(x)+r(x'))^{-1}$, this term will be decreasing in r if, for example, B is $O(1/r^2)$. With these remarks to guide us, we must now return to analyze H, treating as lower order only terms which are $O(1/r^3)$ if they map the ground

state into itself, or are $O(1/r^2)$ if they mix the ground state with excited states. These lower order terms cannot be neglected altogether, but they simply enter as higher terms in our iterative construction of W.

The gauge constraints provide some further simplication in our computations. First we can 'move' to the identity element $k = 1$ in our product space, $SO(3) \times \mathbf{R}^{2d-1} \times SO(d)$ with coordinates (k, Λ, q), by a gauge rotation. From now on, we will restrict our discussion to the $k = 1$ subspace. Further, setting two of the gauge constraints to zero allows us to replace differentiation by X_j, which generates gauge transformations on the bosons, by multiplication by the fermion bilinear which generates gauge transformations on the fermions. For example, in the nine-dimensional case, the fermion bilinear is given by $Q_j := -\frac{1}{2}\psi_{1s}\psi_{js}$. There are similar expressions for the other cases. The remaining gauge constraint, C_1 of the three constraints in (2.2), generates a $U(1)$ subgroup which acts on y. With these considerations in hand, let us turn to the task of getting rid of the massive modes, and obtaining an effective Hamiltonian on the flat directions which we can invert to compute (3.6).

5.3. Constructing the effective Hamiltonian. Let us begin by computing the total contribution of terms that map the ground state to itself. Each of these terms will give rise to an interaction in the effective Hamiltonian of the form, m/r^2, for some m, in a manner described in the beginning of this section. The ground state is of the form $s(q)r^{(d-1)/2}e^{-r|y|^2/2}$, where $s(q)$, the fermion ground state, is actually a section of a bundle determined by the lowest eigenvalue of the fermion term Y_r, since $s(q)$ must satisfy the equation:

$$\{Y_r(q) + 2(d-1)\}s(q) = 0.$$

Therefore, the ground state depends non-trivially on the right angular coordinates, q. We will examine the structure of the fermion ground state in some detail, shortly. Note that the ground state is invariant under the remaining $U(1)$ subgroup of the gauge group. As a first approximation, a general state that we need to consider is a product of the ground state with a wavefunction, $f(r, q)$, along the flat directions. Let us record the various m^i/r^2 contributions to the effective Hamiltonian which acts on f, where m^3, m^5 and m^9 denote the values of the contributions for the three, five and nine-dimensional models, respectively. First, by definition, the term $H_m + rY_r$ vanishes on the ground state.

It is convenient to rewrite y and $\frac{\partial}{\partial y}$ in terms of standard annihilation and creation operators:

$$y = \frac{1}{\sqrt{2r}}(a + a^\dagger),$$
$$\frac{\partial}{\partial y} = \sqrt{\frac{r}{2}}(a - a^\dagger),$$

where $[a, a^\dagger] = 1$. Now we can evaluate the contribution of the derivative terms in H acting on the bosonic ground state,

$$|0\rangle = \left(\frac{r}{\pi}\right)^{(d-1)/2} e^{-r|y|^2/2}.$$

The term in (5.2) with one radial derivative gives,

$$\langle 0|\partial_r|0\rangle = \langle 0|\left(\frac{d-1}{2r} - \frac{y^2}{2}\right)|0\rangle$$
$$= 0.$$

The term with two radial derivatives then gives,

$$\langle 0|\partial_r^2|0\rangle = \frac{1}{r^2}\left[(\frac{d-1}{2})(\frac{d-3}{2}) - 2(\frac{d-1}{2})^2\right] + \frac{1}{4}\langle 0||y|^2|y|^2|0\rangle$$
$$= \frac{1-d}{4r^2}.$$

Let us label this contribution, m_r, then $m_r^3 = 1/4$, $m_r^5 = 1/2$, and $m_r^9 = 1$. After noting that we can replace the X_i^2 terms in (5.2) by the action of the fermion bilinears, Q_i^2, which are just matrices, we see that there are two remaining derivative terms in the Hamiltonian. The first is the $y\frac{\partial}{\partial y}$ term and it is relatively easy to analyze,

$$\langle 0|y\frac{\partial}{\partial y}|0\rangle = -\frac{1}{2}\langle 0|aa^\dagger|0\rangle$$
$$= (1-d).$$

Let us call this contribution, m_y, where $m_y^d = 1 - d$. The sign of this term is critical since it is the only term that maps the ground state to the ground state, and gives a large negative contribution to the net m.

The last derivative term comes from the action of V_i^2 on the ground state, where the V_i generate certain flavor rotations. The q-dependent fermion oscillator ground state is the only part of the full ground state wavefunction that can give a non-vanishing contribution in this case. So, let us now describe the ground state of the fermions in some detail. The real massive fermions, $\{\psi_{2\alpha}, \psi_{3\alpha}\}$, where α runs from 1 to $n = 4, 8$ and 16 for the three, five and nine-dimensional cases, can be arranged into annihilation and creation operators:

$$b_\alpha = \frac{1}{\sqrt{2}}(\psi_{2\alpha} + i\psi_{3\alpha}),$$
$$b_\alpha^\dagger = \frac{1}{\sqrt{2}}(\psi_{2\alpha} - i\psi_{3\alpha}).$$

The operators $b, b^\dagger$ obey the anti-commutation relation, $\{b_\alpha, b_\beta^\dagger\} = \delta_{\alpha\beta}$. Can we construct a fermion state in the kernel of $Y_r + 2(d-1)$? For the moment, let us pick $q = 1$. In this case, $Y_r = 2i\gamma^1_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}$, and we can pick γ^1 to be the diagonal element of the Clifford algebra:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With this choice, $Y_r = 2\sum_{\alpha=1}^{n/2}\left(b_\alpha^\dagger b_\alpha - b_{n/2+\alpha}^\dagger b_{n/2+\alpha}\right)$. Let us choose a fermionic ground state $|0\rangle_F$ which satisfies, $b_\alpha|0\rangle_F = b_{n/2+\alpha}^\dagger|0\rangle_F = 0$ for $\alpha = 1, \ldots, n/2$. This vacuum is then in the kernel of $Y_r + 2(d-1)$ at the point $q = 1$. We should point out that, when restricted to the $(d-1)$-sphere which is the SO(d) orbit of a flat point, the ground state takes values in a flat vector bundle, which is therefore trivial. This means that there is a globally defined fermion ground state wavefunction.

To understand the fermion ground state for arbitrary q and, in particular, to understand the action of V_i^2 on the ground state, we shall study the equivalent problem of its action on the operator which acts as projection onto the kernel of $Y_r + 2(d-1)$. This operator is given by,

$$P(q) = \frac{1}{2\pi i}\oint_\Gamma dz \frac{1}{z - Y_r(q)}, \tag{5.3}$$

using a contour, Γ, which we will take to be a small loop enclosing $-2(d-1)$. This construction of $P(q)$ is readily seen to be correct by diagonalizing $Y_r(q)$, and computing the corresponding contour integral.

Here we should clarify that we are really interested in the action of $P(q)\sum_i V_i^2$; the remainder is $O(1/r^2)$ and does not take the groundstate to itself. These terms will therefore be of interest only as perturbative corrections. The operator $P(q)\sum_i V_i^2$ should act as a scalar $2m_q$ on an appropriately chosen basis of the ground state. Our task is to compute the scalar. Now, $P(q)$ is given by conjugating the projection operator at $q = 1$ with a nonconstant orthogonal matrix, whose columns give a basis of the ground state. We can write this as, $P(q) = O(q)P(1)O(q)^t$. Then $\sum_i V_i^2$ acting on $P(q)$ has terms where $O(q)$ or $O(q)^t$ are twice-differentiated and terms where each is differentiated once. It is easy to show that this last term is annihilated by $P(q)$, and hence is not germane to this calculation. If the remaining term is a multiple of $P(q)$, then the multiple will be $4m_q$, as both $O(q)$ and its conjugate should contribute a factor of $2m_q$ when differentiated. We preface the computation by commenting on how to compute the derivative of $Y_r(q)$. It is enough to compute $V_i q_{1j}$. By a change of coordinates, we then only need to compute, $V_i q_{1j}(1)$. This is given by:

$$V_i q_{1j} = \frac{d}{dt}(qe^{tV_i})_{1j}\Big|_{t=0} = (qV_i)_{1j}.$$

Evaluating at $q = 1$ gives,

$$V_i q_{1j}(1) = (V_i)_{1j} = \delta_{ij}.$$

Now we compute:

$$\sum_i V_i^2 P(q) = \sum_i \frac{2}{2\pi i}\int \frac{1}{z - Y_r(q)} V_i Y_r \frac{1}{z - Y_r(q)} V_i Y_r \frac{1}{z - Y_r(q)} dz + \sum_i \frac{1}{2\pi i}\int \frac{1}{z - Y_r(q)} V_i^2 Y_r \frac{1}{z - Y_r(q)} dz,$$

where Y_r is an eigenfunction of the Laplacian $\sum_i V_i^2$. Hence the second integrand is a simple double pole and therefore integrates to zero. In order to compute the remaining term, it is enough to make a change of coordinates equivalent to taking $q = 1$. Then it is easy to see that $V_i(Y_r)$ takes the $-2(d-1)$ eigenspace of Y_r to the $4 - 2(d-1)$ eigenspace. This gives,

$$P(q)\sum_i \frac{2}{2\pi i}\int \frac{1}{z - Y_r(q)} V_i Y_r \frac{1}{z - Y_r(q)} V_i Y_r \frac{1}{z - Y_r(q)} dz = \sum_i \mathrm{diag}(V_i Y_r)^2 \frac{2}{2\pi i}\int \frac{1}{(z + 2(d-1))}\frac{1}{(z + 2(d-1) - 4)}\frac{1}{(z + 2(d-1))} dz,$$

where $\mathrm{diag}(V_i Y_r)^2 = P(q)(V_i Y_r)^2 V(q)$. Integrating gives $-\sum_i \mathrm{diag}(V_i Y_r)^2/8$.

We can compute this at $q = 1$ where, $(V_i Y_r) = 2i\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}$, and $\sum_i (V_i Y_r)^2/8 = -4\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}\gamma^i_{\alpha'\beta'}\psi_{2\alpha'}\psi_{3\beta'}/8$. The terms which survive $P(q)$ are,

$$-4\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}/8 - 4\gamma^i_{\alpha\beta}\psi_{2\alpha}\psi_{3\beta}\gamma^i_{\beta\alpha}\psi_{2\beta}\psi_{3\alpha}/8 = \gamma^i_{\alpha\beta}\gamma^i_{\alpha\beta}/4 = 4(d-1).$$

So, we finally get that $m_q^3 = 1/2$, $m_q^5 = 2$ and $m_q^9 = 8$.

With the derivative terms out of the way, we can consider the two remaining operators in the Hamiltonian that map the ground state to the ground state. The first is the $O(y^4)$ term in the potential. This gives,

$$\begin{aligned}\langle 0|\sum_{i>j}(y_2^i y_3^j - y_3^i y_2^j)^2|0\rangle &= \langle 0|2\sum_{i>j}(y_2^i y_3^j)^2|0\rangle \\ &= \frac{1}{4r^2}(d-1)(d-2).\end{aligned}$$

Calling this contribution, m_V, we have $m_V^3 = 1/4, m_V^5 = 3/2$, and $m_V^9 = 7$. The last contribution comes from the kinetic term for the two gauge rotation generators, X_2, X_3. Setting the gauge constraints to zero, we can replace the angular Laplacian, X_i^2, by the operator quartic in fermions:

$$\sum_{\alpha=1}^{n}(\psi_{1\alpha}\psi_{2\alpha})^2 + (\psi_{1\alpha}\psi_{3\alpha})^2.$$

The only terms that map the ground state to the ground state are those proportional to the identity. A quick calculation gives the numerical value, $n/2$. Calling this contribution, m_f, we note that $m_f^3 = 1, m_f^5 = 2$ and $m_f^9 = 4$.

If we were to stop at this point, and just consider these diagonal contributions to the effective Hamiltonian, a quick check would show that the net m is non-vanishing. There would therefore be a non-trivial $1/r^2$ interaction in the effective theory. Fortunately, we are not quite finished. First, we can shift the coefficient of the ∂_r term in (5.2), and generate a new $O(1/r^2)$ term by redefining our wavefunctions. The reason this is useful is that on choosing an appropriate coefficient for ∂_r, we can combine the radial derivatives with the V_i angular derivatives to obtain a Laplacian for flat space, together with some $1/r^2$ interaction. This way, we only need to deal with Euclidean coordinates, rather than the messier angular coordinates. We want to shift the coefficient of the ∂_r term from $d+1$ to $d-1$, so we will end up with a free particle Hamiltonian on the d-dimensional moduli space, together with interactions. To do so, we note that:

$$\begin{aligned}-\left(\partial_r^2 + \frac{d+1}{r}\partial_r\right) &= -\left((\partial_r + \frac{1}{r})^2 + \frac{d-1}{r}\partial_r\right) \\ &= -\left((\partial_r + \frac{1}{r})^2 + \frac{d-1}{r}(\partial_r + \frac{1}{r}) - \frac{d-1}{r^2}\right) \\ &= -\frac{1}{r}\left(\partial_r^2 + \frac{d-1}{r}\partial_r - \frac{d-1}{r^2}\right) r,\end{aligned}$$

where we now redefine our wavefunctions, $f(r,q) = \frac{1}{r}\tilde{f}(r,q)$. This gives us a new contribution to m, say $m_c = \frac{d-1}{2}$.

So far, we have found that H acts on the ground state and first excited state in the following way,

$$H = \begin{pmatrix} \frac{1}{2}\Delta_d + m_T^d/r^2 & b^t \\ b & H_{22}\end{pmatrix}.$$

Here, $m_T = m_c + m_f + m_V + m_q + m_y + m_r$ is the total effective interaction that we have found so far, where $m_T^3 = 1, m_T^5 = 4$ and $m_T^9 = 16$. Lower order terms in H_{11} have been

omitted. Let us turn to the form of b, which may change the effective interaction. For example, the terms in b acting on the ground state of order $O(r^{-1/2})$ cannot be neglected. Our initial choice of W_1 is then not diagonal to the requisite order in the basis that we have been using. Does b contain terms of the right order?

It is not hard to check that the only terms that map the ground state to the first excited state, which are of the requisite order are those involving H'_F. This follows from noting that H'_F is proportional to y which is, $\sim a^\dagger/\sqrt{r}$ acting on the ground state. Now we see that we can significantly lower our energy if we define a new ground state $|0\rangle' = (I - \frac{b}{2r})s(q)|0\rangle$. The factor of $2r$ is chosen because acting with b on a state raises its eigenvalue under $H_m + rY_r$ by $2r$. So, $H|0\rangle' = (\frac{1}{2}\Delta_d + m_T^d/r^2 - b^t b/2r)s(q)|0\rangle$ plus lower order terms. The final contribution to m from $b^t b$ is computed in the same way as the other contributions, and we find that $m_b^d = -m_T^d$.

Wonderfully, the $O(1/r^2)$ terms sum to zero! From these considerations, we see that we are reduced to a free-particle calculation. Our choice for an approximation W_{11} to H_{11}^{-1} is then particularly simple: we can just take the free-particle propagator,

$$W_{11} = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot(x-x')}}{k^2}.$$

5.4. Evaluating the boundary contribution. Now that we reached the point where we have a nice simple form for W_{11}, we note that the remainder of the perturbation construction is standard. We will not belabor the reader with the details of this expansion, but just provide some relevant comments. As we observed before, any reasonable construction yields a good approximate H_{22}^{-1}, with a nice error bound. The construction of W, which is perturbative in $1/r$, then follows the outline that we have described earlier in this section. The contribution of the W_{11} term is non-vanishing, but it is clear after extensive, arduous but standard computations, which involve checking powers of r, that any trace involving the rest of W will bring in more powers of r^{-1} than appear in the W_{11} term. These terms will therefore not contribute to the boundary term (3.6), in the limit where $r\to\infty$. We can now restrict ourselves to the free-particle Green's function, W_{11}. The remaining calculation is simple. We have a free particle propagating on $\mathbf{R}^d/\mathbf{Z}_2$. The $\mathbf{Z}_2$ identification comes from the Weyl group action on the Cartan of the gauge group $SU(2)$. Let us take x as coordinates for $\mathbf{R}^d$. The $\mathbf{Z}_2$ action acts as parity, sending $x\to -x$. It also sends the free fermions, $\psi_{1\alpha}$, where $\alpha = 1, \ldots, 2(d-1)$, to minus themselves. The Hilbert space of gauge-invariant wavefunctions is given by:

$$\{f_0(x), f_1(x)\psi_{1\alpha}, f_2(x)\psi_{1\alpha}\psi_{1\beta}, \ldots\}.$$

Each function, f_k, has parity $(-1)^k$. All we need to do is compute the boundary term,

$$-\frac{1}{2}\int_{N_F(r)} \mathrm{tr} e_n(-1)^F Q W_{11},$$

for this system. This becomes an integral over the boundary of the d-dimensional moduli space:

$$\begin{aligned}
&-\frac{1}{2}\int_{S^{d-1}(r)} \mathrm{tr} e_n(-1)^F Q W_{11} \\
&= -\frac{1}{\mathrm{vol}(S^{d-1})2(d-2)}\int_{S^{d-1}(r)} \mathrm{tr} e_n(-1)^F Q \frac{1}{|x-x'|^{d-2}}\Big|_{x'=-x} \\
&= -1/4.
\end{aligned}$$

To close this computation, let us note that the lower bound on the asymptotic behavior of H is given by its lowest value on the modified lowest wavefunctions, where the modification involved the off-diagonal b term. The lower bound is therefore the same as the bound for Δ_d, up to terms of order $O(1/r^3)$. As discussed in section three, this easily leads to the claimed asymptotic lower bound for H.

To summarize: we have found a formula for the index that counts the net number of L^2 ground states in certain quantum mechanical systems, where the potential has flat directions. This involved a study of L^2 index theory for a family of non-Fredholm operators, which allowed us to show that the prescription we presented actually computes the index. For the case of two-particle binding, we have shown that there is a bound state for coincident zero-branes in type IIA string theory. We have also found further evidence that there are no bound states for two-branes twice wrapped on an S^2, and three-branes twice wrapped on an S^3. Note that these models are only special points in the space of theories obtained by deforming the zero-brane quantum mechanics.

The actual computation split into two parts. Computing the principal term involved evaluating the integral (3.5). It would be interesting, and quite non-trivial, to compute this integral for higher rank gauge groups. Even better would be a method for avoiding this integration altogether. The second part of the computation required a study of the propagator for the two particles when they are far apart. Surprisingly, after summing a variety of corrections, this computation reduced to one involving a free particle moving on the moduli space. Undoubtedly, there is a fundamental reason for this simplification, and finding it may also shed light on whether the F^4 term in the effective zero-brane Hamiltonian is protected from corrections. It seems likely that there will be an analogous reduction to a free particle calcuation for other gauge groups. As a further comment, note that if we had studied a system with gauge group $U(1)$ and some charged matter, there would have been no boundary correction, as in the case involving H-monopoles [6].

The sort of decay estimates that we described can probably be used to get a handle on the structure of the ground state wavefunction. What is needed is an upper bound on how fast the wavefunction can decay along the flat directions. They may also lead to a vanishing theorem showing that all ground states in these systems must have a definite fermion number. The index would no longer be just an index, but would then count the total number of ground states. This would allow us to conclude that the zero-brane bound state is unique. Finally, systems involving marginal binding of branes with different dimensions can now be analyzed in much the same way.

Acknowledgement. It is a pleasure to thank G. Jungman, L. Susskind and E. Witten for helpful discussions. The work of S.S. is supported by NSF grant DMS–9627351, while that of M.S. by NSF grant DMS-9505040.

References

1. Polchinski, J.: Phys. Rev. Lett. **75**, 47 (1995)
2. Witten, E.: Nucl. Phys. **B460**, 335 (1996)
3. Sethi, S., Stern, M. and Zaslow, E.: Nucl. Phys. **B457**, 484 (1995)
4. Gauntlett, J. and Harvey, J.: Nucl. Phys. **B463**, 287
5. Sen, A.: Phys. Rev. **D53**, 2874 (1996); Phys. Rev. **D54**, 2964 (1996)
6. Sethi, S. and Stern, M.: Phys. Lett. **B398**, 47 (1997)
7. Banks, T., Fischler, W., Shenker, S.H. and Susskind, L.: Phys. Rev. **D55**, 5112 (1997)

8. de Wit, B., Hoppe, J. and Nicolai, H.: Nucl. Phys. **B305**, 545 (1988);
de Wit, B., Luscher, M.M. and Nicolai, H.: Nucl. Phys. **B320**, 135 (1989);
de Wit, B., Marquard, V. and Nicolai, H.: Commun. Math. Phys. **128**, 39 (1990)
9. Townsend, P.: Phys. Lett. **B373**, 68 (1996)
10. Danielsson, U.H., Ferretti, G., Sundborg, B.: Int. J. Mod. Phys. **A11**, 5463 (1996);
Kabat, D. and Pouliot, P.: Phys. Rev. Lett. **77**, 1004 (1996)
11. Douglas, M.R., Kabat, M.R., Pouliot, P. and Shenker, S.: Nucl. Phys. **B485**, 85 (1997)
12. Susskind, L.: hep-th/9704080
13. Frohlich, J. and Hoppe, J.: hep-th/9701119
14. Lowe, D.: hep-th/9704041
15. Yi, P.: hep-th/9704098
16. Claudson, M. and Halpern, M.: Nucl. Phys. **B250**, 689 (1985)
17. Brink, L., Schwarz, J.H. and Scherk, J.: Nucl. Phys. **B121**, 77 (1977)
18. Simon, B.: Ann. Phys. **146**, 209 (1983)
19. Townsend, P.K.: Phys. Lett. **B350**, 184 (1995)
20. Witten, E.: Nucl. Phys. **B443**, 85 (1995)
21. Strominger, A.: Nucl. Phys. **B451**, 96 (1995)
22. Bershadsky, M., Sadov, V. and Vafa, C.: Nucl. Phys. **B463**, 420 (1996)
23. Sethi, S.: Unpublished
24. Ishibashi, N., Kawai, H., Kitazawa, Y. and Tsuchiya, A.: hep-th/9612115
25. Agmon, S.: *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations*. Princeton, NJ: Princeton University Press, 1982

Communicated by R. H. Dijkgraaf