Commun. Math. Phys. 194, 149 – 175 (1998)

Communications in
Mathematical Physics

Special Quantum Field Theories in Eight and Other Dimensions

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Received: 23 April 1997 / Accepted: 1 October 1997

Abstract: We build nearly topological quantum field theories in various dimensions. We give special attention to the case of eight dimensions for which we first consider theories depending only on Yang–Mills fields. Two classes of gauge functions exist which correspond to the choices of two different holonomy groups in $SO(8)$, namely $SU(4)$ and $Spin(7)$. The choice of $SU(4)$ gives a quantum field theory for a Calabi–Yau fourfold. The expectation values for the observables are formally holomorphic Donaldson invariants. The choice of $Spin(7)$ defines another eight dimensional theory for a Joyce manifold which could be of relevance in M- and F-theories. Relations to the eight dimensional supersymmetric Yang–Mills theory are presented. Then, by dimensional reduction, we obtain other theories, in particular a four dimensional one whose gauge conditions are identical to the non-abelian Seiberg–Witten equations. The latter are thus related to pure Yang–Mills self-duality equations in 8 dimensions as well as to the N=1, D=10 super Yang–Mills theory. We also exhibit a theory that couples 3-form gauge fields to the second Chern class in eight dimensions, and interesting theories in other dimensions.

1. Introduction

Topological quantum field theory (TQFT), or more specifically, cohomological quantum field theory has been extensively studied in two, three and four dimensions. (See e.g. [1, 2] and references therein.) In this article we show that theories which are almost topological also exist in dimensions higher than four. We call them BRSTQFT's instead of TQFT's. We give special attention to the case of Yang–Mills fields in eight dimensions.

A BRSTQFT relies on a Lagrangian which contains as many bosons as fermions, interconnected by a BRST symmetry. The Lagrangian density is locally a sum of d-closed and BRST-exact terms. Starting from classical "topological" invariants, the most crucial point in the construction of the BRSTQFT is the determination of gauge fixing conditions, enforced in a BRST invariant way. In the weak coupling expansion, one interprets the theory as exploring through path integrations all quantum fluctuations around the solutions to the gauge conditions. This provides, eventually, an intuitive way to study the moduli problem associated with the choice of gauge fixing conditions, by computing Green functions defined from the BRST cohomology. Generally, one must distinguish between the ordinary gauge fixing conditions for the ordinary gauge degrees of freedom of forms and the gauge covariant ones which occur when one gauge-fixes a "topological" invariant (i.e, constant on a Pontryagin sector of gauge fields). A BRSTQFT can often be untwisted into a Poincaré supersymmetric theory; we give more examples in this paper. BRSTQFT's are microscopic theories, in the sense that in principle they provide the fundamental fields to study (almost) topological properties. We ask: are their infrared limits describable by effective theories, following the ideas of Seiberg and Witten?

In four dimensions Donaldson [3] used the moduli space of anti-self-dual fields to describe invariants of four manifolds. Witten [4] interpreted these invariants as observables in a topological quantum field theory, twisted $N=2$ supersymmetric Yang–Mills. Baulieu and Singer [5] noted that this TQFT could be obtained from a topological action by the BRST formalism with covariant gauge functions which probe the moduli space of anti-self-dual fields. In this paper, we apply this formalism to higher dimensional cases of self duality; M-theory, F-theory, and low energy limits of string theory have increased the interest in QFTs in dimension greater than four.

Over a decade ago, Corrigan et al [6] classified the cases in which the self-duality equation for Yang–Mills fields in four dimensions could be generalized to higher dimensions. See also Ward [7]. Solutions to these equations are higher dimensional instantons [8, 9]. The generalizations in eight dimensions depend on having the holonomy group reduced from $SO(8)$ to *Spin*(7) or $SU(4)$. See Salamon [10] for background on special holonomy groups.

The third author (IMS) learned about self-duality in eight dimensions for Einstein manifolds and fields associated to the spin bundle from Eric Weinstein in 1990. Weinstein constructed special instantons, computed the dimensions of the corresponding moduli space, and noted the importance of *Spin*(7) and $SU(4)$. For this, and more, see [11].

The geometry for manifolds with holonomy *Spin*(7) can be found in Joyce [12]. For holonomy $SU(4)$, the holomorphic extension of Donaldson Theory is being developed by Donaldson, Joyce, Lewis, and Thomas at Oxford. Their program for extending results in two, three and four dimensions from the real to the complex case is sketched in Donaldson and Thomas [13].

In the first part of this paper we describe two eight dimensional Yang–Mills quantum field theories that reflect the eight dimensional self duality equations found in [6]; we use the geometry developed by the above-mentioned authors to construct the quantum field theory. These theories cannot be called topological for they depend on some geometrical structure of the manifold M_8. For want of a better term, we have called them BRST quantum field theories (BRSTQFT), because they are constructed by starting with a topological action and using the BRST formalism with covariant gauge functions that again probe the moduli space of these new anti-self-dual fields.

When the holonomy group is *Spin*(7) $\subset SO(8)$, we call (M_8, g) a Joyce manifold. Section 2.1.1 gives the geometry needed to construct the BRSTQFT of 2.1.2, which is in turn described geometrically in 2.1.3. Section 2.2 gives a parallel discussion of the holomorphic case, i.e., when the holonomy groups is $SU(4)$. We compare the two cases in Sect. 2.3. We point out in Sect. 2.4 that the J-case is a twist of $D=10$, $N=1$ supersymmetric Yang–Mills theory (SSYM) dimensionally reduced to $D=8$. Since supersymmetries for a curved manifold require covariant constant spinors, there is one remaining supersymmetry; we explain its relation to the topological BRST symmetry.

Having defined pure Yang–Mills BRSTQFT in eight dimensions, we introduce a different theory in Sect. 3 which couples an uncharged 3-form gauge field B_3 to the Yang–Mills field A. We propose as covariant gauge conditions of the coupled systems, the pair of equations

$$F_A = *\Omega \wedge F_A,$$
$$\mathrm{Tr}\,(F_A \wedge F_A)_+ + dB_3 + *dB_3 = 0, \tag{1.1}$$

where Ω is a background closed 4-form. One must be careful here; B_3 is not an ordinary 3-form and dB_3 is not its differential. Rather, B_3 is locally defined, up to an exact 3-form so that dB_3 stands for a closed 4-form. (See the discussion in Sect. 3).

Section 4 discusses other dimensions. When M_{12} is a Calabi–Yau 6-fold, one can define BRSTQFT's and we do so. We reduce our 8D theories to 6D and 4D in sections 4.2 and 4.3, respectively. The H case reduction can be obtained directly on a Calabi–Yau 3-fold by a modification of the methods in Sect. 2.2.

The reduction to 4D is particularly interesting. On the one hand we get a twisted $N = 4$ SSYM of Vafa and Witten [16]. In fact, the H, J cases and the case of M_7, holonomy G_2 theory, reduced to 4D, give the three twists of $N = 4$ SSYM. On the other hand we also get the nonabelian Seiberg–Witten theory. Thus there is a relationship between $N = 4$ SSYM and nonabelian SW theories. The latter theory is obtained from the eight dimensional J theory, with its octonionic structure; the former is obtained from the $N = 1, D = 10$ SSYM theory, by ordinary dimensional reduction. The direct link between the $D = 10$ SSYM theory and the J theory is that the $N = 1, D = 10$ SSYM theory gives by dimensional reduction the $N = 1, D = 8$ SSYM which can be identified with the J theory by a simplest twist, specific to eight dimensions, which interchanges vectors and spinors (Sect. 2.4).

2. Pure Yang–Mills 8 Dimensional Case

The four dimensional Yang–Mills TQFT can be obtained by the BRST formalism. Starting with $p_1 = \frac{1}{8\pi^2}\mathrm{Tr}\ F \wedge F$, one gauge-fixes its invariances with three covariant gauge conditions and one Feynman-Landau gauge condition that probe the moduli space of self-dual curvature fields [5]. These gauge conditions are enforced in a BRST invariant way, by using the 4 gauge freedom of local general infinitesimal variations of the connection A_μ. Put mathematically, we get an elliptic complex $0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2_+ \to 0$, tensored with a Lie algebra $\mathcal{G}$.

In this section we extend this scheme to 8 dimensions when the holonomy group in $SO(8)$ is either $SU(4)$ (the case of a Calabi–Yau 4-fold) or $Spin(7)$ (the case of a Joyce manifold). The 4-D self duality equations must be generalized to

$$\lambda F^{\mu\nu} = \frac{1}{2} T^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{2.1}$$

where λ is a constant (an eigenvalue) and $T^{\mu\nu\rho\sigma}$ is a totally antisymmetric tensor which is generally not invariant under general $SO(D)$ transformations. Rather it is invariant under a subgroup of $SO(D)$. Corrigan et al [6] classified the possible choices of $T^{\mu\nu\rho\sigma}$ up to eight dimensions, where two solutions T are singled out. Indeed, for these cases, the space of 2-forms Λ^2 decomposes into a direct sum and one can thus replace the self-duality condition in four dimensions by the condition that the curvature fields lie in an appropriate summand. The elliptic complex above has an 8-D counterpart: $0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} P_+(\Lambda^2) \to 0$. Moreover, in each case, there is a closed 4-form Ω and one can replace p_1 by

$$\frac{1}{8\pi^2}\Omega \wedge \mathrm{Tr}\,(F \wedge F). \tag{2.2}$$

Since $\int_X \Omega \wedge \mathrm{Tr}\,(F \wedge F)$ is independent of the gauge field A and since the new elliptic complex implies that the number of gauge covariant gauge functions plus Feynman-Landau type gauge condition is eight, one can use the BRST formalism to introduce new (ghosts and ghosts of ghosts) fields and an invariant action. The theory is not topological, because it depends on the reduction of the holonomy group. In the case of the $SU(4)$ reduction, one predicts that the expectation value of the observables depends on the holomorphic structure of X, but not on the choice of the Calabi–Yau metrics. We call these theories BRSTQFT's. We will say the BRSTQFT is of type J for $Spin(7)$ and of type H for $SU(4)$. We will analyze each case. They differ in a subtle way from the point of view of BRST quantization. In the type H case one has 6 independent real covariant gauge conditions which can be seen as three complex 4-D self-duality conditions. We can complete them by a *complex* supress gauge condition which counts for the two missing gauge conditions allowed by the eight freedom in deforming the Yang–Mills field. In the type J case one has seven independent real equations which we can complete by the usual (real) Landau gauge condition. In the former case one has thus a complexification of all ingredients of the 4-D case. In the latter case all fields are real, and the situation is quite like the 4-D case, with the change of the quaternionic structure of the self duality equations in four dimensions into an octonionic one in eight dimensions.

The action we consider will be the BRST invariant gauge fixing of the topological invariant

$$S_0 \;=\; \frac{1}{2}\int_{M_8} \Omega \wedge \mathrm{Tr}\,(F \wedge F), \tag{2.3}$$

where Ω is a fixed closed four form adapted to each case. Depending on the case, we will have six or seven covariant gauge fixing conditions of the type of Eq. (2.1), that we will denote as $\Phi_i = 0$, $1 \le i \le 6$ or 7. That we get an action containing a Yang–Mills part relies on the identity

$$a\sum_i \mathrm{Tr}\,(\Phi_i\Phi_i)\cdot\;(vol) \;=\; -S_0 + \;\mathrm{Tr}\,(F \wedge *F), \tag{2.4}$$

where a is a positive real number (one has different decompositions in the J and H cases). (vol) stands for the volume form. The last term is the action density for the Yang–Mills theory. Hence a solution to $\Phi_i = 0$ gives a stationary point of the eight dimensional Yang–Mills theory. For this reason, the equations $\frac{1}{2}T^{\mu\nu\rho\sigma}F_{\rho\sigma} = \lambda F^{\mu\nu}$, deserve to be called the instanton equation. Notice that one has the correspondence $\Omega_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma\alpha\beta\gamma\delta}T^{\alpha\beta\gamma\delta}$. By adding to S_0 a BRST exact term which generates among other terms $\sum_i(\Phi_i\Phi_i)$, we will thus replace the "topological" invariant $\Omega \wedge \mathrm{Tr}\,(F \wedge F)$ by the standard Yang–Mills Lagrangian $\mathrm{Tr}\,(F \wedge *F)$ plus ghost terms, which constitute the action of the BRSTQFT theory. As explained earlier, the term BRSTQFT seems to us more appropriate than the term TQFT for the resulting theory. Obviously, the remaining gauge invariances must be gauge fixed, which will be done in the same spirit, as in [5].

2.1. Type J case: Joyce Manifold.

2.1.1. Geometrical setup. Recently it has been proposed that the 7 dimensional and the 8 dimensional Joyce manifolds provide a compactification to four dimensions of M-theory and F-theory, respectively [17, 18, 19]. We consider here the 8 dimensional case and call a Joyce manifold an eight dimensional manifold with $Spin(7)$ holonomy [12] [1]. Then $Spin(7)$ acting on $\Lambda^4(M_8)$, the space of 4-forms, leaves invariant a self-dual 4-form $\Omega \not\equiv 0$. Further, Ω is covariantly constant and hence closed. The space of 2-forms $\Lambda^2(M_8)$ splits into $\Lambda^2_{21} \oplus \Lambda^2_+$ with $\dim_{\mathbf{R}} \Lambda^2_+ = 7$. One can see this by noting that $\Lambda^2 \simeq so(8)$ and that $\Lambda^2_{21} \simeq$ Lie algebra of $Spin(7) \subset so(8)$. The splitting can also be obtained as follows: let T be the operator on Λ^2 given by $\tau \to *(\Omega \wedge \tau)$. Then T is self adjoint with eigenvalues +1 and -3, when Ω is scaled. Its eigenspaces are Λ^2_{21} and Λ^2_+, respectively. The ordinary anti-self-dual Yang–Mills fields in four dimensions are now to be replaced by $(P_+F_A) = 0$, where P_+ is the projection of Λ^2 onto Λ^2_+. We next discuss the linearization of this equation.

Let S^+_M and S^-_M (that is, $\mathbf{8_s}$ and $\mathbf{8_c}$ in another notation) denote the chiral and antichiral real (Majorana) spinors for M_8 (M_8 is simply connected and has a unique spin structure). Then the representation of $Spin(7)$ on S^+_M is the direct sum $\mathbf{R} \oplus \mathbf{V}$ (that is, $\mathbf{8_s} = \mathbf{1} \oplus \mathbf{7}$). Let ζ be a covariantly constant spinor field of norm 1 giving the splitting of S^+_M. The representation of $Spin(7)$ on S^-_M is irreducible. Since $S_M \otimes S_M$ is isomorphic to forms, tensoring by ζ identifies spinors with forms. For example, $\Lambda^2(S^+_M) \simeq \Lambda^2(M_8)$; so $\Lambda^2(S^+_M) = \Lambda^2(\mathbf{R} \oplus \mathbf{V}) = \mathbf{V} \wedge \mathbf{V} + \zeta \otimes \mathbf{V}$ gives the splitting into $\Lambda^2_{21} \oplus \Lambda^2_+$. Further $\zeta \otimes S^-_M$ can be identified with $\Lambda^1(M_8)$, that is, $\mathbf{8_v}$. We conclude that the sequence $0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{P_+d} \Lambda^2_+ \to 0$ is an elliptic sequence and $(P_+d + d^*) : \Lambda^1 \to \Lambda^2_+ \oplus \Lambda^0$ is the Dirac operator $\partial\!\!\!/ : S^-_M \to S^+_M$, after the identification of spinors with forms due to ζ.

If P is a principal bundle over M_8 with a compact gauge group G, we can couple forms to its Lie algebra $\mathcal{G}$ by a vector potential A. We have the sequence $0 \to \Lambda^0 \otimes \mathcal{G} \xrightarrow{D_A} \Lambda^1 \otimes \mathcal{G} \xrightarrow{P_+D_A} \Lambda^2_+ \otimes \mathcal{G} \to 0$ which is elliptic when $P_+D^2_A = 0$, i.e. when $P_+F_A = 0$. (Here we have identified the Lie algebra $\mathcal{G}$ with the adjoint Lie algebra bundle over M_8.) In general, $P_+D_A + D^*_A = D\!\!\!\!/_A : \Lambda^1 \otimes \mathcal{G} \to \Lambda^2_+ \otimes \mathcal{G} + \Lambda^0 \otimes \mathcal{G}$ is elliptic. The index of the operator is the virtual dimension of the moduli space $\mathcal{M}_J$ of solutions to the nonlinear equation $P_+F_A = 0$, modulo gauge transformations.

To make contact with the next section, let us remark that $P_+F_A = 0$ determines, in the case of a pure Yang–Mills BRSTQFT, the relevant gauge covariant gauge conditions shown in Eq. (2.1), while D^*_A is the operator related to the Landau-Feynman gauge condition of ordinary gauge degrees of freedom.

More precisely, the BRSTQFT that will be determined shortly is the gauge fixing by BRST techniques of $S_0[A] = \int_{M_8} \Omega \wedge \mathrm{Tr}\,(F \wedge F)$. The latter is independent of A, since it is $8\pi^2 \Omega \cup p_1(P)$ which only depends on the topological charge of A.

The way one gets the Yang–Mills action from the gauge fixing of an invariant is the consequence of the following. If ω is an element of Λ^2, let ω_- and ω_+ be its components on Λ^2_{21} and Λ^2_+. Then $\|\omega\|^2 = \|\omega_+\|^2 + \|\omega_-\|^2$, $\langle \omega_+, \omega_- \rangle = 0$, while

[1] There is another class of Joyce manifolds in seven dimensions [20]. Its holonomy is the exceptional group G_2. Both classes of Joyce manifolds have been studied in superconformal field theory [21, 22].

$$\begin{aligned}\Omega \wedge F \wedge F &= \Omega \wedge (F_+ + F_-) \wedge (F_+ + F_-) \\ &= \Omega \wedge F_+ \wedge F_+ + \Omega \wedge F_- \wedge F_- + \Omega \wedge F_- \wedge F_+ + \Omega \wedge F_+ \wedge F_- \\ &= -3 * F_+ \wedge F_+ + *F_- \wedge F_- + *F_- \wedge F_+ - 3 * F_+ \wedge F_- . \end{aligned} \tag{2.5}$$

Thus

$$\int_{M_8} \mathrm{Tr}\ (\Omega \wedge F \wedge F) = \| F_- \|^2 - 3 \| F_+ \|^2, \tag{2.6}$$

and

$$\| F_A \|^2 = \int_{M_8} \mathrm{Tr}\ (\Omega \wedge F_A \wedge F_A) + 4 \| F_+ \|^2 . \tag{2.7}$$

$\Omega \wedge \Omega$ orients M_8 and is the volume element. Given the topologial sector, we choose Ω so that $\int_{M_8} \mathrm{Tr}\,(\Omega \wedge F_A \wedge F_A) \geq 0$. Then $F_+ = 0$ minimizes the action $\| F_A \|^2$.

To write the BRSTQFT action in physicist's notation, we have to be more explicit. In terms of an orthonormal basis, the self-dual four form is

$$\begin{aligned}\Omega = {} & e_1 \wedge e_2 \wedge e_5 \wedge e_6 + e_1 \wedge e_2 \wedge e_7 \wedge e_8 + e_3 \wedge e_4 \wedge e_5 \wedge e_6 \\ & + e_3 \wedge e_4 \wedge e_7 \wedge e_8 + e_1 \wedge e_3 \wedge e_5 \wedge e_7 - e_1 \wedge e_3 \wedge e_6 \wedge e_8 \\ & - e_2 \wedge e_4 \wedge e_5 \wedge e_7 + e_2 \wedge e_4 \wedge e_6 \wedge e_8 - e_1 \wedge e_4 \wedge e_5 \wedge e_8 \\ & - e_1 \wedge e_4 \wedge e_6 \wedge e_7 - e_2 \wedge e_3 \wedge e_5 \wedge e_8 - e_2 \wedge e_3 \wedge e_6 \wedge e_7 \\ & + e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_5 \wedge e_6 \wedge e_7 \wedge e_8, \end{aligned} \tag{2.8}$$

where e_i $(i = 1, \ldots, 8)$ are vielbein fields.

The operator T defined above can be written as the following $Spin(7)$ invariant fourth rank antisymmetric tensor

$$T^{\mu\nu\rho\sigma} = \zeta^T \gamma^{\mu\nu\rho\sigma} \zeta, \tag{2.9}$$

where $\gamma^{\mu\nu\rho\sigma}$ is the totally antisymmetric product of γ matrices for the $SO(8)$ spinor representation;

$$\gamma^{\mu\nu\rho\sigma} = \frac{1}{4!} \gamma^{[\mu} \gamma^{\nu} \gamma^{\rho} \gamma^{\sigma]}, \tag{2.10}$$

and ζ is the covariantly constant spinor introduced above to identify spinors with forms. This gives another component representation of the four form Ω. To repeat the first paragraph of this section in terms of the fourth rank tensor $T^{\mu\nu\rho\sigma}$, we define an analogue of the instanton equation on the Joyce manifold [6];

$$F^{\mu\nu} = \frac{1}{2} T^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad \text{i.e.} \quad F \in \Lambda^2_+. \tag{2.11}$$

The curvature 2-form $F_{\mu\nu}$ in 8 dimensions has 28 components, whose $Spin(7)$ decomposition is $\mathbf{28} = \mathbf{7} \oplus \mathbf{21}$. (This is made explicit by the eigenspace decomposition of the action of $\frac{1}{2} T^{\mu\nu\rho\sigma}$ in Eq. (2.1) with the eigenvalues $\lambda = -3$ and $\lambda = 1$.)

Equation (2.11) can be written as seven independent equations, showing that the curvature has no components in the former subspace which is 7-dimensional

$$F_{8i} = c_{ijk} F_{jk}, \quad 1 \leq i, j, k \leq 7, \tag{2.12}$$

Equation (2.12) makes the octonionic structure explicit. Indeed, the c_{ijk} are the structure constants for octonions[2] and the eight dimensional tensors $T_{\mu\nu\rho\sigma}$ can be written as [3]

$$\begin{aligned} T_{8ijk} &= c_{ijk}, \quad 1 \leq i,j,k \leq 7 \\ T_{lijk} &= \frac{1}{24}\epsilon^{lijkabc} c_{abc}, \quad 1 \leq i,j,k,l \leq 7. \end{aligned} \tag{2.13}$$

Notice that by construction, the $T_{\mu\nu\rho\sigma}$ are self-dual objects in 8 dimensions. Computed explicitly, Eq. (2.11) is

$$\begin{aligned} \Phi_1 &\equiv F_{12} + F_{34} + F_{56} + F_{78} = 0, \\ \Phi_2 &\equiv F_{13} + F_{42} + F_{57} + F_{86} = 0, \\ \Phi_3 &\equiv F_{14} + F_{23} + F_{76} + F_{85} = 0, \\ \Phi_4 &\equiv F_{15} + F_{62} + F_{73} + F_{48} = 0, \\ \Phi_5 &\equiv F_{16} + F_{25} + F_{38} + F_{47} = 0, \\ \Phi_6 &\equiv F_{17} + F_{82} + F_{35} + F_{64} = 0, \\ \Phi_7 &\equiv F_{18} + F_{27} + F_{63} + F_{54} = 0. \end{aligned} \tag{2.14}$$

In this form, the gauge functions are ready to be used to define the BRSTQFT action.

It is known (see [8, 9, 14]) that at least one instanton solution exists for the 8 dimensional equation $F^{\mu\nu} = \frac{1}{2}T^{\mu\nu\rho\sigma}F_{\rho\sigma}$.[4,5] Finally, Eqs. (2.5)-(2.7) imply

$$4\sum_{i=1}^{7} \mathrm{Tr}\,(\Phi_i\Phi_i)\cdot\ (vol) \ = \ -\Omega \wedge \mathrm{Tr}\,(F \wedge F) + \mathrm{Tr}\,(F \wedge *F). \tag{2.15}$$

2.1.2. Action and observables. In the following all the fields are Lie algebra valued and we will suppress the Lie algebra indices. We use the standard notation (ψ_μ, ϕ) for topological ghost. We also introduce the Faddeev–Popov ghost c to define a completely nilpotent BRST transformation. The topological BRST transformation for the gauge field and the ghost fields is

$$\begin{aligned} sA_\mu &= \psi_\mu + D_\mu c, \quad s\psi_\mu \ = \ -D_\mu\phi - [c,\psi_\mu], \\ sc &= \phi - \frac{1}{2}[c,c], \quad s\phi \ = \ -[c,\phi]. \end{aligned} \tag{2.16}$$

We need as many pairs of the anti-ghost and the auxiliary fields (χ_i, H_i) as topological gauge functions, with the following BRST transformation law;

$$s\chi_i = H_i - [c,\chi_i], \quad sH_i = [\phi,\chi_i] - [c,H_i]. \tag{2.17}$$

One has $1 \leq i \leq 7$. The gauge fixed action at the first stage is

---

[2] If we decompose the octonions into its one dimensional real part and 7 dimensional imaginary part, R^7, then $*_7(\Omega|_{R^7})$ is a 3-form α which determines Cayley multiplication on R^7 by $\alpha(z,y,z) =< x,y,z >$.

[3] In the four dimensional case one has similar equations, with the indices i,j,k running from 1 to 3. Then the coefficients c_{ijk} are the structure constants for quaternions. The holomorphic H case that we will shortly analyze is thus a theory with a complexified quaternionic structure.

[4] It is also known that a solution exists in seven dimensions if one replaces $Spin(7)$ by G_2 (see [15]).

[5] An interesting problem is to find conditions on a curved compact Joyce manifold M_8 so that such instantons exist.

$$
\begin{aligned}
S_1 &= \frac{1}{2}\int_{M_8} \Omega \wedge \mathrm{Tr}\,(F \wedge F) \; + s\left[\frac{1}{2}\int_{M_8} d^8x\sqrt{g}\,\mathrm{Tr}\,(\chi_i\Phi_i + \frac{1}{2}\chi_i H_i)\right] \\
&= \frac{1}{2}\int_{M_8} \Omega \wedge \mathrm{Tr}\,(F \wedge F) \\
&\quad + \frac{1}{2}\int_{M_8} d^8x\sqrt{g}\,\mathrm{Tr}\left(H_i\Phi_i + \frac{1}{2}H_iH_i - \chi_i(D\psi)_i + \frac{1}{2}\phi[\chi_i,\chi_i]\right), \qquad (2.18)
\end{aligned}
$$

where $(D\psi)_i$ is the FP ghost independent part of $s\Phi_i$. Eliminating the auxiliary fields H_i by Eq. (2.15), one recovers the standard Yang–Mills kinetic term

$$
S_1 = \int_{M_8} d^8x\sqrt{g}\,\mathrm{Tr}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \chi_i(D\psi)_i + \frac{1}{2}\phi[\chi_i,\chi_i]\right). \qquad (2.19)
$$

Notice that the fermion terms break the $SO(8)$ global invariance down to G_2, for which the octonion structure coefficient in Eq. (2.12) is an invariant tensor. The gauge fixing and Faddeev–Popov ghost dependence have not been considered yet: the first stage action has still a gauge symmetry in the ordinary sense. To fix it completely we take two more conditions

$$
D \cdot \psi = 0, \quad \partial \cdot A = 0. \qquad (2.20)
$$

(The meaning of the scalar product is the usual one, e.g. $D \cdot \Psi = D_\mu \Psi^\mu$.) Introducing additional fields $(\bar{\phi}, \eta)$ and $(\bar{c}, B)$ with the BRST transformation law,

$$
\begin{aligned}
s\bar{\phi} &= \eta - [c,\bar{\phi}], \quad s\eta = [\phi,\bar{\phi}] - [c,\eta], \\
s\bar{c} &= B - [c,\bar{c}], \quad sB = [\phi,\bar{c}] - [c,B], \qquad (2.21)
\end{aligned}
$$

we write the complete action as

$$
\begin{aligned}
S_2 &= S_1 + s\left[\int_{M_8} d^8x\sqrt{g}\,\mathrm{Tr}\,(\bar{\phi}D\cdot\psi + \bar{c}\partial\cdot A + \frac{1}{2}\bar{c}B)\right] \\
&= \int_{M_8} d^8x\sqrt{g}\,\mathrm{Tr}\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \chi_i(D\psi)_i + \frac{1}{2}\phi[\chi_i,\chi_i]\right. \\
&\quad + \eta D\cdot\psi + \bar{\phi}D\cdot D\phi - \psi\cdot[\bar{\phi},\psi] + B\partial\cdot A + \frac{1}{2}B^2 + \bar{c}\partial\cdot Dc \\
&\quad \left. - \bar{c}\partial\cdot\psi + \partial\cdot A[c,\bar{c}] - \frac{1}{2}\phi[\bar{c},\bar{c}]\right]. \qquad (2.22)
\end{aligned}
$$

A natural set of topological observables is derived from the topological invariants

$$
\frac{1}{2}\int_{M_8} \Omega \wedge \mathrm{Tr}\,(F \wedge F), \quad \int_{M_8} \mathrm{Tr}\,(F \wedge F \wedge F \wedge F). \qquad (2.23)
$$

The method of the descent equation implies a ladder of topological invariants and, for example, gives the following descendants:

$$
\begin{aligned}
\mathcal{O}^{(0)} &= \frac{1}{2}\int_{M_8} \Omega \wedge \mathrm{Tr}\,(F \wedge F),\\
\mathcal{O}^{(1)} &= \int_{\gamma_7} \Omega \wedge \mathrm{Tr}\,(\psi \wedge F),\\
\mathcal{O}^{(2)} &= \int_{\gamma_6} \Omega \wedge \mathrm{Tr}\,(\frac{1}{2}\psi \wedge \psi - \phi \wedge F),\\
\mathcal{O}^{(3)} &= -\int_{\gamma_5} \Omega \wedge \mathrm{Tr}\,(\psi \wedge \phi),\\
\mathcal{O}^{(4)} &= \frac{1}{2}\int_{\gamma_4} \Omega \wedge \mathrm{Tr}\,(\phi \wedge \phi).
\end{aligned}
\tag{2.24}
$$

The descendant $\mathcal{O}^{(k)}$ with ghost number k is an integral over an $(8-k)$ cycle $\gamma_{(8-k)}$.

2.1.3. Geometric interpretation. The virtual dimension of the moduli space $\mathcal{M}_J$ of solutions to $P_+F_A = 0$ is $-index\ \ \not{\partial} \otimes I_{\mathcal{G}}$, i.e., the index of $\not{\partial} \otimes I_{\mathcal{G}} : S^- \otimes \mathcal{G} \to S^+ \otimes \mathcal{G}$. Its value is

$$
-\int_{M_8} \hat{A}(M_8)\ \mathrm{ch}(\mathcal{G}), \tag{2.25}
$$

computable in terms of the relevant characteristic classes. We will discuss the vanishing theorem needed to make the virtual dimension equal to the actual dimension elsewhere.

We can interpret Sect. 2.1.1 geometrically analogous to Sect. 5 in [5]. The BRST equations in this section are the analogues of (7) in [5], and are the structure equations for the universal connection on $\mathcal{A}/\mathcal{G} \times M_8$ with structure group G. The curvature 2-form $\mathcal{F}$ for this universal connection equals $\mathcal{F}_2^0 + \mathcal{F}_1^1 + \mathcal{F}_0^2$, where $\mathcal{F}_{2-i}^i$ is an i-form in the $\mathcal{A}/\mathcal{G}$ direction (ghost number) and a $(2-i)$-form in the M_8 direction. Note that $\mathcal{F}_2^0$ at (A, x) is $F_A(x)$ and $\mathcal{F}_1^1$ assigns to $\tau \in T(\mathcal{A}/\mathcal{G}, A)$ and $v \in T(M_8, v)$ the value $\tau(v) \in \mathcal{G}$, since τ is a 1-form on M_8. Further, $\mathcal{F}_0^2$ on $\tau_1, \tau_2 \in T(\mathcal{M}_J, A)$ is $G(b^*_{\tau_1}(\tau_2))$ where $G = (D_A^* D_A)^{-1}$ on $\Lambda^0 \otimes \mathcal{G}$ and $b_{\tau_1}(f) = [\tau_1, f]$ for $f \in \Lambda^0 \otimes \mathcal{G}$; $b^*_{\tau_1}$ is the adjoint of b_{τ_1}. We restrict $\mathcal{F}$ to $\mathcal{M}_J \times M_8$ and consider $c_2 = \frac{1}{8\pi^2}\mathrm{Tr}\,(\mathcal{F} \wedge \mathcal{F})$ a 4-form on $\mathcal{M}_J \times M_8$. Its expansion contains $\frac{1}{8\pi^2}\mathrm{Tr}\,(\mathcal{F}_1^1 \wedge \mathcal{F}_1^1)$, which has ghost number 2. This 4-form assigns to $\tau_1, \tau_2 \in T(\mathcal{M}_J, A)$ and $v_1, v_2 \in T(M_8, x)$ the value $\frac{1}{8\pi^2}(\mathrm{Tr}\,(\tau_1(v_1)\tau_2(v_2)) - \mathrm{Tr}\,(\tau_1(v_2)\tau_2(v_1))$. Let $\tau_1 \tilde{\wedge} \tau_2$ denote this 2-form on M_8.

Let c^k_{4-k} be the component of c_2 which is of degree k in the $\mathcal{M}_J$ direction and of degree $4-k$ in the M_8 direction. Then $\int_{\gamma_k} \Omega \wedge c^k_{4-k}$ gives a k-form on $\mathcal{M}_J$, when γ_k is a $(8-k)$-cycle on M_8, $k = 0, 1, 2, 3$ or 4. These are the observables $\mathcal{O}^{(k)}$ in Eq. (2.24). Taking products of the forms $\mathcal{O}$ and integrating them over $\mathcal{M}_J$ gives the expectation values of the products of observables. *We are not addressing the central problem of integrating a form over the non compact space* $\mathcal{M}_J$. We can specialize to 6-cycles, or equivalently to 2-forms to get a closer analogy to Donaldson invariants: if $\sigma \in H^2(M_8)$, let Σ_σ be the 2-form on $\mathcal{M}_J$ given by $\Sigma_\sigma(\tau_1, \tau_2) = \int_{M_8} \Omega \wedge \tau_1 \tilde{\wedge} \tau_2 \wedge \sigma$. We get an r-symmetric multi-linear function on $H^2(M_8)$ given by $(\sigma_1, \ldots, \sigma_r) \to \int_{\mathcal{M}_J} \Sigma_{\sigma_1} \wedge \ldots \wedge \Sigma_{\sigma_r}$, if dim $\mathcal{M}_J = 2r$. Of course the issue here is to make these invariants well-defined and to see how they depend on the space of Joyce manifolds modulo diffeomorphisms for a fixed M_8.

2.2. Type H: Calabi–Yau Complex 4-manifold.

2.2.1. Geometrical setup. Suppose now that the holonomy group for (M_8, g) with metric g is $SU(4)$. So M_8 is a complex manifold and we can assume that g is a Calabi–Yau metric with a Kähler 2-form ω. We choose a holomorphic covariantly constant (4,0)-form Ω which trivializes the canonical bundle K. We normalize Ω so that $\Omega \wedge \overline{\Omega}$ is the volume element of M_8. We also choose the trivial $\sqrt{K}$ for the spin structure on M_8.

We know that complex spinors can be identified with forms: $S^{\pm}_M \otimes \mathbf{C} \simeq \Lambda^{0,\,\substack{\text{even}\\ \text{odd}}}$ and the Dirac operator with $\bar\partial + \bar\partial^*$. Real Majorana spinors $S_M \subset S_M \otimes \mathbf{C}$ are the fixed points of a conjugation b on $S_M \otimes \mathbf{C}$. We can identify b with a conjugate linear $*$ operator as follows. For any Calabi–Yau M_{2n}, define $* : \Lambda^{0,p} \to \Lambda^{0,n-p}$ by $\langle \alpha, \beta \rangle = \int_{M_{2n}} \Omega \wedge \alpha \wedge \; *\beta$, where now $\Omega \in \Lambda^{n,0}$. (If one denotes by $*_1$ the usual map on complex manifolds: $\Lambda^{p,q} \to \Lambda^{n-q,n-p}$, then $\overline{*_1}\ \bar{} = \Omega \wedge *$ on $\Lambda^{0,q}$.) When $n = 4$, one can show that conjugation b equals $(-1)^q *$ on $\Lambda^{0,q}$. Consequently, the operator $\bar\partial^* + P_+\bar\partial : \Lambda^{0,1} \to \Lambda^{0,0} + \Lambda^{0,2}_+$ is the Dirac operator from $S^-_M \to S^+_M$. Here $\Lambda^{0,2}_\pm$ is the $\pm$ eigenspace of $*$, $P_\pm$ is the projection of $\Lambda^{0,2}$ on $\Lambda^{0,2}_\pm$; we have identified $\Lambda^{0,1}$ with $\frac{1-*}{2}(\Lambda^{0,1} + \Lambda^{0,3})$ and $\Lambda^{0,0}$ with $\frac{1+*}{2}(\Lambda^{0,0} + \Lambda^{0,4})$. The sequence $\Lambda^0 \xrightarrow{\bar\partial} \Lambda^{0,1} \xrightarrow{P_+\bar\partial} \Lambda^{0,2}_+$ is elliptic and is the linearization of the equation $P_+F_A = 0$, modulo gauge transformations.

Suppose now (E, ρ) is a complex Hermitian vector bundle over M_8 with metric ρ of $\dim_{\mathbf{C}} = N$. If A is a connection for E, we have its covariant differential $D_A : C^\infty(E) \to C^\infty(E \otimes \Lambda^1)$ so that $D_A = \partial_A + \bar\partial_A$ with $\bar\partial_A : C^\infty(E) \to C^\infty(E \otimes \Lambda^{0,1})$. By introducing local complex coordinates z^μ, $\bar\partial_A(ß^I) = (\partial_{\bar\mu} + (A^I_J)_{\bar\mu} ß^J) d\bar z^\mu$, $I, J = 1, \ldots, N$. So $(A^I_J)_{\bar\mu} d\bar z^\mu$ is a (0,1)-form on M_8 with $N \times N$ matrix coefficients.

The 1-form connection A with values in $GL(N, \mathbf{C})$ does not split naturally into $\Lambda^{0,1} + \Lambda^{1,0}$ unless E is holomorphic. A splitting can be obtained by a choice of almost complex structure on the principal bundle. See Bartolomeis and Tian [24]. In any case, the curvature F_A can be decomposed as $F_A = F^{2,0}_A + F^{1,1}_A + F^{0,2}_A$ with $F^{0,2}_A = \bar\partial^2_A$.

For each $\bar\partial$ operator: $C^\infty(E) \to C^\infty(E \otimes \Lambda^{0,1})$, there exists a unique connection A such that (i) A preserves the hermitian metric ρ of E and (ii) $(D_A)^{0,1} = \bar\partial$. Hence, the space $\mathcal{A}_P$ of $\bar\partial$ operators can be identified with the connections of the principal bundle P associated with E, which preserve the Hermitian metric. The group of complex gauge transformations $\mathbf{H}$ acts on the space $\mathcal{A}_P$, because if $h \in \mathbf{H}$, then $h^{-1}\bar\partial h$ is also a $\bar\partial$ operator.

Let $\mathcal{G}$ be $gl(N, \mathbf{C})$. Then the sequence $\Lambda^0 \otimes \mathcal{G} \xrightarrow{\bar\partial_A} \Lambda^{0,1} \otimes \mathcal{G} \xrightarrow{P_+\bar\partial_A} \Lambda^{0,2}_+ \otimes \mathcal{G}$ is still elliptic on the symbol level. We say $\bar\partial_A$ is holomorphic anti-self-dual if $P_+F^{0,2}_A = 0$, in which case the sequence is elliptic. Its index is the index of $\not{\partial} \otimes I_{\mathcal{G}} : S^-_M \otimes \mathcal{G} \to S^+_M \otimes \mathcal{G}$.

Again, the BRSTQFT will be obtained by gauge fixing $S_0 = \int_{M_8} \Omega \wedge \mathrm{Tr}\,(F^{0,2}_A \wedge F^{0,2}_A)$. S_0 is independent of A, because $S_0 = 8\pi^2 \Omega \cup p_1(E)$, since $\Omega \in \Lambda^{4,0}$. When $S_0 \neq 0$, we can normalize Ω further by $e^{i\theta}$, so that S_0 is real and positive.

To verify Eq. (2.4) in the H case, we reduce $\mathcal{G}$ to $u(N)$, using the metric ρ. If $\omega \in \Lambda^{0,2}$ has components $\omega_\pm$ in $\Lambda^{0,2}_\pm$, then $\| \omega \|^2 = \| \omega_+ \|^2 + \| \omega_- \|^2$. And

$$
\begin{aligned}
0 \le S_0 &= \mathrm{Tr} \int_{M_8} \Omega \wedge (F^{0,2}_{A+} + F^{0,2}_{A-}) \wedge (F^{0,2}_{A+} + F^{0,2}_{A-}) \\
&= -\, \| F^{0,2}_{A+} \|^2 + \| F^{0,2}_{A-} \|^2 + i\mathrm{Im}\, \langle F^{0,2}_{A+}, F^{0,2}_{A-} \rangle \\
&= -\, \| F^{0,2}_{A+} \|^2 + \| F^{0,2}_{A-} \|^2 \,.
\end{aligned}
\tag{2.26}
$$

Hence

$$\| F_A^{0,2} \|^2 = 2 \| F_{A^+}^{0,2} \|^2 + S_0. \tag{2.27}$$

So the holomorphic anti-self-dual gauge condition minimizes the action $\| F_A^{0,2} \|^2$ in the topological sector with S_0 fixed.

The $(4,0)$ form Ω can be simply expressed in local coordinates as

$$\Omega = dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4. \tag{2.28}$$

and

$$F^{(0,2)} = d\bar{z}^{\bar{\mu}} d\bar{z}^{\bar{\nu}} F_{\bar{\mu}\bar{\nu}}, \tag{2.29}$$

where

$$F_{\bar{\mu}\bar{\nu}} = \partial_{\bar{\mu}} A_{\bar{\nu}} - \partial_{\bar{\nu}} A_{\bar{\mu}} + [A_{\bar{\mu}}, A_{\bar{\nu}}]. \tag{2.30}$$

also

$$D_{\bar{\mu}} = \partial_{\bar{\mu}} + [A_{\bar{\mu}}, \quad]. \tag{2.31}$$

One has the part of the Bianchi identity

$$D_{[\bar{\mu}} F_{\bar{\nu}\bar{\rho}]} = 0. \tag{2.32}$$

The 3 complex gauge covariant gauge conditions, which count for 6 real conditions on the 8 independent real components contained in $A_{\bar{\mu}}$ are

$${}^{c}F_{\bar{\mu}_1\bar{\mu}_2} + \epsilon_{\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3\bar{\mu}_4} F_{\bar{\mu}_3\bar{\mu}_4} = 0. \tag{2.33}$$

The two other gauge conditions are given by the following complex equation

$$\partial_{\bar{\mu}} {}^{\mathbf{c}}A_{\bar{\mu}} + \frac{1}{2}[A_{\bar{\mu}}, {}^{\mathbf{c}}A_{\bar{\nu}}] = 0. \tag{2.34}$$

If we compute the real and imaginary parts of this condition, they give respectively the Landau gauge condition adn the first of the seven conditions in (2.11). A similar decomposition of (2.33) gives the six other equations in (2.11).

We have now the topological ghost $\Psi_{\bar{\mu}}$ with 4 independent complex components, and we have the ghost gauge condition

$$D^{c}_{\bar{\mu}} \Psi_{\bar{\mu}} = 0. \tag{2.35}$$

(Here and below, we use the left upper symbol $^{\mathbf{c}}$ for complex conjugation.) A consequence of the use of complex gauge transformations is that a complex Faddeev–Popov ghost c must be introduced, with complex ghost of ghost ϕ. Up to the complexification of all fields, we have thus exactly the same field content as the original 4 dimensional Yang–Mills TQFT. This leads us to the BRST algebra that we will shortly display.

2.2.2. Action and observables. From the previous arguments, we must write the BRST algebra in a notation where all fields are *complex* fields and replace the formula of the J case by

$$\begin{aligned} sA_{\bar{\mu}} &= \psi_{\bar{\mu}} + D_{\bar{\mu}} c, \quad s\psi_{\bar{\mu}} = -D_{\bar{\mu}}\phi - [c, \psi_{\bar{\mu}}], \\ sc &= \phi - \frac{1}{2}[c, c], \quad s\phi = -[c, \phi]. \end{aligned} \tag{2.36}$$

In the antighost sector, we have a complex self dual two-form with 3 independant complex components $\chi_{\bar\mu\bar\nu}$ and

$$s\chi_{\bar\mu\bar\nu} = H_{\bar\mu\bar\nu} - [c, \chi_{\bar\mu\bar\nu}], \quad sH_{\bar\mu\bar\nu} = [\phi, \chi_{\bar\mu\bar\nu}] - [c, H_{\bar\mu\bar\nu}]. \tag{2.37}$$

We have also the complexified analogues of the antighosts of the four dimensional Yang–Mills TQFT, with the same transformation laws as in (2.21). Because of complexification, there are in the H case more ghosts than as in the J case. Thus, part of the gauge fixing will consists in setting equal to zero the imaginary parts of the scalar ghosts $c, \phi, \bar\phi, \eta$.

To impose these conditions, and the 3+1 complex gauge conditions Eqs. (2.33) and (2.34), we define

$$Z = \int [DA_{\bar\mu}][D^{\mathbf{c}}A_{\bar\mu}][D\Psi_{\bar\mu}][D^{\mathbf{c}}\Psi_{\bar\mu}][D\kappa_{\bar\mu\bar\nu}][D^{\mathbf{c}}\kappa_{\bar\mu\bar\nu}][DH_{\bar\mu\bar\nu}][D^{\mathbf{c}}H_{\bar\mu\bar\nu}]$$
$$[D\eta][D^{\mathbf{c}}\eta][D\phi][D^{\mathbf{c}}\phi][D\bar\phi][D^{\mathbf{c}}\bar\phi][Dc][D^{\mathbf{c}}c][D\bar c][D^{\mathbf{c}}\bar c][DB][D^{\mathbf{c}}B] \quad \exp S_H, \tag{2.38}$$

$$S_H = \int [\Omega \wedge \mathrm{Tr}\, F^{(0,2)} \wedge F^{(0,2)}]$$
$$\int d^4z d^4\bar z\, s \Big[\mathrm{Tr}\, \Big(\kappa_{\bar\mu\bar\nu}({}^{\mathbf{c}}F_{\bar\mu\bar\nu} + \epsilon_{\bar\mu\bar\nu\bar\rho\bar\sigma}F_{\bar\rho\bar\sigma} + \frac{1}{2}{}^{\mathbf{c}}H_{\bar\mu\bar\nu}) + {}^{\mathbf{c}}\kappa_{\bar\mu\bar\nu}(F_{\bar\mu\bar\nu} + \epsilon_{\bar\mu\bar\nu\bar\rho\bar\sigma}{}^{\mathbf{c}}F_{\bar\rho\bar\sigma} + \frac{1}{2}H_{\bar\mu\bar\nu})$$
$$+ \bar\phi D^{\mathbf{c}}_{\bar\mu}\Psi_{\bar\mu} + {}^{\mathbf{c}}\bar\phi^{\mathbf{c}} D_{\bar\mu}\Psi_{\bar\mu} + Im\,\bar\phi\, Im\, c$$
$$+ \bar c(\partial_{\bar\mu}{}^{\mathbf{c}}A_{\bar\mu} + \frac{1}{2}[A_{\bar\mu}, {}^{\mathbf{c}}A_{\bar\mu}] + \frac{1}{2}{}^{\mathbf{c}}B) + {}^{\mathbf{c}}\bar c({}^{\mathbf{c}}\partial_{\bar\mu}A_{\bar\mu} + \frac{1}{2}[{}^{\mathbf{c}}A_{\bar\mu}, A_{\bar\mu}] + \frac{1}{2}B)\Big)\Big]. \tag{2.39}$$

If we develop the s-exact terms and eliminate the auxiliary fields H and B we get a supersymmetric action starting with Tr $(F \wedge *F)$, because $\frac{1}{4} \parallel F_A \parallel^2 = \parallel F_A^{0,2} \parallel^2 + \frac{1}{4} \parallel \langle F, \omega\rangle \parallel^2$ + topological terms, and a Feynman–Landau gauge fixing term $|\partial\cdot A|^2$. The action of the H case is similar to that of the J case after eliminationof the imaginary parts of $c, \phi, \bar\phi, \eta$ by mean of the equations of motion coming from $s(Im\,\bar\phi\, Im\, c)$. Moreover, if one separate fields in their real and imaginary parts, one finds a mapping between the ghosts of the H and J case (for instance the six antighosts contained in the complex self dual two-form $\kappa_{\bar\mu\bar\nu}$ and the imaginary part of the antighosts $\bar c$ of the H cases can be identified as the seven ghosts κ_i of the J case). Actually, up to this mapping, the actions of the H and J cases are almost identical.

The definition of observables follows from the cocycles obtained by the descent equations, as sketched in the previous section. Their meaning is now discussed.

2.2.3. Geometric interpretation. Let $\widetilde{\mathcal{M}}$ denote $[A \in \mathcal{A}_P]$ with $F_+^{0,2} = 0$. It is invariant under $\mathbf{H}$ (which acts on $\mathcal{G}$ in $\Lambda_+^2 \otimes \mathcal{G}$, but not on Λ_+^2.) Let $\mathcal{M}_H = \widetilde{\mathcal{M}}/\mathbf{H}$. The 3 complex covariant gauge conditions, $\Lambda_+^{0,2} = 0$, probe the moduli space $\mathcal{M}_H$. We remarked earlier that $0 \to \Lambda^0 \otimes \mathcal{G} \xrightarrow{\bar\partial_A} \Lambda^{0,1} \otimes \mathcal{G} \xrightarrow{P_+\bar\partial_A} \Lambda_+^{0,2} \otimes \mathcal{G}$ is an elliptic complex with $\bar\partial_A^* + P_+\bar\partial_A : \Lambda^{0,1}\otimes\mathcal{G} \to \Lambda^0\otimes\mathcal{G} + \Lambda_+^{0,2}\otimes\mathcal{G}$; the elliptic operator $\not\partial_A : S^-\otimes\mathcal{G} \to S^+\otimes\mathcal{G}$. The *complex* gauge condition is $\bar\partial^*\tau = 0$ for $\tau \in \Lambda^{0,1} \otimes \mathcal{G}$.

As before, we get a hermitian vector bundle $\widetilde{E}$ over $\mathcal{M}_H \times M_8$ with connection. One can compute c_2 of $\widetilde{E}$ in terms of its curvature $\mathcal{F}^H$. One has the map T of $H^{0,*}(M_8)$ into forms on $\mathcal{M}_H$ by $\mu \xrightarrow{T} \int_{M_8} \Omega \wedge \mathrm{Tr}\, (\mathcal{F}^H \wedge \mathcal{F}^H) \wedge \mu$.

Formally this gives a multilinear map of $H^{0,*}(M_8) \to \mathbf{C}$ by $\mu_1, \dots, \mu_r \to \int_{\mathcal{M}_H} T(\mu_1) \wedge \dots \wedge T(\mu_r)$. These would be the expectation values of the observables of the BRSTQFT.

As in Sect. 2.1.3. part of c_2 is $\frac{1}{8\pi^2}\mathrm{Tr}\,((\mathcal{F}^H)^1_1 \wedge (\mathcal{F}^H)^1_1)$ with $(\mathcal{F}^H)^1_1(\tau, v) = \tau(v) \in u(N)$. If $\sigma \in H^{0,2}(M_8)$ let T_σ be the 2-form on $\mathcal{M}_H$ given by $\int_{M_8} \Omega \wedge \mathrm{Tr}\,(\tau_1 \wedge \tau_2) \wedge \sigma$, where τ_i, $i = 1, 2$, are (0,1) forms on M_8 with values in $u(N)$. The formal holomorphic Donaldson polynomial is the symmetric r-multilinear function on $H^{0,2}(M_8)$ given by $\sigma_1, \dots, \sigma_r \to \int_{\mathcal{M}_H} T_{\sigma_1} \wedge \dots \wedge T_{\sigma_r}$, when dim $\mathcal{M}_H = 2r$. (Note that if $H^{0,2}(M_8) \not\equiv 0$, then M_8 is hyperKähler because elements of $H^{0,*}$ are covariantly constant.)

It will be very interesting to see when formal integration over $\mathcal{M}_H$ is justified, and when these invariants depend only on the complex structure of M_8, not on the Calabi–Yau metric g, nor the hermitian metric ρ. C. Lewis [12] is investigating the conditions under which $\mathcal{M}_H$ is the set of stable holomorphic vector bundles.

Since the elliptic operator here is $\not\partial$ again, the virtual dimension of $\mathcal{M}_H$ is

$$- \int_{M_8} \hat{A}(M_8)\ \mathrm{ch}\,(\mathcal{G}). \tag{2.40}$$

2.3. Comparison of H and J cases. Under suitable conditions ((E, ρ) a stable[6] vector bundle, for example), one expects that the orbit space of $\mathcal{A}_P$ under the group of complex gauge transformations, will be the same as the sympletic quotient, $\mathcal{A}_P \parallel \mathcal{G}_U$, where $\mathcal{G}_U$ are the gauge transformations on P reduced to the compact group $U(N)$. Since $[A; \langle F^{1,1}_A, \omega\rangle_m = 0,\ m \in M_8]$ is the zeros of the moment map, $\mathcal{A}_P/\mathcal{G}_U$ is the orbit space of this set under $\mathcal{G}_U$.

We replace the condition $P_+(F^{0,2}_A) = 0$ with $F^{0,2}_A \in \Lambda^{0,2}_+ \otimes gl(N, \mathbf{C})$ by the conditions $P_+(F^{0,2}_A) = 0$ and $\langle F^{1,1}_A, \omega\rangle = 0$, where now $F^{0,2}_A \in \Lambda^{0,2} \otimes u(N)$ and $\langle F^{1,1}_A, \omega\rangle \in u(N)$. One should get the same moduli space of solutions.

In the linearization, the sequence $gl(N, \mathbf{C}) \xrightarrow{\bar\partial_A} \Lambda^{0,1} \otimes gl(N, \mathbf{C}) \xrightarrow{P_+\bar\partial_A} \Lambda^{0,2}_+ \otimes gl(N, \mathbf{C}) \to 0$ is replaced by $u(N) \to \Lambda^{0,1} \otimes u(N) \xrightarrow{P_+\bar\partial_A \oplus i_\omega \partial} \Lambda^{0,2}_+ \otimes u(N) \oplus u(N) \to 0$, where $i_\omega\partial : \tau \in \Lambda^{0,1} \otimes u(N) \to \langle \partial\tau, \omega\rangle_m \in u(N)$. The operator $i_\omega\partial$ is the linearization of the 0-momentum condition $\langle F^{1,1}_A, \omega\rangle_m = 0$; it is also the imaginary part of $\bar\partial^*_A : \Lambda^{0,1} \otimes gl(N, \mathbf{C}) \to gl(N, \mathbf{C})$. Thus, with the reduction of $Spin(7)$ holonomy to $Spin(6) = SU(4)$ holonomy, the 7 dimensional Λ^2_+ in the J-case decomposes into the 6 dimensional Λ^2_+ of the H-case plus $\mathbf{R}$.

2.4. Link to twisted supersymmetry. We note that the field content of our Yang–Mills BRSTQFT action in 8 dimensions is similar to that of four dimensional topological Yang–Mills theory. Since four dimensional topological Yang–Mills theory is a twisted version of $D=4,\ N=2$ super Yang–Mills theory and is related by dimensional reduction to the minimal six dimensional supersymmetric Yang–Mills theory, it is natural to expect a similar connection in eight dimensions. This is indeed so; we explain the type J case, although the fields $(c, \bar{c}, B)$ which were employed to impose the Lorentz condition $\partial^\mu A_\mu = 0$, are neglected. The gauge supermultiplet in eight dimensions consists of one gauge field in $\mathbf{8_v}$ (the vector representation), one chiral spinor in $\mathbf{8_s}$, one anti-chiral spinor in $\mathbf{8_c}$ and two scalars [25]. The reduction of the holonomy group to $Spin(7)$

[6] For physicists, one might define (E, ρ) to be stable if it is holomorphic, Einstein-Hermitian, i.e., $F_\rho \cdot \omega$ is a constant multiple of the identity, where F_ρ is the curvature of (E, ρ) relative to its unique ρ-connection.

defines decomposition of the chiral spinor; $\mathbf{8_s} = \mathbf{1} \oplus \mathbf{7}$. Now it is natural to identify A_μ and ψ_μ in our topological theory as $\mathbf{8_v}$ and $\mathbf{8_c}$, respectively. Furthermore χ_i and η just correspond to the chiral spinor $\mathbf{8_s}$ according to the above decomposition. Finally ϕ and $\bar{\phi}$ give the remaining two scalars. This exhausts all the dynamical fields in our action of eight dimensional topological Yang–Mills theory. Though we do not work out the transformation law explicitly, we believe this is a sufficiently convincing argument for the fact that the J case is the $D=8$ SSYM dimensionally reduced from $D=10,\ N=1$ SSYM. The connection between a general supersymmetry transformation and topological BRST transformations is the following: when M_8 is flat, the reduction from $D=10$ is $N=2$ real supersymmetry or $N=1$ complex supersymmetry. For curved manifolds, the only surviving supersymmetries are those depending on covariant constant spinors. In the J case the nilpotent topological BRST symmetry generator is a combination of the real and imaginary parts of the one surviving complex generator of supersymmetry.

As said just above, this supersymmetric Yang–Mills theory in eight dimensions is obtained by dimensional reduction from the $D=10$, $N=1$ super Yang–Mills theory. This suggests a relationship with superstring theory. It has been argued that the effective world volume theory of the D-brane is the dimensional reduction of the ten dimensional super Yang–Mills theory [26]. Thus the BRSTQFT constructed in this section may arise as an effective action of 7-brane theory. In fact Joyce manifolds are discussed in connection with supersymmetric cycles in [27, 28]. Recently in [29], a six dimensional topological field theory of ADHM sigma model is obtained as a world volume theory of D-5 branes. The world volume theory of D-branes could provide a variety of higher dimensional BRSTQFT's.

3. Coupling of the 8D Theory to a 3-Form

For the pure Yang–Mills theory, we have seen that the construction of a BRSTQFT implies a consistent breaking of the $SO(D)$ invariance. This turns out to be quite natural, when closed but not exact forms exist, like the Kähler 2-form on Kähler manifolds or the holomorphic $(n,0)$-form on Calabi–Yau manifolds.

This idea extends to consider BRSTQFT's involving sets of possibly interacting p-form gauge fields with $(p+1)$-form curvatures $G_{p+1} = dB_p + \ldots$, satisfying relevant Bianchi identities. Our point of view is that one must define a system of equations, eventually interpreted in BRSTQFT as gauge conditions, which does not overconstrain the fields. If tensors $T^{\mu_1,\cdots,\mu_{2p+2}}$ of rank $2p+2$; $(2p+2 \leq D)$ exist which are invariant under maximal subgroups of $SO(D)$, we can consider BRSTQFT based on gauge functions of the following type, where λ is a parameter:

$$T^{\mu_1,\ldots,\mu_{2p+2}} G_{\mu_{p+2},\ldots,\mu_{2p+2}} = \lambda G^{\mu_1,\ldots,\mu_{p+1}}. \tag{3.1}$$

Such equations must be understood in a matricial form, since they generally involve several forms B_p, with different values of p. To ensure that the problem is well defined, a first requirement is that Eq. (3.1) has solutions in G_{p+1} for λ different from zero. This algebraic question is in principle straightforward to solve by group theory arguments, although we expect that geometrical arguments should also justify them. Moreover, we must also consider that G_{p+1} is the curvature of a p-form gauge field B_p. Thus, other gauge functions must be introduced, to gauge fix the ordinary gauge freedom of B_p which leave invariant its curvature G_{p+1}. This gives a second requirement, since from the point of view of the quantization, the total number of gauge conditions, the topological ones and the ordinary ones, must be exactly equal to the number of independent components in the gauge field B_p.

To be more precise, the number of ordinary gauge freedom of a p-form gauge field in D dimensions is C_{D-1}^{p-1}: (this amounts to the fact that B_p is truly defined up to a $(p-1)$-form, which is itself defined up to a $(p-2)$-form, and so on.) We should therefore only retain invariant tensors T such that the number of components of B_p equates the rank of the system of linear equations in G presented in Eq. (3.1) plus the number of ordinary gauge freedom in B_p. Obviously, when there are several fields in Eq. (3.1), the counting of independent conditions can become quite subtle, since one must generally combine several equations like Eq. (3.1). For instance, we will display in the next section BRSTQFT theories in dimensions $D < 8$. Their derivation will appear as rather simple, because they all descend by dimensional reduction from the pure Yang–Mills BRSTQFT based on the set of 6 or 7 independent self-duality gauge covariant equations in eight dimensions found in Sect. 2. Without this insight, their derivation would be less obvious.

We now turn to the introduction of a 3-form gauge field in 8 dimensions. In even $D = 2k$ dimensions, Eq. (3.1) has a generic solution for an uncharged $(k-1)$-form gauge field B_{k-1}: assuming the existence of a curvature G_k for B_{k-1}, we can consider the obvious generalization of self-duality equations, $G_k = *G_k$. The number of these conditions is C_{D-1}^{k}. On the other hand, the number of ordinary gauge freedom of a $(k-1)$-form gauge field is $C_{D-1}^{k-2} = C_D^{k-2} - C_D^{k-3} + C_D^{k-4} - \ldots \pm C_D^0$. Thus imposing the ordinary gauge fixing conditions for the $(k-1)$-form gauge field plus the gauge covariant ones, $G_k = *G_k$, gives a number of $C_D^{k-1} = C_{D-1}^{k-2} + C_{D-1}^{k}$ equations, which is equal to the number of arbitrary local deformations of the C_D^{k-1} independent components of the $(k-1)$-form gauge field. We will see that it is possible to generalize the self duality equation satisfied by a $(k-1)$-form gauge field. Moreover, the counting remains correct in the case it has a charge. As an example, in the 8-dimensional theory, a 3-form gauge field has 56 components, with 21 ordinary gauge freedom, while the number of self dual equations involving the 4-form curvature of the 3-form is 35, and one has 56=21+35.

We thus propose as topological gauge conditions for the coupled system made of the Yang–Mills field A and the 3-form gauge field B_3 the following *coupled* equations:

$$\begin{aligned} \lambda F_{\mu\nu} &= T_{\mu\nu\rho\sigma} F_{\rho\sigma}, \\ dB_3 + *(dB_3) + \alpha \mathrm{Tr}\,(F \wedge F)_+ &= 0. \end{aligned} \tag{3.2}$$

α is a real number, possibly quantized, and $\mathrm{Tr}\,(F \wedge F)_+$ denotes the self dual part of $\mathrm{Tr}\,(F \wedge F)$ [7]. Although B_3 is real valued, it interacts with the Yang–Mills connection A, when $\alpha \neq 0$. An octonionic instanton solves the first equation, as shown in [14] and by Eqs. (25), (30), (31) of [15] in the case of $M_8 = S^7 \times \mathbf{R}$. For this solution, the 4-form $\mathrm{Tr}\,(F \wedge F)$ is not self dual.

Given these facts, we are led to define a BRSTQFT in 8 dimensions based on the gauge conditions (3.2), in which a 3-form gauge field is coupled to a Yang–Mills field. The ghost spectrum for the ordinary gauge invariance of the field B_3 generalizes that of the Yang–Mills field, with the following unification between the ghost B_2^1 and the ghosts of ghosts B_1^2 and B_0^3:

$$\widehat{B}_3 = B_3 + B_2^1 + B_1^2 + B_0^3. \tag{3.3}$$

(From now on upper indices mean ghost number and lower indices ordinary form degree.)

[7] Equation (3.2) suggests that the 3-form could be involved in an anomaly compensating mechanism. See sec. 3.1 where we show that Eqs. (3.1) implies $dB_3 = 0$ if M is compact.

The BRST symmetry of the topological Yang–Mills symmetry considered in the previous section satisfies

$$\widehat{A} = A + c, \tag{3.4}$$

$$\widehat{F} = (s + d)\widehat{A} + \frac{1}{2}[\widehat{A}, \widehat{A}] = F_2^0 + \Psi_1^1 + \phi_0^2, \tag{3.5}$$

with the notation $\Psi_1^1 = \Psi_\mu dx^\mu$ and $\phi_0^2 = \phi$.

The gauge symmetry of the 3-form B_3 involves a 2-form infinitesimal parameter associated to B_2^1. We can distinguish however different topological sectors for B_3, which cannot be connected only by these infinitesimal gauge transformations. As an example, B_3 and B_3' can belong to such different sectors, if

$$B_3' = B_3 + \mathrm{Tr}\,(A \wedge dA + \frac{2}{3} A \wedge A \wedge A). \tag{3.6}$$

We thus define the curvature of B_3 as

$$G_4^{(A)} = dB_3 + \mathrm{Tr}\,(F^{(A)} \wedge F^{(A)}), \tag{3.7}$$

where the index (A) means the dependence upon the Yang– Mills field A. Notice that it is not globally possible to eliminate the A dependence of $G_4^{(A)}$ by a field redefinition of B_3 involving the Chern–Simons 3-form.

The topological BRST symmetry of the 3-form gauge field system is defined from

$$\widehat{G}_4 = (s + d)\widehat{B}_3 + \mathrm{Tr}\,(\widehat{F}^{(A)} \wedge \widehat{F}^{(A)}) = G_4 + \mathcal{G}_3^1 + \mathcal{G}_2^2 + \mathcal{G}_1^3 + \mathcal{G}_0^4, \tag{3.8}$$

that is

$$\begin{aligned}(s + d)\,(B_3 + B_2^1 + B_1^2 + B_0^3) + \mathrm{Tr}\,\left((F_2^0 + \Psi_1^1 + \phi_0^2) \wedge (F_2^0 + \Psi_1^1 + \phi_0^2)\right) = \\ dB_3 + \mathrm{Tr}\,(F \wedge F) + \mathcal{G}_3^1 + \mathcal{G}_2^2 + \mathcal{G}_1^3 + \mathcal{G}_0^4.\end{aligned} \tag{3.9}$$

The fields $\mathcal{G}_{4-g}^g$, $g = 1, 2, 3, 4$ are the topological ghosts of B_3. By expansion in ghost number, Eqs. (3.5) and (3.9) define a BRST operation s which, eventually, determines the equivariant cohomology of arbitrary deformations of the Yang–Mills field modulo ordinary gauge transformations and of the 3-form gauge field, modulo the infinitesimal gauge transformations, $\delta B_3 = d\epsilon_2,\ \epsilon_2 \sim \epsilon_2 + d\epsilon_1,\ \epsilon_1 \sim \epsilon_1 + d\epsilon$.

There is a natural topological invariant candidate for the classical part of a BRSTQFT action,

$$I_{top} = \int \left(G_4^{(A)} \wedge G_4^{(A)} + \Omega \wedge \mathrm{Tr}\,(F^{(A)} \wedge F^{(A)})\right). \tag{3.10}$$

Its gauge fixing is a generalization of what we do in the pure Yang–Mills case. The main point is to find the gauge function in the topological sector. The existence of the octonionic instanton, together with an associated moduli space (yet to be explored), indicates that Eq. (3.2) is a good choice [8].

To enforce the gauge function Eq. (3.2), one must introduce a self-dual 4-form antighost $\kappa_{\mu\nu\rho\sigma}$, and consider the following BRST exact action:

[8] Notice that one could also consider a 7-dimensional theory, which is formally related to the BRSTQFT in 8 dimensions as the 3-dimensional Chern–Simons theory is related to the 4-dimensional Yang–Mills TQFT action.

$$S_3 = \int d^8x\ s\left(\kappa^{\mu\nu\rho\sigma}(\partial_{[\mu}B_{\nu\rho\sigma]} + \epsilon^{\mu\nu\rho\sigma\alpha\beta\gamma\delta}\partial_{[\alpha}B_{\beta\gamma\delta]} + \mathrm{Tr}\ F_{[\mu\nu}F_{\rho\sigma]})\right). \qquad (3.11)$$

The remaining conditions are for the usual gauge invariances of forms, whether they are classical or ghost fields. One can choose the following gauge fixing conditions for the longitudinal parts of all ghosts and ghosts of ghosts $\mathcal{G}^g_{4-g}$,

$$\begin{aligned}\partial^\mu \mathcal{G}^1_{3\mu\nu\rho} &= a\Delta^1_{\nu\rho},\\ \partial^\mu \mathcal{G}^2_{2\mu\nu} &= b\Delta^2_{\nu},\\ \partial^\mu \mathcal{G}^3_{1\mu} &= c\Delta^3. \end{aligned}\qquad (3.12)$$

One must also conventionally gauge fix the longitudinal components of $B_{3\mu\nu\rho}$, of the ghosts $B^1_{2\mu\nu}$ and $B^2_{1\mu}$, and of the antighosts. The presence in the r.h.s. of Eq. (3.12) of the cocycles Δ^g_{3-g} stemming from the ghost decomposition of Tr $\widehat{F}\wedge\widehat{F}$ = Tr $(F+\Psi+\phi)\wedge(F+\Psi+\phi)$ is an interesting possibility. It can lead to mass effects in TQFT, when the ghost of ghost ϕ takes a given mean value, depending on the choice of the vacuum in the moduli space, which can be adjusted by suitable choices of the parameters a, b, c.

All these gauge conditions can be enforced in a BRST invariant way, as explained e.g. in [30]. The final result is an action of the following type (including the pure Yang–Mills part discussed in the previous sections)

$$S = \int(\partial_\mu B_{\nu\rho\sigma}\partial^\mu B^{\nu\rho\sigma} + \mathrm{Tr}\ F^{\mu\nu}F_{\mu\nu} + \partial_\mu B_{\nu\rho\sigma}\mathrm{Tr}\ F^{\mu\nu}F^{\rho\sigma}$$
$$+\text{supersymmetric terms}). \qquad (3.13)$$

The observables are defined from all forms $\widehat{O}^g_{8-g}$ occurring in the ghost expansion of the 8-form

$$\widehat{O}_8 = \widehat{G}_4 \wedge \widehat{G}_4. \qquad (3.14)$$

Whether these supersymmetric terms, made of ghost interactions, are linked to Poincaré supersymmetry is an interesting question.

3.1. Mathematical Interpretation. Fix an element of $H^4(M_8, \mathbf{Z})$ and let h_4 denote its harmonic representative. Let ß denote the affine space of all closed 4-forms which represent this cohomology class. Then ß $= h_4 + d\Lambda^3$; strictly speaking ß $= h_4 + d(\Lambda^3/\text{closed 3-forms}) = h_4 + d\delta\Lambda^4$. In any case a tangent vector to ß can be represented as dB_3 with B_3 a 3-form.

There are other ways of describing ß. An element of ß can be represented as a collection of 3-forms $\{B_u\}$, for a collection of coordinate neighborhoods U covering M_8, satisfying $B_u - B_v = dw_{u,v}$ on $u \cap v$. Thus $\{dB_u\}$ gives a well-defined closed form on M_8; to be an element of ß, this 4-form must be cohomologous to h_4. In the earlier part of this section, dB_3 means this element of ß when B_3 is defined locally as $B_{3,u}$; or if B_3 is an ordinary three form, dB_3 is really $h_4 + dB_3$.[9]

Next consider the elliptic complex $0 \to \Lambda^0 \to \Lambda^1 \to \Lambda^2 \to \cdots \to \Lambda^4_+ \to 0$, where $\Lambda^4_\pm$ are the ± 1 eigenspaces of the ordinary $*$ operator on M_8. Remember that in the J-case we also had $0 \to \Lambda^0 \to \Lambda^1 \to \Lambda^2_+ \to 0$ with $\Lambda^2 = \Lambda^2_- \oplus \Lambda^2_+$ of dimensions 21 and 7, respectively. Consider then $0 \to \Lambda^2_- \xrightarrow{d} \Lambda^3 \xrightarrow{d} \Lambda^4_+ \to 0$. We leave the

[9] The theory of gerbes [31] gives a sheaf theoretic description for exhibiting integral cohomology classes, extending the notion of curvature field as an integral 2-cocycle.

reader to check that it is elliptic. (It does not suffice that the dimensions are 21, 56 and 35, respectively.) The linearization of the problem below involves $0 \to \Lambda^0 \otimes \mathcal{G} \xrightarrow{D_A} \Lambda^1 \otimes \mathcal{G} \xrightarrow{D_A} \Lambda^2_+ \otimes \mathcal{G} \to 0$ for connections, and $0 \to \Lambda^2_- \xrightarrow{d} \Lambda^3 \xrightarrow{d} \Lambda^4_+ \to 0$ for 3-forms.

An analogue of the anti-self-dual equations for the pair (A, G) with a connection A and $G \in$ ß is

$$\begin{aligned} &\text{(a)} \quad F_A = * \Omega \wedge F_A, \qquad \text{(i.e. } P_+ F_A = 0) \\ &\text{(b)} \quad (1 + *)G = -\alpha \text{Tr}\,(F_A \wedge F_A)_+. \end{aligned} \tag{3.15}$$

This equation is a mathematical interpretation of (3.2). Note that if a solution $A, G = h_4 + dB_3$ exists for (3.15), then Tr $(F_A \wedge F_A)$ is self-dual and hence harmonic. Hence $(1 + *)(h_4 + dB_3)$ is harmonic. Since $(1 + *)h_4$ is harmonic, so is $(1 + *)dB_3$. Hence $dB_3 = 0$, and $G = h_4$. Note also that the sector ß, i.e. the element chosen in $H^4(M_8, \mathbf{R})$ must have its self-dual part, a multiple of the self-dual element $p_1(P)$.

If we linearize (3.15), we get for $\tau \in T(\mathcal{G})$ and $B_3 \in T$(ß), the equations $P_+^{(2)}(D_A\tau) = 0$ and $P_+^{(4)} dB_3 = 0$, where P_+^j is the projection of $\Lambda^j \to \Lambda^j_+$ $(j = 2, 4)$. We then have a pair of elliptic systems above, with gauge fixing functions $D_A^* \tau = 0$ and $d^* B_3 = 0$, respectively. The covariant gauge functions are given by (3.15).

The candidate for the topological action $S_0(A, G)$ is $\int_{M_8} G \wedge G + \Omega \wedge \text{Tr}\,(F \wedge F)$. Since we now have the covariant gauge functions to probe the moduli space of solutions to (3.15) and we have the gauge fixing functions, we can apply the BRST formalism. We first express S_0 in terms of the norms. From (2.7),

$$\| F_A \|^2 = \int_{M_8} \Omega \wedge \text{Tr}\,(F_A \wedge F_A) + 4 \,\| (F_A)_+ \|^2 \,. \tag{3.16}$$

Also with $G = G_+ + G_-, G_\pm \in \Lambda^4_\pm$, we have $\int_{M_8} G \wedge G = \| G_+ \|^2 - \| G_- \|^2$. Thus one obtains $\| F_A \|^2 + \| G \|^2 = S_0 + 4 \,\| (F_A)_+ \|^2 + 2 \,\| G_+ \|^2$. We know that $\| F_A \|^2$ is minimized when $F_+ = 0$, and that $\| G \|^2$ is minimized when $G = h$. So we get a minimum when (3.15) is satisfied and it equals $S_0 + 16\pi^4\alpha^2 \int_{M_8} p_1^2 = S_0^1$.

In the pure YM case, the natural space was $\mathcal{A}_P/\mathcal{G} \times M_8$ or its subspace $\mathcal{M}_J \times M_8$. Rather than 3-forms on M_8, we need 3-forms on $\mathcal{M}_J \times M_8$ which we write as $\widehat{B}_3 = B_3^0 + B_2^1 + B_1^2 + B_0^3$ (Eq.(3.3), above) with the upper index as the degree in the $\mathcal{M}_J$ direction (ghost number) and the lower index in the M_8 direction. As before s denotes $d_{\mathcal{M}_J}$ so that $(s + d)\widehat{B}_3 = (d_{\mathcal{M}_J} + d_{M_8})\widehat{B}_3 = d_{\mathcal{M}_J}(\widehat{B}_3) + d_{M_8}(\widehat{B}_3)$ is a 4-form with terms in the $\binom{a}{b}$ directions.

4. BRSTQFT's for Other Dimensions Than 8

From many points of view the case $D = 8$ is exceptional. It is of interest, however, to also build BRSTQFT's in other dimensions, by using the BRST quantization of d-closed Lagrangians with gauge functions as in Eq. (3.1). In this section, we first focus on theories with $D < 8$, that we directly obtain by various dimensional reductions in flat space of the J and H theories; we then comment on the cases $D = 12$ and $D = 10$. We will not address the question of observables; their determination is clear from the descent equations which can be derived in all possible cases from the knowledge of the BRST symmetry.

4.1. Dimensional reduction of the Yang–Mills 8D BRSTQFT. In $D=8$, for the J-case, we have seen that there exists a set of seven self-duality equations, on which we have based our BRSTQFT. These equations were complemented with a Landau gauge condition to get a system of 8 independent equations for the 8 components of A_μ. These seven equations can be written as

$$\Phi_i(F_{\mu\nu}(x^\mu)) = 0, \quad 1 \le i \le 7, \quad 1 \le \mu, \nu \le 8. \quad [\,(2.14)] \tag{4.1}$$

Just as one obtains a BRSTQFT action based on Bogomolny equations in 3 dimensions [32], we can define a BRSTQFT in seven dimensions, by standard dimensional reduction on the eighth coordinate; that is, by putting in the above seven equations $x^8 = 0$, $\partial_8 = 0$ and replacing A_8 by a scalar field $\varphi(x^j)$ and F_{i8} by $D_i\varphi(x^j)$. We can then gauge fix the longitudinal part of A_i, with an equation of the following type:

$$\partial_i A_i = [v, \varphi], \tag{4.2}$$

which allows for the case of a massive gauge field A. (Here and in what follows, the constant v defines a direction in the Lie algebra for the Yang–Mills symmetry.) The gauge fixed action will be

$$\int_{M_7} d^7x \left(|F_{ij}|^2 + |D_i\varphi|^2 + |\partial_i A_i - [v, \varphi]\,|^2 + \text{supersymmetric terms} \right). \tag{4.3}$$

This process can be iterated. We can go down from dimension 8 to $8 - n$, by suppressing the dependence on n of the coordinates x^μ. In $D < 8$ dimensions we will have a gauge field with $D = 8-n$ components and a set of n scalar fields $\varphi^p, p = 1, \ldots, n$ which should be considered as Higgs fields. Obviously, the dimensional reduction applies as well to the various ghosts, and the fields φ^p fall into topological BRST multiplets, which, depending on the case, can possibly be interpreted as twisted Poincaré supermultiplets. Moreover, as we will see when $D = n = 4$, there is an interesting option to assign the fields φ^p as elements of other representations, e.g. spinorial ones, of $SO(D)$.

One can also consider the dimensional reduction in the H-case. One can break the symmetry between the coordinates y, z, t, w and their complex conjugates by replacing some of the fields, e.g. Im $A_{\bar{w}}$, by scalar fields.

In all cases, the final theories rely on 8 independent gauge conditions for all fields: 7 for the topological gauge ones plus 1 for the ordinary gauge condition, if one starts from the J case; or 6 for the 3 complex topological gauge conditions plus 2 for the ordinary complex gauge condition, if one starts from the H case.

4.1.1. The case D=6. Since the case $D = 6$ is of great interest in superstring theory, let us display what we get, starting from the H case,

$$\epsilon_{\bar{i}\bar{j}\bar{k}} F_{\bar{j}\bar{k}} = D_{\bar{i}}\varphi, \tag{4.4}$$

$$\partial_{\bar{i}} A_{\bar{i}} = [v, \varphi]. \tag{4.5}$$

This set of gauge functions represents 4 complex equations, for eight degrees of freedom represented by the complex fields $A_{\bar{i}}$ and φ.

If we start from the J case, we have

$$\Phi_i(F_{\mu\nu}(x^\mu), D_\mu\varphi^a(x^\mu)) = 0, \quad 1 \le a \le 2, \quad 1 \le \mu, \nu \le 6, \tag{4.6}$$

possibly complemented by

$$\partial_\mu A_\mu = M_{a,b}\varphi^a\varphi^b + N_{a,b}v^a\varphi^b. \tag{4.7}$$

Notice that a 2-form gauge fields, subjected to the topological invariance $sB_2 = \Psi_2 + \ldots$ can be introduced, still in 6 dimensions, with the topological self-dual gauge condition

$$dB_2 + *(dB_2) + \alpha\text{Tr}\,(AdA + \frac{2}{3}AAA)_+ = 0. \tag{4.8}$$

This possibility is similar to the introduction of a 3-form in $D = 8$.

We can directly build a BRSTQFT in 6 dimensions. First we consider a pure Yang–Mills case, taking the topological gauge fixing condition of the type

$$\lambda F_{\mu\nu} \;=\; \frac{1}{2}T_{\mu\nu\rho\sigma}F^{\rho\sigma}. \tag{4.9}$$

The fourth rank tensor $T_{\mu\nu\rho\sigma}$ is assumed to be invariant under some maximal subgroup of $SO(6)$. According to Corrigan et al [6], only $SO(4)\times SO(2)$ and $U(3)$ allow such an invariant tensor. The first choice corresponds to the case where the 6D manifold is a direct product of a 4D manifold and 2D Riemann surface; $M_6 = M_4 \times \Sigma_2$. The second subgroup is the holonomy group of 6 dimensional Kähler manifolds. In this case we can write down the invariant tensor as the Hodge dual of a Kähler form ω,

$$T_{\mu\nu\rho\sigma} \;=\; (*\,\omega)_{\mu\nu\rho\sigma}. \tag{4.10}$$

The possible eigenvalues λ of (4.9) with the tensor (4.10) are $1, -1$ and -2. The eigenspaces of these eigenvalues give the decomposition of the 15 dimensional representation of $SO(6)$ under its subgroup $SU(3)\times U(1)$; $\mathbf{15} = \mathbf{8} \oplus (\mathbf{3} \oplus \bar{\mathbf{3}}) \oplus \mathbf{1}$.[10] Taking $\lambda = 1$ defines the 8 dimensional subspace given by the following seven linear conditions on $F_{\mu\nu}$, where we use complex indices $a, b = 1, 2, 3$:

$$F_{ab} \;=\; F_{\bar{a}\bar{b}} = 0, \tag{4.11}$$

$$\omega^{a\bar{b}}F_{a\bar{b}} = 0. \tag{4.12}$$

(The last Eq. (4.12) is, e.g., $F_{1\bar{1}} + F_{2\bar{2}} + F_{3\bar{3}} = 0$.) The first condition (4.11) means that the connection is holomorphic. These equations are known as the Donaldson-Uhlenbeck-Yau (DUY) equation for the moduli space of stable holomorphic vector bundles on a Kähler manifold. It also appears in the Calabi–Yau compactification of the heterotic strings. The DUY equation implies the standard second order equation of motion for the Yang–Mills field[11]. In fact, this follows from the following identity in the action density level;

$$\begin{aligned} -\frac{1}{4}\text{Tr}\, F\wedge *F \,+\, \omega\wedge \text{Tr}\,(F\wedge F) \\ = \text{Tr}\left[-\frac{3}{2}g^{a\bar{a}}g^{b\bar{b}}F_{ab}F_{\bar{a}\bar{b}} + (g^{a\bar{b}}F_{a\bar{b}})^2\right], \end{aligned} \tag{4.13}$$

where we have introduced the metric $g_{a\bar{b}}$ for the Kähler form ω. This identity [24] is crucial in constructing a BRST Yang– Mills theory whose classical action is the topological density $\omega \wedge \text{Tr}\,(F\wedge F)$.

From the BRST point of view, one must introduce scalar fields to get a correct balance between the gauge fixing conditions and the field degrees of freedom and to

[10] The usual splitting of $\Lambda^2 \otimes \mathbf{C}$ into $\Lambda^{1,1}\oplus\Lambda^{2,0}\oplus\Lambda^{0,2}$ with $\Lambda^{1,1}$ decomposed into $\lambda\omega\oplus\omega^\perp$, where ω is the Kähler form.

[11] This is a general property of the system (4.9).

recover Eq. (4.4). Given a hermitian connection A for the hermitian vector bundle (E,ρ), Eq. (4.4) says $F_A^{0,2} = (\bar\partial_A)^*\widetilde\varphi = *_1^{-1}\partial_A *_1 \widetilde\varphi$, i.e., $*_1 F_A^{0,2} = \partial_A(*_1, \widetilde\varphi) \in \Lambda^{1,3}\otimes\mathcal{G}$. (See Sect. 2.2.1 for the definition of the operation $*_1$.) When M is a Calabi–Yau 3 fold, let $\varphi = *\widetilde\varphi \in \Lambda^0\otimes\mathcal{G}$ and we get $*F_A^{0,2} = \bar\partial_A\varphi$. Linearization gives the usual elliptic operator, the holomorphic of $\begin{pmatrix}\text{curl} & \text{grad}\\ \text{div} & 0\end{pmatrix}$:

$$\begin{pmatrix}\bar\partial_A & \bar\partial_A^*\\ \bar\partial_A^* & 0\end{pmatrix} : \begin{pmatrix}\Lambda^{0,1}\otimes\mathcal{G}\\ \Lambda^{0,3}\otimes\mathcal{G}\end{pmatrix} \longrightarrow \begin{pmatrix}\Lambda^{0,2}\otimes\mathcal{G}\\ \Lambda^{0,0}\otimes\mathcal{G}\end{pmatrix}. \tag{4.14}$$

Of course what one wants is not (4.4) but $F_A^{0,2} = 0$, Eq. (4.11), the condition that makes E a holomorphic bundle. However, as a consequence of the Bianchi identity, $\bar\partial_A F_A^{0,2} = 0$ and hence (4.4) implies $\bar\partial_A\bar\partial_A^*\widetilde\varphi = 0$, which also implies $\bar\partial_A^*\widetilde\varphi = 0$, when M is compact without boundary. Thus (4.4) implies (4.11); moreover, when M is a Calabi–Yau 3-fold and E is stable, $\bar\partial_A^*\widetilde\varphi = 0$, (equivalent, $\bar\partial_A\varphi = 0$) only happens when φ is a constant multiple of I in $u(N)$. In that sense, the right-hand side of Eq. (4.5) is 0, giving the gauge fixing condition $\partial * \tau = 0,\ \tau\in\Lambda^{0,1}\otimes\mathcal{G}$.

Equation (4.12) is the equation $\langle F,\omega\rangle_m = 0$ (see Sect. 2.3). As stated there, the orbit space under complex gauge transformations should be the same as the symplectic quotient, the orbit space under unitary gauge transformations of the 0-momentum set, i.e., the condition $\langle F,\omega\rangle_m = 0$. Equation (4.13) is a special case of Proposition 3.1 in [24], which we have used previously in Sect. 2.2.2.

The DUY equation can also be obtained from the 6 dimensional supersymmetric Yang–Mills theory on a Calabi–Yau manifold. The supersymmetry transformation laws of the $(N = 1)$ vector multiplet (A_M,Ψ) in 6 dimensions are

$$\begin{aligned}\delta A_M &= i\bar\Xi\Gamma_M\Psi - i\bar\Psi\Gamma_M\Xi,\\ \delta\Psi &= -\frac{i}{2}\Sigma_{MN}\Xi F^{MN},\end{aligned} \tag{4.15}$$

where Γ_M are the gamma matrices and $\Sigma_{MN} = \frac{1}{4}[\Gamma_M,\Gamma_N]$ is the spin representation. On the Calabi–Yau manifold the holonomy group is further reduced to $SU(3)$, which gives a covariantly constant (complex) spinor ζ. In fact this is the very reason why the Calabi–Yau manifold is favorable in the compactification of superstrings to 4 dimensions. We will identify the supersymmetry transformation with $\Xi = \zeta$ as a topological BRST transformation. With this choice of parameter, SUSY transformations are decomposed according to the representations of $SU(3)$. The decomposition of $SO(6)$ vector is $\mathbf{6} = \mathbf{3}\oplus\bar{\mathbf{3}}$ and the chiral spinor decomposes as $\mathbf{4} = \mathbf{3}\oplus\mathbf{1}$. Thus we obtain the following topological BRST transformation law:

$$\begin{aligned}sA_\mu &= \psi_\mu, \quad sA_{\bar\mu} = 0,\\ s\chi &= g^{\mu\bar\mu}F_{\mu\bar\mu}, \quad s\psi_\mu = 0,\\ s\psi_{[\bar\mu\bar\nu]} &= F_{\bar\mu\bar\nu}, \quad s\rho = 0.\end{aligned} \tag{4.16}$$

We should explain how we have “twisted”spinors into ghosts and anti-ghosts. In terms of the covariantly constant spinor ζ which satisfies $\bar\zeta\Gamma_\mu = 0$, we can make the twist as follows;

$$\begin{aligned}\chi &= \bar\zeta\Psi, \quad \bar\psi_{\bar\mu} = \bar\zeta\Gamma_{\bar\mu}\Psi,\\ \psi_{[\bar\mu\bar\nu]} &= \bar\zeta\Gamma_{\bar\mu}\Gamma_{\bar\nu}\Psi, \quad \bar\rho = \epsilon^{\bar\mu\bar\nu\bar\sigma}\bar\zeta\Gamma_{\bar\mu}\Gamma_{\bar\nu}\Gamma_{\bar\sigma}\Psi,\end{aligned} \tag{4.17}$$

where $(\overline{\psi_{\overline{\mu}}}, \overline{\rho})$ are complex conjugates of (ψ_μ, ρ). This is an example of the identification of spinors with forms, explained in Sect. 2.1.1. Looking at the BRST transformations of the anti-ghosts, we recover the DUY equations (4.11, 4.12).

4.1.2. Reduction to a 4-D BRSTQFT; Seiberg–Witten equations. We now turn to the reduction to $D = 4$, which is of special interest, particularly the theory obtained by dimensional reduction of the J theory from $D = 8$ to $D = 4$. We will get a BRSTQFT with gauge conditions identical to the non-Abelian Seiberg–Witten equations, which in turn is also related to the $N = 4, D = 4$ supersymmetric theory.

The main observation is that, in the J case the set of seven equations (2.14) can be separated into 3 plus 4 equations. If we group A_5, A_6, A_7, A_8 into the 4 component field φ^α, $\alpha = 1, 2, 3, 4$, the latter can be interpreted in 4 dimensions as a commuting complex Weyl spinor and $A_\mu = A_1, A_2, A_3, A_4$ as a 4 dimensional vector. The set of the first 3 equations in Eq. (2.14) can now be interpreted as the condition that the self-dual part in 4-D of the curvature of A_μ is equal to a bilinear in φ^α; then, the remaining four equations can be written as Dirac type equations. To be more precise, with the relevant definition of the 4×4 matrices Γ_μ and $\Sigma_{\mu\nu}$, the dimensional reduction down to $D = 4$ of Eq. (2.14) gives

$$\begin{aligned} F_{\mu\nu} + \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} + {}^t\varphi \Sigma_{\mu\nu} \varphi &= 0, \\ D_\mu^{(A)} \Gamma^\mu \varphi &= 0. \end{aligned} \tag{4.18}$$

The consistency of the dimensional reduction from Eq. (2.14) to Eq. (4.18), and the correctness of the $SO(4)$ tensorial properties of all fields, are ensured by the existence of relevant elliptic operators in 8 and 4 dimensions.

The remarkable feature is that the above equations are the non-abelian version of Seiberg–Witten equations. In other words, we have observed that the spinors and vectors of the non-abelian S–W theory get unified in the Yang–Mills field of the J theory.

The generation of a Higgs potential, to break down the symmetry, with a remaining $U(1)$ is in principle possible, by the relevant modifications in the gauge functions, which provide a Higgs potential, function of φ. This is however a subtle issue that we will address elsewhere.

The form of the action after dimensional reduction is just the sum of the bosonic part of the Seiberg–Witten action, plus ghost terms. Its derivation is standard from the knowledge of the gauge function, as a BRST exact term, which enforces the gauge functions.

The link to supersymmetry in 4 dimensions is as follows. The BRSTQFT based on $Spin(7)$ is a twisted version of the $D = 8, N = 1$ theory where the spinor is a complex field counting for $16 = 8 + 8$ independent real components, and one has a complex scalar field in the supersymmetry multiplet. This theory is itself obtained as the dimensional reduction of the $D = 10, N = 1$ super Yang–Mills theory, where the spinor has 16 independent real components. Thus we predict that the theory we get by dimensional reduction to 4 dimensions of BRSTQFT in 8 dimensions is related to twisted versions of the $D = 4, N = 4$ super Yang–Mills theory. For instance, there are 6 scalar fields in the bosonic sector of the theory as presented in the work of Vafa and Witten [16], (see their Eq.(2.1)). In our derivation, these 6 scalar fields are combinations of 4 of the components of the 8-D Yang–Mills field and of the commuting ghost and antighost ϕ and $\bar{\phi}$ of the J theory.

There are actually three ways of twisting the N = 4 SSYM in four dimensions, defined by how $SO(4) \simeq SU(2) \times SU(2)$ is embedded in the R symmetry group[12] $SU(4)$ [16]. They are (i) $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$, (ii) $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2})$ and (iii) $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1})$, where we have indicated how the defining representation of $SU(4)$ decomposes under $SU(2) \times SU(2)$. Taking into account the argument in Sect. 6 of [27], we can see that the cases (i) and (iii) arise from the reduction of type H and J cases, respectively. The remaining case (ii), which is the twist employed by Vafa-Witten [16], is obtained from the 7 dimensional Joyce manifold with G_2 holonomy. On the other hand, we get the non-abelian Seiberg–Witten theory with an adjoint hypermultiplet in the case (iii), which gives the relationship between N = 4 SSYM and non-abelian Seiberg–Witten equation.

We thus conclude that very interesting twists connect the fields of the pure Yang–Mills 8-D BRSTQFT, (obtained by gauge fixing the invariant $\Omega \wedge \mathrm{Tr}\,(F \wedge F)$), the fields which are involved in the four dimensional Seiberg–Witten equations, and the fields of the D=4, N=4 super-Yang–Mills theory.

We note that if one starts from the H case gauge functions, the result of compactifying down to 4 dimensions is just a complexified version of a two dimensional Yang–Mills TQFT, coupled to two scalar fields; it could also be deduced from the dimensional reduction of the 3-dimensional BRSTQFT based on the Bogomolny equations.

4.2. Dimensions larger than 8.

4.2.1. Discussion of the case D=12. A BRSTQFT in 12 dimensions might be a candidate for F-theory. 11-dimensional supergravity, defined on the boundary of a 12 dimensional manifold, emphasizes the relevance of a 3-form gauge field C_3, possibly coupled to a non abelian connection one form A. The most important term $\int_{M_{11}} C_3 \wedge dC_3 \wedge dC_3$ of the 11-dimensional supergravity suggests that one should build a TQFT based on the gauge-fixing of the following invariant [13]:

$$\int_{M_{12}} \Big(dC_3 \wedge dC_3 \wedge dC_3 + dC_3 \wedge dC_3 \wedge P_{inv\ 4}(F) + dC_3 \wedge P_{inv\ 8}(F) + P_{inv\ 12}(F) \Big), \quad (4.19)$$

where $P_{inv\ n}(F)$ are invariant polynomials of degree $n/2$ of the curvature of A, i.e, characteristic classes. Special geometries like hyper or quaternionic Kähler manifolds give natural four-forms. They, their duals (which are 8-forms, and are therefore good candidates to define gauge functions for the curvature of a 3-form in 12 dimensions), and their powers might be used as well here.

It is natural to try and gauge fix these topological actions to get a BRSTQFT. However, we did not find gauge fixing functions for a single uncharged 3-form gauge field in 12 dimensions. Rather, we did find one for a single *charged* 3-form, and another one for a theory with two *uncharged* 3-forms. (See below.)

We could introduce a 5-form gauge field, (not relevant for pure 11-dimensional supergravity), and similar to the 8-dimensional case, consider self-duality conditions for the 6-form curvature of C_5, with a gauge condition of the type

$$dC_5 + *dC_5 + \mathrm{Tr}\,(F \wedge F \wedge F)_+ = 0. \quad (4.20)$$

In the present understanding of superstrings, 5-forms are not so natural; so we will not elaborate further on this case.

[12] The R symmetry is the automorphism of the extended supersymmetry algebra.
[13] Here again dC^3 means $h + dC^3$, where h is the harmonic representative of an element in $H^4(M_{12})$.

When M_{12} is a Calabi–Yau 6-fold, we can do some things in two different theories. In the first theory, we couple a *charged* 3-form B to the Yang–Mills field. (B is valued in the same Lie Algebra as A.) We again use $* : \Lambda^{0,q} \longrightarrow \Lambda^{0,6-q}$, so that $\Lambda^{0,3} \otimes \mathcal{G} = \Lambda^{0,3}_+ \otimes \mathcal{G} + \Lambda^{0,3}_- \otimes \mathcal{G}$. Again $\bar\partial_A F^{0,2} - \bar\partial_A \bar\partial_A {}^* B = 0$ implies for compact manifolds that $\bar\partial_A{}^* B = 0$. The covariant gauge condition is $*F^{0,2} = \bar\partial_A B$, $B \in \Lambda^{0,3}_+ \otimes \mathcal{G}$; equivalently, $F^{0,2} = \bar\partial^*_A B$. So the covariant gauge conditions become the pair $F^{0,2} = 0$ and $\bar\partial_A B = 0$, similar to the Calabi–Yau 3-fold case in Sect. 4.1.1. There, $F^{0,2} = 0$ and $\bar\partial^*_A \widetilde{\varphi} = 0$, with $\widetilde{\varphi} \in \Lambda^{0,3} \otimes \mathcal{G}$. In the present case, $B \in \Lambda^{0,3}_+ \otimes \mathcal{G}$.

The moduli space is a vector bundle over the set of holomorphic bundles for a fixed C^∞ (E, ρ). Each such holomorphic structure gives a unique A with $F^{0,2}_A = 0$. The fiber over A consists of $[B \in \Lambda^{0,3}_+ \otimes \mathcal{G}\ ;\ \bar\partial_A B = 0]$.

The sequence $0 \to \Lambda^{0,0} \otimes \mathcal{G} \xrightarrow{\bar\partial_A} \Lambda^{0,1} \otimes \mathcal{G} \xrightarrow{\bar\partial_A} \Lambda^{0,2} \otimes \mathcal{G} \xrightarrow{\bar\partial_A} \Lambda^{0,3}_+ \otimes \mathcal{G}$ is elliptic at the symbol level; linearization of the covariant gauge condition together with the usual gauge fixing is given by the elliptic operator:

$$\begin{pmatrix} \bar\partial_A & \bar\partial^*_A \\ \bar\partial^*_A & 0 \end{pmatrix} : \begin{pmatrix} \Lambda^{0,1} \otimes \mathcal{G} \\ \Lambda^{0,3}_+ \otimes \mathcal{G} \end{pmatrix} \longrightarrow \begin{pmatrix} \Lambda^{0,2} \otimes \mathcal{G} \\ \Lambda^{0,0} \otimes \mathcal{G} \end{pmatrix}. \tag{4.21}$$

We take as classical "topological" action $S_0[A, B] = \int_{M_{12}} \Omega_6 \wedge \mathrm{Tr}\,(\bar\partial_A B \wedge F_A)$ where Ω_6 is the $(6,0)$ covariant constant form of M_{12}. Since the covariant gauge function is $F^{0,2} - \bar\partial^*_A B$ and since $\langle F^{0,2}, \bar\partial^*_A B\rangle = \int_{M_{12}} \Omega_6 \wedge \mathrm{Tr}\,(F^{0,2} \wedge \bar\partial^*_A B)$, we have $\| F^{0,2} - \bar\partial^*_A B \|^2 = \| F^{0,2} \|^2 + \| \bar\partial^*_A B \|^2 - \langle F^{0,2}, \bar\partial^*_A B\rangle - \langle \bar\partial^*_A B, F^{0,2}\rangle$, that is, $\| F^{0,2} - \bar\partial^*_A B \|^2 = \| F^{0,2} \|^2 + \| \bar\partial^*_A B \|^2 - S_0[A, B] - {}^{\mathbf{c}}S_0[A, B]$. (Remember that $\bar\partial^*_A = *\bar\partial_A *$.) We thus obtain a BRSTQFT whose gauge fixed action will include the term $\| F^{0,2} \|^2 + \| \bar\partial^*_A B \|^2$. Moreover, the condition that $B \in \Lambda^{0,3}_+ \otimes \mathcal{G}$ can be imposed in a BRST invariant by using the ordinary gauge freedom of B [14].

In the second theory, we introduce *two uncharged* 2-form gauge fields B^a_2 and two (non abelian) Yang–Mills fields A^a, with $a = 1$ and 2. We consider the following topological classical action

$$\int_{M_{12}} \epsilon_{ab} \Omega_6 \wedge dB^a_2 \wedge dB^b_2. \tag{4.22}$$

We define the following "holomorphic" gauge conditions, where the complex indices run from 1 to 6,

$${}^{\mathbf{c}}\partial_{[\bar\mu} B^a_{\bar\nu\bar\rho]} + \epsilon^a_b \epsilon_{\bar\mu\bar\nu\bar\rho\bar\alpha\bar\beta\bar\gamma} \partial_{[\bar\alpha} B^b_{\bar\beta\bar\gamma]} = \mathrm{Tr}\,(A^a_{[\bar\mu} \partial_{\bar\nu} A^a_{\bar\rho]} + \frac{2}{3} A^a_{[\bar\mu} A^a_{\bar\nu} A^a_{\bar\rho]}). \tag{4.23}$$

The right-hand side of this equation is the Chern–Simons form of rank 3. The similarity to 8 dimensions is striking, up to the replacement of the even Chern class by the odd Chern–Simons class. Equation (4.23) implies

$$\partial^{\bar\rho} \partial_{[\bar\mu} B^a_{\bar\nu\bar\rho]} = \epsilon^a_b \epsilon_{\bar\mu\bar\nu\bar\rho\bar\alpha\bar\beta\bar\gamma} \mathrm{Tr}\, F^b_{\bar\rho\bar\alpha} F^b_{\bar\beta\bar\gamma}. \tag{4.24}$$

Its solution is the stationary point of the following action:

[14] The (0,3)-form B is valued in the same Lie algebra as the Yang–Mills field. It is thus non abelian and its quantization involves the field anti-field formalism of Batalin and Vilkoviski. We intend to perform elsewhere this rather technical task, which generalizes that sketched at the end of Sect. 3.0.

$$\int_{M_{12}} d^{12}x\ \epsilon_{ab}(\partial_{[\bar\mu} B^a_{\bar\nu\bar\rho]}{}^{\mathbf{c}} \partial_{[\bar\mu} B^b_{\bar\nu\bar\rho]} + \epsilon_{\bar\mu\bar\nu\bar\rho\bar\alpha\bar\beta\bar\gamma}{}^{\mathbf{c}} B^a_{\bar\nu\bar\rho} \mathrm{Tr}\ F^b_{\bar\rho\bar\alpha} F^b_{\bar\beta\bar\gamma} + \text{complex conjugate}). \tag{4.25}$$

Gauge fixing the Lagrangian Eq. (4.22) by the gauge condition Eq. (4.24) provides a BRST invariant action. Its ghost independent and gauge independent part is identical to the action Eq. (4.25).

4.2.2. Other possibilities. In 10 dimensions one could build a BRSTQFT based on a four-form gauge field B_4 and a pair of two gauge field B_2^a, $a = 1, 2$, which naturally fit into the type IIB superstring. All these forms are uncharged, but they can develop non trivial interactions [30]. The curvatures are

$$G_5 = dB_4 + \epsilon_{ab} B_2^a G_3^b, \tag{4.26}$$

$$G_3^a = dB_2^a, \tag{4.27}$$

with Bianchi identities, $dG_5 = \epsilon_{ab} G_3^a G_3^b$ and $dG_3^a = 0$. One can construct from these fields one closed 11-form

$$\Delta_{11} = \epsilon_{ab} G_3^a G_3^b G_5, \tag{4.28}$$

and two 8-forms

$$\Delta_8^a = G_5 G_3^a. \tag{4.29}$$

The role of the invariant forms is obscure, but their existence could signal generalizations of the Green–Schwarz type anomaly cancellation mechanism. The covariant gauge function is

$$dB_4 + *dB_4 + \epsilon_{ab} B_2^a dB_2^b = 0. \tag{4.30}$$

The mixing of forms of various degrees by the gauge functions generalizes that of the 3-form with the Yang–Mills field in the eight dimensional theory of Sect. 3.

5. Conclusion

We have described some new Yang–Mills quantum field theories in dimensions greater than four, using self duality. In eight dimensions we found two BRSTQFT's depending on holonomy $Spin(7)$ (the J-case) or holonomy $SU(4)$ (the H-case). In the J-case, BRST symmetry is what is left of supersymmetry.

The increase in dimension allows us to couple ordinary gauge fields to forms of higher degree. We have given several examples.

Dimensional reduction generates new theories. One of them is a BRSTQFT whose gauge conditions are the non-abelian Seiberg–Witten equations.

In four dimensions, given the self duality condition, there are other ways of deriving the Lagrangian of Witten's topological Yang–Mills theory besides Witten's twist of $N=2$ SSYM and besides BRST [1, 2, 33]. These methods should work equally well in deriving our BRSTQFT Lagrangians for the pure Yang–Mills case.

Finally, as we have indicated earlier, the geometries of the moduli spaces we have probed have not been worked out. Much remains to be done [13]. However, from the lessons learned in four dimensions, it is tempting to hurdle these obstacles and proceed to the corresponding Seiberg–Witten abelian theory. Preliminary investigations indicate that one can compute the Seiberg–Witten invariants, when M_8 is hyperKähler, i.e., when the holonomy group is $Sp(2)$. This case is very similar to the Seiberg–Witten invariants for M_4 when it is Kähler [34].

Acknowledgement. We thank M. Duff for pointing out to us that the octonionic solution in [14] and [15] does not give a self dual $Tr(F \wedge F)$. H.K. would like to thank T. Eguchi and T. Inami for helpful communications. The work of H.K. is supported in part by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture, Japan. L. B. would like to thank the Yukawa Institute where part of this work has been done, and E. Corrigan and H. Nicolai for discussions. IMS would like to thank G. Tian for bringing him up to date on complex geometry. He also benefited from discussions with S. Axelrod, S. Donaldson, R. Thomas, and E. Weinstein. His work is supported in part by a DOE Grant No. DE-FG02-88ER25066.

Note added on July 17, 1997. T.A. Ivanova has called our attention to [14], where instanton solutions are found. B.S. Acharya and M. Loughlin have called our attention to their paper [35] where they discuss self duality for Euclidean gravity when $d \leq 8$. B.S. Acharya, M. Loughlin and B. Spence also discuss self duality in [36]. In their paper, a note added says that their proof of BRST invariance would "seem to conflict" with our theory not being topological. Indeed the theory is *not* topological. They made the corrections in a revised version.

We expand on our assertion. Assume M is a compact oriented simply connected manifold with $\hat{A}$ = 1 and assume M admits a Joyce metric, i.e, a metric with $Spin(7)$ holonomy. The space of Joyce metrics modulo diffeomorphisms isotopic to the identity is of dimension $1 + b^4_-(M)$ (see Theorem D in [20]). It is conceivable that this manifold of Joyce metrics is not connected so that one cannot find a path from one Joyce metric to another with each point of the path a Joyce metric.

The BRST argument for invariance requires a path of Joyce metrics, hence shows formally that the correlation functions are constant on components of the space of Joyce metrics. But the argument does not imply constancy of the correlation functions on all Joyce metrics. This is one reason we chose not to label our J-case QFT a topological quantum field theory.

On the mathematical side the argument analogous to BRST invariance also works formally because the correlation functions come from the second Chern class (see 2.1.3). As we indicated there, to define the analogue of Donaldson invariants (the correlation function precisely), one needs to integrate over the moduli space M_J of self dual connections. To do so, a compactification of M_J is important (work in progress by D. Joyce and C. Lewis).

The H-case (Sect. 2.2.3 in particular) is more complicated. Physicists allow a degeneration of the complex structure to connect one moduli space with another. We do not know how the "holomorphic Donaldson invariants" behave under this degeneration.

References

1. Birmingham, D., Blau, M., Rakowski, M., G. Thompson: Physics Reports **209**, 129 (1991)
2. Cordes, S., Moore, G., Ramgoolam, S.: Lectures on 2D Yang–Mills Theory, Equivariant Cohomology and Topological Field Theories. In: Les Houches Session LXII, hep-th/9411210
3. Donaldson, S.K.: Topology. **29**, 257 (1990)
4. Witten, E.: Commun. Math. Phys. **117**, 353 (1988)
5. Baulieu, L., Singer, I.: Nucl. Phys. B (Proc. Supple.) **5B**, 12 (1988)
6. E. Corrigan, Devchand, C., Fairlie, D.B., Nuyts, J.: Nucl. Phys. **B214**, 452 (1983)
7. Ward, R.S.: Nucl. Phys. **B236**, 381 (1984)
8. Fairlie, D.B., Nuys, J.: J. Phys.**A17**, 2867 (1984)
9. Fubini, S., Nicolai, H.: Phys. Lett. **155B**, 369 (1985)
10. Salamon, S.: *Riemannian Geometry, Holonomy Groups*. Pitman Research Notes in Mathematics Series, 1989

11. Weinstein, E.: Extension of self-dual Yang–Mills equations across the 8th dimension. PHD dissertation, Harvard University Math. Dept, 1992
12. Joyce, D.D.: Invent. Math. **123**, 507 (1996)
13. Donaldson, S.K., Thomas, R.P.: *Gauge Theory in Higher Dimensions*. Oxford preprint (1996)
14. Ivanova, T.A.: Phys. Lett. **B315**, 277 (1993); Ivanova, T.A., Popov, A.D.: Lett. Math. Phys. **24**, 85 (1992); Theor. Math. Phys. **94**, 225 (1993)
15. Günayden, M., Nicolai, H.: Phys. Lett. **B351**, 169 (1995)
16. Vafa, C., Witten, E.: Nucl. Phys. **B431**, 3 (1994)
17. Vafa, C.: Nucl. Phys. **B469**, 403 (1996), hep-th/9602022
18. Papadopoulos, G., Townsend, P.K.: Phys. Lett. **B357**, 300 (1995), hep-th/9506150
19. Acharya, B.S.: N=1 M-Theory-Heterotic Duality in Three-Dimensions and Joyce Manifolds. hep-th/9604133; Dirichlet Joyce Manifolds, Discrete Torsion and Duality. hep-th/9611036
20. Joyce, D.D.: J. Diff. Geom. **43**, 291, 329 (1996)
21. Shatashvili, S., Vafa, C.: Superstrings, Manifolds of Exceptional Holonomy. hep-th/9407025
22. Figueroa-O'Farrill, J.M.: A Note on the Extended Superconformal Algebras Associated with Manifolds of Exceptional Holonomy. hep-th/9609113
23. Günayden, M., Gürsey, F.: J. Math. Phys. **14**, 1651 (1973)
24. DeBartolomeis, P., Tian, G.: J. Diff. Geom. **43**, 231 (1973)
25. Salam, A., Sezgin, E. (Eds), *Supergravities in Diverse Dimensions*. Amsterdam–Singapore: North-Holland/World Scientific, 1989
26. Witten, E.: Nucl. Phys. **B460**, 335 (1996), hep-th/9510135
27. Bershadsky, M., Sadov, V., Vafa, C.: Nucl. Phys. **B463**, 420 (1996), hep-th/9511222
28. Becker, K., Becker, M., Morrison, D.R., Ooguri, H., Oz, Y., Yin, Z.: Supersymmetric Cycles in Exceptional Holonomy Manifolds and Calabi–Yau 4-folds. hep-th/9608116
29. Furuuchi, K., Kunitomo, H., Nakatsu, T.: Topological Field Theory and Second-Quantized Five-Branes. hep-th/9610016
30. Baulieu, L.: *Algebraic quantization of gauge theories*. In: Perspectives in fields and particles, eds. Basdevant-Levy, Cargese Lectures 1983 London: Plenum Press, 1985; Baulieu, L.: Nucl. Phys. **B478**, 431 (1996)
31. Brylinski, J.-L.: *Loop Spaces, Characteristic Classes, Geometric Quantization*. Berlin: Birkhäuser, 1992
32. Baulieu, L., Grossman, B.: Phys. Lett. **214B**, 223 (1988)
33. Atiyah, M., Jeffrey, L.: J. Geom. Phys. **7**, 120 (1990)
34. Witten, E.: Math. Res. Letters **1**, 764 (1994)
35. Acharya, B.S., O'Loughlin, M.: Phys. Rev. **D55**, R4521, (1997), hep-th/9612182
36. Acharya, B.S., O'Loughlin, M., Spence, B.: hep-th/9705138

Communicated by S.-T. Yau