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Classification of Bicovariant Differential Calculi on Quantum Groups (a Representation-Theoretic Approach)

Pierre Baumann, Frédéric Schmitt

U.F.R. de Mathématique, Université Louis Pasteur, 7 rue René Descartes, F-67084 Strasbourg Cedex, France

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Abstract: The restricted dual of a quantized enveloping algebra can be viewed as the algebra of functions on a quantum group. According to Woronowicz, there is a general notion of bicovariant differential calculus on such an algebra. We give a classification theorem of these calculi. The proof uses the notion (due to Reshetikhin and Semenov-Tian-Shansky) of a factorizable quasi-triangular Hopf algebra and relies on results of Joseph and Letzter. On the way, we also give a new formula for Rosso's bilinear form.

Introduction

Let G be a semi-simple connected simply-connected complex Lie group, $\mathfrak{g}$ its Lie algebra, $\mathrm{U}_q\mathfrak{g}$ the quantized enveloping algebra of $\mathfrak{g}$. $\mathrm{U}_q\mathfrak{g}$ is a Hopf algebra. The associated quantum group is an object of non-commutative geometry. According to a point of view due to Woronowicz and developed by Faddeev, Reshetikhin and Takhtadzhyan [F–R–T], one may view the restricted (Hopf) dual $(\mathrm{U}_q\mathfrak{g})^{*\,\mathrm{res}}$ as the algebra $\mathcal{A}_q\mathrm{G}$ of functions on this quantum group. In this way, the Peter–Weyl theorem becomes a definition: the rational representations of the quantum group are the finite-dimensional representations of $\mathrm{U}_q\mathfrak{g}$.

In order to study the differential geometry of quantum groups, Woronowicz [Wo] defined the notion of bicovariant differential calculus. As in the classical case, one needs only to define the differential of functions at the unity point of the quantum group. If $\varepsilon : \mathcal{A}_q\mathrm{G} \to \mathbb{C}(q)$ is the augmentation map, this amounts to take the residual class of functions belonging to $\ker\varepsilon$ modulo a right ideal $\mathcal{R} \subseteq \ker\varepsilon$. In the classical case, one takes $\mathcal{R} = (\ker\varepsilon)^2$. As for quantum groups, it is more important to preserve the group structure than the infinitesimal structure, and one is led to select ideals $\mathcal{R}$ as above by the requirement of a certain invariance condition. In this article, we solve the classification problem for these ideals $\mathcal{R}$, and we give a picture of what they look like.

We now compare our results with previous ones. Rosso [Ro3] showed how to use the quasi-triangular structure of $\mathrm{U}_q\mathfrak{g}$ in order to construct left covariant differential calculi on the quantum group. Modifying this construction, Jurčo [Ju] used the R-matrix in the natural representation of $\mathrm{U}_q\mathfrak{g}$ (and in the dual of it) so as to construct bicovariant differential calculi: he obtained particular cases (when M is the natural $\mathfrak{g}$-module or its dual) of our Theorem 2. (In this spirit, see also [F–P].) As regards classification results, Schmüdgen and Schüler have classified the ideals $\mathcal{R}$ as above, but only when $\mathfrak{g}$ is of classical type, and under restrictive assumptions on $\mathcal{R}$. Most of the results in [S–S1, S–S2] are particular cases of our Theorem 1. For instance, the classification given in Theorem 2.1 of [S–S1] corresponds (in the wording of our theorem) to the ideals $\mathcal{R}$ constructed (up to a twisting character $\chi : {}^{2\mathrm{X}}/_{2\mathrm{Q}} \to \mathbb{C}^\times$, as explained in Sect. 3.3) from the natural $\mathrm{U}_q\mathfrak{sl}_n$-module or its dual.

Let us explain our proof and the contents of our article. Our proof relies on the quasi-triangular structure of $\mathrm{U}_q\mathfrak{g}$. Since the formalism of R-matrices may be justified only for finite dimensional Hopf algebras, we will employ the dual notion of a co-quasi-triangular (c.q.t.) Hopf algebra (see [L–T]): the algebra $\mathcal{A}_q\mathrm{G}$ is c.q.t. We use then a bilinear form on $\mathcal{A}_q\mathrm{G}$, introduced by Reshetikhin and Semenov-Tian-Shansky. As $\mathrm{U}_q\mathfrak{g}$ is a factorizable quasi-triangular Hopf algebra (in the terminology of [R–S]), this pairing is non-degenerate and gives a linear injection $\mathcal{A}_q\mathrm{G} \hookrightarrow \mathrm{U}_q\mathfrak{g} \subseteq (\mathcal{A}_q\mathrm{G})^*$. The image of $\mathcal{R}$ under this map is nearly the annihilator of a $\mathrm{U}_q\mathfrak{g}$-module. It is then easier to discuss what $\mathcal{R}$ may be. The definitions and the proofs of these assertions are given in Sects. 1 and 2. In Sect. 3, we present a contruction of bicovariant differential calculi valid for any factorizable c.q.t. Hopf algebra. Finally we link, in the case of $\mathcal{A}_q\mathrm{G}$, these constructions with our classification result.

Notations.

- Let A be a k-algebra. If M is an A-module, its annihilator is denoted by $\mathrm{ann}_\mathrm{A}\,\mathrm{M}$. If $m \in \mathrm{M}$ and $m^* \in \mathrm{M}^*$ (the k-dual of M), we denote by $\theta_\mathrm{M}(m, m^*)$ the matrix coefficient ($\mathrm{A} \to k, a \mapsto \langle m^*, a \cdot m\rangle$).
- For a Hopf algebra H, we will use Sweedler's notation for coproduct ($\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$) and for coaction on comodules. The sum sign will generally be omitted. We will denote the augmentation and the antipode of H by ε and S respectively.
- Let H be a Hopf algebra, and $\mathrm{H}^{*\,\mathrm{res}}$ be the restricted (Hopf) dual of H. A finite-dimensional left H-module M (with a basis (m_i) and the dual basis (m_i^*) of M^*) can be viewed as a right $\mathrm{H}^{*\,\mathrm{res}}$-comodule with structure map $\delta_\mathrm{R} : (\mathrm{M} \to \mathrm{M} \otimes \mathrm{H}^{*\,\mathrm{res}}, m \mapsto \sum m_i \otimes \theta_\mathrm{M}(m, m_i^*))$.

1. Co-Quasi-Triangular Hopf Algebras

1.1. Some definitions. Let H be a Hopf algebra over a field k. A right crossed bimodule over H (in the sense of Yetter [Ye]) is a k-vector space M, which is also a right H-module, a right H-comodule (with structure map $\delta_\mathrm{R} : (\mathrm{M} \to \mathrm{M} \otimes \mathrm{H}, m \mapsto \sum m_{(0)} \otimes m_{(1)})$), both structures being compatible: $\delta_\mathrm{R}(m \cdot a) = \sum m_{(0)} \cdot a_{(2)} \otimes \mathrm{S}(a_{(1)}) m_{(1)} a_{(3)}$ (for $m \in \mathrm{M}$, $a \in \mathrm{H}$). When M and N are right crossed bimodules over H, $\mathrm{M} \otimes \mathrm{N}$ becomes a right crossed bimodule for the action $(m \otimes n) \cdot a = m \cdot a_{(1)} \otimes n \cdot a_{(2)}$ and the coaction $\delta_\mathrm{R}(m \otimes n) = (m_{(0)} \otimes n_{(0)}) \otimes m_{(1)} n_{(1)}$.

There are two easy examples: we can endow H with the structures: $a \cdot b = ab$ and $\delta_\mathrm{R} : (\mathrm{H} \to \mathrm{H} \otimes \mathrm{H}, a \mapsto a_{(2)} \otimes \mathrm{S}(a_{(1)}) a_{(3)})$. Alternatively, we can put on H the structures $a \cdot b = \mathrm{S}(b_{(1)}) a b_{(2)}$ (right adjoint action) and $\delta_\mathrm{R} : (\mathrm{H} \to \mathrm{H} \otimes \mathrm{H}, a \mapsto a_{(1)} \otimes a_{(2)})$.

When Γ is a bicovariant bimodule (see [Wo]), the space Γ^{L} of left coinvariants is a right crossed bimodule over H. Conversely, any right crossed bimodule over H is the space of left coinvariants of a bicovariant bimodule.

Finally (H still being a Hopf algebra), we endow the tensor product coalgebra $\mathrm{H}^{*\,\mathrm{res}} \otimes \mathrm{H}$ with the product $(f \otimes a)(g \otimes b) = \langle g_{(3)}, a_{(3)}\rangle\langle g_{(1)}, \mathrm{S}(a_{(1)})\rangle(g_{(2)}f \otimes a_{(2)}b)$. We obtain a bialgebra, called Drinfel′d's double of H and denoted by $\mathcal{D}(\mathrm{H})$. (Here $\mathrm{H}^{*\,\mathrm{res}}$ is the standard dual of H, the coproduct is not brought into its opposite.) When M is a right crossed bimodule over H, it is a right $\mathcal{D}(\mathrm{H})$-module for the actions: $m \cdot (f \otimes 1) = \langle f, m_{(1)}\rangle m_{(0)}$, $m \cdot (1 \otimes a) = m \cdot a$.

1.2. Definition of a co-quasi-triangular Hopf algebra. We give the definition of c.q.t. Hopf algebras, by now usual (see [L–T] for historical notes):

Definition 1. *A co-quasi-triangular Hopf algebra is a pair $(\mathcal{A}, \gamma)$, where $\mathcal{A}$ is a Hopf algebra and $\gamma : \mathcal{A} \to \mathcal{A}^{*\,\mathrm{res}}$ is a coalgebra morphism and an algebra antimorphism such that we have the Yang–Baxter equation (or rather the Baxter commutation relations): $a_{(1)}b_{(1)}\langle \gamma a_{(2)}, b_{(2)}\rangle = \langle \gamma a_{(1)}, b_{(1)}\rangle b_{(2)}a_{(2)}$ for all $a, b \in \mathcal{A}$.*

That γ is a coalgebra morphism and an algebra antimorphism gives us that for all $a, b \in \mathcal{A}$, $\langle \gamma a, b\rangle = \langle \gamma \mathrm{S}(a), \mathrm{S}(b)\rangle$. We call $\delta : \mathcal{A} \to \mathcal{A}^*$ the map such that $\langle \delta a, b\rangle = \langle \gamma b, \mathrm{S}(a)\rangle$, for all $a, b \in \mathcal{A}$. Hence we have $\langle \gamma a, b\rangle = \langle \delta b, \mathrm{S}(a)\rangle$. We verify easily that δ takes its values in $\mathcal{A}^{*\,\mathrm{res}}$ and that $(\mathcal{A}, \delta)$ is a c.q.t. Hopf algebra.

If U is a Hopf algebra quasi-triangular for an R-matrix R_{12}, then $\mathrm{U}^{*\,\mathrm{res}}$ becomes a c.q.t. Hopf algebra for the map γ given by: for $a, b \in \mathrm{U}^{*\,\mathrm{res}}$, $\langle \gamma(a), b\rangle = \langle b \otimes a, R_{12}\rangle$, and then $\langle \delta(a), b\rangle = \langle b \otimes a, R_{21}^{-1}\rangle$. This follows from Drinfel′d's classical axioms. For instance, let H be a finite-dimensional Hopf algebra, and $\mathrm{U} = \mathcal{D}(\mathrm{H})$: the dual vector space $\mathrm{H} \otimes \mathrm{H}^*$ of U is the underlying space of the restricted dual of U. If (e_i) is a basis for H, the canonical R-matrix is $\sum (e_i^* \otimes 1) \otimes (1 \otimes e_i) \in \mathrm{U} \otimes \mathrm{U}$. It corresponds to the maps $\gamma : (\mathrm{H} \otimes \mathrm{H}^* \to \mathrm{U}, a \otimes f \mapsto \varepsilon(a) f \otimes 1)$ and $\delta : (\mathrm{H} \otimes \mathrm{H}^* \to \mathrm{U}, b \otimes g \mapsto g(1)\varepsilon \otimes \mathrm{S}^{-1}(b))$ (the antipode of a finite-dimensional Hopf algebra being invertible).

The category of left modules over a quasi-triangular Hopf algebra is braided. The translation in the present formalism is given by the following proposition:

Proposition 1. *Let $(\mathcal{A}, \gamma)$ be a c.q.t. Hopf algebra. If* M *is a right $\mathcal{A}$-comodule, it becomes a right crossed bimodule over $\mathcal{A}$ when endowed with the right module structure given by: for $m \in \mathrm{M}$ and $a \in \mathcal{A}$, $m \cdot a = \langle \gamma a, m_{(1)}\rangle m_{(0)}$. This extra structure is compatible with tensor products of comodules and crossed bimodules.*

Proof. Let $\delta_{\mathrm{R}} : (\mathrm{M} \to \mathrm{M} \otimes \mathcal{A}, m \mapsto m_{(0)} \otimes m_{(1)})$ the structure map for M. Then we have:

$$\begin{aligned} m_{(0)} \cdot a_{(2)} \otimes \mathrm{S}(a_{(1)})m_{(1)}a_{(3)} &= m_{(0)} \otimes \langle \gamma a_{(2)}, m_{(1)}\rangle \mathrm{S}(a_{(1)})m_{(2)}a_{(3)} \\ &= m_{(0)} \otimes \mathrm{S}(a_{(1)})a_{(2)}m_{(1)}\langle \gamma a_{(3)}, m_{(2)}\rangle \\ &= m_{(0)} \otimes m_{(1)}\langle \gamma a, m_{(2)}\rangle \\ &= \delta_{\mathrm{R}}(m \cdot a). \end{aligned}$$

The compatibility with tensor products is a consequence of γ being a coalgebra homomorphism. □

We also note that the antipode of a c.q.t. Hopf algebra is always invertible, the square of its transpose being an inner automorphism of the algebra $\mathcal{A}^*$ (see [Dr2]).

Finally, when $(\mathcal{A},\gamma)$ is a c.q.t. Hopf algebra, we have the maps γ and δ, and Radford [Ra] has shown that (im γ)(im δ) = (im δ)(im γ) is a sub-Hopf-algebra of $\mathcal{A}^{*\,\mathrm{res}}$. This was shown in the early [R–S]: there is a Hopf algebra structure (with invertible antipode) on the tensor product coalgebra $\mathcal{A}\otimes\mathcal{A}$ such that the map $(\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}^{*\,\mathrm{res}}, a\otimes b\mapsto \gamma b\cdot\delta a)$ is a coalgebra morphism and an algebra antimorphism.

Example. In the F.R.T. construction [F–R–T], one considers matrices L^+ and L^-, whose elements lie in im γ and im δ respectively. Then Faddeev, Reshetikhin and Takhtadzhyan defined $\mathrm{U}_q\mathfrak{g}$ to be the algebra (im γ)(im δ).

1.3. The maps I *and* J. We fix in this subsection a c.q.t. Hopf algebra $(\mathcal{A},\gamma)$ over the field k, and note δ the associated map. We define two maps $\mathrm{I} : (\mathcal{A}\to\mathcal{A}^{*\,\mathrm{res}}, a\mapsto \gamma(a_{(1)})\ \mathrm{S}\delta(a_{(2)}))$ and $\mathrm{J} : (\mathcal{A}\to\mathcal{A}^{*\,\mathrm{res}}, a\mapsto \mathrm{S}\delta(a_{(1)})\ \gamma(a_{(2)}))$. Equivalently, we may consider the pairing of two elements $a,b\in\mathcal{A}$: $\langle \mathrm{I}(a),b\rangle = \langle \mathrm{J}(b),a\rangle$. (When $\mathcal{A}$ is the dual of a quasi-triangular Hopf algebra, this pairing is $\langle a\otimes b, R_{21}R_{12}\rangle$.) We have $\mathrm{I}=\mathrm{S}\circ\mathrm{J}\circ\mathrm{S}$ and $\mathrm{J}=\mathrm{S}\circ\mathrm{I}\circ\mathrm{S}$.

We will now state an important property of the map I. $\mathcal{A}^{*\,\mathrm{res}}$ is a left $\mathcal{A}^{*\,\mathrm{res}}\otimes\mathcal{A}^{*\,\mathrm{res}}$-module for the law $(x\otimes y)\cdot z = xz\,\mathrm{S}(y)$. $\mathcal{A}$ is a right crossed bimodule over $\mathcal{A}$ for the structures: $a\cdot b = ab$ and $\delta_{\mathrm{R}} : (\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}, a\mapsto a_{(2)}\otimes \mathrm{S}(a_{(1)})a_{(3)})$, so $\mathcal{A}$ is a right $\mathcal{D}(\mathcal{A})$-module. Let $\Pi : (\mathcal{D}(\mathcal{A})\equiv\mathcal{A}^{*\,\mathrm{res}}\otimes\mathcal{A}\to\mathcal{A}^{*\,\mathrm{res}}\otimes\mathcal{A}^{*\,\mathrm{res}}, x\otimes b\mapsto \gamma(b_{(1)})x_{(1)}\otimes\delta(b_{(2)})x_{(2)})$.

Proposition 2. *In the set-up above,* Π *is an algebra antimorphism. If* $\xi\in\mathcal{D}(\mathcal{A})$ *and* $a\in\mathcal{A}$*, then* $\mathrm{I}(a\cdot\xi)=\Pi(\xi)\cdot\mathrm{I}(a)$.

Proof. That Π is an antimorphism is already in [R–S]. Then, as a consequence of the Yang–Baxter equation, we may write, for $x\in\mathcal{A}^{*\,\mathrm{res}}$ and $a\in\mathcal{A}$, that $\mathrm{S}\gamma(a_{(1)})\langle x,a_{(2)}\rangle = \langle x_{(2)},a_{(1)}\rangle x_{(1)}\mathrm{S}\gamma(a_{(2)})\mathrm{S}(x_{(3)})$. Then we compute, for $\xi = x\otimes b\in\mathcal{D}(\mathcal{A})$:

$$\begin{aligned}
\mathrm{I}(a\cdot\xi) &= \langle x,\mathrm{S}(a_{(1)})a_{(3)}\rangle\ \mathrm{I}(a_{(2)}b)\\
&= \gamma(b_{(1)})\ \langle x,\mathrm{S}(a_{(1)})a_{(4)}\rangle\ \gamma(a_{(2)})\ \mathrm{S}\delta(a_{(3)})\ \mathrm{S}\delta(b_{(2)})\\
&= \gamma(b_{(1)})\ \langle x_{(1)},\mathrm{S}(a_{(1)})\rangle\ \mathrm{S}\gamma\mathrm{S}(a_{(2)})\ \langle x_{(2)},a_{(4)}\rangle\ \mathrm{S}\delta(a_{(3)})\ \mathrm{S}\delta(b_{(2)})\\
&= \gamma(b_{(1)})\ \langle x_{(2)},\mathrm{S}(a_{(2)})\rangle x_{(1)}\mathrm{S}\gamma\mathrm{S}(a_{(1)})\mathrm{S}(x_{(3)})\langle x_{(5)},a_{(3)}\rangle x_{(4)}\mathrm{S}\delta(a_{(4)})\mathrm{S}(x_{(6)})\mathrm{S}\delta(b_{(2)})\\
&= \gamma(b_{(1)})\ \langle x_{(2)},\mathrm{S}(a_{(2)})\rangle\ x_{(1)}\ \mathrm{S}\gamma\mathrm{S}(a_{(1)})\ \langle x_{(3)},a_{(3)}\rangle\ \mathrm{S}\delta(a_{(4)})\ \mathrm{S}(x_{(4)})\ \mathrm{S}\delta(b_{(2)})\\
&= \gamma(b_{(1)})\ x_{(1)}\ \gamma(a_{(1)})\ \mathrm{S}\delta(a_{(2)})\ \mathrm{S}(x_{(2)})\ \mathrm{S}\delta(b_{(2)})\\
&= \Pi(\xi)\cdot\mathrm{I}(a). \qquad \square
\end{aligned}$$

We single out the particular case $b = 1$:

Proposition 3. *We consider* $\mathcal{A}$ *and* $\mathcal{A}^{*\,\mathrm{res}}$ *as left* $\mathcal{A}^{*\,\mathrm{res}}$*-modules for the adjoint action: if* $x,y\in\mathcal{A}^{*\,\mathrm{res}}$ *and* $a\in\mathcal{A}$, $x\cdot a = \langle x,\mathrm{S}(a_{(1)})a_{(3)}\rangle a_{(2)}$ *and* $x\cdot y = x_{(1)}y\,\mathrm{S}(x_{(2)})$*. Then* $\mathrm{I} : \mathcal{A}\to\mathcal{A}^{*\,\mathrm{res}}$ *is a morphism of* $\mathcal{A}^{*\,\mathrm{res}}$*-modules.*

Finally, we give the definition, originally due to Reshetikhin and Semenov-Tian-Shansky [R–S]:

Definition 2. *One says that* $(\mathcal{A},\gamma)$ *is factorizable if the pairing* $(\mathcal{A}\times\mathcal{A}\to k, (a,b)\mapsto \langle\mathrm{I}(a),b\rangle)$ *is non-degenerate.*

Thus $(\mathcal{A},\gamma)$ is factorizable iff the maps I and J are injective. It is possible to show that $(\mathcal{A},\gamma)$ is factorizable iff $(\mathcal{A},\delta)$ is so.

1.4. A related construction. First, let U be a Hopf algebra. It is a left U-module for the adjoint action: $x\cdot y = x_{(1)}y\,\mathrm{S}(x_{(2)})$. We let $\mathrm{F}_\ell(\mathrm{U})$ be the sum of all finite-dimensional

U-submodules of U. It is known [J–L1] that $F_\ell(U)$ is a subalgebra of U, a left coideal in U, and a U-submodule for the left adjoint action. The multiplication in U defines a morphism of left U-modules $F_\ell(U) \otimes F_\ell(U) \to F_\ell(U)$. We can then do the semi-direct product $F_\ell(U) \otimes U$: we obtain an algebra $U \otimes U$ denoting the ordinary tensor product algebra, there is an algebra morphism $(F_\ell(U) \otimes U \to U \otimes U, x \otimes y \mapsto xy_{(1)} \otimes y_{(2)})$. We can make the same constructions on the right: we obtain an algebra $F_r(U)$. If the antipode of U is invertible, the algebra morphism $(U \otimes F_r(U) \to U \otimes U, x \otimes y \mapsto x_{(1)} \otimes x_{(2)}y)$ has the same image as the previous one. Hence this image contains $F_\ell(U) \otimes F_r(U) \subseteq U \otimes U$.

We take now a c.q.t. Hopf algebra $(\mathcal{A}, \gamma)$, with δ, I and J as in the preceding subsection. Let U = $(\mathrm{im}\,\gamma)(\mathrm{im}\,\delta)$ be the minimal sub-Hopf-algebra of $\mathcal{A}^{*\,\mathrm{res}}$ in which γ and δ take their values. We consider on $\mathcal{A}$ and $\mathcal{A}^{*\,\mathrm{res}}$ the $\mathcal{A}^{*\,\mathrm{res}}$-module structures of Proposition 3. By restriction, $\mathcal{A}$ and $\mathcal{A}^{*\,\mathrm{res}}$ are U-modules, and $I : \mathcal{A} \to \mathcal{A}^{*\,\mathrm{res}}$ is a morphism of U-modules. We can see that I takes its values in U, which is a U-submodule of $\mathcal{A}^{*\,\mathrm{res}}$. Further, $\mathcal{A}$ is the sum of its finite-dimensional U-submodules, hence $\mathrm{im}\, I \subseteq F_\ell(U)$.

Proposition 4. *Let* $(\mathcal{A}, \gamma)$ *be a c.q.t. factorizable Hopf algebra, and* I *be the associated map. Let* U *be the sub-Hopf-algebra* $(\mathrm{im}\,\gamma)(\mathrm{im}\,\delta) \subseteq \mathcal{A}^{*\,\mathrm{res}}$. *We suppose that* $\mathrm{im}\, I = F_\ell(U)$. *Then* I *induces a bijection between:*

- *the set of right ideals* $\mathcal{R}$ *of* $\mathcal{A}$*, which are subcomodules for the right coaction* δ_R : $(\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}, a \mapsto a_{(2)} \otimes S(a_{(1)})a_{(3)})$.
- *the set of two-sided ideals* $\mathcal{I}$ *of* $F_\ell(U)$*, which are* U*-submodules for the adjoint action.*

This bijection preserves dimensions, codimensions, and the inclusion ordering in both sets.

Proof. By assumption, $I : \mathcal{A} \to F_\ell(U)$ is a U-module isomorphism. We adopt the notations of Proposition 2. $\mathcal{A}$ is a $\mathcal{D}(\mathcal{A})$-module, and $U \otimes \mathcal{A}$ is (the underlying space of) a sub-Hopf-algebra of $\mathcal{D}(\mathcal{A})$, so we will view $\mathcal{A}$ as a right $U \otimes \mathcal{A}$-module: $1 \otimes \mathcal{A}$ acts on $\mathcal{A}$ by right multiplication, $U^{\mathrm{op}} \otimes 1$ acts on $\mathcal{A}$ by the left adjoint action. The injectivity of I implies that $\mathrm{im}\, J \subseteq U$ separates the points of $\mathcal{A}$: hence the sub-$U \otimes \mathcal{A}$-modules of $\mathcal{A}$ are the right ideals which are subcomodules for the right coaction δ_R.

On the other hand, we let E be the image of the morphism $(F_\ell(U) \otimes U \to U \otimes U, x \otimes y \mapsto xy_{(1)} \otimes y_{(2)})$. U is a $U \otimes U$-module, so is an E-module, and $F_\ell(U)$ is a sub-E-module of U. E contains $F_\ell(U) \otimes F_r(U)$, with $S(F_r(U)) = F_\ell(U)$. Therefore, the sub-E-modules of $F_\ell(U)$ are the two-sided ideals $\mathcal{I}$ which are U-submodules for the adjoint action.

Now the proposition is a consequence of Proposition 2: writing Π as the composition $(F_\ell(U) \otimes \mathcal{A}^{*\,\mathrm{res}} \to \mathcal{A}^{*\,\mathrm{res}} \otimes \mathcal{A}^{*\,\mathrm{res}}, x \otimes y \mapsto xy_{(1)} \otimes y_{(2)}) \circ (\mathcal{A}^{*\,\mathrm{res}} \otimes \mathcal{A} \to F_\ell(U) \otimes \mathcal{A}^{*\,\mathrm{res}}, x \otimes a \mapsto I(a_{(1)}) \otimes \delta(a_{(2)})x)$, and using the assumption $\mathrm{im}\, I = F_\ell(U)$, we can see that E is the image of $U \otimes \mathcal{A}$ through Π. □

2. The Case of the Quantum Coordinate Algebra

2.1. Notations. In this section, we study the preceding constructions in the case where $\mathcal{A}$ is the algebra $\mathcal{A}_q G$ of regular functions on a quantum group.

Let $\mathfrak{g}$ be a finite-dimensional semi-simple split Lie algebra, $\mathfrak{h}$ a splitting Cartan subalgebra, $\{\alpha_1, \ldots, \alpha_\ell\} \subseteq \mathfrak{h}^*$ a basis for the root system, $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subseteq \mathfrak{h}$ the inverse roots, $P \subseteq \mathfrak{h}^*$ and $Q \subseteq \mathfrak{h}^*$ the weight and the root lattices. The choice of an invariant (under Weyl group action) scalar product $(\cdot|\cdot)$ allows us to identify $\mathfrak{h}$ and $\mathfrak{h}^*$, with $\alpha_i = d_i \alpha_i^\vee$, $d_i = \frac{(\alpha_i|\alpha_i)}{2}$. We choose the normalization of $(\cdot|\cdot)$ so that $(\lambda \mid \mu) \in \mathbb{Z}$ whenever λ and μ belong to P. We denote by ρ half the sum of the positive roots, by P_+ the set of dominant weights, and by w_0 the longest element in the Weyl group.

We now choose the following version of $\mathrm{U}_q\mathfrak{g}$: this is a $\mathbb{C}(q)$-algebra (q is generic) generated by E_i, F_i and K_λ ($\lambda \in \mathrm{P}$). The relations are the usual ones among which: $K_\lambda E_i = q^{(\lambda|\alpha_i)} E_i K_\lambda$, $K_\lambda F_i = q^{-(\lambda|\alpha_i)} F_i K_\lambda$, $E_i F_j - F_j E_i = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}}$. The coproduct is given by: $\Delta K_\lambda = K_\lambda \otimes K_\lambda$, $\Delta E_i = E_i \otimes 1 + K_{\alpha_i} \otimes E_i$, $\Delta F_i = 1 \otimes F_i + F_i \otimes K_{-\alpha_i}$. We note S the antipode of $\mathrm{U}_q\mathfrak{g}$. If one chooses a dominant weight λ and a character $\chi : {}^{\mathrm{P}}/_{2\mathrm{Q}} \to \mathbb{C}^\times$, one knows how to construct a simple finite-dimensional $\mathrm{U}_q\mathfrak{g}$-module, in which there is a highest weight vector m_λ such that $K_\mu \cdot m_\lambda = \chi(\mu \bmod 2\mathrm{Q}) q^{(\mu|\lambda)} m_\lambda$. We note $\mathrm{L}_\chi(\lambda)$ such a $\mathrm{U}_q\mathfrak{g}$-module ; when χ is the trivial character, we simply write $\mathrm{L}(\lambda)$, and then $\mathrm{L}_\chi(\lambda) = \mathrm{L}(\lambda) \otimes \mathrm{L}_\chi(0)$.

The matrix coefficients of the representation $\mathrm{L}(\lambda)$ span a linear subspace $\mathrm{C}(\lambda)$ of the restricted dual of $\mathrm{U}_q\mathfrak{g}$, and we let $\mathcal{A}_q\mathrm{G} = \bigoplus_{\lambda \in \mathrm{P}_+} \mathrm{C}(\lambda)$. This is a Hopf subalgebra of $(\mathrm{U}_q\mathfrak{g})^{*\,\mathrm{res}}$. The elements of $\mathcal{A}_q\mathrm{G}$ separate the points of $\mathrm{U}_q\mathfrak{g}$ [J–L1], so that there is an inclusion of $\mathrm{U}_q\mathfrak{g}$ into the dual of $\mathcal{A}_q\mathrm{G}$, actually into the restricted dual of $\mathcal{A}_q\mathrm{G}$. We note S the antipode of $\mathcal{A}_q\mathrm{G}$, which is the restriction to $\mathcal{A}_q\mathrm{G}$ of the transpose of the antipode of $\mathrm{U}_q\mathfrak{g}$.

There is an R-matrix for $\mathrm{U}_q\mathfrak{g}$ [Dr1, Ta, Ga]. We choose the R-matrix with the structure $\sum$(diagonal part)(monomial in F) $\otimes$ (monomial in E). If a and b belong to $\mathcal{A}_q\mathrm{G}$, the number $\langle R_{12}, b \otimes a \rangle \in \mathbb{C}(q)$ is well-defined (thanks to the weight graduation of $\mathrm{U}_q\mathfrak{g}$ and of any finite-dimensional $\mathrm{U}_q\mathfrak{g}$-module), and we can define $\gamma, \delta : \mathcal{A}_q\mathrm{G} \to (\mathcal{A}_q\mathrm{G})^*$ such that $\langle R_{12}, b \otimes a \rangle = \langle \gamma(a), b \rangle = \langle \delta(b), \mathrm{S}(a) \rangle$. $(\mathcal{A}_q\mathrm{G}, \gamma)$ and $(\mathcal{A}_q\mathrm{G}, \delta)$ are c.q.t. Hopf algebras, $\mathrm{im}(\gamma)$ and $\mathrm{im}(\delta)$ are the sub-Hopf-algebras $\mathrm{U}^-\mathrm{U}^0$ and $\mathrm{U}^0\mathrm{U}^+$ of $\mathrm{U}_q\mathfrak{g} \subseteq (\mathcal{A}_q\mathrm{G})^{*\,\mathrm{res}}$ respectively, and $\mathrm{U}_q\mathfrak{g}$ is the sub-Hopf-algebra $(\mathrm{im}\,\gamma)(\mathrm{im}\,\delta) = (\mathrm{im}\,\delta)(\mathrm{im}\,\gamma)$ of $(\mathcal{A}_q\mathrm{G})^{*\,\mathrm{res}}$.

2.2. Factorizability of $\mathcal{A}_q\mathrm{G}$. Let $(\mathcal{A}_q\mathrm{G}, \gamma)$ be the c.q.t. algebra presented above, and δ be the associated map. For all the section, we endow $\mathcal{A}_q\mathrm{G}$ and $\mathrm{U}_q\mathfrak{g}$ with the left adjoint action of $\mathrm{U}_q\mathfrak{g}$, as in Sect. 1.4: in particular, the map $\mathrm{I} : \mathcal{A}_q\mathrm{G} \to \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ is a morphism of left $\mathrm{U}_q\mathfrak{g}$-modules. Joseph and Letzter [J–L1, J–L2] have studied the structure of $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$, and we need the following results:

- If $\lambda \in \mathrm{P}_+$, $K_{-2\lambda}$ generates a finite dimensional $\mathrm{U}_q\mathfrak{g}$-submodule of $\mathrm{U}_q\mathfrak{g}$, and $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g}) = \bigoplus_{\lambda \in \mathrm{P}_+} (\mathrm{U}_q\mathfrak{g} \cdot K_{-2\lambda})$.
- Each block $\mathrm{U}_q\mathfrak{g} \cdot K_{-2\lambda}$ contains a unique one-dimensional $\mathrm{U}_q\mathfrak{g}$-submodule; it defines a unique (up to scalars) element z_λ of the center of $\mathrm{U}_q\mathfrak{g}$.
- $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g}) \subseteq (\mathcal{A}_q\mathrm{G})^*$ separates the points of $\mathcal{A}_q\mathrm{G}$.

The next assertion has been stated in [R–S]:

Proposition 5. $(\mathcal{A}_q\mathrm{G}, \gamma)$ *is a factorizable c.q.t. Hopf algebra, and* $\mathrm{im}\,\mathrm{I} = \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$.

Proof. Let $\lambda \in \mathrm{P}_+$, $\mathrm{L}(\lambda)$ the standard $\mathrm{U}_q\mathfrak{g}$-module, m_λ a highest weight vector, $m_{w_0\lambda}$ a lowest weight vector, (m_i) a basis for $\mathrm{L}(\lambda)$ composed of weight vectors, (m_i^*) the dual basis. We have:

- The matrix element $\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m_{w_0\lambda}^*)$ is the linear form on $\mathrm{U}_q\mathfrak{g}$ given by (in the triangular decomposition $\mathrm{U}^+ \otimes \mathrm{U}^0 \otimes \mathrm{U}^-$ of $\mathrm{U}_q\mathfrak{g}$): $E K_\mu F \mapsto \varepsilon(E) q^{(w_0\lambda|\mu)} \varepsilon(F)$.
- On this element, γ takes the value $K_{w_0\lambda}$ and δ the value $K_{-w_0\lambda}$.

- The image by γ (respectively δ) of the matrix element $\theta_{\mathrm{L}(\lambda)}(m_i, m^*_{w_0\lambda})$ (respectively $\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_i)$) is zero if $i \neq w_0\lambda$.

So we have:

$$\begin{aligned}\mathrm{I}(\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_{w_0\lambda})) &= \gamma((\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_{w_0\lambda}))_{(1)})\ \mathrm{S}(\delta((\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_{w_0\lambda}))_{(2)}))\\ &= \textstyle\sum \gamma(\theta_{\mathrm{L}(\lambda)}(m_i, m^*_{w_0\lambda}))\ \mathrm{S}(\delta(\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_i)))\\ &= \gamma(\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_{w_0\lambda}))\ \mathrm{S}(\delta(\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_{w_0\lambda})))\\ &= K_{2w_0\lambda}.\end{aligned}$$

Hence $\operatorname{im}\mathrm{I}$ is a $\mathrm{U}_q\mathfrak{g}$-submodule of $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ which contains all the $K_{2w_0\lambda}$ ($\lambda \in \mathrm{P}_+$), so $\operatorname{im}\mathrm{I} = \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$. We now want to show that J is injective. If $b \in \ker\mathrm{J}$, then for all $a \in \mathcal{A}_q\mathrm{G}$, $\langle \mathrm{I}(a), b\rangle = \langle \mathrm{J}(b), a\rangle = 0$, so b is null when viewed as a linear form on $\operatorname{im}\mathrm{I} = \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$. Then $b = 0$, because $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ separates the points of $\mathcal{A}_q\mathrm{G}$. Finally, owing to the formula $\mathrm{J} = \mathrm{S}\circ\mathrm{I}\circ\mathrm{S}$ and to the invertibility of S, I is also injective. This concludes the proof of the proposition. □

There is another way to present this result. Rosso [Ro1] introduced a bilinear non-degenerate ad-invariant form on $\mathrm{U}_q\mathfrak{g}$, that Caldero [Ca] writes $(\mathrm{U}_q\mathfrak{g} \times \mathrm{U}_q\mathfrak{g} \to \mathbb{C}(q^{1/2}), (x,y) \mapsto \langle \zeta(x), \mathrm{S}^{-1}(y)\rangle)$, where $\zeta : \mathrm{U}_q\mathfrak{g} \to (\mathrm{U}_q\mathfrak{g})^*$. Rosso's non-degeneracy result is that ζ is injective; Caldero's theorem states that ζ maps $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ onto $\mathcal{A}_q\mathrm{G} \subseteq (\mathrm{U}_q\mathfrak{g})^{*\,\mathrm{res}}$. The triangular behaviour of Rosso's form gives us that $\zeta(K_{2w_0\lambda}) = \theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_{w_0\lambda})$. The ad-invariance of Rosso's form can be translated for ζ: when we restrict ζ to $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ and $\mathcal{A}_q\mathrm{G}$, ζ is a morphism of $\mathrm{U}_q\mathfrak{g}$-modules for the adjoint structures. Now $\mathrm{I}\circ\zeta : \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g}) \to \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ and $\zeta\circ\mathrm{I} : \mathcal{A}_q\mathrm{G} \to \mathcal{A}_q\mathrm{G}$ are morphisms of $\mathrm{U}_q\mathfrak{g}$-modules and fix the respective generators $K_{2w_0\lambda}$ and $\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_{w_0\lambda})$ of these modules. (The fact that $\theta_{\mathrm{L}(\lambda)}(m_{w_0\lambda}, m^*_{w_0\lambda})$ generates the $\mathrm{U}_q\mathfrak{g}$-submodule $\mathrm{C}(\lambda)$ of $\mathcal{A}_q\mathrm{G}$ is equivalent to the fact that $m^*_{w_0\lambda} \otimes m_{w_0\lambda}$ generates the $\mathrm{U}_q\mathfrak{g}$-module $\mathrm{L}(\lambda)^* \otimes \mathrm{L}(\lambda)$.) So we conclude that ζ and I are mutually inverse isomorphisms, and that I is a bijection between $\mathrm{C}(\lambda)$ and $\mathrm{U}_q\mathfrak{g}\cdot K_{2w_0\lambda}$. The analysis also shows the amusing side-result:

Proposition 6. *If $x \in \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$, $y \in \mathrm{U}_q\mathfrak{g}$, then the Rosso form on (x,y) is given by $\langle \mathrm{I}^{-1}(x), \mathrm{S}^{-1}(y)\rangle$, where $\mathrm{I} : (\mathcal{A}_q\mathrm{G} \to \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g}), a \mapsto \langle a \otimes \mathrm{id}_{\mathrm{U}_q\mathfrak{g}}, R_{21}R_{12}\rangle)$ is related to the universal R-matrix and S is the antipode of $\mathrm{U}_q\mathfrak{g}$.*

Remarks. 1. It is also possible to give an heuristic proof of this result, using the canonical R-matrix for Drinfel′d's double and using Rosso's formula for his form [Ro2].

2. In the preceding discussion, we were lying a bit. Caldero's map ζ does not give exactly Rosso's bilinear form, but our formula connecting I and Rosso's form is correct as stated. In our notations, Caldero's map ζ is the inverse of the map $(\mathcal{A}_q\mathrm{G} \to \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g}), a \mapsto \delta(a_{(1)})\ \mathrm{S}\gamma(a_{(2)}))$.

Later, we will need to know the relations between the central elements z_λ defined above. To this aim, we recall Drinfel′d's construction of the center of $\mathrm{U}_q\mathfrak{g}$ [Dr2]. Let $\lambda \in \mathrm{P}_+$ and $t \in \mathcal{A}_q\mathrm{G}$ be the quantum trace in $\mathrm{L}(\lambda)$: for $x \in \mathrm{U}_q\mathfrak{g}$, $\langle t, x\rangle = \mathrm{Tr}_{\mathrm{L}(\lambda)}(K_{2\rho}\,x)$. t is an invariant element for the adjoint action of $\mathrm{U}_q\mathfrak{g}$ in $\mathcal{A}_q\mathrm{G}$, so $\mathrm{I}(t)$ is central, and belongs to $\mathrm{U}_q\mathfrak{g}\cdot K_{2w_0\lambda}$. We choose the normalization of $z_{-w_0\lambda}$ by letting $z_{-w_0\lambda} = \mathrm{I}(t)$. We then have a Mackey-like theorem (which is implicit in [Dr2] and in the thesis of Caldero, Chap. II, 1.2):

Proposition 7. *Let $\mathrm{c}^\nu_{\lambda\mu}$ be the fusion coefficients for $\mathfrak{g}$: $\mathrm{L}(\lambda)\otimes\mathrm{L}(\mu) \simeq \bigoplus_\nu \mathrm{c}^\nu_{\lambda\mu}\mathrm{L}(\nu)$. Then $z_\lambda z_\mu = \sum_\nu \mathrm{c}^\nu_{\lambda\mu} z_\nu$.*

Proof. Let $\mu \in \mathrm{P}_+$. We compute $\mathrm{J}(\theta_{\mathrm{L}(\mu)}(m_\mu, m_\mu^*)) = K_{2\mu}$ (with the help of the formulas $\mathrm{J} = \mathrm{S} \circ \mathrm{I} \circ \mathrm{S}$ and $\mathrm{S}(\theta_{\mathrm{L}(\mu)}(m_\mu, m_\mu^*)) = \theta_{\mathrm{L}(-w_0\mu)}(m_{-\mu}, m_{-\mu}^*)$). Now let $\lambda \in \mathrm{P}_+$ and let t be the quantum trace in $\mathrm{L}(\lambda)$. Let Ψ be the Harish-Chandra morphism from the center of $\mathrm{U}_q\mathfrak{g}$ to U^0 [Ro1]. We want to compute $\Psi(z_{-w_0\lambda})$ on $\mu + \rho$. (Evaluation on $\mu + \rho$ means the algebra homomorphism $(\mathrm{U}^0 \to \mathbb{C}(q), K_\lambda \mapsto q^{(\lambda|\mu+\rho)})$.) The result will be the image of $z_{-w_0\lambda}$ by the central character of $\mathrm{L}(\mu)$. So it is $\langle \mathrm{I}(t), \theta_{\mathrm{L}(\mu)}(m_\mu, m_\mu^*)\rangle = \langle \mathrm{J}\theta_{\mathrm{L}(\mu)}(m_\mu, m_\mu^*), t\rangle = \langle K_{2\mu}, t\rangle = \mathrm{Tr}_{\mathrm{L}(\lambda)}(K_{2\mu}K_{2\rho}) = \mathrm{Tr}_{\mathrm{L}(\lambda)}(K_{2(\mu+\rho)})$. Thus $\Psi(z_{-w_0\lambda})$ equals the sum of $K_{2\nu}$ for ν in the set of weights (with multiplicities) of $\mathrm{L}(\lambda)$. We then use the fact that Ψ is an injective algebra homomorphism. □

We denote by $\mathcal{G}$ the Grothendieck ring of the category of finite-dimensional $\mathrm{U}_q\mathfrak{g}$-modules whose components are modules $\mathrm{L}(\lambda)$, without any twisting character $\chi : {}^{\mathrm{P}}\!/_{2\mathrm{Q}} \to \mathbb{C}^\times$. Let $\mathrm{Z}(\mathrm{U_q}\mathfrak{g})$ be the center of $\mathrm{U}_q\mathfrak{g}$, and $\mathbb{Z}[\mathrm{P}]$ the group algebra of P (with the standard $\mathbb{Z}$-basis denoted by $(e^\nu)_{\nu\in\mathrm{P}}$). The map $(\mathcal{G} \to \mathcal{A}_q\mathrm{G}, [\mathrm{M}] \mapsto \mathrm{Tr}_\mathrm{M}(K_{2\rho} \textbf{—}))$ is a ring homomorphism. If $a, b \in \mathcal{A}_q\mathrm{G}$ are such that $\mathrm{I}(a)$ belongs to the center of $\mathrm{U}_q\mathfrak{g}$, then $\mathrm{I}(ab) = \mathrm{I}(a)\,\mathrm{I}(b)$. As a consequence, the map $\big(\mathcal{G} \to \mathrm{Z}(\mathrm{U}_q\mathfrak{g}), [\mathrm{M}] \mapsto \mathrm{I}(\mathrm{Tr}_\mathrm{M}(K_{2\rho}\textbf{—}))\big)$ is a ring homomorphism. This shows again the statement in Proposition 7, and we can paraphrase the above proof by saying that the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{G} & \longrightarrow & \mathcal{A}_q\mathrm{G} & \xrightarrow{\ \mathrm{I}\ } & \mathrm{Z}(\mathrm{U_q}\mathfrak{g}) \\ {\scriptstyle \mathrm{ch}}\downarrow & & & & \downarrow{\scriptstyle \Psi} \\ \mathbb{Z}[\mathrm{P}] & & \longrightarrow & & \mathrm{U}^0 \end{array}$$

Here $\mathrm{ch} : \mathcal{G} \to \mathbb{Z}[\mathrm{P}]$ is the ring homomorphism which maps a module to its formal character, and the bottom arrow is the map $(\mathbb{Z}[\mathrm{P}] \to \mathrm{U}^0, e^\nu \mapsto K_{2\nu})$.

2.3. A technical result on the representation ring. We have just introduced a Grothendieck ring $\mathcal{G}$: by the classical results of Lusztig and Rosso, $\mathcal{G}$ is naturally isomorphic to the representation ring of $\mathfrak{g}$. The elements $[\mathrm{L}(\lambda)]$ ($\lambda \in \mathrm{P}_+$) form a $\mathbb{Z}$-basis of $\mathcal{G}$ and a $\mathbb{Q}$-basis of $\mathcal{G} \otimes_\mathbb{Z} \mathbb{Q}$.

Proposition 8. *Let* $\lambda \in \mathrm{P}_+$. *Then the ideal of* $\mathcal{R} \otimes_\mathbb{Z} \mathbb{Q}$ *generated by the elements* $[\mathrm{L}(\lambda + \varpi)]$ *(*$\varpi \in \mathrm{P}_+$*) is the whole algebra* $\mathcal{G} \otimes_\mathbb{Z} \mathbb{Q}$.

The proof of this proposition can be skipped without any drawback. Before we give it, we have to state an elementary lemma:

Lemma. *Let* $(\mu^{(1)}, \dots, \mu^{(k)}) \in (\mathbb{C}^\ell)^k$ *be such that their image in* $\left(\mathbb{C}/\mathbb{Z}\right)^\ell$ *are all different, and let* $(P^{(1)}, \dots, P^{(k)}) \in (\mathbb{C}[X_1, \dots, X_\ell])^k$. *If* $\sum_m P^{(m)}(n_1, \dots, n_\ell) \exp(2\pi i \sum_j n_j \mu_j^{(m)}) = 0$ *holds for all* $(n_1, \dots, n_\ell) \in \mathbb{N}^\ell$, *then the polynomials* $P^{(1)}$, *…,* $P^{(k)}$ *are all equal to zero.*

For $\ell = 1$, this lemma states linear independence of elementary solutions of a linear difference equation. The general proof is by induction on ℓ.

Proof of Proposition 8. In this proof, we are in a classical context and we do not identify $\mathfrak{h}$ and $\mathfrak{h}^*$. Let $\mathrm{R} \subseteq \mathfrak{h}^*$ and $\mathrm{R}^\vee \subseteq \mathfrak{h}$ be the direct and inverse root systems, $(\alpha \mapsto \alpha^\vee)$ the canonical bijection between R and $\mathrm{R}^\vee$, and $\mathrm{Q}(\mathrm{R}^\vee) \subseteq \mathfrak{h}$ the root lattice. $\mathrm{P} = \mathrm{P}(\mathrm{R}) \subseteq \mathfrak{h}^*$ is still the weight lattice; we denote by $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\}$ the set of inverse simple roots, and by $\{\varpi_1, \dots, \varpi_\ell\}$ the set of fundamental weights. $R^\vee$ and R define $\mathbb{Q}$-structures on $\mathfrak{h}$ and $\mathfrak{h}^*$, and we can define $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{C}$. The Weyl group W operates on $\mathfrak{h}$ and $\mathfrak{h}^*$, and the affine Weyl group $W_a = W \ltimes Q(R^\vee)$ operates on $\mathfrak{h}$. Let $\mathbb{Z}[P]$ be the $\mathbb{Z}$-algebra of the group P, $\mathbb{Z}[P]^W$ be the set of elements which are invariant under Weyl group action, $\mathrm{ch} : (\mathcal{G} \xrightarrow{\sim} \mathbb{Z}[P]^W)$ be the ring isomorphism "formal character". Finally, we denote by $\varepsilon(w) = \pm 1$ the determinant of an element w of the Weyl group.

For $\mu \in \mathfrak{h}_\mathbb{C}$, let $\mathrm{ev}_\mu : (\mathbb{Z}[P] \to \mathbb{C})$ be the ring morphism which sends a basic element e^ν ($\nu \in P$) to $\exp(2\pi i \langle \mu, \nu \rangle)$, where exp is the complex exponential. This extends to an algebra morphism $\mathrm{ev}_\mu : (\mathbb{C}[P] \to \mathbb{C})$. If $\nu \in P_+$, let f_ν be the map $\big(\mathfrak{h}_\mathbb{C} \to \mathbb{C}, \mu \mapsto \mathrm{ev}_\mu(\mathrm{ch}\, L(\nu))\big)$. We first assert that given any $(x_1, \dots, x_\ell) \in \mathbb{C}^\ell$, there exists $\mu \in \mathfrak{h}_\mathbb{C}$ such that for all $i \in \{1, \dots, \ell\}$, $f_{\varpi_i}(\mu) = x_i$. We view $\mathbb{C}[P]$ as the coordinate ring of the affine variety $(\mathbb{C}^\times)^\ell$, and we view an element $\mu = \sum \mu_i \alpha_i^\vee$ ($\mu_i \in \mathbb{C}$) as the point $(e^{2\pi i \mu_1}, \dots, e^{2\pi i \mu_\ell}) \in (\mathbb{C}^\times)^\ell$. By the Nullstellensatz, it is sufficient to prove that the elements $(\mathrm{ch}\, L(\varpi_i) - x_i e^0)$ ($i = 1, \dots, \ell$) generate a proper ideal in $\mathbb{C}[P]$. This is already true in $\mathbb{C}[P]^W$ by [Bo], Ch. VI, § 3, Théorème 1. The case of $\mathbb{C}[P]$ is given by a standard trick: let $\natural : (\mathbb{C}[P] \to \mathbb{C}[P]^W)$ be the projection onto the trivial homogeneous component in $\mathbb{C}[P]$ for the action of W; $\natural$ is a morphism of $\mathbb{C}[P]^W$-modules, and thus a relation $\sum Q_i \cdot (\mathrm{ch}\, L(\varpi_i) - x_i e^0) = 1$ in $\mathbb{C}[P]$ would give a relation $\sum Q_i^\natural \cdot (\mathrm{ch}\, L(\varpi_i) - x_i e^0) = 1$ in $\mathbb{C}[P]^W$, which is impossible.

We now want to prove a formula for the character $f_\nu(\mu) = \mathrm{ev}_\mu(\mathrm{ch}\, L(\nu))$. We first remark that f_ν is invariant under the action of the affine Weyl group W_a in $\mathfrak{h}_\mathbb{C}$. If the real part $\mathrm{Re}(\mu)$ of μ lies in an open alcove of $\mathfrak{h}_\mathbb{R}$, our formula will just be Weyl's character formula:

$$f_\nu(\mu) = \frac{\sum_{w \in W} \varepsilon(w) \exp(2\pi i \langle w\mu, \nu + \rho \rangle)}{\sum_{w \in W} \varepsilon(w) \exp(2\pi i \langle w\mu, \rho \rangle)}.$$

Writing the denominator as a product over the positive roots:

$$\exp(2\pi i \langle \mu, \rho \rangle) \prod_{\alpha \in R, \alpha \geq 0} (1 - \exp(-2\pi i \langle \mu, \alpha \rangle)),$$

we can see that it is a non-zero complex number. In the general case, we let $T = \{\alpha \in R \mid \mathrm{Re}(\langle \mu, \alpha \rangle) \in \mathbb{Z}\}$: this is a closed symmetric subset of R ([Bo], Ch. VI, § 1, Définition 4), thus T is a root system in the vector space $V_1 \subseteq \mathfrak{h}_\mathbb{R}^*$ that it spans ([Bo], Ch. VI, § 1, Proposition 23). The stabilizer of μ in W_a is generated by the reflections across the affine hyperplanes in which $\mathrm{Re}(\mu)$ lies ([Bo], Ch. V, § 3, Proposition 2), thus $W_1 := \{w \in W \mid \mu - w\mu \in Q(R^\vee)\}$ is precisely the subgroup generated by reflections along $\alpha^\vee$ ($\alpha \in T$), and its restriction to V_1 is the Weyl group of T. Let σ be half the sum of the inverse positive roots of T: $\sigma = \frac{1}{2} \sum_{\alpha \in T, \alpha \geq 0} \alpha^\vee$. In restriction to V_1, σ is the sum of the fundamental weights of the root system $T^\vee$ of V_1^*. Let h be a small real parameter: $\mathrm{Re}(\mu) + h\sigma$ then lies in an open alcove of $\mathfrak{h}_\mathbb{R}$ and we can compute (with a little abuse):

$$\begin{aligned} f_\nu(\mu) &= \lim_{h \to 0} f_\nu(\mu + h\sigma) \\ &= \lim_{h \to 0} \frac{\sum_{w \in W/W_1} \sum_{w_1 \in W_1} \varepsilon(ww_1) \exp(2\pi i \langle w\mu, \nu + \rho \rangle) \exp(2\pi i h \langle w_1 \sigma, w^{-1}(\nu + \rho) \rangle)}{\sum_{w \in W/W_1} \sum_{w_1 \in W_1} \varepsilon(ww_1) \exp(2\pi i \langle w\mu, \rho \rangle) \exp(2\pi i h \langle w_1 \sigma, w^{-1} \rho \rangle)}. \end{aligned}$$

In the sums, we fix $w \in W/W_1$ and compute the sums on w_1: in the numerator for instance, we have an alternating sum of $\exp(2\pi i h \langle w_1 \sigma, w^{-1}(\nu + \rho) \rangle)$, where $w^{-1}(\nu + \rho) \in P(R)$ has to be projected on V_1, as in [Bo], Ch. VI, § 1, Proposition 28. The formula (valid in the group algebra of the weight lattice of $T^\vee$): $\sum_{w_1 \in W_1} \varepsilon(w_1) e^{w_1 \sigma} = e^\sigma \prod_{\alpha \in T, \alpha \geq 0} (1 - e^{-\alpha^\vee})$ then gives:

$$f_\nu(\mu) = \frac{\sum_{w\in \mathrm{W}/\mathrm{W}_1} \varepsilon(w)\exp(2\pi i\langle w\mu, \nu+\rho\rangle)\prod_{\alpha\in \mathrm{T},\alpha\geq 0}\langle\alpha^\vee, w^{-1}(\nu+\rho)\rangle}{\sum_{w\in \mathrm{W}/\mathrm{W}_1} \varepsilon(w)\exp(2\pi i\langle w\mu, \rho\rangle)\prod_{\alpha\in \mathrm{T},\alpha\geq 0}\langle\alpha^\vee, w^{-1}\rho\rangle}.$$

As $\nu+\rho$ and ρ are regular, neither of the products occurring here can be zero. (We will see soon that the denominator cannot be zero.)

We now prove that the ideal of $\mathcal{G}\otimes_{\mathbb{Z}}\mathbb{C}$ generated by the elements $[\mathrm{L}(\lambda+\varpi)]$ ($\varpi\in \mathrm{P}_+$) is the whole algebra $\mathcal{G}\otimes_{\mathbb{Z}}\mathbb{C}$. We consider again [Bo], Ch. VI, § 3, Théorème 1: this time, the isomorphism $\varphi : \mathbb{C}[X_1,\dots,X_\ell]\to\mathbb{C}[\mathrm{P}]^{\mathrm{W}}$ is given by $\varphi(X_i) = \mathrm{ch}\,\mathrm{L}(\varpi_i)$. Composing with the isomorphism $\mathrm{ch} : \mathcal{G}\to\mathbb{Z}[\mathrm{P}]^{\mathrm{W}}$, we can see that $\mathcal{G}\otimes_{\mathbb{Z}}\mathbb{C}$ is a polynomial algebra over $\mathbb{C}$. We suppose by the way of contradiction that the elements $[\mathrm{L}(\lambda+\varpi)]$ ($\varpi\in \mathrm{P}_+$) all belong to some maximal ideal of $\mathcal{G}\otimes_{\mathbb{Z}}\mathbb{C}$. Then, by the Nullstellensatz, there exists a point $(x_1,\dots,x_\ell)\in\mathbb{C}^\ell$ such that for all $\varpi\in \mathrm{P}_+$, $\varphi^{-1}(\mathrm{ch}\,\mathrm{L}(\lambda+\varpi))(x_1,\dots,x_\ell) = 0$. We can find $\mu\in\mathfrak{h}_{\mathbb{C}}$ such that $f_{\varpi_i}(\mu) = x_i$ ($i = 1,\dots,\ell$): then $f_{\lambda+\varpi}(\mu) = 0$ for all $\varpi\in \mathrm{P}_+$. We next use the formula:

$$\begin{aligned} &f_{\lambda+\varpi}(\mu)\ \text{(denominator)}\\ &\quad= \sum_{w\in \mathrm{W}/\mathrm{W}_1}\varepsilon(w)\exp(2\pi i\langle w\mu, \lambda+\varpi+\rho\rangle)\prod_{\alpha\in \mathrm{T},\alpha\geq 0}\langle\alpha^\vee, w^{-1}(\lambda+\varpi+\rho)\rangle,\end{aligned}$$

and write $\varpi = \sum n_i\varpi_i$, where $(n_i)\in\mathbb{N}^\ell$ are any integers. The $w\mu$ ($w\in \mathrm{W}/\mathrm{W}_1$) are all distinct modulo $\mathrm{Q}(\mathrm{R}^\vee)$, and the expressions $\prod_{\alpha\in \mathrm{T},\alpha\geq 0}\langle\alpha^\vee, w^{-1}(\lambda+\varpi+\rho)\rangle$ are nonzero polynomials in $(n_1,\dots,n_\ell)$ (they never vanish indeed). Then the above lemma states that the right-hand side cannot vanish for all $(n_i)\in\mathbb{N}^\ell$. This proves first that the denominator is not null, and second that $f_{\lambda+\sum n_i\varpi_i}(\mu)$ cannot vanish for all $(n_i)\in\mathbb{N}^\ell$. We have reached a contradiction.

To go down to the case of $\mathcal{G}\otimes_{\mathbb{Z}}\mathbb{Q}$ is then easy: we have shown that we can express in $\mathcal{G}\otimes_{\mathbb{Z}}\mathbb{C}$ the unity as a finite sum $1 = \sum x_i[\mathrm{L}(\tau_i)][\mathrm{L}(\nu_i)]$, where $\tau_i\in \mathrm{P}_+$, $\nu_i\in\lambda+\mathrm{P}_+$ and $x_i\in\mathbb{C}$. As the structure constants of $\mathcal{G}\otimes_{\mathbb{Z}}\mathbb{C}$ are integer-valued, this system, viewed as linear equations in (x_i), has a solution in $\mathbb{C}$, so has a solution in $\mathbb{Q}$. □

2.4. Classification of some ideals of $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$. In order to achieve our classification of ideals $\mathcal{R}\subseteq\mathcal{A}_q\mathrm{G}$ in the next section, we must study the ideals $\mathcal{I}\subseteq\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ which are stable by the adjoint action of $\mathrm{U}_q\mathfrak{g}$. The analysis requires the use of the subalgebra V of $\mathrm{U}_q\mathfrak{g}$ generated by $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ and by the elements $K_{2\lambda}$ ($\lambda\in \mathrm{P}_+$).

Joseph and Letzter [J–L1] have shown that V is the subalgebra generated by the elements E_i, $F_iK_{\alpha_i}$ and $K_{2\lambda}$ ($\lambda\in\mathrm{P}$). As it is such a "big" subalgebra of $\mathrm{U}_q\mathfrak{g}$, its representation theory is similar to that of $\mathrm{U}_q\mathfrak{g}$. We will describe it in the next subsection, but in the following proof, we need to know that the annihilator of a finite-dimensional V-module is homogeneous with respect to the Q-graduation of V.

Proposition 9. *The following two properties for a subspace* $\mathcal{I}\subseteq\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ *are equivalent:*

(1) $\mathcal{I}$ *is the annihilator in* $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ *of a finite-dimensional* V*-module;*

(2) $\mathcal{I}$ *is a finite-codimensional two-sided ideal of* $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ *and a* $\mathrm{U}_q\mathfrak{g}$*-submodule of* $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ *for the left adjoint action.*

Proof. We first show that (1) ⇒ (2). If M is a finite-dimensional V-module, its annihilator in V is a finite-codimensional two-sided ideal of V, and is homogeneous w.r.t. the Q-graduation of V. It is then easy to see that $\mathrm{ann}_{\mathrm{V}}\,\mathrm{M}$ is a $\mathrm{U}_q\mathfrak{g}$-submodule of V for the left adjoint action. The annihilator $\mathcal{I} = (\mathrm{ann}_{\mathrm{V}}\,\mathrm{M}) \cap \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ of M in $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ thus satisfies the property (2).

Conversely, let $\mathcal{I} \subseteq \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$, satisfying the property (2). We consider the left regular $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$-module $\mathrm{M} = \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})/\mathcal{I}$. $\mathcal{I}$ is its annihilator, so it is sufficient to show that M extends to a V-module. We thus want to show that the elements $K_{-2\lambda} \in \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ ($\lambda \in \mathrm{P}_+$) map to invertible operators in End(M).

1) M is a finite-dimensional algebra, and is also a left $\mathrm{U}_q\mathfrak{g}$-module (for the adjoint action). The multiplication in M defines a morphism of left $\mathrm{U}_q\mathfrak{g}$-modules: $\mathrm{M} \otimes \mathrm{M} \to \mathrm{M}$. Thus the Q-graduation of M (defined by the structure of $\mathrm{U}_q\mathfrak{g}$-module) is an algebra grading.
2) We fix $\lambda \in \mathrm{P}_+$. We can write $\mathrm{M} = \mathrm{M}_0 \oplus \mathrm{M}_\infty$ (as $\mathbb{C}(q)$-vector space), where $K_{-2\lambda}$ acts nilpotently on M_0 and inversibly on M_∞ (Fitting's decomposition). M_0 and M_∞ are stable by the commutant of $K_{-2\lambda}$ in End(M), so are right ideals of M. If $x \in \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$ is homogeneous w.r.t. the Q-graduation of $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$, x commutes (up to a non-zero scalar) with $K_{-2\lambda}$, so M_0 and M_∞ are stable by left multiplication by x. Thus M_0 and M_∞ are also left ideals of M.
3) We now show that M_0 and M_∞ are $\mathrm{U}_q\mathfrak{g}$-submodules of M.
   (a) Let $\{e_1, \dots, e_k\}$ be the set of central idempotents in M. The elements K_μ ($\mu \in \mathrm{P}$) of $\mathrm{U}_q\mathfrak{g}$ act on M (by the adjoint action) as algebra automorphisms, so permute the elements of the set $\{e_1, \dots, e_k\}$. Hence for each μ, there exists an integer $n \geq 1$ such that $K_{n\mu}$ fixes each e_i. Since M is, as a $\mathrm{U}_q\mathfrak{g}$-module, a direct sum of modules $\mathrm{L}(\nu)$ (without any twisting character χ), and since q is generic, we conclude that $e_1, \dots, e_k$ are fixed by the adjoint action of the elements K_μ.
   (b) Let e be a central idempotent in M. e is of weight zero. We consider the q-exponential $\exp_q(\mathrm{ad}\,E_i) = \sum_{n\geq 0} q^{-d_i n(n-1)/2} \frac{\mathrm{ad}\,E_i^n}{[n]_i!}$ ($i \in \{1, \dots, \ell\}$ fixed). Then $\exp_q(\mathrm{ad}\,E_i)$ is a well defined operator in M. The formula $\Delta(E_i^n) = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_i q^{d_i(n-k)k} E_i^{n-k} K_{\alpha_i}^k \otimes E_i^k$ enables us to see that $\exp_q(\mathrm{ad}\,E_i)(e)$ is an idempotent which we write $e + x$. Then $2ex + x^2 = x$, $x(1 - 2e) = x^2$, $x = x(1-2e)^2 = x^3$. The weights of the Q-homogeneous components of x belong to $\{n\alpha_i \mid n \geq 1\}$; so the weights of the Q-homogeneous components of x^3 belong to $\{n\alpha_i \mid n \geq 3\}$, and the homogeneous component of x of weight α_i is null. We obtain that $(\mathrm{ad}\,E_i)(e) = 0$. Similarly, $(\mathrm{ad}\,F_i)(e) = 0$ for all $i \in \{1, \dots, \ell\}$.
   (c) M_0 and M_∞ are ideals in M generated by central idempotents e_0 and e_∞ respectively. (a) and (b) show that e_0 and e_∞ define the trivial $\mathrm{U}_q\mathfrak{g}$-module. Hence for $x \in \mathrm{M}_0$ and $u \in \mathrm{U}_q\mathfrak{g}$, $u \cdot x = u \cdot (xe_0) = (u_{(1)} \cdot x)(u_{(2)} \cdot e_0) = (u_{(1)} \cdot x)\varepsilon(u_{(2)})e_0 = (u \cdot x)e_0 \in \mathrm{M}_0$. The same holds for M_∞.
4) We first consider the case $\mathfrak{g} = \mathfrak{sl}_2$. We choose naturally $\lambda = \varpi$ the fundamental weight, and write $\mathrm{M}_0 = \mathcal{L}_0/\mathcal{I}$ and $\mathrm{M}_\infty = \mathcal{L}_\infty/\mathcal{I}$. The points 2) and 3) show that $\mathcal{L}_0$ and $\mathcal{L}_\infty$ are two-sided ideals and left $\mathrm{U}_q\mathfrak{g}$-submodules of $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$. By definition of the Fitting decomposition, there exists an integer $n \geq 0$ such that $K_{-2n\varpi} \in \mathcal{L}_\infty$. Hence for all integers $m \geq n$, we have $K_{-2m\varpi} \in \mathcal{L}_\infty$, and thus $z_{m\varpi} \in \mathcal{L}_\infty$. Let $n_0 \geq 0$ be the smallest integer such that for all $m \geq n_0$, $z_{m\varpi} \in \mathcal{L}_\infty$. Proposition 7 and the Clebsch–Gordan theorem show that if $n \geq 1$, $z_{(n+1)\varpi} + z_{(n-1)\varpi} = z_\varpi z_{n\varpi}$. Thus n_0 has to be equal to zero. So $1 = z_0 \in \mathcal{L}_\infty$, $\mathrm{M}_\infty = \mathrm{M}$, and $K_{-2\varpi}$ acts inversibly on M.

5) The general case is solved in the same way. We consider the decomposition of the point 2) and write $\mathrm{M}_0 = \mathcal{L}_0/\mathcal{I}$ and $\mathrm{M}_\infty = \mathcal{L}_\infty/\mathcal{I}$. $\mathcal{L}_0$ and $\mathcal{L}_\infty$ are two-sided ideals and left $\mathrm{U}_q\mathfrak{g}$-submodules of $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$, and there exists an integer $n \geq 0$ such that $K_{-2n\lambda} \in \mathcal{L}_\infty$. If $\varpi \in \mathrm{P}_+$, then $K_{-2(n\lambda+\varpi)} \in \mathcal{L}_\infty$, and thus $z_{n\lambda+\varpi} \in \mathcal{L}_\infty$. Let φ be the $\mathbb{Q}$-algebra morphism $\left(\mathcal{G} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Z}(\mathrm{U_q}\mathfrak{g}), [\mathrm{M}] \mapsto \mathrm{I}(\mathrm{Tr}_{\mathrm{M}}(\mathrm{K}_{2\rho} —))\right)$ considered at the end of Sect. 2.2. Then $\varphi^{-1}(\mathcal{L}_\infty)$ is an ideal of $\mathcal{G} \otimes_{\mathbb{Z}} \mathbb{Q}$, which contains all the elements $[\mathrm{L}(-w_0 n\lambda + \varpi)]$ ($\varpi \in \mathrm{P}_+$). Thus $\varphi^{-1}(\mathcal{L}_\infty) = \mathcal{G} \otimes_{\mathbb{Z}} \mathbb{Q}$ by Proposition 8, and so $1 = \varphi([\mathrm{L}(0)]) \in \mathcal{L}_\infty$, $\mathrm{M}_\infty = \mathrm{M}$, and $K_{-2\lambda}$ acts inversibly on M. □

Remark. This result is a particular case of Proposition 8.4.13 in [Jo]. Accordingly, its proof is shorter than the one of Joseph's theorem, and does not require the knowledge of the inclusions between Verma modules, nor the use of Gel′fand–Kirillov dimensions.

2.5. Classification of some right ideals of $\mathcal{A}_q\mathrm{G}$. The notations $\mathcal{A}_q\mathrm{G}$, $\mathrm{U}_q\mathfrak{g}$, V have the same meaning as in Sects. 2.1 and 2.4. The map $\mathrm{I} : (\mathcal{A}_q\mathrm{G} \xrightarrow{\sim} \mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g}))$ was introduced in Sect. 1.3.

We now specify the structure of the finite-dimensional V-modules: they are completely reducible; each $\mathrm{U}_q\mathfrak{g}$-module $\mathrm{L}_\chi(\lambda)$ (with $\lambda \in \mathrm{P}_+$, $\chi : \mathrm{P}/2\mathrm{Q} \to \mathbb{C}^\times$) is (by restriction) a simple V-module; the V-modules $\mathrm{L}_\chi(\lambda)$ and $\mathrm{L}_\varphi(\mu)$ are isomorphic iff $\lambda = \mu$ and the characters χ, φ restrict to the same character $2\mathrm{P}/2\mathrm{Q} \to \mathbb{C}^\times$. The simple finite-dimensional V-modules will be denoted by $\mathrm{L}_\chi(\lambda)$ with $\lambda \in \mathrm{P}_+$ and $\chi : 2\mathrm{P}/2\mathrm{Q} \to \mathbb{C}^\times$ a character. We finally remark (see [J–L1]) that a simple finite-dimensional V-module is still simple as a $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$-module. Consequently, if (M_i) is a finite family of non-isomorphic finite-dimensional simple V-modules, the natural ring homomorphism $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g}) \to \bigoplus \mathrm{End}\,\mathrm{M}_i$ is surjective.

Theorem 1. 1) *Let* $\mathcal{R}$ *be a finite codimensional right ideal of* $\mathcal{A}_q\mathrm{G}$, *which is a subcomodule of* $\mathcal{A}_q\mathrm{G}$ *w.r.t. the right coaction* $\delta_{\mathrm{R}} : (\mathcal{A}_q\mathrm{G} \to \mathcal{A}_q\mathrm{G} \otimes \mathcal{A}_q\mathrm{G}, a \mapsto a_{(2)} \otimes \mathrm{S}(a_{(1)})a_{(3)})$. *Then there exists a finite-dimensional* V*-module* M *such that* $\mathcal{R} = \mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}\,\mathrm{M})$.

2) *If* M *is a finite-dimensional* V*-module, then* $\mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}\,\mathrm{M})$ *is a finite codimensional right ideal of* $\mathcal{A}_q\mathrm{G}$, *stable by the right coaction* δ_{R}.

3) *If* M *and* N *are finite dimensional* V*-modules, then* $\mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}\,\mathrm{M}) = \mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}\,\mathrm{N})$ *iff* M *and* N *have the same irreducible components.*

4) $\mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}\,\mathrm{M})$ *is included in the augmentation ideal of* $\mathcal{A}_q\mathrm{G}$ *iff* M *contains the trivial* V*-module.*

Proof. 1) and 2) are consequences of Propositions 4 and 9. Let M and N be two finite-dimensional V-modules having the same annihilator in $\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})$. Then $\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}\,\mathrm{M} = \mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}(\mathrm{M} \oplus \mathrm{N})$. Let $\mathrm{M}_1, \ldots, \mathrm{M}_k$ (respectively $\mathrm{M}_1, \ldots, \mathrm{M}_n$) be the distinct irreducible components of M (respectively $\mathrm{M} \oplus \mathrm{N}$). Then we have:

$$\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})/\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}(\mathrm{M}) \simeq \bigoplus_{i=1}^k \mathrm{End}\,\mathrm{M}_i$$

and:

$$\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})/\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}(\mathrm{M} \oplus \mathrm{N}) \simeq \bigoplus_{i=1}^n \mathrm{End}\,\mathrm{M}_i,$$

and so $k = n$: all the irreducible components of N appear in M. 3) follows. 4) can be proved in a similar way, using the fact that the augmentation ideal of $\mathcal{A}_q\mathrm{G}$ is the inverse image by I of the annihilator of the trivial V-module. □

3. Differential Calculi on Quantum Groups

3.1. Woronowicz's definition. Let $\mathcal{A}$ be a Hopf algebra, Γ be a bicovariant bimodule and $\mathrm{d} : \mathcal{A} \to \Gamma$ be a linear map. We say that (Γ, d) is a bicovariant differential calculus on $\mathcal{A}$ if d is a derivation, a morphism of two-sided comodules and if the image of d generates the left $\mathcal{A}$-module Γ. The dimension of the space Γ^{L} of left coinvariants will be supposed to be finite.

When (Γ, d) is a differential calculus over $\mathcal{A}$, we note d^{L} the map $(\mathcal{A} \to \Gamma^{\mathrm{L}}, a \mapsto \mathrm{S}(a_{(1)}) \cdot \mathrm{d}(a_{(2)}))$. The subspace $\mathcal{R} = \ker \mathrm{d}^{\mathrm{L}} \cap \ker \varepsilon$ is a finite-codimensional right ideal of $\mathcal{A}$, and a subcomodule for the right coadjoint coaction $\delta_{\mathrm{R}} : (a \mapsto a_{(2)} \otimes \mathrm{S}(a_{(1)})a_{(3)})$. As shown by Woronowicz, the subspace $\mathcal{R}$ determines (up to isomorphism) the bicovariant differential calculus (Γ, d): we call it the ideal associated to (Γ, d).

Geometrically, $\mathcal{A}$ must be viewed as the algebra of functions over a group G, Γ is the space of 1-forms on G, Γ^{L} is the space of left-G-invariant 1-forms on G, identified with the cotangent space at the unity point of G, and d^{L} maps a function on G to its differential at the unity point.

3.2. A construction of bicovariant differential calculi. Let $\mathcal{A}$ be a c.q.t. Hopf algebra over the field k, and let γ, δ be the associated maps.

We take a finite-dimensional right $\mathcal{A}$-comodule M. We note (m_i) a basis of M, (m_i^*) the dual basis, and R_{ij} the elements of $\mathcal{A}$ such that $\delta_{\mathrm{R}}(m_i) = \sum_j m_j \otimes R_{ji}$. Then $\Delta R_{ji} = \sum_k R_{jk} \otimes R_{ki}$ and $\varepsilon(R_{ji}) = \delta_{ji}$ (Kronecker's symbol). Also, M is a left $\mathcal{A}^*$-module, and the R_{ji} (viewed as linear forms on $\mathcal{A}^*$) are the matrix coefficients $\theta_{\mathrm{M}}(m_i, m_j^*)$ of this module.

Since $(\mathcal{A}, \gamma)$ is c.q.t., M becomes a right crossed bimodule over $\mathcal{A}$ for the action $m_i \cdot a = \sum_j \langle \gamma(a), R_{ji} \rangle m_j$ (Proposition 1). M^* is a right comodule over $\mathcal{A}$ too, for the coaction $\delta_{\mathrm{R}}(m_i^*) = \sum_j m_j^* \otimes \mathrm{S}(R_{ij})$. Using the fact that $(\mathcal{A}, \delta)$ is a c.q.t. Hopf algebra, we may endow M^* with the structure of a right crossed bimodule over $\mathcal{A}$ for the action $m_i^* \cdot a = \sum_j \langle \delta(a), \mathrm{S}(R_{ij}) \rangle m_j^*$. Then, by making the tensor product, we obtain that $\mathrm{End(M)} \simeq \mathrm{M} \otimes \mathrm{M}^*$ is a right crossed bimodule.

We denote by Γ the bicovariant bimodule associated to this right crossed bimodule End(M). As a vector space, Γ is just the tensor product $\mathcal{A} \otimes \mathrm{M} \otimes \mathrm{M}^*$. On the basic elements, the structure maps are:

$$b \cdot (a \otimes m_i \otimes m_j^*) = ba \otimes m_i \otimes m_j^*,$$
$$(a \otimes m_i \otimes m_j^*) \cdot b = \sum_{k,l} ab_{(1)} \otimes \langle \gamma(b_{(2)}), R_{ki} \rangle m_k \otimes \langle \delta(b_{(3)}), \mathrm{S}(R_{j\ell}) \rangle m_\ell^*,$$
$$\delta_{\mathrm{L}}(a \otimes m_i \otimes m_j^*) = a_{(1)} \otimes a_{(2)} \otimes m_i \otimes m_j^*,$$
$$\delta_{\mathrm{R}}(a \otimes m_i \otimes m_j^*) = \sum_{k,l} a_{(1)} \otimes m_k \otimes m_\ell^* \otimes a_{(2)} R_{ki} \mathrm{S}(R_{j\ell}).$$

It follows that the canonical element $X = \sum_i 1 \otimes m_i \otimes m_i^*$ of Γ is left and right coinvariant. The linear map $\mathrm{d} : (\mathcal{A} \to \Gamma, a \mapsto X \cdot a - a \cdot X)$ is then a derivation and a morphism of two-sided comodules.

Theorem 2. 1) *If* $(\mathcal{A}, \gamma)$ *is a factorizable c.q.t. Hopf algebra and if* M *is a simple finite-dimensional non-trivial* $\mathcal{A}$*-comodule, then the above construction gives a bicovariant differential calculus* $\mathrm{d} : (\mathcal{A} \to \Gamma \equiv \mathcal{A} \otimes \mathrm{End(M)})$.

2) *Its associated ideal is* $\mathcal{R} = \mathrm{I}^{-1}(\mathrm{ann}_{\mathcal{A}^*}(k \oplus \mathrm{M}))$, *where* k *is the trivial* $\mathcal{A}^*$*-module.*

Proof. We first compute for $a \in \mathcal{A}$:

$$\begin{aligned} \mathrm{d}(a) &= \textstyle\sum_{k,l} a_{(1)} \langle \mathrm{I}(a_{(2)}), R_{k\ell} \rangle \otimes m_k \otimes m^*_\ell - a_{(1)} \langle a_{(2)}, \delta_{k\ell} \rangle \otimes m_k \otimes m^*_\ell \\ &= \textstyle\sum_{k,l} a_{(1)} \langle \mathrm{I}(a_{(2)}), R_{k\ell} - \delta_{k\ell} \rangle \otimes m_k \otimes m^*_\ell , \end{aligned}$$

and so:

$$\begin{aligned} \mathrm{d}^{\mathrm{L}}(a) &= \textstyle\sum_{k,l} \langle \mathrm{I}(a - \varepsilon(a)), R_{k\ell} \rangle m_k \otimes m^*_\ell \\ &= \textstyle\sum_{k,l} \langle \mathrm{J}(R_{k\ell} - \delta_{k\ell}), a \rangle m_k \otimes m^*_\ell . \end{aligned}$$

The R_{ji} are the matrix coefficients $\theta_{\mathrm{M}}(m_i, m^*_j)$ of the $\mathcal{A}^*$-module M, which is irreducible and non-trivial. Thus, by the Jacobson density theorem, the $(\dim \mathrm{M})^2 + 1$ elements $\{1, R_{ji}\}$ are linearly independent in $\mathcal{A}$. The $(\dim \mathrm{M})^2$ linear forms $\{\mathrm{J}(R_{k\ell} - \delta_{k\ell})\}$ are then linearly independent in $\mathcal{A}^*$, and the formula for $\mathrm{d}^{\mathrm{L}}(a)$ shows that d^{L} maps $\mathcal{A}$ onto $\Gamma^{\mathrm{L}} = \mathrm{End}(\mathrm{M})$. 1) is proved. The same formula shows that $\mathcal{R}$ is the set of elements a in the augmentation ideal of $\mathcal{A}$ such that $\mathrm{I}(a)$ is orthogonal to all the matrix coefficients $R_{k\ell}$ of the $\mathcal{A}^*$-module M. Thus $\mathcal{R} = \ker \varepsilon \cap \mathrm{I}^{-1}(\mathrm{ann}_{\mathcal{A}^*} \mathrm{M}) = \mathrm{I}^{-1}(\mathrm{ann}_{\mathcal{A}^*}(k \oplus \mathrm{M}))$. We have shown 2). □

If we consider now a finite family (M_i) of non-trivial non-isomorphic finite-dimensional simple right $\mathcal{A}$-comodules, we can do the direct sum of such constructions. If $(\mathcal{A}, \gamma)$ is factorizable, then the map $\mathrm{d} : (\mathcal{A} \to \bigoplus(\mathcal{A} \otimes \mathrm{End}\,\mathrm{M}_i))$ is a bicovariant differential calculus. The associated ideal is $\mathrm{I}^{-1}(\mathrm{ann}_{\mathcal{A}^*}(k \oplus \bigoplus \mathrm{M}_i))$.

3.3. The link with the classification theorem. We are now gathering the pieces of our patchwork. According to the statements in Sect. 3.1, Theorem 1 yields a complete classification of bicovariant differential calculi on $\mathcal{A}_q\mathrm{G}$. Morally, they are all given by the construction described in Sect. 3.2.

Proposition 10. *Let* $\mathrm{U}_q\mathfrak{g}$ *and* $\mathcal{A}_q\mathrm{G}$ *be the objects defined in Sect. 2.1. If the root and the weight lattices for* $\mathfrak{g}$ *are equal, all the bicovariant differential calculi on* $\mathcal{A}_q\mathrm{G}$ *can be constructed by the method described in Sect. 3.2.*

Proof. The results in Sect. 2.5 tell us that an ideal $\mathcal{R}$ associated to a bicovariant differential calculus on $\mathcal{A}_q\mathrm{G}$ is a subspace $\mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})} \mathrm{M})$, where M is a V-module containing the trivial V-module. Let $\mathrm{M}_1, \ldots, \mathrm{M}_n$ be the distinct non-trivial irreducible components of M. The assumption on $\mathfrak{g}$ gives us that the M_i are modules $\mathrm{L}(\lambda_i)$ (without any twisting character), and so can be considered as non-trivial non-isomorphic simple right $\mathcal{A}_q\mathrm{G}$-comodules. The construction of Sect. 3.2 for this family of comodules leads to a bicovariant differential calculus whose associated ideal is the inverse image by I of the annihilator of the $(\mathcal{A}_q\mathrm{G})^*$-module $\mathbb{C}(q) \oplus \bigoplus \mathrm{M}_i$. It is $\mathcal{R}$, and the proposition is proved. □

In the remainder of this section, we will discuss what happens when the root and the weight lattices differ. Up to the end of this article, we consider this case. There exist non-trivial characters $\chi : {}^{2\mathrm{P}}/_{2\mathrm{Q}} \to \mathbb{C}^\times$, and for any weight λ, we can look at the ideal $\mathcal{R} = \mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}(\mathbb{C}(q) \oplus \mathrm{L}_\chi(\lambda)))$, and at the associated bicovariant differential calculus. It cannot be constructed by the method of Theorem 2, since $\mathrm{L}_\chi(\lambda)$ is not a right $\mathcal{A}_q\mathrm{G}$-comodule. However, one may notice that the main trick in the construction of Sect. 3.2 consisted in using two different R-matrices, namely R_{12} and R_{21}^{-1}. R_{12} was used to endow the $\mathcal{A}_q\mathrm{G}$-comodule $\mathrm{L}(\lambda)$ with the structure of a right crossed bimodule over $\mathcal{A}_q$G, and R_{21}^{-1} turned the $\mathcal{A}_q$G-comodule $\mathrm{L}(\lambda)^*$ into a right crossed bimodule over $\mathcal{A}_q$G. The tensor product of these right crossed bimodules then gave the bicovariant differential calculus associated to $\mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}(\mathbb{C}(q) \oplus \mathrm{L}(\lambda)))$. When one uses the small freedom allowed in the choice of the R-matrix of $\mathrm{U}_q\mathfrak{g}$ (see [Ga]), one can make similar constructions for the bicovariant differential calculi associated with some of the ideals $\mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}(\mathbb{C}(q) \oplus \mathrm{L}_\chi(\lambda)))$. We will not write all the details, but point out that this is the way followed by Schmüdgen and Schüler for the construction described in [S–S1], Theorem 2.2.

As an example, we now describe explicitly the bicovariant differential calculus associated with the ideal $\mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}(\mathbb{C}(q) \oplus \mathrm{L}_\chi(0)))$. Let $\left(\mathrm{P}/\mathrm{Q}\right)^\wedge$ be the group of characters $\zeta : \mathrm{P}/\mathrm{Q} \to \mathbb{C}^\times$. If ζ is such a character, it extends to a one-dimensional representation $\overline{\zeta}$ of $\mathcal{A}_q$G by letting $\overline{\zeta}(\theta_{\mathrm{L}(\lambda)}(m, m^*)) = \zeta(\lambda \bmod \mathrm{Q})\langle m^*, m\rangle$, and this gives an inclusion of the group $\left(\mathrm{P}/\mathrm{Q}\right)^\wedge$ into the center of $(\mathcal{A}_q\mathrm{G})^{*\,\mathrm{res}}$. Since $(\overline{\zeta} \otimes \mathrm{id}) \circ \delta_{\mathrm{R}} : \mathcal{A}_q\mathrm{G} \to \mathbb{C}(q) \otimes \mathcal{A}_q\mathrm{G}$ is given by $(x \mapsto \overline{\zeta}(x) \otimes 1)$, we can see that the kernel of $\overline{\zeta}$ is a one-codimensional two-sided ideal of $\mathcal{A}_q$G, stable by the right coaction δ_{R}. If ζ is not trivial, the ideal $\mathcal{R} = \ker\varepsilon \cap \ker\overline{\zeta}$ defines a bicovariant differential calculus on $\mathcal{A}_q$G. Putting $\chi : \left(2\mathrm{P}/2\mathrm{Q} \to \mathbb{C}^\times, 2\lambda \bmod 2\mathrm{Q} \mapsto \zeta(\lambda \bmod \mathrm{Q})\right)$, we can check that $\mathcal{R} = \mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}(\mathbb{C}(q) \oplus \mathrm{L}_\chi(0)))$. This construction gives all the one-dimensional differential calculi on $\mathcal{A}_q$G (generalizing the result of [S–S1], Remark 4 after Theorem 2.2).

Finally, let X be an intermediate lattice between P and Q. The matrix coefficients of the irreducible representations of $\mathrm{U}_q\mathfrak{g}$ whose highest weights belong to X span a subalgebra $\mathcal{A}_q\mathrm{G}_\mathrm{X} \subseteq \mathcal{A}_q\mathrm{G}$. These algebras $\mathcal{A}_q\mathrm{G}_\mathrm{X}$ are factorizable c.q.t. Hopf algebras. For instance, $\mathcal{A}_q\mathrm{G}_\mathrm{Q}$ is the algebra of functions on the quantum adjoint group, and $\mathcal{A}_q\mathrm{G} \equiv \mathcal{A}_q\mathrm{G}_\mathrm{P}$ is the algebra of functions on the quantum simply-connected group. Our arguments in Sect. 2.5 show that the indecomposable bicovariant differential calculi on $\mathcal{A}_q\mathrm{G}_\mathrm{X}$ are classified by ideals $\mathcal{R} = \mathcal{A}_q\mathrm{G}_\mathrm{X} \cap \mathrm{I}^{-1}(\mathrm{ann}_{\mathrm{F}_\ell(\mathrm{U}_q\mathfrak{g})}(\mathbb{C}(q) \oplus \mathrm{L}_\chi(\lambda)))$, where $\chi : 2\mathrm{X}/2\mathrm{Q} \to \mathbb{C}^\times$ is a character (extended arbitrarily to a character of the group $2\mathrm{P}/2\mathrm{Q}$). Thus the “twisted” bicovariant differential calculi are non-local, their appearance depending of the choice of X. The bicovariant differential calculi seem localized at the central elements of G_X, that is to say, at the fixed points of G_X under the adjoint action.

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Note added in proof. P. Polo kindly communicated us the following simple proof of Proposition 8. By the formal character isomorphism, $\mathcal{G} \simeq \mathbb{Z}[\mathrm{P}]$. Let $\mathbb{Z}[\mathrm{P}]^\mathrm{W} \subseteq \mathbb{Z}[\mathrm{P}]$ be the subring of W-invariant elements. $\mathbb{Z}[\mathrm{P}]$ is a module of finite type over the noetherian ring $\mathbb{Z}[\mathrm{P}]^\mathrm{W}$, hence we can choose a finite generating set $(e^{\nu_i})_{1\le i\le n}$ from the family $(e^\nu)_{\nu\in\mathrm{P}}$. Take a weight μ such that all $\mu + \nu_i$ are dominant. Let $\lambda =\in \mathrm{P}_+$. Then there exists some $a_i \in \mathbb{Z}[\mathrm{P}]^\mathrm{W}$ such that $e^{-\lambda-\mu} = \sum_i a_i e^{\nu_i}$, hence $1 = \sum_i a_i e^{\lambda+\mu+\nu_i}$. Multiplying this by e^ρ and making the alternating sum over the Weyl group, one abtains that:

$$\mathrm{ch}\,\mathrm{L}(0) = \sum_i a_i\,\mathrm{ch}\,\mathrm{L}(\lambda + \mu + \nu_i).$$

This concludes the proof. Thanks are also due to A. Joseph for some useful comments about this work.

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