Method of Squared Eigenfunction Potentials in Integrable Hierarchies of KP Type

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Abstract: The method of squared eigenfunction potentials (SEP) is developed systematically to describe and gain new information about the Kadomtsev–Petviashvili (KP) hierarchy and its reductions. Interrelation to the τ-function method is discussed in detail. The principal result, which forms the basis of our SEP method, is the proof that any eigenfunction of the general KP hierarchy can be represented as a spectral integral over the Baker–Akhiezer (BA) wave function with a spectral density expressed in terms of SEP. In fact, the spectral representations of the (adjoint) BA functions can, in turn, be considered as defining equations for the KP hierarchy. The SEP method is subsequently used to show how the reduction of the full KP hierarchy to the constrained KP ($\mathsf{cKP}_{r,m}$) hierarchies can be given entirely in terms of linear constraint equations on the pertinent τ-functions. The concept of SEP turns out to be crucial in providing a description of $\mathsf{cKP}_{r,m}$ hierarchies in the language of the universal Sato Grassmannian and finding the non-isospectral Virasoro symmetry generators acting on the underlying τ-functions. The SEP method is used to write down *generalized binary* Darboux-Bäcklund transformations for constrained KP hierarchies whose orbits are shown to correspond to a new Toda model on a *square* lattice. As a result, we obtain a series of new determinant solutions for the τ-functions generalizing the known Wronskian (multi-soliton) solutions. Finally, applications to random matrix models in condensed matter physics are briefly discussed.

1. Introduction

The primary object of this paper is the Kadomtsev–Petviashvili (KP) integrable hierarchy (for comprehensive reviews, see *e.g.* [1, 2]) and its nontrivial reductions generalizing the familiar r-reduction to the $SL(r)$ Korteweg-de Vries (KdV) hierarchy. The KP hierarchy is an infinite-dimensional system which admits different alternative formulations

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and exhibits many types of symmetries. Here we are interested in a formulation based on the notion of *squared eigenfunction potential* and the spectral representations of the underlying eigenfunctions it gives rise to. Because of its connection to vertex operators, many aspects of this theory are algebraic in nature. This allows us to discuss in a systematic manner various symmetries of the hierarchy and applications to a large class of soliton systems obtained from it via symmetry reduction.

The KP hierarchy arises as a set of compatibility conditions for the linear spectral problem involving the pseudo-differential Lax operator $\mathcal{L}$ and the Baker–Akhiezer (BA) wave function $\psi_{BA}(t, \lambda)$. In recent years, the study of integrable systems of KP type has undergone rapid growth due to the applications of the tau-function technique invented by the Kyoto School [3, 4, 5]. The underlying principle of this method is to represent the relevant soliton potentials and Hamiltonian densities in terms of isospectral flows (with evolution parameters $(t) \equiv (t_1 \equiv x, t_2, \ldots)$) of one single function $\tau(t)$ in such a way that $\partial^2 \ln \tau(t)/\partial t_1 \partial t_n$ becomes equal to the coefficient in front of D^{-1} in the pseudo-differential operator expansion of $\mathcal{L}^n$.

In terms of the τ-function, viewed as a function of the infinitely many KP "time"-variables $(t_1 \equiv x, t_2, \ldots)$, the whole KP hierarchy is contained in Hirota's fundamental bilinear identity [5] instead of the infinite system of non-linear partial differential equations derived from the Sato-Wilson Lax operator approach. The τ-function approach bridges the way to several physical applications in view of its direct connection to physical objects, such as correlation and partition functions. Moreover, it allows a coherent treatment of multi-soliton solutions. These solutions of the nonlinear differential equations are generated by the action of the Miwa-Jimbo vertex operator $\widehat{\mathcal{X}}(\lambda, \mu)$ [4] (cf. Eq. (3.1) below) on the τ-function. This vertex operator generates an infinitesimal Bäcklund transformation of the KP hierarchy. The family of all vertex operators constitutes a Lie algebra isomorphic to $GL(\infty)$. The transformation $\tau(t) \rightarrow (\exp(a\widehat{\mathcal{X}}(\lambda, \mu)))\tau(t)$ sends a solution of the KP hierarchy into another solution. In this way, the action of the infinite-dimensional Lie algebra $GL(\infty)$ on the solution space of the KP equation is made explicit via the Bäcklund transformation of "adding one soliton".

This powerful formal machinery embeds many other concrete and useful structures relevant for physical models. Recently a special class of solutions encountered in the matrix models of discrete two-dimensional gravity was realized via the imposition of the Virasoro type of constraints on the underlying τ-function [6, 7] (see also [8]).

A remarkable feature of the KP hierarchy is the existence of the so called additional non-isospectral symmetries which, within the Lax operator formalism, are generated by Orlov-Schulman pseudo-differential operators [9]. The latter are defined as purely pseudo-differential parts of products of powers of the Lax operator $\mathcal{L}$ and its "conjugate" $\mathcal{M}$-operator (cf. Eqs. (5.1),(5.2) below) and their respective flows form the infinite-dimensional Lie algebra $\mathbf{W}_{1+\infty}$[1]. In an important recent development Adler, Shiota, and van Moerbeke [14, 15] (see also [16, 17]) obtained a formula for the KP hierarchy which relates the action of the vertex operator $\widehat{\mathcal{X}}(\lambda, \mu)$ on the τ-function to Orlov-Schulman non-isospectral additional symmetry flows on the BA wave function. The coefficients in the spectral expansion of $\widehat{\mathcal{X}}(\lambda, \mu)$ act as vector fields on the space of τ-functions

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[1] $\mathbf{W}_{1+\infty}$ algebra was originally introduced in physics literature [11] as a nontrivial "large N" limit of the associative, but *non*-Lie, conformal $\mathbf{W_N}$ algebra [12]. It turns out to be isomorphic to the (centrally extended) algebra of differential operators on the circle [13], *i.e.*, the Lie algebra generated by $z^k(\partial/\partial z)^n$ for $k \in \mathbb{Z}$, $n \geq 0$. Let us also recall that the "semiclassical" limit (contraction) $\mathbf{w}_{1+\infty}$ of $\mathbf{W}_{1+\infty}$ is the algebra of area-preserving diffeomorphisms on the cylinder [11].

generating $\mathbf{W}_{1+\infty}$ algebra as well. Hence, the above result relates the $\mathbf{W}_{1+\infty}$ algebra acting on $\tau(t)$ to the centerless $\mathbf{W}_{1+\infty}$ algebra of non-isospectral symmetry flows acting on the BA function $\psi_{BA}(t, \lambda)$.

There exists an alternative to the τ-function method characterization of the KP hierarchy evolution equations in terms of (adjoint) eigenfunctions, *i.e.*, functions whose KP multi-time flows are governed by an infinite set of purely differential operators $\{B_k\}_{k=1}^{\infty}$ (cf. Def.2.1 below). The latter, by virtue of compatibility of the multi-time flows, have to satisfy the so called "zero-curvature" Zakharov–Shabat equations (cf. Eq. 3.44 below). One can then show [18] that all B_k are obtained as purely differential projections of kth powers of a single pseudo-differential operator $\mathcal{L}$, thus leading to the standard Lax formulation of the KP hierarchy.

Overcoming the formal obstacle of having to define a function via an inverse derivative ∂_x^{-1} Oevel succeeded in [19] to associate a well-defined (up to a constant) function – the squared eigenfunction potential (SEP), to a pair of arbitrary eigenfunction and adjoint eigenfunction such that the x-derivative of SEP coincides with the product of the latter eigenfunctions. Consequently, a systematic formalism emerged in [19] for the study of symmetries generated within the KP hierarchy via SEP [20]. In a particular example, when both eigenfunctions defining the SEP are BA functions, the SEP becomes a generating function for the above mentioned additional non-isospectral symmetries of the KP hierarchy [9, 10, 14, 15, 16, 17].

In the SEP framework, the product of any pair of eigenfunction and adjoint eigenfunction, being a x-derivative of SEP, can be viewed as a conserved density within the hierarchy. The transition to the important class of constrained KP hierarchies $\mathsf{cKP}_{r,m}$[2], which are Hamiltonian reductions of the general KP hierarchy and whose Lax operators are given in Eq. (2.20) below, can be effectuated by imposing equality between a linear combination of m ($m \geq 1$) conserved densities of the above mentioned type and the rth ($r \geq 1$) fundamental Hamiltonian density of the KP hierarchy. In such a case, the symmetry generated by SEP (called "ghost" flow) is identified with the rth isospectral flow of the original KP hierarchy.

The principal merit of our work is to reformulate the eigenfunction formalism of the KP hierarchy in a new form called the *squared eigenfunction potential (SEP) method*, namely, to employ SEP as a basic building block in defining the KP hierarchy. The main ingredient of the SEP method is the proof of existence of spectral representation for any eigenfunction involving SEP as an integration kernel (spectral density). A link is then provided between the two alternative formulations of the KP hierarchy: one based on the τ-function and another one based on the SEP method. Furthermore, we apply the SEP method to solve various issues in integrable models of KP type and their applications in physics, among them, deriving new determinant solutions for the τ-function containing the familiar Wronskian (multi-soliton) solutions as simple particular cases, as well as identifying them as possible novel types of joint distribution functions in random matrix models of condensed matter physics.

The plan of the paper is as follows. After reviewing the background material in Sect. 2, we first prove in Sect. 3 that any eigenfunction of the general KP hierarchy can be represented as a spectral integral over the BA wave function with a spectral

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[2] The $\mathsf{cKP}_{r,m}$ integrable hierarchies appeared in different disguises from various parallel developments: (a) symmetry reductions [23, 19, 24] of the full KP hierarchy; (b) abelianization, *i.e.*, free-field realizations, in terms of finite number of fields, of both compatible first and second KP Hamiltonian structures [25, 26]; (c) a method of extracting continuum integrable hierarchies from the generalized Toda-like lattice hierarchies [27] underlying (multi-)matrix models; (d) purely algebraic approach in terms of the zero-curvature equations for the affine Kac-Moody algebras with non-standard gradations [28].

density expressed in terms of SEP. When (at least one) of the eigenfunctions is a BA functions, we obtain a closed expression for the SEP. When both of the eigenfunctions are BA functions, the resulting SEP's are connected straightforwardly to the bilocal vertex operator $\widehat{\mathcal{X}}(\lambda, \mu)$ acting on the τ-function. This association leads to a simple alternative proof for the Adler, Shiota, and van Moerbeke result [14, 15, 16, 17] mentioned above.

A further important observation in Sect. 3 is that the spectral representation equations for the (adjoint) BA functions themselves can be considered as defining equations for the KP hierarchy. In other words, our formalism of spectral representations of KP eigenfunctions can be viewed as an equivalent alternative characterization of the KP hierarchy parallel to Hirota's bilinear identity or Fay's trisecant identity [14].

Our results in the constrained $\mathsf{cKP}_{r,m}$ hierarchy case are as follows. In Sect. 4, using the SEP framework we obtain an equivalent description (parallel to the one within the Lax pseudo-differential operator approach) of $\mathsf{cKP}_{r,m}$ hierarchies entirely in terms of τ-functions only. Namely, we first derive a linear equation for the τ-function (Eq. (4.9) below), involving the bilocal vertex operator $\widehat{\mathcal{X}}(\lambda, \mu)$ and a set of spectral densities, which uniquely constrains the τ-function to belong to the $\mathsf{cKP}_{r,m}$ hierarchy. Furthermore, we provide in Sect. 4 an alternative description of $\mathsf{cKP}_{r,m}$ hierarchies in the language of the universal Sato Grassmannian.

One of the advantages of the SEP approach lies in the fact that it allows for a coherent treatment of the non-isospectral symmetries for KP-type hierarchies. We use this feature in Sect. 5 to demonstrate how the combination of the familiar Orlov-Schulman non-isospectral symmetry flows, operating in the full unconstrained KP hierarchy, together with certain appropriately chosen additional SEP-generated "ghost" symmetry flows [29, 8] gives rise to the correct non-isospectral Virasoro symmetry generators acting on the space of $\mathsf{cKP}_{r,m}$ τ-functions[3].

The SEP method is applied further in Sect. 6 to formulate *generalized multi-step binary* Darboux-Bäcklund (DB) transformation rules of (constrained) KP hierarchies (one-step binary DB transformations with SEP have been introduced previously in ref.[30]). Based on these transformation rules and using the fundamental Fay identities, we derive a series of new determinant solutions for the τ-functions generalizing the known Wronskian (multi-soliton) solutions. The binary DB orbits define a discrete symmetry structure for $\mathsf{cKP}_{r,m}$ hierarchies corresponding to a square lattice. We exhibit the equivalence of these binary DB orbits with a generalized Toda model on a *square* lattice.

Our formalism offers applications to the study of random matrix models in condensed matter physics, which we briefly discuss in Sect. 7, where we also present some discussion of the results and an outlook.

2. Background on General and Constrained KP Hierarchies

The calculus of the pseudo-differential operators (see e.g. [3, 2]) is one of the principal approaches to describe the KP hierarchy of integrable nonlinear evolution equations. In what follows the operator D is such that $[D, f] = \partial f = \partial f/\partial x$ and the generalized Leibniz rule holds:

$$D^n f = \sum_{j=0}^{\infty} \binom{n}{j} (\partial^j f) D^{n-j}, \quad n \in \mathbb{Z}. \tag{2.1}$$

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[3] The standard Orlov-Schulman non-isospectral symmetry flows do *not* preserve the constrained form (2.20) of $\mathsf{cKP}_{r,m}$ hierarchy.

In order to avoid confusion we shall employ the following notations: for any (pseudo-) differential operator A and a function f, the symbol $A(f)$ will indicate application (action) of A on f, whereas the symbol Af will denote just the operator product of A with the zero-order (multiplication) operator f.

In this approach the main object is the pseudo-differential *Lax* operator:

$$L = D^r + \sum_{j=0}^{r-2} v_j D^j + \sum_{i=1}^{\infty} u_i D^{-i}. \tag{2.2}$$

The Lax equations of motion:

$$\frac{\partial L}{\partial t_n} = [L_+^{n/r}, \, L] \quad , \quad n = 1, 2, \dots \tag{2.3}$$

describe isospectral deformations of L. In (2.3) and in what follows the subscripts $(\pm)$ of any pseudo-differential operator $A = \sum_j a_j D^j$ denote its purely differential part $(A_+ = \sum_{j \geq 0} a_j D^j)$ or its purely pseudo-differential part $(A_- = \sum_{j \geq 1} a_{-j} D^{-j})$, respectively. Further, $(t) \equiv (t_1 \equiv x, t_2, \dots)$ collectively denotes the infinite set of evolution parameters (KP "multi-time") from (2.3).

The Lax operator (2.2) can be represented equivalently in terms of the so called *dressing* operator W:

$$W = 1 + \sum_{n=1}^{\infty} w_n D^{-n}; \quad L = W D^r W^{-1} \tag{2.4}$$

whereupon the Lax Eqs. (2.3) become equivalent to the so-called Wilson-Sato equations:

$$\frac{\partial W}{\partial t_n} = L_+^{n/r} W - W D^n = -L_-^{n/r} W. \tag{2.5}$$

A further important object is the Baker–Akhiezer (BA) "wave" function defined via:

$$\psi_{BA}(t, \lambda) = W\big(e^{\xi(t,\lambda)}\big) = w(t, \lambda) e^{\xi(t,\lambda)}; \quad w(t, \lambda) = 1 + \sum_{i=1}^{\infty} w_i(t) \lambda^{-i}, \tag{2.6}$$

where

$$\xi(t, \lambda) \equiv \sum_{n=1}^{\infty} t_n \lambda^n; \quad t_1 = x \tag{2.7}$$

Accordingly, there is also an adjoint BA function:

$$\psi_{BA}^*(t, \lambda) = W^{*-1}\big(e^{-\xi(t,\lambda)}\big) = w^*(t, \lambda) e^{-\xi(t,\lambda)}; \quad w^*(t, \lambda) = 1 + \sum_{i=1}^{\infty} w_i^*(t) \lambda^{-i}. \tag{2.8}$$

The (adjoint) BA functions obey the following linear system:

$$L\big(\psi_{BA}(t, \lambda)\big) = \lambda^r \psi_{BA}(t, \lambda), \quad \frac{\partial}{\partial t_n} \psi_{BA}(t, \lambda) = L_+^{n/r}\big(\psi_{BA}(t, \lambda)\big), \tag{2.9}$$

$$L^*\big(\psi_{BA}^*(t, \lambda)\big) = \lambda^r \psi_{BA}^*(t, \lambda), \quad \frac{\partial}{\partial t_n} \psi_{BA}^*(t, \lambda) = -\big(L^*\big)_+^{n/r}\big(\psi_{BA}^*(t, \lambda)\big).$$

Note that Eqs. (2.3) for the KP hierarchy flows can be regarded as compatibility conditions for the system (2.9).

There exists another equivalent and powerful approach to the KP hierarchy based on one single function of all evolution parameters – the so called tau-function $\tau(t)$ [3] . It is an alternative to using the Lax operator and the calculus of the pseudo-differential operators. The τ-function is related to the BA functions (2.6)–(2.9) via:

$$\psi_{BA}(t,\lambda) = \frac{\tau\big(t - [\lambda^{-1}]\big)}{\tau(t)} e^{\xi(t,\lambda)} = e^{\xi(t,\lambda)} \sum_{n=0}^{\infty} \frac{p_n\left(-[\partial]\right)\tau(t)}{\tau(t)} \lambda^{-n}, \qquad (2.10)$$

$$\psi_{BA}^*(t,\lambda) = \frac{\tau\big(t + [\lambda^{-1}]\big)}{\tau(t)} e^{-\xi(t,\lambda)} = e^{-\xi(t,\lambda)} \sum_{n=0}^{\infty} \frac{p_n\left([\partial]\right)\tau(t)}{\tau(t)} \lambda^{-n}, \qquad (2.11)$$

where

$$[\lambda^{-1}] \equiv \left(\lambda^{-1}, \frac{1}{2}\lambda^{-2}, \frac{1}{3}\lambda^{-3}, \dots\right); \quad [\partial] \equiv \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots\right), \qquad (2.12)$$

and the Schur polynomials $p_n(t)$ are defined through

$$e^{\sum_{l \geq 1} \lambda^l t_l} = \sum_{n=0}^{\infty} \lambda^n p_n(t_1, t_2, \dots). \qquad (2.13)$$

Taking into account (2.10) and (2.6), the expansion for the dressing operator (2.4) becomes:

$$W = \sum_{n=0}^{\infty} \frac{p_n\left(-[\partial]\right)\tau(t)}{\tau(t)} D^{-n}, \quad i.e. \ \ w_1(t) = \mathrm{Res}W = -\partial_x \ln \tau(t). \qquad (2.14)$$

The (adjoint) BA functions enter the fundamental Hirota bilinear identity:

$$\int d\lambda \, \psi_{BA}(t,\lambda)\psi_{BA}^*(t',\lambda) = 0, \qquad (2.15)$$

which generates the entire KP hierarchy. Here and in what follows integrals over spectral parameters are understood as: $\int d\lambda \equiv \oint_0 \frac{d\lambda}{2i\pi} = \mathrm{Res}_{\lambda=0}$.

Let us also recall that the KP hierarchy possesses an infinite set of commuting integrals of motion w.r.t. the compatible first and second fundamental Poisson-bracket structures whose densities $h_{l-1} = \frac{1}{l}\mathrm{Res}L^{l/r}$ are expressed in terms of the τ-function (2.10) as

$$\partial_x \frac{\partial}{\partial t_l} \ln \tau(t) = \mathrm{Res}L^{l/r}. \qquad (2.16)$$

Below we shall be particularly interested in reductions of the full (unconstrained) KP hierarchy (2.2). In this respect, it turns out that a crucial rôle is played by the notions of *(adjoint) eigenfunctions* of KP hierarchy.

Definition 2.1. *The function* Φ *(Ψ) is called an* **(adjoint) eigenfunction** *of the Lax operator L satisfying Sato's flow equation (2.3) if its flows are given by the expressions:*

$$\frac{\partial \Phi}{\partial t_k} = L_+^{k/r}\left(\Phi\right) \quad ; \quad \frac{\partial \Psi}{\partial t_k} = -\left(L^*\right)_+^{k/r}\left(\Psi\right) \qquad (2.17)$$

for the infinitely many times t_k.

Of course, according to (2.9) the (adjoint) BA functions are particular examples of (adjoint) eigenfunctions which, however, satisfy in addition also the corresponding spectral equations.

In what follows a very important rôle will be played by the notion of the so called *squared eigenfunction potential* (SEP). As shown by Oevel [19], for an arbitrary pair of (adjoint) eigenfunctions $\Phi(t)$, $\Psi(t)$ there exists the function $S(\Phi(t), \Psi(t))$, called SEP, which possesses the following characteristics:

$$\frac{\partial}{\partial t_n} S(\Phi(t), \Psi(t)) = \mathrm{Res}\left(D^{-1}\Psi(L^{n/r})_+\Phi D^{-1}\right). \tag{2.18}$$

The argument in [19], proving the existence of $S(\Phi(t), \Psi(t))$, was built on compatibility between isospectral flows as defined in Eq. (2.18) and (2.17).

In particular, for $n = 1$, Eq. (2.18) implies that the space derivative (recall $\partial_x \equiv \partial/\partial t_1$) of $S(\Phi(t), \Psi(t))$ is equal to the product of the underlying eigenfunctions:

$$\partial_x S(\Phi(t), \Psi(t)) = \Phi(t)\Psi(t). \tag{2.19}$$

Remark. Equation (2.18) determines $S(\Phi(t), \Psi(t))$ up to a shift by a trivial constant.

From Eqs. (2.18)–(2.19) one sees that $\Phi(t)\Psi(t)$ is a conserved density of the KP hierarchy. This fact has a special significance for the reduction of the general KP hierarchy to the constrained cKP$_{r,m}$ models (see below).

Definition 2.2. *The constrained KP hierarchy (denoted as* cKP$_{r,m}$*) is given by a Lax operator of the following special form:*

$$L \equiv L_{r,m} = D^r + \sum_{l=0}^{r-2} u_l D^l + \sum_{a=1}^{m} \Phi_a D^{-1}\Psi_a. \tag{2.20}$$

One can easily check that the functions Φ_a, Ψ_a, entering the purely pseudo-differential part of $L_{r,m}$ (2.20), are (adjoint) eigenfunctions of $L_{r,m}$ according to Def.2.1.

The purely pseudo-differential part of arbitrary power of the cKP$_{r,m}$ Lax operator (2.20) has the following explicit form [31]:

$$\left(L^k\right)_- = \sum_{a=1}^{m}\sum_{j=0}^{k-1} L^{k-j-1}(\Phi_a)D^{-1}\left(L^*\right)^j(\Psi_a). \tag{2.21}$$

Formula (2.21) can easily be derived from the simple technical identity involving a product of two pseudo-differential operators of the form $f_i D^{-1} g_i$, $i = 1, 2$:

$$f_1 D^{-1} g_1 f_2 D^{-1} g_2 = f_1 S(f_2, g_1) D^{-1} g_2 - f_1 D^{-1} S(f_2, g_1) g_2, \tag{2.22}$$

where f_i, g_i are pairs of (adjoint) eigenfunctions of some KP Lax operator, with $S(\cdot, \cdot)$ being the corresponding SEP.

Note, that in agreement with Eq. (2.22) the expression $L(\Phi_a)$ in (2.21) with L as in (2.20) explicitly reads: $L(\Phi_a) = L_+(\Phi_a) + \sum_{b=1}^{m} \Phi_b S(\Phi_a, \Psi_b)$. This definition extends naturally to higher powers of L acting on Φ_a as well as higher powers of L^* acting on Ψ_a. Moreover, one can easily show that $L^l(\Phi_a)$ and $L^{*k}(\Psi_a)$ are (adjoint) eigenfunctions of L (2.20) as well.

For three pseudo-differential operators $X_i \equiv f_i D^{-1} g_i$, $i = 1, 2, 3$ the associativity law $(X_1 X_2) X_3 = X_1(X_2 X_3)$ implies the following technical lemma:

Lemma 2.1. *The squared eigenfunction potential* $S(\cdot, \cdot)$ *satisfies:*

$$S(f,g)S(h,k) = S\left(h, kS(f,g)\right) + S\left(fS(h,k), g\right) \tag{2.23}$$

for (adjoint) eigenfunctions f, g, h, k.

3. Spectral Representation of KP Eigenfunctions

Consider the bilocal vertex operator [4]:

$$
\begin{aligned}
\widehat{\mathcal{X}}(\lambda, \mu) &\equiv \frac{1}{\lambda} : e^{\hat{\theta}(\lambda)} : : e^{-\hat{\theta}(\mu)} := \frac{1}{\lambda} e^{\xi(t+[\lambda^{-1}],\mu) - \xi(t,\lambda)} e^{\sum_1^\infty \frac{1}{l}(\lambda^{-l} - \mu^{-l})\frac{\partial}{\partial t_l}} \\
&= -\frac{1}{\mu} e^{\xi(t,\mu) - \xi(t-[\mu^{-1}],\lambda)} e^{\sum_1^\infty \frac{1}{l}(\lambda^{-l} - \mu^{-l})\frac{\partial}{\partial t_l}} + \delta(\lambda, \mu), \tag{3.1}
\end{aligned}
$$

where

$$\hat{\theta}(\lambda) \equiv -\sum_{l=1}^\infty \lambda^l t_l + \sum_{l=1}^\infty \frac{1}{l}\lambda^{-l}\frac{\partial}{\partial t_l}. \tag{3.2}$$

$\xi(t, \lambda)$ is as in (2.7), the columns : ... : indicate Wick normal ordering w.r.t. the creation/annihilation "modes" t_l and $\frac{\partial}{\partial t_l}$, respectively, and the delta-function is defined as

$$\delta(\lambda, \mu) = \frac{1}{\lambda}\frac{1}{1 - \frac{\mu}{\lambda}} + \frac{1}{\mu}\frac{1}{1 - \frac{\lambda}{\mu}}. \tag{3.3}$$

An equivalent representation for $\widehat{\mathcal{X}}(\lambda, \mu)$, using Wick theorem, reads

$$
\begin{aligned}
\widehat{\mathcal{X}}(\lambda, \mu) &= \frac{1}{\lambda - \mu} : e^{\hat{\theta}(\lambda) - \hat{\theta}(\mu)} : \\
&= \frac{1}{\lambda - \mu} e^{\xi(t,\mu) - \xi(t,\lambda)} e^{\sum_1^\infty \frac{1}{l}(\lambda^{-l} - \mu^{-l})\frac{\partial}{\partial t_l}} \quad \text{for} \quad |\mu| \le |\lambda|. \tag{3.4}
\end{aligned}
$$

The vertex operator $\widehat{\mathcal{X}}(\lambda, \mu)$ can be expanded as follows:

$$\widehat{\mathcal{X}}(\lambda, \mu) = \frac{1}{\lambda - \mu}\sum_{l=0}^\infty \frac{(\mu - \lambda)^l}{l!} \sum_{s=-\infty}^\infty \lambda^{-s-l-1}\frac{1}{l+1}\widehat{W}_s^{(l+1)}, \tag{3.5}$$

where the operators $\widehat{W}_s^{(l)}$ span $\mathbf{W}_{1+\infty}$ algebra.

From the standard representation for the (adjoint) Baker–Akhiezer wave function (2.10),(2.11) in terms of the τ-function we deduce the identity:

$$\frac{\widehat{\mathcal{X}}(\lambda, \mu)\tau(t)}{\tau(t)} = \frac{1}{\lambda}\psi_{BA}^*(t, \lambda)\,\psi_{BA}\left(t + [\lambda^{-1}], \mu\right) \tag{3.6}$$

$$= -\frac{1}{\mu}\psi_{BA}(t, \mu)\,\psi_{BA}^*\left(t - [\mu^{-1}], \lambda\right) + \delta(\lambda, \mu). \tag{3.7}$$

Now, recall the Fay identity [14] for τ-functions:

$$(s_0 - s_1)(s_2 - s_3)\tau(t + [s_0] + [s_1])\tau(t + [s_2] + [s_3]) + \text{cyclic}(1, 2, 3) = 0 \tag{3.8}$$

which, in fact, is equivalent to Hirota bilinear identity (2.15). In what follows, we shall often make use of a particular form of (3.8) upon setting $s_0 = 0$, dividing by $s_1 s_2 s_3$ and shifting the KP times $(t) \to (t - [s_2] - [s_3])$:

$$
\begin{aligned}
&\left(s_2^{-1} - s_3^{-1}\right) \tau\left(t + [s_1] - [s_2] - [s_3]\right) \tau(t) \\
&+ \left(s_1^{-1} - s_2^{-1}\right) \tau\left(t - [s_2]\right) \tau\left(t + [s_1] - [s_3]\right) \\
&+ \left(s_3^{-1} - s_1^{-1}\right) \tau\left(t - [s_3]\right) \tau\left(t + [s_1] - [s_2]\right) = 0.
\end{aligned}
\tag{3.9}
$$

Especially, making identification $s_1 = \mu^{-1}$, $s_2 = z^{-1}$ and $s_3 = \lambda^{-1}$ in (3.9) and using (2.10)–(2.11), we arrive at the following useful lemma:

Lemma 3.1. *The truncated Fay identity (3.9) is equivalent to the following bilinear identity for (adjoint) BA functions:*

$$
\frac{1}{\lambda} \widehat{\Delta}_z \left(\psi_{BA}(t, \lambda) \psi_{BA}^*(t - [\lambda^{-1}], \mu) \right) = -\frac{1}{z} \psi_{BA}(t, \lambda) \psi_{BA}^*(t - [z^{-1}], \mu),
\tag{3.10}
$$

where $\widehat{\Delta}_z$ is the shift-difference operator acting on functions depending on the variables $t = (t_1, t_2, ...)$ as follows:

$$
\widehat{\Delta}_z \equiv e^{\sum_1^\infty \frac{1}{l} z^{-l} \partial/\partial t_l} - 1, \quad \widehat{\Delta}_z f(t) = f\left(t - [z^{-1}]\right) - f(t).
\tag{3.11}
$$

The Fay identity (3.8) is also known in its differential version:

$$
\partial_x \left(\frac{\tau\left(t + [\lambda^{-1}] - [\mu^{-1}]\right)}{\tau(t)} \right) =
$$
$$
= (\lambda - \mu) \left(\frac{\tau\left(t + [\lambda^{-1}] - [\mu^{-1}]\right)}{\tau(t)} - \frac{\tau\left(t + [\lambda^{-1}]\right)}{\tau(t)} \frac{\tau\left(t - [\mu^{-1}]\right)}{\tau(t)} \right).
\tag{3.12}
$$

Using (3.4) and multiplying both sides of (3.12) by $\exp\left\{-\xi(t, \lambda) + \xi(t, \mu)\right\}$ we can rewrite it as

$$
\partial_x \left(\frac{\widehat{\mathcal{X}}(\lambda, \mu) \tau(t)}{\tau(t)} \right) = -\psi_{BA}^*(t, \lambda) \psi_{BA}(t, \mu)
\tag{3.13}
$$

or, equivalently, using (3.6) and (3.7):

$$
\partial_x \left(-\frac{1}{\lambda} \psi_{BA}^*(t, \lambda) \psi_{BA}\left(t + [\lambda^{-1}], \mu\right) \right) = \psi_{BA}^*(t, \lambda) \psi_{BA}(t, \mu),
$$
$$
\partial_x \left(\frac{1}{\mu} \psi_{BA}(t, \mu) \psi_{BA}^*\left(t - [\mu^{-1}], \lambda\right) \right) = \psi_{BA}^*(t, \lambda) \psi_{BA}(t, \mu).
\tag{3.14}
$$

Let Φ, Ψ be a pair of an eigenfunction and an adjoint eigenfunction of the general KP hierarchy. Our main statement in this section is:

Proposition 3.1. *Any (adjoint) eigenfunction of the general KP hierarchy possesses a spectral representation of the form:*

$$\Phi(t) = \int d\lambda\, \varphi(\lambda)\, \psi_{BA}(t,\lambda); \quad \Psi(t) = \int d\lambda\, \varphi^*(\lambda)\, \psi^*_{BA}(t,\lambda) \tag{3.15}$$

with spectral densities given by

$$\varphi(\lambda) = \frac{1}{\lambda}\psi^*_{BA}\left(t',\lambda\right)\Phi\left(t' + [\lambda^{-1}]\right); \quad \varphi^*(\lambda) = \frac{1}{\lambda}\psi_{BA}\left(t',\lambda\right)\Psi\left(t' - [\lambda^{-1}]\right), \tag{3.16}$$

where the multi-time $t' = \left(t'_1, t'_2, \ldots\right)$ is taken at some arbitrary fixed value. In other words:

$$\Phi(t) = \int d\lambda\, \psi_{BA}(t,\lambda)\frac{1}{\lambda}\psi^*_{BA}\left(t',\lambda\right)\Phi\left(t' + [\lambda^{-1}]\right), \tag{3.17}$$

$$\Psi(t) = \int d\lambda\, \psi^*_{BA}(t,\lambda)\frac{1}{\lambda}\psi_{BA}\left(t',\lambda\right)\Psi\left(t' - [\lambda^{-1}]\right) \tag{3.18}$$

are valid for arbitrary KP (adjoint) eigenfunctions Φ, Ψ and for an arbitrary fixed multi-time t'. Furthermore, the r.h.s. of (3.17) and (3.18) do not depend on t'.

We will proceed proving the above proposition in two steps. First, let us assume that the (adjoint-)eigenfunctions indeed possess a spectral representation of the form (3.15) with some spectral densities $\varphi^{(*)}(\lambda)$. In such a case the statement of the proposition is contained in a much simpler lemma:

Lemma 3.2. *For (adjoint) eigenfunctions possessing the spectral representation (3.15) their respective spectral densities are given by (3.16). Consequently, in this case Eqs. (3.17) and (3.18) are valid too.*

Proof. Using the spectral representation (3.15) for $\Phi\left(t' + [\lambda^{-1}]\right)$ and substituting it into the right hand side of (3.17), we get:

$$\int d\lambda \int d\mu\, \varphi(\mu)\, \psi_{BA}(t,\lambda)\frac{1}{\lambda}\psi^*_{BA}\left(t',\lambda\right)\psi_{BA}\left(t' + [\lambda^{-1}],\mu\right). \tag{3.19}$$

Recalling (3.7) we can rewrite (3.19) as:

$$\int d\lambda \int d\mu\, \varphi(\mu)\, \psi_{BA}(t,\lambda)\left(\frac{-1}{\mu}\psi_{BA}\left(t',\mu\right)\psi^*_{BA}\left(t' - [\mu^{-1}],\lambda\right) + \delta\left(\lambda,\mu\right)\right)$$

$$= \int d\lambda\, \varphi(\lambda)\, \psi_{BA}(t,\lambda) = \Phi(t). \tag{3.20}$$

where use was made of the fundamental Hirota bilinear identity (2.15). The t'-independence of the r.h.s. of (3.17) and (3.18) will be demonstrated explicitly in the course of the proof of Prop.3.1 given below. □

We are now ready to take a final step of the proof of Prop.3.1 and extend the result of Lemma (3.2) to arbitrary KP (adjoint-)eigenfunctions *without* assuming existence of a spectral representation (3.15). To this end we need to recall the notion of SEP (2.18)–(2.19).

Let $S\left(\Phi(t), \psi^*_{BA}(t,\lambda)\right)$ be the SEP associated with a pair of eigenfunctions $\Phi(t)$ and $\psi^*_{BA}(t,\lambda)$, *i.e.* $\partial_x S\left(\Phi(t), \psi^*_{BA}(t,\lambda)\right) = \Phi(t)\,\psi^*_{BA}(t,\lambda)$. Before proceeding let us

note the following simple property of $S\left(\Phi(t), \psi_{BA}^*(t, \lambda)\right)$, namely, it is an "oscillatory" type of function w.r.t. λ of the form:

$$S\left(\Phi(t), \psi_{BA}^*(t, \lambda)\right) = e^{-\xi(t, \lambda)} \sum_{j=1}^{\infty} s_j(t) \lambda^{-j} = e^{-\xi(t, \lambda)} \left[-\Phi(t)\lambda^{-1} + O(\lambda^{-2})\right] \quad (3.21)$$

$$s_{j+1}(t) = \partial_x s_j(t) - \Phi(t) w_j^*(t), \quad j = 0, 1, \ldots; \quad s_0 \equiv 0 \qquad (3.22)$$

where $w_j^*(t) = \left(\tau(t)\right)^{-1} p_j([\partial])\tau(t)$ are the coefficients in the λ-expansion of $\psi_{BA}^*(t, \lambda)$ (2.11). Indeed, the defining relation (2.18) for the SEP in question:

$$\frac{\partial}{\partial t_n} S\left(\Phi(t), \psi_{BA}^*(t, \lambda)\right) = \mathrm{Res}\left(D^{-1}\psi_{BA}^*(t, \lambda)(L^{n/r})_+\Phi(t)D^{-1}\right) \qquad (3.23)$$

implies the oscillatory form:

$$e^{-\xi(t, \lambda)} \sum_{j=0}^{\infty} s_j(t) \lambda^{-j} \qquad (3.24)$$

of the latter upto an additive constant (recall, that any SEP is defined upto an addition of a constant). Further, taking into account:

$$\partial_x S\left(\Phi(t), \psi_{BA}^*(t, \lambda)\right) = \Phi(t)\, \psi_{BA}^*(t, \lambda) = e^{-\xi(t, \lambda)} \sum_{j=0}^{\infty} \Phi(t) w_j^*(t) \lambda^{-j} \qquad (3.25)$$

and comparing the series in the last equality (3.25) with the ∂_x derivative of the series (3.24), one obtains recurrence relations (3.22). Define now:

$$\widehat{\Phi}\left(t, t'\right) = -\int d\lambda\, \psi_{BA}(t, \lambda)\, S\left(\Phi(t'), \psi_{BA}^*(t', \lambda)\right) \qquad (3.26)$$

We first observe that $\partial\widehat{\Phi}\left(t, t'\right)/\partial t_n' = 0$ due to Eqs. (2.18) and (2.15). Hence $\widehat{\Phi}\left(t, t'\right) = \widehat{\Phi}(t)$ does not depend on the multi-time t'. Moreover, it is obvious from the definition (3.26) that $\widehat{\Phi}(t)$ is an eigenfunction possessing spectral representation of the form (3.15) and, therefore, satisfying the conditions of Lemma (3.2). Consequently, according to (3.17), we have:

$$\widehat{\Phi}(t) = \int d\lambda\, \psi_{BA}(t, \lambda)\frac{1}{\lambda}\psi_{BA}^*\left(t', \lambda\right)\widehat{\Phi}\left(t' + [\lambda^{-1}]\right). \qquad (3.27)$$

Comparing integrands in Eqs. (3.26) and (3.27) we find:

$$\frac{1}{\lambda}\psi_{BA}^*(t, \lambda)\widehat{\Phi}\left(t + [\lambda^{-1}]\right) = -S\left(\Phi(t), \psi_{BA}^*(t, \lambda)\right) + X(t, \lambda), \qquad (3.28)$$

where $X(t, \lambda)$ is a function of the "oscillatory" type (3.24) (taking into account (3.21)–(3.22)):

$$X(t, \lambda) = e^{-\xi(t, \lambda)}\left[\frac{1}{\lambda}\left(\widehat{\Phi}(t) - \Phi(t)\right) + O(\lambda^{-2})\right]. \qquad (3.29)$$

It satisfies:

$$\mathrm{Res}(\psi_{BA}(t, \lambda)X(t', \lambda)) = 0 \qquad (3.30)$$

for arbitrary t'. Inserting (3.29) in (3.30) and taking $t' = t$ we get $\widehat{\Phi}(t) - \Phi(t)$. This concludes the proof of Prop.3.1. $\square$

Corollary 3.1. *Taking into account Prop.3.1, Eqs. (3.13)–(3.14) imply the following relations:*

$$S\left(\psi_{BA}(t,\mu),\psi_{BA}^*(t,\lambda)\right) = -\frac{\widehat{\mathcal{X}}(\lambda,\mu)\,\tau(t)}{\tau(t)}$$

$$= -\frac{1}{\lambda}\psi_{BA}\left(t+[\lambda^{-1}],\mu\right)\psi_{BA}^*(t,\lambda), \qquad (3.31)$$

$$S\left(\psi_{BA}(t,\lambda),\psi_{BA}^*(t,\mu)\right) = -\frac{\widehat{\mathcal{X}}(\mu,\lambda)\,\tau(t)}{\tau(t)} + \delta(\mu,\lambda)$$

$$= \frac{1}{\lambda}\psi_{BA}(t,\lambda)\psi_{BA}^*\left(t-[\lambda^{-1}],\mu\right),$$

$$(3.32)$$

$$S\left(\Phi(t),\psi_{BA}^*(t,\lambda)\right) = -\frac{1}{\lambda}\psi_{BA}^*(t,\lambda)\Phi\left(t+[\lambda^{-1}]\right), \qquad (3.33)$$

$$S\left(\psi_{BA}(t,\lambda),\Psi(t)\right) = \frac{1}{\lambda}\psi_{BA}(t,\lambda)\Psi\left(t-[\lambda^{-1}]\right), \qquad (3.34)$$

where Φ,Ψ are arbitrary (adjoint-)eigenfunctions and $S\left(\cdot,\cdot\right)$ are the corresponding squared eigenfunction potentials. Moreover, we also have the following double spectral density representation for the SEP $S\left(\Phi(t),\Psi(t)\right)$:

$$S\left(\Phi(t),\Psi(t)\right) = -\iint d\lambda\,d\mu\,\varphi^*(\lambda)\varphi(\mu)\frac{1}{\lambda}\psi_{BA}^*(t,\lambda)\psi_{BA}(t+[\lambda^{-1}],\mu)$$

$$= -\iint d\lambda\,d\mu\,\varphi^*(\lambda)\varphi(\mu)\frac{\widehat{\mathcal{X}}(\lambda,\mu)\,\tau(t)}{\tau(t)} \qquad (3.35)$$

Taking into account (3.33)–(3.34), the spectral representations (3.17)–(3.18) become:

$$\Phi(t) = -\int d\lambda\,\psi_{BA}(t,\lambda)\,S\left(\Phi(t'),\psi_{BA}^*(t',\lambda)\right), \qquad (3.36)$$

$$\Psi(t) = \int d\lambda\,\psi_{BA}^*(t,\lambda)\,S\left(\psi_{BA}(t',\lambda),\Psi(t')\right). \qquad (3.37)$$

Remark. Note that the expressions (3.36)–(3.37) applied for (adjoint) BA functions yield:

$$\psi_{BA}(t,\lambda) = -\int d\mu\,\psi_{BA}(t,\mu)S\left(\psi_{BA}(t',\lambda),\psi_{BA}^*(t',\mu)\right),$$

$$\psi_{BA}^*(t,\lambda) = \int d\mu\,\psi_{BA}^*(t,\mu)S\left(\psi_{BA}(t',\mu),\psi_{BA}^*(t',\lambda)\right), \qquad (3.38)$$

which shows that the SEP $S\left(\psi_{BA}(t',\lambda),\psi_{BA}^*(t',\mu)\right)$ can be identified with the Cauchy kernel for each fixed KP multi-time t' (cf. also [21] and references therein, where the above SEP was previously introduced in the context of the Riemann factorization problem, as well as [22] for related discussion within the dispersionless KP hierarchy).

Remark. Going back to the spectral representation Eqs. (3.17)–(3.18), valid for any eigenfunction of the general KP hierarchy, we observe that they can be rewritten as evolution equations w.r.t. the KP multi-time of the following form:

$$\Phi(t) = \hat{U}(t, t')\Phi(t'),$$

$$\hat{U}(t, t') \equiv \int d\lambda \, \psi_{BA}(t, \lambda) \frac{1}{\lambda} \psi_{BA}^* \left(t', \lambda\right) e^{\sum_1^\infty \frac{1}{l}\lambda^{-l}\partial/\partial t'_l}, \qquad (3.39)$$

$$\Psi(t) = \hat{U}^*(t, t')\Psi(t'),$$

$$\hat{U}^*(t, t') \equiv \int d\lambda \, \psi_{BA}^*(t, \lambda) \frac{1}{\lambda} \psi_{BA} \left(t', \lambda\right) e^{-\sum_1^\infty \frac{1}{l}\lambda^{-l}\partial/\partial t'_l}. \qquad (3.40)$$

One can readily verify that:

$$\hat{U}(t, t) = \mathbb{1}, \quad \hat{U}^{-1}(t, t') = \hat{U}(t', t), \quad \hat{U}(t, t') = \hat{U}(t, t'')\,\hat{U}(t'', t'), \qquad (3.41)$$

$$\frac{\partial}{\partial t_l}\hat{U}(t, t') = L_+^{l/r}\hat{U}(t, t'), \quad \frac{\partial}{\partial t'_l}\hat{U}(t, t') = -L_+'^{l/r}\hat{U}(t, t'). \qquad (3.42)$$

From (3.41)–(3.42) we deduce that the evolution operator $\hat{U}(t, t')$ (3.39) can be formally written as a path-ordered exponential:

$$\hat{U}(t, t') = P \exp\left\{\sum_{l=1}^\infty \int_0^1 ds \frac{dt_l}{ds} L_+^{l/r}(t(s))\right\}; \quad t_k(0) = t'_k, \quad t_k(1) = t_k, \quad k = 1, 2, \ldots \qquad (3.43)$$

which precisely agrees with the formal solution of the differential evolution Eqs. (2.17) for the KP eigenfunctions. The r.h.s. of (3.43) is independent of the particular path $\{t_k(s)\}$ connecting the points t' and t in the space of KP multi-times due to the "zero-curvature" Zakharov–Shabat equations:

$$\frac{\partial}{\partial t_k}L_+^{l/r} - \frac{\partial}{\partial t_l}L_+^{k/r} - \left[L_+^{k/r}, L_+^{l/r}\right] = 0. \qquad (3.44)$$

Thus, our SEP method allowed us to find the explicit expression (r.h.s. of the second Eq. (3.39)) for the formal path-ordered exponential (3.43).

Now, it is worthwhile to observe that we can revert the logic of our procedure above, *i.e.*, instead of starting with the Hirota bilinear identity (2.15) (or, equivalently, with the Fay identity (3.8)) as defining the KP hierarchy and deriving from them the spectral representation formalism (3.17)–(3.18) (or (3.36)–(3.37)) for KP eigenfunctions, we can take the spectral representation Eqs. (3.36)–(3.37) as the basic equations defining the KP hierarchy. Namely, we have the following simple:

Proposition 3.2. *Consider a pair of functions $\psi(t, \lambda), \psi^*(t, \lambda)$ of the multi-time $(t_1, t_2, \ldots)$ and the spectral parameter λ of the form $\psi^{(*)}(t, \lambda) = e^{\pm\xi(t,\lambda)} \sum_{j=0}^\infty w_j^{(*)}(t)\lambda^{-j}$ with $w_0^{(*)} = 1$ and $\xi(t, \lambda)$ as in (2.7). Let us assume that $\psi^{(*)}(t, \lambda)$ obey the following spectral identities:*

$$\psi(t, \lambda) = -\int d\mu \, \psi(t, \mu)\, S(t'; \lambda, \mu), \quad \psi^*(t, \lambda) = \int d\mu \, \psi^*(t, \mu)\, S(t'; \mu, \lambda) \qquad (3.45)$$

for two arbitrary multi-times t and t', where by definition the function $S(t; \lambda, \mu)$ is such that $\frac{\partial}{\partial t_1}S(t; \lambda, \mu) = \psi(t, \lambda)\,\psi^*(t, \mu)$. Then, Eqs. (3.45) are equivalent to the Hirota bilinear identity (2.15) and, accordingly, $\psi^{(*)}(t, \lambda)$ become (adjoint) BA functions of the associated KP hierarchy.

To see that Eqs. (3.45) imply the Hirota identity (2.15), it is enough to differentiate both sides of (3.45) w.r.t. t'_1: $0 = \partial \psi(t, \lambda)/\partial t'_1 = -\psi(t', \lambda) \int d\mu \, \psi(t, \mu) \, \psi^*(t', \mu)$. The proof of the inverse statement of the equivalence, namely, that the Hirota bilinear identity (2.15) imply the spectral representation Eqs. (3.45), is contained in the proof of Prop.3.1 above.

Using (2.10)–(2.11), Eqs. (3.33)-(3.34) can be rewritten as:

$$\frac{\tau\left(t + [\lambda^{-1}]\right) \Phi\left(t + [\lambda^{-1}]\right)}{\lambda \tau(t)} e^{-\xi(t, \lambda)} = -S\left(\Phi(t), \psi^*_{BA}(t, \lambda)\right), \qquad (3.46)$$

$$\frac{\tau\left(t - [\lambda^{-1}]\right) \Psi\left(t - [\lambda^{-1}]\right)}{\lambda \tau(t)} e^{\xi(t, \lambda)} = S\left(\psi_{BA}(t, \lambda), \Psi(t)\right). \qquad (3.47)$$

Remark. Spectral representations for eigenfunctions (3.36)–(3.37) as well as identities (3.46)–(3.47) were obtained in a similar form in [32] for the particular case of constrained $\mathsf{cKP}_{r,m}$ hierarchies. Let us specifically emphasize, that all main equations of the present SEP method (3.15)–(3.18), (3.31)–(3.37) and (3.46)–(3.47), derived above, are valid within the general unconstrained KP hierarchy.

Acting with space derivative ∂_x on both sides of (3.46)-(3.47) and shifting the KP time arguments, we get:

$$\frac{\Phi\left(t - [\lambda^{-1}]\right)}{\Phi(t)} - 1 + \lambda^{-1} \partial \ln \Phi(t) = \lambda^{-1} \partial \ln \frac{\tau\left(t - [\lambda^{-1}]\right)}{\tau(t)}, \qquad (3.48)$$

$$\frac{\Psi\left(t + [\lambda^{-1}]\right)}{\Psi(t)} - 1 - \lambda^{-1} \partial \ln \Psi(t) = -\lambda^{-1} \partial \ln \frac{\tau\left(t + [\lambda^{-1}]\right)}{\tau(t)}, \qquad (3.49)$$

which were obtained in [8] by studying the way the τ-function transforms under Darboux-Bäcklund transformations. Taking into consideration that:

$$-\lambda + \lambda \frac{\Phi\left(t - [\lambda^{-1}]\right)}{\Phi(t)} + \partial \ln \Phi(t) = \sum_{n=2}^{\infty} \frac{p_n(-[\partial]) \Phi(t)}{\lambda^{n-1} \Phi(t)} \qquad (3.50)$$

with $p_n(\cdot)$ being the Schur polynomials (2.13), we find that Eq. (3.48) is a generating equation for the following set of equations upon expanding in powers of λ^{-1}:

$$p_n(-[\partial]) \Phi(t) = v_n(t) \Phi(t); \quad n \geq 2, \qquad (3.51)$$
$$v_n(t) \equiv p_{n-1}(-[\partial]) \, \partial \ln \tau(t).$$

Note that $v_n(t)$ are coefficients in the λ-expansion of the generating function $v(t, \lambda)$ [33]:

$$v(t, \lambda) = \sum_{n=1}^{\infty} v_{n+1} \lambda^{-n} \equiv \partial_x \ln \psi_{BA}(t, \lambda) - \lambda = \widehat{\Delta}_\lambda \partial_x \ln \tau(t), \qquad (3.52)$$

where in obtaining the last equality we again used Eqs. (2.10)–(2.11) and notation (3.11). We will later need a slight generalization of (3.52):

$$v^{(l)}(t, \lambda) = \sum_{n=1}^{\infty} \sigma_n^{(l)}(t) \lambda^{-n} \equiv \frac{\partial}{\partial t_l} \ln \psi_{BA}(t, \lambda) - \lambda^l = \widehat{\Delta}_\lambda \frac{\partial}{\partial t_l} \ln \tau(t); \quad l \geq 1. \quad (3.53)$$

Clearly $\sigma_n^{(l)}(t) = p_n(-[\partial]) \, \partial/\partial t_l \ln \tau$ and $v_n(t) = \sigma_{n-1}^{(1)}(t)$, $n \geq 2$. The coefficients $\sigma_n^{(l)}$ enter the basic identity for the KP Lax operator (2.2):

$$\left(L^{l/r} \right)_+ = L^{l/r} + \sum_{n=1}^{\infty} \sigma_n^{(l)} L^{-n/r}. \tag{3.54}$$

Remark. Eqs. (3.51) are, clearly, valid for an arbitrary eigenfunction Φ of the full KP hierarchy. On the other hand, in ref.[3] (see also [24]) Eqs. (3.51) were presented for the special case of $\Phi = \psi_{BA}(t, \lambda)$ as relations equivalent to the standard KP evolution equations $\partial \psi_{BA}(t, \lambda)/\partial t_n = (L^{n/r})_+ \psi_{BA}(t, \lambda)$. In fact, as shown in [33], plugging the BA wave function $\Phi(t) = \psi_{BA}(t, \mu)$ into Eq. (3.48) one easily recovers the differential Fay identity (3.12).

We now define the "ghost" symmetry flows generated by the SEP [9, 19, 24, 8]. Let ∂_α be a vector field, whose action on the Lax operator L and, accordingly, on the dressing operator W, is induced by a set of (adjoint) eigenfunctions Φ_a, Ψ_a, $a \in \{\alpha\}$ through:

$$\partial_\alpha L \equiv \left[\sum_{a \in \{\alpha\}} \Phi_a D^{-1} \Psi_a \, , \, L \right]; \quad \partial_\alpha W \equiv \left(\sum_{a \in \{\alpha\}} \Phi_a D^{-1} \Psi_a \right) W. \tag{3.55}$$

As shown in [19], the corresponding action of the above "ghost" flows on the (adjoint) eigenfunctions Φ, Ψ:

$$\partial_\alpha \Phi = \sum_{a \in \{\alpha\}} \Phi_a S\left(\Phi, \Psi_a\right); \quad \partial_\alpha \Psi = \sum_{a \in \{\alpha\}} S\left(\Phi_a, \Psi\right) \Psi_a \tag{3.56}$$

is compatible with the isospectral evolutions of Φ, Ψ. Furthermore, it is easy to see that

$$\partial_\alpha S\left(\Phi, \Psi\right) = \sum_{a \in \{\alpha\}} S\left(\Phi, \Psi_a\right) S\left(\Phi_a, \Psi\right) \tag{3.57}$$

is compatible with Eq. (3.56).

If $\partial_\beta W \equiv \left(\sum_{b \in \{\beta\}} \Phi_b D^{-1} \Psi_b \right) W$ defines some other "ghost" flow and both flows ∂_α and ∂_β satisfy (3.56), then:

$$\left[\partial_\alpha \, , \, \partial_\beta \right] W = 0, \tag{3.58}$$

as follows from the technical identity (2.22). Equations (3.56) and (3.58) can be compactly expressed by an identity

$$\left[\partial_\alpha - \sum_{a \in \{\alpha\}} \Phi_a D^{-1} \Psi_a \, , \, \partial_\beta - \sum_{b \in \{\beta\}} \Phi_b D^{-1} \Psi_b \right] = 0$$

[20, 19].

Define now $Y(\lambda, \mu) \equiv \psi_{BA}(t, \mu) \, D^{-1} \psi_{BA}^*(t, \lambda)$ (cf. ref.[16]) to be pseudo-differential operator inducing a ghost-flow $\partial_{(\lambda, \mu)} W \equiv Y(\lambda, \mu) W$ according to (3.55). In this case the "SEP" symmetry flow is generated by an infinite combination of $\mathbf{W}_{1+\infty}$ algebra generators [16]. Then, according to Eq. (3.56) the action of this flow on the BA wave function is given by:

$$\widehat{Y}(\lambda, \mu)\big(\psi_{BA}(t,z)\big) \equiv \partial_{(\lambda,\mu)}\big(\psi_{BA}(t,z)\big) = \psi_{BA}(t,\mu)\, S\big(\psi_{BA}(t,z),\psi_{BA}^*(t,\lambda)\big).$$
(3.59)

Further, let us also define the action of the vertex operator $\widehat{\mathcal{X}}(\lambda,\mu)$ on the BA function $\psi_{BA}(t,z)$ as generated by its action (as a vector field) on the ratio of τ-functions entering (2.10):

$$\widehat{\mathcal{X}}(\lambda,\mu)\,\psi_{BA}(t,z) = e^{\xi(t,z)}\frac{\tau(t)\widehat{\mathcal{X}}(\lambda,\mu)\,\tau(t-[z^{-1}]) - \tau(t-[z^{-1}])\widehat{\mathcal{X}}(\lambda,\mu)\,\tau(t)}{\tau^2(t)}.$$
(3.60)

The latter, upon using the shift-difference operator (3.11), can be written as

$$\widehat{\mathcal{X}}(\lambda,\mu)\,\psi_{BA}(t,z) = \psi_{BA}(t,z)\,\widehat{\Delta}_z\frac{\widehat{\mathcal{X}}(\lambda,\mu)\,\tau(t)}{\tau(t)}.$$
(3.61)

Let us stress that, according to (3.59)–(3.61), $\widehat{Y}(\lambda,\mu)$ acts on the BA function as a standard pseudo-differential operator, whereas $\widehat{\mathcal{X}}(\lambda,\mu)$ acts on it as a shift-difference operator.

Now, the above results allow us to give a simple straightforward proof of the following version of the Adler-Shiota-van-Moerbeke proposition [15, 16]. It provides the connection between the form of the *non-isospectral ("additional") symmetries* of KP hierarchies acting on the Lax operators and BA functions [9, 10], on one hand, and their respective form when acting on KP τ-functions, on the other hand.

Corollary 3.2. *With definitions (3.59) and (3.60) it holds:*

$$\widehat{\mathcal{X}}(\lambda,\mu)\,\psi_{BA}(t,z) = \widehat{Y}(\lambda,\mu)\big(\psi_{BA}(t,z)\big).$$
(3.62)

Proof. Indeed, applying (3.7) and Lemma 3.1 to the r.h.s. of (3.61), the latter equation can be rewritten as:

$$\widehat{\mathcal{X}}(\lambda,\mu)\,\psi_{BA}(t,z) = \frac{1}{z}\psi_{BA}(t,z)\,\psi_{BA}(t,\mu)\,\psi_{BA}^*(t-[z^{-1}],\lambda) = \widehat{Y}(\lambda,\mu)\,(\psi_{BA}(t,z)),$$
(3.63)

where in order to arrive at the last equality use was made of (3.32). $\square$

In the literature one often comes across the vertex operator defined as $\widehat{X}(\lambda,\mu) \equiv\; :\exp\big(\hat{\theta}(\lambda) - \hat{\theta}(\mu)\big): \;= (\lambda - \mu)\widehat{\mathcal{X}}(\lambda,\mu)$. In such a notation the expression (3.62) becomes $\widehat{X}(\lambda,\mu) = (\lambda-\mu)\widehat{Y}(\lambda,\mu)$ as in [15, 16].

We conclude this section by proving the following important property of SEP:

Lemma 3.3. *Under shift of the KP times, the squared eigenfunction potential obeys:*

$$S\big(\Phi(t-[\lambda^{-1}]),\Psi(t-[\lambda^{-1}])\big) - S\big(\Phi(t),\Psi(t)\big) = -\frac{1}{\lambda}\Phi(t)\Psi(t-[\lambda^{-1}]), \quad (3.64)$$

$$S\big(\Phi(t+[\lambda^{-1}]),\Psi(t+[\lambda^{-1}])\big) - S\big(\Phi(t),\Psi(t)\big) = \frac{1}{\lambda}\Phi(t+[\lambda^{-1}])\Psi(t). \quad (3.65)$$

Proof. According to (3.35) and (3.31):

$$\widehat{\Delta}_z S\left(\Phi(t), \Psi(t)\right) = \int\int d\lambda \, d\mu \, \phi^*(\lambda)\phi(\mu)\widehat{\Delta}_z S\left(\psi_{BA}(t,\lambda), \psi_{BA}^*(t,\mu)\right), \qquad (3.66)$$

while from Eq. (3.10) we find that:

$$\widehat{\Delta}_z S\left(\psi_{BA}(t,\lambda), \psi_{BA}^*(t,\mu)\right) = -\frac{1}{z}\psi_{BA}(t,\lambda)\psi_{BA}^*(t-[z^{-1}],\mu). \qquad (3.67)$$

Inserting the above identity back in (3.66) gives (3.64). □

After expanding identities (3.64) and (3.65) in power series w.r.t. λ we obtain:

$$p_s(-[\partial])S\left(\Phi(t), \Psi(t)\right) = -\Phi(t)\, p_{s-1}(-[\partial])\Psi(t),$$

$$p_s([\partial])S\left(\Phi(t), \Psi(t)\right) = \Psi(t)\, p_{s-1}([\partial])\Phi(t), \qquad s = 1, 2, \ldots, \qquad (3.68)$$

where $p_s(\cdot)$ are the standard Schur polynomials (2.13).

4. Constraints on $\mathsf{cKP}_{r,m}$ Tau-Functions. Grassmannian Interpretation

From now on we concentrate on studying the class of constrained $\mathsf{cKP}_{r,m}$ hierarchies for which we have:

$$\left(L_{r,m}\right)_- = \sum_{a=1}^{m} \Phi_a D^{-1} \Psi_a \qquad (4.1)$$

according to Eq. (2.20). We first note that the $\mathsf{cKP}_{r,m}$ BA function satisfies, according to (4.1), the following spectral equation:

$$L_{r,m}\psi_{BA}(t,\lambda) = \lambda^r \psi_{BA}(t,\lambda) = (L_{r,m})_+\psi_{BA}(t,\lambda) + \sum_{a=1}^{m} \Phi_a(t)\, S\left(\psi_{BA}(t,\lambda), \Psi_a(t)\right). \qquad (4.2)$$

Due to Eq. (3.34), the latter can be cast in the following form:

$$\lambda^r \psi_{BA}(t,\lambda) = (L_{r,m})_+\psi_{BA}(t,\lambda) + \sum_{a=1}^{m} \frac{1}{\lambda}\Phi_a(t)\Psi_a\left(t - [\lambda^{-1}]\right)\psi_{BA}(t,\lambda)$$

$$= (L_{r,m})_+\psi_{BA}(t,\lambda)$$

$$- \sum_{a=1}^{m}\left[S\left(\Phi_a(t-[\lambda^{-1}]), \Psi_a(t-[\lambda^{-1}])\right) - S\left(\Phi_a(t), \Psi_a(t)\right)\right]\psi_{BA}(t,\lambda), \quad (4.3)$$

where the second equality in (4.3) follows from (3.64). Recalling relation (3.53) we find that

$$\partial\tau(t)/\partial t_r = \sum_{a=1}^{m} S\left(\Phi_a(t), \Psi_a(t)\right)\tau(t). \qquad (4.4)$$

Similarly, using the spectral identity $L_{r,m}^n\psi_{BA}(t,\lambda) = \lambda^{rn}\psi_{BA}(t,\lambda)$ and taking into account relation (2.21) we obtain the following set of differential equations for the $\mathsf{cKP}_{r,m}$ τ-function:

$$\frac{\partial}{\partial t_{rn}} \tau(t) = \left[\sum_{a=1}^{m} \sum_{i=0}^{n-1} S\left(L^{n-1-i}(\Phi_a), L^{*i}(\Psi_a) \right) \right] \tau(t). \tag{4.5}$$

The point we want to stress is that the constraint (4.4) (or, the equivalent relation (4.3)) contains all the information about the cKP system in addition to the regular Hirota bilinear expression for the KP τ-function. This constraint can be given a bilinear form as in [32]. Consider, namely, the expression:

$$\int d\lambda \lambda^r \psi_{BA}(t, \lambda) \psi_{BA}^*(t', \lambda). \tag{4.6}$$

Using (4.3) and Hirota's equation one gets for (4.6):

$$\sum_{a=1}^{m} \Phi_a(t) \int d\lambda \psi_{BA}^*(t', \lambda) \frac{1}{\lambda} \Psi_a(t - [\lambda^{-1}]) \psi_{BA}(t, \lambda) = \sum_{a=1}^{m} \Phi_a(t) \Psi(t') \tag{4.7}$$

where we also used the spectral representation (3.18) with t and t' interchanged. Hence we proved that the constraint (4.4) implies:

$$\sum_{a=1}^{m} \Phi_a(t) \Psi(t') = \int d\lambda \lambda^r \psi_{BA}(t, \lambda) \psi_{BA}^*(t', \lambda) \tag{4.8}$$

which is the well-known bilinear expression for the cKP hierarchy [32] derived here from the simple fundamental τ-function constraint (4.4).

Using the differential Fay identity (3.12), Eqs. (4.5) can be equivalently written in the form:

$$\left\{ \frac{\partial}{\partial t_{rn}} - \left[\frac{\partial}{\partial t_{rn}} , \int \int d\lambda \, d\mu \, (\lambda^r - \mu^r)^{-1} \sum_{a=1}^{m} \varphi_a^*(\lambda) \varphi_a(\mu) \widehat{\mathcal{X}}(\lambda, \mu) \right] \right\} \tau(t) = 0 \tag{4.9}$$

where $\varphi_a^{(*)}(\lambda)$ are the "spectral densities" of the (adjoint) eigenfunctions $\Phi_a(t)$, $\Psi_a(t)$ entering the pseudo-differential part of the $\mathsf{cKP}_{r,m}$ Lax operator (2.20), and also we have used the identity:

$$\left[\frac{\partial}{\partial t_l} , \widehat{\mathcal{X}}(\lambda, \mu) \right] = \left(\mu^l - \lambda^l \right) \widehat{\mathcal{X}}(\lambda, \mu). \tag{4.10}$$

Thus we arrive at the following statement providing an alternative definition of $\mathsf{cKP}_{r,m}$ hierarchies intrinsically in terms of τ-functions:

Proposition 4.1. *Reduction of the full KP hierarchy (2.2) to the* $\mathsf{cKP}_{r,m}$ *hierarchy in terms of Lax operators (2.20) is equivalent to imposing Eqs. (4.9) as constraints on the pertinent τ-functions, where $\varphi_a^{(*)}(\lambda)$ are "spectral densities" of KP (adjoint) eigenfunctions given as in Eqs. (3.16).*

Let us now translate Eq. (4.9) into the language of universal Sato Grassmannian $\mathcal{G}r$ [3, 34]. Consider the hyperplane $\mathcal{W} \in \mathcal{G}r$ defined through a linear basis of Laurent series $\{f_k(\lambda)\}$ in λ in terms of the BA function as generating function $F(t, \lambda)$:

$$\mathcal{W} \equiv \mathrm{span}\langle f_1(\lambda), f_2(\lambda), \ldots \rangle,$$

$$f_k(\lambda) = \frac{\partial^k}{\partial x^k} F(t, \lambda) \bigg|_{x=t_2=t_3=\ldots=0} , \quad F(t, \lambda) = \psi_{BA}(t, \lambda). \tag{4.11}$$

In case of the standard r^{th} KdV reduction, where the corresponding Lax operator $\mathcal{L} = D + \sum_1^\infty u_i D^{-i}$ satisfies $\mathcal{L}^r = \mathcal{L}_+^r$, the latter constraint translates to the Grassmannian language as $\lambda^r \mathcal{W} \subset \mathcal{W}$ [4].

Our aim now is to express the $\mathsf{cKP}_{r,m}$ constraint (4.1) (cf. (2.20)) in the Grassmannian setting. We find from (4.3) that the generating function $F'(t, \lambda)$:

$$F'(t, \lambda) \equiv \left[\lambda^r + \sum_{a=1}^m \sum_{n=1}^\infty \frac{p_n(-[\partial]) S\left(\Phi_a(t), \Psi_a(t)\right)}{\lambda^n} \right] \psi_{BA}(t, \lambda) = (L_{r,m})_+ \psi_{BA}(t, \lambda) \tag{4.12}$$

defines via (4.11) a point $\mathcal{W}'$ of Sato Grassmannian $\mathcal{G}r$:

$$\mathcal{W}' = \text{span}\langle F'(0, \lambda), \partial_x F'(0, \lambda), \partial_x^2 F'(0, \lambda), \ldots \rangle \tag{4.13}$$

which coincides, because of the second equality in (4.12), with the original point $\mathcal{W}$ defined through $F(t, \lambda) = \psi_{BA}(t, \lambda)$ (4.11). Thus, we have[4]:

Proposition 4.2. *Let $S\left(\Phi_a(t), \Psi_a(t)\right)$, $a = 1, \ldots, \mathrm{m}$, be* m *squared eigenfunction potentials (2.18), where Φ_a, Ψ_a are (adjoint-)eigenfunctions of the general KP hierarchy (2.2). Then, the reduction of (2.2) to the $\mathsf{cKP}_{r,m}$ hierarchy (2.20) can be equivalently expressed as a restriction of $\mathcal{G}r$ to a subset whose points (hyperplanes) $\mathcal{W}$ (4.11) are subject to the following constraint:*

$$\left[\lambda^r + \sum_{a=1}^m \widehat{\Delta}_\lambda \, S\left(\Phi_a(t), \Psi_a(t)\right) \right] \mathcal{W} \subset \mathcal{W} \tag{4.14}$$

with $\widehat{\Delta}_\lambda$ as in (3.11) and $S\left(\Phi_a(t), \Psi_a(t)\right)$ being given by (3.35) in terms of the generating function (4.11) of $\mathcal{W}$.

5. Non-Isospectral Virasoro Symmetry for $\mathsf{cKP}_{r,1}$ τ-Functions

The conventional formulation of additional non-isospectral symmetries for the full KP integrable hierarchy [9, 17] is not compatible with the reduction of the latter to the important class of constrained $\mathsf{cKP}_{r,m}$ integrable models. In refs.[29, 8] we solved explicitly the problem of compatibility of the Virasoro part of non-isospectral symmetries with the underlying constraints of $\mathsf{cKP}_{r,m}$ hierarchies within the pseudo-differential Lax operator framework. Our construction in [29, 8] involves an appropriate modification of the standard non-isospectral symmetry flows, acting on the space of $\mathsf{cKP}_{r,m}$ Lax operators, by adding a set of additional "ghost symmetry" flows (of the type appearing in Eq. (3.55)). In this section, we derive the explicit form of the action of the correct modified Virasoro non-isospectral symmetries as flows on the space of $\mathsf{cKP}_{r,m}$ τ-functions. Note that the corresponding result for the full unconstrained KP hierarchy has been previously obtained in [15, 16, 17].

To this end, let us first recall that the standard *additional (non-isospectral) symmetries* [9, 17] are defined as vector fields on the space of general KP Lax operators (2.2) or, alternatively, on the dressing operators (2.4), through their flows as follows:

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[4] For a different criteria characterizing $\mathsf{cKP}_{r,m}$ hierarchies within the Sato Grassmannian framework, see refs.[35, 36].

$$\bar{\partial}_{k,n}L = -\Big[\big(M^n L^k\big)_- \,,\, L\Big]$$
$$= \Big[\big(M^n L^k\big)_+ \,,\, L\Big] + nM^{n-1}L^k; \quad \bar{\partial}_{k,n}W = -\big(M^n L^k\big)_- W. \quad (5.1)$$

Here M is a pseudo-differential operator "canonically conjugated" to L such that:

$$\Big[L\,,\, M\Big] = \mathbb{1}, \quad \frac{\partial}{\partial t_l}M = \Big[L_+^{1/r}\,,\, M\Big]. \quad (5.2)$$

Within the Sato-Wilson dressing operator formalism, the M-operator can be expressed in terms of dressing of the "bare" $M^{(0)}$ operator:

$$M^{(0)} = \sum_{l \geq 1}\frac{l}{r}t_l D^{l-r} = X_{(r)} + \sum_{l \geq 1}\frac{l+r}{r}t_{r+l}D^l; \quad X_{(r)} \equiv \sum_{l=1}^{r}\frac{l}{r}t_l D^{l-r} \quad (5.3)$$

conjugated to the "bare" Lax operator $L^{(0)} = D^r$.

    The additional symmetry flows (5.1) commute with the usual KP hierarchy isospectral flows given in (2.3). However, they do not commute among themselves, instead they form a centerless $\mathbf{W}_{1+\infty}$ algebra (see e.g. [17]). One finds that the Lie algebra of operators $\bar{\partial}_{k,n}$ is isomorphic to the Lie algebra generated by $-z^k(\partial/\partial z)^n$. Especially for $n = 1$ this becomes an isomorphism to the centerless Virasoro algebra $\bar{\partial}_{k,1} \sim -\mathcal{L}_{k-1}$, with $[\mathcal{L}_l\,,\, \mathcal{L}_k] = (l-k)\mathcal{L}_{l+k}$.

    As demonstrated in [29, 8], the conventional non-isospectral flows (5.1) do *not* preserve the space of cKP$_{r,m}$ Lax operators given by (2.20). In particular, for the Virasoro non-isospectral symmetry algebra the transformed Lax operator $\bar{\partial}_{k,1}L$ belongs to a *different* class of constrained KP hierarchies – cKP$_{r,m(k-1)}$ (when $k \geq 3$). The solution to this problem is provided by the following [29, 8]:

Proposition 5.1. *The correct non-isospectral symmetry flows for the* cKP$_{r,m}$ *hierarchies (2.20), spanning the Virasoro algebra, are given by:*

$$\partial_k^* L \equiv \Big[-\big(ML^k\big)_- + X_{k-1}^{(1)}\,,\, L\Big], \quad (5.4)$$

i.e., with the following isomorphism $\mathcal{L}_{k-1} \sim -\big(ML^k\big)_- + X_{k-1}^{(1)}$, *where* $X_{k-1}^{(1)}$ *are ghost-symmetry generating operators (cf. (3.55)) defined as:*

$$X_k^{(1)} = \sum_{i=1}^{m}\sum_{j=0}^{k-1}\left(j - \frac{1}{2}(k-1)\right)L^{k-1-j}(\Phi_i)D^{-1}\big(L^*\big)^j(\Psi_i); \quad k \geq 1. \quad (5.5)$$

    Since (auto-)Darboux-Bäcklund transformations of cKP$_{r,m}$ hierarchies (see next section) play a fundamental rôle for finding exact solutions, as well as in establishing the link between cKP$_{r,m}$ integrable models and (multi-)matrix models, it is natural to impose the additional condition of commutativity of the non-isospectral symmetries with the Darboux-Bäcklund transformations. The latter condition was shown in refs.[29, 8] to be satisfied only by the subclass cKP$_{r,1}$ of constrained KP hierarchies (it is precisely cKP$_{r,1}$ hierarchies which provide the integrability structure of discrete multi-matrix models [8]). Therefore, in the rest of this section we restrict our attention to cKP$_{r,1}$ models.

Consider the modified non-isospectral Virasoro symmetry flows (5.4) acting on the dressing operator of the $\mathsf{cKP}_{r,1}$ hierarchy:

$$\partial_k^* W = - \left(ML^k\right)_- W + X_{k-1}^{(1)} W. \tag{5.6}$$

Taking the operator residuum on both parts of (5.6) we obtain:

$$\partial_k^* \tau(t) = \frac{1}{2r}\widehat{W}_{r(k-1)}^{(2)}\tau(t) + \left[\sum_{j=0}^{k-2}\left(\frac{1}{2}(k-2)-j\right)S\left(L^{k-2-j}(\Phi), L^{*j}(\Psi)\right)\right]\tau(t). \tag{5.7}$$

In deriving Eq. (5.7) we used the expression for $X_{k-1}^{(1)}$ (5.5) together with the differential Fay identity (3.12) as well as:

$$\begin{aligned}
\mathrm{Res}\left(M^l L^k\right) &= Res_\lambda\left[\left(M^l L^k\right)\left(\psi_{BA}(t,\lambda)\right)\psi_{BA}^*(t,\lambda)\right]\\
&= \frac{1}{r^l}Res_\lambda\left(\lambda^{kr-l(r-1)}\frac{\partial^l}{\partial\lambda^l}\psi_{BA}(t,\lambda)\,\psi_{BA}^*(t,\lambda)\right)\\
&= -\partial_x\left[\frac{1}{\tau(t)}Res_\lambda\left(\frac{1}{r^l}\mu^{kr-l(r-1)}\frac{\partial^l}{\partial\mu^l}\widehat{\mathcal{X}}(\lambda,\mu)\mid_{\mu=\lambda}\right)\tau(t)\right]\\
&= \partial_x\left(\frac{1}{r^l(l+1)}\frac{\widehat{W}_{(k-l)r}^{(l+1)}\tau(t)}{\tau(t)}\right).
\end{aligned} \tag{5.8}$$

In the chain of the identities in (5.8) we took into account Dickey's formula for $\left(M^l L^k\right)_-$ [16] (first equality in (5.8)), Eq. (3.13) (third equality in (5.8)), and formula (3.5) for $\widehat{\mathcal{X}}(\lambda,\mu)$ to arrive at the last equality above. The Virasoro operator in the first term on the r.h.s. of (5.7) comes from the standard Orlov-Schulman non-isospectral symmetry flow and reads explicitly (for $k \geq -1$):

$$\widehat{W}_k^{(2)} = 2\sum_{l\geq 1}lt_l\frac{\partial}{\partial t_{l+k}} - (k+1)\frac{\partial}{\partial t_k} + \sum_{l=1}^{k-1}\frac{\partial^2}{\partial t_l \partial t_{k-l}}. \tag{5.9}$$

We now express the second additional "ghost-flow" term on the r.h.s. of (5.7) as a differential operator acting on $\tau(t)$ of a form similar to (5.9). The starting point are the differential Eqs. (4.5) obeyed by the $\mathsf{cKP}_{r,1}$ τ-function, wherefrom we get for the second-order derivatives:

$$\begin{aligned}
\frac{1}{\tau(t)}\frac{\partial^2\tau(t)}{\partial t_{rl}\partial t_{rn}} &= \sum_{i=0}^{n-1}\left[S\left(L^{n+l-1-i}(\Phi), L^{*i}(\Psi)\right) - S\left(L^{n-1-i}(\Phi), L^{*i+l}(\Psi)\right)\right]\\
&+ \sum_{i=0}^{n-1}\sum_{j=0}^{l-1}\left[S\left(L^{n-1-i}(\Phi), L^{*i}(\Psi)\right)S\left(L^{l-1-j}(\Phi), L^{*j}(\Psi)\right)\right.\\
&\left.- S\left(L^{n-1-i}(\Phi), L^{*j}(\Psi)\right)S\left(L^{l-1-j}(\Phi), L^{*i}(\Psi)\right)\right]. \tag{5.10}
\end{aligned}$$

In obtaining relation (5.10) we made use of the following lemma:

Lemma 5.1. *The relation:*

$$\frac{\partial}{\partial t_{nr}} S(f,g) = S\left(L^n(f),g\right) - S\left(f,L^{*n}(g)\right)$$

$$- \sum_{i=0}^{n-1} S\left(L^{n-1-i}(\Phi),g\right) S\left(f,L^{*i}(\Psi)\right) \qquad (5.11)$$

holds for f an eigenfunction and g an adjoint eigenfunction of the Lax operator $L \equiv L_{r,1} = L_+ + \Phi D^{-1}\Psi$ belonging to the $\mathsf{cKP}_{r,1}$ hierarchy.

Proof. We are going to show that $\partial/\partial t_{nr}\, S(f,g) = \mathrm{Res}\left(D^{-1}g(L^n)_+ f D^{-1}\right)$ is equal to the right-hand side of Eq. (5.11) (up to a constant). We first apply $\partial/\partial t_{mr}$ on the left hand side of Eq. (5.11). This yields

$$\frac{\partial}{\partial t_{mr}}\frac{\partial}{\partial t_{nr}} S(f,g) = \frac{\partial}{\partial t_{nr}}\frac{\partial}{\partial t_{mr}} S(f,g) = -\mathrm{Res}\left(D^{-1}(L^*)^n_+(g) L^m f D^{-1}\right)$$

$$+ \mathrm{Res}\left(D^{-1}g L^m (L)^n_+(f) D^{-1}\right). \qquad (5.12)$$

After making the substitutions:

$$(L)^n_+(f) = L^n(f) - \sum_{i=0}^{n-1} L^{n-1-i}(\Phi) S\left(f,L^{*i}(\Psi)\right), \qquad (5.13)$$

$$(L^*)^n_+(g) = L^{*n}(g) + \sum_{i=0}^{n-1} L^{*i}(\Psi) S\left(L^{n-1-i}(\Phi),g\right), \qquad (5.14)$$

where use was made of (2.21), we obtain agreement with the result of applying $\partial/\partial t_{mr}$ on the right-hand side of Eq. (5.11) and using Eq. (2.18) as well as Lemma 2.1 $\quad\square$

Using Eqs. (4.5),(5.10) we obtain:

$$\sum_{j=0}^{k-2}\left(\frac{1}{2}(k-2)-j\right) S\left(L^{k-2-j}(\Phi),L^{*j}(\Psi)\right) = \frac{1}{2\tau(t)}\sum_{l=1}^{k-2}\frac{\partial^2\tau}{\partial t_{rl}\partial t_{r(k-1-l)}}. \qquad (5.15)$$

Collecting (5.9) and (5.15), the final form of the $\mathsf{cKP}_{r,1}$ non-isospectral Virasoro symmetry flows reads:

$$\partial^*_k \tau(t) = \left[\frac{1}{r}\sum_{l\geq 1} l t_l \frac{\partial}{\partial t_{l+r(k-1)}} - \frac{r(k-1)+1}{2r}\frac{\partial}{\partial t_{r(k-1)}}\right.$$

$$\left. + \frac{1}{2r}\sum_{l=1}^{r(k-1)-1}\frac{\partial^2}{\partial t_l \partial t_{r(k-1)-l}} + \frac{1}{2}\sum_{l=1}^{k-2}\frac{\partial^2}{\partial t_{rl}\partial t_{r(k-1-l)}}\right]\tau(t). \qquad (5.16)$$

In particular, for $\mathsf{cKP}_{1,1}$ hierarchies the Virasoro non-isospectral symmetry takes the form (Eq. (5.16) for $r = 1$):

$$\partial^*_k \tau(t) = \left[\sum_{l\geq 1} l t_l \frac{\partial}{\partial t_{l+k-1}} - \frac{k}{2}\frac{\partial}{\partial t_{k-1}} + \sum_{l=1}^{k-2}\frac{\partial^2}{\partial t_l \partial t_{k-1-l}}\right]\tau(t). \qquad (5.17)$$

Concluding this section it is instructive to point out the relation of (5.17) with the so-called Virasoro constraints in conventional discrete matrix models [37, 7] spanning the Borel subalgebra of the Virasoro algebra:

$$\mathcal{L}_s^{(N)} Z_N = 0, \quad s \geq -1, \tag{5.18}$$

$$\mathcal{L}_s^{(N)} = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k+s}} + 2N \frac{\partial}{\partial t_s} + \sum_{k=1}^{s-1} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{s-k}}, \quad s \geq 1, \tag{5.19}$$

$$\mathcal{L}_0^{(N)} = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + N^2; \quad \mathcal{L}_{-1}^{(N)} = \sum_{k=2}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}} + N t_1. \tag{5.20}$$

Here Z_N denotes the one-matrix model partition function with N indicating the size of the corresponding (Hermitian) random matrix. On one hand, it can be identified as a τ-function of the semi-infinite one-dimensional Toda lattice model [37]. On the other hand, from the point of view of continuum integrable systems it was shown in [38, 8] to be $Z_N = \tau^{(N,0)}$, *i.e.*, N^{th} member of the Darboux-Bäcklund orbit on the subspace of $\mathsf{cKP}_{1,1}$ τ-functions starting from the "free" initial $\tau^{(0,0)} = 1$ (see the next section for more details about Darboux-Bäcklund orbits on constrained KP hierarchies). Comparing (5.19)–(5.20) with (5.17) one finds:

$$\mathcal{L}_{-1}^{(N)} = \partial_0^* + N t_1; \quad \mathcal{L}_0^{(N)} = \partial_1^* + N^2,$$

$$\mathcal{L}_{k-1}^{(N)} = \partial_k^* + (2N + k/2)\partial/\partial t_{k-1}, \quad k \geq 2. \tag{5.21}$$

6. Binary Darboux-Bäcklund Orbits on $\mathsf{cKP}_{r,m}$ Hierarchies. Toda Square-Lattice Model

Let us recall the form of the Darboux-Bäcklund and adjoint-Darboux-Bäcklund transformations which preserve the constrained form of the $\mathsf{cKP}_{r,m}$ hierarchy Lax operator (2.20) [38, 8], *i.e.*, we shall discuss *auto*-Darboux-Bäcklund transformations for $\mathsf{cKP}_{r,m}$ hierarchies (for a general discussion of DB transformations of generic KP hierarchies without the requirement of preserving specific classes of constrained KP hierarchies, see refs.[39, 19]):

$$L \to \widetilde{L} = \widetilde{L}_+ + \sum_{i=1}^{m} \widetilde{\Phi}_i D^{-1} \widetilde{\Psi}_i = T_a L T_a^{-1}, \quad T_a = \Phi_a D \Phi_a^{-1}, \tag{6.1}$$

$$L \to \bar{L} = \bar{L}_+ + \sum_{i=1}^{m} \bar{\Phi}_i D^{-1} \bar{\Psi}_i = \bar{T}_b^{*-1} L \bar{T}_b^*, \quad \bar{T}_b = \Psi_b D \Psi_b^{-1}. \tag{6.2}$$

Here Φ_a, Ψ_b are (adjoint) eigenfunctions entering L_- (2.20) with fixed indices a, b which henceforth will be assumed $a \neq b$. Accordingly, for BA functions, the τ-function, eigenfunctions and their respective spectral "densities", the DB transformations (6.1) imply:

$$\widetilde{\Phi}_a = T_a L(\Phi_a), \quad \widetilde{\Psi}_a = \frac{1}{\Phi_a},$$

$$\widetilde{\Phi}_i = T_a(\Phi_i), \quad \widetilde{\Psi}_i = T_a^{*-1}(\Psi_i) = -\frac{1}{\Phi_a} S(\Phi_a, \Psi_i), \quad , i \neq a, \tag{6.3}$$

$$\widetilde{\psi}_{BA}(t,\lambda) = \frac{1}{\lambda}T_a(\psi_{BA}(t,\lambda)); \quad \widetilde{\tau}(t) = \Phi_a(t)\tau(t), \tag{6.4}$$

$$\widetilde{\psi}_{BA}^*(t,\lambda) = \lambda T_a^{*\,-1}(\psi_{BA}^*(t,\lambda)) = -\lambda\frac{1}{\Phi_a(t)}S(\Phi_a(t),\psi_{BA}^*(t,\lambda)), \tag{6.5}$$

$$\widetilde{\varphi}_a(\lambda) = \lambda^{1+r}\varphi_a(\lambda); \quad \widetilde{\varphi}_i(\lambda) = \lambda\varphi_i(\lambda), \quad \widetilde{\varphi}_i^*(\lambda) = \lambda^{-1}\varphi_i^*(\lambda), \quad i \neq a. \tag{6.6}$$

For the adjoint DB transformations (6.2) we have:

$$\bar{\Phi}_b = -\frac{1}{\Psi_b}, \quad \bar{\Psi}_b = \bar{T}_b L^*(\Psi_b),$$

$$\bar{\Phi}_i = \bar{T}_b^{*\,-1}(\Phi_i) = -\frac{1}{\Psi_b}S(\Phi_i,\Psi_b), \quad \bar{\Psi}_i = \bar{T}_b(\Psi_i), \quad i \neq b, \tag{6.7}$$

$$\bar{\psi}_{BA}(t,\lambda) = -\lambda\bar{T}_b^{*\,-1}(\psi_{BA}(t,\lambda)) = \lambda\frac{1}{\Psi_b(t)}S(\psi_{BA}(t,\lambda),\Psi_b(t)), \tag{6.8}$$

$$\bar{\psi}_{BA}^*(t,\lambda) = -\frac{1}{\lambda}\bar{T}_b(\psi_{BA}^*(t,\lambda); \quad \bar{\tau}(t) = \Psi_b(t)\tau(t), \tag{6.9}$$

$$\bar{\varphi}_b^*(\lambda) = -\lambda^{1+r}\varphi_b^*(\lambda); \quad \bar{\varphi}_i(\lambda) = -\frac{1}{\lambda}\varphi_i(\lambda), \quad \bar{\varphi}_i^*(\lambda) = -\lambda\varphi_i^*(\lambda), \quad i \neq b. \tag{6.10}$$

We shall use the double superscript (n,k) to indicate the iteration of n successive Darboux-Bäcklund transformations (6.3) plus k successive adjoint-Darboux-Bäcklund transformations (6.7). One can easily show that the result does not depend on the particular order these transformations are performed. Therefore, the set of all (n,k) DB transformations, called *generalized binary* DB transformations in what follows, defines a discrete symmetry structure on the space of $\mathsf{cKP}_{r,m}$ hierarchies corresponding to a two-dimensional lattice. Let us note that the $(1,1)$ binary DB transformations were previously introduced in ref.[30].

For one-step binary-DB transformed τ- and BA functions we get:

$$\tau^{(1,1)}(t) = -S\left(\Phi_a^{(0,0)}(t),\Psi_b^{(0,0)}(t)\right)\tau^{(0,0)}(t), \tag{6.11}$$

$$\psi_{BA}^{(1,1)}(t,\lambda) = \left[1 - \frac{1}{\lambda}\frac{\Phi_a^{(0,0)}(t)\Psi_b^{(0,0)}(t-[\lambda^{-1}])}{S\left(\Phi_a^{(0,0)}(t),\Psi_b^{(0,0)}(t)\right)}\right]\psi_{BA}^{(0,0)}(t,\lambda)$$

$$= \frac{S\left(\Phi_a^{(0,0)}(t-[\lambda^{-1}]),\Psi_b^{(0,0)}(t-[\lambda^{-1}])\right)}{S\left(\Phi_a^{(0,0)}(t),\Psi_b^{(0,0)}(t)\right)}\psi_{BA}^{(0,0)}(t,\lambda), \tag{6.12}$$

$$\psi_{BA}^{*\,(1,1)}(t,\lambda) = \left[1 + \frac{1}{\lambda}\frac{\Psi_b^{(0,0)}(t)\Phi_a^{(0,0)}(t+[\lambda^{-1}])}{S\left(\Phi_a^{(0,0)}(t),\Psi_b^{(0,0)}(t)\right)}\right]\psi_{BA}^{*\,(0,0)}(t,\lambda)$$

$$= \frac{S\left(\Phi_a^{(0,0)}(t+[\lambda^{-1}]),\Psi_b^{(0,0)}(t+[\lambda^{-1}])\right)}{S\left(\Phi_a^{(0,0)}(t),\Psi_b^{(0,0)}(t)\right)}\psi_{BA}^{*\,(0,0)}(t,\lambda), \tag{6.13}$$

where in the second equalities in (6.12) and (6.13) we used again (3.33)–(3.34) and (3.64)–(3.65). Combining Eq. (6.11) with Eq. (4.5) for $n = 1$ we find the following transformation formula for the squared eigenfunction potentials:

$$\sum_{i=1}^{m} S\left(\Phi_i^{(1,1)}, \Psi_i^{(1,1)}\right) - \sum_{i=1}^{m} S\left(\Phi_i^{(0,0)}, \Psi_i^{(0,0)}\right) = \frac{\partial}{\partial t_r} \ln\left(-S\left(\Phi_a^{(0,0)}(t), \Psi_b^{(0,0)}(t)\right)\right).$$
(6.14)

Let us recall again that here and below $a \neq b$ are *fixed* indices of Lax (adjoint) eigenfunctions.

Remark. Comparing formula (6.11) with (3.35) we see that one-step binary-DB transformations on τ-functions are nothing but the well-known Sato Bäcklund transformations [4] (i.e., those generated by the bilocal vertex operator $\widehat{\chi}(\lambda, \mu)$ (3.1)) "averaged" with appropriate spectral densities.

Introducing short-hand notations:

$$\chi_a^{(i)}(t) \equiv \left(L^{(0,0)}\right)^i (\Phi_a^{(0,0)}(t)) \quad , \quad \chi_b^{*(i)}(t) \equiv \left(L^{(0,0)}\right)^{*i}(\Psi_b^{(0,0)}(t))$$

$$S_{ab}^{(i,j)}(t) \equiv S\left(\left(L^{(0,0)}\right)^i (\Phi_a^{(0,0)}(t)), \left(L^{(0,0)}\right)^{*j}(\Psi_b^{(0,0)}(t))\right) \qquad (6.15)$$

we have:

Proposition 6.1. *The following determinant formulae hold for the n-step binary DB transformed quantities:*

$$\frac{\tau^{(n,n)}(t)}{\tau^{(0,0)}(t)} = (-1)^n \prod_{j=0}^{n-1} S\left(\Phi_a^{(j,j)}(t), \Psi_b^{(j,j)}(t)\right) = (-1)^n \det_n \left\| S_{ab}^{(i-1,j-1)}(t) \right\|, \quad (6.16)$$

$$\Phi_a^{(n,n)}(t) = (-1)^n \frac{\det_{n+1} \left\| \begin{matrix} S_{ab}^{(i-1,j-1)}(t) & \chi_a^{(i-1)}(t) \\ S_{ab}^{(n,j-1)}(t) & \chi_a^{(n)}(t) \end{matrix} \right\|}{\det_n \left\| S_{ab}^{(i-1,j-1)}(t) \right\|}, \quad (6.17)$$

$$\Psi_a^{(n,n)}(t) = (-1)^n \frac{\det_{n+1} \left\| \begin{matrix} S_{ab}^{(i-1,j-1)}(t) & S_{ab}^{(i-1,n)}(t) \\ \chi_b^{*(j-1)}(t) & \chi_b^{*(n)}(t) \end{matrix} \right\|}{\det_n \left\| S_{ab}^{(i-1,j-1)}(t) \right\|}, \quad (6.18)$$

$$S\left(\Phi_a^{(n,n)}(t), \Psi_b^{(n,n)}(t)\right) = \frac{\det_{n+1} \left\| S_{ab}^{(i-1,j-1)}(t) \right\|}{\det_n \left\| S_{ab}^{(i-1,j-1)}(t) \right\|}, \quad (6.19)$$

where $\chi_a^{(i)}, \chi_b^{*(i)}$ and $S_{ab}^{(i,j)}$ are defined in (6.15), and the matrix indices i, j run from 1 to n or $n+1$ according to the indicated sizes of the determinants.

Formulae (6.17)–(6.19) can be further generalized to:

$$\left(L^{(n,n)}\right)^s \left(\Phi_a^{(n,n)}(t)\right) = (-1)^n \frac{\det_{n+1} \left\| \begin{matrix} S_{ab}^{(i-1,j-1)}(t) & \chi_a^{(i-1)}(t) \\ S_{ab}^{(s+n,j-1)}(t) & \chi_a^{(s+n)}(t) \end{matrix} \right\|}{\det_n \left\| S_{ab}^{(i-1,j-1)}(t) \right\|}, \quad (6.20)$$

$$\left(L^{(n,n)*}\right)^s \left(\Psi_b^{(n,n)}(t)\right) = (-1)^n \frac{\det_{n+1}\left\|\begin{matrix} S_{ab}^{(i-1,j-1)}(t) & S_{ab}^{(i-1,s+n)}(t) \\ \chi_b^{*(j-1)}(t) & \chi_b^{*(s+n)}(t) \end{matrix}\right\|}{\det_n\left\|S_{ab}^{(i-1,j-1)}(t)\right\|}, \quad (6.21)$$

$$S\left(\left(L^{(n,n)}\right)^l(\Phi_a^{(n,n)}), \left(L^{(n,n)*}\right)^s(\Psi_b^{(n,n)})\right)$$

$$= \frac{\det_{n+1}\left\|\begin{matrix} S_{ab}^{(i-1,j-1)}(t) & S_{ab}^{(i-1,s+n)}(t) \\ S_{ab}^{(l+n,j-1)}(t) & S_{ab}^{(l+n,s+n)}(t) \end{matrix}\right\|}{\det_n\left\|S_{ab}^{(i-1,j-1)}(t)\right\|}, \quad (6.22)$$

We are now ready, with the help of (6.20)–(6.22), to write down the generalization of Eq. (6.16) for the cKP τ-function subject to an arbitrary (n,k) binary DB transformation:

Proposition 6.2. *The general discrete binary Darboux-Bäcklund orbit on the space of* cKP$_{r,m}$ τ-functions, generated by a fixed pair of (adjoint) eigenfunctions Φ_a, Ψ_b and *starting from arbitrary "initial"* $\tau^{(0,0)}$, *consists of the following elements* $\tau^{(n,k)}$:

$$\frac{\tau^{(n,k)}}{\tau^{(0,0)}} = \left((-1)^k \det_k\left\|S_{ab}^{(i-1,j-1)}\right\|\right)^{-(n-k-1)} \times$$

$$W_{n-k}\left[\det_{k+1}\left\|\begin{matrix} S_{ab}^{(i-1,j-1)} & \chi_a^{(i-1)} \\ S_{ab}^{(k,j-1)} & \chi_a^{(k)} \end{matrix}\right\|, \ldots, \det_{k+1}\left\|\begin{matrix} S_{ab}^{(i-1,j-1)} & \chi_a^{(i-1)} \\ S_{ab}^{(n-1,j-1)} & \chi_a^{(n-1)} \end{matrix}\right\|\right], \quad (6.23)$$

$$\text{for } n \geq k \quad , \quad i,j = 1,\ldots,k$$

$$\frac{\tau^{(n,k)}(t)}{\tau^{(0,0)}(t)} = \left((-1)^n \det_n\left\|S_{ab}^{(i-1,j-1)}\right\|\right)^{-(k-n-1)} \times$$

$$W_{k-n}\left[\det_{n+1}\left\|\begin{matrix} S_{ab}^{(i-1,j-1)} & S_{ab}^{(i-1,n)} \\ \chi_b^{*(j-1)} & \chi_b^{*(n)} \end{matrix}\right\|, \ldots, \det_{n+1}\left\|\begin{matrix} S_{ab}^{(i-1,j-1)} & S_{ab}^{(i-1,k-1)} \\ \chi_b^{*(j-1)} & \chi_b^{*(k-1)} \end{matrix}\right\|\right], \quad (6.24)$$

$$\text{for } n \leq k \quad , \quad i,j = 1,\ldots,n$$

where $\chi_a^{(i)}, \chi_b^{*(i)}$ *and* $S_{ab}^{(i,j)}$ *are as in (6.15),*
and $W_k[f_1,\ldots,f_k] = \det\left\|\partial^{\alpha-1}f_\beta\right\|_{\alpha,\beta=1,\ldots,k}$ *indicates the Wronskian determinant of a set of functions* $\{f_1,\ldots,f_k\}$.

Remark. Note that the entries in the Wronskians in Eqs. (6.23)–(6.24) are determinants themselves.

Remark. In the Appendix we write down the explicit expressions for the cKP$_{r,m}$ τ-functions on the most general discrete binary Darboux-Bäcklund orbit generated via successive (adjoint) DB transformations (6.3)–(6.10) w.r.t. an arbitrary set of (adjoint) eigenfunctions.

In the simple case of a "free" initial system with $L^{(0,0)} = D$ which, accordingly, is characterized by:

$$\tau^{(0,0)}(t) = 1 \quad ; \quad \frac{\partial}{\partial t_n}\chi_a^{(i)} = \partial_x^n \chi_a^{(i)} \quad , \quad \frac{\partial}{\partial t_n}\chi_b^{*(j)} = -(-\partial_x)^n \chi_b^{*(j)}, \quad (6.25)$$

formula (6.16) reproduces the Fredholm determinant expression $\tau = \det \left\| \delta_{ij} + a_{ij} \right\|$ for the τ-function with $a_{ij} \equiv \int_{-\infty}^{x} \chi_a^{(i)} \chi_b^{*(j)} dy$ [40, 41]. Namely, it follows that:

$$\frac{\partial}{\partial t_n} \left(\delta_{ij} + a_{ij} \right) = \sum_{l=1}^{n} \partial_x^{l-1} \chi_a^{(i)} (-\partial_x)^{n-l} \chi_b^{*(j)} = \mathrm{Res} \left(D^{-1} \chi_b^{*(j)} D^n \chi_a^{(i)} D^{-1} \right). \quad (6.26)$$

The latter allows us to identify $\delta_{ij} + a_{ij}$ with $S\left(\chi_a^{(i)}, \chi_b^{*(j)} \right)$ and establishes connection between the above special case of (6.16) and the Fredholm determinant expressions for the τ-functions of refs.[40, 41].

Now, we shall show that the (n,k) binary Darboux-Bäcklund orbit of the $\mathsf{cKP}_{r,2}$ hierarchy defines a two-dimensional Toda square-lattice system which describes two coupled ordinary two-dimensional Toda-lattice models corresponding to the horizontal $(n,0)$ and the vertical $(0,k)$ one-dimensional sublattices of the (n,k) binary DB square-lattice. Namely, consider:

$$\partial_x \frac{\partial}{\partial t_r} \ln \tau^{(n,k)} = \mathrm{Res} L^{(n,k)} = \Phi_1^{(n,k)} \Psi_1^{(n,k)} + \Phi_2^{(n,k)} \Psi_2^{(n,k)} = \frac{\Phi_1^{(n,k)}}{\Phi_1^{(n-1,k)}} - \frac{\Psi_2^{(n,k)}}{\Psi_2^{(n,k-1)}},$$
$$(6.27)$$

where we used (6.3) and (6.7). Taking into account the expressions for the (adjoint-)DB transformed τ-functions (6.4) and (6.9), *i.e.*

$$\Phi_1^{(n,k)} = \frac{\tau^{(n+1,k)}}{\tau^{(n,k)}} \quad , \quad \Psi_2^{(n,k)} = \frac{\tau^{(n,k+1)}}{\tau^{(n,k)}}, \quad (6.28)$$

Eq. (6.27) can be rewritten in the form:

$$\partial_x \frac{\partial}{\partial t_r} \ln \tau^{(n,k)} = \frac{\tau^{(n+1,k)} \tau^{(n-1,k)} - \tau^{(n,k+1)} \tau^{(n,k-1)}}{\left(\tau^{(n,k)} \right)^2}, \quad (6.29)$$

or, equivalently:

$$\partial_x \frac{\partial}{\partial t_r} \tau^{(n,k)} - \partial_x \tau^{(n,k)} \frac{\partial}{\partial t_r} \tau^{(n,k)} = \tau^{(n+1,k)} \tau^{(n-1,k)} - \tau^{(n,k+1)} \tau^{(n,k-1)}. \quad (6.30)$$

Similarly, Eq. (6.27) can be represented as a system of coupled equations of motion for $\Phi_1^{(n,k)}$ and $\Psi_2^{(n,k)}$ using again (6.28):

$$\partial_x \frac{\partial}{\partial t_r} \ln \Phi_1^{(n,k)} = \left(\frac{\Phi_1^{(n+1,k)}}{\Phi_1^{(n,k)}} - \frac{\Phi_1^{(n,k)}}{\Phi_1^{(n-1,k)}} \right) - \left(\frac{\Psi_2^{(n+1,k)}}{\Psi_2^{(n+1,k-1)}} - \frac{\Psi_2^{(n,k)}}{\Psi_2^{(n,k-1)}} \right), \quad (6.31)$$

$$\partial_x \frac{\partial}{\partial t_r} \ln \Psi_2^{(n,k)} = - \left(\frac{\Psi_2^{(n,k+1)}}{\Psi_2^{(n,k)}} - \frac{\Psi_2^{(n,k)}}{\Psi_2^{(n,k-1)}} \right) + \left(\frac{\Phi_1^{(n,k+1)}}{\Phi_1^{(n-1,k+1)}} - \frac{\Phi_1^{(n,k)}}{\Phi_1^{(n-1,k)}} \right). \quad (6.32)$$

When $\Psi_2^{(n,k)}$ vanishes the remaining equations for $\Phi_1^{(n,k)}$ reduce for a fixed k to the equations of motion for the well-known Toda model on one-dimensional lattice w.r.t. n (and *vice versa* if $\Phi_1^{(n,k)} = 0$).

7. Discussion and Outlook. Relation to Random Matrix Models

In this paper we provided a new version of the eigenfunction formulation of the KP hierarchy, called the *squared eigenfunction potential (SEP) method*, where the SEP plays a rôle of a basic building block. The principal ingredient of the SEP method is the proof of existence of spectral representation for *any* KP eigenfunction as a spectral integral over the (adjoint) BA function with spectral density explicitly given in terms of a SEP. It was pointed out that the spectral representations of the (adjoint) BA functions themselves (being particular examples of KP eigenfunctions) can, in turn, serve as defining relations for the whole KP hierarchy parallel to Hirota fundamental bilinear identity or Fay identity.

The SEP method was subsequently employed to solve various issues in integrable hierarchies of KP type both of conceptual, as well as more applied character. Many, previously unrelated, recent developments in the theory of the τ-function of the KP hierarchy gained from being described by the present formalism. As one of the important illustrations of how our method works, we have shown how the SEP, acting on the manifold of wave functions $\psi_{BA}(t, \lambda)$ by generating non-isospectral symmetry algebra, lifts to a vertex operator acting on τ-functions. This reproduced in the SEP setting the results of [14, 15, 16, 17].

We have also employed the SEP construction in the context of Hamiltonian reductions of KP hierarchy providing:

– description of the reductions of the general KP hierarchy to the constrained $\mathsf{cKP}_{r,m}$ hierarchies entirely in terms of linear constraint equations on the pertinent τ-functions;
– description of constrained $\mathsf{cKP}_{r,m}$ hierarchies in the language of the universal Sato Grassmannian;
– obtaining the explicit form of the non-isospectral Virasoro symmetry generators acting on the $\mathsf{cKP}_{r,m}$ τ-functions.

The achieved progress should result in further clarification of the status of the $\mathsf{cKP}_{r,m}$ hierarchies and their connection to the underlying fermionic field language. It would also be interesting to look for the signs of the affine $\hat{sl}(r + m + 1)$ symmetry encountered in construction of the $\mathsf{cKP}_{r,m}$ models by the generalized Drinfeld-Sokolov method associated to affine Kac-Moody algebras [28].

Furthermore, as a principal application, the SEP method was used to derive a series of new determinant solutions for the τ-functions of (constrained) KP hierarchies which generalize the familiar Wronskian (multi-soliton) solutions. These new solutions belong to a new type of *generalized binary* Darboux-Bäcklund orbits which, in turn, were shown to correspond to a novel Toda model on a *square* lattice. An important task for future study is to find a closed Lagrangian description of this new Toda square-lattice model.

Finally, let us briefly describe another potential physical application of the present approach.

Using the spectral representation for (adjoint) eigenfunctions (3.15) together with (2.10)–(2.11), as well as the following form of the Fay identity for τ-functions [14]:

$$\det_n \left\| \frac{\tau\left(t + [\lambda_i^{-1}] - [\mu_j^{-1}]\right)}{(\lambda_i - \mu_j)\tau(t)} \right\|$$

$$= (-1)^{\frac{n(n-1)}{2}} \frac{\prod_{i>j} \left(\lambda_i - \lambda_j\right)\left(\mu_i - \mu_j\right)}{\prod_{i,j}\left(\lambda_i - \mu_j\right)} \frac{\tau\left(t + \sum_l[\lambda_l^{-1}] - \sum_l[\mu_l^{-1}]\right)}{\tau(t)}, \quad (7.1)$$

we obtain an equivalent "spectral" representation for $\tau^{(n,n)}(t)$ (6.16):

$$
\begin{aligned}
&\tau^{(n,n)}(t) \\
&= \frac{(-1)^{\frac{n(n-1)}{2}}}{(n!)^2} \int \prod_{j=0}^{n-1} d\lambda_j \, d\mu_j \prod_{i>j} (\lambda_i - \lambda_j) (\lambda_i^r - \lambda_j^r) \prod_{j=0}^{n-1} \left(\varphi_b^{*(0,0)}(\lambda_j) e^{-\xi(t,\lambda_j)} \right) \times \\
&\quad \frac{1}{\prod_{i,j}(\lambda_i - \mu_j)} \prod_{i>j} (\mu_i - \mu_j) (\mu_i^r - \mu_j^r) \\
&\quad \prod_{j=0}^{n-1} \left(\varphi_a^{(0,0)}(\mu_j) e^{\xi(t,\mu_j)} \right) \tau^{(0,0)}\left(t + \sum_l [\lambda_l^{-1}] - \sum_l [\mu_l^{-1}] \right).
\end{aligned} \tag{7.2}
$$

Following [42], we can interpret the τ-function (7.2) as a partition function of certain random multi-matrix ensemble with the following joint distribution function of eigenvalues:

$$
Z_n[\{t\}] \equiv const \, \tau^{(n,n)}(t) = \int \prod_{j=0}^{n-1} d\lambda_j \, d\mu_j \, \exp\left\{ -H(t; \{\lambda\}, \{\mu\}) \right\}, \tag{7.3}
$$

$$
\begin{aligned}
H(t; \{\lambda\}, \{\mu\}) \equiv \sum_j \left(\bar{H}_1(\lambda_j) + H_1(\mu_j) \right) + \sum_{i>j} \left(H_2(\lambda_i, \lambda_j) + H_2(\mu_i, \mu_j) \right) \\
+ \sum_{i,j} \widetilde{H}_2(\lambda_i, \mu_j) + H_n(\{\lambda\}, \{\mu\}),
\end{aligned} \tag{7.4}
$$

where the one-, two- and many-body Hamiltonians read, respectively:

$$
H_1(\lambda) = -\ln \varphi^{(0,0)}(\lambda) - \xi(t, \lambda), \quad \bar{H}_1(\lambda) = -\ln \varphi^{*(0,0)}(\lambda) + \xi(t, \lambda), \tag{7.5}
$$

$$
H_2(\lambda_i, \lambda_j) = -\ln \left(\lambda_i - \lambda_j \right)^2 - \ln\left(\sum_{s=0}^{r-1} \lambda_i^s \lambda_j^{r-1-s} \right), \quad \widetilde{H}_2(\lambda, \mu) = \ln(\lambda - \mu), \tag{7.6}
$$

$$
H_n(\{\lambda\}, \{\mu\}) = -\ln \tau^{(0,0)}\left(t + \sum_l [\lambda_l^{-1}] - \sum_l [\mu_l^{-1}] \right). \tag{7.7}
$$

The physical implications of the above new type of joint distribution function (7.3)–(7.7) deserves further study especially regarding critical behavior of correlations. The emerging new interesting features of (7.3)–(7.7), absent in the joint distribution function derived from the conventional two-matrix model [42], are as follows:

(a) the second attractive term in the two-body potential H_2 (7.6) (both for λ- and μ-"particles") dominating at very long distances over the customary repulsive first term;

(b) an additional two-body attractive potential $\widetilde{H}_2$ (7.6) between each pair of λ- and μ-"particles"

(c) a genuine many-body potential H_n (7.7).

One of the most important issues here is to exhibit the explicit form of the generalized multi-matrix model behind (7.3)–(7.7).

A. Appendix: The Most General $\mathsf{cKP}_{r,m}$ Binary Darboux-Bäcklund Orbit

Let us first introduce a few convenient compact notations for Wronskian and related Wronskian-like determinants:

$$W_k \equiv W_k\,[f_1,\ldots,f_k] = \det\left\|\partial^{\alpha-1}f_\beta\right\| \quad , \quad W_k(f) \equiv W_{k+1}\,[f_1,\ldots,f_k,f], \quad \text{(A.1)}$$

$$\widetilde{W}_{k+1}(f) \equiv \widetilde{W}\,[f_1,\ldots,f_{k+1};f] \equiv \det_{k+1}\left\|\begin{matrix}\partial^{\alpha-1}f_\beta & \partial^{\alpha-1}f_{k+1}\\ S\left(f_\beta,f\right) & S(f_{k+1},f)\end{matrix}\right\|, \quad \text{(A.2)}$$

where the matrix indices $\alpha,\beta = 1,\ldots,k$ and, as above, $\partial_x S\left(f_\beta,f\right) = f_\beta\,f$. The Wronskian(-like) determinants (A.1)–(A.2) obey the following useful identities:

$$\partial\left(\frac{W_{k-1}(f)}{W_k}\right) = \frac{W_k(f)\,W_{k-1}}{W_k^2};\quad \partial\left(\frac{\widetilde{W}_{k+1}(f)}{W_k}\right) = -\frac{\widetilde{W}_k(f)\,W_{k+1}}{W_k^2}, \quad\quad \text{(A.3)}$$

where the first one is known as Jacobi expansion theorem (see, *e.g.* [43]), whereas the second identity in (A.3) can be easily verified via induction. Equations (A.3) imply in turn the identities:

$$T_k\cdots T_1(f) = \frac{W_k(f)}{W_k} \quad , \quad T_k^{*\,-1}\cdots T_1^{*\,-1}(f) = -\frac{\widetilde{W}_k(f)}{W_k} \quad\quad \text{(A.4)}$$

with

$$T_j = \frac{W_j}{W_{j-1}}D\frac{W_{j-1}}{W_j} \quad , \quad T_j^{*\,-1} = -\frac{W_{j-1}}{W_j}D^{-1}\frac{W_j}{W_{j-1}}. \quad\quad \text{(A.5)}$$

Now we can use Eqs. (A.4) to derive explicit expressions for the (adjoint) eigenfunctions and τ-functions of $\mathsf{cKP}_{r,m}$ hierarchies generated via successive (adjoint) DB transformations (6.3)–(6.10) w.r.t. an arbitrary set of (adjoint) eigenfunctions of the "initial" $\mathsf{cKP}_{r,m}$ Lax operator $L = L_{r,m}$ (2.20). We shall denote the latter arbitrary successive (adjoint) DB transformations by the following double-vector superscript:

$$\left(\vec{n},\vec{k}\right) \equiv ((n_1,\ldots,n_m),(k_1,\ldots k_m)) \quad\quad \text{(A.6)}$$

indicating n_1 successive DB transformations w.r.t. Φ_1 etc., until n_m DB transformations w.r.t. Φ_m and, similarly, k_1 successive adjoint-DB transformations w.r.t. Ψ_1 etc., until k_m adjoint-DB transformations w.r.t. Ψ_m. Specifically, we have:

$$\Phi_a^{\left(\vec{n},\vec{0}\right)} = (-1)^{\sum_{a+1}^m n_j}$$
$$\frac{W\left[\chi_1^{(0)},\ldots,\chi_1^{(n_1-1)},\ldots,\chi_a^{(0)},\ldots,\chi_a^{(n_a-1)},\chi_a^{(n_a)},\ldots,\chi_m^{(0)},\ldots,\chi_m^{(n_m-1)}\right]}{W\left[\chi_1^{(0)},\ldots,\chi_1^{(n_1-1)},\ldots,\chi_a^{(0)},\ldots,\chi_a^{(n_a-1)},\ldots,\chi_m^{(0)},\ldots,\chi_m^{(n_m-1)}\right]},$$
$$\text{(A.7)}$$

$$\frac{\tau^{\left(\vec{n},\vec{0}\right)}}{\tau^{\left(\vec{0},\vec{0}\right)}} = W\left[\chi_1^{(0)},\ldots,\chi_1^{(n_1-1)},\ldots,\chi_m^{(0)},\ldots,\chi_m^{(n_m-1)}\right], \quad\quad \text{(A.8)}$$

$$\Psi_a^{(\vec{n},\vec{0})} =$$

$$\begin{cases} (-1)^{\sum_{a+1}^m n_j} \dfrac{W\left[\chi_1^{(0)},\ldots,\chi_1^{(n_1-1)},\ldots,\chi_a^{(0)},\ldots,\chi_a^{(n_a-2)},\ldots,\chi_m^{(0)},\ldots,\chi_m^{(n_m-1)}\right]}{W\left[\chi_1^{(0)},\ldots,\chi_1^{(n_1-1)},\ldots,\chi_a^{(0)},\ldots,\chi_a^{(n_a-1)},\ldots,\chi_m^{(0)},\ldots,\chi_m^{(n_m-1)}\right]} , & \text{for } n_a \geq 1 \\[12pt] -\dfrac{\widetilde{W}\left[\chi_1^{(0)},\ldots,\chi_1^{(n_1-1)},\ldots,\chi_m^{(0)},\ldots,\chi_m^{(n_m-1)};\Psi_a^{(\vec{0},\vec{0})}\right]}{W\left[\chi_1^{(0)},\ldots,\chi_1^{(n_1-1)},\ldots,\chi_m^{(0)},\ldots,\chi_m^{(n_m-1)}\right]} , & \text{for } n_a = 0 \end{cases}$$

$$\tag{A.9}$$

where the functions $\chi_a^{(s)}$ are the same as in (6.15) with the superscripts $(0,0)$ replaced with the corresponding double-vector ones $\left(\vec{0},\vec{0}\right)$. Equations (A.7)–(A.8) already appeared in [29] (see also refs.[44]), whereas Eq. (A.9) is derived via iterative application of the second identity in (A.4) and taking into account (A.7).

Now, performing arbitrary successive adjoint-DB transformations on $\tau^{(\vec{n},\vec{0})}$ (A.8) according to the second Eq. (6.9) upon using the first identity in (A.4) and inserting there the explicit expressions (A.9), we arrive at the following:

Proposition A.1. *The most general discrete binary Darboux-Bäcklund orbit on the space of* cKP$_{r,m}$ *τ-functions is built-up of the following elements:*

$$\frac{\tau^{(\vec{n},\vec{k})}}{\tau^{(\vec{0},\vec{0})}} = \left(-W\left[\chi_1^{(0)},\ldots,\chi_1^{(n_1-1)},\ldots,\chi_a^{(0)},\ldots,\chi_a^{(n_a-1)}\right]\right)^{-\sum_{a+1}^m k_j} \times$$

$$W\left[\Delta_{(0,a+1)}^{\vec{n}},\ldots,\Delta_{(k_{a+1}-1,a+1)}^{\vec{n}},\ldots,\Delta_{(0,m)}^{\vec{n}},\ldots,\Delta_{(k_m-1,m)}^{\vec{n}}\right] \quad \text{(A.10)}$$

$$\Delta_{(l,s)}^{\vec{n}} \equiv \widetilde{W}\left[\chi_1^{(0)},\ldots,\chi_1^{(n_1-1)},\ldots,\chi_a^{(0)},\ldots,\chi_a^{(n_a-1)};\chi_s^{*(l)}\right] \quad \text{(A.11)}$$

where

$$\left(\vec{n},\vec{k}\right) = ((n_1,\ldots,n_a,0,\ldots,0),(0,\ldots,0,k_{a+1},\ldots,k_m)) \quad ; \quad a = 0,1,\ldots,m$$

$$\tag{A.12}$$

and, furthermore, notations (6.15) and (A.2) are employed.

Remark. The reason for the zero entries in the labels (A.12) of the most general binary DB transformations, preserving the spaces of cKP$_{r,m}$ hierarchies (2.20), lies in the fact that any pair of two successive (adjoint-)DB transformations w.r.t. Φ_a, Ψ_a, *i.e.* both with the *same* index, is equivalent to an identity transformation as one can easily conclude by combining the second equation in (6.3) with the second equations in (6.4) and (6.9).

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