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Hidden Symmetries of the Principal Chiral Model Unveiled

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Abstract: By relating the two-dimensional U(N) Principal Chiral Model to a simple linear system we obtain a free-field parametrisation of solutions. Obvious symmetry transformations on the free-field data give symmetries of the model. In this way all known "hidden symmetries" and Bäcklund transformations, as well as a host of new symmetries, arise.

1. Introduction

The definition of *complete integrability* for field theories remains rather imprecise. One usually looks for structures analogous to those existing in completely integrable hamiltonian systems with finitely many degrees of freedom, such as a Lax–pair representation or conserved quantities equal in number to the number of degrees of freedom. A very transparent notion of integrability is that completely integrable nonlinear systems are actually simple linear systems in disguise. For example, the Inverse Scattering Transform for two dimensional integrable systems such as the KdV equation establishes a correspondence between the nonlinear flow for a potential and a constant–coefficient linear flow for the associated scattering data. Similarly, the twistor transform for the self-dual Yang-Mills equations converts solutions of nonlinear equations to holomorphic data in twistor space; and for the KP hierarchy Mulase has explicitly proven complete integrability by performing a transformation to a constant–coefficient linear system [11]. In all these examples, a map is constructed between solutions of a simple, automatically–consistent linear system and the nonlinear system in question. This is distinct from the Lax–pair notion of linearisation, with the nonlinear system in question arising as the consistency condition for a linear system.

Just as the dynamics of completely integrable systems gets trivialised in an auxiliary space, it seems that the confusing plethora of symmetry transformations of these systems arise naturally from obvious transformations on the initial data of the associated linear systems. This idea has been exploited recently by one of us [16] for the KdV hierarchy: A linearisation of KdV, mimicking Mulase's for the KP hierarchy, was used to give a unified description of all known symmetries.

The central feature of Mulase's construction is a group G on which the relevant linear flow acts. The group G (or at least a dense subset thereof) is assumed to be factorisable into two subgroups G_+ and G_-. For the KP hierarchy G is a group of pseudo-differential operators. For KdV and for the two-dimensional Principal Chiral Model (PCM), as we shall see in this paper, G is a "loop group" of smooth maps from a contour $\mathcal{C}$ in the complex λ plane to some group H. This has subgroups G_- (resp. G_+) of maps analytic inside (resp. outside) $\mathcal{C}$. Mulase notes that any flow on G induces flows on $G_\pm$, but the flows on the factors induced by a simple linear flow on G can be complicated and nonlinear. This is the genesis of nonlinear integrable hierarchies; complete integrability is just a manifestation of the system's linear origins. The universality of this kind of construction was noticed by Haak *et al* [8].

We consider on G the linear system

$$d\,U = \Omega\,U, \tag{1}$$

where d is the exterior derivative on the base space $\mathcal{M}$ of the hierarchy, U is a G-valued function on $\mathcal{M}$ and Ω a 1-form on $\mathcal{M}$ with values in G_+. Consistency (Frobenius integrability) of this system requires $d\Omega = \Omega \wedge \Omega$. In fact for KP, KdV and PCM we have the stronger condition $d\Omega = \Omega \wedge \Omega = 0$, and (1) has the general solution

$$U = e^M U_0\,; \quad dM = \Omega, \quad U_0 \in G. \tag{2}$$

The initial data U_0 determines a solution of the linear system, and hence a solution of the associated nonlinear hierarchy. A hierarchy is specified by a choice of G with a factorisation and a choice of one-form Ω.

The purpose of this paper is to provide a description of the two-dimensional Principal Chiral Model in the general framework of Mulase's scheme. We show that for the appropriate group G, and a choice of one-form Ω within a certain class, solutions of Eq. (1) give rise to solutions of PCM. Thus there is a map giving, for each allowed choice of Ω and each choice of initial data U_0, a solution of PCM. The allowed choices of Ω are parametrised by free fields. The known hidden symmetries and Bäcklund transformations of PCM all have their origins in natural field-independent transformations of U_0. We also reveal other symmetries, corresponding to other transformations of U_0 as well as to transformations of the free fields in Ω.

We were motivated to reconsider the symmetries of PCM by a recent paper of Schwarz [17], in which infinitesimal hidden symmetries were reviewed. However the mystery surrounding their origin remained. Further, Schwarz's review did not encompass the work of Uhlenbeck [19] or previous work on finite Bäcklund transformations [9]. We wish to present all these results in a unified framework and to lift the veil obscuring the nature of these symmetries.

2. The Principal Chiral Model

The defining equations for the U(N) PCM on two-dimensional Minkowski space $\mathcal{M}$ with (real) light-cone coordinates x^+, x^- are

$$\begin{aligned}\partial_- A_+ &= \tfrac{1}{2}[A_+, A_-],\\ \partial_+ A_- &= \tfrac{1}{2}[A_-, A_+],\end{aligned} \tag{3}$$

where $A_\pm$ take values in the Lie algebra of U(N), i.e. they are $N \times N$ antihermitian matrices. Considering the sum and difference of the two equations in (3) yields the alternative "conserved current" form of the PCM equations

$$\partial_- A_+ + \partial_+ A_- = 0 \, , \tag{4}$$

together with the zero-curvature condition

$$\partial_- A_+ - \partial_+ A_- + [A_-, A_+] = 0 \, . \tag{5}$$

The latter has pure–gauge solution

$$A_\pm = g^{-1} \partial_\pm g \, , \tag{6}$$

where g takes values in U(N). Substituting this into (4) yields the familiar harmonic map equation

$$\partial_-(g^{-1}\partial_+ g) + \partial_+(g^{-1}\partial_- g) = 0. \tag{7}$$

This is manifestly invariant under the "chiral" transformation $g \mapsto a \; g \; b$, for a and b constant U(N) matrices. At some fixed point x_0 in space-time, we may choose $g(x_0) = I$, the identity matrix. The chiral symmetry then reduces to

$$g \mapsto b^{-1} \; g \; b. \tag{8}$$

There is a further invariance of the equations under the transformation

$$g \mapsto g^{-1} \, . \tag{9}$$

Equation (3) has obvious solutions [21]

$$A_+ = A(x^+) \, , \quad A_- = B(x^-) \, , \tag{10}$$

respectively left- and right–moving diagonal matrices, i.e. taking values in the Cartan subalgebra. (This type of solution is familiar from WZW models and for commuting matrices the Eqs. (3) indeed reduce to WZW equations.) In greater generality, the PCM equations imply that the spectrum of A_+ (resp. A_-) is a function of x^+ (resp. x^-) alone. Thus general solutions take the form:

$$\begin{aligned} A_+ &= s_0(x^+, x^-) A(x^+) s_0^{-1}(x^+, x^-) \\ A_- &= \widetilde{s}_0(x^+, x^-) B(x^-) \widetilde{s}_0^{-1}(x^+, x^-), \end{aligned} \tag{11}$$

where $A(x^+)$ and $B(x^-)$ are antihermitian diagonal matrices, and $s_0(x^+, x^-)$, $\widetilde{s}_0(x^+, x^-)$ are unitary. For given $A(x^+), B(x^-)$, we have seen that there exists at least one such solution, that with $s_0 = \widetilde{s}_0 = I$. We shall see in the next section that a solution $A_\pm$ of the PCM is determined by the diagonal matrices $A(x^+)$ and $B(x^-)$, together with another free field; and our construction leads to solutions of precisely the form (11). Moreover, we shall prove in Sect. 6 that hidden symmetries and Bäcklund transformations act on the space of solutions with given $A(x^+)$ and $B(x^-)$.

3. Construction of Solutions

In this section we give the formulation of the PCM in the framework of Mulase's general scheme. Let us begin by defining a one-form on two-dimensional Minkowski space $\mathcal{M}$ with coordinates (x^+, x^-),

$$\Omega = -\frac{A(x^+)}{1+\lambda}dx^+ - \frac{B(x^-)}{1-\lambda}dx^- . \tag{12}$$

Here $A(x^+), B(x^-)$ are arbitrary diagonal antihermitian matrices, depending only on x^+, x^- respectively. Clearly,

$$d\Omega = \Omega \wedge \Omega = 0 , \tag{13}$$

so that the linear equation

$$d\,U = \Omega\, U \tag{14}$$

is manifestly Frobenius–integrable. The general solution is

$$U(x^+, x^-, \lambda) = e^{M(x^+,x^-,\lambda)}\, U_0(\lambda) , \tag{15}$$

$$M(x^+, x^-, \lambda) = -\frac{1}{1+\lambda}\int_{x_0^+}^{x^+} A(y^+)dy^+ - \frac{1}{1-\lambda}\int_{x_0^-}^{x^-} B(y^-)dy^- ,$$

where U_0, the initial condition, is a free (unconstrained) element of the group G in which U takes values. We need to specify this group.

Remarks. 1) Since A, B are anti-hermitian, hermitian–conjugation of (14) yields

$$dU(\lambda)^\dagger = -U(\lambda)^\dagger \Omega(\lambda^*),$$

whereas U^{-1} satisfies

$$dU^{-1}(\lambda) = -U^{-1}(\lambda)\Omega(\lambda).$$

We therefore obtain the condition

$$U^\dagger(\lambda^*) = U^{-1}(\lambda). \tag{16}$$

2) Ω has poles at $\lambda = \pm 1$, so it is analytic everywhere in the λ-plane including the point at ∞, except in two discs with centres at $\lambda = \pm 1$. We therefore introduce a contour $\mathcal{C}$, the union of two small contours $\mathcal{C}_\pm$ around $\lambda = \pm 1$ (such that $\lambda = 0$ remains outside both of them), dividing the λ-plane into two distinct regions: the "outside" $\{|\lambda - 1| > \delta\} \cap \{|\lambda + 1| > \delta\}$ and the "inside" $\{|\lambda - 1| < \delta\} \cup \{|\lambda + 1| < \delta\}$, where $\delta < 1$ is some small radius.

Definition. *G is the group of smooth maps $V = V(\lambda)$ from the contour $\mathcal{C}$ to $GL(N, \mathbb{C})$ satisfying the condition $V^\dagger(\lambda^*) = V^{-1}(\lambda)$.*

We are going to pretend that there exists a Birkhoff factorisation $G = G_- \; G_+$, where G_- denotes the group of maps analytic inside $\mathcal{C}$ and G_+ denotes the group of maps analytic outside $\mathcal{C}$ *and* equal to the identity at $\lambda = \infty$. The corresponding Lie algebra decomposition is $\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_+$. This factorisation is definitely a pretence; but the point is that sufficiently many elements of G do factor this way so that the results we will obtain using this factorisation do hold. For a more precise discussion we refer to [19, 8].

We now have the spaces in which the objects in (14),(15) take values. Clearly, Ω is a one-form on $\mathcal{M}$ with values in the Cartan subalgebra of the Lie algebra $\mathcal{G}_+$. The matrix $U = U(x^+, x^-, \lambda)$ is a map from $\mathcal{M}$ to G and $U_0(\lambda)$ is an element of G (independent of $x^\pm$).

Consider a solution U of (14). Assuming the existence of a Birkhoff factorisation for U, we can write

$$U = S^{-1}\, Y\ , \tag{17}$$

where $S^{-1} : \mathcal{M} \to G_-$ and $Y : \mathcal{M} \to G_+$. Now, applying the exterior derivative on both sides and using (14) yields

$$S\Omega S^{-1} = -dSS^{-1} + dYY^{-1}. \tag{18}$$

$S\Omega S^{-1}$, which takes values in the Lie algebra $\mathcal{G}$, decomposes into its components in the $\mathcal{G}_-$ and $\mathcal{G}_+$ subalgebras. The above equation allows us to write separate equations for the projections:

$$\begin{aligned}(S\Omega S^{-1})_- &= -dSS^{-1},\\ (S\Omega S^{-1})_+ &= dYY^{-1}.\end{aligned} \tag{19}$$

Here the suffix notation denotes the projection of an element of $\mathcal{G}$ into $\mathcal{G}_\pm$. We introduce a one-form Z taking values in $\mathcal{G}_+$,

$$Z = dYY^{-1} = (S\Omega S^{-1})_+. \tag{20}$$

Now, since S takes values in G_-, it is analytic at $\lambda = \pm 1$ and has two power-series representations, converging in discs with centres at $\lambda = \pm 1$, viz.

$$S = \sum_{n=0}^{\infty} s_n(x^+, x^-)(1+\lambda)^n = \sum_{n=0}^{\infty} \widetilde{s}_n(x^+, x^-)(1-\lambda)^n, \tag{21}$$

where the coefficients $s_0(x^+, x^-), \widetilde{s}_0(x^+, x^-)$ are U(N)-valued matrices. Inserting these expansions in $(S\Omega S^{-1})$, we see that only the s_0 and $\widetilde{s}_0$ terms survive the projection to the $\mathcal{G}_+$ subalgebra, yielding

$$Z = (S\Omega S^{-1})_+ \ = \ -\frac{s_0 A(x^+) s_0^{-1}}{1+\lambda} dx^+ - \frac{\widetilde{s}_0 B(x^-) \widetilde{s}_0^{-1}}{1-\lambda} dx^-. \tag{22}$$

Define

$$A_+ = s_0 A(x^+) s_0^{-1}\ , \quad A_- = \widetilde{s}_0 B(x^-) \widetilde{s}_0^{-1}. \tag{23}$$

These satisfy the PCM equations (3).

The proof is immediate. From (20),

$$d\, Z = \ Z \wedge Z. \tag{24}$$

Inserting the form (22) in this equation yields

$$\frac{\partial_+ A_-}{1-\lambda} - \frac{\partial_- A_+}{1+\lambda} + \frac{1}{2}\left(\frac{1}{1-\lambda} - \frac{1}{1+\lambda}\right)[A_+, A_-] = 0 \, .$$

Since Y takes values in G_+, for consistency this equation needs to hold for all values of λ away from ± 1. In other words, the coefficients of $\frac{1}{1-\lambda}$ and $\frac{1}{1+\lambda}$ must be separately zero. This yields precisely the two equations in (3) as integrability conditions.

Note that the solutions (23) have precisely the form (11). We have seen that for given diagonal matrices $A(x^+)$ and $B(x^-)$, *a solution of the linear field–independent system (14) determines a solution of the PCM in the spectral class of A and B.*

In fact the general solution of (14) takes the form (15), where the e^M factor contains only spectral information (i.e. A, B). Everything else is encoded in the free element $U_0(\lambda) \in G$. So the freely–specifiable data $\{A(x^+), B(x^-), U_0(\lambda)\}$ corresponds to a solution of the PCM. Given any choice of these three fields, a solution of the PCM can be constructed in the following stages:

(a) Construct the corresponding $U(x^+, x^-, \lambda)$ from (15).

(b) Perform the factorisation (17) to obtain $S(x^+, x^-, \lambda)$.

(c) Perform the two expansions (21) to extract the coefficients $s_0(x^+, x^-)$ and $\widetilde{s}_0(x^+, x^-)$.

(d) Insert these in (23) to obtain a solution of the PCM.

Note that this procedure is purely algebraic, though the factorisation may not be very easy to perform in practice. However, it is clear that for any choice of $A(x^+), B(x^-)$ (which is tantamount to fixing the spectral class of $A_\pm$), every $U_0(\lambda) \in G$ corresponds to a solution of the PCM. In fact there is a large redundancy, for a right–multiplication

$$U_0 \mapsto U_0 k_+ \; ; \quad k_+ \in G_+ \tag{25}$$

corresponds to a right-multiplication $U \mapsto U k_+$, which does nothing to alter the S^{-1} factor in (17). PCM solutions therefore correspond to G_+ orbits in G, or equivalently, $U_0(\lambda)$'s from the Grassmannian G/G_+. This correspondence is, however, still redundant: Consider a left–multiplication by a diagonal matrix analytic inside $\mathcal{C}$,

$$U_0 \mapsto h_- \, U_0 \; ; \quad h_- \in G_{0,-} \; , \text{ the maximal torus of } G_- . \tag{26}$$

Since this commutes with the diagonal e^M, it corresponds to a transformation $S^{-1} \mapsto h_- \; S^{-1}$. However, since h_- is a diagonal matrix, the $A_\pm$ in (23) do not notice this transformation; they are invariant. The correct space of U_0's corresponding to solutions of (3) in each spectral class of $A_\pm$ is therefore the double coset $G_{0,-}\backslash G/G_+$. In particular, natural transformations of $U_0(\lambda)$ preserving this double coset correspondence induce symmetry transformations on the space of PCM solutions.

4. The Extended Solution

The fact that the consistency condition (24) with Z given by (22) yields the PCM equations is well known. Writing (20) in more familiar form,

$$d\,Y = Z\,Y \, ,$$

it is precisely the PCM Lax-pair [14, 21],

$$\left(\partial_+ + \frac{1}{1+\lambda}A_+\right)Y = 0,$$
$$\left(\partial_- + \frac{1}{1-\lambda}A_-\right)Y = 0\ . \tag{27}$$

It is easy to check that the Y we have defined above has all the properties required of a solution of this pair of equations:

1. As a function of λ, the only singularities of Y on the entire λ-plane including the point at ∞ are at $\lambda = \pm 1$.
2. The solution of the system (27) is easily seen to satisfy the reality condition (16)
$$Y^\dagger(\lambda^*) = Y^{-1}(\lambda). \tag{28}$$
3. There is an invariance of the Lax system: $Y(x,\lambda) \mapsto Y(x,\lambda)f(\lambda)$, which is usually fixed by setting
$$Y(x_0,\lambda) = I\ , \tag{29}$$
for some fixed point x_0. This invariance corresponds to right–multiplications (25) of U_0 and the condition (29) corresponds to choosing a representative point on the G_+ orbit of U_0 in G.
4. At $\lambda = \infty$, $\partial_+ Y = \partial_- Y = 0$, so $Y(x,\lambda = \infty)$ is a constant and using (29) we obtain
$$Y(x,\lambda = \infty) = I. \tag{30}$$
5. The system (27) yields the expressions
$$A_+ = (1+\lambda)Y\partial_+Y^{-1}\ , \quad A_- = (1-\lambda)Y\partial_-Y^{-1}, \tag{31}$$
which together with (29) and (6) imply that
$$Y(x,\lambda = 0) = g^{-1}. \tag{32}$$

We have already seen that the $A_\pm$ solving (3) may be recovered from power series expansions around $\lambda = \pm 1$ of the S^{-1} factor of U using the expressions (23). We now see that solutions may equally be obtained from the Y factor using (32) and (6). We can also obtain solutions from the Y factor by expanding around $\lambda = \infty$. Denoting the leading terms consistently with (30),

$$Y(x,\lambda) = I + \frac{f(x)}{\lambda} + \ldots, \tag{33}$$

where $f(x)$ is antihermitian, the $\lambda = \infty$ limit of (31) yields the expressions

$$A_\pm = \mp\partial_\pm f\ , \tag{34}$$

which identically satisfy (4) and shift the dynamical description to (5) instead, which acquires the form

$$\partial_-\partial_+ f + \frac{1}{2}[\partial_- f, \partial_+ f] = 0\ . \tag{35}$$

This equation is known as the "dual formulation" of the harmonic map equation (7). A $Y(x,\lambda)$ obtained from the factorisation procedure automatically yields a solution of this equation on expansion around $\lambda = \infty$. We therefore see that the factorisation (17) produces a $Y(x,\lambda)$ which interpolates between the dual descriptions of PCM solutions; yielding a U(N)–valued solution g^{-1} of Eq. (7) on evaluation at $\lambda = 0$ and a Lie-algebra-valued solution f of the alternative equation (35) on development around $\lambda = \infty$. The G_+–valued $Y(x,\lambda)$ thus encapsulates these dual descriptions of chiral fields and this field was aptly named the *extended solution* of the PCM by Uhlenbeck [19].

We shall later need information about the next-to-leading-order term in the expansion of Y around $\lambda = 0$. If we substitute

$$Y = (I + \lambda\varphi)g^{-1} + O(\lambda^2), \tag{36}$$

where φ is a Lie-algebra-valued field, into (31), and use (6), we obtain the following first-order equation for φ:

$$\partial_\pm\varphi + [A_\pm\ ,\ \varphi] = \pm A_\pm. \tag{37}$$

The consistency condition for this is just (4).

Reflecting the G_+–valued extended solution $Y(x,\lambda)$, there is also the G_-–valued $S(x,\lambda)$, which clearly also describes some extension of the PCM solution given by the expression (23). Using $dSS^{-1} = -(S\Omega S^{-1})_- = -(S\Omega S^{-1}) + (S\Omega S^{-1})_+$, we find the following flows for the components of S, which we shall need later:

$$\partial_+ s_n = s_{n+1}A - A_+ s_{n+1}, \tag{38}$$

$$\partial_- s_n = \sum_{r=0}^{n} \frac{s_r B - A_- s_r}{2^{n-r+1}}, \tag{39}$$

$$\partial_+ \widetilde{s}_n = \sum_{r=0}^{n} \frac{\widetilde{s}_r A - A_+ \widetilde{s}_r}{2^{n-r+1}}, \tag{40}$$

$$\partial_- \widetilde{s}_n = \widetilde{s}_{n+1}B - A_- \widetilde{s}_{n+1}. \tag{41}$$

Using (23) and these equations for $n = 0$ yields the interesting flow equations:

$$\begin{aligned} \partial_+ A_+ &= s_0\ \partial_+ A\ s_0^{-1}\ +\ [A_+, [A_+\ ,\ s_1 s_0^{-1}]], \\ \partial_- A_- &= \widetilde{s}_0\ \partial_- B\ \widetilde{s}_0^{-1}\ +\ [A_-, [A_-\ ,\ \widetilde{s}_1 \widetilde{s}_0^{-1}]]. \end{aligned} \tag{42}$$

5. Symmetry Transformations Unveiled

Non-space-time symmetry transformations of the PCM were traditionally derived using mainly guesswork inspired by analogies with other integrable models like the sine-Gordon model. Their origin remained largely veiled in mystery and they were therefore called "hidden symmetries". Previous discussions of them have recently been reviewed by Schwarz [17] and Uhlenbeck [19]. In the framework of the present paper there is nothing "hidden" about these symmetries. As we shall see, in terms of the free-field data $U_0(\lambda), A(x^+), B(x^-)$, the veil hiding these symmetries is entirely lifted: the most natural field-independent transformations of these free fields, which preserve their analyticity properties in their respective independent variables, induce the entire array of known symmetry transformations of PCM fields and more. Moreover, the algebraic structure of the symmetry transformations is completely transparent when acting on the free-field data, and there is no need to compute commutators and check closure using the complicated action of the symmetries on physical fields. The physical fields automatically carry representations of all the symmetry actions on the free-field data.

In this section we classify PCM symmetry transformations according to the corresponding transformations of the free fields. The formulas for the induced transformations on the extended solutions Y, on the chiral fields g and on the potentials $A_\pm$ will be derived in the next section.

5.1. Symmetry transformations of U_0. We first list symmetry transformations which leave $A(x^+)$ and $B(x^-)$ unchanged.

5.1.1. Right dressings. Right-actions by elements of the G_+ subgroup (25) have already been seen to correspond to trivial redundancies and have already been factored out. This leaves the possibility of right–multiplying U_0 by an element of G_-,

$$U_0 \mapsto U_0 k_- \; ; \quad k_- \in G_- \; . \tag{43}$$

Such transformations fall into the following classes:

a) $k_- = b$, a constant (i.e. an element of U(N)). This may easily be seen to induce the transformations $Y \mapsto b^{-1} Y b$ and $g \mapsto b^{-1} g b$, i.e. the symmetry (8).

b) If we take $k_- = \left(I + \frac{N(\mu)}{\lambda - \mu} \pi \right)$, having a simple pole at a single point $\lambda = \mu$ outside $\mathcal{C}$ (here $N(\mu)$ is a λ-independent matrix), the transformations induced on the chiral fields are precisely the Bäcklund transformations of [9, 13].

c) We are presently considering the U(N) PCM. For the GL(N,$\mathbb{C}$) PCM we could consider finite transformations with k_- in a triangular subgroup of G_-. Such transformations induce the explicit transformations discussed by Leznov [10]. We will not go into details of this.

d) General $k_-(\lambda)$ infinitesimally close to the identity. This is a realisation of the algebra $\mathcal{G}_-$ on the free-field $U_0(\lambda)$ and is a remarkably transparent way of expressing the action of the celebrated loop algebra of hidden symmetries [6] of the PCM. The precise structure of this algebra has not been properly identified before.

e) General finite $k_-(\lambda)$. This finite version of the infinitesimal symmetries in d) reproduces (modulo some details) the loop group action on chiral fields g and on extended maps Y given by Uhlenbeck in Sect. 5 of [19].

5.1.2. Left dressings. Left actions on U_0 by elements of $G_{0,-}$ have already been pointed out to leave the associated solution of the PCM invariant (see (26)). We wish to consider only left actions on U_0 that descend to the double coset $G_{0,-} \backslash G / G_+$, i.e. actions by elements that commute with $G_{0,-}$. Thus we have only the transformations

$$U_0 \mapsto h_+ U_0; \quad h_+ \in G_{0,+}. \tag{44}$$

This is the action of an infinite-dimensional abelian group, which has not yet appeared in the literature. The infinitesimal version of this gives an infinite set of mutually commuting flows also commuting with the PCM flow. This is the PCM hierarchy.

5.1.3. Reparametrisations of $U_0(\lambda)$. These are transformations generated by λ-diffeo morphisms

$$U_0(\lambda) \mapsto U_0(\lambda + \epsilon(\lambda)). \tag{45}$$

General reparametrisations can move $\mathcal{C}_{\pm}$ to curves that do not enclose ± 1. The easiest way to prevent this is to restrict the diffeomorphisms to those that fix ± 1. For infinitesimal diffeomorphisms this condition is not strictly necessary. It turns out however that the infinitesimal diffeomorphisms fixing ± 1 are technically simpler (in terms of their action on g, Y) and these give (modulo a detail that will be explained) the "half Virasoro" algebra described in [17]. We show how this can be extended to a full centreless Virasoro algebra.

The only finite reparametrisations of the λ-plane preserving ± 1 are

$$U_0(\lambda) \mapsto U_0\left(\frac{a\lambda + b}{b\lambda + a}\right) , \quad a^2 + b^2 = 1. \tag{46}$$

These induce the S^1 action of sect. 7 of [19].

5.2. Symmetry transformations of $A(x^+), B(x^-)$*.* We now consider symmetries that keep U_0 fixed. For symmetries acting just on $A(x^+)$ it is natural to consider

a) Shifts $A(x^+) \mapsto A(x^+) + \alpha(x^+)$, where $\alpha(x^+)$ is a diagonal antihermitian matrix.
b) Rescalings $A(x^+) \mapsto \rho(x^+)A(x^+)$ where $\rho(x^+)$ is a scalar function.
c) Reparametrisations $A(x^+) \mapsto A(x^+ + \epsilon(x^+))$.

There are other possibilities. Similar symmetries exist for $B(x^-)$. All these symmetries are new.

5.3. Other symmetry transformations. Two other symmetries of PCM should be mentioned. The first is a particularly significant combination of an action on U_0 with an action on A, B. The second is not strictly within the class of symmetries we have been considering, as it acts on the coordinates as well as the fields.

5.3.4. Inversion. The transformation

$$U_0(\lambda) \mapsto U_0(\lambda^{-1}) \quad \text{and} \quad (A, B) \mapsto (-A, -B) \tag{47}$$

may easily be seen to induce the inversion symmetry (9).

5.3.5. Lorentz transformations. The transformation

$$\begin{gathered} U_0 \text{ invariant}, \quad A \mapsto \theta_+ A, \quad B \mapsto \theta_- B \\ x^{\pm} \mapsto \theta_{\pm}^{-1} x^{\pm} \end{gathered} \tag{48}$$

induces the residual Lorentz transformations in light cone coordinates

$$A_{\pm} \mapsto \theta_{\pm} A_{\pm} , \quad x^{\pm} \mapsto \theta_{\pm}^{-1} x^{\pm}. \tag{49}$$

We can also consider more general reparametrisations of $x^{\pm}$.

6. Induced Symmetries of PCM Fields

As we have already claimed, natural transformations on the free–field data, $U_0(\lambda)$, $A(x^+)$, $B(x^-)$ induce, through Birkhoff factorisation, rather complicated transformations on the PCM fields $Y(x,\lambda)$, $g(x)$, $A_\pm(x)$; and (field–independent) representations of symmetry algebras induce (field-dependent) representations on the PCM fields. In this section we prove this for the intereresting and not immediately obvious cases listed in the previous section. We also comment on the relation with previous results in the literature.

6.1. Right dressings. Consider the transformation induced by (43) on $U(x,\lambda)$.

$$U = S^{-1}\,Y \;\mapsto\; U_{new} = S^{-1}\,Yk_-. \tag{50}$$

Birkhoff factorisation of Yk_- yields (in the obvious notation)

$$U_{new} = S^{-1}\,(Yk_-)_-(Yk_-)_+ = S^{-1}_{new}\,Y_{new}. \tag{51}$$

In other words, we have the symmetry transformation

$$Y \;\mapsto\; (Yk_-)_+\,, \tag{52}$$

which is just the representation of G_- described by Uhlenbeck in Sect. 6 of [19] (except that she uses a subgroup of G_-). We can equivalently write

$$Y \;\mapsto\; (Yk_-Y^{-1})_+Y\,. \tag{53}$$

Now writing $k_- = I + \epsilon(\lambda)$ with $\epsilon(\lambda) \in \mathcal{G}_-$ an infinitesimal parameter, we obtain the infinitesimal version of this,

$$Y \;\mapsto\; \left(I + (Y\epsilon(\lambda)Y^{-1})_+\right)Y\,. \tag{54}$$

We note that this directly gives the generating function of [4] for these transformations, which was originally obtained by extrapolation from the leading terms in a power series expansion [6]. The $\mathcal{G}_+$ projection corresponds to taking the singular part at $\lambda = \pm 1$. This may be done using a contour integral, so that this transformation takes the form

$$Y(x,\lambda) \;\mapsto\; \left(I + \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{Y(x,\lambda')\epsilon(\lambda')Y^{-1}(x,\lambda')}{\lambda'-\lambda}d\lambda'\right)\,Y(x,\lambda)\,. \tag{55}$$

Here $\mathcal{C}_\pm$ are oriented counter-clockwise around ± 1. The transformation for g may be read off by taking the $\lambda \to 0$ limit, yielding the form of the transformation given in [18, 17],

$$g \mapsto g\left(I - \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{Y(x,\lambda')\epsilon(\lambda')Y^{-1}(x,\lambda')}{\lambda'}d\lambda'\right)\,. \tag{56}$$

The parameter of this infinitesimal transformation, $\epsilon(\lambda)$ is an arbitrary infinitesimal $\mathcal{G}_-$ element. In particular, if we introduce a basis $\{T^a\}$ for the Lie algebra of antihermitian matrices, we can take $\epsilon(\lambda)$ proportional to $\lambda^r T^a$, $r \in \mathbf{Z}$. This gives an infinite set of transformations, which we denote J^a_r, and which satisfy the commutation relations

$$[J^a_r, J^b_s] = \sum_c f^{ab}_c J^c_{r+s}, \tag{57}$$

where the f_c^{ab} are the structure constants defined by $[T^a, T^b] = \sum_c f_c^{ab} T^c$. Although the commutation relations of a centreless Kac-Moody algebra thus appear, this is *not* sufficient to identify the symmetry algebra $\mathcal{G}_-$ with a centreless Kac-Moody algebra. We illustrate this in two ways: first we show that in $\mathcal{G}_-$ there exist certain linear relations absent in a Kac-Moody algebra, and second we show that in $\mathcal{G}_-$ the J_r^a are not a spanning set.

The crucial point is that although we can certainly try to expand elements of $\mathcal{G}_-$ in Laurent series, and finite sums of matrices of the form $\lambda^r T^a$ are certainly in $\mathcal{G}_-$, the natural way to expand an element of $\mathcal{G}_-$ is in a Taylor series in $\lambda + 1$ (or alternatively in $\lambda - 1$). Taking $\epsilon(\lambda)$ in (56) proportional to $(\lambda + 1)^n T^a$, for $n \geq 0$, we can define a set of transformations K_n^a satisfying the relations

$$[K_n^a, K_m^b] = \sum_c f_c^{ab} K_{n+m}^c \qquad n, m \geq 0. \tag{58}$$

Considering the expansion of λ^r in powers of $\lambda + 1$ (valid in $|\lambda + 1| < \delta$), we find that the J_r^a are expressed as linear combinations of the K_n^a in the following way:

$$J_r^a = \begin{cases} \sum_{n=0}^{r} (-1)^{n+r} \binom{r}{n} K_n^a & r \geq 0 \\ \sum_{n=0}^{\infty} (-1)^r \binom{n-r-1}{-r-1} K_n^a & r < 0 \end{cases} . \tag{59}$$

It is straightforward, using standard formulae for sums of binomial coefficients (see for example [7]), to check that these linear combinations, by virtue of (58), imply the commutation relations (57). The relation between the J_r^a for non-negative r can be inverted: we find

$$K_n^a = \sum_{r=0}^{n} \binom{n}{r} J_r^a \; . \tag{60}$$

Now, if our symmetry algebra were indeed a Kac-Moody algebra with generators J_r^a satisfying (57), we would be able to define the algebra elements K_n^a (which certainly exist as symmetry generators) from the J_r^a's with non-negative r using (60). When we substitute (60) into the infinite sum in (59) we find that we cannot reorder the summations to express this infinite sum as a linear combination of the J_r^a's with $r \geq 0$. In other words, this infinite sum is not in the Kac-Moody algebra. We thus have our first distinction between a Kac-Moody algebra and $\mathcal{G}_-$: In a Kac-Moody algebra the elements K_n^a and the elements J_r^a for $r < 0$ need to be linearly independent, whereas in the PCM symmetry algebra $\mathcal{G}_-$ they are linearly dependent via the relationship given in (59).

The second distinction is that in $\mathcal{G}_-$, unlike in a regular Kac-Moody algebra, the elements $\{J_r^a\}$ are not a spanning set. Elements of $\mathcal{G}_-$ need to be analytic inside $\mathcal{C}$. There are therefore elements of $\mathcal{G}_-$ that do not have Laurent expansions in powers of λ; consider for example an $\epsilon(\lambda)$ proportional to ln λ, defined with a cut from 0 to ∞ along half of the imaginary axis. Now, the reader may be concerned that we have claimed that $\mathcal{G}_-$ is spanned by the K_n^a, that the relationship between the K_n^a and the J_r^a for $r \geq 0$ is invertible, but that the J_r^a (and therefore certainly the J_r^a for $r \geq 0$) are not a spanning set for $\mathcal{G}_-$. There is absolutely no contradiction here. As we have seen above, the relationship between the K_n^a and the J_a^r for $r \geq 0$ implies that finite linear combinations of the K_n^a can be written as linear combinations of the J_r^a for $r \geq 0$, but for infinite linear combinations of the K_n^a this is not the case. However, it does suggest that we should be able in some sense to approximate elements of $\mathcal{G}_-$ given by infinite sums of the K_n^a's by finite sums of the J_r^a, which are equivalent to finite sums of the K_n^a. This is indeed the case, as follows from a classical theorem in complex analysis, Runge's theorem (see, for example, [15]). Runge's theorem implies the remarkable fact that a function analytic on an arbitrary finite union of non-intersecting open discs can be approximated uniformly and to any accuracy on any closed subset of the union by a polynomial. In particular, this implies that elements in $\mathcal{G}_-$ can be approximated uniformly and to any accuracy on $\{|\lambda - 1| < \delta\} \cup \{|\lambda + 1| < \delta\}$ by a finite linear combination of the J_r^a for $r \geq 0$.

To conclude this section we note that the contour integral in (56) is easily evaluated when $\epsilon(\lambda)$ is proportional to λ^r: For $r < 0$ the integral is evaluated by shrinking $\mathcal{C}$ to a contour around 0; for $r > 0$ to a contour around ∞; and for $r = 0$ to a pair of contours around 0 and ∞.

6.2. The Bäcklund transformation. The element $k_- \in G_-$ in (43) can clearly have all variety of singularities *outside* $\mathcal{C}$. Trying to give k_- just one simple pole at the point $\lambda = \mu$ outside $\mathcal{C}$, suggests the natural form [21]

$$k_-(\lambda, \mu) = \left(I + \frac{N(\mu)}{\lambda - \mu} \right) . \tag{61}$$

For the satisfaction of the reality condition (16) for elements of G_- we require that $N^\dagger = \frac{NN^\dagger}{\mu - \bar{\mu}} = -N$. These conditions are satisfied by $N = (\mu - \bar{\mu})\pi$, if π is a projector satisfying $\pi^2 = \pi = \pi^\dagger$. Such transformations thus correspond to finite right-dressing transformation of the particular form

$$U_0 \mapsto U_0 \left(I + \frac{\mu - \bar{\mu}}{\lambda - \mu} \pi \right) . \tag{62}$$

Note that k_- in fact has a singularity at $\lambda = \bar{\mu}$ as well, since $(I - \pi)$ has zero determinant. Using (50) we obtain the transformation

$$U \mapsto S^{-1} \left(I + \frac{\mu - \bar{\mu}}{\lambda - \mu} Y(\lambda) \pi Y^{-1}(\lambda) \right) Y(\lambda) . \tag{63}$$

In order to factorise the middle factor, we introduce a hermitian projector $P = P^\dagger = P^2$, independent of λ (but not of $x^\pm$). Using this we see that

$$\begin{aligned} &\left(I + \tfrac{\mu - \bar{\mu}}{\lambda - \mu} \, Y(\lambda) \pi Y^{-1}(\lambda) \right) Y(\lambda) \\ &= \left(I + \tfrac{\mu - \bar{\mu}}{\lambda - \mu} \, P \right) \left(I + \tfrac{\bar{\mu} - \mu}{\lambda - \bar{\mu}} \, P \right) \left(I + \tfrac{\mu - \bar{\mu}}{\lambda - \mu} \, Y(\lambda) \pi Y^{-1}(\lambda) \right) Y(\lambda) \\ &= \left(I + \tfrac{\mu - \bar{\mu}}{\lambda - \mu} \, P \right) \left(I + \tfrac{\mu - \bar{\mu}}{\lambda - \mu} \, (I - P) \, Y(\lambda) \pi + \tfrac{\bar{\mu} - \mu}{\lambda - \bar{\mu}} \, P Y(\lambda) \, (I - \pi) \right) . \end{aligned}$$

To have an acceptable factorisation, all we need now is that the right-hand factor above be regular outside $\mathcal{C}$. Specifically, we require regularity at μ and $\bar{\mu}$, which yields algebraic conditions relating the projectors P and π, viz.

$$(I - P) Y_\mu \pi = 0 \, , \quad P Y_\mu (I - \pi) = 0,$$

where Y_μ denotes $Y(\lambda)$ evaluated at $\lambda = \mu$. If we write $\pi = v(v^\dagger v)^{-1} v^\dagger$ (see [9]), these equations are solved by the expression

$$P = Y_\mu v \left(v^\dagger Y_\mu^\dagger Y_\mu v \right)^{-1} v^\dagger Y_\mu^\dagger \, .$$

Now we can read-off the induced transformation rules for Y and g. These are just the known PCM Bäcklund transformations [9, 13, 21].

6.3. Left dressings. Here we consider in detail the left dressings (44). Matrices $h_+ \in G_{0,+}$ commute with M, so such transformations act by left multiplication on U, i.e. $U \mapsto h_+ U = h_+ S^{-1} Y = S^{-1}(Sh_+ S^{-1})Y$. Hence the action on Y is given by

$$Y \mapsto (h_+ S^{-1})_+ Y = (Sh_+ S^{-1})_+ Y. \tag{64}$$

For an infinitesimal transformation $h_+ = I + \epsilon$, $\epsilon \in \mathcal{G}_{0,+}$ and we have

$$\begin{aligned} Y &\mapsto \left(I + (S\epsilon S^{-1})_+\right) Y \\ &= \left(I + \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{S(\lambda')\epsilon(\lambda')S^{-1}(\lambda')}{\lambda' - \lambda} d\lambda'\right) Y, \end{aligned} \tag{65}$$

implying

$$g \mapsto g\left(I - \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{S(\lambda')\epsilon(\lambda')S^{-1}(\lambda')}{\lambda'} d\lambda'\right). \tag{66}$$

In general ϵ has the form

$$\epsilon(\lambda) = \sum_{n=1}^{\infty}\left(\frac{\alpha_n}{(1+\lambda)^n} + \frac{\widetilde{\alpha}_n}{(1-\lambda)^n}\right), \tag{67}$$

where the $\alpha_n, \widetilde{\alpha}_n$ are constant infinitesimal diagonal matrices. The integral in (66) is evaluated by computing the residues of the integrand at $\lambda' = \pm 1$. For example, the case $\alpha_1 \neq 0$ with all other $\alpha_n, \widetilde{\alpha}_n$ zero yields the transformation rules

$$\begin{aligned} g^{-1}\delta g &= -s_0\alpha_1 s_0^{-1}, \\ \delta A_+ &= \left[A_+, [s_1 s_0^{-1}, s_0\alpha_1 s_0^{-1}]\right], \\ \delta A_- &= -\tfrac{1}{2}[A_-, s_0\alpha_1 s_0^{-1}]. \end{aligned} \tag{68}$$

Similarly, if $\alpha_2 \neq 0$ with all other $\alpha_n, \widetilde{\alpha}_n$ zero we find

$$\begin{aligned} g^{-1}\delta g &= -\left(s_0\alpha_2 s_0^{-1} + [s_1 s_0^{-1}, s_0\alpha_2 s_0^{-1}]\right), \\ \delta A_+ &= \left[A_+, [s_2 s_0^{-1}, s_0\alpha_2 s_0^{-1}] - [s_1 s_0^{-1}, s_0\alpha_2 s_0^{-1}] s_1 s_0^{-1}\right], \\ \delta A_- &= -\left[A_-, \tfrac{1}{4}s_0\alpha_2 s_0^{-1} + \tfrac{1}{2}[s_1 s_0^{-1}, s_0\alpha_2 s_0^{-1}]\right]. \end{aligned} \tag{69}$$

The formulae for $\delta A_\pm$ are computed using the variation of the relation (6),

$$\delta A_\pm = \partial_\pm(g^{-1}\delta g) + [A_\pm, g^{-1}\delta g], \tag{70}$$

and Eqs. (38)-(41). The latter also allow one to check directly that the above transformations are indeed infinitesimal symmetries, i.e. that $\partial_-\delta A_+ + \partial_+\delta A_- = 0$.

Now considering the sector of PCM in which $A = \alpha_1$, independent of x^+, we see that the ∂_+-derivations of $A_\pm$ given by (3) and (42) are effected by the transformations (68). So left dressing transformations with only α_1 non-zero correspond to x^+ translations in this sector. Similarly the transformations (69) can be seen to be related to coordinate translations in an extended system (described in the appendix) belonging to a *hierarchy* associated to the PCM. Whenever an infinite dimensional abelian symmetry algebra (like $\mathcal{G}_{0,+}$) is identified in a system, it is possible to define a corresponding hierarchy. Traditionally, for each generator in the algebra a coordinate is introduced and the flow in each coordinate is defined as the infinitesimal action of the corresponding symmetry. In our formulation there is an alternative way to define a PCM hierarchy. Instead of working on a space $\mathcal{M}$ with coordinates (x^+, x^-), we work on a larger space $\mathcal{M}$ with $2P$ coordinates $(x_1^+, \ldots, x_P^+, x_1^-, \ldots, x_P^-)$ and replace the Ω of (12) by

$$\Omega = -\sum_{n=1}^{P} \left(\frac{A_n(x_n^+) dx_n^+}{(1+\lambda)^n} + \frac{B_n(x_n^-) dx_n^-}{(1-\lambda)^n} \right), \tag{71}$$

where the $A_n(x_n^+), B_n(x_n^-)$ are all antihermitian diagonal matrices, each depending on only one coordinate. The associated nonlinear equations are again the equations $dZ = Z \wedge Z$, where $Z = (S\Omega S^{-1})_+$ and S is a map from $\mathcal{M}$ to G_-. For the case $P = 2$ we write out this system of equations in full in the appendix. Another possibility of obtaining a hierarchy within our framework is to enlarge $\mathcal{M}$ to a space with $2NP$ coordinates (x_n^{a+}, x_n^{a-}), $1 \leq n \leq P$, $1 \leq a \leq N$, and taking

$$\Omega = -\sum_{n=1}^{P} \sum_{a=1}^{N} \left(\frac{A_n^a(x_n^{a+}) H^a dx_n^{a+}}{(1+\lambda)^n} + \frac{B_n^a(x_n^{a-}) H^a dx_n^{a-}}{(1-\lambda)^n} \right), \tag{72}$$

where $\{H^a\}$, $a = 1, \ldots, N$ is a basis for the algebra of antihermitian, diagonal $N \times N$ matrices. In this hierarchy, left dressings on U_0 correspond precisely to coordinate translations in the sector with the scalar functions A_n^a, B_n^a constant.

The physical or geometric significance of these PCM hierarchies remains to be understood. An alternative approach to defining a PCM hierarchy was given in [1].

6.4. The Virasoro symmetry. In this section we consider the symmetries of PCM associated with reparametrisations of $U_0(\lambda)$. We consider the infinitesimal reparametrisations $U_0(\lambda) \to U_0(\lambda + \epsilon_m \lambda^{m+1})$, where the ϵ_m are infinitesimal parameters and $m \in \mathbf{Z}$, or, equivalently, variations $\delta U_0 = \epsilon_m \lambda^{m+1} U_0'(\lambda)$. The prime denotes differentiation with respect to λ.

These variations give rise to a centreless Virasoro algebra of infinitesimal symmetries of PCM. In [17] Schwarz documents the existence of "half" of this algebra. Schwarz's symmetries are associated with reparametrisations that fix the points $\lambda = \pm 1$. We shall see that from a technical standpoint these are simpler to handle than the full set of symmetries. But there is also a fundamental reason to make such a restriction. If we were to consider finite reparametrisations, we would need to ensure that the contour $\mathcal{C}$ remains qualitatively unchanged. The simplest way to do this is to require the points $\lambda = \pm 1$ to be fixed. In [19] Uhlenbeck identifies an S^1 symmetry of PCM. It is a simple exercise to check that this symmetry corresponds, in our formalism, to global reparametrisations of the λ-plane fixing the points ± 1, i.e. Möbius transformations of the form

$$\lambda \to \frac{a\lambda + b}{b\lambda + a}, \qquad a^2 + b^2 = 1. \tag{73}$$

At the level of infinitesimal symmetries, however, the need to fix ± 1 is really superfluous, and so we find a full Virasoro algebra of symmetries. But as we have said above, the symmetries fixing ± 1 are technically easier, which is why Schwarz was able to identify them, and also for the more general symmetries we can be quite certain that there exists no exponentiation.

With this introduction, we consider the variations $\delta_m U_0 = \epsilon_m \lambda^{m+1} U_0'(\lambda)$. These manifestly realise the algebra $[\delta_m, \delta_n] = (n-m)\delta_{n+m}$. This realisation descends to the physical fields. Using $U_0 = e^{-M} S^{-1} Y$ we have the chain of implications

$$\delta_m U_0 = \epsilon_m \lambda^{m+1}(-M' e^{-M} S^{-1} Y - e^{-M} S^{-1} S' S^{-1} Y + e^{-M} S^{-1} Y'), \quad (74)$$

$$\begin{aligned} \delta_m U &= e^M \delta_m U_0 \\ &= \epsilon_m \lambda^{m+1}(-M' S^{-1} Y - S^{-1} S' S^{-1} Y + S^{-1} Y'), \end{aligned} \quad (75)$$

$$\begin{aligned} \delta_m S &= -(S \delta_m U Y^{-1})_- S \\ &= -\epsilon_m \left(\lambda^{m+1}(-S M' S^{-1} - S' S^{-1} + Y' Y^{-1})\right)_- S, \end{aligned} \quad (76)$$

$$\begin{aligned} \delta_m Y &= (S \delta_m U Y^{-1})_+ Y \\ &= \epsilon_m \left(\lambda^{m+1}(-S M' S^{-1} + Y' Y^{-1})\right)_+ Y. \end{aligned} \quad (77)$$

In the last equation we have used the fact that for all m, $\lambda^{m+1} S' S^{-1}$ takes values in $\mathcal{G}_-$. Of the remaining two terms, the first has a $\mathcal{G}_+$ piece originating in the double pole of M' at $\lambda = \pm 1$. To explicitly compute this is a simple exercise. For the second term, we use a contour integral formula for the projection. We thus arrive at the final result

$$\begin{aligned} \delta_m Y\, Y^{-1} = \epsilon_m \Bigg(& \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\mu^{m+1} Y'(\mu) Y^{-1}(\mu)}{\mu - \lambda} d\mu \\ & + (-1)^m \left(s_0 \textstyle\int A\, s_0^{-1}\right) \left(\frac{1}{(1+\lambda)^2} - \frac{m+1}{1+\lambda}\right) + \frac{(-1)^m}{1+\lambda} \left[s_1 s_0^{-1}, \left(s_0 \textstyle\int A\, s_0^{-1}\right)\right] \\ & + \left(\widetilde{s}_0 \textstyle\int B\, \widetilde{s}_0^{-1}\right) \left(\frac{1}{(1-\lambda)^2} - \frac{m+1}{1-\lambda}\right) + \frac{1}{1-\lambda} \left[\widetilde{s}_1 \widetilde{s}_0^{-1}, \left(\widetilde{s}_0 \textstyle\int B\, \widetilde{s}_0^{-1}\right)\right] \Bigg). \end{aligned} \quad (78)$$

Here $\int A$ and $\int B$ are shorthand for $\int_{x_0^+}^{x^+} A(y^+) dy^+$ and $\int_{x_0^-}^{x^-} B(y^-) dy^-$ respectively. The g transformations are read off by setting λ to zero. In the expression for $\delta_m g$, the contour integral term is evaluated, depending on the value of m, by shrinking $\mathcal{C}$ to a contour around either 0 or ∞. Explicitly for the SL(2) subalgebra of the Virasoro algebra, we obtain (omitting the overall infinitesimal parameters),

$$\begin{aligned} g^{-1}\delta_{-1} g &= \phi + (s_0 \textstyle\int A s_0^{-1}) - (\widetilde{s}_0 \int B \widetilde{s}_0^{-1}) + \left[s_1 s_0^{-1}, (s_0 \int A s_0^{-1})\right] - \left[\widetilde{s}_1 \widetilde{s}_0^{-1}, (\widetilde{s}_0 \int B \widetilde{s}_0^{-1})\right], \\ g^{-1}\delta_0 g &= -\left[s_1 s_0^{-1}, (s_0 \textstyle\int A s_0^{-1})\right] - \left[\widetilde{s}_1 \widetilde{s}_0^{-1}, (\widetilde{s}_0 \int B \widetilde{s}_0^{-1})\right], \\ g^{-1}\delta_1 g &= f - (s_0 \textstyle\int A s_0^{-1}) + (\widetilde{s}_0 \int B \widetilde{s}_0^{-1}) + \left[s_1 s_0^{-1}, (s_0 \int A s_0^{-1})\right] - \left[\widetilde{s}_1 \widetilde{s}_0^{-1}, (\widetilde{s}_0 \int B \widetilde{s}_0^{-1})\right]. \end{aligned}$$

We see that in these formulae, not only do the leading coefficients $s_0, s_1, \widetilde{s}_0, \widetilde{s}_1$ in the expansions of S appear, but also the fields ϕ and f, coefficients in the expansions of Y around 0 and ∞ respectively (see Sect. 4). The work required to check directly that these, or any of the δ_m's, are symmetries is formidable, but we again emphasize that the advantage of the present framework is that such direct checks are not necessary in order to prove that the physical fields carry a representation of the full centreless Virasoro algebra.

Schwarz [17] has previously found half a Virasoro algebra. We observe that if we define transformations $\Delta_m = \delta_{m+1} - \delta_{m-1}$ a substantial simplification takes place, yielding the formula

$$\Delta_m g = -\epsilon_m g\bigg(\frac{1}{2\pi i}\int_{\mathcal{C}}\mu^{m-1}(\mu^2-1)Y'(\mu)Y^{-1}(\mu)d\mu \\ + 2(-1)^m(s_0\int A\, s_0^{-1}) - 2(\widetilde{s}_0\int B\,\widetilde{s}_0^{-1})\bigg). \tag{79}$$

We will see in Sect. 6 (see Eqs. (83),(84)) that the second and third terms in the above expression are individually symmetries of PCM that mutually commute and commute with all the symmetries being considered here. Removing these terms gives exactly the "half-Virasoro" symmetries of [17]

$$\widetilde{\Delta}_m g = -\epsilon_m g\frac{1}{2\pi i}\int_{\mathcal{C}}\mu^{m-1}(\mu^2-1)Y'(\mu)Y^{-1}(\mu)d\mu, \qquad m\in\mathbf{Z}. \tag{80}$$

Thus we see the precise nature of Schwarz's symmetries as combinations of reparametrisations preserving the points $\lambda = \pm 1$ with certain simple symmetries that act on the A, B fields but leave U_0 invariant. Taking the appropriate combinations we see that for the simplest Schwarz symmetry $\widetilde{\Delta}_0$,

$$g^{-1}\widetilde{\Delta}_0 g = \phi - f, \tag{81}$$

and using (34) and (37),

$$\widetilde{\Delta}_0 A_\pm = \mp 2A_\pm + [A_\pm, f]. \tag{82}$$

This is easily checked to be a symmetry. The symmetry Δ_0 acts on the physical fields in a much more complicated way:

$$\begin{aligned} g^{-1}\Delta_0 g &= \phi - f - 2(s_0{\textstyle\int} A s_0^{-1}) + 2(\widetilde{s}_0{\textstyle\int} B\widetilde{s}_0^{-1}),\\ \Delta_0 A_+ &= -4A_+ + [A_+, f] - 2\left[[s_1 s_0^{-1}, A_+], (s_0{\textstyle\int} A s_0^{-1})\right] + \left[A_+, (\widetilde{s}_0{\textstyle\int} B\widetilde{s}_0^{-1})\right],\\ \Delta_0 A_- &= 4A_- + [A_-, f] + 2\left[[\widetilde{s}_1\widetilde{s}_0^{-1}, A_-], (\widetilde{s}_0{\textstyle\int} B\widetilde{s}_0^{-1})\right] - \left[A_-, (s_0{\textstyle\int} A s_0^{-1})\right]. \end{aligned}$$

6.5. Transformations of the free fields $A(x^+), B(x^-)$. Following the by now familiar reasoning, an infinitesimal transformation $A(x^+) \mapsto A(x^+) + \delta A(x^+)$ induces the following transformations on Y, g, A_+, A_-:

$$\begin{aligned} \delta Y &= -\frac{(s_0\int \delta A\, s_0^{-1})}{1+\lambda}\, Y,\\ \delta g &= g\,(s_0\int\delta A\, s_0^{-1}),\\ \delta A_+ &= s_0\,\delta A\, s_0^{-1} - \left[A_+, \left[s_1 s_0^{-1}, (s_0\int\delta A\, s_0^{-1})\right]\right],\\ \delta A_- &= \left[A_-, (s_0\int\delta A\, s_0^{-1})\right]. \end{aligned}$$

Here we have written $\int\delta A$ as shorthand for $\int_{x_0^+}^{x^+}\delta A(y^+)dy^+$. As expected, the spectrum of A_- remains invariant, while that of A_+ is shifted. Using the flow equations for s_0, s_1, it is easy to check that these are genuine symmetries, i.e. that $\partial_-\delta A_+ + \partial_+\delta A_- = 0$.

There are a variety of possibilities for $\delta A(x^+)$. If $\{H^a\}$, $a = 1\ldots N$, is a basis of the algebra of antihermitian diagonal matrices, we can consider variations $\delta A(x^+) \sim (x^+)^m H^a, a = 1,\ldots,N, m\in\mathbf{Z}$. This gives a loop algebra of symmetries, corresponding to translations of $A(x^+)$. Taking $\delta A(x^+) \sim (x^+)^m A'(x^+)$, $m\in\mathbf{Z}$, gives a centreless Virasoro algebra of symmetries, corresponding to reparametrizations of $A(x^+)$. Taking

$\delta A(x^+) \sim (x^+)^m A(x^+)$, $m \in$ **Z**, gives an infinite dimensional abelian symmetry algebra corresponding to x^+-dependent rescalings of $A(x^+)$. Clearly these symmetries are not independent: The latter two families can be written in terms of the first family, but the generators are then *field dependent* combinations of the generators of the first family.

Analogous sets of symmetries can be obtained from infinitesimal variations of $B(x^-)$.

The simple variation $\delta A(x^+) = \epsilon A(x^+)$, where ϵ is a constant infinitesimal parameter, yields the symmetry

$$\delta g = \epsilon g(s_0 \int A \, s_0^{-1}), \tag{83}$$

whereas the transformation $B \mapsto (1 + \zeta)B$, where ζ is also an infinitesimal parameter, yields

$$\delta g = \zeta g(\widetilde{s}_0 \int B \, \widetilde{s}_0^{-1}). \tag{84}$$

These transformations were used in Sect. (6) to make contact between our Virasoro symmetries and those of [17].

7. Concluding Remarks

We have seen that formulating the nonlinear equations of motion (3) of the PCM in the form of the simple linear system (1) makes the precise nature of their integrability completely transparent. It yields a novel free-field parametrisation of the space of solutions, which we have used to classify all the symmetries of on-shell PCM fields in terms of natural transformations on the free-field data. The confusing cacophony of symmetry transformations in the literature is thereby seen to arise in the most natural fashion imaginable. We have thus demonstrated that this notion of complete integrability, previously applied to traditional soliton systems, like the KP, NLS and KdV hierarchies, encompasses the Lorentz–invariant PCM field theories. We believe that this notion of integrability is a universal one and we expect a clarification of the nature of the integrability of the self-dual Yang-Mills and self-dual gravity equations by similarly reformulating the twistor constructions for these systems. Indeed Crane [3] has already discussed a loop group of symmetries in terms of an action on free holomorphic data in twistor space.

Our construction raises many questions.

1) Standard integrable soliton systems exhibit multiple hamiltonian structures and infinite numbers of conservation laws, both these phenomena being symptoms of their integrability. These phenomena ought to have a natural explanation in terms of the associated simple linear systems (free-field data). For the PCM, some work on such structures exists [5].

2) The free-field parametrisation of solutions of PCM should play a critical role in the quantisation of the theory. What is the relation with standard quantisations? (The PCM can be quantised in different ways, using either the field f or the field g as fundamental, giving different results [12].) How are we to understand quantum integrability?

3) There is a large body of related mathematical work, mostly focusing on the enumeration and construction of solutions of the PCM in Euclidean space (for recent references see [2]). Most of our formalism goes through for the case of Euclidean space, but the reality conditions are different, and a little harder to handle. An important class of solutions are the *unitons* [19, 20]. These correspond, up to the need for right dressings by G_+ elements, to Y's with finite order poles at one of the two points ± 1, and regular elsewhere. We wonder: What are the corresponding U_0's? (The work of Crane on self-dual

Yang-Mills [3] may have an analog.) Is there a natural geometric understanding of our construction? Or a relation with the constructions of [20] or [2]?
4) Is there a geometric interpretation of our PCM hierarchy?

Appendix. The PCM Hierarchy

In Sect. 6.3 we have described a procedure to generate a PCM hierarchy. In this appendix we illustrate this procedure by obtaining the simplest integrable extension of the PCM equation. We use the Ω given in (71) for $P = 2$. Using $Z = (S\Omega S^{-1})_+$ we obtain the following form for Z:

$$Z = -\left(\frac{A_+ dx_1^+}{1+\lambda} + \left(\frac{B_+}{(1+\lambda)^2} + \frac{[C_+, B_+]}{1+\lambda}\right) dx_2^+ \right.$$
$$\left. + \frac{A_- dx_1^-}{1-\lambda} + \left(\frac{B_-}{(1-\lambda)^2} + \frac{[C_-, B_-]}{1-\lambda}\right) dx_2^- \right).$$

The six fields $A_+, B_+, C_+, A_-, B_-, C_-$ are defined in terms of the coefficients of S and the free fields $A_1(x_1^+), A_2(x_2^+), B_1(x_1^-), B_2(x_2^-)$. They depend on the four coordinates $x_1^+, x_2^+, x_1^-, x_2^-$ and are constrained in virtue of their defining relations thus: A_+ commutes with B_+, A_- commutes with B_- and the spectra of A_+, B_+, A_-, B_- depend only on $x_1^+, x_2^+, x_1^-, x_2^-$ respectively. If we nevertheless ignore these constraints and simply substitute the above form for Z into $dZ = Z \wedge Z$, we find:

1. $[A_+, B_+] = [A_-, B_-] = 0$.
2. The following system of evolution equations for A_+, B_+, A_-, B_-:

$$\begin{aligned}
\partial_{2+}A_+ &= -\tfrac{1}{2}[A_+, [[B_+, C_+], C_+]] - [B_+, \partial_{1+}C_+ + \tfrac{1}{2}[[A_+, C_+], C_+]],\\
\partial_{1-}A_+ &= \tfrac{1}{2}[A_+, A_-],\\
\partial_{2-}A_+ &= \tfrac{1}{2}[A_+, \tfrac{1}{2}B_- + [C_-, B_-]],\\
\partial_{1+}B_+ &= [B_+, [A_+, C_+]],\\
\partial_{1-}B_+ &= \tfrac{1}{2}[B_+, A_-],\\
\partial_{2-}B_+ &= \tfrac{1}{2}[B_+, \tfrac{1}{2}B_- + [C_-, B_-]],\\
\partial_{1+}A_- &= \tfrac{1}{2}[A_-, A_+],\\
\partial_{2+}A_- &= \tfrac{1}{2}[A_-, \tfrac{1}{2}B_+ + [C_+, B_+]],\\
\partial_{2-}A_- &= -\tfrac{1}{2}[A_-, [[B_-, C_-], C_-]] - [B_-, \partial_{1-}C_- + \tfrac{1}{2}[[A_-, C_-], C_-]],\\
\partial_{1+}B_- &= \tfrac{1}{2}[B_-, A_+],\\
\partial_{2+}B_- &= \tfrac{1}{2}[B_-, \tfrac{1}{2}B_+ + [C_+, B_+]],\\
\partial_{1-}B_- &= [B_-, [A_-, C_-]].
\end{aligned}$$

   These evidently imply that the spectra of A_+, B_+, A_-, B_- depend only on x^{1+}, x^{2+}, x^{1-}, x^{2-} respectively, as required.
3. The following evolution equations for C_+, C_-:

$$
\begin{aligned}
\partial_{1+}C_- &= -\tfrac{1}{4}A_+ - \tfrac{1}{2}[A_+, C_-],\\
\partial_{2+}C_- &= -\tfrac{1}{8}B_+ + \tfrac{1}{4}([C_-, B_+] - [C_+, B_+]) + \tfrac{1}{2}[C_-, [C_+, B_+]],\\
\partial_{1-}C_+ &= -\tfrac{1}{4}A_- - \tfrac{1}{2}[A_-, C_+],\\
\partial_{2-}C_+ &= -\tfrac{1}{8}B_- + \tfrac{1}{4}([C_+, B_-] - [C_-, B_-]) + \tfrac{1}{2}[C_+, [C_-, B_-]].
\end{aligned}
$$

(In fact, from the $dZ = Z \wedge Z$ equation, both of the C_- evolutions appear commutated with B_+ and both of the C_+ evolutions appear commutated with B_-.)

This system is a 4-dimensional integrable system, but its physical or geometric interpretation is not immediately apparent. It has a variety of interesting reductions apart from the reduction to PCM by setting $B_- = B_+ = 0$. We can consistently reduce by taking $A_- = B_-$ or $A_+ = B_+$ or both. Or we can take just $B_- = 0$ (or $B_+ = 0$) in which case the x_2^- (or x_2^+) dependence becomes trivial. For all these reductions, and the full system as well, the methods of this paper give a free-field parametrisation of solutions.

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