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Quantization of Coset Space σ-Models Coupled to Two-Dimensional Gravity

D. Korotkin[*], **H. Samtleben**[**]

II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany

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Abstract: The mathematical framework for an exact quantization of the two-dimensional coset space σ-models coupled to dilaton gravity, that arise from dimensional reduction of gravity and supergravity theories, is presented. Extending previous results [49] the two-time Hamiltonian formulation is obtained, which describes the complete phase space of the model in the isomonodromic sector. The Dirac brackets arising from the coset constraints are calculated. Their quantization allows to relate exact solutions of the corresponding Wheeler–DeWitt equations to solutions of a modified (Coset-)Knizhnik-Zamolodchikov system.

On the classical level, a set of observables is identified, that is complete for essential sectors of the theory. Quantum counterparts of these observables and their algebraic structure are investigated. Their status in alternative quantization procedures is discussed, employing the link with Hamiltonian Chern–Simons theory.

1. Introduction

It is an important class of physical theories, that admit the formulation as a gravity coupled coset space σ-model after dimensional reduction to two dimensions. Including pure gravity and Kaluza-Klein theories as well as extended supergravity theories, in 3+1 dimensions they are described by a set of scalar and vector fields coupled to gravity, where the scalar fields already form a non-linear σ-model. Further reduction is achieved by imposing additional symmetries – manifest by assuming two additional commuting Killing vector fields, for example corresponding to the study of axisymmetric stationary models.

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[*] On leave of absence from Steklov Mathematical Institute, Fontanka, 27, St.Petersburg 191011, Russia

[**] Present address: Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Schlaatzweg 1, D-14473 Potsdam, Germany

This reduction to effectively two dimensions leads to a non-linear σ-model in an enlarged coset space, coupled to two-dimensional gravity and a dilaton field. The arising additional scalar fields that contribute to parametrizing the coset space are a remnant of the original vector fields and of components of the former higher-dimensional metric. For general reason, related to boundedness of the energy, it is the maximal compact subgroup H of G that is divided out in the coset. The first reduction of this type, discovered for pure gravity [33], leads to the simplest coset space $SL(2,\mathbb{R})/SO(2)$. It was generalized up to the case of maximally extended $N = 8$ supergravity, where the $E_{8(+8)}/SO(16)$ arises [40, 41]. The general proceeding was analyzed in [13, 55].

In [47–49] a program was started to perform an exact quantization of these dimensionally reduced gravity models. Progress has been achieved using methods and techniques similar to those developed in the theory of flat space integrable systems [24, 26, 46]. Despite the fact that dimensional reduction via additional symmetries represents an essential truncation of the theory, these so-called midi-superspace models under investigation are sufficiently complicated to justify the hope that their exact quantization might provide insights into fundamental features of a still outstanding quantized theory of gravitation. In particular and in contrast to previously exactly quantized mini-superspace models, they exhibit an infinite number of degrees of freedom, which is broadly accepted to be a sine qua non for any significant model of quantum gravity (compare [52, 5] for a discussion of this point in the context of related models). One of the final purposes of this approach is the identification of exact quantum states, whose classical limit corresponds to the known classical solutions. For pure gravity this includes the quantum analogue of the Kerr solution describing the rotating black hole; for extended supergravities recently discovered corresponding solutions have been of particular interest exhibiting fundamental duality symmetries [17, 16], such that their exact quantum counterparts should shed further light onto the role of these symmetries in a quantized theory.

The main ideas of the new framework are the following: Exploiting the integrability of the model, new fundamental variables have been identified (certain components of the flat connection of the auxiliary linear system continued into the plane of the spectral parameter), in terms of which the “right” and “left” moving sectors have been completely decoupled [47]. The quantization is further performed in the framework of a generalized “two-time” Hamiltonian formalism, i.e. these sectors are quantized independently. The whole procedure has been established in that sector of the theory, where the new fundamental connection exhibits simple poles at fixed singularities.

In the present paper we achieve the consistent general formulation of the desired coset-models in this approach. So far the formalism was mainly elaborated in the technically simplified principal model, where the coset G/H had been replaced by the group G itself. For the coset model the phase space spanned by the new variables is too large and must be restricted by proper constraints. Their canonical treatment requires a Dirac procedure, which effectively reduces the degrees of freedom. It leads to a consistent analogous Hamiltonian formulation of the coset model allowing canonical quantization. Exact quantum states are shown to be in correspondence to solutions of a modified (Coset-)Knizhnik-Zamolodchikov system. Moreover, the formalism is kept general as long as possible, without restricting to the simple pole sector. In particular, we completely extend it to the case of connections with poles of arbitrary high order at fixed singularities, which span the isomonodromic sector of the theory. Generalization of the scheme to the full phase space is sketched in Appendix A.

The other main result of this paper is the identification of classical and quantum observables. For the above mentioned simple pole sector, these sets are complete. Natural candidates for classical observables are the monodromies of the fundamental connection in the plane of the variable spectral parameter. We determine their (quadratic) Poisson structure. After quantization of the connection quantum counterparts of these monodromy matrices are identified as monodromies of certain higher-dimensional KZ systems. Following Drinfeld [22] their algebraic structure may be determined to build some quasi-associative braided bialgebra. The classical limit of this structure coincides with the Poisson algebra of the classical monodromies found above. In this sense, complete consistency of the picture is established. The weakened coassociativity leads to a quantum algebra of observables with operator-valued structure constants. This might have been avoided by directly quantizing the regularized classical algebra of monodromies, as is common in Chern–Simons theory [2, 3], instead of recovering quantum monodromies in the picture of the quantized connection. We discuss this link and its consequences.

The treatment of observables is performed in great detail for the simplified principal model mentioned above. This is for the sake of clarity of the presentation, since the arising difficulties in the coset case deserve an extra study in the sequel. However, the main tools and strategies that will finally be required can already and more clearly be developed and used in this context. The modifications required for the coset model are clarified afterwards.

The paper is organized as follows. In Chap.2 we start by introducing the known linear system associated to the model and describe the related on-shell conformal symmetry. A short summary and generalization of the results from [47, 49] about the classical treatment of the principal model is given without restricting to the simple pole sector. The link to Hamiltonian Chern–Simons theory is discussed, where the same holomorphic Poisson structure is obtained by symplectic reduction of the complexified phase space in a holomorphic gauge fixing. This link in particular enables us to relate the status of observables in both theories. Observables in terms of monodromy matrices are identified; their Poisson structure is calculated and discussed. The technical part of the calculation is shifted into Appendix B.

Chapter 3 treats the quantization of the principal model. We first briefly repeat the quantization of the simple pole sector of this model [48, 49]. Quantum analogues of the monodromy matrices are defined. Their algebraic structure and its classical limit are determined and shown to be consistent with the classical results. The alternative treatment in Chern–Simons theory and the identification of quantum observables in these approaches are discussed. In Chap.4 we finally present the generalization of the formalism to the coset models. A Hamiltonian formulation in terms of modified fundamental variables is provided. The coset constraints are explicitly solved by a Dirac procedure. Furthermore, we quantize the simple pole sector of the coset model, showing that solutions of a modified Knizhnik-Zamolodchikov system identify physical quantum states, i.e. exact solutions of the Wheeler–DeWitt equations. We close with a sketch of how to employ the whole machinery to the simplest case of pure four-dimensional axisymmetric stationary gravity. In particular, the existence of normalizable quantum states is shown. Chapter 5 briefly summarizes the open problems for future work.

2. Principal σ-Model Coupled to Two-Dimensional Dilaton Gravity

The model to be studied in this paper is described by the two-dimensional Lagrangian

$$\mathcal{L} = e\rho\Big(R + h^{\mu\nu}\mathrm{tr}[\partial_\mu g g^{-1}\partial_\nu g g^{-1}]\Big)\ . \tag{2.1}$$

Here, $h_{\mu\nu}$ is the $2D$ ("worldsheet") metric, $e = \sqrt{|\det h|}$, R is the Gaussian curvature of $h^{\mu\nu}$, $\rho \in \mathbb{R}$ is the dilaton field and g takes values in some real coset space G/H, where H is the maximal compact subgroup of G. The currents $\partial_\mu g g^{-1}$ therefore live in a fixed faithful representation of the algebra $\mathfrak{g}$ on some auxiliary d_0-dimensional space V_0. It is well known that this type of model arises from the dimensional reduction of higher dimensional gravities [13, 55], e.g. from $4D$ gravity in the presence of two commuting Killing vectors [12]. In the latter case which describes axisymmetric stationary gravity, the relevant symmetric space is $G/H = SL(2,\mathbb{R})/SO(2)$.

Let us first briefly describe further reduction of the Lagrangian (2.1) by means of gauge fixing and state the resulting equations of motion. The residual freedom of coordinate transformations can be used to achieve conformal gauge of the $2D$ metric $h_{\mu\nu}$:

$$h_{\mu\nu} dx^\mu dx^\nu = h(z,\bar{z}) dz d\bar{z} \,,$$

with world-sheet coordinates $z, \bar{z}$, which reduces the Lagrangian to

$$\mathcal{L} = \rho \Big(hR + \mathrm{tr}[g_z g^{-1} g_{\bar{z}} g^{-1}] \Big) \,. \tag{2.2}$$

In this gauge the Gaussian curvature takes the form $R = (\log h)_{z\bar{z}}/h$. The equation of motion for ρ derived from (2.2)

$$\rho_{z\bar{z}} = 0 \tag{2.3}$$

is solved by $\rho(z,\bar{z}) = \mathrm{Im}\,\xi(z)$, where $\xi(z)$ is a (locally) holomorphic function. Then the equations of motion for g coming from (2.2) read

$$\big((\xi - \bar{\xi}) g_z g^{-1}\big)_{\bar{z}} + \big((\xi - \bar{\xi}) g_{\bar{z}} g^{-1}\big)_z = 0 \,. \tag{2.4}$$

We can further specialize the gauge by identifying $\xi, \bar{\xi}$ with the worldsheet coordinates. Then (2.4) turns into

$$\big((\xi - \bar{\xi}) g_\xi g^{-1}\big)_{\bar{\xi}} + \big((\xi - \bar{\xi}) g_{\bar{\xi}} g^{-1}\big)_\xi = 0 \,. \tag{2.5}$$

The equations of motion for the conformal factor are derived from the original Lagrangian (2.1):

$$(\log h)_\xi = \frac{\xi - \bar{\xi}}{4} \mathrm{tr}(g_\xi g^{-1})^2 \quad \text{and c.c.} \tag{2.6}$$

Throughout this whole chapter we will for above mentioned reasons of clarity investigate the simplified model, where the symmetric space G/H is replaced by the group G itself. We will refer to this plainer model as the **principal model**.

2.1. Linear system and on-shell conformal symmetry of the model. The starting point of our treatment is the following well-known linear system associated to Eqs. (2.5) [10, 54]:

$$\frac{d\Psi}{d\xi} = \frac{g_\xi g^{-1}}{1-\gamma}\Psi \,, \qquad \frac{d\Psi}{d\bar{\xi}} = \frac{g_{\bar{\xi}} g^{-1}}{1+\gamma}\Psi \,, \tag{2.7}$$

where γ is the spacetime-coordinates dependent "variable spectral parameter"

$$\gamma = \frac{2}{\xi - \xi}\left\{ w - \frac{\xi + \bar{\xi}}{2} \pm \sqrt{(w-\xi)(w-\bar{\xi})} \right\} \,, \tag{2.8}$$

or alternatively $w \in \mathbb{C}$ may be interpreted as a hidden "constant spectral parameter"; $\Psi(w, \xi, \bar{\xi})$ is a $G_{\mathbb{C}}$-valued function. The variable spectral parameter γ lives on the twofold covering of the complex w-plane, the transition between the sheets being performed by $\gamma \mapsto \frac{1}{\gamma}$. It satisfies

$$\frac{\partial\gamma}{\partial\xi} = \frac{\gamma}{\xi - \bar{\xi}} \frac{1+\gamma}{1-\gamma} , \qquad \frac{\partial\gamma}{\partial\bar{\xi}} = \frac{\gamma}{\bar{\xi} - \xi} \frac{1-\gamma}{1+\gamma} , \tag{2.9}$$

such that in (2.7) it is

$$\frac{d}{d\xi} = \frac{\partial}{\partial\xi} + \frac{\gamma}{\xi - \bar{\xi}} \frac{1+\gamma}{1-\gamma} \frac{\partial}{\partial\gamma} , \qquad \frac{d}{d\bar{\xi}} = \frac{\partial}{\partial\bar{\xi}} + \frac{\gamma}{\bar{\xi} - \xi} \frac{1-\gamma}{1+\gamma} \frac{\partial}{\partial\gamma} . \tag{2.10}$$

The linear system (2.7) exists due to the following on-shell Möbius symmetry of equations of motion.[1]

Theorem 2.1. *Let $g(z, \bar{z})$, $\rho(z, \bar{z}) = \mathrm{Im}\xi(z)$ and $h(z, \bar{z})$ be some solution of (2.3), (2.4), (2.6) and Ψ be the related solution of the linear system (2.7). Then*

$$\sigma^w[g] \equiv \Psi^{-1}\left(\frac{1}{\gamma}\right)\Psi(\gamma) , \qquad \sigma^w[\xi] \equiv \frac{w\xi(z)}{w - \xi(z)} , \qquad \sigma^w[h] \equiv h , \tag{2.11}$$

also solve (2.4), (2.6).

Proof. We have

$$\sigma^w[g_\xi g^{-1}] = \sqrt{\frac{w - \bar{\xi}}{w - \xi}}\,\Psi^{-1}\left(\frac{1}{\gamma}\right) g_\xi g^{-1} \Psi\left(\frac{1}{\gamma}\right) ,$$
$$\sigma^w[g_{\bar{\xi}} g^{-1}] = \sqrt{\frac{w - \xi}{w - \bar{\xi}}}\,\Psi^{-1}\left(\frac{1}{\gamma}\right) g_{\bar{\xi}} g^{-1} \Psi\left(\frac{1}{\gamma}\right) .$$

Now fulfillment of (2.4), (2.6) may be checked by straightforward calculation. □

The transformations σ^w form a one-parametric abelian subgroup of the group $SL(2, \mathbb{R})$ of conformal transformations. We have

$$\sigma^{w_1}\sigma^{w_2} = \sigma^{w_3} , \qquad \frac{1}{w_1} + \frac{1}{w_2} = \frac{1}{w_3} .$$

The full Möbius group may be obtained combining transformations σ^w with the (essentially trivial) transformations

$$\xi(z) \mapsto a\xi(z) + b , \qquad g(z) \mapsto g(z) ,$$

which obviously leave the equations of motion invariant. As a result the action of an arbitrary $SL(2, \mathbb{R})$ Möbius transformation σ on a solution of the equations of motion is

$$\xi(z) \mapsto \sigma[\xi] \equiv a\frac{w\xi(z)}{w - \xi(z)} + b , \qquad g(z, \bar{z}) \mapsto \sigma[g] \equiv \Psi^{-1}\left(\frac{1}{\gamma}\right)\Psi(\gamma) , \tag{2.12}$$

[1] A similar symmetry exists in the theory of Bianchi surfaces [11].

leaving h invariant. In addition to the Möbius symmetry (2.12) the model possesses the symmetry corresponding to an arbitrary holomorphic change of the worldsheet coordinate z (this symmetry disappears if we identify z with ξ). Combining this symmetry with (2.12) reveals the following Möbius symmetry of Eq. (2.5)

$$g(\xi,\bar{\xi}) \mapsto \sigma[g]\left(\frac{w(\xi-b)}{aw+\xi-b}, \frac{w(\bar{\xi}-b)}{aw+\bar{\xi}-b}\right) , \tag{2.13}$$

$$h(\xi,\bar{\xi}) \mapsto h\left(\frac{w(\xi-b)}{aw+\xi-b}, \frac{w(\bar{\xi}-b)}{aw+\bar{\xi}-b}\right) . \tag{2.14}$$

Infinitesimally, the symmetry (2.13) is a subalgebra of the Virasoro symmetry of (2.5) [42].

Note 2.1. It is known that the Ernst equation (2.4) for $SL(2,\mathbb{R})/SO(2)$ may be rewritten as a fourth order differential equation in terms of the conformal factor h. The transformation (2.14) shows that this equation is, in contrast to the Ernst equation itself, Möbius invariant in the $\xi,\bar{\xi}$-plane.

2.2. Two-time Hamiltonian formulation of the principal model. Here we present a generalized version of the "two-time" Hamiltonian formalism of the principal σ-model proposed in [47, 48]. It is the strategy to define a new set of fundamental variables by means of exploiting the corresponding linear system. These variables may be equipped with a Poisson structure such that a two-time Hamiltonian formulation of the model is achieved.

2.2.1. New fundamental variables and the isomonodromic sector. The main objects we are going to consider as fundamental variables in the sequel are certain components of the following one-form:

Definition 2.1. *Let $\Psi(\gamma,\xi,\bar{\xi})$ be a solution of the linear system (2.7). Then the $\mathfrak{g}$-valued one-form $\boldsymbol{A}$ is defined as*

$$\boldsymbol{A} := d\Psi\Psi^{-1} . \tag{2.15}$$

In particular, we are interested in the components

$$\boldsymbol{A} = A^{\gamma}d\gamma + A^{\xi}d\xi + A^{\bar{\xi}}d\bar{\xi} = A^{w}dw + \tilde{A}^{\xi}d\xi + \tilde{A}^{\bar{\xi}}d\bar{\xi} , \tag{2.16}$$

where $(\gamma,\xi,\bar{\xi})$ and $(w,\xi,\bar{\xi})$ respectively are considered to be independent variables. In the sequel we shall use the shortened notation $A \equiv A^{\gamma}$.

Moreover, we will restrict our study to that sector of the theory, where A is a single-valued meromorphic function of γ, i.e. that also $\boldsymbol{A}$ is single-valued and meromorphic in γ. A solution Ψ of (2.7) with this property is called **isomonodromic**, *as its monodromies in the γ-plane then have no w-dependence due to (2.15).*

Further on, we immediately get the following relations:

Lemma 2.1. *The relation of the original field g to A is given by*

$$g_{\xi}g^{-1} = \frac{2}{\xi-\bar{\xi}}A(\gamma,\xi,\bar{\xi})\Big|_{\gamma=1} , \qquad g_{\bar{\xi}}g^{-1} = \frac{2}{\bar{\xi}-\xi}A(\gamma,\xi,\bar{\xi})\Big|_{\gamma=-1} , \tag{2.17}$$

as a corollary of (2.7) and (2.10). Moreover, the linear system (2.7) and definition (2.16) imply

$$
\begin{aligned}
A^w &= \frac{\partial \gamma}{\partial w} A, \\
\tilde{A}^\xi &= \frac{2A(1)}{(\xi - \bar{\xi})(1-\gamma)} , \qquad \tilde{A}^{\bar{\xi}} = \frac{2A(-1)}{(\xi - \bar{\xi})(1+\gamma)} , \\
A^\xi &= \frac{2A(1) - \gamma(1+\gamma)A(\gamma)}{(\xi - \bar{\xi})(1-\gamma)} , \qquad A^{\bar{\xi}} = \frac{2A(-1) + \gamma(1-\gamma)A(\gamma)}{(\xi - \bar{\xi})(1+\gamma)} .
\end{aligned}
\tag{2.18}
$$

□

Note 2.2. In the sequel $A(\gamma)$ will be exploited as the basic fundamental variable. At this point we should stress the difference between the real group G (with algebra $\mathfrak{g}$) entering the physical models and the related complexified group $G_{\mathbb{C}}$ (with algebra $\mathfrak{g}_{\mathbb{C}}$). Namely, it is $A(\gamma \in \mathbb{C}) \in \mathfrak{g}_{\mathbb{C}}$, whereas we will additionally impose the "imaginary cut" $\mathrm{i}A(\gamma \in \mathrm{i}\mathbb{R}) \in \mathfrak{g}$. Since $A(\gamma)$ is a (locally) holomorphic function, this implies

$$A(\bar{\gamma}) = -A^*(-\gamma) , \tag{2.19}$$

where * denotes the anti-linear conjugation on $\mathfrak{g}_{\mathbb{C}}$ defined by the real form $\mathfrak{g}$. Together with (2.17) this ensures $g \in G$.

Note 2.3. The linear system (2.7) admits the normalization

$$\Psi(\gamma = \infty) = I , \tag{2.20}$$

which implies regularity of A at infinity:

$$A_\infty := \lim_{\gamma \to \infty} \gamma A(\gamma) = 0 . \tag{2.21}$$

Furthermore, (2.7) implies an additional relation between the original field g and the Ψ-function:

$$\Psi(\gamma = 0) = gC_0 , \tag{2.22}$$

where C_0 is a constant matrix in the isomonodromic sector.

The definition of $\boldsymbol{A}$ as pure gauge (2.15) implies integrability conditions on its components, which in particular give rise to the following closed system for $A(\gamma)$:

$$\frac{\partial A}{\partial \xi} = [A^\xi, A] + \frac{\partial A^\xi}{\partial \gamma} , \qquad \frac{\partial A}{\partial \bar{\xi}} = [A^{\bar{\xi}}, A] + \frac{\partial A^{\bar{\xi}}}{\partial \gamma} . \tag{2.23}$$

The main advantage of the system (2.23) in comparison with the original equations of motion in terms of g (2.5) is that the dependence on ξ and $\bar{\xi}$ is now completely decoupled. Once the system (2.23) is solved, it is easy to check that Eqs. (2.17) are compatible and the field g restored by means of them satisfies (2.5).

The remaining set of equations of the principal model (2.6), which concern the conformal factor h, may be rewritten taking into account (2.17) as the following constraints:

$$\mathcal{C}^\xi := -(\log h)_\xi + \frac{1}{\xi - \bar{\xi}} \mathrm{tr} A^2(1) = 0 , \quad \mathcal{C}^{\bar{\xi}} := -(\log h)_{\bar{\xi}} + \frac{1}{\bar{\xi} - \xi} \mathrm{tr} A^2(-1) = 0 . \tag{2.24}$$

2.2.2. Poisson structure and Hamiltonians. The described decoupling of ξ and $\bar{\xi}$ dependence allows to treat the system (2.23), (2.24) in the framework of a manifestly covariant two-time Hamiltonian formalism, where the field $A(\gamma)$, the "times" ξ, $\bar{\xi}$ and the fields $(\log h)_\xi$, $(\log h)_{\bar{\xi}}$ are considered as new basic variables. The spirit of the generalized "several-times" Hamiltonian formalism is described for example in [44, 18].

For this purpose we equip $A(\gamma)$ with the following (equal $\xi, \bar{\xi}$) Poisson structure:

Definition 2.2. *Define the Poisson bracket on* $A(\gamma) \equiv A^a(\gamma) t_a$ *as:*

$$\left\{ A^a(\gamma)\,,\, A^b(\mu) \right\} = -f^{abc}\,\frac{A^c(\gamma) - A^c(\mu)}{\gamma - \mu}\;, \tag{2.25}$$

f^{abc} *being the structure constants of* $\mathfrak{g}$.[2]

The relations

$$\left\{ A(\gamma)\,,\, \frac{1}{\xi - \bar{\xi}} \mathrm{tr} A^2(1) \right\} = \left[A^\xi(\gamma)\,,\, A(\gamma) \right]\;, \tag{2.26}$$
$$\left\{ A(\gamma)\,,\, \frac{1}{\bar{\xi} - \xi} \mathrm{tr} A^2(-1) \right\} = \left[A^{\bar{\xi}}(\gamma)\,,\, A(\gamma) \right]\;,$$

compared with the equations of motion (2.23) give rise to

Definition 2.3. *We call the* $(\xi, \bar{\xi})$*-dynamics that is generated by*

$$H^\xi := \frac{1}{\xi - \bar{\xi}} \mathrm{tr} A^2(1)\;, \qquad H^{\bar{\xi}} := \frac{1}{\bar{\xi} - \xi}\, \mathrm{tr} A^2(-1)\;, \tag{2.27}$$

the **implicit time dependence** *of the fields. The remaining* $(\xi, \bar{\xi})$*-dynamics is referred to as* **explicit time dependence**.

In fact, the motivation for this definition arises from [47, 48], where it has been shown that in essential sectors of the theory (simple pole singularities in the connection A), it is possible to identify a complete set of explicitly time-independent variables. They may be treated as canonical variables then, such that H^ξ and $H^{\bar{\xi}}$ serve as complete Hamiltonians. This will be illustrated and generalized in the next subsections for the isomonodromic sector of the theory, where $A(\gamma)$ is assumed to be a meromorphic function of γ.

The extension of this framework to the whole phase space of arbitrary connections A, that is strongly inspired from the treatment of the simple pole case, is sketched in Appendix A. The variables $A(\gamma)$ themselves are explicitly time-dependent in general according to (2.23) and (2.26).

Note 2.4. The quantities

$$B(w) = A^w(\gamma) + A^w\left(\frac{1}{\gamma}\right) \equiv \frac{\partial \gamma}{\partial w}\left(A(\gamma) - \frac{1}{\gamma^2} A\left(\frac{1}{\gamma}\right)\right) \tag{2.28}$$

build a rather simple set of explicitly time-independent variables, carrying half of the degrees of freedom of the full phase space. This may be checked by straightforward calculation. Moreover, (2.25) implies

$$\{B^a(w), B^b(v)\} = -f^{abc}\,\frac{B^c(w) - B^c(v)}{w - v}\;. \tag{2.29}$$

[2] Assuming $\mathfrak{g}$ to be semisimple, the existence of the symmetric Killing-form enables us to arbitrarily pull up and down the algebra indices.

Note 2.5. From the mathematical point of view, (2.25) is a rather natural structure [26], even though it is not canonically derived from the Lagrangian (2.1). It may however be obtained from an alternative Chern–Simons Lagrangian formulation of the model, as is sketched in the following section. Comparison to the conventional Poisson structure of (2.1) should be worked out on the space of observables, where due to spacetime-diffeomorphism invariance no principal difference between one- and two-time structures appears.

In order to gain a Hamiltonian description for the total $(\xi, \bar{\xi})$-dependence of the fields, we employ a full covariant treatment by additionally introducing conjugate momenta for the canonical "time" variables ξ and $\bar{\xi}$.

Definition 2.4. *Define the (equal $\xi, \bar{\xi}$) Poisson bracket*

$$\left\{\xi, -(\log h)_\xi\right\} = \left\{\bar{\xi}, -(\log h)_{\bar{\xi}}\right\} = 1 \ , \tag{2.30}$$

where in the sense of a covariant theory only the explicit appearance of $\xi, \bar{\xi}$ (compare Def. 2.3) is covered by treating these previous "times" as additional canonical variables, which obey the bracket (2.30).

This identification of the conjugate momenta for the explicitly appearing times with the logarithmic derivatives of the conformal factor is motivated from the Lagrangian (2.2) [56]. It implies that the dynamics in ξ and $\bar{\xi}$ directions is completely given by the Hamiltonian constraints $\mathcal{C}^\xi$ and $\mathcal{C}^{\bar{\xi}}$ defined in (2.24), i.e. for any functional F we have

$$\frac{dF}{d\xi} = \{F, \mathcal{C}^\xi\} \ , \qquad \frac{dF}{d\bar{\xi}} = \{F, \mathcal{C}^{\bar{\xi}}\} \ . \tag{2.31}$$

The remaining equations of motion (2.24) mean weak vanishing of the Hamiltonians. This phenomena always arises in the framework of covariant Hamiltonian formalism when time is treated as canonical variable in its own right canonically conjugated to the Hamiltonian [35]; it is a standard way to take into account possible reparametrization of the time variable.

2.2.3. First order poles. In this simplest case considered in [47, 49] we assume that $A(\gamma)$ has only simple poles, i.e.

$$A(\gamma) = \sum_{j=1}^{N} \frac{A_j(\xi, \bar{\xi})}{\gamma - \gamma_j} \ , \tag{2.32}$$

where according to (2.7) all γ_j should satisfy (2.9), i.e. $\gamma_j = \gamma(w_j, \xi, \bar{\xi})$, $w_j \in \mathbb{C}$. Then the equations of motion (2.23) yield

$$\frac{\partial A_j}{\partial \xi} = \frac{2}{\xi - \bar{\xi}} \sum_{k \neq j} \frac{[A_k,\ A_j]}{(1-\gamma_k)(1-\gamma_j)} \ , \qquad \frac{\partial A_j}{\partial \bar{\xi}} = \frac{2}{\bar{\xi} - \xi} \sum_{k \neq j} \frac{[A_k,\ A_j]}{(1+\gamma_k)(1+\gamma_j)} \ , \tag{2.33}$$

and the Poisson brackets (2.25) and (2.30) reduce to

$$\{A_i^a, A_j^b\} = \delta_{ij} f^{abc} A_j \ , \tag{2.34}$$

$$\begin{aligned} \{A_j, (\log h)_\xi\} &= \{A_j, (\log h)_{\bar{\xi}}\} \ = \ 0 \ , \\ \{\gamma_j, (\log h)_\xi\} &= -\partial_\xi \gamma_j \ , \\ \{\gamma_j, (\log h)_{\bar{\xi}}\} &= -\partial_{\bar{\xi}} \gamma_j \ , \end{aligned} \tag{2.35}$$

i.e. in this case, the residues A_j together with the set of (hidden constant) positions of the singularities $\{w_j\}$ give the full set of explicitly time-independent variables.

2.2.4. Higher order poles. We can also generalize the described formulation to the case, where $A(\gamma)$ has higher order poles in the γ-plane:

$$A(\gamma) = \sum_{j=1}^{N} \sum_{k=1}^{r_j} \frac{A_j^k(\xi, \bar{\xi})}{(\gamma - \gamma_j)^k} \ . \tag{2.36}$$

The Poisson structure (2.25) in terms of A_j^k has the following form:

$$\{(A_i^k)^a, (A_j^l)^b\} = \begin{cases} \delta_{ij} f^{abc} (A_j^{k+l-1})^c & \text{for } k + l - 1 \leq r_j \\ 0 & \text{for } k + l - 1 > r_j \end{cases} \ , \tag{2.37}$$

building a set of mutually commuting truncated half affine algebras.

However, it turns out that for $r_j > 1$ the variables A_j^k for $k = 1, \dots r_j - 1$ have non-trivial Poisson brackets with $(\log h)_\xi$ and $(\log h)_{\bar{\xi}}$, and, therefore, are not explicitly time-independent. The problem of identification of explicitly time-independent variables can be solved in the following way. Consider

$$A^w(\gamma) = \frac{\partial \gamma}{\partial w} A(\gamma) \ ,$$

which as a function of w is meromorphic on the twofold covering of the w-plane. Parametrize the local expansion of A^w around one of its singularities γ_j as

$$A^w(\gamma) = \sum_{k=1}^{r_j} \frac{A_j^{(w)k}}{(w - w_j)^k} + \mathcal{O}((w - w_j)^0) \qquad \text{for} \quad \gamma \sim \gamma_j \ . \tag{2.38}$$

We can now formulate

Theorem 2.2. *The coefficients $A_j^{(w)k}$ of the local expansion of A^w have no explicit time dependence, i.e.*

$$\partial_\xi A_j^{(w)k} = \{A_j^{(w)k}, H^\xi\} \ , \qquad \partial_{\bar{\xi}} A_j^{(w)k} = \{A_j^{(w)k}, H^{\bar{\xi}}\} \ . \tag{2.39}$$

They satisfy the same Poisson structure as the A_j^k (2.37):

$$\left\{(A_i^{(w)k})^a \ , \ (A_j^{(w)l})^b\right\} = \begin{cases} \delta_{ij} f^{abc} (A_j^{(w)k+l-1})^c & \textit{for } k + l - 1 \leq r_j \\ 0 & \textit{for } k + l - 1 > r_j \end{cases} \ . \tag{2.40}$$

Proof. Let us first prove (2.39). From (2.25) and the definition of H^ξ it follows that

$$\begin{aligned} \{A^w(\gamma), H^\xi\} &= \left\{\partial_w \gamma A(\gamma) \ , \ \frac{2\mathrm{tr} A^2(1)}{(\xi - \bar{\xi})}\right\} \\ &= \frac{\partial_w \gamma}{(\xi - \bar{\xi})} \left[\frac{2A(1)}{1 - \gamma}, A(\gamma)\right] \ = \ [\tilde{A}^\xi(\gamma), A^w(\gamma)] \ , \end{aligned}$$

whereas from (2.15) the ξ-dynamics of A^w is determined to be

$$\partial_\xi A^w \ = \ [\tilde{A}^\xi(\gamma), A^w(\gamma)] + \partial_w \tilde{A}^\xi(\gamma) \ = \ [\tilde{A}^\xi(\gamma), A^w(\gamma)] + \partial_w \gamma \frac{2A(1)}{(1 - \gamma)^2} \ .$$

As the last term is regular in $\gamma = \gamma_j$, comparison of the two previous lines shows that the ξ-dependence of the coefficients in the w-expansion around these points is completely generated by H^ξ, which proves (2.39).

To show the Poisson structure (2.40), one has to consider the corresponding coefficients of singularities in (2.25). For $i \neq j$, the result follows directly from (2.37), as $A_j^{(w)k}$ is a function of $A_j^l, l=1,\dots,r_j$ only, such that locality remains. For $i=j$, one may first extract from (2.25) the behavior of $\{A^w(\gamma), A^w(\mu)\}$ around $\gamma \sim \gamma_j$:

$$\{(A^w)^a(\gamma),(A^w)^b(\mu)\} \;=\; -\partial_w\gamma\partial_v\mu f^{abc}\frac{A^c(\gamma)-A^c(\mu)}{\gamma-\mu} \;\sim\; f^{abc}\frac{(A^w)^c(\gamma)}{\mu-\gamma}\partial_v\mu\ ,$$

to then further study the asymptotical behavior $\mu \sim \gamma$:

$$\{(A^w)^a(\gamma),(A^w)^b(\mu)\} \;\sim\; f^{abc}\frac{(A^w)^c(\gamma)}{v-w}\ ,$$

such that (2.40) for $i=j$ follows in the same way, as does (2.37) from (2.25). □

Thus, also in this case we have succeeded in identifying a complete set of canonical explicitly time-independent variables.

Note 2.6. Comparing (2.36) with (2.38) shows that the $A_j^{(w)k}$ are related to the A_j^l by means of explicit recurrent relations that may be derived, expanding (2.36) in $(w-w_j)$. Then $A_j^{(w)k}$ is a function of A_j^l with $k \leq l \leq r_j$. In particular, the residues of highest order are related by

$$\left(\frac{\partial\gamma_j}{\partial w_j}\right)^{r_j-1} A_j^{(w)r_j} = A_j^{r_j}\ ,$$

which explains for example, why this difference was not relevant in the case of simple poles in the last subsection.

2.3. The link to Hamiltonian Chern–Simons theory. The treatment of the principal model of dimensionally reduced gravity in the previous section was inspired by the fact that the equations of motion were obtained as compatibility conditions (2.23) of special linear systems. The interpretation of these equations as zero curvature conditions suggests a link with Chern–Simons theory whose equations of motion also state the vanishing of some curvature. The Chern–Simons gauge connection then lives on a space locally parametrized simultaneously by the spectral parameter γ and one of the true space time coordinates playing the role of time.

The relevant Chern–Simons action reads

$$S = \frac{k}{4\pi}\int_M \mathrm{tr}[\boldsymbol{A}d\boldsymbol{A} - \frac{2}{3}\boldsymbol{A}^3]\ , \tag{2.41}$$

where $\boldsymbol{A}$ is a connection on a trivial G principal bundle over the 3-dimensional manifold M. In the case of interest here, the manifold M is the direct product of the Riemann surface Σ, on which the spectral parameter γ lives, and the real axis, which is interpreted as time. For this configuration, Chern–Simons theory is known to have a Hamiltonian formulation. Choosing proper boundary conditions on the connection, the action may be rewritten in the form

$$S = -\frac{k}{4\pi}\int_M \mathrm{tr}[A\partial_t A]dt + \frac{k}{2\pi}\int_M \mathrm{tr}[A^0(dA - A^2)]dt \;. \tag{2.42}$$

The connection has been split $\boldsymbol{A} = A + A^0 dt$ into spatial and time components, where A^0 now plays the role of a Lagrangian multiplier for the constraint

$$F = dA - A^2 = 0 \;. \tag{2.43}$$

Usually, A^0 is gauged to zero which leads to static components A. In particular, any singularities of the connection are time-independent in this case and treated by inserting static Wilson lines in the action (2.42) [61, 23]. A nontrivial and somewhat singular gauge for A^0 must be chosen, to derive the equations of motion of the described principal model of dimensionally reduced gravity.

The further required holomorphic reduction of Chern–Simons theory can still be described for arbitrary gauge fixing of A^0, as the results will be valid in any gauge.

2.3.1. Holomorphic reduction and Poisson bracket of the connection. For the following we first complexify the phase space and thereby also the gauge group. This enlarged gauge freedom may be used for a holomorphic gauge fixing then.

Denoting the spatial coordinates which locally parametrize Σ by $\gamma = x{+}\mathrm{i}y, \overline{\gamma} = x{-}\mathrm{i}y$, defining the measure as $\frac{k}{4\pi}dxdy \equiv \frac{-2\mathrm{i}\kappa}{4\pi}dx\mathrm{d}y = \frac{\kappa}{4\pi}d\gamma d\overline{\gamma}$ and splitting the remaining dynamical parts of $\boldsymbol{A}$ into $A = A^{\gamma}d\gamma + A^{\bar{\gamma}}\mathrm{d}\overline{\gamma}$, the action (2.42) implies the Poisson structure

$$\{A^{\gamma,a}(\gamma,\overline{\gamma}), A^{\bar{\gamma},b}(\mu,\overline{\mu})\} = -\frac{\mathrm{i}\pi}{\kappa}\delta^{ab}\delta^{(2)}(\gamma-\mu) \;, \tag{2.44}$$

where here and in the following the δ-function is understood as a real two-dimensional δ-function: $\delta^{(2)}(x+\mathrm{i}y) \equiv \frac{i}{2}\delta(x)\delta(y)$, normalized such that $\int d\gamma d\overline{\gamma}\delta^{(2)}(\gamma) = 1$.

This Poisson structure corresponds to the Atiyah-Bott symplectic form on the space of smooth connections on the Riemann surface Σ [6]:

$$\Omega = \frac{k}{4\pi}\mathrm{tr}\int_\Sigma \delta \mathrm{A} \wedge \delta \mathrm{A} \;.$$

The flatness constraints (2.43) are of the first class with respect to this bracket:

$$\{F^a(\gamma,\overline{\gamma}), F^b(\mu,\overline{\mu})\} = \frac{\mathrm{i}\pi}{\kappa} f^{abc}F^c(\gamma)\delta^{(2)}(\gamma-\mu) \;,$$

where f^{abc} are the total antisymmetric structure constants of $\mathfrak{g}_{\mathbb{C}}$. These constraints generate the canonical gauge transformations

$$A \mapsto gAg^{-1} + dgg^{-1} \;, \tag{2.45}$$

which leave the symplectic structure invariant.

The phase space of the original theory is therefore reduced to the space of flat connections $A(\gamma,\overline{\gamma})$ modulo the action of the complex gauge group (2.45). If the singularities of the connection A are restricted to simple poles, this phase space is for instance completely described by the monodromies of the connection. As a first step to explicitly reduce the number of degrees of freedom, we will fix the gauge freedom (2.45) in A, by demanding

$$A^{\bar{\gamma}} = 0 \;, \tag{2.46}$$

which makes flatness of $A(\gamma,\overline{\gamma})$ turn into holomorphy of the surviving component $A^{\gamma}(\gamma)$.

Note 2.7. The existence of corresponding gauge transformations is a nontrivial problem. In general, when $A^{\bar{\gamma}}$ is gauged away, $A^{\gamma}\mathrm{d}\gamma$ becomes a connection on a nontrivial bundle over Σ. On Riemann surfaces of higher genus, this form of gauge generically leads to multivalued holomorphic quantities exhibiting certain twist properties [50]. On the Riemann sphere the gauge transformations preserving single-valuedness of $A^{\gamma}d\gamma$ at least exist on a dense subspace of connections [6, 31]. For the purpose here, strictly speaking we a priori restrict the phase space to the class of functions on the punctured sphere that allow this gauge fixing. This includes e.g. all the connections with the curvature exhibiting δ-function singularities treated in [23] (gauge fixed to holomorphic connections with simple poles) as well as connections with higher order derivatives of δ-functions in the curvature.

This gauge fixing of first-class constraints changes the Poisson structure according to Dirac [19], leading to

Theorem 2.3. *Let the Poisson structure (2.44) for the connection*

$$A(\gamma,\bar{\gamma}) \equiv A^{\gamma,a}(\gamma,\bar{\gamma})t_a d\gamma + A^{\bar{\gamma},a}(\gamma,\bar{\gamma})t_a d\bar{\gamma}$$

be restricted by the constraints (2.43) and (2.46). Then the Dirac bracket for the surviving holomorphic components $A^a(\gamma) \equiv A^{\gamma,a}(\gamma)$ is given by

$$\{A^a(\gamma), A^b(\mu)\}^* = \frac{1}{2\kappa} f^{abc} \frac{A^c(\gamma) - A^c(\mu)}{\gamma - \mu} \,. \tag{2.47}$$

In this context, the holomorphic structure (2.47) has first been proposed by Fock and Rosly [28].

Proof. The bracket between the constraints and the gauge-fixing condition is of the form

$$\{F^a(\gamma), A^{\bar{\gamma},b}(\mu)\} = \frac{\mathrm{i}\pi}{\kappa}\delta^{ab}\partial_{\bar{\gamma}}\delta^{(2)}(\gamma-\mu) + \frac{\mathrm{i}\pi}{\kappa} f^{abc} A^{\bar{\gamma},c}(\gamma)\delta^{(2)}(\gamma-\mu)\,. \tag{2.48}$$

On the constraint surface (2.46) this matrix can be inverted using $\partial_{\bar{\gamma}}\frac{1}{\gamma} = -2\pi\mathrm{i}\delta^{(2)}(\gamma)$, which follows from the inhomogeneous Cauchy theorem. The Dirac bracket for the remaining holomorphic variables $A^{\gamma}(\gamma)$ then is

$$\begin{aligned}
&\{A^{\gamma,a}(\gamma), A^{\gamma,b}(\mu)\}^* \\
&\quad = -\sum_{m,n}\int dx d\bar{x} dy d\bar{y} \\
&\qquad \Big(\{A^{\gamma,a}(\gamma), F^m(x)\}\left(\{F^m(x), A^{\bar{\gamma},n}(y)\}\right)^{-1}\{A^{\bar{\gamma},n}(y), A^{\gamma,b}(\mu)\} \\
&\qquad + \{A^{\gamma,a}(\gamma), A^{\bar{\gamma},n}(y)\}\left(\{A^{\bar{\gamma},n}(y), F^m(x)\}\right)^{-1}\{F^m(x), A^{\gamma,b}(\mu)\}\Big) \\
&\quad = -\frac{\mathrm{i}\pi}{\kappa}\sum_m \int dx d\bar{x} dy d\bar{y} \\
&\qquad \bigg(\left(\delta^{am}\partial_x\delta^{(2)}(x-\gamma) + f^{mac}A^{\gamma,c}(x)\delta^{(2)}(x-\gamma)\right)\frac{\delta^{mb}\delta^{(2)}(y-\mu)}{2\pi i(x-y)} \\
&\qquad - \left(\delta^{bm}\partial_x\delta^{(2)}(x-\mu) + f^{mbc}A^{\gamma,c}(x)\delta^{(2)}(x-\mu)\right)\frac{\delta^{am}\delta^{(2)}(\gamma-y)}{2\pi i(x-y)}\bigg) \\
&\quad = \frac{1}{2\kappa} f^{abc}\frac{A^{\gamma,c}(\gamma) - A^{\gamma,c}(\mu)}{\gamma-\mu}\,. \qquad \square
\end{aligned}$$

Note 2.8. For convenience in concrete calculations we still give this result in tensor notation, as is explicitly explained in [26], where the relation of (2.47) to the corresponding current algebra is discussed. This structure may be put into the form

$$\{A(\gamma) \stackrel{\otimes}{,} A(\mu)\} = [r(\gamma-\mu), A(\gamma)\otimes I + I\otimes A(\mu)] \ , \tag{2.49}$$

with the classical r-matrix $r(\gamma) = -\frac{1}{2\kappa}\frac{\Omega}{\gamma}$, where $\Omega = t^a \otimes t_a$ is represented as $d_0^2 \times d_0^2$ matrix here. For the simplest but important case $\mathfrak{g}=\mathfrak{sl}(2)$, it is $\Omega = \frac{1}{2} I\otimes I + \Pi$, with Π being the 4×4 permutation operator. The matrix $r(\gamma)$ satisfies the classical Yang-Baxter equation with spectral parameter

$$[r^{12}(\gamma-\mu), r^{13}(\gamma) + r^{23}(\mu)] + [r^{13}(\gamma),\ r^{23}(\mu)] = 0 \ . \tag{2.50}$$

In shortened notation, (2.49) reads

$$\{A(\gamma)^0, A(\mu)^{\bar{0}}\} = [r(\gamma-\mu), A(\gamma)^0 + A(\mu)^{\bar{0}}] \ , \tag{2.51}$$

with $A(\gamma)^0 := A(\gamma)\otimes I \ , \quad A(\mu)^{\bar{0}} := I \otimes A(\mu) \ .$

Note 2.9. In the framework of canonical and geometric quantization of Chern–Simons theory [61, 7, 23, 31], the variables A^γ and $A^{\bar{\gamma}}$ are – according to (2.44) – considered and treated as canonically conjugated coordinate and momentum, respectively. After the holomorphic gauge fixing the surviving variable $A(\gamma) = A^\gamma(\gamma)$ resembles – according to (2.47) – a combination of angular momenta.

Note 2.10. The flatness constraints (2.43) have not been totally fixed by the choice of gauge (2.46). Apparently this gauge still admits holomorphic gauge transformations, which on the sphere reduce to constant gauge transformations. This freedom may also be seen from the appearance of $\partial_{\bar{\gamma}}$ in the matrix of constraint brackets (2.48), which actually prevents its strict invertibility. This implies the surviving of the (global) first-class part of the flatness constraint F, which for meromorphic A in the parametrization (2.36) is

$$\int F^a(\gamma) d\gamma d\overline{\gamma} = \int \partial_{\bar{\gamma}} A^a(\gamma) d\gamma d\overline{\gamma} = -2\pi \mathrm{i} \sum_i (A_i^1)^a = -2\pi\mathrm{i} A_\infty^a \ , \tag{2.52}$$

where $A_\infty = A_\infty^a t_a$, compare (2.21). Obviously, A_∞^a is a generator of constant gauge transformations in the bracket (2.47).

2.3.2. Embedding the principal model. In this holomorphic structure of Chern–Simons theory the link to the principal model can be established. As a first fact, note that the Dirac bracket (2.47) for $\kappa = -\frac{1}{2}$ equals the Poisson structure (2.25) that was used for the Hamiltonian formulation of the principal model.

The equations of motion from Chern–Simons action (2.41) read

$$\partial_t A^\gamma = \partial_\gamma A^0 + [A^\gamma, A^0] \ , \tag{2.53}$$

leading to trivial dynamics in the gauge $A^0 = 0$, whereas for t being replaced by ξ and the special (singular) choice of gauge

$$A^0(\gamma) := A^\xi(\gamma) = \frac{2A^\gamma(1) - \gamma(1+\gamma)A^\gamma(\gamma)}{(\xi - \xi)(1-\gamma)} \ ,$$

one exactly recovers the equations of motion (2.23).

Finally the surviving first-class constraints (2.52) that are due to former flatness on the sphere gain a definite physical meaning in the principal model of dimensionally reduced gravity. Arising there equivalently as regularity conditions in $\gamma \sim \infty$ (2.21), they are directly related to the asymptotical flatness of the corresponding solution g of Einstein's equations (2.5). As first-class constraints in different pictures [12], they generate respectively the Matzner-Misner or the Ehlers symmetry transformations of the model.

Their actual role as a physical gauge transformation related to the local Lorentz transformations becomes manifest in the proper treatment of the coset model below, see Subsect.4.

2.4. The algebra of observables. A consistent treatment of the theory and in particular the ability to extract classical and quantum predictions from the theoretical framework requires the identification of a complete set of observables. In the model as presented so far, observables can be defined in the sense of Dirac as objects that have vanishing Poisson bracket with all the constraints including the Hamiltonian constraints (2.24), which even play the most important role here. In the two-time formalism this condition shows the observables to have no total dependence on ξ and $\bar{\xi}$. This is a general feature of a covariant theory, where time dynamics is nothing but unfolding of a gauge transformation, and observables are the gauge invariant objects.

Regarding the connection $A(\gamma)$ as fundamental variables of the theory, the natural objects to build observables from are the monodromies of the linear system (2.15). They may be equivalently characterized as

$$\Psi(\gamma) \mapsto \Psi(\gamma)M_l \ , \qquad \text{for } \gamma \text{ running along the closed path } l \ , \qquad (2.54)$$

or

$$M_l = P \exp\left(\oint_l A(\gamma) d\gamma\right) \ .$$

These objects naturally have no total $(\xi, \bar{\xi})$-dependence; in the isomonodromic sector we treat, the w-dependence is also absent.

For simple poles let us denote by $M_i \equiv M_{l_i}$ the monodromies corresponding to the closed paths l_i which respectively encircle the singularities γ_i and touch in one common basepoint. From the local behavior of $\Psi(\gamma)$ around $\gamma = \gamma_i$,

$$\Psi(\gamma) = G_i \Big(I + \mathcal{O}(\gamma - \gamma_i)\Big)(\gamma - \gamma_i)^{T_i} C_i \ ,$$

one also extracts the relations

$$A_i = G_i T_i G_i^{-1}, \quad M_i = C_i^{-1} \mathrm{e}^{2\pi \mathrm{i} T_i} C_i \ . \qquad (2.55)$$

The remaining constraint of the theory which should have vanishing Poisson bracket with the observables is the generator of the constant gauge transformations (2.52), under which the monodromies transform by a common constant conjugation. This justifies

Definition 2.5. *In the case, where the connection $A(\gamma)$ exhibits only simple poles at fixed singularities w_j and with fixed eigenvalues of A_j, we call the set of Wilson loops*

$$\left\{ \mathrm{tr} \prod_k M_{i_k} \middle| k, (i_1, \ldots, i_k) \right\} \tag{2.56}$$

the **set of observables***.*

Note 2.11. For these connections $A(\gamma)$, the corresponding monodromies together with the position of the singularities and the eigenvalues of A_j generically already carry the complete information. (It is necessary to add the set of eigenvalues of A_j – i.e. the matrices T_j or the Casimir operators of the algebra respectively – to the set of monodromies, since from the monodromies only the exponentials of these eigenvalues can be extracted.) In the presence of higher order poles in the connection, additional scattering data – so-called Stokes multipliers – are required to uniquely specify the connection [39].

The generic case, in which the whole information is contained in the above data, is precisely defined by the fact that no eigenvalues of the monodromy matrices coincide [38, 39]. In particular, this excludes the case of multisolitons, where the monodromies equal $\pm I$.

The algebraic structure of the observables (2.56) is inherited from the Poisson structure on the corresponding connection $A(\gamma)$.

Before we explicitly describe this structure, let us briefly comment on the relation to Chern–Simons theory, where quite similarly the Poisson bracket (2.44) provides a Poisson structure on gauge invariant objects.

2.4.1. Observables in Chern–Simons theory. In Chern–Simons theory on the punctured sphere, the set of observables is also built from the monodromy matrices. Note that since in the usual gauge $A^0 = 0$ the Hamiltonian constraint is absent, observables are identified as gauge invariant objects, where this is invariance under local (γ-dependent) gauge transformations. Fixing this gauge freedom by holomorphic gauge as described above, the Dirac bracket (2.47) is now a structure on the reduced phase space of holomorphic connections $A(z)$ modulo the action of *constant* gauge transformations.

It has been explained in [2] that the canonical bracket (2.44) does not define a unique structure on monodromy matrices due to arising ambiguities from the singularities of this bracket (see also [59]). However, on gauge invariant objects, built from traces of arbitrary products of monodromy matrices, these ambiguities vanish [28, 1]. Hence the strategy there is to postulate some structure on the monodromy matrices which reduces to the proper one [34] on gauge invariant objects.

The holomorphic Dirac bracket (2.47) allows the calculation also for the monodromies themselves, as we shall show in the following. To relate this result to [28, 2], note that in general the original Poisson bracket and reduced Dirac bracket coincide on quantities of first class in Dirac terminology, i.e. here on gauge invariant objects. In this sense the holomorphic reduction finally leads to the same result on the space of observables.

2.4.2. Poisson structure of monodromy matrices. The holomorphic Poisson structure (2.47) defines a Poisson structure on the monodromy matrices M_j. The result is summarized in the following

Theorem 2.4. *Let $A(\gamma)$ be a connection on the punctured plane $\gamma \setminus \{\gamma_1, \ldots, \gamma_N\}$, equipped with the Poisson structure*

$$\left\{A(\gamma)^{0}, A(\mu)^{\bar{0}}\right\} = \frac{1}{\gamma-\mu}\left[\Omega, A(\gamma)^{0} + A(\mu)^{\bar{0}}\right] . \tag{2.57}$$

Let further Ψ be defined as a solution of the linear system

$$\partial_\gamma \Psi(\gamma) = A(\gamma)\Psi(\gamma) , \tag{2.58}$$

normalized at a fixed basepoint s_0,

$$\Psi(s_0) = I , \tag{2.59}$$

and denote by $M_1, \ldots, M_N$ the monodromy matrices of Ψ corresponding to a set of paths with endpoint s_0, which encircle $\gamma_1, \ldots, \gamma_N$, respectively. Ensure holomorphy of Ψ at ∞ by the first-class constraint

$$A_\infty = \lim_{\gamma\to\infty} \gamma A(\gamma) = 0 . \tag{2.60}$$

Then, in the limit $s_0 \to \infty$, the Poisson structure of the monodromy matrices is given by

$$\left\{M_i^{0}, M_i^{\bar{0}}\right\} = \mathrm{i}\pi\left(M_i^{\bar{0}}\,\Omega\,M_i^{0} - M_i^{0}\,\Omega\,M_i^{\bar{0}}\right) , \tag{2.61}$$

$$\left\{M_i^{0}, M_j^{\bar{0}}\right\} = \mathrm{i}\pi\left(M_i^{0}\,\Omega\,M_j^{\bar{0}} + M_j^{\bar{0}}\,\Omega\,M_i^{0} - \Omega\,M_i^{0}M_j^{\bar{0}} - M_i^{0}M_j^{\bar{0}}\,\Omega\right) , \quad \textit{for} \;\; i<j , \tag{2.62}$$

where the paths defining the monodromy matrices M_j are ordered with increasing j with respect to the distinguished path $[s_0 \to \infty]$.

At this point several comments on the result of this theorem are in order, whereas the proof is postponed to Appendix B.

Note 2.12. The first-class constraint (2.60) generates constant gauge transformations of the connection A in the Poisson structure (2.57). For the connections of the type (2.36) this reduces to the constraint (2.52). In terms of the monodromy matrices, holomorphy of Ψ at ∞ is reflected by

$$M_\infty \equiv \prod M_i = I , \tag{2.63}$$

which in turn is a first-class constraint and generates the action of constant gauge transformations on the monodromy matrices in the structure (2.61) and (2.62). The ordering of this product is fixed to coincide with the ordering that defines (2.62).

The gauge transformation behavior of the fields explicitly reads

$$\left\{A_\infty^{0}\,,\,A_j^{\bar{0}}\right\} = \left[\Omega\,,\,A_j^{\bar{0}}\right] , \tag{2.64}$$

$$\left\{M_\infty^{0}\,,\,M_j^{\bar{0}}\right\} = \mathrm{i}\pi\Big(M_\infty^{0}\Omega M_j^{\bar{0}} - M_j^{\bar{0}}\Omega M_\infty^{0} - \Omega M_\infty^{0}M_j^{\bar{0}} + M_\infty^{0}M_j^{\bar{0}}\Omega\Big) .$$

This transformation law is further inherited by arbitrary products $M = \prod_k M_{j_k}$ of monodromies, where on the constraint surface $M_\infty = I$ it takes the form

$$\left\{ M_\infty^0, M^{\bar{0}} \right\} = -2\pi \mathrm{i} \left[\Omega \, , \, M^{\bar{0}} \right] \; , \tag{2.65}$$

resembling (2.64).

The generators of gauge transformations build the algebra

$$\left\{ A_\infty^0, A_\infty^{\bar{0}} \right\} = \left[\Omega, A_\infty^{\bar{0}} \right] \; , \tag{2.66}$$

or

$$\left\{ M_\infty^0 , \; M_\infty^{\bar{0}} \right\} = \mathrm{i}\pi \left(M_\infty^{\bar{0}} \, \Omega \, M_\infty^0 - \; M_\infty^0 \, \Omega \, M_\infty^{\bar{0}} \right) , \tag{2.67}$$

in terms of A_∞ and M_∞ respectively. In fact, the algebras (2.66) and (2.67) turn out to be isomorphic: the quadratic bracket (2.67) linearizes if the Casimirs are split out.

As mentioned, we will further be interested in gauge invariant objects, which are now identified by their vanishing Poisson bracket with (2.63) and which are therefore invariant under a global common conjugation of all monodromies. Note that this includes invariance under gauge transformations with gauge parameters (conjugation matrices) that have nonvanishing Poisson bracket with the monodromies themselves. In accordance with Definition 2.5, the structure (2.61), (2.62) implies

$$\{ M_\infty, \mathrm{tr} M \} = 0 \tag{2.68}$$

for an arbitrary product of monodromies M.

Note 2.13. The evident asymmetry of (2.62) with respect to the interchange of i and j is due to the fact that the monodromy matrices are defined by the homotopy class of the path, which connects the encircling path with the basepoint in the punctured plane. This gives rise to a cyclic ordering of the monodromies.

The distinguished path $[s_0 \to \infty]$ breaks and thereby fixes this ordering, as is explicitly illustrated in Fig.3 in Appendix B below. It is remnant of the so-called eyelash that enters the definition of the analogous Poisson structure in the combinatorial approach [28, 1, 2], being attached to every vertex and representing some freedom in this definition. However, the choice of another path $[s_0 \to \infty]$ simply corresponds to a global conjugation by some product of monodromy matrices: a shift of this eyelash by j steps corresponds to the transformation

$$M_k \to (M_1 \ldots M_j)^{-1} M_k (M_1 \ldots M_j) \; .$$

Therefore the restricted Poisson structure on gauge invariant objects is independent of this path.

Note 2.14. A seeming obstacle of the structure (2.61), (2.62) is the violation of Jacobi identities. Actually, this results from heavily exploiting the constraint (2.60) in the calculation of the Poisson brackets. As therefore these brackets are valid only on the first-class constraint surface (2.63), Jacobi identities can not be expected to hold in general.

However, the same reasoning shows [58], that the structure (2.61), (2.62) restricts to a Poisson structure fulfilling Jacobi identities on the space of gauge invariant objects. On this space, the structure reduces to the original Goldman bracket [34] and coincides with the restrictions of previously found and studied structures on the monodromy matrices [28]:

$$\left\{ M_i^0 \, , \, M_i^{\bar{0}} \right\} = M_i^{\bar{0}} r_+ M_i^0 + M_i^0 r_- M_i^{\bar{0}} - r_- M_i^0 M_i^{\bar{0}} - M_i^0 M_i^{\bar{0}} r_+ \; , \tag{2.69}$$

$$\left\{ M_i^0 \, , \, M_j^{\bar{0}} \right\} = M_i^0 r_+ M_j^{\bar{0}} + M_j^{\bar{0}} r_+ M_i^0 - r_+ M_i^0 M_j^{\bar{0}} - M_i^0 M_j^{\bar{0}} r_+ \; , \qquad \text{for} \;\; i < j \; ,$$

where r_+ and $r_- := -\Pi r_+ \Pi$ are arbitrary solutions of the classical Yang-Baxter equation

$$[r^{12}, r^{23}] + [r^{12}, r^{13}] + [r^{13}, r^{23}] = 0 \ , \tag{2.70}$$

and the symmetric part of r_+ is required to be $\mathrm{i}\pi\Omega$. Setting $r_+ \equiv \mathrm{i}\pi\Omega$, (2.69) reduces to (2.61), (2.62) such that our structure is in some sense the skeleton, which may be dressed with additional freedom that vanishes on gauge invariant objects. On the space of monodromy matrices themselves, introduction of r-matrices may be considered as some regularization to restore associativity, whereas the fact that Ω itself does not satisfy the classical Yang-Baxter equation is equivalent to (2.61), (2.62) not obeying Jacobi identities.

In the Poisson structure (2.69), the generator of gauge transformations $M_\infty \equiv \prod_i M_i$ has the following Poisson brackets with any monodromy M_k:

$$\left\{ M^0_\infty, M^{\bar{0}}_k \right\} = M^{\bar{0}}_k r_+ M^0_\infty - M^{\bar{0}}_k M^0_\infty r_- - r_+ M^0_\infty M^{\bar{0}}_k + M^0_\infty r_- M^{\bar{0}}_k \ , \tag{2.71}$$

which entails the same Poisson bracket of M_∞ with an arbitrary product of monodromies $M \equiv \prod_k M_{j_k}$. On the constraint surface $M_\infty = I$, taking into account $r_+ - r_- = 2\mathrm{i}\pi\Omega$, this again implies (2.65), such that M_∞ again generates the constant gauge transformations.

Note 2.15. The subset of observables

$$\{\mathrm{tr}[(M_i)^m] | i, m\} \cup \{w_i | i\} \tag{2.72}$$

commutes with the whole set of observables.

For the positions of the singularities this follows just trivially from the Poisson structure (2.25), whereas the eigenvalues of the monodromy matrices are related to the eigenvalues of the corresponding residues A_i (2.55), which in turn provide the Casimir operators of the mutually commuting algebras (2.34). This subset of commuting variables thus parametrizes the symplectic leaves of (2.61), (2.62).

Note 2.16. For our treatment of the coset model below, the following additional structure will be of importance. There is an involution $\tilde{\eta}$ on the set of observables, defined by the cyclic shift $M_i \mapsto M_{i \pm n}$, where $N = 2n$ is the total number of monodromies. The crucial observation is now that this involution is an automorphism of the Poisson structure on the algebra of observables:

$$\{\tilde{\eta}(X_1), \tilde{\eta}(X_2)\} = \tilde{\eta}(\{X_1, X_2\}) \ , \tag{2.73}$$

for X_1, X_2 being traces of arbitrary products of monodromy matrices. This is a corollary of Note 2.13, as it follows from the invariance of the Poisson structure on gauge invariant objects with respect to a shift of the eyelash that defines the ordering of monodromy matrices.

Like every involution, $\tilde{\eta}$ defines a grading of the algebra into its eigenspaces of eigenvalue ± 1. In particular, the even part forms a closed subalgebra.

3. Quantization of the Principal Model

3.1. Quantization in terms of the connection. The quantization of the model looks especially natural in the isomonodromic sector with only simple poles. This has been performed in [48, 49], as we shall briefly summarize. In this case straightforward quantization of the linear Poisson brackets (2.34) leads to the following commutation relations:

$$[A_i^a, A_j^b] = \mathrm{i}\hbar\delta_{ij} f^{abc} A_j , \tag{3.1}$$

$$[\xi, (\log h)_\xi] = [\bar\xi, (\log h)_{\bar\xi}] \ = \ -\mathrm{i}\hbar \ , \tag{3.2}$$

$$[\bar\xi, (\log h)_\xi] = [\xi, (\log h)_{\bar\xi}] \ = \ 0.$$

According to (3.2), representing ξ and $\bar\xi$ by multiplication operators, one can choose

$$(\log h)_\xi = \mathrm{i}\hbar\frac{\partial}{\partial\xi} \ , \qquad (\log h)_{\bar\xi} = \mathrm{i}\hbar\frac{\partial}{\partial\bar\xi} \, . \tag{3.3}$$

From (3.1), the residues A_j can be represented according to

$$A_j^a = \mathrm{i}\hbar t_j^a \ , \tag{3.4}$$

which acts on a representation V_j of the algebra $\mathfrak{g}_\mathbb{C}$.

Thus the quantum state $\psi(\xi,\bar\xi)$ in a sector with given singularities should depend on $(\xi,\bar\xi)$ and live in the tensor-product $V^{(N)} := V_1 \otimes \ldots \otimes V_N$ of N representation spaces. Denote the dimension of V_j by d_j, such that $d := \dim V^{(N)} = \prod d_j$.

3.1.1. Wheeler–DeWitt equations and Knizhnik-Zamolodchikov system. The whole "dynamics" of the theory is now encoded in the constraints (2.24), which accordingly play the role of the Wheeler–DeWitt equations here:

$$\mathcal{C}^\xi \psi = \mathcal{C}^{\bar\xi}\psi = 0 \ , \tag{3.5}$$

which can be written out in explicit form using (2.24), (2.27), (3.3) and (3.4):

$$\frac{\partial\psi}{\partial\xi} = \frac{\mathrm{i}\hbar}{\xi-\bar\xi}\sum_{k\neq j}\frac{\Omega_{jk}}{(1-\gamma_j)(1-\gamma_k)}\ \psi \ , \tag{3.6}$$

$$\frac{\partial\psi}{\partial\bar\xi} = \frac{\mathrm{i}\hbar}{\bar\xi-\xi}\sum_{k\neq j}\frac{\Omega_{jk}}{(1+\gamma_j)(1+\gamma_k)}\ \psi \ ,$$

where $\Omega_{jk} := t_j^a \otimes t_k^a$ is the symmetric 2-tensor of $\mathfrak{g}$, acting nontrivially only on V_j and V_k.

The other constraint that restricts the physical states arrives from (2.52); its meaning was sketched in Subsect. 2.3.2. In the quantized sector it is reflected by

$$\left(\sum_j t_j^a\right)\psi(\xi,\bar\xi) = 0 \ . \tag{3.7}$$

The general solution of the system (3.6) is not known. However, these equations turn out to be intimately related to the Knizhnik-Zamolodchikov (KZ) system [45]:

$$\frac{\partial\varphi_{\text{KZ}}}{\partial\gamma_j} = \mathrm{i}\hbar\sum_{k\neq j}\frac{\Omega_{jk}}{\gamma_j-\gamma_k}\varphi_{\text{KZ}} \ , \tag{3.8}$$

with an $V^{(N)}$-valued function $\varphi_{\text{KZ}}(\gamma_j)$:

Theorem 3.1. *If $\varphi_{\rm KZ}$ is a solution of (3.8) obeying the constraint (3.7), and the γ_j depend on $(\xi, \bar{\xi})$ according to (2.8), then*

$$\psi = \prod_{j=1}^{N} \left(\frac{\partial \gamma_j}{\partial w_j} \right)^{\frac{1}{2} {\rm i}\hbar \Omega_{jj}} \varphi_{\rm KZ} \tag{3.9}$$

solves the constraint (Wheeler–DeWitt) equations (3.6).

The Casimir operator Ω_{jj} defined above is assumed to act diagonal on the states; for $\mathfrak{g}$=$\mathfrak{sl}(2)$ for example, this is simply $\Omega_{jj} = \frac{1}{2} s_j(s_j - 2)$, classifying the representation.

Theorem 3.1 and the proof were obtained in [48]. The task of solving (3.6) reduces to the solution of (3.8).

Note 3.1. The γ_j dependence of the quantum states, introduced in Theorem 3.1, can be understood as just a formal dependence, which covers the $(\xi, \bar{\xi})$-dependence of these states. However, one may also split up this dynamics into several commuting flows generated by the corresponding operators from (3.8). The full set of solutions of (3.8) then may be interpreted as a "γ_j-evolution operator," describing this dynamics. In some sense [49] this quantum operator resembles the classical τ-function introduced in [38].

Note 3.2. We have described how the solution of the Wheeler–DeWitt equations is related to the solution of the KZ system (3.8) in the sector of the theory, where the connection has only simple poles. It is therefore natural to suppose that the quantization of the higher pole sectors that were classically presented in Subsect. 2.2.4 is achievable in a similar way and will moreover reveal a link to the higher order KZ systems, which were introduced in [57] in the quantization of isomonodromic deformations with exactly the Poisson structure (2.37) on the residues.

Note 3.3. For definiteness it is convenient to assume pure imaginary singularities $\gamma_j \in {\rm i}\mathbb{R}$ (i.e. $w_j \in \mathbb{R}$). Then classically $A_j \in \mathfrak{g}$ and quantized they carry representations of $\mathfrak{g}$ itself, not of $\mathfrak{g}_{\mathbb{C}}$.

3.2. Quantum algebra of monodromy matrices.

3.2.1. Quantum monodromies. Having quantized the connection $A(\gamma)$ as described in the previous section, it is a priori not clear how to identify quantum operators corresponding to the classical monodromy matrices in this picture. As they are classically highly nonlinear functions of the A_j, arbitrarily complicated normal-ordering ambiguities may arise in the quantum case.

The first problem is the definition of the quantum analogue of the classical Ψ–function. Its $d_0 \times d_0$ matrix entries are now operators on the d-dimensional representation space $V^{(N)}$. We choose here a simple convention, replacing the classical linear system

$$\partial_\gamma \Psi(\gamma) = A(\gamma)\Psi(\gamma) \tag{3.10}$$

by formally the same one, where all the arising matrix entries are operators now, i.e. (3.10) remains valid for higher dimensional matrices A and Ψ. We have thereby fixed the operator ordering on the right-hand side in what seems to be the most natural way. In the same way, we define the quantum monodromy matrices:

Definition 3.1. *The* **quantum monodromy matrix** M_j *is defined to be the r.h.s. monodromy matrix of the (higher dimensional)* **quantum linear system** *(3.10):*

$$\Psi(\gamma) \mapsto \Psi(\gamma) M_j \qquad \text{for } \gamma \text{ encircling } \gamma_j \ , \tag{3.11}$$

where the quantum Ψ*-function is normalized as*

$$\Psi(\gamma) = \left(I + \mathcal{O}\left(\frac{1}{\gamma}\right)\right) \gamma^{-A_\infty} \qquad \text{around } \ \gamma \sim \infty \ . \tag{3.12}$$

Note 3.4. The normalization condition (3.12) generalizes the one we chose in the classical case (2.59) where the basepoint s_0 was sent to infinity. This generalization is necessary, because the constraint (2.60) is not fulfilled as an operator identity in the quantum case, which means that the quantum Ψ-function as an operator is definitely singular at $\gamma = \infty$ with the behavior (3.12). Only its action on physical states, which are by definition annihilated by the constraint (2.52) may be put equal to the identity for $\gamma = \infty$.

For proceeding further we now make use of an interesting observation of [57], relating the KZ systems with N and $N+1$ insertions by means of the quantum linear system (3.10). We state this as

Theorem 3.2. *Let* $\varphi(\gamma_1, \dots, \gamma_N)$ *be the evolution operator of the KZ system*

$$\partial_j \varphi = \mathrm{i}\hbar \sum_{k \neq j} \frac{\Omega_{jk}}{\gamma_j - \gamma_k} \varphi \ ,$$

and $\Phi(\gamma_0, \dots, \gamma_N)$ *be the corresponding evolution operator of the KZ system with an additional insertion at* $N = 0$*. Then* $\Psi(\gamma_0, \dots, \gamma_N) := (I \otimes \varphi^{-1})\Phi$ *satisfies the following system of equations:*

$$\partial_0 \Psi = \mathrm{i}\hbar \sum_{j=1}^{N} \frac{t_0^a \otimes (\varphi t_j^a \varphi^{-1})}{\gamma_0 - \gamma_j} \Psi \ , \tag{3.13}$$
$$\partial_j \Psi = -\mathrm{i}\hbar \frac{t_0^a \otimes (\varphi t_j^a \varphi^{-1})}{\gamma_0 - \gamma_j} \Psi \ .$$

The proof is obtained by a simple calculation. □

Consider the relations (3.13). Together with the remarks of Note 3.1, it follows that this Ψ just obeys the proper quantum linear system (3.10) in a Heisenberg picture: the $(\xi, \bar{\xi})$-dependence of the operators A_j is generated by conjugation with the evolution-operator φ. For the definition of the quantum Ψ-function it is the Heisenberg picture which provides the most natural framework, as only in this picture implicit and explicit $(\xi, \bar{\xi})$-dependence of operators are treated more or less on the same footing. Thus one may identify

$$A_j = \mathrm{i}\hbar t_0^a \otimes (\varphi t_j^a \varphi^{-1}) \ .$$

The operators t_0^a play the role of the classical representation t^a acting on the auxiliary space V_0, which is already required for the formulation of the classical linear system. In this sense, the KZ system with $N+1$ insertions combines the classical linear system with the quantum equations of motion that are described by the KZ system with N

insertions. The additional insertion γ_0 then plays the role of γ. We shall use this link to gain information about the algebraic structure of the quantum monodromy matrices.

3.2.2. Quantum group structure. We now start from the representation of the quantum Ψ-function due to Theorem 3.2:

$$\Psi(\gamma, \gamma_1, \dots, \gamma_N) = \Big(I \otimes \varphi^{-1}(\gamma_1, \dots, \gamma_N)\Big)\Phi(\gamma, \gamma_1, \dots, \gamma_N)\ . \tag{3.14}$$

This shows in particular that the quantum monodromy matrices of the principal model defined in (3.11) equal the corresponding monodromies of the KZ system with $N+1$ insertions. To obtain their algebraic structure, we employ a deep result of Drinfeld about the relation between the monodromies of the KZ connection and the braid group representations induced by certain quasi-bialgebras [21, 22]. Before we state these relations, we have to briefly describe the induced braid group representations.

The KZ system that is of interest here, is

$$\partial_j \Phi = \mathrm{i}\hbar \sum_{k \neq j} \frac{\Omega_{jk}}{\gamma_j - \gamma_k} \Phi\ ,$$

with $j = 0, \dots, N$, which, as explained, in a formal sense combines the classical and the quantum degrees of freedom, the function Φ living in $V^{(N+1)} := V_0 \otimes V^{(N)}$. This system naturally induces a representation of monodromy matrices, which may canonically be lifted to a braid group representation [43]. However, for our purpose, it is sufficient to remain on the level of the monodromy representation, which we denote by ρ_{KZ}.

We further have to briefly mention two algebraic structures, which are standard examples for braided quasi-bialgebras, where for details and exact definitions we refer to [22, 43]. Let us denote by $\mathcal{U}_\hbar$ the so-called Drinfeld-Jimbo quantum enveloping algebra associated with $\mathfrak{g}$ [20, 37]. This is a braided bialgebra, which includes the existence of a comultiplication Δ, a counit ϵ and a universal R-matrix $R_{\mathcal{U}} \in \mathcal{U}_\hbar \otimes \mathcal{U}_\hbar$, obeying several conditions of which the most important here is the (quantum) Yang-Baxter equation

$$R_{\mathcal{U}}^{12} R_{\mathcal{U}}^{13} R_{\mathcal{U}}^{23} = R_{\mathcal{U}}^{23} R_{\mathcal{U}}^{13} R_{\mathcal{U}}^{12}\ . \tag{3.15}$$

The matrix $R_{\mathcal{U}}$ can in principle be explicitly given, but is of a highly complicated form. It is Drinfeld's achievement to relate this structure to a braided quasi-bialgebra $\mathcal{A}_\hbar$, where the nontriviality of the R-matrix is essentially shifted into an additional element $\phi_{\mathcal{A}} \in \mathcal{A}_\hbar \otimes \mathcal{A}_\hbar \otimes \mathcal{A}_\hbar$, the so-called associator, which weakens the coassociativity. The R-matrix of $\mathcal{A}_\hbar$ is simply $R_{\mathcal{A}} = e^{-\pi\hbar\Omega}$, where $\Omega := t^a \otimes t_a$ is the symmetric 2-tensor of $\mathfrak{g}$. This R-matrix satisfies a weaker form of (3.15), the quasi-Yang-Baxter equation

$$R_{\mathcal{A}}^{12} \phi_{\mathcal{A}}^{312} R_{\mathcal{A}}^{13} (\phi_{\mathcal{A}}^{-1})^{132} R_{\mathcal{A}}^{23} \phi_{\mathcal{A}}^{123} = \phi_{\mathcal{A}}^{321} R_{\mathcal{A}}^{23} (\phi_{\mathcal{A}}^{-1})^{231} R_{\mathcal{A}}^{13} \phi_{\mathcal{A}}^{213} R_{\mathcal{A}}^{12}\ . \tag{3.16}$$

The algebras $\mathcal{U}_\hbar$ and $\mathcal{A}_\hbar$ are isomorphic as braided quasi-bialgebras [22].

There is a standard way, in which braided quasi-bialgebras induce representations of the braid group. Each simple braid σ_i is represented as

$$\rho(\sigma_i) := \phi_i^{-1} \Pi^{i,i+1} R^{i,i+1} \phi_i\ , \tag{3.17}$$

where Π is the permutation operator and ϕ_i is defined as $\phi_i := \Delta^{(i+1)}(\phi) \otimes I^{\otimes(N-i-2)}$ with $\Delta^{(1)} := 1$, $\Delta^{(2)} := \mathrm{Id}$ and $\Delta^{(i+1)} := (\Delta \otimes \mathrm{Id}^{\otimes i})\Delta^{(i)}$. We will denote the restrictions of these representations of the algebras $\mathcal{U}_\hbar$ and $\mathcal{A}_\hbar$ on the monodromies, which are built from products of simple braids, by $\rho_{\mathcal{U}}$ and $\rho_{\mathcal{A}}$ respectively.

Now we have collected all the ingredients to state the result of Drinfeld as:

Theorem 3.3. *The monodromy representation of the KZ system equals the described monodromy representation of the braided quasi-bialgebra $\mathcal{A}_\hbar$, which in turn is equivalent to the monodromy representation of the braided bialgebra $\mathcal{U}_h$. This means, that there is an automorphism u on $V^{(N+1)}$, such that*

$$\rho_{\mathrm{KZ}} = \rho_{\mathcal{A}} = u \rho_{\mathcal{U}} u^{-1} \ . \tag{3.18}$$

For the proof we refer to the original literature [22] or to the textbook of Kassel [43]. We should stress that in this construction the deformation parameter of the quantum group structure coincides with the true Planck constant $\hbar$. □

3.2.3. Quantum algebra and classical limit. It was our aim to describe the algebraic structure of the quantum monodromy matrices defined in (3.11). By Theorem 3.2 these monodromy matrices have been identified among the monodromies of the KZ system with $N+1$ insertions as the monodromies of the additional point γ_0 encircling the other insertions. Exploiting the consequences of Theorem 3.3 now, the quantum algebra of the monodromy matrices $M_1, \ldots, M_N$ is given by:

Theorem 3.4. *The matrices M_j from (3.11) satisfy*

$$\begin{aligned} R_- M_i^0 R_-^{-1} M_i^{\bar{0}} &= M_i^{\bar{0}} R_+ M_i^0 R_+^{-1} \ , \\ R_+ M_i^0 R_+^{-1} M_j^{\bar{0}} &= M_j^{\bar{0}} R_+ M_i^0 R_+^{-1} \ , \qquad \textit{for } i < j \ , \end{aligned} \tag{3.19}$$

where these relations are understood in a fixed representation of the $d_0 \times d_0$ matrix entries of the monodromy matrices on the tensor-product $V^{(N)} = \bigotimes_j V_j$. The R-matrices $R_\pm$ are

$$R_- := u_{\bar{0}} R_{\mathcal{U}}^{-1} u_0^{-1} \ , \qquad R_+ := \Pi R_-^{-1} \Pi \ , \tag{3.20}$$

where $R_{\mathcal{U}}$ is the universal R-matrix of $\mathcal{U}_\hbar$ mentioned above, u_0 is some automorphism on $V_0 \otimes V^{(N)}$ and $u_{\bar{0}}$ is the corresponding one on $V_{\bar{0}} \otimes V^{(N)}$. The classical limit of these R-matrices is given by

$$R_\pm = I \otimes I \ \pm \ (\mathrm{i}\hbar)(\mathrm{i}\pi\Omega) \ + \ \mathcal{O}_\pm(\hbar^2) \ . \tag{3.21}$$

Note 3.5. The relations (3.19) are to be understood as follows. The notation requires two copies 0 and $\bar{0}$ of the classical auxiliary space V_0. While the standard R-matrices $R_{\mathcal{U}}$ and $R_{\mathcal{A}}$ live on these classical spaces only, R_- and R_+ also act nontrivially on the quantum representation space $V^{(N)}$, due to conjugation with the automorphisms $u_0, u_{\bar{0}}$.

Proof of Theorem 3.4. Consider the monodromy representation (3.17) corresponding to the coassociative bialgebra $\mathcal{U}$. The monodromy M_j for $\gamma = \gamma_0$ encircling γ_j is thereby represented as

$$\rho_{\mathcal{U}}(M_j) = (R_{\mathcal{U}}^{-1})^{01} (R_{\mathcal{U}}^{-1})^{02} \ldots R_{\mathcal{U}}^{j0} R_{\mathcal{U}}^{0j} R_{\mathcal{U}}^{0,j-1} \ldots R_{\mathcal{U}}^{01} \ , \tag{3.22}$$

such that it is just a matter of sufficiently often exploiting the Yang-Baxter equation (3.15) to explicitly show that the relations (3.19) hold for $\rho_{\mathcal{U}}(M_j)$ with $R_- := R_{\mathcal{U}}^{-1}$, $R_+ := \Pi R_-^{-1} \Pi$. Theorem 3.3 further implies the conjugation of the R-matrices with the automorphism u in order to extend the result to the representation ρ_{KZ}, in which the monodromies from (3.11) were recovered.

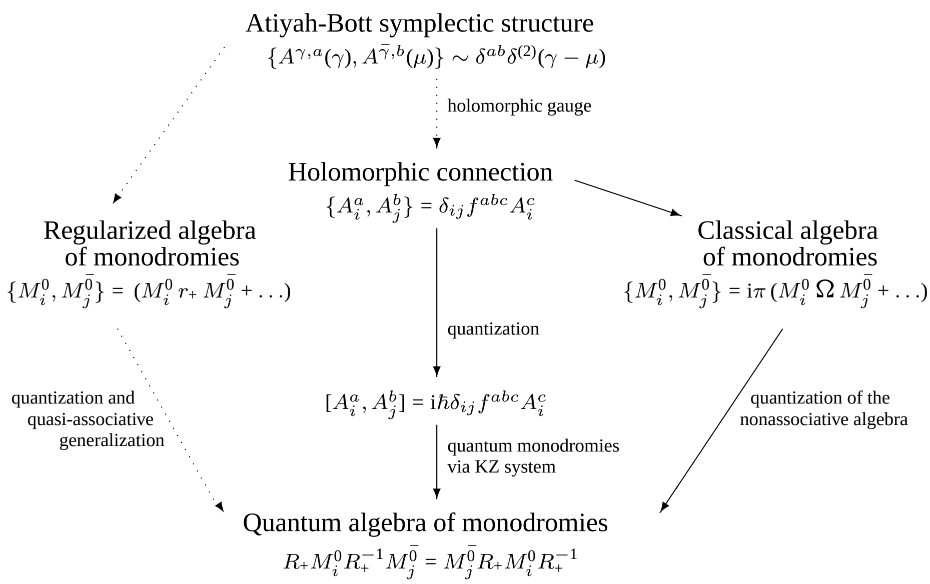
To further prove the asymptotic behavior (3.21), it is not enough to know the classical limit of $R_{\mathcal{U}}$ – which is a classical r-matrix simply – since the semiclassical expansion of the automorphisms $u_0, u_{\bar{0}}$ must be taken into account. For this reason, we additionally have to use the other part of Theorem 3.3, which relates the representations ρ_{KZ} and $\rho_{\mathcal{A}}$. The relations (3.19) for the $\rho_{\mathcal{A}}(M_j)$ hold with $R_- := R_{\mathcal{A}}^{-1}, R_+ := \Pi R_-^{-1} \Pi$ in a generalized form, modified by certain conjugations with the nontrivial associator $\phi_{\mathcal{A}}$. The semiclassical expansion of the associator is given by [43]:

$$\phi_{\mathcal{A}} = I \otimes I \otimes I \; + \; \mathcal{O}(\hbar^2) \; , \tag{3.23}$$

which implies that the term of order $\hbar$ in the semiclassical expansion of (3.19) is determined by the corresponding one in $R_{\mathcal{A}} = e^{-\pi\hbar\Omega}$, which yields (3.21).

The last point to be ensured is that the normalization of the quantum monodromies (3.12) around $\gamma \sim \infty$ coincides with the normalization chosen in the definition of the KZ monodromies [21] in certain asymptotic regions of the space of $(\gamma, \gamma_1 \; \ldots, \gamma_N)$, up to the order $\hbar$. The proof of this fact goes along the same line as the proof of (3.23). □

We have now established the quantum algebra of the quantum monodromy matrices by identifying the corresponding operators inside the picture of the quantized holomorphic connection $A(\gamma)$. The classical limit of this algebra equals exactly the classical algebra of monodromy matrices (2.61), (2.62). Hence, we have shown the "commutativity" of the (classical and quantum) links between the connection and the monodromies with the corresponding quantization procedures. Let us sketch this in the following diagram:



Note 3.6. The dotted lines in this diagram depict the link to the usual way quantum monodromies have been treated. This was done by directly quantizing their classical algebra, which is derived from the original symplectic structure of the connection up to certain degrees of gauge freedom: for later restriction on gauge invariant objects, this

algebra may be described with an arbitrary classical r-matrix, as was sketched in Note 2.14. A direct quantization of this structure is provided by a structure of the form (3.19), where the quantum R-matrices live in the classical spaces only and admit the classical expansion $R_\pm = I + i\hbar r_\pm + \mathcal{O}_\pm(\hbar^2)$ [1, 2].

Note 3.7. In contrast to this quantum algebra which underlies (2.69), in (3.19) the R-matrices – due to the automorphisms $u_0, u_{\bar{0}}$ – also act nontrivially on the quantum representation space. Their classical matrix entries may be considered as operator-valued, meaning that the quantum algebra can be treated alternatively as nonassociative or as “soft.” This is in some sense the quantum reason for the fact, that the classical algebra (2.61), (2.62) fails to satisfy Jacobi identities. However, note that (3.19) only describes the R-matrix in any fixed representation of the monodromies; for a description of the abstract algebra, compare the quasi-associative generalization in [2, 3], which provides the link between the quantum structure described in the previous note and (3.19).

3.2.4. Quantum observables. Let us discuss now the quantum observables, i.e. operators commuting with all the constraints. In analogy with the classical case it is clear that all monodromies of the quantum linear system (3.11) commute with the Hamiltonian constraints. Therefore, it remains to get rid of the gauge freedom (2.63), i.e. to identify functions of monodromies commuting with quantum generators of the gauge transformations. In the classical case the gauge transformations were generated by matrix entries of the matrix A_∞ or, equivalently, of the matrix $M_\infty - I$. The straightforward quantization of the classical algebra of gauge transformations generated by A_∞ (2.66) is

$$[A^a_\infty, A^b_\infty] = f^{ab}_c A^c_\infty \; , \tag{3.24}$$

i.e. coincides with $\mathfrak{g}$. In terms of M_∞, the algebra of the same gauge transformations according to (3.19) reads

$$R_- M^0_\infty R_-^{-1} M^{\bar{0}}_\infty = M^{\bar{0}}_\infty R_+ M^0_\infty R_+^{-1} \; . \tag{3.25}$$

The set of quantum observables is characterized as the set of operator-valued functions F of components of monodromies M_j which commute with all components of A_∞:

$$[F(\{M_j\}), A^a_\infty] = 0 \; . \tag{3.26}$$

Recall that in the classical case observables were just traces of arbitrary products of monodromies M_j. At the moment the quantum analog of this representation is not clear. One should suppose that there is a similar situation to the case we would have arrived at by directly quantizing the algebra of monodromies, mentioned in Note 3.6.

In this case, which has been studied in the combinatorial quantization of Chern–Simons theory [2, 3], the R-matrices live in the classical spaces only and the transformation behavior of arbitrary products of monodromies M under gauge transformations generated by M_∞ reads

$$R_- M^0 R_-^{-1} M^{\bar{0}}_\infty = M^{\bar{0}}_\infty R_+ M^0 R_+^{-1} \; .$$

Introducing the quantum trace $\mathrm{tr}_q M$ with characteristic relations

$$\mathrm{tr}^0_q R^{0\bar{0}} M^0 (R^{0\bar{0}})^{-1} = \mathrm{tr}_q M^0 \; , \tag{3.27}$$

we see that the operators $\mathrm{tr}_q M$ commute with the components of M_∞:

$$[\mathrm{tr}_q M, M^0_\infty] = 0 \ . \tag{3.28}$$

Therefore, the quantum group generated by M_∞:

$$R_- M^0_\infty R_-^{-1} M^{\bar{0}}_\infty = M^{\bar{0}}_\infty R_+ M^0_\infty R_+^{-1} \tag{3.29}$$

in this approach plays the role of algebra of gauge transformations.

It appears a difference of this approach with the approach which we mainly follow in this paper: instead of the Lie group G generated by the algebra (3.24), the role of the gauge group is played by its quantum deformation (3.29). A question therefore remains: what is the proper quantum gauge group of a consistent quantum theory, the group G itself or its quantum deformation G_q?

Note 3.8. With the notation of the quantum trace at hand, the quantum analogue of Note 2.15 can be formulated. From the abstract algebraic point of view – beyond the presented concrete representation of the quantum monodromies – the quantum trace of powers of the M_j build the center of the free algebra defined by (3.19) and may thus be fixed according to the classical values.

4. Coset Model

In this final chapter we will explain, how to modify the previously presented scheme in order to treat the coset models, which actually arise from physical theories. The field g is required to take values in a certain representation system of the coset space G/H, where H is the maximal compact subgroup of G.

This subgroup may be characterized by an involution η of G as the subgroup, which is invariant under η. The involution can further be lifted to the algebra $\mathfrak{g}$, e.g. $\eta(X) = -X^t$ for $X \in \mathfrak{g} = \mathfrak{sl}(N)$. The algebra $\mathfrak{g}$ is thereby split into its eigenspaces with eigenvalues ± 1, which are denoted by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, the subgroup H underlying $\mathfrak{h}$. In terms of the involution, the field g is restricted to satisfy:

$$g\eta(g) = I \ , \tag{4.1}$$

which defines the special choice of a representation system of the coset space.

4.1. Classical treatment. Classically speaking, the Poisson structure for the G/H-valued model may be obtained from the previously described Poisson structure for the principal G-valued model by implementing additional constraints.

These constraints were discussed in detail in [49] and may be equivalently formulated in terms of the function Ψ or of the connection A:

$$\eta\left(\Psi\left(\frac{1}{\gamma}\right)\right)^{-1} g^{-1}\Psi(\gamma) = C_0 \ , \tag{4.2}$$

$$A(\gamma) + \frac{1}{\gamma^2} g\eta\left(A\left(\frac{1}{\gamma}\right)\right) g^{-1} = 0 \ . \tag{4.3}$$

The first line is a consequence of (4.1) with $C_0 = C_0(w)$ from (2.22) also satisfying $C_0\eta(C_0) = I$ now. Studying the monodromies of Ψ shows that in the isomonodromic sector, C_0 must be gauged to a constant matrix, using the freedom of the right-hand side multiplication of the solution of (2.7). This can be seen from Eq. (4.36) below. Derivation of (4.2) with respect to γ then yields (4.3).

An unpleasant feature of these constraints is that they explicitly contain the field g, which in this framework is not among the fundamental variables. To avoid this difficulty, it is convenient to slightly modify the Hamiltonian formalism of the principal model. Namely, let us relax the normalization condition $\Psi(\gamma=\infty) = I$, which was imposed in (2.20) before and consider the function $\hat{\Psi}$ related to Ψ by a G-valued gauge transformation $\mathcal{V}$ instead:

$$\hat{\Psi} := \mathcal{V}(\xi,\bar{\xi})\Psi \,. \tag{4.4}$$

Then it is $\hat{\Psi}(\gamma=\infty) = \mathcal{V}$ and $gC_0 = \mathcal{V}^{-1}\hat{\Psi}(\gamma=0)$, such that the coset constraint (4.1) may be rewritten as:

$$g = \mathcal{V}^{-1}\eta(\mathcal{V}) \,. \tag{4.5}$$

The modified function $\hat{\Psi}$ now satisfies the linear system

$$\frac{d\hat{\Psi}}{d\xi} = \left(-\frac{1+\gamma}{1-\gamma}P_+ + Q_+\right)\hat{\Psi}, \qquad \frac{d\hat{\Psi}}{d\xi} = \left(-\frac{1-\gamma}{1+\gamma}P_- + Q_-\right)\hat{\Psi}, \tag{4.6}$$

with $(\xi,\bar{\xi})$-dependent matrices $P_\pm \in \mathfrak{k}$ and $Q_\pm \in \mathfrak{h}$ which can be reconstructed from $\mathcal{V}$ on the coset constraint surface (4.5):

$$\mathcal{V}_\xi\mathcal{V}^{-1} = P_+ + Q_+ \,, \qquad \mathcal{V}_{\bar{\xi}}\mathcal{V}^{-1} = P_- + Q_- \,.$$

Note 4.1. In the coset model the Möbius symmetry (2.11) appears in especially simple form [8]:

$$\mathcal{V} \mapsto \hat{\Psi}(\gamma)\,, \quad P_+ \mapsto \sqrt{\frac{w-\bar{\xi}}{w-\xi}}P_+\,, \quad P_- \mapsto \sqrt{\frac{w-\xi}{w-\xi}}P_-\,, \quad h \mapsto h\,.$$

In complete analogy to the principal model, we further introduce

Definition 4.1. *Define the connection* $\hat{A}$ *by*

$$\hat{A}(\gamma) := \partial_\gamma\hat{\Psi}(\gamma)\hat{\Psi}^{-1}(\gamma)\,. \tag{4.7}$$

The constraint of regularity at infinity then reads

$$\hat{A}_\infty := \lim_{\gamma\to\infty}\gamma\hat{A}(\gamma) = 0\,. \tag{4.8}$$

The relations (2.17) between the original fields and the connection $\hat{A}$ take the following form:

$$\frac{1}{\xi-\bar{\xi}}\hat{A}(\gamma,\xi,\bar{\xi})\Big|_{\gamma=1} = -P_+\,, \qquad \frac{1}{\xi-\bar{\xi}}\hat{A}(\gamma,\xi,\bar{\xi})\Big|_{\gamma=-1} = -P_-\,. \tag{4.9}$$

Hence, the coset constraints (4.5) are equivalent to

$$\hat{A}(\pm1) = -\eta\left(\hat{A}(\pm1)\right)\,, \tag{4.10}$$

which is implied by (4.3). Let us stress again that the originally equivalent coset constraints (4.1), (4.5) or (4.10) are lifted to (4.3) due to the special choice of $C_0 = \text{const}$ in the isomonodromic sector.

The constraints (4.2) and (4.3) take simpler forms in terms of the new variables $\hat{\Psi}$ and $\hat{A}$, since the field g is absorbed now:

$$\eta\left(\hat\Psi\left(\frac{1}{\gamma}\right)\right)^{-1}\hat\Psi(\gamma) = C_0, \tag{4.11}$$

$$\hat A(\gamma) + \frac{1}{\gamma^2}\eta\left(\hat A\left(\frac{1}{\gamma}\right)\right) = 0\ . \tag{4.12}$$

The first of these equations is a sign of the invariance of the linear system (4.6) on the coset constraint surface under the extended involution η^∞, introduced in [12]:

$$\eta^\infty(\hat\Psi(\gamma)) := \eta\left(\hat\Psi\left(\frac{1}{\gamma}\right)\right)\ , \tag{4.13}$$

but is difficult to handle due to the unknown matrix C_0. The latter form (4.12) of the constraint admits a complete treatment as will be described below. Note that the constraint of regularity at infinity (4.8) is already contained in (4.12) and is thereby naturally embedded in the coset constraints.

The set of constraints (4.12) is complete and consistent in the following sense:

Lemma 4.1. *The coset constraints (4.12) are invariant under $(\xi, \bar\xi)$-translation on the constraint surface.*

Proof. The total ξ-dependence of $\hat A$ can be extracted from (2.23) to be

$$\begin{aligned}\frac{d}{d\xi}\hat A(\gamma) &= \mathcal{V}[A^\xi(\gamma), A(\gamma)]\mathcal{V}^{-1} + [\mathcal{V}_\xi\mathcal{V}^{-1}, \hat A(\gamma)] + \mathcal{V}\frac{\partial A^\xi(\gamma)}{\partial\gamma}\mathcal{V}^{-1}\\ &= \left[\frac{-2P_+}{1-\gamma}, \hat A(\gamma)\right] + \left[(P_+ + Q_+), \hat A(\gamma)\right]\\ &\quad - \frac{2P_+}{(1-\gamma)^2} + \frac{\gamma^2 - 2\gamma - 1}{(\xi - \bar\xi)(1-\gamma)^2}\hat A(\gamma) - \frac{\gamma(1+\gamma)}{(\xi-\bar\xi)(1-\gamma)}\partial_\gamma\hat A(\gamma)\ .\end{aligned}$$

Together with $\frac{d}{d\xi}\left(f\left(\frac{1}{\gamma}\right)\right) = \left(-\frac{d}{d\xi}f\right)\left(\frac{1}{\gamma}\right)$ for any function $f(\gamma)$, which follows from the structure of γ_ξ, a short calculation reveals that on the constraint surface (4.12) it is

$$\frac{d}{d\xi}\left(\hat A(\gamma) + \frac{1}{\gamma^2}\eta\left(\hat A\left(\frac{1}{\gamma}\right)\right)\right) \approx -\gamma_\xi\ \frac{d}{d\gamma}\left(\hat A(\gamma) + \frac{1}{\gamma^2}\eta\left(\hat A\left(\frac{1}{\gamma}\right)\right)\right) \approx 0\ .$$

□

In a Hamiltonian formulation these constraints therefore have weakly vanishing Poisson bracket with the full Hamiltonian, which is required for a consistent treatment. Let us now briefly present the Hamiltonian formulation of the coset model in terms of the new variables.

4.1.1. Poisson structure and Hamiltonian formulation. The definition of the connection $\hat A$ already implies the relation

$$\hat A(\gamma) = \mathcal{V}A(\gamma)\mathcal{V}^{-1}, \tag{4.14}$$

such that from (2.23) one extracts the equations of motion for these new variables:

$$
\begin{aligned}
\frac{\partial \hat{A}}{\partial \xi} &= \mathcal{V}[A^{\xi}, A]\mathcal{V}^{-1} + \mathcal{V}\frac{\partial A^{\xi}}{\partial \gamma}\mathcal{V}^{-1} + [\mathcal{V}_{\xi}\mathcal{V}^{-1}, \hat{A}], \\
\frac{\partial \hat{A}}{\partial \bar{\xi}} &= \mathcal{V}[A^{\bar{\xi}}, A]\mathcal{V}^{-1} + \mathcal{V}\frac{\partial A^{\bar{\xi}}}{\partial \gamma}\mathcal{V}^{-1} + [\mathcal{V}_{\bar{\xi}}\mathcal{V}^{-1}, \hat{A}]\ .
\end{aligned} \tag{4.15}
$$

In analogy with the principal model, this motivates

Definition 4.2. *Define on $\hat{A}(\gamma)$ the following Poisson structure:*

$$
\left\{\hat{A}^a(\gamma), \hat{A}^b(\mu)\right\}_{\mathcal{V}} = -f^{abc}\frac{\hat{A}^c(\gamma) - \hat{A}^c(\mu)}{\gamma - \mu}\ , \tag{4.16}
$$

and denote by **implicit time-dependence** *the $(\xi, \bar{\xi})$-dynamics, that is generated by*

$$
\begin{aligned}
\hat{H}^{\xi} &:= \frac{1}{\xi - \bar{\xi}}\mathrm{tr}\hat{A}^2(1) - \mathrm{tr}[\hat{A}_{\infty}(\partial_{\xi}\mathcal{V}\mathcal{V}^{-1})]\ , \\
\hat{H}^{\bar{\xi}} &:= \frac{1}{\bar{\xi} - \xi}\mathrm{tr}\hat{A}^2(-1) - \mathrm{tr}[\hat{A}_{\infty}(\partial_{\bar{\xi}}\mathcal{V}\mathcal{V}^{-1})]\ ,
\end{aligned} \tag{4.17}
$$

on the constraint surface (4.8). The remaining explicit time-dependence is then defined to be generated in analogy to (2.30).

Note 4.2. The Poisson structures (4.16) are certainly different for different $\mathcal{V}$ and, therefore, are different from (2.25), that was introduced in the principal model. However, this previous treatment may be embedded in the following way. The structures (4.16) and (2.25) are certainly equivalent if we restrict them to the functionals of $\hat{A}$ that are invariant with respect to the choice of $\mathcal{V}$, i.e. invariant with respect to the transformations

$$
\hat{A} \mapsto \theta^{-1}\hat{A}\theta\ , \tag{4.18}
$$

with arbitrary $\theta \in G$. These were the gauge transformations in the principal model, generated by (2.21). Hence, on the set of observables of the principal model, the different Poisson structures coincide. Correspondingly, the action of H^{ξ} and $\hat{H}^{\xi}$ from (2.27) and (4.17) respectively differs only by the unfolding of such a gauge transformation.

For the coset model it is important to note that the gauge freedom (4.18) is restricted to H-valued matrices θ, since only that part of the constraint (4.8) remains first-class here and generates gauge transformations. This is part of the result of Theorem 4.1 below.

4.1.2. Solution of the constraints. Given a set of constraints (4.12) and a Poisson structure (4.16), the canonical procedure is due to Dirac [19]. The constraints are separated into first and second class constraints, of which the latter are explicitly solved – which changes the Poisson bracket into the Dirac bracket – whereas the former survive in the final theory.

In the case at hand, the essential part of the constraints is of the second class, such that the Poisson structure has to be modified and only a small part of the constraints survives as first-class constraints. We state the final result as

Theorem 4.1. *The Dirac procedure for treating the constraints (4.12) in the Poisson structure (4.16) yields the following Dirac bracket for the connection $\hat{A}$:*

$$\left\{\hat{A}^a(\gamma), \hat{A}^b(\mu)\right\}^*_{\mathcal{V}} = -\frac{1}{2} f^{abc} \frac{\hat{A}^c(\gamma) - \hat{A}^c(\mu)}{\gamma - \mu} + \frac{1}{2} f^{a\eta(b)c} \frac{\hat{A}^c(\gamma)}{\mu - \frac{1}{\gamma}} + \frac{1}{2} f^{\eta(a)bc} \frac{\hat{A}^c(\mu)}{\gamma - \frac{1}{\mu}} , \tag{4.19}$$

where the notation of indices means a choice of basis with $t^{\eta(a)} \equiv \eta(t^a)$. The bracket for the logarithmic derivatives of the conformal factor remains unchanged:

$$\left\{\xi, -(\log h)_\xi\right\}^*_{\mathcal{V}} = \left\{\bar{\xi}, -(\log h)_{\bar{\xi}}\right\}^*_{\mathcal{V}} = 1 . \tag{4.20}$$

The structure is compatible with the (now strong) identity

$$\hat{A}(\gamma) + \frac{1}{\gamma^2}\eta\left(\hat{A}\left(\frac{1}{\gamma}\right)\right) = \frac{1}{\gamma}\hat{A}_\infty = \frac{1}{\gamma}\eta(\hat{A}_\infty) , \tag{4.21}$$

such that compared with (4.12) it remains the first-class constraint

$$\hat{A}_\infty + \eta(\hat{A}_\infty) = 0 . \tag{4.22}$$

Proof. The main idea of the proof is the separation of the variables $\hat{A}(\gamma)$ into weakly commuting halves:

$$\Phi_1(\gamma) := \hat{A}(\gamma) + \frac{1}{\gamma^2}\eta\left(\hat{A}\left(\frac{1}{\gamma}\right)\right) - \frac{1}{\gamma}\hat{A}_\infty ,$$
$$\Phi_2(\gamma) := \hat{A}(\gamma) - \frac{1}{\gamma^2}\eta\left(\hat{A}\left(\frac{1}{\gamma}\right)\right) - \frac{1}{\gamma}\hat{A}_\infty ,$$

with

$$\left\{\Phi_1^a(\gamma), \Phi_2^b(\mu)\right\}_{\mathcal{V}} \approx 0 \tag{4.23}$$

on the constraint surface (4.12), as follows from (4.16) by direct calculation, using the fact that η is an automorphism: $f^{abc} = f^{\eta(a)\eta(b)\eta(c)}$.

The whole constraint surface is spanned by $\Phi_1 = 0$ and $\hat{A}_\infty = 0$, whereas Φ_2 covers the remaining degrees of freedom. Since Φ_1 and Φ_2 contain respectively $\hat{A}_\infty \mp \eta(\hat{A}_\infty)$, the relations (4.23) show that $\hat{A}_\infty + \eta(\hat{A}_\infty)$ is a first-class constraint of the theory.

If we further explicitly solve the second-class constraints $\Phi_1 = 0$, the commutativity (4.23) implies that the Poisson bracket of Φ_2 remains unchanged by the Dirac procedure:

$$\left\{\Phi_2^a(\gamma), \Phi_2^b(\mu)\right\}^*_{\mathcal{V}} = \left\{\Phi_2^a(\gamma), \Phi_2^b(\mu)\right\}_{\mathcal{V}} .$$

Moreover, the Dirac bracket is by construction compatible with the vanishing of Φ_1:

$$\{\Phi_1^a(\gamma), \, . \,\}^*_{\mathcal{V}} = 0 .$$

These facts may be used to easily calculate the Dirac bracket of the original variables $\hat{A}(\gamma)$ without explicitly inverting any matrix of constraint brackets. With the decomposition

$$\hat{A}(\gamma) = \frac{1}{2}\Phi_1(\gamma) + \frac{1}{2}\Phi_2(\gamma) + \frac{1}{2\gamma}(\hat{A}_\infty + \eta(\hat{A}_\infty)) + \frac{1}{2\gamma}(\hat{A}_\infty - \eta(\hat{A}_\infty)) ,$$

the result is obtained. The bracket (4.20) follows from the calculations performed in Lemma 4.1, which imply the vanishing Poisson bracket between $(\log h)_\xi$ and the constraints. □

4.1.3. Final formulation and symmetries of the theory. Let us summarize the final status of the theory and the relation of the new fundamental variables $\hat{A}(\gamma)$ to the original fields $\mathcal{V}$ and g respectively. We further discuss how the local and global symmetries of the original fields become manifest in this formulation.

The formulation in terms of the new variables $\hat{A}(\gamma)$ is completely described in Theorem 4.1, where their modified Poisson structure is given. The solved constraints (4.21) may be considered to be valid strongly.

The remaining first-class constraint (4.22) generates the transformation

$$\hat{A} \mapsto \chi^{-1} \hat{A} \chi \,, \tag{4.24}$$

with $\chi \in H$. According to (4.9), the field $\mathcal{V}$ transforms as

$$\mathcal{V} \mapsto \chi \mathcal{V} \,. \tag{4.25}$$

The relation (4.5) on the coset constraint surface shows that the field g does not feel this transformation. The gauge transformations generated by (4.22) are the manifestation of a really physical gauge freedom in the decomposition of the metric into some vielbein; they are remnant of the gauge freedom of local Lorentz transformations in general relativity. This freedom may be fixed to choose some special gauge for the vielbein field $\mathcal{V}$.

Note 4.3. It is important to notice that the second term in the modified Hamiltonians $\hat{H}^\xi, \hat{H}^{\bar\xi}$ from (4.17), that makes them differ from $H^\xi, H^{\bar\xi}$ from (2.27) becomes a pure gauge generator after the presented solution of the constraints. This is due to the fact that $\hat{A}_\infty \in \mathfrak{h}$ according to (4.21). Since $\mathfrak{h}$ and $\mathfrak{k}$ are orthogonal with respect to the Cartan-Killing form, the action of H^ξ and $\hat{H}^\xi$ just differs by $\mathfrak{h}$-conjugation and thus by a gauge transformation of the coset model.

The field $\hat{A}$ now does not contain the complete information about the original field $\mathcal{V}$, but only the currents $\mathcal{V}_\xi\mathcal{V}^{-1}, \mathcal{V}_{\bar\xi}\mathcal{V}^{-1}$, which may be extracted from $\hat{A}(\pm 1)$ by means of (4.9). At first sight, one might get the impression that in contrast to (2.17), the relations (4.9) do not even contain the full information about these currents. However, if the gauge freedom (4.25) in $\mathcal{V}$ is fixed, the currents may be uniquely recovered from (4.9). For $\mathfrak{g} = \mathfrak{sl}(N)$ for example, usually a triangular gauge of $\mathcal{V}$ is chosen, such that $\mathcal{V}_\xi\mathcal{V}^{-1}$ is recovered from its symmetric part $2P_+ = (\mathcal{V}_\xi\mathcal{V}^{-1}) + (\mathcal{V}_\xi\mathcal{V}^{-1})^t$.

The field $\mathcal{V}$ moreover is determined only up to right multiplication $\mathcal{V} \mapsto \mathcal{V}\theta$ from the currents $\mathcal{V}_\xi\mathcal{V}^{-1}, \mathcal{V}_{\bar\xi}\mathcal{V}^{-1}$. This is a (global) symmetry of the theory, under which the field g according to (4.5) transforms as

$$g \mapsto \theta^{-1} g \eta(\theta) \,. \tag{4.26}$$

For axisymmetric stationary $4D$ gravity these are the so-called Ehlers transformations. They are obviously a symmetry of the original equations of motion (2.5).

The new variables $\hat{A}(\gamma)$ are invariant under these global transformations, which become only manifest in the transition to the original fields. The related $\hat{\Psi}$-function transforms due to its normalization at ∞ as

$$\hat{\Psi} \mapsto \hat{\Psi}\theta \,, \tag{4.27}$$

as well as the auxiliary matrix C_0, which is related to $\hat{\Psi}(\gamma = 0)$:

$$C_0 \mapsto \eta(\theta)^{-1} C_0 \theta \ . \tag{4.28}$$

Thereby, we have made explicit the global and local symmetries of the original fields in the new framework.

4.1.4. First order poles. Let us evolve the previous result for the case of simple poles of $\hat{A}(\gamma)$. We again parametrize $\hat{A}(\gamma)$ by its singularities and residues:

$$\hat{A}(\gamma) = \sum_{j=1}^{N} \frac{\hat{A}_j}{\gamma - \gamma_j} \ . \tag{4.29}$$

Thus

$$\hat{A}_j = \mathcal{V} A_j \mathcal{V}^{-1} \ . \tag{4.30}$$

Their equations of motion read

$$\begin{aligned} \frac{\partial \hat{A}_j}{\partial \xi} &= \frac{2}{\xi - \bar{\xi}} \sum_{k \neq j} \frac{[\hat{A}_k,\ \hat{A}_j]}{(1-\gamma_k)(1-\gamma_j)} + [\mathcal{V}_\xi \mathcal{V}^{-1}, \hat{A}_j] \ , \\ \frac{\partial \hat{A}_j}{\partial \bar{\xi}} &= \frac{2}{\bar{\xi} - \xi} \sum_{k \neq j} \frac{[\hat{A}_k,\ \hat{A}_j]}{(1+\gamma_k)(1+\gamma_j)} + [\mathcal{V}_{\bar{\xi}} \mathcal{V}^{-1}, \hat{A}_j] \ , \end{aligned} \tag{4.31}$$

and are completely generated by the Hamiltonians $\hat{H}^{\xi}$ and $\hat{H}^{\bar{\xi}}$ from (4.17).

Theorem 4.1 now implies

Corollary 4.1. *Let $\hat{A}$ be parametrized as in (4.29). After the Dirac procedure, the following identities hold strongly:*

$$\gamma_j = \frac{1}{\gamma_{j+n}} \ , \tag{4.32}$$

$$\hat{A}_j = \eta(\hat{A}_{j+n}) \ , \tag{4.33}$$

where $N = 2n$. They may be explicitly checked to also commute with the full Hamiltonian constraints $\mathcal{C}^{\xi}, \mathcal{C}^{\bar{\xi}}$. The remaining degrees of freedom are therefore covered by the γ_j and $\hat{A}_j$ for $1 \le j \le n$, which are equipped with the Dirac bracket:

$$\left\{ \hat{A}_i^a, \hat{A}_j^b \right\}_{\mathcal{V}}^{*} = \frac{1}{2} \delta_{ij} f^{abc} \hat{A}_j^c \ . \tag{4.34}$$

The remaining first-class constraint is

$$\frac{1}{2}\left(\hat{A}_\infty + \eta(\hat{A}_\infty)\right) = \sum_{j=1}^{n} \hat{A}_j + \eta\left(\sum_{j=1}^{n} \hat{A}_j\right) = 0 \ . \tag{4.35}$$

This solution of the constraints in the case of first order poles may alternatively be carried out in terms of the monodromies M_j. As was mentioned above, in the presence of only simple poles, the variables A_j are generically (see Note 2.11) completely defined by the monodromies M_j.

Assuming that (4.32) is fulfilled, the coset constraints in the form (4.11) are equivalent to

$$M_{j+n} - C_0^{-1}\eta(M_j)C_0 = 0 \ . \tag{4.36}$$

There are two important points that this form of the constraints exhibits. First, it shows the necessity to choose the matrix C_0 to be constant in the isomonodromic sector. Moreover, it uniquely relates the ordering of the monodromy matrices fixed for calculation of its Poisson brackets in Theorem 2.4 to the ordering defined by (4.32). This results from choosing the corresponding paths pairwise symmetric under $\gamma \mapsto \frac{1}{\gamma}$.

The goal is now to calculate the Dirac bracket between monodromies M_j with respect to (4.33), or, equivalently, with respect to (4.36). One way is clearly to repeat the calculation of Theorem 2.4 using the Dirac bracket (4.19) instead of the Poisson bracket (2.25). However, we can alternatively determine the Dirac bracket from simple symmetry arguments avoiding direct calculation at least for objects that are invariant under G-valued gauge transformations (i.e. traces of arbitrary products of M_j).

The involution η^∞ introduced by (4.13) acts on M_j according to (4.11) as follows:

$$\eta^\infty(M_j) = C_0\eta(M_{j+n})C_0^{-1} \ . \tag{4.37}$$

Therefore, the set of all G-invariant functionals of M_j may be represented as

$$M_{\mathrm{S}} \oplus M_{\mathrm{AS}} \ , \tag{4.38}$$

where the set M_{S} contains functionals which are invariant with respect to η^∞ and M_{AS} contains functionals changing the sign under the action of η^∞. Since η is an automorphism of the structure (2.61), (2.62), the definition of η^∞ in (4.37) implies, taking into account Note 2.16:

$$\{M_{\mathrm{S}}, M_{\mathrm{S}}\} \subseteq M_{\mathrm{S}} \ , \quad \{M_{\mathrm{S}}, M_{\mathrm{AS}}\} \subseteq M_{\mathrm{AS}} \ , \quad \{M_{\mathrm{AS}}, M_{\mathrm{AS}}\} \subseteq M_{\mathrm{S}} \ . \tag{4.39}$$

The constraints (4.36) are equivalent to vanishing of all functionals from M_{AS}; therefore the part of G-invariant variables surviving after the Dirac procedure is contained in M_{S}. The former Poisson bracket on M_{S} coincides with the Dirac bracket.

Note 4.4. The treatment of coset constraints in terms of the monodromies presented above is invariant with respect to change of $\mathcal{V}$ since the monodromies of $\hat{\Psi}$ are. Therefore, this treatment also works in the former Poisson structure (2.25).

4.2. Quantum coset model. The quantization of the coset model goes along the same line as the quantization of the principal model described above. We again restrict to the first order pole sector of the theory, although generalization to the whole isomonodromic sector should be achievable according to Note 3.2.

Having solved the constraints, the remaining degrees of freedom are the singularities γ_j, the residues $\hat{A}_j$ for $j = 1, \ldots, n$ and the logarithmic derivatives of the conformal factor h. They may be represented as in (3.3) and (3.4) again. The quantum representation space is $V^{(n)} := V_1 \otimes \ldots \otimes V_n$.

The Wheeler–DeWitt equations (3.5) take the form:

$$\begin{aligned} \frac{\partial\psi}{\partial\xi} &= \frac{\mathrm{i}\hbar}{\xi - \bar{\xi}} \left\{ \sum_{j,k} \frac{1+\gamma_j\gamma_k}{(1-\gamma_j)(1-\gamma_k)}\, \Omega_{jk} - \sum_{j,k} \frac{\gamma_j+\gamma_k}{(1-\gamma_j)(1-\gamma_k)}\, \tilde{\Omega}_{jk} \right\} \psi \ , \\ \frac{\partial\psi}{\partial\bar{\xi}} &= \frac{\mathrm{i}\hbar}{\bar{\xi} - \xi} \left\{ \sum_{j,k} \frac{1+\gamma_j\gamma_k}{(1+\gamma_j)(1+\gamma_k)}\, \Omega_{jk} + \sum_{j,k} \frac{\gamma_j+\gamma_k}{(1+\gamma_j)(1+\gamma_k)}\, \tilde{\Omega}_{jk} \right\} \psi \ , \end{aligned} \tag{4.40}$$

with

$$\Omega_{jk} = t^a_j \otimes t^a_k \qquad \tilde{\Omega}_{jk} := t^{\eta(a)}_j \otimes t^a_k \ .$$

Additionally, the physical states have to be annihilated by the first-class constraint (4.22):

$$\left(\sum_j t^a_j + \sum_j t^{\eta(a)}_j \right) \psi(\xi, \bar{\xi}) = 0 \ . \tag{4.41}$$

The result of Theorem 3.1 is modified to establish a link to solutions of what we will refer to as the **Coset-KZ system**:

$$\frac{\partial \varphi_{\text{CKZ}}}{\partial \gamma_j} = \mathrm{i}\hbar \left\{ \sum_{k \neq j} \frac{1 + \gamma_k/\gamma_j}{\gamma_j - \gamma_k} \, \Omega_{jk} + \sum_k \frac{\gamma_k + 1/\gamma_j}{\gamma_j \gamma_k - 1} \, \tilde{\Omega}_{jk} \right\} \varphi_{\text{CKZ}} \ . \tag{4.42}$$

The relation between solutions of the Wheeler–DeWitt equations and solutions of the Coset-KZ system is now explicitly given by

Theorem 4.2. *If φ_{CKZ} is a solution of (4.42) obeying the constraint (4.41), and the γ_j depend on $(\xi, \bar{\xi})$ according to (2.8), then*

$$\psi = \prod_{j=1}^{n} \left(\gamma_j^{-1} \frac{\partial \gamma_j}{\partial w_j} \right)^{\mathrm{i}\hbar \Omega_{jj}} \varphi_{\text{CKZ}} \tag{4.43}$$

solves the constraint (Wheeler–DeWitt) Eqs. (4.40).

This may directly be calculated in analogy to (3.9). □

The procedure of identifying observables may be outlined just as in the case of the principal model, where this was described in great detail. Again the monodromies of the quantum linear system are the natural candidates for building observables and contain a complete set for the simple pole sector. In analogy to Theorem 3.2 they should be identified with the monodromies of a certain higher-dimensional Coset-KZ system with an additional insertion playing the role of the classical γ. The actual observables are generated from combinations of matrix entries of these monodromies that commute with the constraint (4.41). From general reasoning according to the classical procedure, relevant objects turn out to be the combinations of G-invariant objects, that are also invariant under the involution η_∞.

4.3. Application to dimensionally reduced Einstein gravity. Let us finally sketch how the previous formalism and results work for the case of axisymmetric stationary $4D$ gravity. In this case, the Lagrangian of general relativity is known to reduce to (2.1) with the field g taking values in $SL(2, \mathbb{R})$ as a symmetric 2×2 matrix; its symmetry corresponds to the coset constraint (4.1).

Most of the physically reasonable solutions of the classical theory – among them in particular the Kerr solution – lie in the isomonodromic sector and are described by first order poles at purely imaginary singularities in the connection. The quantization of this sector may be performed within the framework of this paper. According to (3.4) and Note 3.3 the residues $\hat{A}_j$ are represented as

$$\hat{A}_j \equiv \mathrm{i}\hbar \begin{pmatrix} \frac{1}{2}h_j & e_j \\ f_j & -\frac{1}{2}h_j \end{pmatrix} , \tag{4.44}$$

where h_j, e_j and f_j are the Chevalley generators of $\mathfrak{sl}(2,\mathbb{R})$.

Due to its non-compactness, $\mathfrak{sl}(2,\mathbb{R})$ admits no finite dimensional unitary representations, but several series of infinite dimensional representations. The study of the classical limit singles out the principal series, as was discussed in [49]. The representation space consists of complex functions $f(\zeta)$ on the real line with the ordinary $L^2(\mathbb{R})$ scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}} \overline{f_1(\zeta)} f_2(\zeta) d\zeta , \tag{4.45}$$

and the anti-hermitian operators act as

$$h_j \equiv 2\zeta_j \partial_j + s_j, \qquad e_j \equiv \zeta_j^2 \partial_j + s_j \zeta_j , \qquad f_j \equiv -\partial_j . \tag{4.46}$$

The spin s_j takes values $s_j = 1 + \mathrm{i}q_j$ with a continuous parameter $q_j \in \mathbb{R}$.

The surviving first-class constraint (4.41) now takes a simple form:

Lemma 4.2. *A solution $f(\zeta_1, \ldots, \zeta_n)$ of the constraint (4.41) is of the form*

$$f(\zeta_1, \ldots, \zeta_n) = \prod_j (\zeta_j^2 + 1)^{-\frac{1}{2}s_j} F(\tilde{\zeta}_1, \ldots, \tilde{\zeta}_n) , \tag{4.47}$$

with $\tilde{\zeta}_j := \frac{\zeta_j + \mathrm{i}}{\zeta_j - \mathrm{i}}$ and

$$\left(\sum_j \frac{\partial}{\partial \tilde{\zeta}_j} \right) F = 0 . \tag{4.48}$$

This follows by direct calculation. □

The prefactor in (4.47) is exactly sufficient for convergence of the integral, such that for finiteness of the norm, it is sufficient to demand boundedness of F which is a function on the product of $(n-1)$ circles S^1. In contrast to the analogous $\mathfrak{sl}(2,\mathbb{R})$ representation of the principal model, where solutions of finite norm are absent due to several redundant integration variables, a convergency factor here comes out for free. This interestingly resembles the fact that the general reason for dividing out the maximal compact subgroup in the physical coset models corresponds to avoiding unboundedness of the energy in the theory.

It remains to solve the Coset-KZ system in this representation. Although the general solution for $\mathfrak{sl}(2,\mathbb{R})$ is not known, one might be able to obtain explicit results for a small number of insertions. The Kerr solution for instance, which is of major interest, requires only two classical insertions $\gamma_1, \gamma_2 \in \mathrm{i}\mathbb{R}$. In this case, we may exploit Theorem 4.2 and Lemma 4.2 to explicitly reduce the WDW equation to a second order differential equation in two variables. Let V_1 and V_2 be two representations from the principal series of $\mathfrak{sl}(2,\mathbb{R})$ fixed by s_1 and s_2 and parametrize the quantum state $\psi(\xi, \bar{\xi}) \in V_1 \otimes V_2$ according to:

$$\psi(\xi, \bar{\xi}, \zeta_1, \zeta_2) = (\zeta_1^2 + 1)^{-\frac{1}{2}s_1} (\zeta_2^2 + 1)^{-\frac{1}{2}s_2} \left(\frac{\gamma_1}{\gamma_1^2 - 1} \right)^{\Delta_1} \left(\frac{\gamma_2}{\gamma_2^2 - 1} \right)^{\Delta_2} F(\gamma, \zeta) , \tag{4.49}$$

with

$$\Delta_1 \equiv \frac{\mathrm{i}}{2}\hbar s_1(s_1-2)\ , \quad \Delta_2 \equiv \frac{\mathrm{i}}{2}\hbar s_2(s_2-2)\ ,$$
$$\gamma \equiv \frac{\gamma_1+1}{\gamma_1-1}\frac{\gamma_2-1}{\gamma_2+1} \in S^1\ , \qquad \zeta \equiv \frac{\zeta_1+\mathrm{i}}{\zeta_1-\mathrm{i}}\frac{\zeta_2-\mathrm{i}}{\zeta_2+\mathrm{i}} \in S^1\ .$$

After some calculation the WDW equation then becomes

$$\partial_\gamma F(\gamma,\zeta) \ = \ \mathrm{i}\hbar D_{s_1,s_2}(\gamma)\ \ F(\gamma,\zeta)\ , \tag{4.50}$$

with

$$\begin{aligned} D_{s_1,s_2}(\gamma) = \Bigg\{ &\frac{1}{\gamma-1}\left[2\zeta(\zeta-1)^2\partial_\zeta^2 + \left(2(\zeta-1)^2 + (s_1+s_2)(\zeta^2-1)\right)\partial_\zeta + \frac{\zeta^2+1}{2\zeta}s_1s_2\right] \\ &- \frac{1}{\gamma+1}\left[2\zeta(\zeta+1)^2\partial_\zeta^2 + \left(2(\zeta+1)^2 + (s_1+s_2)(\zeta^2-1)\right)\partial_\zeta + \frac{\zeta^2+1}{2\zeta}s_1s_2\right] \\ &+\frac{4}{\gamma}\left(\zeta^2\partial_\zeta^2 + \zeta\partial_\zeta\right) \Bigg\}\ . \end{aligned} \tag{4.51}$$

This form e.g. suggests expansion into a Laurent series in ζ on S^1 leading to recurrent differential equations in γ for the coefficients. Further study of this equation should be a subject of future work.

Note 4.5. Equation (4.50) reduces to a Painlevé equation when the principal series representation of $\mathfrak{sl}(2,\mathbb{R})$ is formally replaced by the fundamental representation of $\mathfrak{g}=\mathfrak{su}(2)$. In the study of four-point correlation-functions in Liouville theory a similar generalization of the hypergeometric differential equation appeared [62].

5. Outlook

We have completed the classical two-time Hamiltonian formulation of the coset model for the isomonodromic sector and sketched a continuous extension in Appendix A. For the quantum theory it remains the problem of consistent quantization of the total phase space including a proper understanding of the structures (A.8). The most important physical problem in the investigated model is the description of states corresponding to quantum black holes. One may certainly hope to extract first insights from a closer study of the exact isomonodromic quantum states of the coset model identified in the last chapter, in particular from the study of Eq. (4.50).

An open problem is the link of the employed two-time Hamiltonian formalism with the conventional one. To rigorously relate the different Poisson structures, the canonical approach should be compared to our model after a Wick rotation into the Lorentzian case. This corresponds to a dimensional reduction of spatial dimensions only, such that the model would describe colliding plane or cylindrical waves rather than stationary black holes. It is further reasonable to suspect that proper comparison of the different Poisson structures can only be made on the set of observables, see also Note 2.5. Recent progress in the canonical approach has been stated in [51], where in particular the canonical algebraic structures of the observables have been revealed. However, so far the canonical and the isomonodromic approaches appear to favor different characteristic observables, which still remain to be related.

As another possibility to compare our treatment with canonical approaches, the relation to further restricted and already studied models should be investigated. Of major interest in this context would be for instance the relation to the Einstein-Rosen solutions, investigated and quantized in [52, 5], where imposing of additional hypersurface orthogonality of the Killing vector fields reduces the phase space to "one polarization," yet maintaining an infinite number of degrees of freedom.

An additional interesting field of future research descends from the link to broadly studied two-dimensional dilaton gravity (see e.g. [14, 32, 9, 27]), further allowing to extract information about the black hole thermodynamics. Further relevance of the investigated model appeared in certain sectors of string theory [30, 53].

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A. Extension Beyond the Isomonodromic Sector

The treatment of the isomonodromic sector presented in this paper allows a rather natural extension to the full phase space. This general scheme recalls a continuous version of the simple pole sector treated in Subsect.2, which in turn may be understood as a discrete embedding into the former. We will again first describe the scheme for the principal model and then discuss the modifications required for the coset model, see also [56].

A.1. Principal model. We start from a simply-connected domain Ω in the $\xi, \bar{\xi}$-plane, symmetric with respect to conjugation $\xi \mapsto \bar{\xi}$, where the classical solution $g(\xi, \bar{\xi})$ is assumed to be non-singular. This regularity is reflected by corresponding properties of the related Ψ-function in the w-plane. It is holomorphic and invertible in a (ring-like) domain D of the Riemann surface $\mathcal{L}$ of the function $\sqrt{(w-\xi)(w-\bar{\xi})}$ bounded by contours l and l^σ, where σ is the involution $\gamma \mapsto 1/\gamma$ interchanging the w-sheets of $\mathcal{L}$.

To simplify the following formulas we further assume the spectral parameter current $A(\gamma)$ to be holomorphic on the whole second sheet of $\mathcal{L}$, such that it may be represented inside of l (we denote this simply-connected domain by D_0) by a Cauchy integral over l:

$$A(\mu) = \oint_l \frac{\mathcal{A}(w, \xi, \bar{\xi})dw}{\gamma(w) - \mu} \ , \tag{A.1}$$

which is the continuous analog of the simple pole ansatz (2.32) in the isomonodromic sector; $\mathcal{A}(w), w \in l$ is a density corresponding to the residues A_j from (2.32).

From (A.1), $\mathcal{A}(w)$ is not uniquely defined by the values of $A(\gamma)$, $\gamma \in D_0$, in particular, it may not coincide with the boundary values of $A(\gamma)$ on l. To fix $\mathcal{A}(w)$, we postulate the following deformation equations which are a continuous version of the discrete deformation Eqs. (2.33):

$$\frac{\partial \mathcal{A}(w)}{\partial \xi} = \frac{2}{\xi - \bar{\xi}} \oint_l \frac{[\mathcal{A}(v), \ \mathcal{A}(w)]}{(1-\gamma(v))(1-\gamma(w))} dv \ , \tag{A.2}$$
$$\frac{\partial \mathcal{A}(w)}{\partial \bar{\xi}} = \frac{2}{\bar{\xi} - \xi} \oint_l \frac{[\mathcal{A}(v), \ \mathcal{A}(w)]}{(1+\gamma(v))(1+\gamma(w))} dv \ , \qquad w \in l \ .$$

It is easy to check that (A.2) together with (A.1) imply the deformation Eqs. (2.23) for $A(\gamma)$.

The Poisson structure on $\mathcal{A}(w)$ is also a direct continuous analog of (2.34):

$$\{\mathcal{A}^a(w), \mathcal{A}^b(v)\} = -f^{abc}\mathcal{A}^c(w)\delta(w-v)\ , \qquad w, v \in l, \tag{A.3}$$

where $\delta(w)$ is a one-dimensional δ-function living on the contour l (and should, strictly speaking, be defined as $\frac{ds}{dw}\delta(s)$ with an arbitrary affine parameter s along l). This structure in turn induces the proper holomorphic bracket (2.25) for $A(\gamma)$:

$$\begin{aligned}\{A^a(\gamma(w)), A^b(\gamma(v))\} &= -f^{abc}\oint_l \frac{\mathcal{A}^c(w')dw'}{(\gamma(w')-\gamma(w))(\gamma(w')-\gamma(v))} \\ &= -f^{abc}\frac{A^c\gamma((w)) - A^c(\gamma(v))}{\gamma(w)-\gamma(v)}\ .\end{aligned}$$

The nice feature of $\mathcal{A}(w)$ in contrast to $A(\gamma)$ is that $\mathcal{A}(w)$ (as its discrete analog A_j) is explicitly $(\xi, \bar{\xi})$ independent, i.e. the whole dependence of $\mathcal{A}(w)$ on ξ and $\bar{\xi}$ is generated by the Hamiltonians (2.27) (note that the points $\gamma = \pm 1$ lie inside of D_0):

$$H^{\xi} = \frac{1}{\xi - \bar{\xi}}\mathrm{tr}\left[\oint_l \frac{\mathcal{A}(w)dw}{1-\gamma(w)}\right]^2\ , \qquad H^{\bar{\xi}} = \frac{1}{\xi - \bar{\xi}}\mathrm{tr}\left[\oint_l \frac{\mathcal{A}(w)dw}{1+\gamma(w)}\right]^2\ . \tag{A.4}$$

We may now also identify a continuous family of observables, generalizing the construction of Sect.2. Define $A(\gamma)$ inside and outside of D_0 by the Cauchy formula (A.1) and construct the related functions $\Psi_{\mathrm{in}}(\gamma \in D_0)$ and $\Psi_{\mathrm{out}}(\gamma \notin D_0)$ according to $\Psi_\gamma\Psi^{-1} = A(\gamma)$. Then the continuous monodromy matrix

$$M(w) \equiv \Psi_{\mathrm{out}}(w)\Psi_{\mathrm{in}}^{-1}(w)\ , \qquad w \in l \tag{A.5}$$

is $(\xi, \bar{\xi})$-independent, since both Ψ_{in} and Ψ_{out} satisfy the linear system (2.7). Calculations similar to those in Appendix B yield the following Poisson brackets for $M(w)$:

$$\begin{aligned}\{M^0(v), M^{\bar{0}}(w)\} = \mathrm{i}\pi\Big(&-M^0(v)\,\Omega\, M^{\bar{0}}(w) + M^{\bar{0}}(w)\,\Omega\, M^0(v) \\ &+ \Omega\, M^0(v)M^{\bar{0}}(w) - M^0(v)M^{\bar{0}}(w)\,\Omega\Big)\ , \\ \text{for}\quad v \le w\ ,&\quad v, w \in l\ ,\end{aligned} \tag{A.6}$$

where the points of contour l are ordered with respect to a fixed point w_0, playing the role of the eyelash in the discrete case.

The brackets (A.6), are again valid up to the first-class constraint generated by

$$A_\infty = \oint_l \mathcal{A}(w)dw\ , \tag{A.7}$$

and therefore satisfy Jacobi identities only being restricted to the gauge-invariant objects.

Again there appear two fundamental ways of quantization. In terms of $\mathcal{A}$, (A.3) would be replaced by a possibly centrally extended affine algebra. Alternatively, the Poisson algebra of observables (A.6) may be quantized directly after regularization analogously to (2.69):

$$\begin{aligned}\{M^0(v), M^{\bar{0}}(w)\} = &-M^0(v)\, r_+\, M^{\bar{0}}(w) + M^{\bar{0}}(w)\, r_-\, M^0(v) \\ &+ r_-\, M^0(v)M^{\bar{0}}(w) - M^0(v)M^{\bar{0}}(w)\, r_+ \quad v \le w, \quad v, w \in l\ ,\end{aligned}$$

leading to:

$$R_- M^0(w) R_-^{-1} M^{\bar{0}}(v) = M^{\bar{0}}(v) R_+ M^0(w) R_+^{-1}, \qquad v \leq w \; . \tag{A.8}$$

Embedding of the isomonodromic sector into the presented extension looks especially simple if all the singularities $\gamma_1, \ldots, \gamma_N$ are assumed to belong to the contour l. The density $\mathcal{A}(w)$ is then parametrized as

$$\mathcal{A}(w) = -\sum_{j=N}^{n} A_j \delta(w - w_j) \; , \tag{A.9}$$

where the residues A_j are the same as in (4.29). The Poisson structure (A.3) is the directly inherited from (2.34) and (A.9):

$$\begin{aligned} \{\mathcal{A}^a(w),\ \mathcal{A}^b(v)\} &= \sum_{j=1}^{N} f^{abc} A_j \delta(w - w_j)\delta(v - w_j) \\ &= -f^{abc}\mathcal{A}^c(v)\delta(v - w) \; . \end{aligned}$$

The monodromy $M(w)$ here is a step function on l with jumps at $w = w_j$. Fixing the eyelash between γ_N and γ_1 it is

$$M(w) = M_1 \ldots M_j \; , \qquad \text{for} \quad w \in]\gamma_j, \gamma_{j+1}[\; .$$

Note A.1. Isomonodromic solutions with higher order poles are embedded into the general scheme by inserting higher order derivatives of δ-functions into (A.9). The definition (A.1) already shows that the proper object in this case is the connection $A^w = \frac{\partial \gamma}{\partial w} A$, in accordance with the results from Subsect.2.

Note A.2. The representation (A.1) gains a well known meaning when the model is truncated to a real scalar field g, where $\mathcal{A}(w)$ becomes independent of $\xi, \bar{\xi}$ and the equation of motion (2.5) reduces to the Euler-Darboux equation

$$\partial_\xi \partial_{\bar{\xi}} \phi - \frac{\partial_\xi \phi - \partial_{\bar{\xi}} \phi}{2(\xi - \bar{\xi})} = 0 \; , \tag{A.10}$$

for $\phi = \log g$. Solutions of this equation may be represented as [15]

$$\phi = \oint_l \frac{f(w) dw}{\sqrt{(w - \xi)(w - \bar{\xi})}} \; , \tag{A.11}$$

with $2\pi \mathrm{i} f(w) \equiv \phi(\xi = \bar{\xi} = w)$ defined on the axis $\xi = \bar{\xi}$ and continued analytically. After differentiating in ξ and integrating by parts in w, this representation takes the form

$$\partial_\xi \phi = \frac{2}{\xi - \bar{\xi}} \oint_l \frac{f'(w) dw}{\sqrt{(w - \xi)(w - \bar{\xi})}} \; ,$$

and thus equals (2.17) with $A(\pm 1)$ defined by (A.1) after identification of $f'(w)$ and $\mathcal{A}(w)$.

A.2. Coset model. In analogy to the discrete case, the coset model is most conveniently described in terms of modified variables

$$\hat{\mathcal{A}} = \eta(\mathcal{V})\mathcal{A}\eta(\mathcal{V}^{-1}) \ .$$

Due to the symmetry (4.12) between the values of $\hat{A}(\gamma)$ on different sheets of $\mathcal{L}$, we can no longer assume $\hat{A}(\gamma)$ to be holomorphic in D_0, but have to replace the l by $l \cup l^\sigma$ enclosing D in the formulas of the last section. The coset constraints in terms of $\hat{\mathcal{A}}(w)$ take the form

$$\hat{\mathcal{A}}(w) = \eta\big(\hat{\mathcal{A}}(w^\sigma)\big), \qquad w \in l \ , \tag{A.12}$$

and allow rather simple solution via a Dirac procedure, such that the phase space is reduced to the values of $\hat{\mathcal{A}}(w)$ on l only, equipped with the Dirac bracket

$$\{\hat{\mathcal{A}}^a(w), \hat{\mathcal{A}}^b(v)\}^*_{\mathcal{V}} = -\frac{1}{2} f^{abc} \hat{\mathcal{A}}^c(w)\delta(w-v) \ , \qquad v, w \in l \ . \tag{A.13}$$

Via the Cauchy representation (A.1) on the contour $l \cup l^\sigma$, this bracket further gives the previously derived Dirac bracket (4.19) on $\hat{A}(\gamma)$. It remains the $\mathfrak{h}$-valued first class constraint

$$\oint_l \Big(\hat{\mathcal{A}}(w) + \eta(\hat{\mathcal{A}}(w))\Big) dw = 0 \ ,$$

generalizing (4.22). The Hamiltonians finally also take the form (A.4) with l being replaced by $l \cup l^\sigma$. In terms of the observables $M(w)$, restriction to the coset leads to

$$M(w^\sigma) = C_0^{-1} \eta\Big(M(w)\Big) C_0 \ , \qquad w \in l \ ,$$

with some constant matrix C_0 playing the same role as in (4.36).

B. Poisson Structure of Monodromy Matrices

This appendix is devoted to the proof of Theorem 2.4, which was obtained in collaboration with H. Nicolai.[3] For simplicity of the presentation, we give the calculation for the case, where the Casimir element Ω differs from the permutation operator Π by some scalar multiple of the identity only, which is the case for $\mathfrak{g} = \mathfrak{sl}(N, \mathbb{R})$ for example. The procedure may easily be extended (concerning the notation mainly) to the general case.

Here, the Poisson-structure of the connection is given by

$$\{A(\gamma) \overset{\otimes}{,} A(\mu)\} = \frac{1}{\gamma - \mu} \left[\Pi, A(\gamma) \otimes I + I \otimes A(\mu)\right] ,$$

and the statement to be proven reads:

$$\{M_i \overset{\otimes}{,} M_i\} = \mathrm{i}\pi \left[\, \Pi, M_i M_i \otimes I \,\right] , \tag{B.1}$$

$$\{M_i \overset{\otimes}{,} M_j\} = \mathrm{i}\pi\Pi \Big(M_j M_i \otimes I + I \otimes M_i M_j - M_i \otimes M_j - M_j \otimes M_i \Big) \ , \quad \text{for } \ i < j \ . \tag{B.2}$$

We first calculate the Poisson structure of matrix entries of the function Ψ at different points s_1 and s_2. These points are defined on the Riemann surface given by Ψ by paths,

[3] After completion we learned about related results in [4, 36].

connecting them to a common base-point s_0, at which Ψ is taken to be normalized according to (2.59). The limit $s_0 \to \infty$ will be treated later on.

For the calculation, we make use of the standard formula

$$\{\Psi(s_1) \stackrel{\otimes}{,} \Psi(s_2)\} = \Big(\Psi(s_1) \otimes \Psi(s_2)\Big) \int_{s_0}^{s_1} \int_{s_0}^{s_2} d\mu_1 d\mu_2 \times \left[\Big(\Psi^{-1}(\mu_1) \otimes \Psi^{-1}(\mu_2)\Big) \Big\{A(\mu_1) \stackrel{\otimes}{,} A(\mu_2)\Big\} \Big(\Psi(\mu_1) \otimes \Psi(\mu_2)\Big)\right] ,$$

where the integrand may be rewritten as

$$\frac{\Pi}{\mu_2 - \mu_1} \Big(\partial_{\mu_1} + \partial_{\mu_2}\Big)\Big(\Psi^{-1}(\mu_2)\Psi(\mu_1) \otimes \Psi^{-1}(\mu_1)\Psi(\mu_2)\Big) .$$

This expression is completely regular, even for $\mu_1 = \mu_2$. However, if the appearance of the derivation operators is exploited by partial integration, the integrals will split up into parts that exhibit singularities in coinciding points $\mu_1 = \mu_2$. Thus, we restrict to distinguished endpoints s_1 and s_2, choosing the defining paths $[s_0 \to s_1]$ and $[s_0 \to s_2]$ nonintersecting in the punctured plane from the very beginning. Singularities remain in the common endpoints of the paths at s_0. As a regularization, one of these coinciding endpoints is shifted by a small (complex) amount ϵ that is put to zero afterwards. Then, partial integration can be carried out properly, leaving only boundary terms, that lead to surviving simple line integrals, whereas the remaining double integrals cancel exactly. The arising singularities in $\epsilon = 0$ regularize each other such that the sum is independent of the way, ϵ tends to zero. In a comprehensive form, the result may be stated as

Theorem B.1. *Let s_1 and s_2 be different points on the punctured plane, defined as points on the covering by nonintersecting paths $[s_0 \to s_1]$ and $[s_0 \to s_2]$ with common basepoint s_0 at which Ψ is normalized. Then, the Poisson bracket between matrix entries of $\Psi(s_1)$ and $\Psi(s_2)$ is given by*

$$\begin{aligned}\{\Psi(s_1) \stackrel{\otimes}{,} \Psi(s_2)\} = \Big(\Psi(s_1) \otimes \Psi(s_2)\Big) \times \\ \Bigg\{ \int_{s_0}^{s_2} d\mu \frac{\Pi}{\mu - s_1} \Big(\Psi^{-1}(\mu)\Psi(s_1) \otimes \Psi^{-1}(s_1)\Psi(\mu)\Big) \\ - \int_{s_0}^{s_1} d\mu \frac{\Pi}{\mu - s_2} \Big(\Psi^{-1}(s_2)\Psi(\mu) \otimes \Psi^{-1}(\mu)\Psi(s_2)\Big) \\ + \int_{s_0}^{s_2} d\mu \frac{1}{\mu - s_0} \Big[\Pi\, ,\, \Psi(\mu) \otimes \Psi^{-1}(\mu)\Big] \\ + \lim_{\epsilon \to 0} \left(\int_{s_2}^{s_0-\epsilon} + \int_{s_0+\epsilon}^{s_1} \right) d\mu \frac{\Pi}{\mu - s_0} \Big(\Psi(\mu) \otimes \Psi^{-1}(\mu)\Big) \Bigg\} .\end{aligned} \tag{B.3}$$

This expression is regular and independent of the limit procedure. □

Note B.1. The result of the regularization is the complete fixing of the relative directions of the paths $[s_0 \to s_1]$ and $[s_0 \to s_2]$ approaching the basepoint s_0, that is determined by the form in which ϵ arises in the last term in (B.3). In other words, the path $[s_1 \to s_0 \to s_2]$ must pass through the basepoint s_0 straightforwardly, as is indicated in Fig.1.

The result of Theorem B.1 may be further simplified in the limit $s_0 \to \infty$, where the third term of (B.3) vanishes:

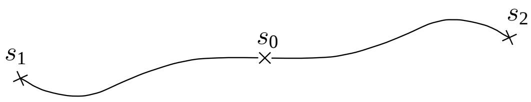

Fig. 1. Choice of paths

Lemma B.1. *For a fixed point s on the punctured plane and $\Psi(\gamma)$ holomorphic at $\gamma = \infty$, it is*

$$\lim_{s_0 \to \infty} \left(\int_{s_0}^{s} d\mu \, \frac{1}{\mu - s_0} \, \left[\Pi, \Psi(\mu) \otimes \Psi^{-1}(\mu) \right] \right) = 0 \, . \tag{B.4}$$

The proof is obtained by estimating the integrand as a holomorphic function of γ and s_0. □

To proceed in calculating the Poisson bracket between monodromy matrices, we choose points s_1, s_2, s_3 and s_4, pairwise coinciding on the punctured plane as $s_1 \sim s_2$ and $s_3 \sim s_4$, but distinguished on the covering and defining the monodromy matrices M_i and M_j:

$$\Psi(s_2) = \Psi(s_1) M_i \, , \qquad \Psi(s_4) = \Psi(s_3) M_j \, . \tag{B.5}$$

Then, (B.3) leads to:

$$\begin{aligned} \{M_i \stackrel{\otimes}{,} M_j\} = \; & (M_i \otimes M_j) \left[\int_{s_4 \to s_0 \to s_2} d\mu \, \frac{\Pi}{\mu - s_0} \, \Big(\Psi(\mu) \otimes \Psi^{-1}(\mu) \Big) \right] \\ + \; & \left[\int_{s_3 \to s_0 \to s_1} d\mu \, \frac{\Pi}{\mu - s_0} \, \Big(\Psi(\mu) \otimes \Psi^{-1}(\mu) \Big) \right] (M_i \otimes M_j) \\ - \; & (I \otimes M_j) \left[\int_{s_4 \to s_0 \to s_1} d\mu \, \frac{\Pi}{\mu - s_0} \, \Big(\Psi(\mu) \otimes \Psi^{-1}(\mu) \Big) \right] (M_i \otimes I) \\ - \; & (M_i \otimes I) \left[\int_{s_3 \to s_0 \to s_2} d\mu \, \frac{\Pi}{\mu - s_0} \, \Big(\Psi(\mu) \otimes \Psi^{-1}(\mu) \Big) \right] (I \otimes M_j) \, , \end{aligned} \tag{B.6}$$

which is understood in the limit $\epsilon \to 0$ and $s_0 \to \infty$ and for paths $[s_j \to s_0 \to s_i]$, $i = 1, 2; j = 3, 4$, chosen fixed and in accordance with the conditions of Theorem B.1 and Note B.1.

Proof of (B.1). Consider first the case $i = j$. Then a proper choice of paths is illustrated in Fig.2.

The expression (B.6) allows to put $s_1 = s_3$ and $s_2 = s_4$ and to split the integration paths into paths encircling s_0 and γ_i, respectively:

$$\begin{aligned} \{M_i \stackrel{\otimes}{,} M_i\} = \; & (M_i \otimes M_i) X - X (M_i \otimes M_i) - (M_i \otimes I) X (I \otimes M_i) \\ & + (I \otimes M_i) X (M_i \otimes I) + (I \otimes M_i) Y (M_i \otimes I) - (M_i \otimes I) Y (I \otimes M_i) \, , \end{aligned}$$

with

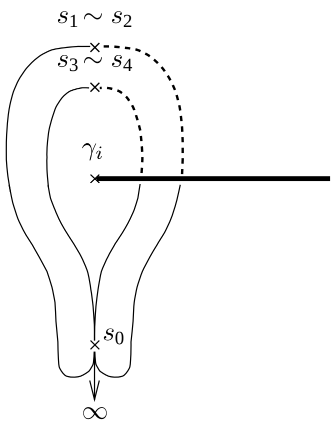

Fig. 2. Choice of paths for $\{M_i \overset{\otimes}{,} M_i\}$

$$X = \frac{1}{2}\oint_{s_0} d\mu \, \frac{\Pi}{\mu - s_0} \, \Big(\Psi(\mu) \otimes \Psi^{-1}(\mu)\Big) \, ,$$
$$Y = \int_{s_1}^{s_2} d\mu \, \frac{\Pi}{\mu - s_0} \, \Big(\Psi(\mu) \otimes \Psi^{-1}(\mu)\Big) \, .$$

The path of the integral Y neither passes through s_0 nor intersects the path $[s_0 \to \infty]$; such that this integral vanishes in the limit $s_0 \to \infty$. This choice of path uniquely determines the orientation of the remaining paths in X, which encircle s_0. The corresponding integrals can be easily evaluated due to Cauchy's theorem and single-valuedness of the integrands. This proves formula (B.1). □

Proof of (B.2). This case is treated in complete analogy. A suitable form of the paths is shown in Fig.3, which in particular illustrates the asymmetric position of the paths defining respectively M_i and M_j, with respect to the marked path $[s_0 \to \infty]$.

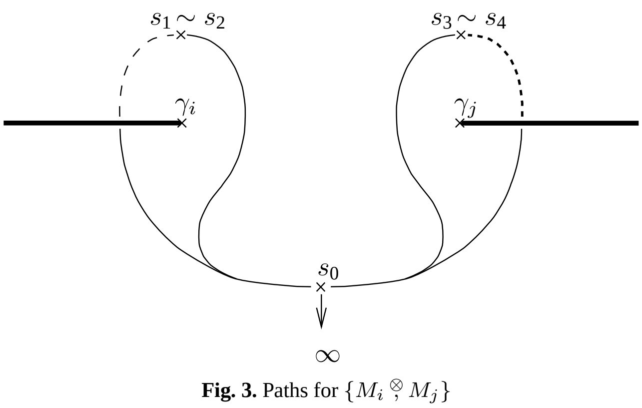

Fig. 3. Paths for $\{M_i \overset{\otimes}{,} M_j\}$

Similar reasoning as above yields

$$\{M_i \overset{\otimes}{,} M_j\} = -(M_i \otimes M_j)X - X(M_i \otimes M_j) \\ + (M_i \otimes I)X(I \otimes M_j) + (I \otimes M_j)X(M_i \otimes I) \,, \qquad \text{(B.7)}$$

where again several integrals have already vanished in the limit $s_0 \to \infty$. Evaluating the remaining terms proves formula (B.2). □

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