Commun. Math. Phys. 191, 467 – 492 (1998)

Communications in
Mathematical Physics

Coalgebra Bundles*

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Received: 22 February 1996 / Accepted: 29 May 1997

Abstract: We develop a generalised theory of bundles and connections on them in which the role of gauge group is played by a coalgebra and the role of principal bundle by an algebra. The theory provides a unifying point of view which includes quantum group gauge theory, embeddable quantum homogeneous spaces and braided group gauge theory, the latter being introduced now by these means. Examples include ones in which the gauge groups are the braided line and the quantum plane.

1. Introduction

In a recent paper [Brz96b] it was shown by the first author that a generalisation of the quantum group principal bundles introduced in [BM93] is needed if one wants to include certain "embeddable" quantum homogeneous spaces, such as the full family of quantum two-spheres of Podleś [Pod87]. A one-parameter specialisation of this family was used in [BM93] in construction of the q-monopole, but the general members of the family do not have the required canonical fibering. The required generalised notion of quantum principal bundles proposed in [Brz96b], also termed a C-Galois extension (cf. [Sch92]), consists of an algebra P, a coalgebra C with a distinguished element e and a right action of P on $P \otimes C$ satisfying certain conditions. In the present paper we develop a version of such "coalgebra principal bundles" based on a map $\psi : C \otimes P \to P \otimes C$ and $e \in C$, and giving now a theory of connections on them.

Another motivation for the paper is the search for a generalisation of gauge theory powerful enough to include braided groups [Maj91, Maj93b, Maj93a] as the gauge group. Although not quantum groups, braided groups do have at least a coalgebra and hence can be covered in our theory. We describe the main elements of such a braided

* Research supported by the EPSRC grant GR/K02244

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principal bundle theory as arising in this way. This is a first step towards a theory of braided-Lie algebra valued gauge fields, Chern-Simons and Yang-Mills actions, to be considered elsewhere.

As well as providing a unifying point of view which includes our previous quantum group gauge theory [BM93], the theory of embeddable homogeneous spaces [Brz96b] and braided group gauge theory, our coalgebra bundles have their own characteristic properties. In particular, the axioms obeyed by ψ involve the algebra and coalgebra in a symmetrical way, opening up the possibility of an interesting self-duality of the construction. This becomes manifest when we are given a character κ on P; then we have also the possibility of a dual "algebra principal bundle", corresponding in the finite-dimensional case to a coalgebra principal bundle with the fibre P^*, total space C^* and the structure map ψ^*. This is a new phenomenon which is not possible within the realm of ordinary (non-Abelian) gauge theory. Moreover, the axioms obeyed by ψ correspond in the finite-dimensional case to the factorisation of an algebra into $P^{\rm op}C^*$, which is a common situation [Maj90]. Indeed, all bicrossproduct quantum groups [Maj90] provide a dual pair of examples.

Finally, we note that some steps towards a theory of fibrations based on algebra factorisations have appeared independently in [CKM94], including topological considerations which may be useful in further work. However, we really need the present coalgebra treatment for our infinite-dimensional algebraic examples, for our treatment of differential calculus and in order to include quantum and braided group gauge theories. We demonstrate the various stages of our formalism on some concrete examples based on the braided line and quantum plane.

Preliminaries. All vector spaces are taken over a field k of generic characteristic and all algebras have the unit denoted by 1. C denotes a coalgebra with the coproduct $\Delta : C \to C \otimes C$ and the counit $\epsilon : C \to k$ which satisfy the standard axioms. For the coproduct we use the Sweedler notation

$$\Delta c = c_{(1)} \otimes c_{(2)}, \qquad \Delta^2 c = (\Delta \otimes \mathrm{id}) \circ \Delta c = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}, \quad \text{etc.},$$

where $c \in C$, and the summation sign and the indices are suppressed.

A vector space P is a right C-comodule if there exists a map $\Delta_R : P \to P \otimes C$, such that $(\Delta_R \otimes \mathrm{id}) \circ \Delta_R = (\mathrm{id} \otimes \Delta) \circ \Delta_R$, and $(\mathrm{id} \otimes \epsilon) \circ \Delta_R = \mathrm{id}$. For Δ_R we use the explicit notation

$$\Delta_R u = u^{(\bar{0})} \otimes u^{(\bar{1})},$$

where $u \in P$ and $u^{(\bar{0})} \otimes u^{(\bar{1})} \in P \otimes C$ (summation understood). For $e \in C$, we denote by P_e^{coC} the vector subspace of P of all elements $u \in P$ such that $\Delta_R u = u \otimes e$.

H denotes a Hopf algebra with product $\mu : H \otimes H \to H$, unit 1, coproduct $\Delta : H \to H \otimes H$, counit $\epsilon : H \to k$ and antipode $S : H \to H$. We use Sweedler's sigma notation as before. Similarly as for a coalgebra, we can define right H-comodules. We say that a right H-comodule P is a right H-comodule algebra if P is an algebra and Δ_R is an algebra map.

If P is an algebra then by $\Omega^n P$ we denote the P-bimodule of universal n-forms on P, which is defined as $\Omega^n P = \Omega^1 P \otimes_P \cdots \otimes_P \Omega^1 P$ (n-fold tensor product over P). By the natural identification $P \otimes_P P = P$ we have [Con85, Kar87]

$$\Omega^n P = \{\omega \in P^{\otimes n+1} : \ \forall i \in \{1, \dots, n\}, \ \ \mu_i \omega = 0\},$$

where μ_i denotes a multiplication in P acting on the i and $i+1$ factors in $P^{\otimes n+1}$. $\Omega P = \bigoplus_{n=0} \Omega^n P$, where $\Omega^0 P = P$, is a differential algebra with the universal differential

$\mathrm{d} : P \to \Omega^1 P$, $\mathrm{d}u = 1 \otimes u - u \otimes 1$. When extended to $\Omega^n P \subset P^{\otimes n+1}$, d explicitly reads:

$$\mathrm{d}\left(u^0 \otimes u^1 \otimes \ldots \otimes u^n\right) = \sum_{k=0}^{n+1} (-1)^k u^0 \otimes \ldots \otimes u^{k-1} \otimes 1 \otimes u^k \otimes \ldots \otimes u^n. \qquad (1)$$

$\Omega^n(P)$ denotes a bimodule of n-forms on P obtained from $\Omega^n P$ as an appropriate quotient.

Finally, if C is a coalgebra and P is an algebra then we define a convolution product $*$ in the space of linear maps $C \to P$ by $f * g(c) = f(c_{(1)})g(c_{(2)})$, where $f, g : C \to P$ and $c \in C$. The map $f : C \to P$ is said to be convolution invertible if there is a map $f^{-1} : C \to P$ such that $f * f^{-1} = f^{-1} * f = \eta \circ \epsilon$, where $\eta : k \to P$ is given by $\eta : \alpha \mapsto \alpha 1$.

In addition, we will also discuss examples based on the theory of braided groups [Maj91, Maj93b, Maj93a] and the theory of bicrossproduct and double cross product and Hopf algebras [Maj90, Maj94b], due to the second author. Chapters 6.2,7.2,9 and 10 of the text [Maj95] contain full details on these topics.

2. Coalgebra ψ-Principal Bundles

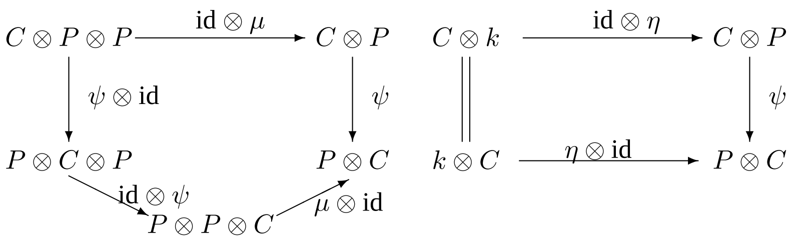
In this paper we will be dealing with a particular formulation of C-Galois extensions or generalised quantum principal bundles. This formulation is more tractable than the one in [Brz96b], allowing us to develop a theory of connections for it in the next section. Yet, it is general enough to include all our main examples of interest. Our data is the following:

Definition 2.1. *We say that a coalgebra C and an algebra P are* entwined *if there is a map $\psi : C \otimes P \to P \otimes C$ such that*

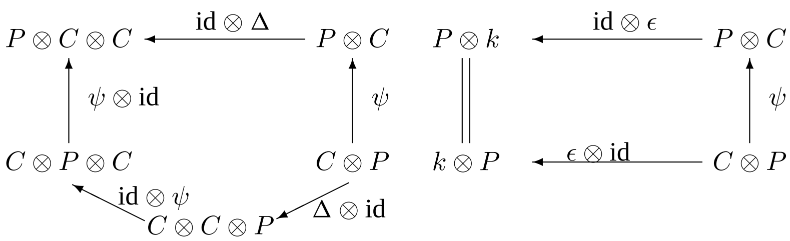
$$\psi \circ (\mathrm{id} \otimes \mu) = (\mu \otimes \mathrm{id}) \circ \psi_{23} \circ \psi_{12}, \quad \psi(c \otimes 1) = 1 \otimes c, \quad \forall c \in C \qquad (2)$$

$$(\mathrm{id} \otimes \Delta) \circ \psi = \psi_{12} \circ \psi_{23} \circ (\Delta \otimes \mathrm{id}), \quad (\mathrm{id} \otimes \epsilon) \circ \psi = \epsilon \otimes \mathrm{id}, \qquad (3)$$

where μ denotes multiplication in P, and $\psi_{23} = \mathrm{id} \otimes \psi$ etc. Explicitly, we require that the following diagrams commute:



(4)



(5)

where η is the unit map $\eta : \alpha \mapsto \alpha 1$.

In the finite-dimensional case this is exactly equivalent by partial dualisation to the requirement that $\tilde{\psi} : C^* \otimes P^{\mathrm{op}} \to P^{\mathrm{op}} \otimes C^*$ is an algebra factorisation structure (which is part of the theory of Hopf algebra double cross products [Maj90]). This is made precise at the end of the section, where it provides a natural way to obtain examples of such ψ.

Proposition 2.2. *Let C, P be entwined by ψ. For every group-like element $e \in C$ we have the following:*

1. *For any positive n, $P^{\otimes n}$ is a right C-comodule with the coaction $\Delta_R^n = \psi_{nn+1} \circ \psi_{n-1n} \circ \ldots \circ \psi_{12} \circ (\eta_C \otimes \mathrm{id}^n) \equiv \overset{\longleftarrow n}{\psi} \circ (\eta_C \otimes \mathrm{id}^n)$, where $\eta_C : k \to C$, $\alpha \mapsto \alpha e$.*
2. *The coaction Δ_R^{n+1} restricts to a coaction on $\Omega^n P$.*
3. *$M = P_e^{coC} = \{u \in P\ ;\ \ \Delta_R^1 u = u \otimes e\}$ is a subalgebra of P.*
4. *The linear map $\chi_M : P \otimes_M P \to P \otimes C$, $u \otimes_M v \mapsto u\psi(e \otimes v)$ is well-defined. If χ_M is a bijection we say that we have a ψ-principal bundle $P(M, C, \psi, e)$.*

Proof. We write $\psi(c \otimes u) = \sum_\alpha u_\alpha \otimes c^\alpha$ and henceforth we omit the summation sign. In this notation, the conditions (4) and (5) are

$$(uv)_\alpha \otimes c^\alpha = u_\alpha v_\beta \otimes c^{\alpha\beta}, \qquad 1_\alpha \otimes c^\alpha = 1 \otimes c, \tag{6}$$

$$u_\alpha \otimes c^\alpha{}_{(1)} \otimes c^\alpha{}_{(2)} = u_{\alpha\beta} \otimes c_{(1)}{}^\beta \otimes c_{(2)}{}^\alpha, \qquad \epsilon(c^\alpha)u_\alpha = \epsilon(c)u, \tag{7}$$

for all $u, v \in P$ and $c \in C$.

1. The map Δ_R^n is given explicitly by

$$\Delta_R^n(u^1 \otimes \ldots \otimes u^n) = u^1_{\alpha_1} \otimes \ldots \otimes u^n_{\alpha_n} \otimes e^{\alpha_1 \ldots \alpha_n}.$$

Hence

$$\begin{aligned}(\Delta_R^n \otimes \mathrm{id})\Delta_R^n\ \ (u^1 \otimes \ldots \otimes u^n) &= u^1_{\alpha_1\beta_1} \otimes \ldots \otimes u^n_{\alpha_n\beta_n} \otimes e^{\beta_1 \ldots \beta_n} \otimes e^{\alpha_1 \ldots \alpha_n} \\ &= u^1_{\alpha_1\beta_1} \otimes \ldots \otimes u^n_{\alpha_n\beta_n} \otimes e_{(1)}{}^{\beta_1 \ldots \beta_n} \otimes e_{(2)}{}^{\alpha_1 \ldots \alpha_n} \\ &= u^1_{\alpha_1} \otimes u^2_{\alpha_2\beta_2} \otimes \ldots \otimes u^n_{\alpha_n\beta_n} \otimes e^{\alpha_1}{}_{(1)}{}^{\beta_2 \ldots \beta_n} \otimes e^{\alpha_1}{}_{(2)}{}^{\alpha_2 \ldots \alpha_n} \\ &= \ldots = u^1_{\alpha_1} \otimes \ldots \otimes u^n_{\alpha_n} \otimes e^{\alpha_1 \ldots \alpha_n}{}_{(1)} \otimes e^{\alpha_1 \ldots \alpha_n}{}_{(2)} \\ &= (\mathrm{id}^n \otimes \Delta)\Delta_R^n(u^1 \otimes \ldots \otimes u^n),\end{aligned}$$

where we used the group-like property of e to derive the second equality and then we used the condition (5) n times to obtain the penultimate one. We also have

$$\begin{aligned}(\mathrm{id}^n \otimes \epsilon)\Delta_R^n(u^1 \otimes \ldots \otimes u^n) &= u^1_{\alpha_1} \otimes \ldots \otimes u^n_{\alpha_n}\epsilon(e^{\alpha_1 \ldots \alpha_n}) \\ &= u^1_{\alpha_1} \otimes \ldots \otimes u^{n-1}_{\alpha_{n-1}} \otimes u^n \epsilon(e^{\alpha_1 \ldots \alpha_{n-1}}) \\ &= \ldots = \epsilon(e)u^1 \otimes \ldots \otimes u^n \\ &= u^1 \otimes \ldots \otimes u^n,\end{aligned}$$

where we have first used the condition (5) n-times and then the group-like property of e. Hence Δ_R^n is a coaction.

2. If $\sum_i u^i\otimes v^i \in \ker\mu$ then $(\mu\otimes\mathrm{id})\sum_i \Delta_R^2(u^i\otimes v^i) = \sum_i u^i_\alpha v^i_\beta\otimes e^{\alpha\beta} = \sum_i (u^i v^i)_\alpha\otimes e^\alpha = 0$, using (6). Hence the coaction preserves $\Omega^1 P$. Similarly for $\Omega^n P$.
3. Here $M = \{u \in P | u_\alpha\otimes e^\alpha = u\otimes e\}$, and if $u, v \in M$, then $(uv)_\alpha\otimes e^\alpha = u_\alpha v_\beta\otimes e^{\alpha\beta} = uv_\beta\otimes e^\beta = uv\otimes e$ as well, using (6).
4. It is easy to see that χ_M is well-defined as a map from $P\otimes_M P$. Thus, if $x\in M$ we have $\chi_M(u, xv) = u(xv)_\alpha\otimes e^\alpha = ux_\alpha v_\beta\otimes e^{\alpha\beta} = uxv_\beta\otimes e^\beta = \chi_M(ux, v)$, using (6). □

We remark that parts 3 and 4 also follow from the theory of C-Galois extensions of [Brz96b], for $P(M, C, \psi, e)$ is such an extension. The required right action of P on $P\otimes C$ is given by $(\mu\otimes\mathrm{id})\circ\psi_{23} : P\otimes C\otimes P \to P\otimes C$.

Example 2.3. Let H be a Hopf algebra and P be a right H-comodule algebra. The linear map $\psi : H\otimes P \to P\otimes H$ defined by $\psi : c\otimes u \to u^{\bar{(0)}}\otimes cu^{\bar{(1)}}$ entwines H, P. Therefore a quantum group principal bundle $P(M, H)$ with universal differential structure as in [BM93] is a ψ-principal bundle $P(M, H, \psi, 1)$.

Proof. For any $c\in H$ and $u\in P$ we have $u_\alpha\otimes c^\alpha = u^{\bar{(0)}}\otimes cu^{\bar{(1)}}$. Clearly $1_\alpha\otimes c^\alpha = 1\otimes c$. We compute

$$u_\alpha v_\beta\otimes c^{\alpha\beta} = u^{\bar{(0)}} v_\beta\otimes (cu^{\bar{(1)}})^\beta = u^{\bar{(0)}} v^{\bar{(0)}}\otimes cu^{\bar{(1)}} v^{\bar{(1)}} = (uv)^{\bar{(0)}}\otimes c(uv)^{\bar{(1)}} = (uv)_\alpha\otimes c^\alpha,$$

hence the condition (4) is satisfied. Furthermore, $\epsilon(c_\alpha)u^\alpha = \epsilon(cu^{\bar{(1)}})u^{\bar{(0)}} = \epsilon(c)u$ and

$$\begin{aligned} u_{\alpha\beta}\otimes c_{(1)}{}^\beta\otimes c_{(2)}{}^\alpha &= u^{\bar{(0)}}{}_\beta\otimes c_{(1)}{}^\beta\otimes c_{(2)}u^{\bar{(1)}} = u^{\bar{(0)}\bar{(0)}}\otimes c_{(1)}u^{\bar{(0)}\bar{(1)}}\otimes c_{(2)}u^{\bar{(1)}} \\ &= u^{\bar{(0)}}\otimes (cu^{\bar{(1)}})_{(1)}\otimes (cu^{\bar{(1)}})_{(2)} = u_\alpha\otimes c^\alpha{}_{(1)}\otimes c^\alpha{}_{(2)}, \end{aligned}$$

so that the condition (5) is also satisfied. Clearly the induced coaction in Proposition 2.2 coincides with the given coaction of H. □

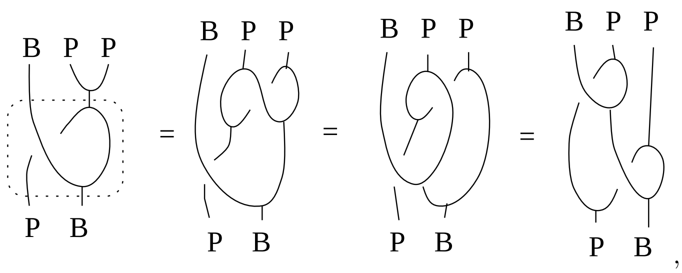
We can easily replace H here by one of the braided groups introduced in [Maj91, Maj93b]. To be concrete, we suppose that our braided group B lives in a k-linear braided category with well-behaved direct sums, such as that of modules over a quasitriangular Hopf algebra or comodules over a dual-quasitriangular Hopf algebra. This background quantum group does not enter directly into the braided group formulae but rather via the braiding Ψ which it induces between any objects in the category. We refer to [Maj93a] for an introduction to the theory and for further details. In particular, a right braided B-module algebra P means a coaction $P \to P\underline{\otimes} B$ in the category which is an algebra homomorphism to the braided tensor product algebra [Maj91]

$$(u\otimes b)(v\otimes c) = u\Psi(b\otimes v)c. \tag{8}$$

The coproduct $\underline{\Delta} : B \to B\underline{\otimes} B$ of a braided group is itself a homomorphism to such a braided tensor product.

Example 2.4. Let B be a braided group with braiding Ψ and P a right braided B-comodule algebra. The linear map $\psi : B\otimes P \to P\otimes B$ defined by $\psi : c\otimes u \to \Psi(c\otimes u^{\bar{(0)}})u^{(1)}$ entwines B, P. If the induced map χ_M is a bijection we say that the associated ψ-principal bundle $P(M, B, \psi, 1)$ is a *braided group principal bundle*, and denote it by $P(M, B, \Psi)$.

Proof. This is best done diagrammatically by the technique introduced in [Maj91]. Thus, we write $\Psi = \times$ and products by $\mu = \curlyvee$. We denote coactions and coproducts by $\curlywedge$. The proof of the main part of (4) is then the diagram:



where box is ψ as stated, in diagrammatic form. The first equality is the assumed homomorphism property of the braided coaction $\curlywedge$. The second equality is associativity of the product in B, and the third is functoriality of the braiding, which we use to push the diagram into the right form. The minor condition is immediate from the axioms of a braided comodule algebra and the properties of the unit map $\eta : \underline{1} \to P$. Here $\underline{1}$ denotes the trivial object for our tensor product and necessarily commutes with the braiding in an obvious way (such that $\underline{1}$ is denoted consistently by omission). For the proof of (5) we ask the reader to reflect the diagram in a mirror about a horizontal axis (i.e. view it up-side-down and from behind) and then reverse all braid crossings (restoring them all to $\times$). The result is the diagrammatic proof for the main part of (5) if we relabel the product of P as the coproduct of B and relabel the product of B as the right coaction of B on P. The minor part of (5) is immediate from properties of the braided counit. $\square$

Example 2.5. Let H be a Hopf algebra and $\pi : H \to C$ a coalgebra surjection. If $\ker \pi$ is a minimal right ideal containing $\{u - \epsilon(u) | u \in M\}$ then $\psi : C \otimes H \to H \otimes C$ defined by $\psi(c \otimes u) = u_{(1)} \otimes \pi(v u_{(2)})$ entwines C, H, where $u \in H$, $c \in C$ and $v \in \pi^{-1}(c)$, and we have a ψ-principal bundle $H(M, C, \psi, \pi(1))$ in the setting of Proposition 2.2, denoted $H(M, C, \psi, \pi)$. Hence the generalised bundles over embeddable quantum homogeneous spaces in [Brz96b] are examples of ψ-principal bundles.

Proof. In this case $u_\alpha \otimes c^\alpha = u_{(1)} \otimes \pi(w u_{(2)})$, for any $u \in H$, $c \in C$ and $w \in \pi^{-1}(c)$. Clearly $1_\alpha \otimes c^\alpha = 1 \otimes c$. We compute

$$u_\alpha v_\beta \otimes c^{\alpha\beta} = u_{(1)} v_\alpha \otimes \pi(w u_{(2)})^\alpha = u_{(1)} v_{(1)} \otimes \pi(w u_{(2)} v_{(2)}) = (uv)_\alpha \otimes c^\alpha,$$

where $w \in \pi^{-1}(c)$. Hence condition (4) is satisfied. Furthermore, we have $\epsilon(c^\alpha) u_\alpha = \epsilon(\pi(w u_{(2)})) u_{(1)} = \epsilon(c) u$ and

$$\begin{aligned} u_{\alpha\beta} \otimes c_{(1)}{}^\beta \otimes c_{(2)}{}^\alpha &= u_{(1)_\alpha} \otimes \pi(w_{(1)} u_{(2)}) \otimes c_{(2)}{}^\alpha = u_{(1)} \otimes \pi(w_{(1)} u_{(2)}) \otimes \pi(w_{(2)} u_{(3)}) \\ &= u_{(1)} \otimes \pi(w u_{(2)})_{(1)} \otimes \pi(w u_{(2)})_{(2)} = u_\alpha \otimes c^\alpha{}_{(1)} \otimes c^\alpha{}_{(2)}, \end{aligned}$$

where again $w_{(1)} \in \pi^{-1}(c_{(1)})$ and $w_{(2)} \in \pi^{-1}(c_{(2)})$. Therefore condition (5) is also satisfied.

Some concrete examples of coalgebra bundles over quantum embeddable homogeneous spaces may be found in [Brz96b] (cf. [DK94]).

We note that diagrams (4) and (5) are dual to each other in the following sense. The diagrams (5) may be obtained from diagrams (4) by interchanging μ with Δ, η with ϵ and P with C, and by reversing the arrows. With respect to this duality property the axioms for the map ψ are self-dual. Therefore we can dualise Proposition 2.2 to obtain the following:

Proposition 2.6. *Let C, P be entwined by $\psi : C\otimes P \to P\otimes C$. For every algebra character $\kappa : P \to k$ we have the following:*

1. *For any positive integer n, $C^{\otimes n}$ is a right P-module with the action $\triangleleft^n = (\kappa\otimes \mathrm{id}^n)\circ \psi_{12}\circ\psi_{23}\circ\ldots\circ\psi_{nn+1} = (\kappa\otimes\mathrm{id}^n)\circ \overrightarrow{\psi}^n$.*
2. *The action $\triangleleft^n$ maps $\Delta^n(C)$ to itself.*
3. *The subspace $I_\kappa = \mathrm{span}\{c\triangleleft^1 u - c\kappa(u) | c\in C,\ u\in P\}$ is a coideal. Hence $M = C/I_\kappa$ is a coalgebra. We denote the canonical surjection by $\pi_\kappa : C\to M$.*
4. *There is a map $\zeta^M : C\otimes P \to C\otimes^M C$ defined by $\zeta^M(c\otimes u) = c_{(1)}\otimes^M c_{(2)}\triangleleft^1 u$, where $C\otimes^M C = \mathrm{span}\{c\otimes d\in C\otimes C | c_{(1)}\otimes \pi_\kappa(c_{(2)})\otimes d = c\otimes\pi_\kappa(d_{(1)})\otimes d_{(2)}\}$ is the cotensor product under M. If ζ^M is a bijection, we say that $C(M, P, \psi, \kappa)$ is a dual ψ-principal bundle.*

Proof. 1. The explicit action is

$$(c_n\otimes\cdots\otimes c_1)\triangleleft^n u = c_n^{\alpha_n}\otimes\cdots\otimes c_1^{\alpha_1}\kappa(u_{\alpha_1\cdots\alpha_n}).$$

Then clearly

$$\begin{aligned}((c_n\otimes\cdots\otimes c_1)\triangleleft^n u)\triangleleft^n v &= c_n^{\alpha_n\beta_n}\otimes\cdots\otimes c_1^{\alpha_1\beta_1}\kappa(u_{\alpha_1\cdots\alpha_n}v_{\beta_1\cdots\beta_n})\\ &= c_n^{\alpha_n}\otimes c_{n-1}^{\alpha_{n-1}\beta_{n-1}}\otimes\cdots\otimes c_1^{\alpha_1\beta_1}\kappa((u_{\alpha_1\cdots\alpha_{n-1}}v_{\beta_1\cdots\beta_{n-1}})_{\alpha_n})\\ &= \cdots = c_n^{\alpha_n}\otimes\cdots\otimes c_1^{\alpha_1}\kappa((uv)_{\alpha_1\cdots\alpha_n}) = (c_n\otimes\cdots\otimes c_1)\triangleleft^n(uv)\end{aligned}$$

for all $c_i\in C$ and $u, v\in P$. We used (6) repeatedly.

2. We have $(c_{(1)}\otimes c_{(2)})\triangleleft^2 u = c_{(1)}{}^\beta\otimes c_{(2)}{}^\alpha\kappa(u_{\alpha\beta}) = c^\alpha{}_{(1)}\otimes c^\alpha{}_{(2)}\kappa(u_\alpha) = \Delta(c\triangleleft^1 u)$ by (7), and similarly for higher $\Delta^n(C)$.
3. Explicitly, $I_\kappa = \mathrm{span}\{c^\alpha\kappa(u_\alpha) - c\kappa(u) | c\in C,\ u\in P\}$. But using (7) we have $\Delta(c^\alpha\kappa(u_\alpha) - c\kappa(u)) = c_{(1)}{}^\beta\otimes c_{(2)}{}^\alpha\kappa(u_{\alpha\beta}) - c_{(1)}\otimes c_{(2)}\kappa(u) = c_{(1)}\otimes(c_{(2)}{}^\alpha\kappa(u_\alpha) - c_{(2)}\kappa(u)) + (c_{(1)}{}^\beta\kappa(u_{\alpha\beta}) - c_{(1)}\kappa(u_\alpha))\otimes c_{(2)}{}^\alpha \in C\otimes I_\kappa + I_\kappa\otimes C$. Hence I_κ is a coideal.
4. The stated map $\zeta^M(c\otimes u) = c_{(1)}\otimes c_{(2)}{}^\alpha\kappa(u_\alpha)$ has its image in $C\otimes^M C$ since

$$c_{(1)}\otimes\pi_\kappa(c_{(2)})\otimes c_{(3)}{}^\alpha\kappa(u_\alpha) = c_{(1)}\otimes\pi_\kappa(c_{(2)}{}^\beta)\kappa(u_{\alpha\beta})\otimes c_{(3)}{}^\alpha$$

using (7) and $\pi_\kappa(I_\kappa) = 0$. By dimensions in the finite-dimensional case, it is natural to require that this is an isomorphism. $\square$

This is also an example of a dual version of the theory of C-Galois extensions. The proposition is dual to Proposition 2.2 in the sense that all arrows are reversed. In concrete terms, if P, C are finite-dimensional then $\psi^* : P^*\otimes C^*\to C^*\otimes P^*$ and $\kappa\in P^*$ make $C^*(M^*, P^*, \psi^*, \kappa)$ a ψ^*-principal bundle. Here $M^* = \{f\in C^* | (\kappa\otimes f)\circ\psi = f\otimes\kappa\}$. If C, P are entwined and we have both $e\in C$ and $\kappa : P\to k$, we can have both a ψ-principal bundle and a dual one at the same time. An obvious example, in the setting of Example 2.3, is $P = C = H$ a Hopf algebra and $\psi(c\otimes u) = u_{(1)}\otimes cu_{(2)}$ by the coproduct.

Then Proposition 2.2 with $e = 1$ gives a quantum principal bundle with $M = k$ and right coaction given by the coproduct. On the other hand, Proposition 2.6 with $\kappa = \epsilon$ gives a dual bundle with action by right multiplication.

Finally, we note that there is a close connection with the theory of factorisation of (augmented) algebras introduced in [Maj90, Maj94b] as part of a factorisation theory of Hopf algebras. According to this theory, a factorisation of an algebra X into subalgebras A, B (so that the product $A \otimes B \to X$ is a linear isomorphism) is equivalent to a *factorisation structure* $\tilde{\psi} : B \otimes A \to A \otimes B$ with certain properties. It was also shown that when A, B are augmented by algebra characters then the factorisation structure induces a right action of A on B and a left action of B on A, respectively.

Proposition 2.7. *Let C be finite-dimensional. Then an entwining structure $\psi : C \otimes P \to P \otimes C$ is equivalent by partial dualisation to a factorisation structure $\tilde{\psi} : C^* \otimes P^{\mathrm{op}} \to P^{\mathrm{op}} \otimes C^*$. In the augmented case, the induced coaction Δ_R^1 and action $\triangleleft^1$ in Propositions 2.2 and 2.6 are the dualisations of the actions induced by the factorisation.*

Proof. We use the notation $\tilde{\psi}(f \otimes u) = u_i \otimes f^i$ say, for $f \in C^*$ and $u \in P$. The equivalence with ψ is by $u_i\langle f^i, c\rangle = u_\alpha\langle f, c^\alpha\rangle$, where $\langle\ ,\ \rangle$ denotes the evaluation pairing. It is easy to see that ψ entwines C, P iff $\tilde{\psi}$ obeys [Maj94b]cf. [Maj90]

$$\tilde{\psi} \circ (\mu \otimes \mathrm{id}) = (\mathrm{id} \otimes \mu) \circ \tilde{\psi}_{12} \circ \tilde{\psi}_{23}, \quad \tilde{\psi}(f \otimes 1) = 1 \otimes f, \tag{9}$$

$$\tilde{\psi} \circ (\mathrm{id} \otimes \mu) = (\mu \otimes \mathrm{id}) \circ \tilde{\psi}_{23} \circ \tilde{\psi}_{12}, \quad \tilde{\psi}(1 \otimes u) = u \otimes 1 \tag{10}$$

for all $f \in C^*$ and $u \in P^{\mathrm{op}}$. Thus, the first of these is $u_i\langle c, (fg)^i\rangle = u_\alpha\langle c^\alpha, fg\rangle = u_\alpha\langle {c^\alpha}_{(1)}, f\rangle\langle {c^\alpha}_{(2)}, g\rangle = u_{\alpha\beta}\langle {c_{(1)}}^\beta, f\rangle\langle {c_{(2)}}^\alpha, g\rangle = u_{\alpha i}\langle c_{(1)}, f^i\rangle\langle {c_{(2)}}^\alpha, g\rangle = u_{ji}\langle c_{(1)}, f^i\rangle \langle c_{(2)}, g^j\rangle$ using (7). Similarly for (10) using (6), provided we remember to use the opposite product on P. Such data $\tilde{\psi}$ is equivalent by [Maj94b, Maj90] to the existence of an algebra X factorising into $P^{\mathrm{op}}C^*$. Given such X we recover $\tilde{\psi}$ by $uc = \mu \circ \tilde{\psi}(c \otimes u)$ in X, and conversely, given $\tilde{\psi}$ we define $X = P^{\mathrm{op}} \underline{\otimes} C^*$ as in (8), but with $\tilde{\psi}$. Also from this theory, if we have κ an algebra character on P^{op} (or on P) then $\triangleleft = (\kappa \otimes \mathrm{id}) \circ \tilde{\psi}$ is a right action of P^{op} on C^*, which clearly dualises to the right action of P on C in Proposition 2.6. Similarly, if e is a character on C^* then $\triangleright = (\mathrm{id} \otimes e) \circ \tilde{\psi}$ is a left action of C^* on P^{op} (or on P) which clearly dualises to the right coaction of C in Proposition 2.2. $\square$

An obvious setting in which factorisations arise is the braided tensor product (8) of algebras in braided categories [Maj91, Maj93b, Maj93a], with $\tilde{\psi} = \Psi$ the braiding. Thus if $A \underline{\otimes} B$ is a braided tensor product of algebras (e.g. of module algebras under a background quantum group) we can look for a suitable dual coalgebra B^* in the category and the corresponding entwining ψ of B^*, A^{op}. This provides a large class of entwining structures.

Another source is the theory of double cross products $G \bowtie H$ of Hopf algebras in [Maj90]. These factorise as Hopf algebras and hence, in particular, as algebras. In this context, Proposition 2.7 can be combined with the result in [Maj90, Sect. 3.2] that the double cross product is equivalent by partial dualisation to a *bicrossproduct* $H^* \blacktriangleright\!\!\triangleleft G$. These bicrossproduct Hopf algebras (also due to the second author) provided one of the first general constructions for non-commutative and non-cocommutative Hopf algebras, and many examples are known.

Proposition 2.8. *Let $C \blacktriangleright\!\!\triangleleft P^{\mathrm{op}}$ be a bicrossproduct bialgebra [Maj90, Sect. 3.1], where P^{op}, C are bialgebras suitably (co)acting on each other. Then C, P are entwined by*

$$\psi(c \otimes u) = u_{(1)}{}^{(\bar{0})} \otimes u_{(1)}{}^{(\bar{1})}(u_{(2)} \triangleright c).$$

Here $\triangleright$ is the left action of P^{op} on C and $u^{(\bar{0})} \otimes u^{(\bar{1})}$ is the right coaction of C on P^{op}, as part of the bicrossproduct construction.

Proof. We derive this result under the temporary assumption that C is finite-dimensional. Thus the bicrossproduct is equivalent to a double cross product $P^{\mathrm{op}} \bowtie C^*$ with actions $\triangleright, \triangleleft$ defined by $f \triangleright u = u^{(\bar{0})}\langle f, u^{(\bar{1})}\rangle$ and $\langle u \triangleright c, f\rangle = \langle c, f \triangleleft u\rangle$ for all $f \in C^*$. Then $\tilde{\psi}$ for this factorisation is $\tilde{\psi}(f \otimes u) = f_{(1)} \triangleright u_{(1)} \otimes f_{(2)} \triangleleft u_{(2)}$ according to [Maj90, Maj94b]. The correspondence in Proposition 2.7 then gives ψ as stated. Once the formula for ψ is known, one may verify directly that it entwines C, P given the compatibility conditions between the action and coaction of a bicrossproduct in [Maj90, Sect. 3.1]. □

Now we describe trivial ψ-principal bundles and gauge transformations in them.

Proposition 2.9. *Let P and C be entwined by ψ as in Definition 2.1 and let e be a group-like element in C. Assume the following data:*

1. A map $\psi^C : C \otimes C \to C \otimes C$ such that

$$(\mathrm{id} \otimes \Delta) \circ \psi^C = \psi^C_{12} \circ \psi^C_{23} \circ (\Delta \otimes \mathrm{id}), \quad (\mathrm{id} \otimes \epsilon) \circ \psi^C = \epsilon \otimes \mathrm{id}, \tag{11}$$

and $\psi^C(e \otimes c) = \Delta c$, for any $c \in C$;

2. A convolution invertible map $\Phi : C \to P$ such that $\Phi(e) = 1$ and

$$\psi \circ (\mathrm{id} \otimes \Phi) = (\Phi \otimes \mathrm{id}) \circ \psi^C. \tag{12}$$

Then there is a ψ-principal bundle over $M = P_e^{coC}$ with structure coalgebra C and total space P. We call it the trivial ψ-principal bundle $P(M, C, \Phi, \psi, \psi^C, e)$ *associated to our data, with trivialisation Φ.*

Proof. The proof of the proposition is similar to the proof that the trivial quantum principal bundle in [BM93, Example 4.2] is in fact a quantum principal bundle. First we observe that the map

$$\Theta : M \otimes C \to P, \qquad x \otimes c \mapsto x\Phi(c)$$

is an isomorphism of linear spaces. Explicitly the inverse is given by

$$\Theta^{-1} : u \mapsto u^{(\bar{0})}\Phi^{-1}(u^{(\bar{1})}{}_{(1)}) \otimes u^{(\bar{1})}{}_{(2)},$$

where $\Phi^{-1} : C \to P$ is a convolution inverse of Φ, i.e.

$$\Phi^{-1}(c_{(1)})\Phi(c_{(2)}) = \Phi(c_{(1)})\Phi^{-1}(c_{(2)}) = \epsilon(c)1.$$

To see that the image of the above map is in $M \otimes C$ we first notice that (12) implies that $\Delta^1_R \circ \Phi = (\Phi \otimes \mathrm{id}) \circ \Delta$ and that

$$\psi(c_{(1)} \otimes \Phi^{-1}(c_{(2)})) = \Phi^{-1}(c) \otimes e. \tag{13}$$

Therefore for any $u \in P$,

$$\Delta^1_R(u^{(\bar{0})}\Phi^{-1}(u^{(\bar{1})})) = u^{(\bar{0})}\psi(u^{(\bar{1})}{}_{(1)} \otimes \Phi^{-1}(u^{(\bar{1})}{}_{(2)})) = u^{(\bar{0})}\Phi^{-1}(u^{(\bar{1})}) \otimes e,$$

and thus $u^{\bar{(0)}}\Phi^{-1}(u^{\bar{(1)}}) \in M$. Then it is easy to prove that the above maps are inverses to each other.

We remark that Θ is in fact a left M-module and a right C-comodule map, where the coaction in $M \otimes C$ is given by $x \otimes c \mapsto x \otimes c_{(1)} \otimes c_{(2)}$. Moreover $\psi \circ (\mathrm{id} \otimes \Theta) = (\Theta \otimes \mathrm{id}) \circ \psi^C_{23} \circ \psi_{12}$.

The proof that χ_M in this case is a bijection follows exactly the method used in the proof of [BM93, Example 4.2] and thus we do not repeat it here. □

Next, we consider gauge transformations.

Definition 2.10. *Let $P(M, C, \Phi, \psi, \psi^C, e)$ be a trivial ψ-principal bundle as in Proposition 2.9. We say that a convolution invertible map $\gamma : C \to M$ such that $\gamma(e) = 1$ is a* gauge transformation *if*

$$\psi^C_{23} \circ \psi_{12} \circ (\mathrm{id} \otimes \gamma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = (\gamma \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \psi^C. \tag{14}$$

Proposition 2.11. *If $\gamma : C \to M$ is a gauge transformation in $P(M, C, \Phi, \psi, \psi^C, e)$ then $\Phi' = \gamma * \Phi$, where $*$ denotes the convolution product is a trivialisation of $P(M, C, \Phi, \psi, \psi^C, e)$. The set of all gauge transformations in $P(M, C, \Phi, \psi, \psi^C, e)$ is a group with respect to the convolution product. We say that two trivialisations Φ and Φ' are* gauge equivalent *if there exists a gauge transformation γ such that $\Phi' = \gamma * \Phi$.*

Proof. Clearly Φ' is a convolution invertible map such that $\Phi'(e) = 1$. To prove that it satisfies (12) we first introduce the notation

$$\psi^C(b \otimes c) = c_A \otimes b^A \qquad \text{(summation assumed)},$$

in which the condition (14) reads explicitly

$$\gamma(c_{(1)})_\alpha \otimes c_{(2)A} \otimes b^{\alpha A} = \gamma(c_{A(1)}) \otimes c_{A(2)} \otimes b^A,$$

and then compute

$$\begin{aligned}\psi \circ (\mathrm{id} \otimes \Phi')(b \otimes c) &= \psi(b \otimes \gamma(c_{(1)})\Phi(c_{(2)})) = \gamma(c_{(1)})_\alpha \Phi(c_{(2)})_\beta \otimes b^{\alpha\beta}\\ &= \gamma(c_{(1)})_\alpha \Phi(c_{(2)A}) \otimes b^{\alpha A} = \gamma(c_{A(1)})\Phi(c_{A(2)}) \otimes b^A\\ &= (\Phi' \otimes \mathrm{id}) \circ \psi^C(b \otimes c).\end{aligned}$$

This proves the first part of the proposition.

Assume now that γ_1, γ_2 are gauge transformations. Then

$$\begin{aligned}(\gamma_1(c_{(1)})\gamma_2(c_{(2)}))_\alpha \otimes c_{(3)A} \otimes b^{\alpha A} &= \gamma_1(c_{(1)})_\alpha \gamma_2(c_{(2)})_\beta \otimes c_{(3)A} \otimes b^{\alpha\beta A}\\ &= \gamma_1(c_{(1)})_\alpha \gamma_2(c_{(2)A(1)}) \otimes c_{(2)A(2)} \otimes b^{\alpha A}\\ &= \gamma_1(c_{A(1)})\gamma_2(c_{A(2)}) \otimes c_{A(3)} \otimes b^A.\end{aligned}$$

Therefore $\gamma_1 * \gamma_2$ is a gauge transformation too. Clearly ϵ is a gauge transformation and thus provides the unit. Finally, to prove that if γ is a gauge transformation then so is γ^{-1}, we observe that if $\gamma_3 = \gamma_1 * \gamma_2$ and γ_2 are gauge transformations then so is γ_1. Indeed, if $\gamma_1 * \gamma_2$ is a gauge transformation then

$$(\gamma_1(c_{(1)})\gamma_2(c_{(2)}))_\alpha \otimes c_{(3)A} \otimes b^{\alpha A} = \gamma_1(c_{A(1)})\gamma_2(c_{A(2)}) \otimes c_{A(3)} \otimes b^A,$$

but since γ_2 is a gauge transformation, we obtain

$$\gamma_1(c_{(1)})_\alpha \gamma_2(c_{(2)}{}_{A(1)}) \otimes c_{(2)}{}_{A(2)} \otimes b^{\alpha A} = \gamma_1(c_{A(1)})\gamma_2(c_{A(2)}) \otimes c_{A(3)} \otimes b^A.$$

Applying γ_2^{-1} to the second factor in the tensor product and then multiplying the first two factors we obtain

$$\gamma_1(c_{(1)})_\alpha \otimes c_{(2)}{}_A \otimes b^{\alpha A} = \gamma_1(c_{A(1)}) \otimes c_{A(2)} \otimes b^A,$$

i.e. γ_1 is a gauge transformation as stated. Now applying this result to $\gamma_3 = \epsilon$ and $\gamma_2 = \gamma$ we deduce that γ^{-1} is a gauge transformation as required. This completes the proof of the proposition. □

Although the existence of the map ψ^C as in Proposition 2.9 is not guaranteed for all coalgebras, the map ψ^C exists in most of the examples discussed in this section:

Example 2.12. For a quantum principal bundle $P(M, H)$ as in Example 2.3, we define

$$\psi^H(b \otimes c) = c_{(1)} \otimes bc_{(2)},$$

for all $b, c \in H$. Then (2.9)–(2.11) reduces to the theory of trivial quantum principal bundles and their gauge transformations in [BM93].

Proof. It is easy to see by standard Hopf algebra calculations that (11) is satisfied by the bialgebra axiom for $H = C$ in this case. Moreover, (12) reduces to Φ being an intertwiner of Δ_R with Δ. The condition (14) is empty. This recovers the setting introduced in [BM93]. □

In the braided case we use the above theory to arrive at a natural definition of trivial braided principal bundle:

Example 2.13. For a braided principal bundle $P(M, B, \Psi)$ as in Example 2.4, we define a trivialisation as a convolution-invertible unital morphism $\Phi : B \to P$ in the braided category such that $\Delta_R \circ \Phi = (\Phi \otimes \mathrm{id}) \circ \underline{\Delta}$, where Δ_R is the braided right coaction of B on P. We define a gauge transformation as a convolution-invertible unital morphism $\gamma : B \to M$, acting on trivialisations by the convolution product $*$. This is a trivial ψ-principal bundle with

$$\psi^B(b \otimes c) = \Psi(b \otimes c_{\underline{(1)}})c_{\underline{(2)}},$$

where $\underline{\Delta} c = c_{\underline{(1)}} \otimes c_{\underline{(2)}}$ is the braided group coproduct.

Proof. This time, (11) follows from the braided-coproduct homomorphism property of a braided group [Maj91]. From this and the form of ψ, we see that (12) becomes $\Delta_R \circ \Phi(c) = ((\Phi \otimes \mathrm{id}) \circ \Psi(b \otimes c_{\underline{(1)}}))c_{\underline{(2)}}$. Setting $b = e$ gives the condition stated on Φ because the braiding with $e = 1$ is always trivial. Assuming the stated condition, (12) then becomes $\Phi(c_{\underline{(1)}}) \otimes bc_{\underline{(2)}} = ((\Phi \otimes \mathrm{id}) \circ \Psi(b \otimes c_{\underline{(1)}}))c_{\underline{(2)}}$, which is equivalent (by replacing $c_{\underline{(2)}}$ by $c_{\underline{(2)}} \otimes \underline{S}c_{\underline{(3)}}$ and multiplying, where $\underline{S}$ is the braided antipode) to

$$(\Phi \otimes \mathrm{id}) \circ \Psi = \Psi \circ (\mathrm{id} \otimes \Phi).$$

When all our constructions take place in a braided category, this is the functoriality property implied by requiring that Φ is a morphism in the category. The theory of trivial ψ-bundles only requires this functoriality condition itself. Similarly, we compute the gauge condition (14) using $\psi(b \otimes \gamma(c)) = \Psi(b \otimes \gamma(c))$ because $\gamma(c) \in M$, and operate on it by replacing $c_{(2)}$ by $c_{(2)} \otimes \underline{S}c_{(3)}$ and multiplying. Then it reduces to $\Psi_{23} \circ \Psi_{12} \circ (\mathrm{id} \otimes \gamma \circ \underline{\Delta}) = (\gamma \circ \underline{\Delta} \otimes \mathrm{id}) \circ \Psi$. Since $\underline{\Delta}$ is a morphism, we see (by applying the braided group counit) that the gauge condition (14) is equivalent to

$$(\gamma \otimes \mathrm{id}) \circ \Psi = \Psi \circ (\mathrm{id} \otimes \gamma).$$

As before, this is naturally implied by requiring that γ is a morphism in our braided category. It is clear that the convolution product $*$ preserves the property of being a morphism since $\underline{\Delta}$ and Δ_R are assumed to be morphisms. □

For a ψ-principal bundle over a quantum homogeneous space as in Example 2.5, we can define a trivialisation if, for example, the map

$$\psi^C(b \otimes c) = \pi(v_{(1)}) \otimes \pi(uv_{(2)}), \tag{15}$$

where $u \in \pi^{-1}(b)$, $v \in \pi^{-1}(c)$ is well-defined. Then a trivialisation of the bundle is a convolution-invertible map $\Phi : C \to H$ obeying $\Phi \circ \pi(1) = 1$ and

$$\Phi(c)_{(1)} \otimes \pi(u\Phi(c)_{(2)}) = \Phi \circ \pi(v_{(1)}) \otimes \pi(uv_{(2)}) \tag{16}$$

for all $c \in C, u \in H$, and $v \in \pi^{-1}(c)$. Taking $u = 1$ requires, in particular, the natural intertwiner condition $(\Phi \otimes \mathrm{id}) \circ \Delta = \Delta_R \circ \Phi$. There is, similarly, a condition on gauge transformations γ obtained from (14). Hence our formulation of trivial ψ-principal bundles covers all the main sources of ψ-principal bundles discussed in this section.

We conclude this section with some explicit examples of ψ-principal bundles.

Example 2.14. Let H be a quantum cylinder $A_q^{2|0}[x^{-1}]$, i.e. a free associative algebra generated by x, x^{-1} and y subject to the relations $yx = qxy$, $xx^{-1} = x^{-1}x = 1$, with a natural Hopf algebra structure:

$$\Delta x^{\pm 1} = x^{\pm 1} \otimes x^{\pm 1}, \qquad \Delta y = 1 \otimes y + y \otimes x, \quad etc. \tag{17}$$

Consider a right ideal J in H generated by $x - 1$ and $x^{-1} - 1$. Clearly, J is a coideal and therefore $C = A_q^{2|0}[x^{-1}]/J$ is a coalgebra and a canonical epimorphism $\pi : H \to C$ is a coalgebra map. C is spanned by the elements $c_n = \pi(y^n)$, $n \in \mathbb{Z}_{\geq 0}$, and the coproduct and the counit are given by

$$\Delta c_n = \sum_{k=0}^{n} \binom{n}{k}_q c_k \otimes c_{n-k}, \qquad \epsilon(c_n) = 0. \tag{18}$$

We are in the situation of Example 2.5 and thus we have the entwining structure $\psi : C \otimes H \to H \otimes C$, which explicitly computed comes out as

$$\psi(c_l \otimes x^m y^n) = \sum_{k=0}^{n} \binom{n}{k}_q q^{l(k+m)} x^m y^k \otimes c_{n+l-k}, \tag{19}$$

where

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!},$$

and

$$[n]_q! = [n]_q \cdots [2]_q[1]_q, \quad [0]_q! = 1, \qquad [n]_q = 1 + q + \ldots + q^{n-1}.$$

From this definition of ψ one easily computes the right coaction of C on H as well as the fixed point subalgebra $M = k[x, x^{-1}]$, i.e. the algebra of functions on a circle. By Example 2.5 we have just constructed a generalised quantum principal bundle $A_q^{2|0}[x^{-1}](k[x, x^{-1}], C, \psi, c_0)$.

Finally we note that the above bundle is trivial in the sense of Proposition 2.9. The trivialisation $\Phi : C \to A_q^{2|0}[x^{-1}]$ and its inverse Φ^{-1} are defined by

$$\Phi(c_n) = y^n, \qquad \Phi^{-1}(c_n) = (-1)^n q^{n(n-1)/2} y^n. \tag{20}$$

One can easily check that the map Φ satisfies the required conditions. Explicitly, the map $\psi^C : C \otimes C \to C \otimes C$ reads

$$\psi^C(c_m \otimes c_n) = \sum_{k=0}^{n} \binom{n}{k}_q q^{km} c_k \otimes c_{m+n-k}.$$

Therefore

$$\begin{aligned}\psi \circ (\mathrm{id} \otimes \Phi)(c_m \otimes c_n) = \psi(c_m \otimes y^n) &= \sum_{k=0}^{n} \binom{n}{k}_q q^{km} y^k \otimes c_{m+n-k} \\ &= (\Phi \otimes \mathrm{id}) \circ \psi^C(c_m \otimes c_n).\end{aligned}$$

Since the bundle discussed in this example is trivial, we can compute its gauge group. One easily finds that a convolution invertible map $\gamma : C \to k[x, x^{-1}]$ satisfies condition (14) if and only if $\gamma(c_n) = \Gamma_n x^n$ (no summation), where $n \in \mathbb{Z}_{\geq 0}$, $\Gamma_n \in k$ and $\Gamma_0 = 1$. Therefore the gauge group is equivalent to the group of sequences $\Gamma = (1, \Gamma_1, \Gamma_2, ...)$ with the product given by

$$(\Gamma \cdot \Gamma')_n = \sum_{k=0}^{n} \binom{n}{k}_q \Gamma_k \Gamma'_{n-k}.$$

For the simplest example of a braided principal bundle, one can simply take any braided group B and any algebra M in the same braided category. Then the braided tensor product algebra $P = M \underline{\otimes} B$, along with the definitions

$$\Delta_R = \mathrm{id} \otimes \underline{\Delta}, \quad \Phi(b) = 1 \otimes b, \quad \Phi^{-1}(b) = 1 \otimes \underline{S} b \tag{21}$$

put us in the setting of Examples 2.4 and 2.13. Note first that Δ_R is a coaction (the tensor product of the trivial coaction and the right coregular coaction) and makes P into a braided comodule algebra. Moreover, the induced map

$$\chi_M(m \otimes b \otimes n \otimes c) = m \Psi(b \otimes n) c_{\underline{(1)}} \otimes c_{\underline{(2)}}$$

for $m, n \in M$, $b, c \in B$, is an isomorphism $P \otimes_M P \to P \otimes P$; it has inverse

$$\chi_M^{-1}(m \otimes b \otimes c) = m \otimes b \underline{S} c_{\underline{(1)}} \otimes 1 \otimes c_{\underline{(2)}}.$$

It is also clear that Φ is a trivialisation. This is truly a trivial braided principal bundle because P is just a (braided) tensor product algebra.

Example 2.15. Let $B = k[c]$ be the braided line generated by c with braiding $\Psi(c \otimes c) = qc \otimes c$ and the linear coproduct $\underline{\Delta} c = c \otimes 1 + 1 \otimes c$. It lives in the braided category Vec_q of $\mathbb{Z}$-graded vector spaces with braiding $q^{\deg(\ )\deg(\ )}$ times the usual transposition. Here $\deg(c) = 1$. Let $M = k[x, x^{-1}]$ be viewed as a $\mathbb{Z}$-graded algebra as well, with $\deg(x) = 1$. Then $P = k[x, x^{-1}]\underline{\otimes} k[c]$ is a trivial braided principal bundle with the coaction and trivialisation

$$\Delta_R(x^m \otimes c^n) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}_q x^m \otimes c^k \otimes c^{n-k}, \quad \Phi(c^n) = 1 \otimes c^n. \tag{22}$$

As a ψ-principal bundle, this example clearly coincides with the preceding one, albeit constructed quite differently: we identify $c^n = c_n$ and $y = 1 \otimes c$, and note that in the braided tensor product algebra $k[x, x^{-1}]\underline{\otimes} k[c]$ we have the product $(1 \otimes c)(x \otimes 1) = \Psi(c \otimes x) = q(x \otimes 1)(1 \otimes c)$, i.e. $P = A_q^{2|0}[x^{-1}]$. It is also clear that the coproduct deduced in (18) can be identified with the braided line coproduct which is part of our initial data here. This particular braided tensor product algebra P is actually the algebra part of the bosonisation of $B = k[c]$ viewed as living in the category of comodules over $k[x, x^{-1}]$ as a dual-quasitriangular Hopf algebra (see [Maj95, p. 510]), and becomes in this way a Hopf algebra. This bosonisation is the Hopf algebra H which was part of the initial data in the preceding example. Finally, gauge transformations γ from the braided point of view are arbitrary degree-preserving unital maps $k[c] \to k[x, x^{-1}]$, i.e. given by the group of sequences Γ as found before.

This example demonstrates the strength of braided group gauge theory; even the most trivial braided quantum principal bundles may be quite complicated when constructed by more usual Hopf algebraic means. On the other hand, the following embeddable quantum homogeneous space does not appear to be of the braided type, nor (as far as we know) a trivial bundle.

Example 2.16. Let P be the algebra of functions on the quantum group $GL_q(2)$. This is generated by elements α, β, γ, δ and D subject to the relations

$$\alpha\beta = q\beta\alpha, \qquad \alpha\gamma = q\gamma\alpha, \qquad \alpha\delta = \delta\alpha + (q - q^{-1})\beta\gamma, \qquad \beta\gamma = \gamma\beta,$$
$$\beta\delta = q\delta\beta, \qquad \gamma\delta = q\delta\gamma, \qquad (\alpha\delta - q\beta\gamma)D = D(\alpha\delta - q\beta\gamma) = 1.$$

Let C be a vector space spanned by $c_{m,n}$, $m \in \mathbb{Z}_{>0}$, $n \in \mathbb{Z}$ with the coalgebra structure

$$\Delta(c_{i,j}) = \sum_{k=0}^{m} q^{k(m-k)} \begin{pmatrix} m \\ k \end{pmatrix}_{q^{-2}} c_{k,n} \otimes c_{m-k,n+k}, \qquad \epsilon(c_{m,n}) = \delta_{m0}.$$

Let the linear map $\psi : C \otimes P \to P \otimes C$ be given by

$$\begin{aligned} &\psi(c_{i,j} \otimes \alpha^k \gamma^l \beta^m \delta^n D^r) \\ &= \sum_{s=0}^{m} \sum_{t=0}^{n} \begin{pmatrix} m \\ s \end{pmatrix}_{q^{-2}} \begin{pmatrix} n \\ t \end{pmatrix}_{q^{-2}} q^{(m-s)(s+t-l)+(n-t)t-i(k+l-t-s)} \times \\ &\quad \alpha^{k+m-s} \gamma^{l+n-t} \beta^s \delta^t D^r \otimes c_{i+m+n-s-t, j-r+t+s}. \end{aligned} \tag{23}$$

Then ψ entwines P with C. Furthermore if we take $e = c_{0,0}$ then the fixed point subalgebra P_e^C is generated by $1, \alpha, \gamma$ and hence it is isomorphic to $A_{1/q}^{2|0}$ and there is a ψ-principal bundle $P(A_{1/q}^{2|0}, C, \psi, e)$.

Proof. The algebra $GL_q(2)$ can be equipped with the standard Hopf algebra structure

$$\Delta\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\otimes\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \qquad S\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = D\begin{pmatrix}\delta & -q^{-1}\beta\\ -q\gamma & \alpha\end{pmatrix},$$

$\epsilon(\alpha) = \epsilon(\delta) = 1$, $\epsilon(\beta) = \epsilon(\gamma) = 0$. We define a surjection $\pi : GL_q(2) \to C$ by

$$\pi(\alpha^k\beta^l\gamma^m\delta^n D^r) = \delta_{m0}q^{ln}c_{l,n-r}.$$

In Sect. 5 of [Brz96b] it is shown that π is a coalgebra map and that the data $H = GL_q(2)$, π, C satisfy requirements of Example 2.5. Therefore we have a ψ-principal bundle with ψ as in Example 2.5. Written explicitly this ψ is exactly as in Eq. (23).

In [Brz96b] it is also noted that the coalgebra C can be equipped with the algebra structure of $A^{2|0}_{q^{-2}}[x^{-1}]$ by setting $c_{m,n} = q^{-mn}x^n y^m$. The coproduct in C is then the same as in Example 2.14, Eq. (17). □

3. Connections in the Universal Differential Calculus Case

From the first assertion of Proposition 2.2 we know that the natural coaction $\Delta_R = \Delta_R^1$ of C on P extends to the coaction of C on the tensor product algebra $P^{\otimes n}$ for any positive integer n. Still most importantly this coaction can be restricted to $\Omega^n P$ by the second assertion of Proposition 2.2. Therefore the coalgebra C coacts on the algebra of universal forms on P. The universal differential structure on P is covariant with respect to the coaction Δ_R^n in the following sense:

Proposition 3.1. *Let P, C, ψ and e be as in Proposition 2.2. Let* $\mathrm{d} : P \to \Omega^1 P$ *be the universal differential,* $\mathrm{d}u = 1\otimes u - u\otimes 1$ *extended to the whole of* ΩP *as in the Preliminaries. Then* $\overleftarrow{\psi}^{n} \circ (\mathrm{id}\otimes \mathrm{d}) = (\mathrm{d}\otimes\mathrm{id})\circ \overleftarrow{\psi}^{n-1}$ *for any integer $n > 1$. Therefore* $\Delta_R^n \circ (\mathrm{id}\otimes\mathrm{d}) = (\mathrm{d}\otimes\mathrm{id})\circ\Delta_R^{n-1}$.

Proof. We take $\upsilon = \sum_i u^{0,i}\otimes u^{1,i}\otimes\ldots\otimes u^{n,i} \in \Omega^n P$ (i.e., any adjacent product vanishes). Using conditions (4), and the explicit form of $\mathrm{d}\upsilon$ (1), for any $c\in C$ we compute

$$\begin{aligned}
&\overleftarrow{\psi}^{n+2}(c\otimes \mathrm{d}\upsilon)\\
&=\sum_{k=0}^{n+1}(-1)^k\sum_i u^{0,i}_{\alpha_0}\otimes\ldots\otimes u^{k-1,i}_{\alpha_{k-1}}\otimes 1_\beta\otimes u^{k,i}_{\alpha_k}\otimes\ldots\otimes u^{n,i}_{\alpha_n}\otimes c^{\alpha_0\ldots\alpha_{k-1}\beta\alpha_k\ldots\alpha_n}\\
&=\sum_{k=0}^{n+1}(-1)^k\sum_i u^{0,i}_{\alpha_0}\otimes\ldots\otimes u^{k-1,i}_{\alpha_{k-1}}\otimes 1\otimes u^{k,i}_{\alpha_k}\otimes\ldots\otimes u^{n,i}_{\alpha_n}\otimes c^{\alpha_0\ldots\ldots\alpha_n}\\
&=(\mathrm{d}\otimes\mathrm{id})\circ\overleftarrow{\psi}^{n+1}(c\otimes\upsilon). \qquad \square
\end{aligned}$$

To discuss a theory of connections in $P(M, C, \psi, e)$ it is important that the horizontal one forms $P\Omega^1 MP$ be covariant under the action of Δ_R^2 or, more properly, $\overleftarrow{\psi}^{2}$. The following lemma gives a criterion for the covariance of horizontal one-forms.

Lemma 3.2. *For a ψ-principal bundle $P(M, C, \psi, e)$ assume that $\psi(C\otimes M) \subset M\otimes C$. Then $\overleftarrow{\psi}^{2}(C \otimes P\Omega^1 MP) \subset P\Omega^1 MP \otimes C$.*

Proof. Using (6) one easily finds that for any $u, v \in P$, $x, y \in M$ and $c \in C$,

$$\overleftarrow{\psi}^{2}(c \otimes ux \otimes yv) = u_\alpha x_\beta \otimes y_\gamma v_\delta \otimes c^{\alpha\beta\gamma\delta}.$$

If we assume further that $x_\beta, y_\gamma \in M$ then the result follows. □

We will see later that the hypothesis of Lemma 3.2 is automatically satisfied for braided principal bundles of Example 2.4. In contrast, it is not necessarily satisfied for ψ-bundles on quantum embeddable homogeneous spaces of Example 2.5. For example, one can easily check that it is satisfied for the bundle discussed in Example 2.16. On the other hand the ψ-principal bundle over the quantum hyperboloid, which is an embeddable homogeneous space of $E_q(2)$ [BCGST96] fails to fulfil requirements of Lemma 3.2.

The covariance of ΩP and $P\Omega^1 MP$ enables us to define a connection in $P(M, C, \psi, e)$ in a way similar to the definition of a connection in a quantum principal bundle $P(M, H)$ (compare [BM93]).

Definition 3.3. *Let $P(M, C, \psi, e)$ be a generalised quantum principal bundle such that $\psi(C \otimes M) \subset M \otimes C$. A connection in $P(M, C, \psi, e)$ is a left P-module projection $\Pi : \Omega^1 P \to \Omega^1 P$ such that $\ker \Pi = P\Omega^1 MP$ and $\overleftarrow{\psi}^{2}(\mathrm{id} \otimes \Pi) = (\Pi \otimes \mathrm{id})\overleftarrow{\psi}^{2}$.*

It is clear that for a usual quantum principal bundle $P(M, H)$, Definition 3.3 coincides with the definition of a connection given in [BM93]. Thus, the condition in Lemma 3.2 always holds for ψ as in Example 2.3, while $\overleftarrow{\psi}^{2}(\mathrm{id} \otimes \Pi) = (\Pi \otimes \mathrm{id})\overleftarrow{\psi}^{2}$ if and only if $\Delta_R^2 \Pi = (\Pi \otimes \mathrm{id})\Delta_R^2$, which was the condition in [BM93].

In what follows we assume that the condition in Lemma 3.2 is satisfied. A connection Π in $P(M, C, \psi, e)$ can be equivalently described as follows. First we define a map $\phi : C \otimes P \otimes \ker \epsilon \to P \otimes \ker \epsilon \otimes C$ by the commutative diagram

$$\begin{array}{ccc} C \otimes \Omega^1 P & \xrightarrow{\ \overleftarrow{\psi}^{2}\ } & \Omega^1 P \otimes C \\ \big\downarrow{\scriptstyle \mathrm{id}\otimes\chi} & & \big\downarrow{\scriptstyle \chi\otimes\mathrm{id}} \\ C \otimes P \otimes \ker\epsilon & \xrightarrow{\ \ \phi\ \ } & P \otimes \ker\epsilon \otimes C \end{array}$$

where $\chi(u \otimes v) = u\psi(e \otimes v)$. The map ϕ is clearly well-defined. Indeed, because χ_M is a bijection, $\ker \chi = P\Omega^1 MP$ and then $\overleftarrow{\psi}^{2}(C \otimes \ker \chi) \subset \ker \chi \otimes C$, by Lemma 3.2. Therefore $\phi(0) = 0$.

By definition of $P(M, C, \psi, e)$ we have a short exact sequence of left P-module maps

$$0 \to P\Omega^1 MP \to \Omega^1 P \xrightarrow{\chi} P \otimes \ker \epsilon \to 0. \qquad (24)$$

The exactness of the above sequence is clear since the fact that χ_M is bijective implies that χ is surjective and ker $\chi = P\Omega^1 MP$. By definition, χ intertwines $\overleftarrow{\psi}^2$ with ϕ.

Proposition 3.4. *The existence of a connection Π in $P(M, C, \psi, e)$ is equivalent to the existence of a left P-module splitting $\sigma : P \otimes \ker \epsilon \to \Omega^1 P$ of the above sequence such that $\overleftarrow{\psi}^2 \circ (\mathrm{id} \otimes \sigma) = (\sigma \otimes \mathrm{id}) \circ \phi$.*

Proof. Clearly the existence of a left P-module projection is equivalent to the existence of a left P-module splitting. It remains to check the required covariance properties. Assume that σ has the required properties, then

$$\overleftarrow{\psi}^2 \circ (\mathrm{id} \otimes \Pi) = \overleftarrow{\psi}^2 \circ (\mathrm{id} \otimes \sigma) \circ (\mathrm{id} \otimes \chi)$$
$$= (\sigma \otimes \mathrm{id}) \circ \phi \circ (\mathrm{id} \otimes \chi) = (\sigma \circ \chi \otimes \mathrm{id}) \circ \overleftarrow{\psi}^2 = (\Pi \otimes \mathrm{id}) \circ \overleftarrow{\psi}^2 .$$

Conversely, if Π has the required properties, then one easily finds that

$$\overleftarrow{\psi}^2 \circ (\mathrm{id} \otimes \sigma \circ \chi) = (\sigma \otimes \mathrm{id}) \circ \phi \circ (\mathrm{id} \otimes \chi).$$

Since χ is a surjection the required property of σ follows. □

To each connection we can associate its connection one-form $\omega : \ker \epsilon \to \Omega^1 P$ by setting $\omega(c) = \sigma(1 \otimes c)$. [1] Similarly to the quantum bundle case of [BM93] we have

Proposition 3.5. *Let Π be a connection on $P(M, C, \psi, e)$. Then, for all $c \in \ker \epsilon$, the connection 1-form $\omega : \ker \epsilon \to \Omega^1 P$ has the following properties:*

1. $\chi \circ \omega(c) = 1 \otimes c$,
2. *For any $b \in C$, $\overleftarrow{\psi}^2 (b \otimes \omega(c)) = c^{(1)}{}_\alpha c^{(2)}{}_{\beta\gamma} \omega(e^\gamma) \otimes b^{\alpha\beta}$, where $c^{(1)} \otimes_M c^{(2)}$ (summation understood) denotes the* **translation map** $\tau(c) = \chi_M^{-1}(1 \otimes c)$ *in $P(M, C, \psi, e)$.*

Conversely, if ω is any linear map $\omega : \ker \epsilon \to \Omega^1 P$ obeying conditions 1-2, then there is a unique connection $\Pi = \mu \circ (\mathrm{id} \otimes \omega) \circ \chi$ in $P(M, C, \psi, e)$ such that ω is its connection 1-form.

Proof. For any $b \otimes u \otimes c \in C \otimes P \otimes \ker \epsilon$ the map ϕ is explicitly given by

$$\phi(b \otimes u \otimes c) = u_\alpha c^{(1)}{}_\beta c^{(2)}{}_{\gamma\delta} \otimes e^\delta \otimes b^{\alpha\beta\gamma}.$$

Therefore if ω is a connection one-form then

$$\overleftarrow{\psi}^2 (b \otimes \omega(c)) = \overleftarrow{\psi}^2 \circ (\mathrm{id} \otimes \sigma)(b \otimes 1 \otimes c)$$
$$= \sigma(c^{(1)}{}_\alpha c^{(2)}{}_{\beta\gamma} \otimes e^\gamma) \otimes b^{\alpha\beta}$$
$$= c^{(1)}{}_\alpha c^{(2)}{}_{\beta\gamma} \omega(e^\gamma) \otimes b^{\alpha\beta}.$$

Conversely, if $\omega : \ker \epsilon \to \Omega^1 P$ satisfies condition 1 then $\sigma = (\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \omega)$ gives a left P-module splitting of (24). Furthermore, Condition 2 implies

[1] We can equivalently think of a connection 1-form as a map $C \to \Omega^1 P$ given by $\omega(c - e\epsilon(c))$. This was the point of view adopted in [BM93].

$$\begin{aligned}(\sigma\otimes\mathrm{id})\circ\phi(b\otimes u\otimes c) &= \sigma(u_\alpha c^{(1)}{}_\beta c^{(2)}{}_{\gamma\delta}\otimes e^\delta)\otimes c^{\alpha\beta\gamma}\\ &= u_\alpha c^{(1)}{}_\beta c^{(2)}{}_{\gamma\delta}\,\omega(e^\delta)\otimes c^{\alpha\beta\gamma}\\ &= u_\alpha \overleftarrow{\psi}^2(b^\alpha\otimes\omega(c)) = (\mu\otimes\mathrm{id})\circ\overleftarrow{\psi}^3(b\otimes u\otimes\omega(c))\\ &= \overleftarrow{\psi}^2(b\otimes u\omega(c)) = \overleftarrow{\psi}^2\circ(\mathrm{id}\otimes\sigma)(b\otimes u\otimes c). \qquad\square\end{aligned}$$

Example 3.6. For a quantum principal bundle $P(M,H)$, Condition 2 in Proposition 3.5 is equivalent to the Ad_R-covariance of ω.

Proof. Using the definition of ψ in Example 2.3 one finds

$$\begin{aligned}c^{(1)}{}_\alpha c^{(2)}{}_{\beta\gamma}\otimes e^\gamma\otimes b^{\alpha\beta} &= c^{(1)}{}_\alpha c^{(2)}{}_\beta{}^{(\bar 0)}\otimes c^{(2)}{}_\beta{}^{(\bar 1)}\otimes b^{\alpha\beta}\\ &= c^{(1)(\bar 0)}c^{(2)}{}_\beta{}^{(\bar 0)}\otimes c^{(2)}{}_\beta{}^{(\bar 1)}\otimes b^\beta c^{(1)(\bar 1)}\\ &= c^{(1)(\bar 0)}c^{(2)(\bar 0)(\bar 0)}\otimes c^{(2)(\bar 0)(\bar 1)}\otimes bc^{(1)(\bar 1)}c^{(2)(\bar 1)}\\ &= \chi_M(c^{(1)(\bar 0)}\otimes_M c^{(2)(\bar 0)})\otimes bc^{(1)(\bar 1)}c^{(2)(\bar 1)}.\end{aligned}$$

From the covariance properties of the translation map [Brz96a] it then follows that

$$c^{(1)}{}_\alpha c^{(2)}{}_{\beta\gamma}\otimes e^\gamma\otimes b^{\alpha\beta} = \chi_M(\tau(c_{(2)}))\otimes b(Sc_{(1)})c_{(3)} = 1\otimes c_{(2)}\otimes bS(c_{(1)})c_{(3)}.$$

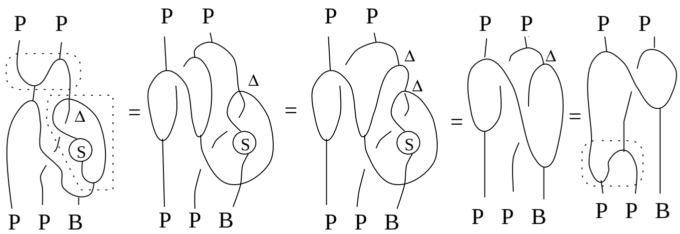
This also follows from covariance of χ_M as intertwining Δ_R^2 projected to $P\otimes_M P$ with the tensor product coaction $\Delta_R^1\otimes Ad_R$ on $P\otimes H$. Hence Condition 2 may be written as

$$\overleftarrow{\psi}^2(b\otimes\omega(c)) = \omega(c_{(2)})\otimes b(Sc_{(1)})c_{(3)}$$

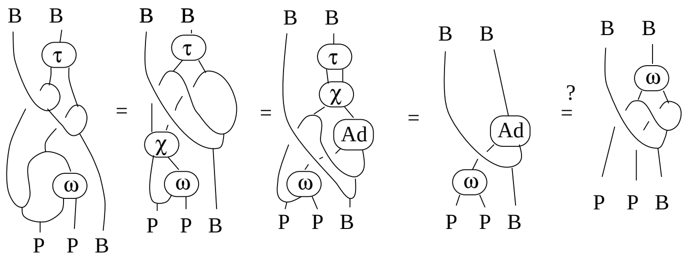
which is equivalent to $\Delta_R^2\circ\omega = (\omega\otimes\mathrm{id})\circ Ad_R$. $\square$

Example 3.7. For a braided group principal bundle $P(M,B,\Psi)$ in Example 2.4, Lemma 3.2 holds. Moreover, Condition 2 in Proposition 3.5 is equivalent to Ad_R-covariance of ω, where Ad_R is the braided adjoint coaction as in [Maj94a].

Proof. The braided group adjoint action is studied extensively in [Maj94a] as the basis of a theory of braided Lie algebras; we turn the diagrams up-side-down for the braided adjoint coaction and its properties (or see earlier works by the second author). Firstly, $\psi(B\otimes M)\subset M\otimes B$ is immediate since by properties of $e=1$, $\Psi(B\otimes M)\subset M\otimes B$. Also clear is that Δ_R^1 coincides with the given braided coaction of B on P and Δ_R^2 coincides with the braided tensor product coaction on $P\otimes P$. Δ_R^2 projects to a coaction on $P\otimes_M P$ by Lemma 3.2. We show first that $\chi_M: P\otimes_M P\to P\otimes B$ intertwines this coaction with the braided tensor product coaction $\Delta_R^1\otimes Ad_R$. We work with representatives in $P\otimes P$ and use the notation [Maj93a] as in the proof of Example 2.4. Branches $\wedge$ labelled Δ are the coproduct of B; otherwise they are the given coaction of B on P. Thus,



where the upper box on the left is χ_M and the lower box is the braided adjoint coaction Ad_R. S denotes the braided antipode of B. The tensor product $\Delta^1_R \otimes Ad_R$ uses the braiding and the product of B according to the theory of braided groups [Maj93b]. The first equality uses the homomorphism property of the given coaction of B on P. The second uses the comodule axiom. The third identifies an "antipode loop" and cancels it (using associativity and coassociativity, and the braided antipode axioms). The fourth equality uses the comodule axiom in reverse and also pushes the diagram into the form where we recognise the braided tensor product coaction Δ^2_R followed by χ_M. Using this intertwining property of χ_M, we write the right hand side of Condition 2 in Proposition 3.5 as



where $\tau = \chi_M^{-1}(1 \otimes (\ ))$. The left hand side $\overleftarrow{\psi}^2(b \otimes \omega(c))$ is shown on the right hand side of the diagram (using associativity of the product in B). Hence equality is equivalent to $\Delta^2_R \circ \omega = (\omega \otimes \mathrm{id}) \circ Ad_R$. $\square$

We remark that in the framework with C^* in place of C as explained in Proposition 2.7, we can use for C^* braided groups of enveloping algebra type, in particular $U(\mathcal{L})$ associated to a braided-Lie algebra $\mathcal{L}$ in [Maj94a] with braided-Lie bracket based on the properties of the braided adjoint action. In this case one could take $\omega \in \mathcal{L} \otimes \Omega^1 P$ with the corresponding covariance properties. Using the braided Killing form also in [Maj94a] one has the possibility (for the first time) to write down scalar Lagrangians built functorially from ω and its curvature. On the other hand, for a theory of trivial bundles (in order to have familiar formulae for gauge fields on the base) one needs to restrict trivialisations and gauge transforms in such a way that ω retains its values in $\mathcal{L}$. This aspect requires further work, to be developed elsewhere.

Example 3.8. Consider $H(M, C, \pi)$, the ψ-principal bundle associated to an embeddable quantum homogeneous space in Example 2.5. Assume that $\psi(C \otimes M) \subset M \otimes C$. Condition 2 in Proposition 3.5 is equivalent to

$$\Delta^2_R \circ \omega \circ \pi = (\omega \otimes \mathrm{id}) \circ (\pi \otimes \pi) \circ Ad_R. \tag{25}$$

In particular, this implies that any linear inclusion $i : M \to H$ such that $\pi \circ i = \mathrm{id}$ and $\epsilon(c) = \epsilon \circ i(c)$ gives rise to the canonical connection 1-form $\omega(c) = (Si(c)_{(1)})\mathrm{d}i(c)_{(2)}$, provided that

$$(\mathrm{id} \otimes \pi) \circ Ad_R \circ i = (i \otimes \mathrm{id}) \circ (\pi \otimes \pi) \circ Ad_R \circ i.$$

Proof. In this case $\psi(c \otimes v) = v_{(1)} \otimes \pi(uv_{(2)})$, and $\tau(c) = Su_{(1)} \otimes_M u_{(2)}$, for any $c \in C$, $v \in H$ and $u \in \pi^{-1}(c)$. Also $e = \pi(1)$. The transformation property of ω now reads

$$\overleftarrow{\psi}^2 (b \otimes \omega(c)) = (Su_{(1)})_\alpha u_{(2)_{\beta\gamma}} \omega(\pi(1)^\gamma) \otimes \pi(v)^{\alpha\beta}$$
$$= (Su_{(2)}) u_{(3)_{\beta\gamma}} \omega(\pi(1)^\gamma) \otimes \pi(vSu_{(1)})^\beta$$
$$= (Su_{(2)}) u_{(3)_\gamma} \omega(\pi(1)^\gamma) \otimes \pi(v(Su_{(1)})u_{(4)})$$
$$= (Su_{(2)}) u_{(3)} \omega(\pi(u_{(4)})) \otimes \pi(v(Su_{(1)})u_{(5)})$$
$$= \omega(\pi(u_{(2)})) \otimes \pi(v(Su_{(1)})u_{(3)}),$$

where $v \in \pi^{-1}(b)$. Choosing $v = 1$ we obtain property (25). The converse is obviously true. $\square$

Before we describe some concrete examples of connections we construct connections in the trivial ψ-bundles of Proposition 2.9.

Proposition 3.9. *Let $P(M, C, \Phi, \psi, \psi^C, e)$ be a trivial coalgebra ψ-principal bundle such that $\psi(C \otimes M) \subset M \otimes C$. Let $\beta : C \to \Omega^1 M$ be a linear map, $\beta(e) = 0$ and such that*

$$\psi^C_{34} \circ \psi_{23} \circ \psi_{12} \circ (\mathrm{id} \otimes \beta \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta) = (\beta \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \psi^C. \qquad (26)$$

Then the map $\omega : \ker \epsilon \to \Omega^1 P$,

$$\omega = \Phi^{-1} * \mathrm{d}\Phi + \Phi^{-1} * \beta * \Phi \qquad (27)$$

is a connection one-form in $P(M, C, \Phi, \psi, \psi^C, e)$. In particular for $\beta = 0$ we have a trivial connection in $P(M, C, \Phi, \psi, \psi^C, e)$.

Proof. To prove the proposition we will show that ω satisfies conditions specified in Proposition 3.5. Firstly, however, we observe that the translation map in $P(M, C, \Phi, \psi, \psi^C, e)$ is given by

$$\tau(c) = \Phi^{-1}(c_{(1)}) \otimes_M \Phi(c_{(2)}). \qquad (28)$$

Indeed, a trivial computation shows that $\chi_M(\tau(c)) = 1 \otimes c$, as required. The same computation shows that for any $c \in \ker \epsilon$,

$$\chi(\Phi^{-1}(c_{(1)})\mathrm{d}\Phi(c_{(2)}) + \Phi^{-1}(c_{(1)})\beta(c_{(2)})\Phi(c_{(3)})) = \chi(\Phi^{-1}(c_{(1)}) \otimes \Phi(c_{(2)})) = 1 \otimes c,$$

and therefore Condition 1 of Proposition 3.5 is satisfied by ω.

Now we prove that Condition 2 of Proposition 3.5 holds for $\Phi^{-1} * \mathrm{d}\Phi$ and $\Phi^{-1} * \beta * \Phi$ separately. For the former the left hand side of Condition 2 reads

$$LHS = \overleftarrow{\psi}^2 (b \otimes \Phi^{-1}(c_{(1)}) \otimes \Phi(c_{(2)})) = \Phi^{-1}(c_{(1)})_\alpha \otimes \Phi(c_{(2)})_\beta \otimes b^{\alpha\beta}$$
$$= \Phi^{-1}(c_{(1)})_\alpha \otimes \Phi(c_{(2)_A}) \otimes b^{\alpha A}$$

On the other hand we use the definition of τ (28) and the properties of Φ to write the right-hand side of condition 2 as follows:

$$RHS = \Phi^{-1}(c_{(1)})_\alpha \Phi(c_{(2)})_{\beta\gamma} \Phi^{-1}(e^\gamma{}_{(1)}) \otimes \Phi(e^\gamma{}_{(2)}) \otimes b^{\alpha\beta}$$
$$= \Phi^{-1}(c_{(1)})_\alpha \Phi(c_{(2)})_{\beta\gamma\delta} \Phi^{-1}(e^\delta) \otimes \Phi(e^\gamma) \otimes b^{\alpha\beta}$$
$$= \Phi^{-1}(c_{(1)})_\alpha \Phi(c_{(2)_A})_{\gamma\delta} \Phi^{-1}(e^\delta) \otimes \Phi(e^\gamma) \otimes b^{\alpha A}$$
$$= \Phi^{-1}(c_{(1)})_\alpha \Phi(c_{(2)_{A(1)}}) \Phi^{-1}(c_{(2)_{A(2)}}) \otimes \Phi(c_{(2)_{A(3)}}) \otimes b^{\alpha A}$$
$$= \Phi^{-1}(c_{(1)})_\alpha \otimes \Phi(c_{(2)_A}) \otimes b^{\alpha A} = LHS.$$

To compute the action of $\overleftarrow{\psi}^2$ on the second part of ω we will use the shorthand notation

$$\overleftarrow{\psi}^2(b \otimes \rho) = \rho_{\underline{\alpha}} \otimes b^{\underline{\alpha}},$$

for any $b \in C$ and $\rho \in \Omega^1 P$. In this notation Eq. (26) explicitly reads

$$\beta(c_{(1)})_{\underline{\alpha}} \otimes c_{(2)A} \otimes b^{\underline{\alpha}A} = \beta(c_{A(1)}) \otimes c_{A(2)} \otimes b^A.$$

Using the similar steps as in computation of the action of $\overleftarrow{\psi}^2$ on the first part of ω we find that the right hand side of Condition 2 reads

$$\Phi^{-1}(c_{(1)})_\alpha \beta(c_{(2)A(1)})\Phi(c_{(2)A(2)}) \otimes b^{\alpha A},$$

while the left hand side is

$$\Phi^{-1}(c_{(1)})_\alpha \beta(c_{(2)})_{\underline{\alpha}} \Phi(c_{(3)A}) \otimes b^{\alpha\underline{\alpha}A} = \Phi^{-1}(c_{(1)})_\alpha \beta(c_{(2)A(1)})\Phi(c_{(2)A(2)}) \otimes b^{\alpha A}.$$

From Proposition 3.5 we now deduce that ω is a connection one-form as stated. □

Using similar arguments as in [BM93] we can easily show that the behaviour of β under gauge transformations is exactly the same as in the case of quantum principal bundles. For example, if we make a gauge transformation of Φ, $\Phi \mapsto \gamma * \Phi$ and then view ω in this new trivialisation then the local connection one-from β will undergo the gauge transformation

$$\beta \mapsto \gamma^{-1} * \mathrm{d}\gamma + \gamma^{-1} * \beta * \gamma. \tag{29}$$

As before, we can specialise this theory to our various sources of ψ-principal bundles. For quantum principal bundles we recover the formalism in [BM93]. For braided principal bundles we make a computation similar to the one for γ in Example 2.13, finding that (26) is naturally ensured by requiring that $\beta : B \to \Omega^1 M$ is a morphism in our braided category. Then the same formulae (27) and transformation law (29) etc. apply in the braided case. Indeed, they do not involve any braiding directly.

Now we construct explicit examples of connections in one of the bundles described at the end of Sect. 2.

Example 3.10. Consider the quantum cylinder bundle $A_q^{2|0}[x^{-1}](k[x, x^{-1}], k[c], \psi, 1)$ in Example 2.14. Then $\psi(k[c] \otimes k[x, x^{-1}]) \subset k[x, x^{-1}] \otimes k[c]$. The most general connection of the type described in Proposition 3.9 has the form

$$\begin{aligned}\omega(c^n) &= \sum_{k=0}^{n-1}(-1)^k \binom{n}{k}_q q^{k(k-1)/2} y^k \mathrm{d}y^{n-k}\\ &+ \sum_i \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^k q^{k((k-1)/2+i)} \binom{n}{m}_q \binom{m}{k}_q \Gamma_{i,m-k} x^i y^k (\mathrm{d}x^{m-k-i}) y^{n-m},\end{aligned} \tag{30}$$

where for all $i \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$, $\Gamma_{n,i} \in k$, $\Gamma_{0,i} = 0$.

Proof. If we set $n = 0$ in formula (19) then we find $\psi(c^l \otimes x^m) = q^{lm} x^m \otimes c^l$ and the first assertion holds. This assertion also follows from Example 3.7. We identify $C = k[c]$ by $c_n = c^n$, as a certain (braided) coalgebra.

It is an easy exercise to check that a map $\beta : k[c] \to \Omega^1 k[x, x^{-1}]$ satisfies condition (26) if and only if

$$\beta(c^n) = \sum_i \Gamma_{n,i} x^i \mathrm{d}x^{n-i}, \tag{31}$$

where $i \in \mathbb{Z}$, $\Gamma_{n,i} \in k$, $\Gamma_{0,i} = 0$. Now writing the explicit definition of trivialisation Φ (20), and the coproduct of c^n (18) we see that ω in (30) is as in (27) with β given by (31). $\square$

From the braided bundle point of view in Example 2.15 on the same bundle, we work in the braided category of $\mathbb{Z}$-graded spaces and are allowed for β any degree-preserving that vanishes on 1. This immediately fixes it in the form (31), and hence ω from (27).

4. Bundles with General Differential Structures

Let $P(M, C, \psi, e)$ be a ψ-principal bundle as in Proposition 2.2. Let $\mathcal{N}$ be a subbimodule of $\Omega^1 P$ such that $\overleftarrow{\psi}^2(C \otimes \mathcal{N}) \subset \mathcal{N} \otimes C$. The map $\overleftarrow{\psi}^2$ induces a map $\overleftarrow{\psi}^2_{\mathcal{N}} : C \otimes \Omega^1 P/\mathcal{N} \to \Omega^1 P/\mathcal{N} \otimes C$ and $\mathcal{N}$ defines a right-covariant differential structure $\Omega^1(P) = \Omega^1 P/\mathcal{N}$ on P. We say that $\Omega^1(P)$ is a differential structure on $P(M, C, \psi, e)$.

Definition 4.1. *Let $P(M, C, \psi, e)$ be a coalgebra ψ-principal bundle and let $\psi(C \otimes M) \subset M \otimes C$. Assume that $\mathcal{N} \subset \Omega^1 P$ defines a differential structure $\Omega^1(P)$ on $P(M, C, \psi, e)$. A connection in $P(M, C, \psi, e)$ is a left P-module projection $\Pi : \Omega^1(P) \to \Omega^1(P)$ such that* $\ker \Pi = P\Omega^1(M)P$ *and* $\overleftarrow{\psi}^2_{\mathcal{N}} \circ (\mathrm{id} \otimes \Pi) = (\Pi \otimes \mathrm{id}) \circ \overleftarrow{\psi}^2_{\mathcal{N}}$.

Similarly as for the universal differential calculus case, a connection in $P(M, C, \psi, e)$ can be described by its connection one-form. First we consider the vector space $\mathcal{M} = (P \otimes \ker \epsilon)/\chi(\mathcal{N})$ with a canonical surjection $\pi_{\mathcal{M}} : P \otimes \ker \epsilon \to \mathcal{M}$. Since χ is a left P-module map, $\chi(\mathcal{N})$ is a left P-sub-bimodule of $P \otimes \ker \epsilon$. Therefore $\mathcal{M}$ is a left P-module and $\pi_{\mathcal{M}}$ is a left P-module map. The action of P on $\mathcal{M}$ is defined by

$$u \cdot v = \sum_i \pi_{\mathcal{M}}(uv_i \otimes c^i),$$

where $u \in P$, $v \in \mathcal{M}$ and $\sum_i u_i \otimes c^i \in \pi_{\mathcal{M}}^{-1}(v)$. We denote $\Lambda = \pi_{\mathcal{M}}(1 \otimes \ker \epsilon)$. The left P-module structure of $\mathcal{M}$ implies that for every element $v \in \mathcal{M}$, there exist $u_i \in P$ and $\lambda^i \in \Lambda$ such that $v = \sum_i u_i \cdot \lambda^i$. Therefore there is a natural surjection $P \otimes \Lambda \to \mathcal{M}$.

We assume that $\psi(C \otimes M) \subset M \otimes C$, and hence the map ϕ can be defined. For any $u \in P$, $c \in M$ and $b \in C$ we have

$$\phi(b \otimes u \otimes c) = \phi(b \otimes \chi(n)) = (\chi \otimes \mathrm{id}) \circ \overleftarrow{\psi}^2(b \otimes n) \in \chi(\mathcal{N}) \otimes C,$$

where $n \in \mathcal{N}$ is such that $\chi(n) = u \otimes c$. We used the fact that $\overleftarrow{\psi}^2(C \otimes \mathcal{N}) \subset \mathcal{N} \otimes C$.

Therefore we can define a map $\phi_{\mathcal{N}} : C \otimes \mathcal{M} \to \mathcal{M} \otimes C$ by the diagram

$$
\begin{array}{ccccc}
C \otimes P \otimes \ker \epsilon & \xrightarrow{\mathrm{id} \otimes \pi_{\mathcal{M}}} & C \otimes \mathcal{M} & \longrightarrow & 0 \\
\downarrow \phi & & \downarrow \phi_{\mathcal{N}} & & \\
P \otimes \ker \epsilon \otimes C & \xrightarrow{\pi_{\mathcal{M}} \otimes \mathrm{id}} & \mathcal{M} \otimes C & \longrightarrow & 0
\end{array}
$$

The map χ induces a map $\chi_{\mathcal{N}} : \Omega^1(P) \to \mathcal{M}$ by the commutative diagram

$$
\begin{array}{ccccc}
\Omega^1 P & \xrightarrow{\pi_{\mathcal{N}}} & \Omega^1(P) & \longrightarrow & 0 \\
\downarrow \chi & & \downarrow \chi_{\mathcal{N}} & & \\
P \otimes \ker \epsilon & \xrightarrow{\pi_{\mathcal{M}}} & \mathcal{M} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}
$$

Clearly, $\chi_{\mathcal{N}}$ is a left P-module map, i.e., $\chi_{\mathcal{N}}(u\mathrm{d}v) = u \cdot \chi_{\mathcal{N}}(\mathrm{d}v)$. We can use the map $\chi_{\mathcal{N}}$ to obtain another description of $\phi_{\mathcal{N}}$.

Lemma 4.2. *The following diagram*

$$
\begin{array}{ccc}
C \otimes \Omega^1(P) & \xrightarrow{\overleftarrow{\psi}^2_{\mathcal{N}}} & \Omega^1(P) \otimes C \\
\downarrow \mathrm{id} \otimes \chi_{\mathcal{N}} & & \downarrow \chi_{\mathcal{N}} \otimes \mathrm{id} \\
C \otimes \mathcal{M} & \xrightarrow{\phi_{\mathcal{N}}} & \mathcal{M} \otimes C
\end{array}
$$

is commutative.

Proof. We take any $v \in \Omega^1(P)$, $c \in C$ and $\tilde{v} \in \pi_{\mathcal{N}}^{-1}(v)$ and compute

$$\phi_{\mathcal{N}} \circ (\mathrm{id} \otimes \chi_{\mathcal{N}})(c \otimes v) = \phi_{\mathcal{N}} \circ (\mathrm{id} \otimes \pi_{\mathcal{M}})(c \otimes \chi(\tilde{v})) = (\pi_{\mathcal{M}} \otimes \mathrm{id}) \circ \phi(c \otimes \chi(\tilde{v}))$$
$$= (\pi_{\mathcal{M}} \otimes \mathrm{id}) \circ (\chi \otimes \mathrm{id}) \circ \overleftarrow{\psi}^2 (c \otimes \tilde{v})$$
$$= (\chi_{\mathcal{N}} \otimes \mathrm{id}) \circ (\pi_{\mathcal{N}} \otimes \mathrm{id}) \circ \overleftarrow{\psi}^2 (c \otimes \tilde{v}) = (\chi_{\mathcal{N}} \otimes \mathrm{id}) \circ \overleftarrow{\psi}^2_{\mathcal{N}}(c \otimes v). \qquad \square$$

Using arguments similar to the proof of Example 4.11 of [BM93] and the definition of a coalgebra ψ-principal bundle $P(M, C, \psi, e)$ we deduce that

$$0 \to P\Omega^1(M)P \to \Omega^1(P) \stackrel{\chi_{\mathcal{N}}}{\to} \mathcal{M} \to 0 \tag{32}$$

is a short exact sequence of left P-module maps.

Proposition 4.3. *A connection in $P(M, C, \psi, e)$ with differential structure induced by $\mathcal{N}$ is equivalent to a left P-module splitting $\sigma_{\mathcal{N}}$ of the sequence (32), such that*

$$(\sigma_{\mathcal{N}} \otimes \mathrm{id}) \circ \phi_{\mathcal{N}} = \overleftarrow{\psi}^2_{\mathcal{N}} \circ (\mathrm{id} \otimes \sigma_{\mathcal{N}}).$$

Proof. We use Lemma 4.2 to deduce the covariance properties of $\chi_{\mathcal{N}}$ and then preform calculation similar to the proof of Proposition 3.4. $\square$

To each connection Π we can associate its connection one form $\omega : \Lambda \to \Omega^1(P)$ by $\omega(\lambda) = \sigma_{\mathcal{N}}(\lambda)$.

Similarly to the case of universal differential structure, one proves

Proposition 4.4. *Let Π be a connection in $P(M, C, \psi, e)$ with differential structure defined by $\mathcal{N} \subset \Omega^1 P$. Then, for all $\lambda \in \Lambda$ the connection 1-form $\omega : \Lambda \to \Omega^1(P)$ has the following properties:*

1. *$\chi_{\mathcal{N}} \circ \omega(\lambda) = \lambda$,*
2. *For any $b \in C$, $\overleftarrow{\psi}^2_{\mathcal{N}}(b \otimes \omega(\lambda)) = \tilde{c}^{(1)}{}_{\alpha}\tilde{c}^{(2)}{}_{\beta\delta}\omega(\pi_{\mathcal{M}}(1 \otimes e^{\delta})) \otimes b^{\alpha\beta}$, where $\tilde{c}^{(1)} \otimes_M \tilde{c}^{(2)}$ denotes the translation map $\chi_M^{-1}(1 \otimes \tilde{c})$, and $\tilde{c} \in \ker \epsilon$ is such that $\pi_{\mathcal{M}}(1 \otimes \tilde{c}) = \lambda$.*

Conversely, if $\mathcal{M}$ is isomorphic to $P \otimes \Lambda$ as a left P-module and ω is any linear map $\omega : \Lambda \to \Omega^1(P)$ obeying Conditions 1–2, then there is a unique connection $\Pi = \mu \circ (\mathrm{id} \otimes \omega) \circ \chi_{\mathcal{N}}$ in $P(M, C, \psi, e)$ such that ω is its connection 1-form.

In the setting of [BM93] the condition $P \otimes \Lambda = \mathcal{M}$ is always satisfied for quantum principal bundles, and $\Lambda = \ker \epsilon/\mathcal{Q}$, where $\mathcal{Q}$ is an Ad_R-invariant right ideal in $\ker \epsilon$ that generates the bicovariant differential structure on the structure quantum group H as in [Wor89]. The detailed analysis of braided group principal bundles with general differential structures will be presented elsewhere. Here we remark only that it seems natural to assume that $\mathcal{M} = P\underline{\otimes}\Lambda$ and then choose Λ to be the space dual to the braided Lie algebra $\mathcal{L}$ as discussed in Sect. 3. This choice of Λ is justified by the fact that from the properties of the maps ϕ and $\phi_{\mathcal{N}}$ it follows that the space Λ is invariant under the braided adjoint coaction (cf. Example 3.7).

We complete this section with an explicit example of differential structures and connections on the quantum cylinder bundle in Example 2.14 (cf. Example 3.10).

Example 4.5. We consider the quantum cylinder bundle of Example 2.14 (cf. Example 2.15) and we work with differential structures on $A_q^{2|0}$ classified in [BDR92]. Using the definition of ψ (19) one easily finds that there are two differential structures for which the covariance condition $\overrightarrow{\psi}^2(k[c] \otimes \mathcal{N}) \subset \mathcal{N} \otimes k[c]$ is satisfied. The subbimodules $\mathcal{N}$ are generated by

$$(1 + s)x \otimes x - x^2 \otimes 1 - 1 \otimes x^2,$$
$$y \otimes x - qxy \otimes 1 - q1 \otimes xy + qx \otimes y$$
$$(1 + q)y \otimes y - y^2 \otimes 1 - 1 \otimes y^2,$$

where $s \in k$ is a free parameter, in the first case, and by

$$(1 + q)x \otimes x - x^2 \otimes 1 - 1 \otimes x^2,$$
$$y \otimes x - xy \otimes 1 - q1 \otimes xy + x \otimes y,$$
$$(1 + q)y \otimes y - y^2 \otimes 1 - 1 \otimes y^2,$$

in the second case. In both cases the modules of 1-forms $\Omega^1(A_q^{2|0})$ are generated by the exact one-forms $\mathrm{d}x$ and $\mathrm{d}y$. Definitions of the $\mathcal{N}$ imply the following relations in $\Omega^1(A_q^{2|0})$

$$x\mathrm{d}x = s\mathrm{d}xx, \quad x\mathrm{d}y = q^{-1}\mathrm{d}yx, \quad y\mathrm{d}x = q\mathrm{d}xy, \quad y\mathrm{d}y = q\mathrm{d}yy,$$

in the first case, and

$$x\mathrm{d}x = q\mathrm{d}xx, \quad x\mathrm{d}y = \mathrm{d}yx, \quad y\mathrm{d}x = q\mathrm{d}xy + (q-1)\mathrm{d}xy, \quad y\mathrm{d}y = q\mathrm{d}yy,$$

in the second one. In both cases $(A_q^{2|0}[x^{-1}] \otimes \ker \epsilon)/\chi(\mathcal{N}) = A_q^{2|0}[x^{-1}] \otimes \Lambda$, where Λ is a one-dimensional vector space spanned by $\lambda = \pi_{\mathcal{M}}(1 \otimes c)$ and can be therefore identified with a subspace of $k[c]$ spanned by c. Also in both cases the most general connection is given by

$$\Pi(\mathrm{d}x) = 0, \qquad \Pi(\mathrm{d}y) = \mathrm{d}y + \alpha\mathrm{d}x,$$

where $\alpha \in k$, and extended to the whole of $\Omega^1(A_q^{2|0}[x^{-1}])$ as a left $A_q^{2|0}[x^{-1}]$-module map. The corresponding connection one form reads

$$\omega(\lambda) = \mathrm{d}y + \alpha\mathrm{d}x.$$

The bundle is trivial and this connection can be described by the map $\beta : k[c] \to \Omega(k[x, x^{-1}])$ as in Proposition 3.9 (cf. Eq. (31)) with $\beta(c^n) = 0$ if $n \neq 1$ and $\beta(c) = \alpha\mathrm{d}x$.

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Communicated by A. Connes