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# A Steady-State Quantum Euler–Poisson System for Potential Flows

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**Abstract:** A potential flow formulation of the hydrodynamic equations with the quantum Bohm potential for the particle density and the current density is given. The equations are selfconsistently coupled to Poisson's equation for the electric potential. The stationary model consists of nonlinear elliptic equations of degenerate type with a quadratic growth of the gradient. Physically motivated Dirichlet boundary conditions are prescribed. The existence of solutions is proved under the assumption that the electric energy is small compared to the thermal energy. The proof is based on Leray-Schauder's fixed point theorem and a truncation method. The main difficulty is to find a uniform lower bound for the density. For sufficiently large electric energy, there exists a generalized solution (of a simplified system), where the density vanishes at some point. Finally, uniqueness of the solution is shown for a sufficiently large scaled Planck constant.

## 1. Introduction

The evolution of a fluid or gas is governed by the hydrodynamic equations [20]

$$\frac{\partial n}{\partial t} + \operatorname{div} J = 0, \tag{1.1}$$

$$\frac{\partial J}{\partial t} + \operatorname{div}\left(\frac{J \otimes J}{n} + P\right) - nF = W. \tag{1.2}$$

The first equation expresses the conservation of mass where $n$ is the particle density and $J$ the particle current density. The second equation expresses the conservation of momentum where $P = (P_{ij})$ denotes the pressure tensor, $F$ the sum of the external forces, and $W$ the momentum relaxation term. The $i^{\text{th}}$ component of div $(J \otimes J/n + P)$ is given by

$$\sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left( \frac{J_i J_j}{n} + P_{ij} \right),$$

where $d \geq 1$ is the space dimension.

We consider an isothermal or isentropic quantum fluid of charged particles. In particular, the pressure tensor is assumed to be of the form $P = \big(\delta_{ij} r(n)\big)$, where $\delta_{ij}$ is the Kronecker symbol. The pressure function $r$ is given by the particle density, i.e. $r(n) = T_o n$ in the isothermal case and $r(n) = T_o n^\beta$ in the isentropic case, where $\beta > 1$ and $T_o$ is a (scaled) temperature constant. In the isothermal case, the fluid temperature $T$ is equal to $T_o$; in the isentropic case we get $T = T_o n^{\beta-1}$. We assume that the external force is the gradient of the sum of the electric potential $V$, the external potential $V_{ext}$, and the quantum Bohm potential

$$Q = \delta^2 \frac{1}{\sqrt{n}} \Delta \sqrt{n},$$

$\delta > 0$ being the scaled Planck constant. The external potential models (interior) quantum wells. Equations (1.1)–(1.2) are coupled to Poisson's equation for the electric potential,

$$\lambda^2 \Delta V = n - C(x). \tag{1.3}$$

Here, $\lambda$ denotes the scaled Debye length, and $C(x)$ models fixed background ions. Finally, the relaxation term is given by $W = -\alpha J$, where $\alpha > 0$ is the inverse of the scaled relaxation time. With these assumptions the quantum hydrodynamic equations can be formulated as

$$\frac{\partial n}{\partial t} + \operatorname{div} J = 0, \tag{1.4}$$

$$\frac{\partial J}{\partial t} + \operatorname{div}\left( \frac{J \otimes J}{n} \right) + \nabla r(n) - n\nabla(V + V_{ext}) - \delta^2 n \nabla \left( \frac{\Delta \sqrt{n}}{\sqrt{n}} \right) = -\alpha J. \tag{1.5}$$

The primary application of the quantum hydrodynamic equations to date has been in analyzing the flow of electrons in quantum semiconductor devices, like resonant tunneling diodes [10]. Very similar model equations have been used in other areas of physics, e.g. in superfluidity [22] and in superconductivity [6].

The quantum Euler–Poisson system (1.3)–(1.5) has been justified in [1, 10, 12, 13, 14]. It can be derived from a moment expansion of the Wigner-Boltzmann equations [10] or from a mixed state Schrödinger–Poisson system [12]. In particular, the single state Schrödinger-Poisson system

$$i\varepsilon \frac{\partial \psi}{\partial t} = -\frac{\varepsilon^2}{2} \Delta \psi + (V + V_{ext})\psi, \quad \lambda^2 \Delta V = |\psi|^2 - C(x)$$

is equivalent (for appropriate "smooth" solutions) to the irrotational zero temperature flow equations

$$\frac{\partial n}{\partial t} + \operatorname{div} J = 0,$$

$$\frac{\partial J}{\partial t} + \operatorname{div}\left( \frac{J \otimes J}{n} \right) - n\nabla(V + V_{ext}) - \frac{\varepsilon^2}{2} n \nabla \left( \frac{\Delta \sqrt{n}}{\sqrt{n}} \right) = 0$$

and Poisson's equation (1.3) (see [21, 14]). These equations are known as Madelung's fluid equations [22]. The expression "irrotational" means that the current density can be written as $J = n\nabla S$, where $S$ is called a phase or quantum Fermi potential. The equivalence of the two models follows from the definitions $n = |\psi|^2$, $\psi = \sqrt{n}\exp(iS/\varepsilon)$ and $J = n\nabla S$. We note that for finite relaxation times $\alpha > 0$, there is no equivalence to a Schrödinger-Poisson system, even not in the mixed state.

In this paper we study the steady-state equations

$$\operatorname{div} J = 0, \tag{1.6}$$

$$\operatorname{div}\left(\frac{J\otimes J}{n}\right) + \nabla r(n) - n\nabla(V + V_{ext}) - \delta^2 n\nabla\left(\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = -\alpha J, \tag{1.7}$$

$$\lambda^2\Delta V = n - C \tag{1.8}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) occupied by the fluid. The main assumption is that we consider a potential flow, i.e. we assume that the particle current can be written as

$$J = n\nabla S$$

with the quantum Fermi potential $S$ (see above). This means that the velocity $J/n = \nabla S$ is assumed to be irrotational. It is physically reasonable to assume that $n > 0$ holds in the device. Since $\operatorname{div}(J\otimes J/n) = \frac{1}{2}n\nabla|\nabla S|^2$ we can rewrite (1.7) as

$$n\nabla\left(\frac{1}{2}|\nabla S|^2 + T_o h(n) - V - V_{ext} - \delta^2\frac{\Delta\sqrt{n}}{\sqrt{n}}\right) = -\alpha n\nabla S, \tag{1.9}$$

where

$$h(n) = \frac{1}{T_o}\int_1^n \frac{r'(s)}{s}ds \tag{1.10}$$

is the enthalpy function. In the isothermal case, $h(n) = \log(n)$ holds; for isentropic states, we have $h(n) = (\beta/(\beta-1))(n^{\beta-1} - 1)$ for $\beta > 1$. Notice that the electric potential and the quantum Fermi potential are fixed only up to additional constants. Since $n > 0$, Eq. (1.9) implies

$$\frac{1}{2}|\nabla S|^2 + T_o h(n) - V - V_{ext} - \delta^2\frac{\Delta\sqrt{n}}{\sqrt{n}} + \alpha S = 0.$$

The integration constant can be assumed to be zero by choosing a reference point for the electric potential. For the analysis it is convenient to use $w = \sqrt{n}$ as a variable. Then (1.6), (1.8), and (1.9) can be written as

$$\delta^2\Delta w = w(\tfrac{1}{2}|\nabla S|^2 + T_o h(w^2) - V - V_{ext} + \alpha S), \tag{1.11}$$

$$\operatorname{div}(w^2\nabla S) = 0, \tag{1.12}$$

$$\lambda^2\Delta V = w^2 - C \qquad \text{in } \Omega. \tag{1.13}$$

Physically relevant boundary conditions for $w$, $S$, and $V$ will be specified later.

The fluid models (1.6)–(1.8) or (1.11)–(1.13) have been studied in some special situations. For vanishing convective and quantum terms the problem (1.6)–(1.8) is known as the isentropic drift-diffusion model used for semiconductor devices [17, 18, 24]. The quantum drift-diffusion model (zero convective term, $\delta > 0$) has been investigated in [2]. The classical potential flow hydrodynamic equations ($\delta = 0$) are analyzed in, e.g. [5, 7, 9]. In the paper [29] the existence for the one-dimensional stationary quantum hydrodynamic equations (1.6)–(1.8) with non-standard boundary conditions is investigated. The steady-state system (1.11)–(1.13) in several space dimensions is studied here mathematically for the first time.

In the analysis of (1.11)–(1.13), two main difficulties arise. The elliptic equation (1.12) is, a priori, of degenerate type with a non-standard (since non-local) degeneracy. We will show, however, that the solution $w$ is strictly positive and therefore, (1.12) becomes strictly elliptic. Every solution $(w, S, V)$ of (1.11)–(1.13) with positive $w$ is a solution of the problem (1.6)–(1.8) with $n = w^2$, $J = n\nabla S$.

Another difficulty arises due to the term $|\nabla S|^2$ on the right hand side of (1.11), stemming from the convective term in (1.6). This difficulty also appears in the thermistor problem (see [4, 27]). However, we have to apply different techniques than used in the thermistor problem.

To derive the boundary conditions we make physically relevant hypotheses. The boundary data are assumed to be the superposition of the thermal equilibrium functions $(n_{eq}, S_{eq}, V_{eq})$ and the applied potential $U(x)$:

$$n = n_{eq}, \quad S = S_{eq} + U, \quad V = V_{eq} + U \quad \text{on } \partial\Omega.$$

The thermal equilibrium state is defined by $J = 0$ or, equivalently, $S = \text{const.}$ (as $n > 0$). By fixing the reference point for $S$ (and $S_{eq}$) we can suppose that $S_{eq} = 0$. We assume further that the total space charge $C - n_{eq}$ vanishes at the boundary and that no quantum effects occur on $\partial\Omega$, i.e. $\Delta\sqrt{n_{eq}}/\sqrt{n_{eq}} = 0$. Finally, $V_{ext} = 0$ on $\partial\Omega$, since $V_{ext}$ is introduced to model interior quantum wells. We get from (1.11)

$$0 = \frac{1}{2}|\nabla S_{eq}|^2 + T_o h(n_{eq}) - V_{eq} + \alpha S_{eq}$$

or, since $S_{eq} = 0$,

$$V_{eq} = T_o h(n_{eq}) \qquad \text{on } \partial\Omega.$$

Therefore we get the Dirichlet boundary conditions

$$w = w_o, \quad S = S_o, \quad V = V_o \qquad \text{on } \partial\Omega \tag{1.14}$$

with

$$w_o = \sqrt{C}, \quad S_o = U, \quad V_o = T_o h(C) + U. \tag{1.15}$$

It is the aim of this paper to show the existence and uniqueness of solutions to (1.11)–(1.14). More precisely, we prove in Sect. 2 that there exists a solution $(w, S, V)$ to (1.11)–(1.14) with $\nabla S \in L^\infty(\Omega)$ under the assumption that the temperature constant $T_o$ is large enough (isothermal and isentropic case) or that the boundary Fermi potential $S_o$ is small enough in some norm (isothermal case). This means that the electric energy which is connected with the applied potential $U$ (and hence with $S_o$) has to be much smaller than the thermal energy, in some sense. For the proof we first replace Eq. (1.12) by $\operatorname{div}(\max(m, w)^2\nabla S) = 0$ ($m > 0$) which is uniformly elliptic. By means of Leray-Schauder's fixed point theorem, the existence of a solution to the truncated problem will be shown. For this solution the density $w$ turns out to be strictly positive. So we get a solution to the original problem (1.11)–(1.14) by choosing the truncation parameter $m > 0$ smaller than the lower bound of $w$.

We need the smallness assumption on the data in the proof of the positivity of $w$. We do not know if the existence of solutions can be proved without this assumption. In the stationary thermistor problem which is formally related to the quantum hydrodynamic model, it is well known that there exist solutions only if the applied potential is "small" enough (for the precise conditions see [27]). Furthermore, in the one-dimensional case it is possible to show the non-existence of solutions for "large" applied voltages [3, 4]. We recall that the thermistor problem reads

$$\operatorname{div}(k(w)\nabla w) = -\sigma(w)|\nabla S|^2,$$
$$\operatorname{div}(\sigma(w)\nabla S) = 0,$$

where $w$ and $S$ have here the meaning of the temperature and electric potential, respectively.

In the simulation of semiconductor tunneling devices where a variant of the presented quantum fluid model has been used, numerical results indicate that the density can be extremely small compared to, e.g., the boundary density, for values of the applied voltage $U$ far from the thermal equilibrium (e.g. $n_{min} = 10^{-4}$, $n|_{\partial\Omega} = 1$; see [10]). It is not clear if there is a lower bound for the density for all $U$ and if yes, how it can be controlled. The positivity property of $w$ is connected to the regularity for $S$. Indeed, we show that $w$ is strictly positive if and only if the gradient of $S$ is bounded (Sect. 3). For ultra-small devices, Eqs. (1.11)–(1.14) can be replaced asymptotically by a simplified system [19]. We show that there exists a solution of this (one-dimensional) system, where the density vanishes at some point. However, the solution is discontinuous and therefore, it is only defined in a generalized sense (see Sect. 3).

There exists at most one solution to (1.11)–(1.14) if the scaled Planck constant $\delta$ is sufficiently large (Sect. 4). For $\delta = 0$, there exist situations where the problem has more than one solution [11].

## 2. Existence of Solutions

In this section we prove the existence of solutions to (1.11)–(1.14) with general Dirichlet boundary data. The following assumptions are needed:

(A1) $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain with boundary $\partial\Omega \in C^{1,1}$.
(A2) $h \in C^0(0, \infty)$ is a non-decreasing function satisfying

$$\lim_{x\to\infty} h(x) = +\infty, \quad \lim_{x\to 0+} xh(x^2) < +\infty.$$

(A3) $w_o \in W^{2,p}(\Omega)$ for $p > d/2$, $\inf_{\partial\Omega} w_o > 0$; $S_o \in C^{1,\gamma}(\overline{\Omega})$ with $\gamma = 2 - d/p$; $V_o \in H^1(\Omega) \cap L^\infty(\Omega)$; $C, V_{ext} \in L^\infty(\Omega)$.

The constants $\alpha$, $\delta$, $\lambda$, and $T_o$ are assumed to be positive. We call a function $h \in C^0(0, \infty)$ satisfying (A2) *isothermal* if $h(0+) = -\infty$ and *isentropic* if $h(0+) < 0$. The enthalpy function $h(s) = \log(s)$ is isothermal. Furthermore, the enthalpy $h(s) = (\beta/(\beta - 1))(s^{\beta-1} - 1)$ is isentropic.

The main results of this section are the following theorems:

**Theorem 2.1.** *Let (A1)–(A3) hold and let $h$ be isothermal. Then there exists $\varepsilon > 0$ such that if*

$$\|S_o\|_{C^{1,\gamma}(\overline{\Omega})} \leq \varepsilon \quad \textit{or} \quad T_o \geq 1/\varepsilon,$$

*then there exists a solution $(w, S, V)$ of (1.11)–(1.14) satisfying, for some $\underline{w} > 0$,*

$$w \in W^{2,p}(\Omega), \quad S \in C^{1,\gamma}(\overline{\Omega}), \quad V \in H^1(\Omega) \cap L^\infty(\Omega), \tag{2.1}$$

$$w(x) \geq \underline{w} > 0 \quad \textit{in } \Omega. \tag{2.2}$$

**Theorem 2.2.** *Let (A1)–(A3) hold and let $h$ be isentropic. Then there exists $\varepsilon > 0$ such that if $T_o \geq 1/\varepsilon$ then there exists a solution $(w, S, V)$ of (1.11)–(1.14) satisfying (2.1)–(2.2).*

Notice that we are assuming boundary data which are independent of the parameter $T_o$. The case of the boundary functions (1.15) can also be treated, see Remark 2.5.

First we prove that there exists a solution of a truncated system. For this, define $s_K = \max(0, \min(s, K))$ and $t_m(s) = \max(m, s)$ for $s \in \mathbb{R}$ and $0 < m \leq K$. Throughout this section (A1)–(A3) are assumed to hold. Consider

$$\delta^2 \Delta w = w_K(\tfrac{1}{2}|\nabla S|^2 + T_o h(w_K^2) - V - V_{ext} + \alpha S), \tag{2.3}$$

$$\operatorname{div}(t_m(w_K)^2 \nabla S) = 0, \tag{2.4}$$

$$\lambda^2 \Delta V = w_K w - C \qquad \text{in } \Omega, \tag{2.5}$$

$$w = w_o, \quad S = S_o, \quad V = V_o \qquad \text{on } \partial\Omega. \tag{2.6}$$

The proof of existence of solutions to this truncated system is based on the following a priori estimates.

**Lemma 2.3.** *Let $(w, S, V)$ be a weak solution to (2.3)–(2.6). Then there exist constants $\overline{w}$, $\underline{S}$, $\overline{S}$, $\underline{V}$, $\overline{V}$, and $c_1(m)$ such that*

$$0 \leq w(x) \leq \overline{w}, \quad -\underline{S} \leq S(x) \leq \overline{S}, \quad -\underline{V} \leq V(x) \leq \overline{V} \qquad \textit{in } \Omega, \tag{2.7}$$

$$\|w\|_{2,p,\Omega} \leq c_1(m). \tag{2.8}$$

Here, $\|\cdot\|_{2,p,\Omega}$ denotes the norm of the Sobolev space $W^{2,p}(\Omega)$. The precise dependence of the above bounds on the data is needed in the uniqueness proof in Sect. 4 and is stated here for future reference:

$$\underline{S} = -\inf_{\partial\Omega} S_o, \quad \overline{S} = \sup_{\partial\Omega} S_o, \tag{2.9}$$

$$\overline{V} = \sup_{\partial\Omega} V_o + c(\Omega, d, \lambda)\|C\|_{0,\infty,\Omega}, \tag{2.10}$$

$$\overline{w} = \max\left(\|w_o\|_{0,\infty,\partial\Omega}, w_1(\overline{V}, \underline{S}, T_o, h)\right), \tag{2.11}$$

$$\underline{V} = -\inf_{\partial\Omega} V_o + c(\Omega, d, \lambda)\left(\|C\|_{0,\infty,\Omega} + \overline{w}^2\right), \tag{2.12}$$

where $c(\Omega, d, \lambda) > 0$ and $w_1 = w_1(\overline{V}, \underline{S}, T_o, h) > 0$ is such that $h(w_1^2) \geq (\overline{V} + \|V_{ext}\|_{0,\infty,\Omega} + \alpha\underline{S})/T_o$.

*Proof. First step.* $L^\infty$ *estimates for* $w$, $S$, *and* $V$. First observe that, using $w^- = \min(0, w) \in H_0^1(\Omega)$ as test function in (2.3), it follows

$$w(x) \geq 0 \quad \text{a.e. in } \Omega. \tag{2.13}$$

The maximum principle gives the bounds

$$-\underline{S} = \inf_{\partial\Omega} S_o \leq S(x) \leq \sup_{\partial\Omega} S_o = \overline{S} \qquad \text{in } \Omega. \tag{2.14}$$

Next we show that $V$ is uniformly bounded in $L^\infty(\Omega)$. Let $U_o = \sup_{\partial\Omega} V_o$, $U \geq U_o$, and take $(V - U)^+ = \max(0, V - U)$ as a test function in (2.5). Then

$$\begin{aligned} \lambda^2 \int_\Omega |\nabla(V - U)^+|^2 &= -\int w_K w (V - U)^+ + \int C(V - U)^+ \\ &\leq \int C(V - U)^+ \\ &\leq c\|(V - U)^+\|_{1,2,\Omega} (\text{meas}\,(V > U))^{1/2}. \end{aligned} \tag{2.15}$$

Here and in the following, $c$, $c_i$ denote positive constants only depending on the given data. Let $r > 2$ be such that the embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ is continuous. It is well known that for $W > U$,

$$(\text{meas}\,(V > W))^{1/r}(W - U) \leq c(\Omega)\|(V - U)^+\|_{1,2,\Omega}$$

holds [25, Ch. 4]. Therefore we get from (2.15), for $W > U \geq U_o$,

$$\text{meas}\,(V > W) \leq \frac{c}{(W - U)^r} (\text{meas}\,(V > U))^{r/2}.$$

Since $r/2 > 1$, we can apply Stampacchia's Lemma (see [26, Ch. 2.3] or [25, Ch. 4]) to get

$$V(x) \leq \overline{V} \stackrel{\text{def}}{=} U_o + c(\Omega, d, \lambda)\|C\|_{0,\infty,\Omega}, \tag{2.16}$$

where $c(\Omega, d, \lambda) > 0$.

Before we can find a lower bound for $V$, we prove that $w$ is bounded from above (independently of $K$). For this set $\overline{V}_{ext} = \|V_{ext}\|_{0,\infty,\Omega}$, let $\overline{w} \geq \|w_o\|_{0,\infty,\partial\Omega}$ and $K > \overline{w}$ and use $(w - \overline{w})^+$ as a test function in (2.3):

$$\begin{aligned} \delta^2 \int_\Omega |\nabla(w - \overline{w})^+|^2 &= -\frac{1}{2}\int w_K (w - \overline{w})^+ |\nabla S|^2 \\ &\quad - \int w_K (w - \overline{w})^+ T_o (h(w^2) - h(\overline{w}^2)) \\ &\quad + \int w_K (w - \overline{w})^+ (V + V_{ext} - T_o h(\overline{w}^2) - \alpha S) \\ &\leq \int w_K (w - \overline{w})^+ (\overline{V} + \overline{V}_{ext} - T_o h(\overline{w}^2) + \alpha \underline{S}), \end{aligned}$$

using (A2), (2.14) and (2.16). Since $h(s) \to \infty$ as $s \to \infty$, there exists $\overline{w} \geq \|w_o\|_{0,\infty,\partial\Omega}$ such that $h(\overline{w}^2) \geq (\overline{V} + \overline{V}_{ext} + \alpha\underline{S})/T_o$. This implies

$$w(x) \leq \overline{w} \qquad \text{a.e. in } \Omega. \tag{2.17}$$

Now use $(-V-U)^+$ with $U \geq U_o = -\inf_{\partial\Omega} V_o$ as test function in (2.5) to get

$$\begin{aligned} \lambda^2 \int_\Omega |\nabla(-V-U)^+|^2 &\leq \int (w_K w - C)(-V-U)^+ \\ &\leq c \int (-V-U)^+, \end{aligned}$$

where $c > 0$ depends on $C$ and $\overline{w}$. Using Stampacchia's method as above allows to conclude that

$$V(x) \geq -\underline{V} \stackrel{\text{def}}{=} -U_o - c(\Omega, d, \lambda)(\|C\|_{0,\infty,\Omega} + \overline{w}^2).$$

*Second step. $H^1$ estimate for $w$.* Use $w - w_o$ as test function in (2.3) to obtain

$$\begin{aligned} \delta^2 \int_\Omega |\nabla w|^2 &= \delta^2 \int \nabla w \cdot \nabla w_o - \frac{1}{2} \int w_K w |\nabla S|^2 + \frac{1}{2} \int w_K w_o |\nabla S|^2 \\ &\quad - T_o \int w_K (w - w_o) h(w_K^2) + \int w_K w V - \int w_K w_o V \\ &\quad + \int w_K (w - w_o) V_{ext} - \alpha \int w_K (w - w_o) S. \end{aligned} \tag{2.18}$$

With the test functions $V - V_o$ and $S - S_o$ in (2.5), (2.4) respectively, we get on the one hand

$$\begin{aligned} \int_\Omega w_K w V &= -\lambda^2 \int |\nabla V|^2 + \lambda^2 \int \nabla V \cdot \nabla V_o + \int V_o w_K w + \int C(V - V_o) \\ &\leq -\frac{\lambda^2}{2} \int |\nabla V|^2 + \lambda^2 \int |\nabla V_o|^2 + c \int w_K w + c, \end{aligned}$$

using Young's and Poincaré's inequalities; on the other hand, we have for $K > \overline{w}$,

$$\begin{aligned} m \int_\Omega w_K |\nabla S|^2 &\leq \int t_m(w_K)^2 |\nabla S|^2 \leq \int t_m(w_K)^2 |\nabla S_o|^2 \\ &\leq \overline{w}^2 \int |\nabla S_o|^2. \end{aligned}$$

Therefore we can estimate (2.18) as follows:

$$\begin{aligned} \frac{\delta^2}{2} \int_\Omega |\nabla w|^2 &\leq \frac{\delta^2}{2} \int |\nabla w_o|^2 + \frac{\overline{w}^2}{m} c(w_o, S_o) - T_o \int w_K w h(w_K^2) \\ &\quad + c \int |w_K h(w_K^2)| - \frac{\lambda^2}{4} \int |\nabla V|^2 + c(\lambda) \int w_K^2 \\ &\quad + c \int w_K w + c \\ &\leq c(m, \overline{w}). \end{aligned}$$

*Third step. $W^{2,p}$ estimate for $w$.* The following elliptic estimate holds [15, Thm. 8.33 and 8.34]:

$$\|S\|_{C^{1,\varepsilon}(\overline{\Omega})} \leq c_2 \|S_o\|_{C^{1,\varepsilon}(\overline{\Omega})} \qquad \text{for all } 0 < \varepsilon \leq \gamma, \tag{2.19}$$

where $c_2 > 0$ depends on $\Omega$, $d$, $m$, and the $C^{0,\varepsilon}(\overline{\Omega})$ norm of $t_m(w_K)^2$. It can be seen from the proof of this estimate that

$$c_2 = c_3(\Omega, d)c_4(m)\|t_m(w_K)^2\|_{C^{0,\varepsilon}(\overline{\Omega})}.$$

Furthermore, we have the elliptic estimate

$$\|w\|_{2,p,\Omega} \leq c_5\big(\|w_o\|_{2,p,\Omega} + \|w(\tfrac{1}{2}|\nabla S|^2 + h(w_K^2) - V - V_{ext} + \alpha S)\|_{0,p,\Omega}\big),$$

where $c_5 > 0$ depends on $\Omega$, $d$ and $\delta$ [15, 9.15 and 9.17]. Hence, using (2.19) for $\varepsilon = \gamma/2$,

$$\begin{aligned}
\delta^2\|w\|_{2,p,\Omega} &\leq c(1 + \|S\|^2_{1,2p,\Omega}) \leq c(1 + \|S\|^2_{C^{1,\gamma/2}(\overline{\Omega})}) \\
&\leq c(1 + \|w^2\|^2_{C^{0,\gamma/2}(\overline{\Omega})}) \leq c(1 + \overline{w}^2\|w\|^2_{C^{0,\gamma/2}(\overline{\Omega})}) \\
&\leq c(1 + \|w\|_{C^{0,\gamma}(\overline{\Omega})}) \\
&\leq \frac{\delta^2}{2}\|w\|_{2,p,\Omega} + c(\delta, m).
\end{aligned}$$

In the last step we have used the interpolation inequality

$$\|w\|_{C^{0,\gamma}(\overline{\Omega})} \leq \varepsilon\|w\|_{2,p,\Omega} + c(\varepsilon)\|w\|_{0,\infty,\Omega},$$

which follows from the facts that the embedding $W^{2,p}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega})$ is compact (since $p > d/2$) and the embedding $C^{0,\gamma}(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$ is continuous [28, p. 365]. The constant $c(\delta, m)$ in the above estimate depends on $\Omega$, $d$, $\delta$, $m$, $\overline{w}$, and $\underline{V}$. We obtain finally

$$\|w\|_{2,p,\Omega} \leq 2c(\delta, m)/\delta^2 = c_1(m).$$

**Lemma 2.4.** *There exists a solution* $(w, S, V)$ *of*

$$\delta^2\Delta w = w(\tfrac{1}{2}|\nabla S|^2 + T_o h(w^2) - V - V_{ext} + \alpha S), \tag{2.20}$$

$$\operatorname{div}(t_m(w)^2\nabla S) = 0, \tag{2.21}$$

$$\lambda^2\Delta V = w^2 - C \qquad \text{in } \Omega, \tag{2.22}$$

$$w = w_o, \quad S = S_o, \quad V = V_o \qquad \text{on } \partial\Omega, \tag{2.23}$$

*such that* $w \in W^{2,p}(\Omega)$, $S \in C^{1,\gamma}(\overline{\Omega})$, $V \in H^1(\Omega) \cap L^\infty(\Omega)$, *and* $w(x) \geq 0$ *in* $\Omega$.

*Proof.* We use a fixed point argument. Let $u \in C^{0,\gamma}(\overline{\Omega})$. Let $V \in H^1(\Omega)$ be the unique solution of

$$\lambda^2\Delta V = u_K u - C \quad \text{in } \Omega, \qquad V = V_o \quad \text{on } \partial\Omega,$$

and let $S \in H^1(\Omega)$ be the unique solution of

$$\operatorname{div}(t_m(u_K)^2\nabla S) = 0 \quad \text{in } \Omega, \qquad S = S_o \quad \text{on } \partial\Omega.$$

As in the proof of Lemma 2.3, we see that $V \in L^\infty(\Omega)$. Since $t_m(u_K)^2$ is Hölder continuous of order $\gamma$, we get $S \in C^{1,\gamma}(\overline{\Omega})$ [15, Thm. 8.34]. Finally, let $w \in H^1(\Omega)$ be the unique solution of

$$\delta^2 \Delta w = \sigma u_K(\tfrac{1}{2}|\nabla S|^2 + T_o h(u_K^2) - V - V_{ext} + \alpha S) \quad \text{in } \Omega,$$
$$w = \sigma w_o \quad \text{on } \partial\Omega,$$

with $\sigma \in [0,1]$. The right-hand side of this elliptic problem lying in $L^\infty(\Omega)$, we conclude $w \in W^{2,p}(\Omega)$ and, since $p > d/2$, $w \in C^{0,\gamma}(\overline{\Omega})$. Thus the fixed point operator $T : C^{0,\gamma}(\overline{\Omega}) \times [0,1] \to C^{0,\gamma}(\overline{\Omega})$, $(u,\sigma) \mapsto w$, is well defined. It holds $T(u,0) = 0$ for $u \in C^{0,\gamma}(\overline{\Omega})$. Estimates similarly as in the proof of Lemma 2.3 give the bound

$$\|w\|_{2,p,\Omega} \le c$$

for all $w \in C^{0,\gamma}(\overline{\Omega})$ satisfying $T(w,\sigma) = w$, where $c > 0$ is independent of $w$ and $\sigma$. Standard arguments show that $T$ is continuous and compact, noting the compactness of the embedding $W^{2,p}(\Omega) \hookrightarrow C^{0,\gamma}(\overline{\Omega})$. We can apply Leray-Schauder's fixed point theorem to get a solution $(w,S,V)$ of (2.3)–(2.6). Choosing $K > \overline{w}$ (see (2.7)), this tripel is also a solution of (2.20)–(2.23).

*Proof of Theorems 2.1 and 2.2.* We rewrite the elliptic estimate (2.19) for $\varepsilon = \gamma$:

$$\|S\|_{C^{1,\gamma}(\overline{\Omega})} \le c_3(\Omega,d)c_4(m)\|t_m(w)^2\|_{C^{0,\gamma}(\overline{\Omega})}\|S_o\|_{C^{1,\gamma}(\overline{\Omega})}.$$

It holds $c_4(m) \to \infty$ as $m \to 0+$. Now,

$$\|t_m(w)^2\|_{C^{0,\gamma}(\overline{\Omega})} \le c(\overline{w})\|w\|_{C^{0,\gamma}(\overline{\Omega})} \le c(\overline{w})\|w\|_{2,p,\Omega} \le c_5.$$

From the proof of Lemma 2.3 it can be seen that $c_5 = c_6(\overline{w})c_7(m)$ with $c_6(\overline{w}) \to \infty$ as $\overline{w} \to \infty$ and $c_7(m) \to \infty$ as $m \to 0+$. The bound $\overline{w}$ depends on $T_o$ such that $\overline{w} \to \infty$ as $T_o \to 0+$ (see (2.11)). Thus we can write

$$\|S\|^2_{C^{1,\gamma}(\overline{\Omega})} \le \frac{c_0}{f(T_o)g(m)}\|S_o\|^2_{C^{1,\gamma}(\overline{\Omega})}, \tag{2.24}$$

where $f$ and $g$ are positive continuous non-decreasing functions in $[0,\infty)$ such that $f(T_o) \to 0$ as $T_o \to 0+$, $f(T_o) > 0$ as $T_o \to \infty$, and $g(m) \to 0$ as $m \to 0+$. The constant $c_0 > 0$ does not depend on $S_o$, $T_o$, or $m$.

Let $0 < m < \inf_{\partial\Omega} w_o$ and take $(w-m)^- = \min(0, w-m)$ as test functions in (2.20). Then, using (A2), (2.24), and (2.7),

$$\begin{aligned}
\delta^2 \int_\Omega |\nabla(w-m)^-|^2 &= -\int w(w-m)^- T_o(h(w^2) - h(m^2)) \\
&\quad - \int w(w-m)^-(\frac{1}{2}|\nabla S|^2 + T_o h(m^2) - V - V_{ext} + \alpha S) \\
&\le \int w(-(w-m)^-)\Big(\frac{c_0}{f(T_o)g(m)}\|S_o\|^2_{C^{1,\gamma}(\overline{\Omega})} \\
&\quad + T_o h(m^2) + \underline{V} + \overline{V}_{ext} + \alpha\overline{S}\Big),
\end{aligned}$$

where $\overline{V}_{ext} = \|V_{ext}\|_{0,\infty,\Omega}$. The constant $c_8(T_o) \stackrel{\text{def}}{=} \underline{V} + \overline{V}_{ext} + \alpha\overline{S}$ depends on $T_o$ through $\underline{V}$ such that $c_8(T_o)$ can be taken to be non-increasing as $T_o$ increases (see (2.11)–(2.12)). Then

$$\delta^2 \int_\Omega |\nabla(w-m)^-|^2 \le \left(I_1 + \frac{T_o}{g(m)} I_2\right) \int w(-(w-m)^-), \tag{2.25}$$

where

$$I_1 = \frac{1}{2} T_o h(m^2) + c_8(T_o),$$
$$I_2 = \frac{c_0}{T_o f(T_o)} \|S_o\|^2_{C^{1,\gamma}(\overline{\Omega})} + \frac{1}{2} g(m) h(m^2).$$

*First case:* Let $h$ be isothermal. For arbitrary $T_o > 0$, let $\underline{w} \in (0, \inf_{\partial\Omega} w_o)$ be such that $h(\underline{w}^2) \le -2c_8(T_o)/T_o$ (using (A2)). This implies, for $m = \underline{w}$, that $I_1 \le 0$. Set $A = -\frac{1}{2} g(\underline{w}) h(\underline{w}^2) > 0$ and $\varepsilon^2 = A T_o f(T_o)/c_0$. Then, for $m = \underline{w}$ and $\|S_o\|_{C^{1,\gamma}(\overline{\Omega})} \le \varepsilon$, we obtain

$$I_2 \le \frac{c_0}{T_o f(T_o)} \varepsilon^2 - A \le 0.$$

Taking into account (2.25) we conclude that $w \ge \underline{w}$ in $\Omega$.

For arbitrary $S_o$, take $m = \underline{w} \in (0, \inf_{\partial\Omega} w_o)$ such that $h(\underline{w}^2) \le -2c_8(1)$ and let $A$ be defined as above. Choose $T_1 \ge 1$ such that $T_1 f(T_1) \ge c_0 \|S_o\|^2_{C^{1,\gamma}(\overline{\Omega})}/A$. Then we have for all $T_o \ge T_1$, since $T \mapsto c_8(T)/T$ is non-increasing,

$$h(\underline{w}^2) \le -2c_8(1) \le -2c_8(T_o)/T_o,$$

and hence $I_1 \le 0$. Since the function $T \mapsto T f(T)$ is increasing, we obtain

$$\frac{c_0}{T_o f(T_o)} \|S_o\|^2_{C^{1,\gamma}(\overline{\Omega})} \le \frac{c_0}{T_1 f(T_1)} \|S_o\|^2_{C^{1,\gamma}(\overline{\Omega})} \le A,$$

by definition of $T_1$. This implies $I_2 \le 0$ and $w \ge \underline{w}$ in $\Omega$.

*Second case:* Let $h$ be isentropic. Let $\underline{w} \in (0, \inf_{\partial\Omega} w_o)$ be such that $h(\underline{w}^2) < 0$, and let $T_2 \ge 1$ be such that $T_2 \ge -2c_8(1)/h(\underline{w}^2) > 0$ and $T_2 f(T_2) \ge c_0 \|S_o\|^2_{C^{1,\gamma}(\overline{\Omega})}/A$, where $A$ is defined as in the first case. Taking $m = \underline{w}$ and $T_o \ge T_2$, we get $I_1 \le 0$ and $I_2 \le 0$.

We conclude the proof by taking the truncation parameter $m = \underline{w}$ in (2.21).

*Remark 2.5.* We have assumed that the boundary functions $w_o$, $S_o$, and $V_o$ do not depend on the parameters, e.g. $T_o$. However, if we take $V_o = T_o h(C) + U(x)$ (see (1.15)), the above arguments also apply. Indeed, let $C_o > 0$ be such that $h(C_o) = 0$ and choose a scaling of the variables and functions such that $\inf_{\partial\Omega} C \ge C_0$ (this does not affect $T_o$). Then, for isothermal or isentropic functions, $h(\inf_{\partial\Omega} C) \ge 0$. This implies $\underline{V} = -T_o \inf_{\partial\Omega} h(C) + U \le U$, and the constant $c_8(T_o)$ can be taken non-increasing as $T_o$ increases. Note that now $\overline{V}$ also depends on $T_o$, but in such a way that the property $\overline{w} \to \infty$ as $T_o \to 0+$ remains valid.

*Remark 2.6.* Using a relaxation scaling as in [23], i.e. defining the rescaled variables $\hat{n} = n$, $\hat{S} = \alpha S = S/\tau$, $\hat{V} = V$, where $\tau = 1/\alpha$ is the scaled relaxation time, we get from (1.11)–(1.12) the equations

$$\delta^2 \Delta \hat{w} = \hat{w}\left(\frac{\tau^2}{2} |\nabla \hat{S}|^2 + T_o h(\hat{w}^2) - \hat{V} - V_{ext} + \hat{S}\right),$$
$$\mathrm{div}(\hat{w}^2 \nabla \hat{S}) = 0.$$

One may expect that the diffusive term $T_o h(\hat{w}^2)$ dominates the convective term $(\tau^2/2)|\nabla \hat{S}|^2$ for sufficiently small $\tau > 0$, which would give the existence of solutions by the presented method, for fixed $T_o$. However, we also have to transform the boundary function $\hat{S}_o = S_o/\tau = U/\tau$, and it is easy to see that then the convective term is not necessarily "small" for small relaxation times. Choosing different boundary conditions, namely $S_o = U/\alpha$, the above rescaling gives $\hat{S}_o = U$, and the estimates of the presented proofs lead to an existence result for sufficiently small $\tau > 0$ (see [8]).

*Remark 2.7.* It would be very interesting to study the small dispersion limit $\delta \to 0$ and the relaxation time limit $\tau \to 0$. However, the $W^{2,p}(\Omega)$ norm of $w$ and therefore, the lower bound $\underline{w}$ depend on $\delta$ such that $\underline{w} \to 0$ as $\delta \to 0$. Moreover, it seems difficult to identify the limits of the nonlinear functions. Concerning the relaxation time limit, it can be seen that $c_8(T_o) \to \infty$ as $\tau \to 0$ (see the proof of Theorems 2.1 and 2.2), and hence, $\underline{w} \to 0$ as $\tau \to 0$. Taking the boundary conditions discussed in Remark 2.6, we expect, however, that the limit $\tau \to 0$ can be performed (see [8]). For the small dispersion limit in thermal equilibrium states, we refer to [11]. The relaxation time limit $\tau \to 0$ of the hydrodynamic equations (i.e. $\delta = 0$ in (1.7)) is performed in [23].

## 3. Positivity and Non-Positivity Properties

We show in this section that the existence of a uniform lower bound for the density $w$ is related to the regularity of the gradient of $S$. Furthermore, we construct a generalized one-dimensional solution of a simplified problem, where the density $w$ vanishes at some point. For this solution, the Fermi potential $S$ is discontinuous.

Let (A1)–(A3) hold and let $h$ be isothermal or isentropic.

**Proposition 3.1.** *Let $(w, S, V) \in (H^1(\Omega) \cap L^\infty(\Omega))^3$ be a weak solution to (1.11)–(1.14) with $S \in W^{1,\infty}(\Omega)$. Then there exists $m > 0$ such that*

$$w(x) \geq m > 0 \qquad \textit{in } \Omega.$$

*Proof.* First let $h$ be isentropic. Then the function

$$f = \tfrac{1}{2}|\nabla S|^2 + T_o h(w^2) - V - V_{ext} + \alpha S$$

is bounded in $\Omega$. Since $w \geq 0$, we can apply Harnack's inequality [15, p. 199] to

$$\delta^2 \Delta w = wf$$

to conclude that for all subsets $\omega \subset\subset \Omega$,

$$\sup_\omega w \leq c(\omega) \inf_\omega w. \tag{3.1}$$

Now suppose that $w$ vanishes in some non-empty set $\omega_o \subset\subset \Omega$. Let $\omega_n \subset\subset \Omega$ be a sequence of sets with $\omega_o \subset \omega_n$ and $\omega_n \to \Omega$ as $n \to \infty$ in the set theoretic sense. Then (3.1) gives $w = 0$ in $\omega_n$ and, in the limit $n \to \infty$, $w = 0$ in $\Omega$. This contradicts the positivity of $w_o$ on $\partial\Omega$.

If $h$ is isothermal, we proceed as in [2]. Consider $\omega_o = \{w = 0\} \subset \Omega$. Since $wf \in L^\infty(\Omega)$, $w$ is continuous, hence $\omega_o$ is relatively closed in $\Omega$. Suppose that $\omega_o$ is nonvoid and choose $x_o \in \omega_o$. Then $wf \leq 0$ in a ball $B(x_o) \subset \Omega$ and $\Delta w \leq 0$ in $B(x_o)$. As the function $w$ assumes its nonnegative infimum 0 in $B(x_o)$, it follows that $w = 0$ in

$B(x_o)$. Thus $\omega_o$ is relatively open in $\Omega$. This implies $\omega_o = \Omega$ or $\omega_o = \emptyset$. By the positivity of $w_o$, we conclude that $w > 0$ in $\Omega$.

The existence of a uniform lower bound $m > 0$ for $w$ now follows from the continuity of $w$ in $\overline{\Omega}$.

**Corollary 3.2.** *Let $(w, S, V)$ be a weak solution to (1.11)–(1.14). Then*

$$w(x) \geq m > 0 \quad \text{a.e. in } \Omega \quad \text{if and only if} \quad S \in W^{1,\infty}(\Omega).$$

Now we consider the following simplified system in $\Omega = (0,1) \subset \mathbb{R}$:

$$\delta^2 w_{xx} = \tfrac{1}{2} w (S_x)^2 \quad \text{in } \Omega, \qquad w(0) = 1,\ w(1) = 1, \tag{3.2}$$

$$J_x = (w^2 S_x)_x = 0 \quad \text{in } \Omega, \qquad S(0) = 0,\ S(1) = U_o, \tag{3.3}$$

It can be seen that Eqs. (1.11)–(1.12) reduce to (3.2)–(3.3) for very small domains (after an appropriate asymptotic limit; see [19]). We only consider $U_o \in [0, \sqrt{2}\delta\pi]$. To solve (3.2)–(3.3) we have to distinguish the cases $U_o < \sqrt{2}\delta\pi$ and $U_o = \sqrt{2}\delta\pi$.

We say that $(w, S) \in H^1(\Omega) \times L^\infty(\Omega)$ is a *generalized solution* to (3.2)–(3.3) with $S(1) = U_o$ if there exists a sequence of weak solutions $(w_\varepsilon, S_\varepsilon) \in (H^1(\Omega))^2$ of (3.2)–(3.3) with $S(1) = U_\varepsilon$ and $U_\varepsilon \to U_o$ as $\varepsilon \to 0$ such that

$$w = \lim_{\varepsilon \to 0} w_\varepsilon, \quad S = \lim_{\varepsilon \to 0} S_\varepsilon \qquad \text{in the } L^2(\Omega) \text{ sense},$$

and for all $\phi \in H_0^1(\Omega)$ it holds

$$\lim_{\varepsilon \to 0} \delta^2 \int_\Omega (w_\varepsilon)_x \phi_x = - \lim_{\varepsilon \to 0} \frac{1}{2} \int_\Omega w_\varepsilon (S_\varepsilon)_x^2 \phi,$$
$$\lim_{\varepsilon \to 0} \int_\Omega w_\varepsilon^2 (S_\varepsilon)_x \phi_x = 0.$$

**Proposition 3.3.** *(i) Let $0 \leq U_o < \sqrt{2}\delta\pi$. Then there exists a smooth solution $(w, S) \in \left(C^2(\overline{\Omega})\right)^2$ of (3.2)–(3.3) such that*

$$w(x) \geq c(U_o) > 0 \qquad \text{in } \Omega.$$

*(ii) If $U_o = \sqrt{2}\delta\pi$ then there exists a generalized solution $(w, S) \in H^1(\Omega) \times L^\infty(\Omega)$ of (3.2)–(3.3) such that $w(\frac{1}{2}) = 0$.*

*Proof.* Let $U_o = \sqrt{2}\delta\pi$ and let $U_\varepsilon < \sqrt{2}\delta\pi$ be a sequence such that $U_\varepsilon \to U_o$ as $\varepsilon \to 0$. Set $\sigma_\varepsilon = U_\varepsilon / \sqrt{2}\delta$. A computation shows that

$$w_\varepsilon(x) = \left((1-2x)^2 + 2(1+\cos\sigma_\varepsilon)x(1-x)\right)^{1/2},$$
$$S_\varepsilon(x) = \sqrt{2}\delta \arccos \frac{1-(1-\cos\sigma_\varepsilon)x}{w_\varepsilon(x)}, \qquad x \in [0,1],$$

solve (3.2)–(3.3) with $S_\varepsilon(1) = U_\varepsilon$. Furthermore,

$$w_\varepsilon^2(x)(S_\varepsilon)_x(x) = \sqrt{2}\delta \sin\sigma_\varepsilon \tag{3.4}$$

and

$$w_\varepsilon(x) \geq \sqrt{\frac{1}{2}(1+\cos\sigma_\varepsilon)} > 0 \qquad \text{in } \Omega.$$

In the limit $\varepsilon \to 0$ we get $\cos\sigma_\varepsilon \to -1$ and

$$\begin{aligned} w_\varepsilon(x) \to w(x) &= |1-2x| && \text{in } L^2(\Omega), \\ S_\varepsilon(x) &\to \sqrt{2}\delta\, H(x) && \text{in } L^2(\Omega) \quad (\varepsilon \to 0), \end{aligned}$$

where $H(x) = 0$ for $x \in (0, 1/2)$ and $H(x) = \pi$ for $x \in (1/2, 1)$. Taking into account (3.4) we obtain for all $\phi \in H_0^1(\Omega)$,

$$\begin{aligned} &-\frac{1}{2}\int_\Omega w_\varepsilon (S_\varepsilon)_x^2 \phi = \delta^2 \int_\Omega (w_\varepsilon)_x \phi_x \to \delta^2 \int_\Omega w_x \phi_x = 4\delta^2 \phi\Big(\frac{1}{2}\Big), \\ &\int_\Omega w_\varepsilon^2 (S_\varepsilon)_x \phi_x = \sqrt{2}\delta \sin\sigma_\varepsilon \int_\Omega \phi_x \to 0 \qquad (\varepsilon \to 0). \end{aligned}$$

Therefore, $(w, S)$ is a generalized solution to (3.2)–(3.3).

## 4. Uniqueness of Solutions

Uniqueness of solutions follows under the assumption that the scaled Planck constant $\delta$ is large enough. If $\delta = 0$, there exists more than one solution of the thermal equilibrium state (i.e. $J = 0$; see [11]).

**Theorem 4.1.** *Let (A1)–(A3) hold and let $h$ be isothermal or isentropic. Then there exists $\delta_o > 0$ such that if $\delta \geq \delta_o$, there exists at most one solution $(w, S, V)$ to (1.11)–(1.14) satisfying (2.1)–(2.2).*

*Proof.* Let $(w_1, S_1, V_1)$ and $(w_2, S_2, V_2)$ be two solutions to (1.11)–(1.14) satisfying (2.1)–(2.2). Take $w_1 - w_2$ as a test function in the difference of the Eqs. (1.11) satisfied by $w_1$, $w_2$, respectively, to get

$$\begin{aligned} \delta^2 \int_\Omega |\nabla(w_1 - w_2)|^2 &= -\frac{1}{2}\int (w_1|\nabla S_1|^2 - w_2|\nabla S_2|^2)(w_1 - w_2) \\ &\quad + \int (w_1 V_1 - w_2 V_2)(w_1 - w_2) \\ &\quad - \int T_o (w_1 h(w_1^2) - w_2 h(w_2^2))(w_1 - w_2) \\ &\quad - \alpha \int (w_1 S_1 - w_2 S_2)(w_1 - w_2) \\ &\quad + \int V_{ext}(w_1 - w_2)^2 \\ &= I_1 + \cdots + I_5. \end{aligned} \tag{4.1}$$

The weak formulation of the difference of (1.12) for $S_1$, $S_2$, respectively, reads

$$\int w_1^2 \nabla(S_1 - S_2) \cdot \nabla\phi = -\int (w_1^2 - w_2^2)\nabla S_2 \cdot \nabla\phi$$

for all $\phi \in H_0^1(\Omega)$. Taking $\phi = S_1 - S_2$ we obtain

$$\begin{aligned}\underline{w}^2 \int_\Omega |\nabla(S_1 - S_2)|^2 &\le \int w_1^2 |\nabla(S_1 - S_2)|^2 \\ &= -\int (w_1^2 - w_2^2)\nabla S_2 \cdot \nabla(S_1 - S_2) \\ &\le 2\overline{w}\|w_1 - w_2\|_{0,2}\|\nabla S_2\|_{0,\infty}\|\nabla(S_1 - S_2)\|_{0,2}\end{aligned}$$

which implies

$$\|\nabla(S_1 - S_2)\|_{0,2} \le (2\overline{w}/\underline{w}^2)\|w_1 - w_2\|_{0,2}\|\nabla S_2\|_{0,\infty}. \tag{4.2}$$

Now we are able to estimate $I_1, \ldots, I_5$:

$$\begin{aligned}I_1 &= -\frac{1}{2}\int (w_1\nabla(S_1 - S_2)\cdot\nabla(S_1 + S_2) + (w_1 - w_2)|\nabla S_2|^2)(w_1 - w_2) \\ &\le \big((\overline{w}/\underline{w})^2 + 1\big)\|\nabla S_2\|_{0,\infty}(\|\nabla S_1\|_{0,\infty} + \|\nabla S_2\|_{0,\infty})\|w_1 - w_2\|_{0,2}^2,\end{aligned}$$

using (4.2). The integral $I_2$ is estimated by using (1.13):

$$\begin{aligned}I_2 &= \frac{1}{2}\int ((w_1 - w_2)^2(V_1 + V_2) + (w_1^2 - w_2^2)(V_1 - V_2)) \\ &= \frac{1}{2}\int (w_1 - w_2)^2(V_1 + V_2) - \frac{\lambda^2}{2}\int |\nabla(V_1 - V_2)|^2 \\ &\le \overline{V}\|w_1 - w_2\|_{0,2}^2.\end{aligned}$$

The monotonicity of $h$ implies

$$\begin{aligned}I_3 &= -T_o\int (w_1(h(w_1^2) - h(w_2^2))(w_1 - w_2) + (w_1 - w_2)^2 h(w_2^2)) \\ &\le -T_o h(\underline{w}^2)\|w_1 - w_2\|_{0,2}^2.\end{aligned}$$

Finally, we can estimate the integral $I_4$ employing Poincaré's inequality and (4.2):

$$\begin{aligned}I_4 &= -\alpha\int (w_1(S_1 - S_2)(w_1 - w_2) + (w_1 - w_2)^2 S_2) \\ &\le \alpha\big(c(\Omega)(\overline{w}/\underline{w})^2\|\nabla S_2\|_{0,\infty} + \overline{S}\big)\|w_1 - w_2\|_{0,2}^2.\end{aligned}$$

Let $K = \|\nabla S_1\|_{0,\infty} + \|\nabla S_2\|_{0,\infty}$. Then we get from (4.1),

$$\left(\delta^2 - 2K^2\left(\frac{\overline{w}^2}{\underline{w}^2} + 1\right) - \overline{V} - \overline{V}_{ext} + T_o h(\underline{w}^2) - \alpha\left(\frac{\overline{w}^2}{\underline{w}^2}K + \overline{S}\right)\right)\|w_1 - w_2\|_{0,2}^2 \le 0. \tag{4.3}$$

Only $K$ depends on $\delta$ (via the $W^{2,p}(\Omega)$ norm of $w$; see the third step of the proof of Lemma 2.3) such that $K$ remains bounded as $\delta \to \infty$. Therefore there exists $\delta_o > 0$ such that if $\delta \ge \delta_o$, then (4.3) implies

$$\|w_1 - w_2\|_{0,2}^2 \le 0.$$

Hence $w_1 = w_2$ in $\Omega$. Finally, we infer $S_1 = S_2$ from (4.2) and $V_1 = V_2$ from (1.13).

*Remark 4.2.* There exists at most one weak solution $(w, S, V)$ in the class of functions satisfying $w, V \in H^1(\Omega) \cap L^\infty(\Omega)$, $w(x) \geq m > 0$ in $\Omega$, and (only) $S \in W^{1,q}(\Omega)$, where $q = d$ if $d \geq 3$, $q > 2$ if $d = 2$ and $q = 2$ if $d = 1$, under the assumption that the scaled Planck constant $\delta > 0$ is large enough. The proof of this result is similar to the proof in [16].

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