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Instability and Stability of Rolls in the Swift–Hohenberg Equation*

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Dedicated to Professor K. Kirchgässner on the occasion of his sixty-fifth birthday

Abstract: We develop a method for the stability analysis of bifurcating spatially periodic patterns under general nonperiodic perturbations. In particular, it enables us to detect sideband instabilities. We treat in all detail the stability question of roll solutions in the two–dimensional Swift–Hohenberg equation and derive a condition on the amplitude and the wave number of the rolls which is necessary *and* sufficent for stability. Moreover, we characterize the set of those wave vectors $\sigma \in \mathbb{R}^2$ which give rise to unstable perturbations.

1. Introduction

The bifurcation of periodic patterns for partial differential equations on unbounded domains attracted a lot of attention within the last decade, especially concerning stability aspects. Often stability of bifurcating patterns is studied with respect to perturbations of related symmetry classes. However, for practical purposes it is also important to have stability with respect to general nonperiodic perturbations. To tackle this problem the theory of sideband instabilities was devised starting with the pioneering work of Eckhaus [Eck65]. Yet this theory remained purely formal, due to its usage of multiple scaling arguments.

Only very few rigorous results where obtained at that time, as for instance in [KiS69], where instability of bifurcating roll–type solutions in the Navier–Stokes equations was proven whenever the period is not the one which is associated to the critical Reynolds number. However, the Eckhaus criterion for instability was mathematically justified only twenty–five years later: first, for scalar model problems in [CoE90, Mie95] and then for the Navier–Stokes equation in [KvW97, Mi97b].

A more general method, called the *principle of reduced instability*, was developed in [Mie95, BrM96] which then was applied to the Benjamin–Feir instability of sur-

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face waves on a fluid layer of finite depth [BrM95] and to the sideband instabilities of convection rolls in the Rayleigh–Bénard problem [Mi97b]. This principle of reduced instability employs local arguments in the set of wave vectors, and is thus ideally suited to detect sideband instabilities. However, it only provides sufficient conditions for linear instability and is not able to give stability results. It is the purpose of this work to show how necessary conditions for stability can be provided.

Problems with more than one unbounded direction display a more complex behavior and are less well understood. First rigorous (in)stability results for the two–dimensional Swift–Hohenberg equation (SHE) were obtained in [Mie95] and [Kuw96]. The former work gives sufficient conditions for instability while the latter also establishes sufficient conditions for stability. However, there remained a region in parameter space where no result could be obtained, see Remark 2 after Theorem 3.2. Here we generalize the principle of reduced instability such that it provides stability results also. In particular, we are able to derive condition (1.2) below which is necessary *and* sufficient for stability. Moreover, for the case of instability we can characterize the set of those wave vectors $\sigma \in \mathbb{R}^2$ which give rise to unstable modes.

We will explain the main philosophy of the method in Sect. 2 and work out a first example in Sects. 3 and 4, namely the sideband instabilities for the roll patterns in the SHE:

$$\partial_t u \; = \; -(1+\Delta)^2 u + \varepsilon u - u^3, \quad t \geq 0,\; x \in \mathbb{R}^2. \tag{1.1}$$

There are roll solutions $u(t,x) = U_{\varepsilon,\kappa}(kx_1) = \sqrt{4(\varepsilon - \kappa^2)/3}\cos(kx_1) + \mathcal{O}(|\varepsilon - \kappa^2|^{3/2})$ which are independent of (t, x_2) and periodic in x_1 with period $2\pi/k$. For notational convenience we throughout use the parameter $\kappa = k^2 - 1$. These solutions exist for all $\varepsilon \in (\kappa^2, \varepsilon_0]$ for some small positive ε_0.

However, some of these roll patterns are unstable: There are two curves $\kappa = K_Z(\varepsilon) = \mathcal{O}(\varepsilon^2)$ and $\varepsilon = E_E(\kappa) = 3\kappa^2 + \mathcal{O}(|\kappa|^3)$ such that the rolls with $\varepsilon < E_E(\kappa)$ are *Eckhaus unstable* and that the rolls with $\kappa < K_Z(\varepsilon)$ are *zigzag unstable*. These bounds were known on the formal level for more than 25 years, see [Eck65] and [Bus71] for the first studies. Exploiting the ideas introduced in [KiS69] it is possible to prove instability of the rolls with $\varepsilon \in (\kappa^2, (1+c_0)\kappa^2)$ for some small c_0, yet the Eckhaus bound is $c_0 = 2$. A more general theory was developed in [CoE90, KvW97] for the Eckhaus criterion and [Kuw96, Mie95, Mi97b] for both cases. The novel result of the present work is that we are able to show that these conditions are not only sufficient for instability but also necessary:

$$U_{\varepsilon,\kappa} \text{ is linearly stable if and only if } \quad \kappa \geq K_Z(\varepsilon) \text{ and } \varepsilon \geq E_E(\kappa). \tag{1.2}$$

This result is stated in Sect. 3 and proved in Sect. 4. Since $K_Z(\varepsilon) = -\varepsilon^2/512 + \mathcal{O}(\varepsilon^3)$ we conclude that there are stable rolls with $\kappa < 0$. As far as we know this result is new.

For comparison we derive, in Sect. 3, the sideband instabilities of the roll solution $A_{\varepsilon,\kappa}(x_1) = \sqrt{\varepsilon - \kappa^2}\, e^{ikx_1}$, with $k = \sqrt{1+\kappa}$, of the complex SHE,

$$\partial_t A \; = \; -(1+\Delta)^2 A + \varepsilon A - |A|^2 A, \quad t \geq 0,\; x \in \mathbb{R}^2,$$

where $A(t,x) \in \mathbb{C}$. This stability problem is easier as it reduces to a purely algebraic one. Lengthy algebraic manipulations yield

$$A_{\varepsilon,\kappa} \text{ is linearly stable if and only if } \quad \kappa \geq 0 \text{ and } \varepsilon \geq \kappa^2 \frac{6+7\kappa}{2+3\kappa}.$$

To establish result (1.2) we generalize the principle of reduced instability in Sect. 2. There we use a general setting for arbitrary elliptic operators, however in this introduction we give the ideas for the SHE only. Our notion of stability for $U_{\varepsilon,k}$ is always spectral stability, that is, we have to study the spectral problem

$$\lambda v = B_{\varepsilon,\kappa} v \quad \text{where } B_{\varepsilon,\kappa} v \stackrel{\text{def}}{=} -(1+\Delta)^2 v + \varepsilon v - 3U_{\varepsilon,\kappa}^2(kx_1)v.$$

The main difference from classical approaches is that we allow v to lie in $W^{n,\infty}(\mathbb{R}^2)$ rather than restricting it to the space $H^4(\mathcal{T}_{2\pi})$, containing only the patterns with the same periodicity as $U_{\varepsilon,\kappa}$. (We continue to use $\mathcal{T}_\alpha = \mathbb{R}/_{\alpha\mathbb{Z}}$ for the one–dimensional torus of length α.) Following [MiS95] we use the more general space $L^2_{lu}(\mathbb{R}^2)$, the Banach space of uniformly local L^2 functions, see Sect. 2 for the definition. The methods developed there imply that SHE defines a global semiflow in $L^2_{lu}(\mathbb{R}^2)$. Using the results in [Sca94] we immediately conclude nonlinear instability of $U_{\varepsilon,\kappa}$ if it is spectrally unstable. In the case of spectral stability the nonlinear stability is less understood. For the one–dimensional case (no x_2–dependence) local nonlinear stability in $L^2(\mathbb{R})$ is proved in [Sch96] (for $\kappa = 0$), but the case $L^2_{lu}(\mathbb{R})$ and the two–dimensional problem are still open.

We may treat $B_{\varepsilon,\kappa}$ as operator on $L^2_{lu}(\mathbb{R}^2)$ or $L^2(\mathbb{R}^2)$ with domain of definition $H^4_{lu}(\mathbb{R}^2)$ or $H^4(\mathbb{R}^2)$, respectively. The first variant allows us especially to study so–called Bloch waves v given in the form $v(x) = e^{i(k\sigma_1 x_1 + \sigma_2 x_2)}V(\xi)$ with $\xi = kx_1$, $V \in X = H^4(\mathcal{T}_\pi)$, and wave vector $\sigma \in \mathbb{R}^2$. We use the fact that $3U^2_{\varepsilon,\kappa}(kx_1)$, the only x_1–dependent coefficient of $B_{\varepsilon,\kappa}$, has period π/k since $U_{\varepsilon,\kappa}(\xi+\pi) = -U_{\varepsilon,\kappa}(\xi)$. The main point is that the whole stability question in $L^2_{lu}(\mathbb{R}^2)$ or $L^2(\mathbb{R}^2)$ can be reduced to the study of Bloch waves. Such results are well known for Schrödinger operators with periodic potentials, cf. [ReS78], and were generalized to reaction diffusion problems in [Sca94] and to the Navier–Stokes equations in [Sca95].

Because of $V \in H^4(\mathcal{T}_\pi)$ it suffices to consider wave vectors σ only in $\mathcal{T}^* = \mathcal{T}_2 \times \mathbb{R}$ and for given σ we are left with a spectral problem for $V \in X$:

$$\lambda V = B(\varepsilon,\kappa,\sigma)V \stackrel{\text{def}}{=} -(1+(1+\kappa)(\partial_\xi + i\sigma_1)^2 - \sigma_2^2)^2 V + \varepsilon V - 3U^2_{\varepsilon,\kappa}V. \tag{1.3}$$

The operators $B(\varepsilon,\kappa,\sigma)$ are called Bloch operators. The essential feature is the following spectral identity:

$$L^2\text{–spec}(B_{\varepsilon,\kappa}) = L^2_{lu}\text{–spec}(B_{\varepsilon,\kappa}) = \text{closure}\Big(\bigcup_{\sigma\in\mathcal{T}^*}\text{spec}(B(\varepsilon,\kappa,\sigma))\Big). \tag{1.4}$$

We establish this result for general elliptic operators in Appendix A in a short, self–contained way.

For the above–mentioned general theory no smallness assumption on the non–constant parts of the coefficients in the operator $B_{\varepsilon,\kappa}$ was needed. However, for the analysis of the spectra of each $B(\varepsilon,\kappa,\sigma)$ we heavily rely on the fact, that we are dealing with small perturbations from a homogeneous state, that is, $\|U^2_{\varepsilon,\kappa}\|_\infty = \mathcal{O}((\varepsilon-\kappa^2))$. Thus, we are able to study the Bloch operators $B(\varepsilon,\kappa,\sigma)$ as small perturbations of $B(0,0,\sigma)$, which have constant coefficients. For each $\sigma \in \mathbb{R}^2$ the linear spectral problem (1.3) can be attacked by the Liapunov–Schmidt reduction with a splitting $V = V_0 + V_1$ according to the kernel of $B(0,0,\sigma)$. We find reduced finite–dimensional spectral problems

$$0 = b(\varepsilon,\kappa,\sigma,\lambda)V_0 \overset{\text{def}}{=} P\big[B(\varepsilon,\kappa,\sigma) - \lambda I\big]\big(V_0 + \mathcal{V}(\varepsilon,\kappa,\sigma,\lambda)V_0\big), \qquad (1.5)$$

where $V_1 = \mathcal{V}(\ldots)V_0$ defines the associated reduction.

It is important to note that we can handle general large wave vectors σ; only the eigenvalue parameter $\lambda \in \mathbb{C}$ needs to be small. However, since the spectrum of $B_{0,0}$ is equal to $(-\infty, 0] \subset \mathbb{C}$, classical perturbation arguments (cf. [Kat76]) show that possible unstable modes can only occur for $|\lambda| = \mathcal{O}((\varepsilon-\kappa^2))$. In our case $X_0 = PX$ is, depending on σ, one– or two–dimensional and thus $b(\varepsilon,\kappa,\sigma,\lambda)$ corresponds to a scalar or a 2×2–matrix. The control of spec$(B(\varepsilon,\kappa,\sigma))$ is now managed by solving

$$0 = \Lambda(\varepsilon,\kappa,\sigma,\lambda) \overset{\text{def}}{=} \det b(\varepsilon,\kappa,\sigma,\lambda)$$

for λ as a function of $(\varepsilon,\kappa,\sigma)$.

Our method allows us to characterize the set $\mathcal{S}_{\varepsilon,\kappa}$ of unstable wave vectors for the state $U_{\varepsilon,\kappa}$:

$$\mathcal{S}_{\varepsilon,\kappa} = \{\, \sigma \in \mathcal{T}^* \;:\; B(\varepsilon,k,\sigma) \text{ has an eigenvalue } \lambda \text{ with } \operatorname{Re}\lambda > 0 \,\}.$$

In Sect. 4 we give all curves in the (ε,κ) plane where the topological structure of $\mathcal{S}_{\varepsilon,\kappa}$ changes. Moreover, we point out some differences between the sets $\mathcal{S}_{\varepsilon,\kappa}$ and its counterpart $\mathcal{S}^A_{\varepsilon,\kappa}$ for the rolls $A_{\varepsilon,\kappa}$ in the complex SHE.

The knowledge of the sets $\mathcal{S}_{\varepsilon,k}$ can in fact be used to study the stability of the solution $U_{\varepsilon,k}$ on finite domains $\Omega = (0, 2\pi N/k) \times (0, 2\pi L)$, where $N \in \mathbb{N}$ and $L > 0$, with periodic boundary conditions. Considering functions with such periodicity the stability analysis has to be restricted to perturbations having wave vectors σ with $\sigma_1 N,\ \sigma_2 L \in \mathbb{Z}$. Under this periodicity assumption we have stability if and only if

$$\mathcal{S}_{\varepsilon,\kappa} \cap \{\, (n/N, l/L) \in \mathcal{T}^* \;:\; n = 0,\ldots,2N-1,\ l \in \mathbb{Z} \,\} = \emptyset.$$

Thus, it is possible to rederive and refine the results in [Kuw96] by using the characterization of the set $\mathcal{S}_{\varepsilon,\kappa}$ given in the present work.

2. General Theory

We consider systems of partial differential equations which are posed over unbounded physical domains $\mathcal{Q} = \mathbb{R}^d \times \Sigma$ with variables $(x,z) \in \mathbb{R}^d \times \Sigma$. We assume for simplicity the form

$$\partial_t u = A_\mu(\partial_x)u + N(\mu, \partial_x, u) \quad \text{in } \mathcal{Q} = \mathbb{R}^d \times \Sigma, \qquad (2.1)$$

where $u = u(t,x,z) \in \mathbb{R}^n$ is the state variable, $A_\mu(\partial_x)$ is an elliptic operator of order $2m$ in the (x,z) variables and incorporates the boundary conditions $\mathcal{B}u = 0$ on $\partial\mathcal{Q} = \mathbb{R}^d \times \partial\Sigma$. The cross–section Σ is a bounded domain in $\mathbb{R}^s$ with Lipschitz boundary, and the vector $\mu \in \mathbb{R}^p$ denotes all parameters. The problem is translational invariant (no x–dependence) while dependence on the cross–sectional variable z is allowed but not explicitly displayed.

Our aim is to study the linearized stability of a given stationary spatially periodic pattern $\widetilde{u}_\mu$ of (2.1) under general nonperiodic perturbations. The linearization at $\widetilde{u}_\mu$ reads

$$\partial_t v = B_\mu(\partial_x)v \quad \text{with } B_\mu(\partial_x) \overset{\text{def}}{=} A_\mu(\partial_x) + D_u N(\mu, \partial_x, \widetilde{u}_\mu). \qquad (2.2)$$

To study (2.1) in a large function space which contains all sufficiently smooth bounded functions we define the uniformly local L^2 space as in [MiS95]: Let

$$\widetilde{L}^2_{lu}(\mathcal{Q}) = \{\, u \in L^2_{\text{loc}}(\mathcal{Q}) \;:\; \|u\|_{lu} < \infty \,\},$$
$$\|u\|^2_{lu} = \sup\{\, \textstyle\int_{\mathcal{Q}} T_y\rho(x)|u(x,z)|^2 dx\, dz \;:\; y \in \mathbb{R}^d \,\},$$

where $\rho : \mathbb{R}^d \to [0,\infty)$ is a suitable bounded and integrable weight function and T_y is the translation operator with $T_y v(x) = v(x-y)$. For definiteness we choose the weight $\rho(x) = e^{-|x|}$. The final L^2_{lu} uniformly local L^2–space is given by

$$L^2_{lu}(\mathcal{Q}) = \{\, u \in \widetilde{L}^2_{lu}(\mathcal{Q}) \;:\; \|T_y u - u\|_{lu} \to 0 \text{ for } y \to 0 \,\}.$$

As usual we define Sobolev spaces $H^k_{lu}(\mathcal{Q})$ by asking that all partial derivatives in (x,z) up to order k lie in $L^2_{lu}(\mathcal{Q})$. Then, $H^k_{lu}(\mathcal{Q})$ is densely contained in $L^2_{lu}(\mathcal{Q})$, and the classical space $L^2(\mathcal{Q})$ is continuously embedded in $L^2_{lu}(\mathcal{Q})$ but not dense.

The linear operator B_μ can now be defined on two spaces:

$$\begin{aligned} \widehat{B}_\mu &: D(\widehat{A}_0) = \{\, u \in H^{2m}(\mathcal{Q}) \;:\; \mathcal{B}u = 0 \text{ on } \partial\mathcal{Q} \,\} \to L^2(\mathcal{Q}),\; u \mapsto B_\mu(\partial_x)u; \\ \widetilde{B}_\mu &: D(\widetilde{A}_0) = \{\, u \in H^{2m}_{lu}(\mathcal{Q}) \;:\; \mathcal{B}u = 0 \text{ on } \partial\mathcal{Q} \,\} \to L^2_{lu}(\mathcal{Q}),\; u \mapsto B_\mu(\partial_x)u. \end{aligned} \tag{2.3}$$

The stationary periodic pattern $\widetilde{u}_\mu$ lies in $H^{2m}_{lu}(\mathcal{Q})$. Its stability analysis can first be done with respect to perturbations in $L^2(\mathcal{Q})$, but finally we will show that the spectrum of the linearization around the periodic pattern is the same considered in $L^2(\mathcal{Q})$ and in $L^2_{lu}(\mathcal{Q})$.

For the spectral analysis of B_μ we exploit the fact that B_μ has periodic coefficients via $D_u N(\mu, \partial_x, \widetilde{u}_\mu(x))$. Using the translation operators T_y this periodicity is characterized by the lattice group $\mathcal{L} \subset \mathbb{R}^d$ such that $B_\mu T_\ell = T_\ell B_\mu$ for all $\ell \in \mathcal{L}$. In some cases, see e.g. the SHE in Sect. 3, the lattice group $\mathcal{L}$ is larger than $\widetilde{\mathcal{L}} = \{\, y \in \mathbb{R}^d \;:\; T_y\widetilde{u}_\mu = \widetilde{u}_\mu \,\}$, which is the translation group of $\widetilde{u}_\mu$, but $\widetilde{\mathcal{L}} \subset \mathcal{L}$ always holds. Restricting the functions in $L^2_{lu}(\mathcal{Q})$ to the subclass with the given lattice group $\mathcal{L}$ we obtain as natural space

$$L^2_{lu}(\mathcal{Q})/_{\mathcal{L}} = \{\, u \in L^2_{lu}(\mathcal{Q}) \;:\; T_\ell u = u \text{ for all } \ell \in \mathcal{L} \,\},$$

which is easily identifiable with $L^2(\mathcal{Q}/_{\mathcal{L}})$ where $\mathcal{Q}/_{\mathcal{L}} = \mathcal{T} \times \Sigma$ is the periodicity domain and $\mathcal{T} = \mathbb{R}^d/_{\mathcal{L}}$.

For the wave vectors the dual lattice group $\mathcal{L}^* \subset \mathbb{R}^d$ is relevant. It is given by

$$\mathcal{L}^* = \{\, h \in \mathbb{R}^d \;:\; h \cdot \ell \in 2\pi\mathbb{Z} \text{ for all } \ell \in \mathcal{L} \,\}.$$

Throughout we assume that $\mathcal{L}$ contains d linearly independent vectors and that the connected components of $\mathcal{L}$ are $\widetilde{d}$–dimensional, then $\mathcal{T}$ is a $(d-\widetilde{d})$–dimensional torus. Under these conditions on $\mathcal{L}$, the dual lattice $\mathcal{L}^*$ is discrete and contained in a $(d-\widetilde{d})$–dimensional subspace. By choosing appropriate coordinates in $\mathbb{R}^d$ we can arrange things such that $\mathcal{L} = (2\pi\mathbb{Z})^{d-\widetilde{d}} \times \mathbb{R}^{\widetilde{d}} \subset \mathbb{R}^d$. Then, $\mathcal{T} = (\mathcal{T}_{2\pi})^{d-\widetilde{d}} \times \{0\}$, $\mathcal{L}^* = \mathbb{Z}^{d-\widetilde{d}} \times \{0\}$, and $\mathcal{T}^* = (\mathcal{T}_1)^{d-\widetilde{d}} \times \mathbb{R}^{\widetilde{d}}$, where $\mathcal{T}_\alpha = \mathbb{R}/_{\alpha\mathbb{Z}}$ is the one–dimensional torus of length α.

The main idea is to reduce the spectral analysis in $L^2(\mathcal{Q})$ to the space $L^2(\mathcal{Q}/_{\mathcal{L}})$ by using the Bloch decomposition which is also called the direct integral, cf. [ReS78], XIII.16. It is given by the isomorphism $\mathcal{D} : L^2(\mathcal{T}^*, L^2(\mathcal{Q}/_{\mathcal{L}})) \to L^2(\mathcal{Q})$ with

$$\mathcal{D}(U)(x,z) = \int_{\sigma \in \mathcal{T}^*} e^{i\sigma\cdot x} U(\sigma, x, z)\, d\sigma, \tag{2.4}$$

and satisfying $\|\mathcal{D}(U)\|^2_{L^2(\mathcal{Q})} = (2\pi)^d \big(\text{vol}(\mathcal{T})\big)^{-1} \int_{\sigma\in\mathcal{T}^*} \|U(\sigma,\cdot)\|^2_{L^2(\mathcal{Q}/_{\mathcal{L}})} d\sigma$. For more details we refer to [ReS78] and to Appendix A.

We define the closed subspaces

$$X_\sigma = \{\, e^{i\sigma\cdot x}U \;:\; U \in L^2(\mathcal{Q}/_{\mathcal{L}})\,\} \subset L^2_{lu}(\mathcal{Q}),$$

such that (2.4) tells us that $L^2(\mathcal{Q})$ can be understood as the direct L^2–product of all the spaces X_σ. It is clear that each X_σ is left invariant under the action of $\widetilde{B}_\mu$, and we are able to define the Bloch operators $B(\mu,\sigma) : D(B) \subset L^2(\mathcal{Q}/_{\mathcal{L}}) \to L^2(\mathcal{Q}/_{\mathcal{L}})$ as follows

$$B(\mu,\sigma)U = e^{-i\sigma\cdot x}\widetilde{B}_\mu(\partial_x)[e^{i\sigma\cdot x}U] = B_\mu(i\sigma + \partial_x)U, \tag{2.5}$$

where $D(B) = \{\, u \in H^{2m}(\mathcal{Q}/_{\mathcal{L}}) \;:\; \mathcal{B}u = 0\,\}$ does not depend on σ if the boundary operators $\mathcal{B}$ do not contain tangential derivatives (i.e. ∂_x).

The family of Bloch operators allows us to gain full control over the operator $B_\mu(\partial_x)$. In fact, assuming that the resolvents $(B(\mu,\sigma) - \lambda I)^{-1} : L^2(\mathcal{Q}/_{\mathcal{L}}) \to L^2(\mathcal{Q}/_{\mathcal{L}})$ exist for all $\sigma \in \mathcal{T}^*$ with their norm uniformly bounded, we have

$$(\widehat{B}_\mu - \lambda I)^{-1}f = \mathcal{D}\big[B(\mu,\cdot)F(\cdot)\big], \quad \text{where } F = \mathcal{D}^{-1}f. \tag{2.6}$$

See Lemma A.3 for the exact statement.

In such a way it is possible to reduce the set of perturbations in $L^2(\mathcal{Q})$ to the space $L^2(\mathcal{Q}/_{\mathcal{L}})$ while $\sigma \in \mathcal{T}^*$ appears as an additional parameter. If we are able to control the perturbations for all $\sigma \in \mathcal{T}^*$ simultaneously, then we are able to decide on stability. Note that no assumption on self–adjointness is needed for this theory. The only important fact is that we are in a Hilbert space setting, which enables us to use the Bloch decomposition. In Appendix A we show that all this can be made rigorous for general elliptic operators with suitable boundary conditions. The following result is provided there.

Theorem 2.1. *Let $B_\mu(\partial_x)$ be an elliptic operator on $\mathcal{Q}$ with $\mathcal{L}$–periodic coefficients and $\mathcal{B}$ a boundary operator on $\partial\mathcal{Q}$ satisfiying conditions A.2. Define $\widehat{B}_\mu(\partial_x)$, $\widetilde{B}_\mu(\partial_x)$ according to* (2.3) *on $L^2(\mathcal{Q})$ and $L^2_{lu}(\mathcal{Q})$, respectively, and the Bloch operators $B(\mu,\sigma)$ according to* (2.5)*. Then we have*

$$\text{spec}(\widetilde{B}_\mu(\partial_x)) = \text{spec}(\widehat{B}_\mu(\partial_x)) = \text{closure}\Big(\bigcup_{\sigma\in\mathcal{T}^*}\text{spec}(B(\mu,\sigma))\Big). \tag{2.7}$$

Remarks.

1. The spectra of $\widetilde{B}_\mu$ and $\widehat{B}_\mu$ are the same as sets, however the type of spectrum usually differs dramatically. In fact, it is easy to see that $\bigcup_\sigma \text{spec}(B(\mu,\sigma))$ is contained in spec$(\widetilde{B}_\mu)$) as point spectrum. Observe that from $B(\mu,\sigma)U = \lambda U$ immediately $\widetilde{B}_\mu[e^{i\sigma\cdot x}U] = \lambda e^{i\sigma\cdot x}U \in L^2_{lu}(\mathcal{Q})$ follows. For the operator $\widehat{B}_\mu$ these points are not necessarily in the point spectrum, since $e^{i\sigma\cdot x}U \notin L^2(\mathcal{Q})$.

2. Another difference appears when approaching the spectrum from inside the resolvent set. For instance, if $\widetilde{B}_\mu$ is self–adjoint we have

$$\|(\widetilde{B}_\mu - \lambda I)^{-1}\|_{L^2(\mathcal{Q})\to L^2(\mathcal{Q})} = C\Big(\text{dist}(\lambda, \text{spec}(\widetilde{B}_\mu))\Big)^{-n}$$

with $C = n = 1$. However, the blow up for the operator $(\widehat{B}_\mu - \lambda I)^{-1}$ might be much worse, i.e. with $C \geq 1$ and $n \geq 1$. This question plays an important role if spectral stability has to be improved to linearized stability. Then, we want to estimate the semigroup $(e^{\widetilde{B}_\mu t})_{t\geq 0}$ or $(e^{\widehat{B}_\mu t})_{t\geq 0}$ for large t. Under the additional assumption that B_μ is a sectorial operator, one obtains $\|e^{B_\mu t}\| \leq \widetilde{C}(1 + t^{n-1})e^{\nu t}$ for $t \geq 1$.

3. In our application we cannot expect exponential stability since the spectrum always contains the origin $\lambda = 0$ if the periodic solution $\widetilde{u}_\mu$ is non–constant. This is easily seen since some partial derivative $\partial_{x_j}\widetilde{u}_\mu$ is nonzero and it is in the kernel of $\widetilde{B}_\mu(\partial_x)$.

Thus, it remains to study the spectra of the Bloch operators $B(\mu, \sigma)$. For elliptic operators $B_\mu(\partial_x)$ the Bloch operators are also elliptic and they are defined on the bounded spatial domain $\mathcal{Q}/_{\mathcal{L}} = \mathcal{T} \times \Sigma$. Hence, they are Fredholm operators of index zero with compact resolvent. In order to analyze the spectrum we assume further on that we are in a bifurcation situation, where the stationary periodic pattern $\widetilde{u}_\mu$ is small. Then, it is natural to assume that $u = 0$ is stable for $\mu = 0$. If $u = 0$ would be unstable, then small $\widetilde{u}_\mu$ could not gain stability. Thus, our main assumption on system (2.1) is that $A_0(\partial_x)$ is an elliptic operator on $L^2(\mathcal{Q})$ which is spectrally stable.

More precisely, our method can only work when the spectrum of $A_0(\partial_x)$ is contained in a set $S_g = \{\, \lambda \in \mathbb{C} \;:\; \mathrm{Re}\,\lambda \leq -g(|\mathrm{Im}\,\lambda|) \,\}$ where $g : [0, \infty) \to [0, \infty)$ satisfies $g(0) = 0$ and $g(t) \geq g(s) > 0$ for $t > s > 0$. The reason for this spectral bound is that our method involves perturbation arguments. Linearization around a small solution $(\mu, \widetilde{u}_\mu)$ leads to the linear operator $B_\mu = A_\mu(\partial_x) + D_u N(\mu, \partial_x, \widetilde{u}_\mu)$ with $\delta(\mu) = \|(B_\mu - A_0)(A_0 - I)^{-1}\| \to 0$ for $|\mu| \to 0$. Hence, standard perturbation arguments (see [Kat76]) show that the distance of the spectrum of B_μ from that of A_0 is less than $\delta(\mu)$. Our assumption spec$(A_0) \subset S_g$ now implies that the spectrum of B_μ is contained in $\{\, \lambda \in \mathbb{C} \;:\; \mathrm{dist}(\lambda, S_g) \leq \delta(\mu) \,\}$. Thus, we immediately conclude that for $\mu \to 0$ the unstable part (i.e., $\mathrm{Re}\,\lambda > 0$) of the spectrum of B_μ is contained in a small neighborhood of zero. More precisely, for each $\varepsilon > 0$ there is a μ_0 such that for all μ with $|\mu| \leq \mu_0$ the spectrum of B_μ is contained in $\{\, \lambda \in \mathbb{C} \;:\; \mathrm{Re}\,\lambda < 0 \text{ or } |\lambda| \leq \varepsilon \,\}$.

Our method is exactly devised to study the spectrum close to $\lambda = 0$ in the case that $\widetilde{u}_\mu$ is a small spatially periodic steady state of (2.1). We are not able to control large λ nor large solutions $\widetilde{u}_\mu$ since our analysis is based in the exact control of the operator $A_0(\partial_x)$, which can be obtained by Fourier transform with respect to $x \in \mathbb{R}^d$.

For $\mu = 0$ we know that the spectrum of $B(0, \sigma)$ is contained in $\{0\} \cup \{\, \lambda \in \mathbb{C} \;:\; \mathrm{Re}\,\lambda < 0 \,\}$. The kernel is finite–dimensional and depends on σ. The general a–priori estimate (A.9) tells us that for large $\sigma \in \mathcal{T}^*$ the kernel is trivial, so that only a compact set $\mathcal{S}_0$ of wave vectors σ can be important, i.e. $\mathcal{S}_0 = \{\, \sigma \in \mathcal{T}^* \;:\; \dim \mathrm{kernel}(B(0, \sigma)) > 0 \,\}$. Considering now general small μ we immediately see that we only have to control the operators in a neighborhood of $\mathcal{S}_0$. In fact, defining the set $\mathcal{S}_\mu$ of unstable wave vectors as

$$\mathcal{S}_\mu = \{\, \sigma \in \mathcal{T}^* \;:\; B(\mu, \sigma) \text{ has an eigenvalue } \lambda \text{ with } \mathrm{Re}\,\lambda > 0 \,\}, \tag{2.8}$$

perturbation theory for operators with compact resolvent implies $\mathrm{dist}(\mathcal{S}(\mu), \mathcal{S}_0) \to 0$ for $\mu \to 0$.

Thus, it remains to control the finitely many eigenvalues of $B(\mu, \sigma)$ for $\mu \approx 0$ and $\sigma \approx \sigma_0 \in \mathcal{S}_0$. This, we can do with the help of the *Liapunov–Schmidt reduction* applied to the linear eigenvalue problem

$$\mathcal{K}(\mu, \sigma, \lambda)U \stackrel{\text{def}}{=} B(\mu, \sigma)U - \lambda U = 0. \tag{2.9}$$

It is our aim to find nontrivial solutions of this equation, and we do this by treating it as a bifurcation problem. Although this is a perturbation problem for linear operators we use the Liapunov–Schmidt reduction since it is so closely related to the typical way of establishing the bifurcation result for the nonlinear problem, cf. [Mie95, Mi97b].

The main point is that it is sufficient to consider small λ as was shown above. Hence, for $\sigma_0 \in \mathcal{S}_0$ fixed, $(\mu, \sigma - \sigma_0, \lambda)$ can be treated as a small bifurcation parameter in (2.9). For $(\mu, \sigma, \lambda) = (0, \sigma_0, 0)$ we find splittings $D(B) = X_0(\sigma_0) \oplus X_1(\sigma_0)$ and $L^2(\mathcal{Q}/_{\mathcal{L}}) = Y_0(\sigma_0) \oplus Y_1(\sigma_0)$ such that $X_0(\sigma_0)$ is the finite–dimensional kernel of $\mathcal{K}(0, \sigma_0, 0) = B(0, \sigma_0)$ and $Y_1(\sigma_0)$ its range. Since the Fredholm index of $B(\mu, \sigma)$ is 0, the dimensions of Y_0 and X_0 are the same. Decompose $U = U_0 + U_1$ with $U_j \in X_j$, $F = F_0 + F_1$ with $F_j \in Y_j$, and let $P : L^2(\mathcal{Q}/_{\mathcal{L}}) \to L^2(\mathcal{Q}/_{\mathcal{L}})$ be the projection with $PF = F_0$. Then, $\mathcal{K}(\mu, \sigma, \lambda)U = 0$ is equivalent to

$$P\mathcal{K}(\mu, \sigma, \lambda)(U_0 + U_1) = 0, \quad (I - P)\mathcal{K}(\mu, \sigma, \lambda)(U_0 + U_1) = 0,$$

where the second relation can be inverted for $(\mu, \sigma - \sigma_0, \lambda)$ sufficiently small in order to obtain $U_1 = \mathcal{U}(\mu, \sigma, \lambda)U_0$. Inserting this result into the first equation we are left with the reduced spectral problem

$$\widetilde{\mathcal{K}}(\mu, \sigma, \lambda)U_0 \stackrel{\text{def}}{=} P\mathcal{K}(\mu, \sigma, \lambda)(U_0 + \mathcal{U}(\mu, \sigma, \lambda)U_0) = 0. \tag{2.10}$$

This reduced problem is no longer linear in λ, however, it is finite–dimensional with $\widetilde{\mathcal{K}}(\mu, \sigma, \lambda) : X_0(\sigma_0) \to Y_0(\sigma_0)$. Equation (2.10) has nontrivial solutions U_0 if and only if

$$\Lambda(\mu, \sigma, \lambda) \stackrel{\text{def}}{=} \det \widetilde{\mathcal{K}}(\mu, \sigma, \lambda) = 0.$$

We note that σ has to be close to $\sigma_0 \in \mathcal{S}_0$. By compactness it is sufficient to do this reduction for finitely many σ_0, where the subspaces $X_0(\sigma_0)$ and $Y_0(\sigma_0)$ can change dramatically: generally, even the dimension will change.

The present approach does not only provide a tool to decide on stability or instability of the periodic pattern. It also gives a way to describe the set of unstable wave vectors quite precisely. Analyzing the problems $\Lambda(\mu, \sigma, \lambda) = 0$ we obtain information on the set $\mathcal{S}_\mu$, cf. (2.8). Moreover, it is possible to find those wave vectors $\sigma \in \mathcal{S}_\mu$ which correspond to those λ having the largest real part. Such characterizations of $\mathcal{S}_\mu$ are important in the theory of pattern formation.

One special case attracted a lot of attention over the last thirty years, namely those of sideband instabilities. This phenomenon is now easily identified in the present context with the situation when $\mathcal{S}_\mu$ is contained in a small neighborhood of $\sigma = 0$, but $\sigma = 0$ itself is not in $\mathcal{S}_\mu$. We will discover such sideband instabilities in the next section.

3. The Real and Complex Swift–Hohenberg Equation

We work out the details of the method for a simple model problem showing the same theoretical behavior as many other pattern forming systems. The two–dimensional Swift–Hohenberg equation (SHE) is given by

$$u_t = -(1 + \Delta)^2 u + \varepsilon u - u^3, \qquad \text{for } t > 0,\ x \in \mathcal{Q} = \mathbb{R}^2, \tag{3.1}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ is the Laplace operator. The linearization at zero admits the solutions $v(t, x) = e^{\lambda t + i(k_1 x_1 + k_2 x_2)}$ with $\lambda(k_1, k_2) = -(1 - k_1^2 - k_2^2)^2 + \varepsilon$. Hence, $u \equiv 0$ is weakly unstable with unstable modes having wave vectors with $k_1^2 + k_2^2 \approx 1$.

The basic patterns of interest are so–called rolls, which are independent of time and of x_2 (after a suitable rotation), and periodic in x_1. Taking the period in x_1 to be $2\pi/k$ with $k = \sqrt{1+\kappa}$ we are looking for a solution u of (3.1) in the form $u(t,x) = U(\xi)$ where $\xi = kx_1 \in \mathcal{T}_{2\pi} = \mathbb{R}/_{2\pi\mathbb{Z}}$. The problem for U reads

$$0 = \mathcal{N}(\varepsilon,\kappa,U) \stackrel{\text{def}}{=} -(1+(1+\kappa)\partial_\xi^2)^2U + \varepsilon U - U^3, \quad U \in H^4(\mathcal{T}_{2\pi}), \qquad (3.2)$$

where $\mathcal{N} : \mathbb{R}^2 \times H^4(\mathcal{T}_{2\pi}) \to L^2(\mathcal{T}_{2\pi})$ is an analytical mapping.

From [Mie95] (see also [CoE90], Thm. 17.1) we have the following result on the existence of steady roll patterns.

Theorem 3.1. *There is an $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0]$ and all $\kappa \in (-\sqrt{\varepsilon},\sqrt{\varepsilon})$ there is a unique small solution $U = U_{\varepsilon,\kappa} \in H^4(\mathcal{T}_{2\pi})$ of* (3.2) *which is even in ξ and positive at $\xi = 0$. This solution has the expansion*

$$U_{\varepsilon,\kappa}(\xi) = a_1\cos\xi + a_3\cos(3\xi) + \mathcal{O}(\widetilde{a}^5) \text{ for } (\varepsilon,\kappa) \to 0, \qquad (3.3)$$

where $\widetilde{a} = \widetilde{a}(\varepsilon,\kappa) = \sqrt{4(\varepsilon-\kappa^2)/3}$ and

$$a_1 = \tfrac{1}{\pi}\int_0^{2\pi} U_{\varepsilon,\kappa}(\xi)\cos\xi\,d\xi = \widetilde{a} + \widetilde{a}^3/512 + \mathcal{O}(\widetilde{a}^4),$$
$$a_3 = \tfrac{1}{\pi}\int_0^{2\pi} U_{\varepsilon,\kappa}(\xi)\cos(3\xi)\,d\xi = -\widetilde{a}^3/256 + \mathcal{O}(\widetilde{a}^4).$$

Moreover, $U_{\varepsilon,\kappa}(\pi+\xi) = -U_{\varepsilon,\kappa}(\xi)$.

In light of Sect. 2 we say that the solution $\widetilde{u}_{\varepsilon,\kappa}(x) = U_{\varepsilon,\kappa}(kx_1)$ is (spectrally) unstable, if there exists $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda > 0$ and a nontrivial smooth bounded function v such that

$$\lambda v = -(1+\Delta)^2 v + \big(\varepsilon - 3\widetilde{u}^2_{\varepsilon,\kappa}\big)v.$$

The following necessary and sufficient stability criterion is derived in the next section together with precise information on the set $\mathcal{S}_{\varepsilon,\kappa}$ of unstable wave vectors.

Theorem 3.2. *There is a positive ε_1, and there are curves $\kappa = K_Z(\varepsilon)$ and $\varepsilon = E_E(\kappa)$, satisfying the expansions*

$$K_Z(\varepsilon) = -\varepsilon^2/512 + \mathcal{O}(\varepsilon^3), \quad E_E(\kappa) = 3\kappa^2 - \kappa^3 + \mathcal{O}(|\kappa|^4),$$

such that the roll solution $U_{\varepsilon,\kappa}$ with $\varepsilon \in (0,\varepsilon_1]$ and $|\kappa| \le \sqrt{\varepsilon}$ is stable if and only if

$$\varepsilon \ge E_E(\kappa) \quad \text{and} \quad \kappa \ge K_Z(\varepsilon). \qquad (3.4)$$

Remarks.

1. The bound $\varepsilon \ge E_E(\kappa)$ is called the Eckhaus criterion (cf. [Eck65]), which contains the universal factor 3: rolls exist for $\varepsilon > \kappa^2$ but the rolls are stable only for $\varepsilon \ge 3\kappa^2 + \mathcal{O}(|\kappa|^3)$. The bound $\kappa \ge K_Z(\varepsilon)$ is the zigzag instability bound, see [Bus71] for a first discussion.

2. Our results are sharper than those in [CoE90], Thm. 20.1+2 and [Kuw96]. Reformulating the latter results in our notation gives a statement as follows: there are curves $K_Z^1(\varepsilon) < 0 < K_Z^2(\varepsilon)$ and $E_E^1(\kappa) < E_E^2(\kappa)$ with $K_Z^2(\varepsilon) - K_Z^1(\varepsilon) = \mathcal{O}(\varepsilon^\alpha)$ and $E_E^2(\kappa) - E_E^1(\kappa) = \mathcal{O}(|\kappa|^\beta)$ for suitable $\alpha > 1$ and $\beta > 2$ such that stability can be concluded if $\varepsilon \ge E_E^2(\kappa)$ and $\kappa \ge K_Z^2(\varepsilon)$ whereas instability holds if either $\varepsilon < E_E^1(\kappa)$ or $\kappa < K_Z^1(\varepsilon)$. Hence, small tongues around the exact boundaries remained where no conclusion could be made.

3. There are parameters (κ,ε) with $\kappa < 0$ such that the roll $U_{\varepsilon,\kappa}$ is stable. Moreover, all small rolls with $\kappa = 0$ are stable.

We postpone the proof of this result to Sect. 4 and study first a somewhat similar problem which is much easier as no Liapunov–Schmidt reduction is necessary. But nevertheless it shows the ideas and technicalities in the discussion of the algebraic eigenvalue problem. The complex SHE is given by

$$\partial_t A = -(1+\Delta)^2 A + \varepsilon A - |A|^2 A, \quad t \geq 0,\ x \in \mathbb{R}^2, \tag{3.5}$$

where $A(t,x,y) \in \mathbb{C}$. In contrast to the real SHE this problem has an additional symmetry group, namely the phase invariance $A \mapsto e^{i\alpha} A$ for $\alpha \in \mathcal{T}_{2\pi}$.

Obviously, the real SHE is contained in (3.5) by restricting to real–valued A. We will study the stability of the explicitly known family of stationary roll solutions given by

$$A(x) = re^{i(\alpha + k_1 x_1 + k_2 x_2)}, \quad \text{where } r^2 = \varepsilon - (1 - k_1^2 - k_2^2)^2. \tag{3.6}$$

Using the rotational invariance we may assume $(k_1, k_2) = (k, 0)$ with $k = \sqrt{1+\kappa}$ and denote by $A_{\varepsilon,\kappa}$ the unique solution in (3.6) with $\alpha = 0$. These solutions are not related to the previously studied $U_{\varepsilon,\kappa}$, which are, of course, also stationary solutions of (3.5).

To study the stability of $A_{\varepsilon,\kappa}$ we consider the linearization of (3.5) around this steady state:

$$\partial_t B = -(1+\Delta)^2 B + \varepsilon B - 2|A_{\varepsilon,\kappa}|^2 B - A_{\varepsilon,\kappa}^2 \overline{B}, \tag{3.7}$$

where $\overline{B}$ is the complex conjugate of B. We let $B = (w_1 + iw_2)e^{ikx_1}$ with $w_1, w_2 \in \mathbb{R}$ and arrive at the constant coefficient problem

$$\partial_t \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} L_4 + \varepsilon - 3r^2 & 4kL_2\partial_{x_1} \\ -4kL_2\partial_{x_1} & L_4 + \varepsilon - r^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \tag{3.8}$$

where $L_4 = -L_2^2 + 4(1+\kappa)\partial_{x_1}^2$ and $L_2 = \Delta - \kappa$. This linear system can be solved completely by Fourier transform. Looking for solutions in the form $w = e^{\lambda t + i(k(\sigma_1 - 1)x_1 + \sigma_2 x_2)} W$ with constant $W \in \mathbb{C}^2$ we obtain the algebraic problem

$$\begin{pmatrix} \rho + c - \lambda & i\nu \\ -i\nu & \rho - \lambda \end{pmatrix} W = 0, \tag{3.9}$$

where

$$\begin{aligned} \rho &= -(\kappa + (1+\kappa)(\sigma_1 - 1)^2 + \sigma_2^2)^2 - 4(1+\kappa)^2(\sigma_1 - 1)^2 + \kappa^2, \\ \nu &= -4(1+\kappa)(\sigma_1 - 1)(\kappa + (1+\kappa)(\sigma_1 - 1)^2 + \sigma_2^2), \text{ and } c = -2(\varepsilon - \kappa^2). \end{aligned} \tag{3.10}$$

Since roll solutions only exist for $\varepsilon > \kappa^2$ we always have $c < 0$. Note that we have shifted back the vector σ by $(-1, 0)$ to account for the factor e^{ikx_1} in the ansatz $B = (w_1 + iw_2)e^{ikx_1}$.

The two eigenvalues λ obtained from solving (3.9) are real and can be expressed explicitly by solving $\lambda^2 - (2\rho + c)\lambda + \rho(\rho + c) - \nu^2 = 0$. Our aim is to characterize the unstable wave vectors σ, where at least one eigenvalue is positive. This is the case if and only if either (i) or (ii) hold, where

$$\text{(i)}\ \ \rho + c/2 > 0 \quad \text{and} \ \ \text{(ii)}\ \ \rho(\rho + c) - \nu^2 < 0. \tag{3.11}$$

To analyze these conditions in more detail we use the abbreviations

$$\widetilde{s} = k^2(\sigma_1 - 1)^2, \quad t = \sigma_2^2, \quad \text{and } \mu = \rho + 8(1+\kappa)\widetilde{s}.$$

In this notation conditions (3.11) take the form

$$\begin{aligned} &\text{(i)}\ \ \mu-(\varepsilon-\kappa^2)-8(1+\kappa)\widetilde{s}>0,\\ &\text{(ii)}\ \ \mu^2-2(\varepsilon-\kappa^2)\mu+16(\varepsilon-2\kappa^2)(1+\kappa)\widetilde{s}<0. \end{aligned} \tag{3.12}$$

Of course, only such $(\widetilde{s},\mu)$ are allowed which can be obtained from $(\widetilde{s},t)\in[0,\infty)^2$, namely

$$0\le\widetilde{s}<\infty \quad \text{and} \quad \mu\le g(\kappa,\widetilde{s}) \stackrel{\text{def}}{=} \begin{cases} q(\kappa,\widetilde{s}) & \text{if } \widetilde{s}>-\kappa,\\ \kappa^2+4(1+\kappa)\widetilde{s} & \text{if } \widetilde{s}\in[0,-\kappa]; \end{cases},$$

where $q(\kappa,\widetilde{s})=-\widetilde{s}^2+2(2+\kappa)\widetilde{s}$. Then, $t=-\widetilde{s}-\kappa\pm\sqrt{\kappa^2+4(1+\kappa)\widetilde{s}-\mu}$, where the minus sign is only allowed if $q(\kappa,\widetilde{s})\le\mu\le\kappa^2+4(1+\kappa)\widetilde{s}$.

Condition (i) in (3.12) can only hold if $\kappa\in[-\sqrt{\varepsilon},-\sqrt{\varepsilon}/2]$, namely in the region

$$A_1=\left\{\,(\widetilde{s},\mu)\ :\ 0\le\widetilde{s}<\tfrac{2\kappa^2-\varepsilon}{4(1+\kappa)},\ \varepsilon-\kappa^2+8(1+\kappa)\widetilde{s}<\mu<\kappa^2+4(1+\kappa)\widetilde{s}\,\right\}.$$

For condition (ii) we first consider the case $\kappa\in[0,\sqrt{\varepsilon}]$. The instability set is characterized by the intersection of the sets

$$\begin{aligned} A_2&=\left\{\,(\widetilde{s},\mu)\in[0,\infty)\times\mathbb{R}\ :\ \mu\le g(\kappa,\widetilde{s})\,\right\} \quad \text{and}\\ A_3&=\left\{\,(\widetilde{s},\mu)\in[0,\infty)\times\mathbb{R}\ :\ \mu^2-2(\varepsilon-\kappa^2)\mu+16(\varepsilon-2\kappa^2)(1+\kappa)\widetilde{s}<0\,\right\}. \end{aligned}$$

Both regions are bounded by a parabola which contains the origin. Checking their position it is immediate that $A_2\cap A_3$ is nonempty if and only if the slope of ∂A_2 in the origin is larger than that of ∂A_3. This gives the stability condition $2(2+\kappa)\le 8(\varepsilon-2\kappa^2)(1+\kappa)/(\varepsilon-\kappa^2)$, which is the classical Eckhaus criterion:

$$\varepsilon\ge E_E^C(\kappa)\stackrel{\text{def}}{=}\kappa^2\frac{6+7\kappa}{2+3\kappa}=3\kappa^2-\kappa^3+3\kappa^4/2+\mathcal{O}(|\kappa|^5).$$

For $\varepsilon<E_E^C(\kappa)$ we have a nontrivial intersection $A_2\cap A_3$, which changes its type when $\varepsilon\approx 2\kappa^2$. For $\varepsilon\in(E_3^C(\kappa),E_E^C(\kappa))$ the set $A_2\setminus A_3$ has one connected component while for $\varepsilon\in[\kappa^2,E_3^C(\kappa)]$ the set $A_2\setminus A_3$ has two connected components: one above the line $\mu=0$ and one below. The boundary $\varepsilon=E_3^C(\kappa)$ is determined by the condition that the boundaries of A_2 and A_3 touch each other in a point $(\widetilde{s},\mu)\approx(4,0)$. We find the expansion

$$E_3^C(\kappa)=2\kappa^2+\frac{1}{64}\kappa^4+\mathcal{O}(|\kappa|^5).$$

The analysis of the case $\kappa\in[-\sqrt{\varepsilon},0)$ is more involved, since the set A_2 is enlarged due to the fact that $g(\kappa,\widetilde{s})>q(\kappa,\widetilde{s})$ for $\widetilde{s}<-\kappa$. Now the intersection $A_2\cap A_3$ is always nontrivial and hence instability is concluded. To characterize the intersection we note that $A_2\setminus A_3$ consists of one or two connected components for $\varepsilon>E_3^C(\kappa)$ or $\varepsilon\in[\kappa^2,E_3^C(\kappa)]$ respectively. Moreover,

$$A_2\cap A_3=\left\{\,(\widetilde{s},\mu)\ :\ \mu\le\kappa^2+4(1+\kappa)\widetilde{s},\ m_-(\varepsilon,\kappa,\widetilde{s})<\mu<m_+(\varepsilon,\kappa,\widetilde{s})\,\right\},$$

where $m_\pm(\varepsilon,\kappa,\widetilde{s})=\varepsilon-\kappa^2\pm\sqrt{(\varepsilon-\kappa^2)^2+16(2\kappa^2-\varepsilon)(1+\kappa)\widetilde{s}}$.
For $\kappa\in[-\sqrt{\varepsilon},-\sqrt{2\varepsilon/3})$ the bound m_+ lies below the straight line $\mu=m_0(\widetilde{s})=\kappa^2+4(1+\kappa)\widetilde{s}$ for small $\widetilde{s}$. However, the set A^* lying between m_0 and m_+ is contained inside the region A_1, where $\rho+c/2>0$. Hence, A^* characterizes those σ for which both eigenvalues $\lambda_{1,2}$ are positive.

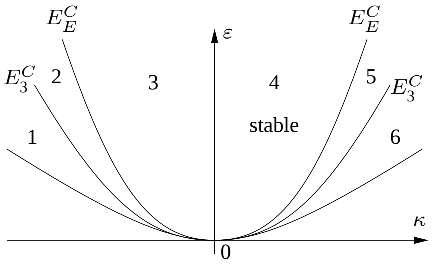

Fig. 3.1. The regions R_1^C to R_6^C for the complex SHE

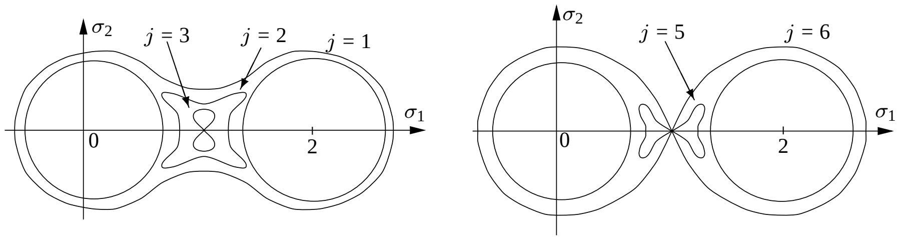

Fig. 3.2. The set $\mathcal{S}^C_{\varepsilon,\kappa}$ of unstable wave vectors for $A_{\varepsilon,\kappa}$ with $(\varepsilon,\kappa) \in R_j^C$

For the interpretation of the above results in terms of σ we recall that $A_2 \cap A_3$ always lies in a strip of width $\mathcal{O}(\varepsilon)$ around the $\widetilde{s}$–axis. Moreover, the line $\mu = 0$ corresponds in the case $k = 1$ to the two circles

$$\mathcal{S}_0^C = \{\, \sigma \in \mathbb{R}^2 \;:\; \sigma_1^2 + \sigma_2^2 = 1 \text{ or } (\sigma_1 - 2)^2 + \sigma_2^2 = 1 \,\}.$$

For a given solution $A_{\varepsilon,\kappa}$ with $\varepsilon \in [\kappa^2, \varepsilon_0)$ of (3.5) we define the instability set

$$\mathcal{S}^C_{\varepsilon,\kappa} = \{\, \sigma \in \mathbb{R}^2 \;:\; \text{ either (i) or (ii) hold } \,\}.$$

Using the semidistance $\text{dist}(A, B) = \sup\{\, \inf\{\, |a - b| \;:\; b \in B \,\} \;:\; a \in A \,\}$ for $A, B \subset \mathbb{R}^2$ we have the following results.

Theorem 3.3. *There is a positive ε_0 and curves $\varepsilon = E_E^C(\kappa)$ and $\varepsilon = E_3^C(\kappa)$ in the form as given above such that for a roll solution $A_{\varepsilon,\kappa}$ with $\varepsilon \in (\kappa^2, \varepsilon_0)$ of the complex SHE* (3.5) *the following holds.*

(a) $(1, 0) \notin \mathcal{S}^C_{\varepsilon,\kappa}$*, and* $(\sigma_1, \sigma_2) \in \mathcal{S}^C_{\varepsilon,\kappa}$ *implies* $(2 - \sigma_1, \sigma_2),\ (\sigma_1, -\sigma_2) \in \mathcal{S}^C_{\varepsilon,\kappa}$.

(b) $\text{dist}(\mathcal{S}_{\varepsilon,\kappa}, \mathcal{S}_0^C) = \mathcal{O}(\sqrt{\varepsilon})$ *for* $\varepsilon \to 0$.

(c) The solution $A_{\varepsilon,\kappa}$ is stable (i.e., $\mathcal{S}^C_{\varepsilon,\kappa} = \emptyset$) if and only if $\kappa \geq 0$ and $\varepsilon \geq E_E^C(\kappa)$.

(d) On the curve $\varepsilon = E_3^C(\kappa)$ the boundary of $\mathcal{S}_{\varepsilon,\kappa}$ has a pair of double points on the line $\sigma_2 = 0$ close to $\sigma_1 = -1$ and $\sigma_1 = 3$.

The curves $\varepsilon = E_E^C(\kappa)$, $\varepsilon = E_3^C(\kappa)$ and $\kappa = 0$ divide the region $\varepsilon \geq \kappa^2$ into six regions R_j^C, see Fig. 3.1. The boundaries between R_j^C and R_{j+1}^C are exactly those curves where the topological structure of $\mathcal{S}^C_{\varepsilon,\kappa}$ changes. We depict the different shapes in Fig. 3.2.

The cases $\kappa = \sqrt{\varepsilon}$ and $\kappa = -\sqrt{\varepsilon}$ can be given explicitly:

$$\begin{aligned}\mathcal{S}^C_{\varepsilon,\varepsilon} &= \{\, \sigma \in \mathbb{R}^2 \;:\; \exists\, \beta \in \{0,2\} \text{ such that } \tfrac{1-\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}} < (\sigma_1+\beta)^2 + \tfrac{\sigma_2^2}{1+\sqrt{\varepsilon}} < 1 \,\},\\ \mathcal{S}^C_{\varepsilon,-\varepsilon} &= \{\, \sigma \in \mathbb{R}^2 \;:\; \exists\, \beta \in \{0,2\} \text{ such that } 1 < (\sigma_1+\beta)^2 + \tfrac{\sigma_2^2}{1+\sqrt{\varepsilon}} < \tfrac{1+\sqrt{\varepsilon}}{1-\sqrt{\varepsilon}} \,\}.\end{aligned} \tag{3.13}$$

In both cases we have two annuli of radii close to 1 and thickness $2\sqrt{\varepsilon} + \mathcal{O}(\varepsilon)$. For $\kappa = \sqrt{\varepsilon}$ the annuli touch each other in $\sigma^* = (1,0)$, while for $\kappa = -\sqrt{\varepsilon}$ they overlap such that σ^* remains an isolated point in the complement of $\mathcal{S}^C_{\varepsilon,-\varepsilon}$.

For later reference we consider the case $\varepsilon < 2\kappa^2$ such that the boundary of $\mathcal{S}^C_{\varepsilon,\kappa}$ for $\sigma \approx (2,1)$ has two branches $\sigma_2 = \Sigma_{+,-}(\varepsilon,\kappa,\sigma_1)$ with the expansions

$$\Sigma_{+,-}(\varepsilon,\kappa,\sigma_1) = \alpha_{+,-}(\varepsilon,\kappa) + \beta_{+,-}(\varepsilon,\kappa)(\sigma_1-2) + \mathcal{O}(|\sigma_1-2|^2), \tag{3.14}$$

where $\alpha_{+,-} = 1 \pm \frac{1}{2}\sqrt{2\kappa^2-\varepsilon} + \mathcal{O}(\varepsilon)$ and $\beta_{+,-} = \mp \frac{(\varepsilon-\kappa^2)^2}{32\sqrt{2\kappa^2-\varepsilon}} + \mathcal{O}(\varepsilon^2)$.

4. On the Set of Unstable Wave Vectors

We return to the real SHE and study the set $\mathcal{S}_{\varepsilon,\kappa}$ of the unstable wave vectors associated to the roll $U_{\varepsilon,\kappa}$. In showing $\mathcal{S}_{\varepsilon,\kappa} = \emptyset$ we prove Theorem 3.2. The linearization of (3.1) around the roll solution $U_{\varepsilon,\kappa}$ given in (3.3) defines the full operator

$$\begin{aligned}&\widehat{B}_{\varepsilon,\kappa}(\partial_\xi) : H^4(\mathbb{R}^2) \subset L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)\\ &\widehat{B}_{\varepsilon,\kappa}(\partial_\xi)v = -(1+k^2\partial_\xi^2+\partial_{x_2}^2)^2 v + \Big(\varepsilon - 3U_{\varepsilon,\kappa}(\xi)^2\Big)v.\end{aligned}$$

Of course we can also consider the operator $\widetilde{B}_{\varepsilon,\kappa} : H^4_{lu}(\mathbb{R}^2) \subset L^2_{lu}(R^2) \to L^2_{lu}(R^2)$, which is defined by the same formula.

The basic state $U_{\varepsilon,\kappa}$ is 2π–periodic, however the coefficient $\varepsilon - 3U^2_{\varepsilon,\kappa}$ is π–periodic in ξ, since $U_{\varepsilon,\kappa}(\xi+\pi) = -U_{\varepsilon,\kappa}(\xi)$. Hence, it is advantageous to work with the lattice group $\mathcal{L} = \pi\mathbb{Z}$ rather than with $\widetilde{\mathcal{L}} = 2\pi\mathbb{Z}$, which is the translation group of $U_{\varepsilon,\kappa}$. We apply the abstract theory of Sect. 2 (using the coordinates (ξ,x_2)) with $d=2$, $\widetilde{d}=1$, $\mathcal{Q} = \mathbb{R}^2$, $\mathcal{L} = \pi\mathbb{Z}\times\mathbb{R}$, $\mathcal{L}^* = 2\mathbb{Z}\times\{0\}$, $\mathcal{T} = \mathbb{R}^2/_{\mathcal{L}} = \mathcal{T}_\pi \times \{0\}$, and $\mathcal{T}^* = \mathbb{R}^2/_{\mathcal{L}^*} = \mathcal{T}_2\times\mathbb{R}$.

The Bloch operator family is given by

$$\begin{aligned}&B(\varepsilon,\kappa,\sigma) : H^4(\mathcal{T}_\pi) \subset L^2(\mathcal{T}_\pi) \to L^2(\mathcal{T}_\pi),\\ &B(\varepsilon,\kappa,\sigma)V = -(1+k^2(\partial_\xi+i\sigma_1)^2-\sigma_2^2)^2 V + \Big(\varepsilon - 3U^2_{\varepsilon,\kappa}(\xi)\Big)V,\end{aligned}$$

where $(\varepsilon,\kappa,\sigma) \in \mathbb{R}^4$. Here B is even in σ_2, and the operator $B(\varepsilon,\kappa,\sigma_1+m,\sigma_2)$, $m \in 2\mathbb{Z}$, is unitary equivalent to $B(\varepsilon,\kappa,\sigma)$, since it is connected to $B(\varepsilon,\kappa,\sigma)$ by the transformation $V(\xi) \mapsto e^{im\xi}V(\xi)$. Moreover, there are two reflection symmetries given by

$$(R_1V)(\xi) = V(-\xi) \quad \text{and } (R_2V)(\xi) = \overline{V}(\xi). \tag{4.1}$$

In both cases we have $R_j^{-1} = R_j$ and $B(\varepsilon,\kappa,\sigma) = R_jB(\varepsilon,\kappa,(-\sigma_1,\sigma_2))R_j$. Hence, it is sufficient to study the case $\sigma \in [0,1]\times[0,\infty)$ which is only one quarter of $\mathcal{T}^* = \mathcal{T}_2\times\mathbb{R}$.

All the Bloch operators $B(\varepsilon,\kappa,\sigma)$ are selfadjoint, which is helpful but not essential for our theory. We strongly use the fact that the operators are small perturbations of $B^\kappa(\sigma) \overset{\text{def}}{=} B(\kappa^2,\kappa,\sigma)$ which is trivially analyzed as it has constant coefficients: $B^\kappa(\sigma)\phi_m = (\mu_m(\kappa,\sigma)+\kappa^2)\phi_m$ with $\phi_m(\xi) = e^{im\xi}$, $m \in 2\mathbb{Z}$, and

$$\mu_m(\kappa,\sigma) = -(1-(1+\kappa)(m+\sigma_1)^2-\sigma_2^2)^2. \tag{4.2}$$

Thus, for $\sigma_1 \in [0,1]$ we have the explicit upper bound

$$\int_0^\pi (B^\kappa(\sigma)V)\,\overline{V}\,d\xi \le -\beta(\kappa,\sigma)\|V\|^2_{L^2(\mathcal{T}_\pi)},$$

where $\beta(\kappa,\sigma) = \min\{(1-(1+\kappa)\sigma_1^2-\sigma_2^2)^2, (1-(1+\kappa)(2-\sigma_1)^2-\sigma_2^2)^2\} - \kappa^2$. Since $\kappa \approx 0$ we immediately identify the dangerous set

$$\mathcal{S}_0 = \{\,\sigma \in \mathcal{T}^* = \mathcal{T}_2 \times \mathbb{R} \;:\; \sigma_1^2+\sigma_2^2 = 1 \text{ or } (\sigma_1-2)^2+\sigma_2^2 = 1\,\}.$$

Hence, if σ is bounded away from $\mathcal{S}_0$, we obtain a good bound on the spectrum of B. Choosing a small $\delta \in (0,1]$ independent of $(\varepsilon,\kappa,\sigma)$ we define the set of good wave vectors as

$$G_\delta = \{\,\sigma \in [0,1]\times[0,\infty) \;:\; \text{dist}(\sigma,\mathcal{S}_0) \ge \delta\,\}.$$

For $\sigma \in G_\delta$ we have $\beta(\sigma,\kappa) \ge \delta^2/2$ for all sufficiently small κ. For general small $\varepsilon \ge \kappa^2$ we have $\|B(\varepsilon,\kappa,\sigma)-B^\kappa(\sigma)\|_{L^2\to L^2} = \|\varepsilon-\kappa^2-3U^2_{\varepsilon,\kappa}\|_\infty \le \widetilde{a}^2$ for sufficiently small ε_0. Thus, for $\sigma \in G_\delta$ we derive the estimate

$$\int_0^\pi (B(\varepsilon,\kappa,\sigma)V)\,\overline{V}\,d\xi \le -(\delta^2/2-\widetilde{a}^2)\|V\|^2_{L^2(\mathcal{T}_\pi)} \le -(\delta^2/2-2\varepsilon)\|V\|^2_{L^2(\mathcal{T}_\pi)}.$$

This shows that we may choose the width δ of the good set G_δ to be of order $\sqrt{\varepsilon}$, e.g., $\delta = 3\sqrt{\varepsilon}$. However, for our purposes it suffices to fix a small δ independent of ε.

It remains to study $B(\varepsilon,\kappa,\sigma)$ in the dangerous parts close to the circle $\sigma_1^2+\sigma_2^2 = 1$. To this end we distinguish the two regions

$$C_1 = \{\,\sigma \in [0,1]\times[0,2] \;:\; \text{dist}(\sigma,\mathcal{S}_0) \le \delta, \text{ and } \sigma_2 \ge \sqrt{\delta}\,\},$$
$$C_2 = \{\,\sigma \in [1-2\delta,1]\times[0,\sqrt{\delta}] \;:\; \text{dist}(\sigma,\mathcal{S}_0) \le \delta\,\}.$$

The operator $B^\kappa(\sigma)$ has only one small eigenvalue for $\sigma \in C_1$, while for $\sigma \in C_2$ there are two small eigenvalues. It suffices to control the movement of these small eigenvalues only, since all other eigenvalues are bounded away from the imaginary axis.

Region C_1. For $\sigma \in C_1$ the eigenfunction $\phi_0(\xi) \equiv 1$ is the only eigenfunction for $B^\kappa(\sigma)$ associated to a small eigenvalue, namely $\lambda_0 = -(1-k^2\sigma_1^2-\sigma_2^2)^2+\kappa^2$. The associated eigenvalue of $B(\varepsilon,\kappa,\sigma)$ is constructed by Liapunov–Schmidt reduction of the eigenvalue problem $BV-\lambda V = 0$. To this end we define $P_1V = \frac{1}{\pi}\int_0^\pi V\phi_0 d\xi\,\phi_0$ which is the orthogonal projection in $L^2(\mathcal{T}_\pi)$ onto span$\{\phi_0\}$, and write the eigenvalue problem as

$$P_1[B(\varepsilon,\kappa,\sigma)-\lambda I][\alpha_0\phi_0+V_2] = 0,$$
$$(I-P_1)[B(\varepsilon,\kappa,\sigma)-\lambda I][\alpha_0\phi_0+V_2] = 0, \quad \text{where } P_1V_2 = 0.$$

Since $B^\kappa(\sigma)$ is invertible on $(I-P_1)L^2(\mathcal{T}_\pi)$, the second equation can be solved for $V_2 = \mathcal{V}(\varepsilon,\kappa,\sigma,\lambda)\alpha_0$ yielding the expansion

$$\mathcal{V}(\varepsilon,\kappa,\sigma,\lambda) = \tfrac{3\widetilde{a}^2}{4}\Big(\tfrac{1}{\mu_2+\varepsilon-\lambda}\phi_2(\xi) + \tfrac{1}{\mu_{-2}+\varepsilon-\lambda}\phi_{-2}(\xi)\Big) + \mathcal{O}(\widetilde{a}^4),$$

where μ_m is defined in (4.2) and the error term $\mathcal{O}(\widetilde{a}^4)$ is uniform in bounded sets for (σ, λ), e.g., $|\sigma| \le 3$ and $|\lambda| \le 1$. Inserting the result in the first equation we obtain the reduced spectral problem $b_0(\varepsilon, \kappa, \sigma, \lambda)\alpha_0\phi_0 = 0$ with

$$\begin{aligned} b_0(\varepsilon, \kappa, \sigma, \lambda) &= \mu_0 + \varepsilon - \lambda - \tfrac{3}{\pi} \int_0^\pi U_{\varepsilon,\kappa}^2 (\phi_0 + \mathcal{V})\, \phi_0 d\xi \\ &= \mu_0 + \varepsilon - \lambda - \tfrac{3a_1^2}{2} - \tfrac{9\widetilde{a}^4}{16}\left(\tfrac{1}{\mu_2+\varepsilon-\lambda} + \tfrac{1}{\mu_{-2}+\varepsilon-\lambda}\right) + \mathcal{O}(\widetilde{a}^6). \end{aligned}$$

The small eigenvalue λ is determined by solving $b_0(\varepsilon, \kappa, \sigma, \lambda) = 0$. To discuss the sign of λ it is convenient to use polar coordinates

$$\sigma = (\sigma_1, \sigma_2) = \sqrt{1 + r}\ \left(\tfrac{1}{k} \sin\gamma, \cos\gamma\right),$$

where the region C_1 corresponds to $\gamma \in [0, \pi/2 - \sqrt{\delta}]$ and $|r| \le \delta$. We obtain

$$\lambda = \lambda_0(\varepsilon, \kappa, r, \gamma) = \varepsilon - \frac{3}{2}a_1^2 - r^2 + \frac{9}{128}\widetilde{a}^4 \frac{1 + \sin^2\gamma}{\cos^4\gamma} + \mathcal{O}(\widetilde{a}^4(\widetilde{a} + |r|)). \tag{4.3}$$

For $\varepsilon > 2\kappa^2 + \mathcal{O}(|\kappa|^3)$ we always have $\lambda \le 0$, while for smaller ε there is a band of unstable σ of width $\mathcal{O}(\sqrt{\varepsilon})$ around the circle $|\sigma| = 1$.

Region C_2. We are now in the situation of $\sigma \approx \sigma^* = (1, 0)$, where $B^\kappa(\sigma)$ has the critical eigenfunctions $\phi_0 \equiv 1$ and $\phi_{-2}(\xi) = e^{-i2\xi}$. In fact, this is the realm of classical sideband instability as discussed in [Mie95]. There, the analysis was done in a space of functions which are 2π–periodic in ξ such that our region C_2 corresponds to $\sigma \approx 0$ there (as σ_1 is taken modulo 1). There the instability result of Theorem 3.2 was already derived, yet for our stability proof we have to repeat and improve upon these calculations.

To be compatible with the calculations in [Mie95] we use the basis functions $U_1(\xi) = e^{-i\xi}\cos\xi$ and $U_2(\xi) = e^{-i\xi}\sin\xi$ and set $\widehat{\sigma} = \sigma - \sigma^* = (\sigma_1 - 1, \sigma_2)$. Letting $P_2 V = \frac{2}{\pi}\int_0^\pi V\overline{U}_1 d\xi U_1 + \frac{2}{\pi}\int_0^\pi V\overline{U}_2 d\xi U_2$ and $V = \beta_1 U_1 + \beta_2 U_2 + V_1$ with $P_2 V_1 = 0$, the equation $(I - P_2)[B(\ldots) - \lambda I]V = 0$ can be solved uniquely for $V_1 = \mathcal{V}(\varepsilon, \kappa, \sigma, \lambda)\beta = \mathcal{O}(\widetilde{a}^2|\beta|)$, for all sufficiently small $(\varepsilon, \kappa, \widehat{\sigma}, \lambda)$. Again the estimate follows easily from the fact that the coupling only occurs through the term $-3U_{\varepsilon,\kappa}^2 V$.

Inserting this expansion into $P_2[B(\ldots) - \lambda I]V = 0$ leads to the reduced eigenvalue problem. It is given by a 2×2–matrix, which depends nonlinearly on λ:

$$m(\varepsilon, \kappa, \sigma, \lambda)\beta = \mathcal{P}[B(\varepsilon, \kappa, \sigma) - \lambda I][\beta_1 U_1 + \beta_2 U_2 + \mathcal{V}(\varepsilon, \kappa, \sigma, \lambda)\beta],$$

where $\mathcal{P} : V \mapsto \frac{2}{\pi}(\int_0^\pi V\,\overline{U}_1 d\xi, \int_0^\pi V\,\overline{U}_2 d\xi) \in \mathbb{C}^2$. This gives

$$m(\varepsilon, \kappa, \sigma, \lambda) = \begin{pmatrix} \rho + c(\varepsilon, \kappa) - \lambda & i\nu \\ -i\nu & \rho - \lambda \end{pmatrix} + \widetilde{a}^4 \begin{pmatrix} \mathcal{O}(|\widehat{\sigma}|^2 + |\lambda|) & \mathcal{O}(|\widehat{\sigma}_1|) \\ \mathcal{O}(|\widehat{\sigma}_1|) & \mathcal{O}(|\widehat{\sigma}|^2 + |\lambda|) \end{pmatrix}, \tag{4.4}$$

with $\rho = (\mu_1 + \mu_{-1})/2$ and $\nu = (\mu_1 - \mu_{-1})/2$ from (3.10) and $c(\varepsilon, \kappa) = -3\widetilde{a}^2/2 + \mathcal{O}(\widetilde{a}^4)$.

Of course, m is Hermitian and each entry is even in σ_2. Two additional facts in this expansion are nontrivial. Firstly, the symmetries R_1 and R_2 in (4.1) show that the diagonal elements are even in $\widehat{\sigma}_1 = \sigma_1 - 1$ while $m_{12} = -\overline{m}_{21}$ is odd in $\widehat{\sigma}_1$. Secondly, $m(\varepsilon, \kappa, \sigma^*, 0)$ takes the form $\left(\begin{smallmatrix} c(\varepsilon,\kappa) & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where the lower diagonal element vanishes as it corresponds to the eigenvalue $\lambda = 0$ associated to the translational mode $\partial_\xi U_{\varepsilon,\kappa} = -\widetilde{a}\sin\xi + \mathcal{O}(\widetilde{a}^3)$ (compare to Lemma 5.3 in [Mi97b]).

In fact, we need a more refined expansion which follows from determining the term of order $\widetilde{a}^2$ in $\mathcal{V}$:

$$\begin{aligned} m_{11}(\varepsilon,\kappa,\sigma,\lambda) &= \rho + \varepsilon - \lambda - \tfrac{9}{4}a_1^2 - \tfrac{3}{2}a_1 a_3 + \eta_+ + \mathcal{O}(\widetilde{a}^6),\\ m_{12}(\varepsilon,\kappa,\sigma,\lambda) &= -m_{21}(\varepsilon,\kappa,\sigma,\lambda) = i\big[\nu + \eta_- + \mathcal{O}(\widetilde{a}^6)\big],\\ m_{22}(\varepsilon,\kappa,\sigma,\lambda) &= \rho + \varepsilon - \lambda - \tfrac{3}{4}a_1^2 + \tfrac{3}{2}a_1 a_3 + \eta_+ + \mathcal{O}(\widetilde{a}^6), \end{aligned} \tag{4.5}$$

where $\eta_\pm = \frac{9}{32}\widetilde{a}^4\big[(\mu_2 + \varepsilon - \lambda)^{-1} \pm (\mu_{-4} + \varepsilon - \lambda)^{-1}\big]$.

In order to study which wave vectors are stable we have to find λ from

$$\Lambda(\varepsilon,\kappa,\sigma,\lambda) = \det m(\varepsilon,\kappa,\sigma,\lambda) = 0.$$

Applying Weierstraß' preparation theorem (see [ChH82], Ch. 2.6) we have

$$\Lambda(\varepsilon,\kappa,\sigma,\lambda) = \Lambda_0(\varepsilon,\kappa,\sigma,\lambda)\big[\lambda^2 + n_1(\varepsilon,\kappa,\sigma)\lambda + n_0(\varepsilon,\kappa,\sigma)\big],$$

where Λ_0, n_1, and n_0 are analytical functions with $\Lambda_0(0,0,0,0) = 1$ and

$$n_1 = -2\rho - c + \mathcal{O}(\varepsilon^2|\widehat{\sigma}|^2) \quad \text{and } n_0 = \rho(\rho + c) - \nu^2 + \mathcal{O}\big(\varepsilon^2|\widehat{\sigma}|^2(\sqrt{\varepsilon} + |\widehat{\sigma}|^2)]\big).$$

On the one hand, we know that m is Hermitian implying that both eigenvalues are real. Hence, without calculation we always have $n_1^2 \geq 4n_0$. On the other hand, these two eigenvalues have negative real part if and only if $n_0 \geq 0$ and $n_1 \geq 0$. Since $n_1(0,0,\sigma^*) > 0$ and since n_1 can only change sign when $n_0 < 0$, we conclude that it suffices to consider the condition $n_0 \geq 0$ when we are only interested in the question whether $U_{\varepsilon,\kappa}$ is stable or not. However, for the subsequent calculation of $\mathcal{S}_{\varepsilon,\kappa}$ we need to consider both conditions $n_1 \geq 0$ and $n_0 \geq 0$. Since n_0 is a positive multiple of the determinant of $m(\varepsilon,\kappa,\sigma,0)$ the stability condition is $\det m(\varepsilon,\kappa,\sigma,0) \geq 0$ for all $\sigma \in C_1 \cup C_2$.

From (4.5) we obtain the following expansion for $\sigma \approx \sigma^*$.

Lemma 4.1. *Let $s = (\sigma_1 - 1)^2$, $t = \sigma_2^2$, and $M(\varepsilon,\kappa,\sigma) = \det m(\varepsilon,\kappa,\sigma,0)$. Then, we have*

$$M = \mu_{0,1}t + \mu_{1,0}s + \mu_{0,2}t^2 + \mu_{1,1}st + \mu_{0,3}t^3 + \mu_{2,0}s^2 + \mu_{1,2}st^2 + \mu_{0,4}t^4 + \mathcal{O}((s+t^2)^{5/2}),$$

where

$$\begin{aligned} \mu_{0,1}(\varepsilon,\kappa) &= c(\varepsilon,\kappa)\big[-2\kappa - (\varepsilon - \kappa^2)^2/256 + \mathcal{O}(\varepsilon^{5/2})\big],\\ \mu_{1,0}(\varepsilon,\kappa) &= 8(\varepsilon - 3\kappa^2) + 4\kappa(5\varepsilon - 13\kappa^2) + \mathcal{O}(\varepsilon^2),\\ \mu_{0,2}(\varepsilon,\kappa) &= 2(\varepsilon + \kappa^2) + \mathcal{O}(\varepsilon^2),\\ \mu_{1,1}(\varepsilon,\kappa) &= -16\kappa + 4(\varepsilon - 7\kappa^2) + \mathcal{O}(\varepsilon^{3/2}),\\ \mu_{0,3}(\varepsilon,\kappa) &= 4\kappa + (\varepsilon - \kappa^2)^2/128 + \mathcal{O}(\varepsilon^{5/2}),\\ \mu_{2,0}(\varepsilon,\kappa) &= 16 + 48\kappa + 2(\varepsilon + 25\kappa^2) + \mathcal{O}(\varepsilon^{3/2}),\\ \mu_{1,2}(\varepsilon,\kappa) &= -8 - 4\kappa + 4\kappa^2 + \mathcal{O}(\varepsilon^{3/2}),\\ \mu_{0,4}(\varepsilon,\kappa) &= 1 + \mathcal{O}(\varepsilon^{3/2}). \end{aligned} \tag{4.6}$$

This expansion is suitable to discuss the set of unstable wave vectors in region $C_2 \subset [1 - 2\delta, 1] \times [0, \sqrt{\delta}]$. While instability is easily obtained from the signs of $\mu_{1,0}$ and $\mu_{0,1}$, it will be a rather delicate task to prove the stability result.

Proof of Theorem 3.2. The expansion $\det m = \mu_{0,1}t + \mu_{1,0}s + \mathcal{O}(s^2 + t^2)$ immediately leads to instability if either $\mu_{0,1}$ or $\mu_{1,0}$ in (4.6) is negative. Thus, defining K_Z and E_E

via $\mu_{1,0}(E_E(\kappa), k) = 0$ and $\mu_{0,1}(\varepsilon, K_Z(\varepsilon)) = 0$, the instability result follows by choosing suitably small σ_1 or σ_2, respectively.

To establish stability we need to conclude $\mathcal{S}_{\varepsilon,\kappa} = \emptyset$ for those (ε, κ) satisfying (3.4). In that region we have $\varepsilon \le 2\widetilde{a}^2 \le 3\varepsilon$ such that the error terms in (4.6) can be expressed in terms of powers of $\widetilde{a}$. At first we note that the intersection of $\mathcal{S}_{\varepsilon,\kappa}$ with C_1 is empty since $\varepsilon \ge E_E(\kappa) = 3\kappa^2 + \mathcal{O}(|\kappa|^3)$ clearly implies $\varepsilon \ge E_1(\kappa) = 2\kappa^2 + \mathcal{O}(|\kappa|^3)$. Thus, it remains to consider region C_2. As argued before of Lemma 4.1 it suffices to show that $\det m(\varepsilon, \kappa, \sigma, 0) \ge 0$ in a neighborhood of σ^* which is independent of (ε, κ).

Before employing Lemma 4.1 we recall the expansion (4.4) which gives

$$M(\varepsilon, \kappa, \sigma) = \rho(\rho + c) - \nu^2 + \mathcal{O}\big(\widetilde{a}^4[\widetilde{a}s + t^2 + s^2]\big). \tag{4.7}$$

We define $\widehat{M}(\varepsilon, \kappa, \sigma) = \sum_{2j+l\le 4} \mu_{j,l}(\varepsilon, \kappa)s^j t^l$ and find

$$R(\varepsilon, \kappa, \sigma) \stackrel{\text{def}}{=} M(\varepsilon, \kappa, \sigma) - \widehat{M}(\varepsilon, \kappa, \sigma) = \mathcal{O}(\widetilde{a}^4(s + t^2)^{5/2}).$$

Our aim is to use positivity of $\widehat{M}$ in order to show $2|R| \le \widehat{M}$ in a neighborhood of σ^*. This estimate is subtle, since $\widehat{M}$ is degenerate for $\varepsilon = 0$, namely $\widehat{M}(0, 0, \sigma) = (4s - t^2)^2$.

For estimating $\widehat{M}$ from below we use

$$\mu_{1,2} \ge -2\sqrt{\mu_{2,0}\mu_{0,4}} \quad \text{and} \quad -\mu_{1,1}\sqrt{\mu_{0,4}} \le \mu_{0,3}\sqrt{\mu_{2,0}}.$$

Both inequalities hold (after cancellation of the leading order terms) for sufficiently small (ε, κ) satisfying the stability criterion (3.4). Whence,

$$\widehat{M}(\varepsilon, \kappa, \sigma) \ge \mu_{0,1}t + \mu_{1,0}s + \mu^* t^2 + \Big(\sqrt{\mu_{0,4}}t^2 - \sqrt{\mu_{2,0}}s - \tfrac{\mu_{1,1}}{2\sqrt{\mu_{2,0}}}t\Big)^2,$$

where

$$\mu^*(\varepsilon, \kappa) = \mu_{0,2} - \mu_{1,1}^2/(4\mu_{2,0}) = 2(\varepsilon - \kappa^2) + \mathcal{O}(\widetilde{a}^3) \ge \widetilde{a}^2$$

for sufficiently small $\widetilde{a}$.

For all small $s, t \ge 0$ we obtain $\widehat{M} \ge \mu_{0,1}t + \mu_{1,0}s + \widetilde{a}^2 t^2$. Additionally, for $s \ge t$ we have $\sqrt{\mu_{2,0}}\, s - \sqrt{\mu_{0,4}}\, t^2 - \frac{\mu_{1,1}}{2\sqrt{\mu_{2,0}}}\, t \ge s$, implying $\widehat{M} \ge \mu_{0,1}t + \mu_{1,0}s + \widetilde{a}^2 t^2 + s^2$. Together with the previous estimate, this gives $\widehat{M}(\varepsilon, \kappa, \sigma) \ge \mu_{0,1}t + \mu_{1,0}s + \frac{\widetilde{a}^2}{2}(s^2 + t^2)$. Since $R = \mathcal{O}(\widetilde{a}^4|\widehat{\sigma}|^5)$ we conclude that for all small (ε, κ) which satisfy the stability criterion we have

$$M(\varepsilon, \kappa, \sigma) \ge \mu_{0,1}\sigma_1^2 + \mu_{1,0}\sigma_2^2 + \frac{\widetilde{a}^2}{4}|\widehat{\sigma}|^4 \quad \text{for all } \sigma \in C_2. \tag{4.8}$$

This proves Theorem 3.2. □

In the unstable case it is desirable to describe the set $\mathcal{S}_{\varepsilon,\kappa}$ of unstable wave vectors σ. The analyses for C_1 and C_2 provides us with a lot of information. To formulate the results correctly it is useful to consider the full set of wave vectors, namely $(\sigma_1, \sigma_2) \in \mathcal{T}^* = \mathcal{T}_2 \times \mathbb{R}$, where $\mathcal{T}_2 = \mathbb{R}/_{2\mathbb{Z}}$. Recall that the identification of σ_1 with $\sigma_1 + m$, where $m \in 2\mathbb{Z}$, is due to the fact that $B(\varepsilon, \kappa, (\sigma_1 + m, \sigma_2))$ is unitary equivalent to $B(\varepsilon, \kappa, \sigma)$. The critical set $\mathcal{S}_0 = \{\, \sigma \in [0, 2) \times \mathbb{R} \,:\, |\sigma| = 1 \text{ or } |(\sigma_1 - 2, \sigma_2)| = 1 \,\}$ considered as a set in the cylinder $\mathcal{T}^*$ consists of only *one* circle which is wrapped around the cylinder once, touching itself in σ^*.

We have the following results.

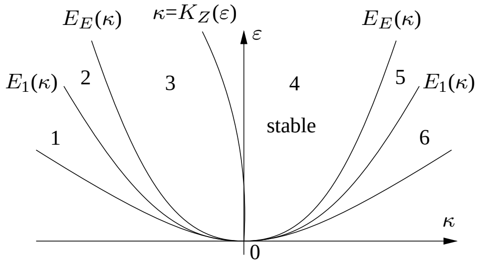

Fig. 4.3. The regions R_1 to R_6 for the real SHE

Theorem 4.2. *There exists an $\varepsilon_0 > 0$ such that for all (ε, κ) with $0 \le \varepsilon \le \varepsilon_0$ and $\kappa^2 \le \varepsilon$ the set $\mathcal{S}_{\varepsilon,\kappa} \subset \mathcal{T}^*$ of unstable wave vectors for the roll solution $u = U_{\varepsilon,\kappa}$ has the following properties.*

(a) $\sigma^* = (1, 0) \notin \mathcal{S}_{\varepsilon,\kappa}$, *and* $\sigma \in \mathcal{S}_{\varepsilon,\kappa}$ *implies* $(\sigma_1, -\sigma_2), (2 - \sigma_1, \sigma_2) \in \mathcal{S}_{\varepsilon,\kappa}$.

(b) *dist* $(\mathcal{S}_{\varepsilon,\kappa}, \mathcal{S}_0) = \mathcal{O}(\sqrt{\varepsilon})$ *for* $\varepsilon \to 0$.

(c) There is a curve $\varepsilon = E_1(\kappa)$ *with the expansion*

$$E_1(\kappa) = 2\kappa^2 + \tfrac{11}{96}\kappa^4 + \mathcal{O}(|\kappa|^5),$$

such that for $\varepsilon = E_1(\kappa)$ *the boundary of* $\mathcal{S}_{\varepsilon,\kappa}$ *has a pair of double points on the line* $\sigma_1 = 0$ *near* $\sigma_2 = \pm 1$.

(d) For $\widehat{\kappa} \in (-1/\sqrt{2}, 1/\sqrt{2})$ *there exists a constant C such that the estimate*

$$\mathcal{S}_{\varepsilon,\widehat{\kappa}\sqrt{\varepsilon}} \subset [-C\sqrt{\varepsilon}, C\sqrt{\varepsilon}] \times [-C\varepsilon^{1/4}, C\varepsilon^{1/4}]$$

holds.

Part (c) is obtained by studying the behavior of λ in region C_1, where formula (4.3) holds. The curve E_1 is obtained by solving $\lambda_0(\varepsilon, \kappa, r, 0) = 0$ and $\partial_r \lambda_0(\varepsilon, \kappa, r, 0) = 0$, that is, we search for a double zero in $\sigma_2 = \sqrt{1 + r}$ on the symmetry line $\sigma_1 = 0$ ($\gamma = 0$).

The curves $\varepsilon = E_E(\kappa)$, $\varepsilon = E_1(\kappa)$, and $\kappa = K_Z(\varepsilon)$ divide the region $\varepsilon \ge \kappa^2$ into six regions R_1 to R_6, see Fig. 4.1. In each of these regions the shape of the set of unstable wave vectors $\mathcal{S}_{\varepsilon,\kappa}$ can be derived from the above analysis. The curves separating R_j and R_{j+1} stand for a topological change in the structure of $\mathcal{S}_{\varepsilon,k}$.

In R_4 the rolls $U_{\varepsilon,\kappa}$ are stable, i.e., $\mathcal{S}_{\varepsilon,\kappa} = \emptyset$. In R_3 and R_5 the set $\mathcal{S}_{\varepsilon,\kappa}$ consists of two simply connected components such that the boundary is a figure 8 with the double point in σ^*. In R_2 the set $\mathcal{S}_{\varepsilon,\kappa}$ is homeomorphic to a pointed disc, namely a disc–shaped region where the interior point σ^* is taken away. Schematic drawings of the boundary of $\mathcal{S}_{\varepsilon,k}$ are given in Fig. 4.2 for each of the regions R_j.

We now mention a few differences between the stability analyses for the rolls $U_{\varepsilon,\kappa}$ in the real SHE (3.1) and the rolls $A_{\varepsilon,\kappa}$ for the complex SHE (3.5). For this purpose we define the factorization mapping

$$J : \begin{cases} \mathbb{R}^2 & \to \mathcal{T}^* = \mathcal{T}_2 \times \mathbb{R}, \\ (\sigma_1, \sigma_2) & \mapsto (\sigma_1 \bmod 2, \sigma_2). \end{cases}$$

Thus, we can compare $\mathcal{S}_{\varepsilon,\kappa}$ with $\mathcal{S}^A_{\varepsilon,\kappa} = J\mathcal{S}^C_{\varepsilon,\kappa}$, which means that we have to interpret the results from Sect. 3 taking σ_1 modulo 2.

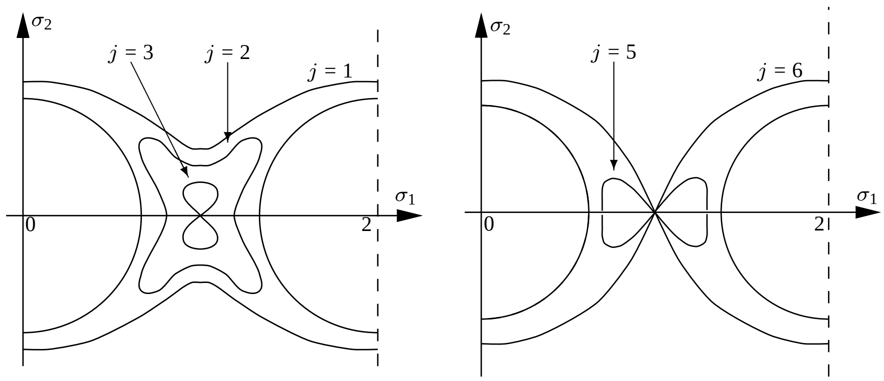

Fig. 4.4. The set $\mathcal{S}_{\varepsilon,\kappa}$ of unstable wave vectors for $U_{\varepsilon,\kappa}$ with $(\varepsilon,\kappa)\in R_j$.

As a first result we find, due to (4.7) and $\widetilde{\alpha}=0$, that $\mathcal{S}_{\varepsilon,\pm\sqrt{\varepsilon}}=\mathcal{S}^A_{\varepsilon,\pm\sqrt{\varepsilon}}$ which is explicitly given in (3.13). However, the number of unstable modes in the complex case is twice the number in the real case, e.g., for $\sigma\approx\sigma^*$ the complex case has the four unstable modes $\binom{1}{0}$, $\binom{0}{1}$, $\binom{1}{i}e^{2i\xi}$, and $\binom{1}{-i}e^{-2i\xi}$, whereas the real case has only two unstable modes, namely ϕ_0 and ϕ_2.

We easily find the counterpart of the curve E_1. It means that the topological structure of $\mathcal{S}^A_{\varepsilon,\kappa}$ changes since the boundary meets itself at points near $(0,\pm1)$. For the complex SHE this occurs when the boundary of $\mathcal{S}^C_{\varepsilon,\kappa}$ touches $\sigma_1=2$. Using $\widetilde{s}=(1+\kappa)(\sigma_1-1)^2$ we simply have to insert $\widetilde{s}=1+\kappa$ into (3.12) (ii) (with <0 replaced by $=0$) and solve for a double zero in μ, giving $E^A_1(\kappa)=2\kappa^2+\frac{\kappa^4}{16}-\frac{\kappa^5}{8}+\mathcal{O}(|\kappa|^6)$. The difference between the real and the complex SHE is that the boundary of $\mathcal{S}^A_{E^A_1(\kappa),\kappa}$ touches itself on the line $\sigma_1=0$ whereas the boundary of $\mathcal{S}_{E_1(\kappa),\kappa}$ has a double point. Moreover, for smaller ε the boundary of $\mathcal{S}_{\varepsilon,\kappa}$ is smooth close to $\sigma=(0,\pm1)$, whereas in the complex case the boundary of $\mathcal{S}^A_{\varepsilon,\kappa}$ has corners on the line $\sigma_1=0$ which follows from the expansions (3.14) where $\beta_{+,-}$ are nonzero.

The shape of $\mathcal{S}_{\varepsilon,\kappa}$ inside the region C_2 is in fact similar to $\mathcal{S}^A_{\varepsilon,\kappa}$. This follows from (4.7) and the scaling $(\varepsilon,\kappa,\sigma)=(a^4,a^2\widehat{\kappa},(a^2\widehat{\sigma}_1,a\widehat{\sigma}_2))$ giving

$$M(a^4,a^2\widehat{\kappa},(a^2\widehat{\sigma}_1,a\widehat{\sigma}_2))=a^8\widehat{\Lambda}(\widehat{\kappa},\widehat{\sigma})+\mathcal{O}(a^{10}).$$

Because of (4.7) the limit function $\widehat{\Lambda}$ is the same for the real and the complex SHE and thus determines for each $\widehat{\kappa}\in[-1,1]$ the shape of $\mathcal{S}_{a^4,a^2\widehat{\kappa}}$ and $\mathcal{S}^C_{a^4,a^2\widehat{\kappa}}$ to lowest order.

Remark. Instead of working in the space $L^2(\mathcal{T}_\pi)$ we could also have used $L^2(\mathcal{T}_{2\pi})$ by ignoring the difference in the minimal periods of $U_{\varepsilon,\kappa}$ and $U^2_{\varepsilon,\kappa}$. We would encounter a completely similar analysis with wave vectors σ lying in $\widetilde{\mathcal{T}}^*=\mathcal{T}_1\times\mathbb{R}$. In fact, the above results can easily be transferred to that case by using the mapping $J_2:\mathcal{T}^*\to\widetilde{\mathcal{T}}^*;\sigma\mapsto(\sigma_1 \bmod 1,\sigma_2)$. The critical set $\widetilde{\mathcal{S}}_0=J_2\mathcal{S}_0$ is still one circle, but now it is wrapped around the cylinder twice such that additional intersections at $\sigma=(1/2,\pm\sqrt{3}/2)$ appear. Thus, the sets $\widetilde{\mathcal{S}}_{\varepsilon,\kappa}=J_2\mathcal{S}_{\varepsilon,\kappa}$ will undergo an additional topological change along a curve $\varepsilon=$

$E_2(\kappa)$ which lies slightly above the curve $\varepsilon = E_1(\kappa)$. When the boundary of $\mathcal{S}_{\varepsilon,\kappa}$ touches the lines $|\sigma_1 - 1| = 1/2$ this corresponds to a touching of the boundary of $\widetilde{\mathcal{S}}_{\varepsilon,\kappa}$ with itself. Employing (4.3) with $\gamma = \pi/6$ yields the expansion $E_2(\kappa) = 2\kappa^2 + 77\kappa^4/288 + \mathcal{O}(|\kappa|^5)$.

A. Elliptic Operators

Let A be a differential operator of order $2m$ given in the form

$$(A(\partial_x)u)(x,z) = \sum_{|p|+|q|\le 2m} a_{p,q}(x,z)(D_x^p D_z^q u)(x,z) \quad \text{for } (x,z) \in \mathcal{Q} \tag{A.1}$$

together with m–dimensional boundary operator $\mathcal{B} = (B_1, \ldots, B_m)$ with

$$(B_l u)(x,z) = \sum_{|q|\le 2m-1} b_{l,q}(x,z)(D_z^q u)(x,z), \quad \text{for } l = 1, \ldots, m, \tag{A.2}$$

where $(x,z) \in \partial\mathcal{Q} = \mathbb{R}^d \times \partial\Sigma$. We assume that $\partial\Sigma$ is of class C^{2m} and that A is uniformly strongly elliptic on $\mathcal{Q}$ and $(A, \mathcal{B})$ satisfies the complementing condition for each $(x,z) \in \partial\mathcal{Q}$, see [ReR92], Ch. 8.4, for the definitions. For simplicity, we do not allow for tangential derivatives in the boundary operators B_l. Additionally, we assume that all coefficient matrices $a_{p,q}(x,z),\ b_{p,q}(x,z) \in \mathbb{R}^{n\times n}$ are bounded together with their first $2m$ derivatives. (Our main interest lies in the case of periodic coefficients, where uniformity and boundedness are trivial.)

We define the L^2–based operator $\widehat{A} : D(\widehat{A}) \subset L^2(\mathcal{Q}) \to L^2(\mathcal{Q})$ via

$$D(\widehat{A}) \stackrel{\text{def}}{=} \{\, u \in H^{2m}(\mathcal{Q}) \ : \ \mathcal{B}u = 0 \text{ on } \partial\mathcal{Q} \,\}, \quad \widehat{A}u = A(\partial_x)u, \tag{A.3}$$

and similarly the L^2_{lu}–based operator $\widetilde{A} : D(\widetilde{A}) \subset L^2_{lu}(\mathcal{Q}) \to L^2_{lu}(\mathcal{Q})$ via

$$D(\widetilde{A}) \stackrel{\text{def}}{=} \{\, u \in H^{2m}_{lu}(\mathcal{Q}) \ : \ \mathcal{B}u = 0 \text{ on } \partial\mathcal{Q} \,\}, \quad \widetilde{A}u = A(\partial_x)u. \tag{A.4}$$

We simply write $A : D(A) \subset X \to X$ in order to denote both cases simultaneously. Moreover, X^k denotes $H^k(\mathcal{Q})$ or $H^k_{lu}(\mathcal{Q})$, respectively. The associated norms are written as $\|u\|_k$ and $\|u\|_{k,lu}$, where the subscript $k = 0$ is dropped. The general theory of elliptic operators (see [ReR92], Thm. 8.31) provides the a–priori regularity estimate

$$\|u\|_{X^{2m}} \le C(\|Au\|_X + \|u\|_X) \quad \text{for all } u \in D(A), \tag{A.5}$$

where C is independent of u.

In order to relate the cases $u \in L^2(\mathcal{Q})$ and $u \in L^2_{lu}(\mathcal{Q})$ with each other, we use the weight function $w(x) = \cosh(|x|)$ on $\mathbb{R}^d$ and the scalar α to define the operators

$$A_{\alpha,y}u = w(\cdot - y)^{-\alpha} A(\partial_x)[w(\cdot - y)^\alpha u] = A(\partial_x + \alpha \tfrac{\tanh(|x-y|)}{|x-y|}(x-y))u\,.$$

Mostly we omit the index y. Applying this transformation to the boundary operators has no effect; hence for all $\alpha \in \mathbb{R}$ we obtain elliptic opertors $A_\alpha : D(A) \subset X \to X$. Moreover, there is a constant C such that for all $\alpha \in [-1,1]$ and $y \in \mathbb{R}^d$ we have the estimate

$$\|(A_{\alpha,y} - A)u\|_X \le C_1|\alpha|\, \|u\|_{X^{2m-1}} \quad \text{for all } u \in D(A). \tag{A.6}$$

Thus, if A is invertible from X into $D(A) \subset X^{2m}$, then for sufficiently small $|\alpha|$ the operator A_α is also invertible and satisfies $A_\alpha^{-1} f = w^{-\alpha} A^{-1}[w^\alpha f]$. This follows by combining (A.5) and (A.6).

The weight w allows us to go from $L^2_{lu}(\mathcal{Q})$ to $L^2(\mathcal{Q})$ or vice versa via $u \mapsto w^{-\alpha} u$ by using the following simple characterizations.

Lemma A.1. *Let $\alpha > 0$ and w as above.*

(a) Let $w^{-\alpha} u \in L^2(\mathcal{Q})$. Then $u \in L^2_{lu}(\Omega)$ if and only if there exists a $C > 0$ such that $\|w(\cdot - y)^{-\alpha} u\| \le C$ for all $y \in \mathbb{R}^d$.

(b) There is a constant C_α such that for all $u \in L^2_{lu}(\mathcal{Q})$

$$\|u\|_{lu} \le C_\alpha \sup\{\, \|w(\cdot - y)^{-\alpha} u\| \ :\ y \in \mathbb{R}^d \,\} \le C_\alpha^2 \|u\|_{lu}.$$

(c) A function $u \in L^2_{lu}(\mathcal{Q})$ lies in $L^2(\mathcal{Q})$ if and only if $\sum_{n \in \mathbb{Z}^d} \|\chi_n u\|^2_{lu} < \infty$, where the partition of unity χ_n, $n \in \mathbb{Z}^d$, is given by $\chi_n(x) = 1$ for $x \in [n, n + \eta)$ and 0 otherwise (here $\eta = (1, \ldots, 1) \in \mathbb{Z}^d$).

For a proof we refer to Lemma C.1 in [Mi97a]. We now obtain the first main result.

Theorem A.2. *Let the elliptic operator $A(\partial_x)$ from* (A.1) *satisfy the assumptions from above and let $\widehat{A}$ and $\widetilde{A}$ be the operators defined in* (A.3) *and* (A.4)*. Then, $\widehat{A}$ is invertible if and only if $\widetilde{A}$ is invertible and moreover, spec($\widetilde{A}$) = spec($\widehat{A}$).*

Proof. We assume that $\widehat{A}$ is invertible. Thus we know that there exists an $\alpha > 0$ such that $A_{\alpha,y}$ is invertible from $L^2(\mathcal{Q})$ into $D(\widehat{A})$ for any $y \in \mathbb{R}^d$ with a bound C_2 not depending on y. For $f \in L^2_{lu}(\mathcal{Q})$ we know $w^{-\alpha} f \in L^2(\mathcal{Q})$ such that $u = \widetilde{A}^{-1} f = w^\alpha A_\alpha^{-1}[w^{-\alpha} f]$ is well defined. Using Lemma A.1b) we obtain $\|w^{-\alpha}(\cdot - y)u\|_{2m} \le C_2 \|w^{-\alpha}(\cdot - y) f\| \le C_2 C_\alpha \|f\|_{lu}$. Thus, we have proved finiteness of the norm and the operator $\widetilde{A}^{-1}$ maps $L^2_{lu}(\mathcal{Q})$ into $\widetilde{H}^{2m}_{lu}(\mathcal{Q})$.

We still have to establish the continuity of the translates $y \mapsto T_y u$. To this end we use that the coefficients of $A(\partial_x)$ are uniformly continuous and define the operators A^y which is obtained by using the translated coefficients $T_y a_{pq}$. Then, $\|(A^y - A)u\|_{lu} \le \gamma(|y|)\|u\|_{2m,lu}$, where $\gamma(t) \to 0$ for $t \to 0$. Obviously, $T_y u$ satisfies $\widetilde{A}^y T_y u = T_y f$ if and only $\widetilde{A} u = f$. Hence, applying $\widetilde{A}^{-1}$ to the equality $\widetilde{A}(u - T_y u) + (\widetilde{A} - \widetilde{A}^y) T_y u = f - T_y f$, we obtain the desired estimate $\|u - T_y u\|_{2m,lu} \le C[\gamma(|y|)\|u\|_{2m,lu} + \|f - T_y f\|_{lu}]$.

For the opposite direction we now assume that $\widetilde{A} : D(\widetilde{A}) \to L^2_{lu}(\mathcal{Q})$ is invertible. There is a $\alpha > 0$ such that $\widetilde{A}_{\alpha,y}$ is invertible with bound C_3 for any $y \in \mathbb{R}^d$. We use the partition χ_n as in Lemma A.1(c) and define, for any $f \in L^2(\mathcal{Q})$, the functions $f_n = \chi_n f$ and $u_n = \widetilde{A}^{-1} f_n = w^{-\alpha}(\cdot - n) A^{-1}_{-\alpha,n}[w(\cdot - n)^\alpha f_n]$. Thus, we obtain for each $n, m \in \mathbb{Z}^d$ the estimate

$$\begin{aligned}
\|\chi_m u_n\|^2 &= \int_{[m,m+\eta)} w(\cdot - n)^{-2\alpha} |A^{-1}_{-\alpha,n} g_n|^2 dx \\
&\le \sup\{\, w(x - n)^{-\alpha} \ :\ x \in [m, m + \eta) \,\} \int_{\mathbb{R}^d} w(\cdot - n)^{-\alpha} |A^{-1}_{-\alpha,n} g_n|^2 dx \\
&\le C e^{-\alpha|n-m|} \|A^{-1}_{-\alpha,n} g_n\|^2_{lu},
\end{aligned}$$

where $g_n(x) = w(x - n)^\alpha f_n(x)$ satisfies $\|g_n\|_{lu} \le C\|f_n\|$. Thus, we can define the function u via $\chi_m u = \sum_{n \in \mathbb{Z}^d} \chi_m u_n$, where the sum converges in L^2. To show that this u lies in fact in $L^2(\mathcal{Q})$ we employ Young's inequality for convolutions applied to the sequences $(\|\chi_m u\|)_m$ and $(\|f_n\|)_n$ which satisfy the convolutional estimate $\|\chi_m u\| \leq \sum_{n\in\mathbb{Z}^d} Ce^{-\alpha/2|n-m|}\|f_n\|$. Thus, we obtain

$$\|u\| = \Big(\sum_m \|\chi_m u\|^2\Big)^{1/2} \leq \sum_{p\in\mathbb{Z}^d} Ce^{-\alpha|p|/2}\Big(\sum_n \|f_n\|^2\Big)^{1/2} = C_2\|f\|,$$

such that $u = \widehat{A}^{-1}f \in L^2(\mathcal{Q})$, and the invertibility of $\widehat{A}$ is established.

Applying the result on the invertibility to $\widetilde{A} - \lambda I$ and $\widehat{A} - \lambda I$ we conclude that the resolvent sets of $\widetilde{A}$ and $\widehat{A}$ are equal. But this is the desired result on the spectra. □

The above result holds for general elliptic operators without any periodicity assumption. We now return to operators where the Bloch decomposition is available. Every function $u \in L^2(\mathcal{Q})$ can be decomposed into $(U(\sigma,\cdot))_{\sigma\in\mathcal{T}^*} \in L^2(\mathcal{T}^*, L^2(\mathcal{Q}/_{\mathcal{L}}))$ via the direct integral $u = \mathcal{D}(U) = \int_{\sigma\in\mathcal{T}^*} e^{i\sigma\cdot x}U(\sigma,\cdot)d\sigma$. Recall the notations from Sect. 2: $\mathcal{Q} = \mathbb{R}^d\times\Sigma$, $\mathcal{T} = \mathbb{R}^d/_{\mathcal{L}}$, and $\mathcal{Q}/_{\mathcal{L}} = \mathcal{T}\times\Sigma$. The integral $\mathcal{D}(U)$ has to be understood in the $L^2(\mathcal{Q})$–sense, see [ReS78]. For example, for simple functions $U(\sigma,\cdot) = \sum_{j=1}^N \chi_{A_j}(\sigma)U_j$ with $A_j \subset \mathcal{T}^*$ and $U_j \in L^2(\mathcal{Q}/_{\mathcal{L}})$ we have

$$u(x) = \mathcal{D}(U)(x) = \sum_{j=1}^N v_j(x)U_j(x) \quad \text{with } v_j(x) = \int_{\sigma\in A_j} e^{i\sigma\cdot x}d\sigma,$$

where $v_j \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and $U_j \in L^2(\mathcal{Q}/_{\mathcal{L}}) \subset L^2_{lu}(\mathcal{Q})$ such that $v_jU_j \in L^2(\mathcal{Q})$ is well–defined.

The inverse of $\mathcal{D}$ can be constructed by using the inverse of the classical Fourier transform in the x–variable,

$$(\mathcal{F}u)(k,z) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^{d/2}}\int_{x\in\mathbb{R}^d} e^{-ik\cdot x}u(x,z)\,dx.$$

Setting $k = \sigma + \ell$ with $\sigma \in \mathcal{T}^*$ and $\ell \in \mathcal{L}$ we immediately find $u = \mathcal{D}(U)$ with

$$U(\sigma,x,z) = \frac{1}{(2\pi)^{d/2}}\sum_{\ell\in\mathcal{L}} e^{i\ell\cdot x}(\mathcal{F}u)(\sigma+\ell,z).$$

Using Parseval's identity for $U(\sigma,\cdot)$ we obtain the norm relation

$$\begin{aligned}\|u\|^2 = \|\mathcal{F}u\|^2 &= \int_{\sigma\in\mathcal{T}^*}\sum_{\ell\in\mathcal{L}}\int_{z\in\Sigma} |(\mathcal{F}u)(\sigma+\ell,z)|^2dz\,d\sigma\\ &= \frac{(2\pi)^d}{\text{vol}(\mathcal{T})}\int_{\sigma\in\mathcal{T}^*}\|U(\sigma,\cdot)\|^2_{L^2(\mathcal{Q}/_{\mathcal{L}})}d\sigma.\end{aligned}$$

This shows that $\mathcal{D}$ defines an isomorphism between $L^2(\mathcal{Q})$ and $L^2(\mathcal{T}^*, L^2(\mathcal{Q}/_{\mathcal{L}}))$.

Additionally, we have the following characterization. A direct integral $u = \mathcal{D}(U)$ lies in $H^k(\mathcal{Q})$ if and only if $U(\sigma,\cdot) \in H^k(\mathcal{Q}/_{\mathcal{L}})$ for a.e. $\sigma \in \mathcal{T}^*$ and

$$\int_{\sigma\in\mathcal{T}^*}\Big\{(1+|\sigma|^{2k})\|U(\sigma,\cdot)\|^2_{L^2(\mathcal{Q}/_{\mathcal{L}})} + \|U(\sigma,\cdot)\|^2_{H^k(\mathcal{Q}/_{\mathcal{L}})}\Big\}d\sigma < \infty. \tag{A.7}$$

Assume now that an elliptic operator $A = A(\partial_x)$ and a boundary operator $\mathcal{B}$ with the properties from above are given such that A has periodic coefficients with periodicity lattice $\mathcal{L}$. Applying $A(\partial_x)$ to a Bloch wave leads to the definition of the Bloch operators $B(\sigma) : D(B) \subset L^2(\mathcal{Q}/_{\mathcal{L}}) \to L^2(\mathcal{Q}/_{\mathcal{L}})$ with

$$B(\sigma)U = e^{-i\sigma\cdot x}A(\partial_x)[e^{i\sigma\cdot x}U] = A(i\sigma + \partial_x)U, \tag{A.8}$$

where $D(B) = \{\, u \in H^{2m}(\mathcal{Q}/_{\mathcal{L}}) \;:\; \mathcal{B}u = 0 \,\}$ does not depend on σ since the boundary operator $\mathcal{B}$ does not contain tangential derivatives.

Inserting Bloch waves $u = e^{i\sigma\cdot x}U$ into the regularity estimate (A.5) we obtain the a–priori estimate

$$(1+|\sigma|^{2m})\|U\|_{L^2(\mathcal{Q}/_{\mathcal{L}})}+\|U\|_{H^{2m}(\mathcal{Q}/_{\mathcal{L}})} \le C(\|B(\mu,\sigma)U\|_{L^2(\mathcal{Q}/_{\mathcal{L}})}+\|U\|_{L^2(\mathcal{Q}/_{\mathcal{L}})}) \tag{A.9}$$

for any $\sigma \in \mathcal{T}^*$ and $U \in D(B)$, where C is independent of σ and U.

Lemma A.3. *Let $\widehat{A} : D(\widehat{A}) \subset L^2(\mathcal{Q}) \to L^2(\mathcal{Q})$ be given as above with $\mathcal{L}$–periodic coefficients and associated Bloch operators $B(\sigma)$. Then, $\widehat{A}$ has a bounded inverse $\widehat{A}^{-1} : L^2(\mathcal{Q}) \to D(\widehat{A})$ if and only if all $B(\sigma)$, $\sigma \in \mathcal{T}^*$, have a bounded inverse $B(\sigma)^{-1} : L^2(\mathcal{Q}/_{\mathcal{L}}) \to D(B)$ with*

$$b = \sup\{\, \|B(\sigma)^{-1}\|_{L^2(\mathcal{Q}/_{\mathcal{L}})\to L^2(\mathcal{Q}/_{\mathcal{L}})} \;:\; \sigma \in \mathcal{T}^* \,\} < \infty.$$

If $b < \infty$ then $\|A^{-1}\|_{L^2(\mathcal{Q})\to L^2(\mathcal{Q})} = b$ and $A^{-1} = \mathcal{D}B(\cdot)^{-1}\mathcal{D}^{-1}$.

Proof. Assume that $\widehat{A}$ is invertible. By Theorem A.2 we know that also $\widetilde{A}$ is invertible on $L^2_{lu}(\mathcal{Q})$. Since Bloch waves lie in $L^2_{lu}(\mathcal{Q})$ the inverse of the Bloch operators is given by $B(\sigma)^{-1}F = e^{-i\sigma\cdot x}\widetilde{A}^{-1}[e^{i\sigma\cdot x}F]$ and there is a constant C such that for all $F \in L^2(\mathcal{Q}/_{\mathcal{L}})$,

$$\begin{aligned}&\|B(\sigma)^{-1}F\|_{L^2(\mathcal{Q}/_{\mathcal{L}})} \le C\|B(\sigma)^{-1}F\|_{L^2_{lu}(\mathcal{Q})} = C\|e^{-i\sigma\cdot x}\widetilde{A}^{-1}[e^{i\sigma\cdot x}F]\|_{L^2_{lu}(\mathcal{Q})}\\ &\le C^2\|e^{i\sigma\cdot x}F\|_{L^2_{lu}(\mathcal{Q})} \le C^3\|F\|_{L^2(\mathcal{Q}/_{\mathcal{L}})}.\end{aligned}$$

This proves the 'only if' part.

For the opposite assertion insert $U = B(\sigma)^{-1}F$ into (A.9) giving

$$(1 + |\sigma|^{2m})\|U\|_{L^2(\mathcal{Q}/_{\mathcal{L}})} + \|U\|_{H^{2m}(\mathcal{Q}/_{\mathcal{L}})} \le C(1 + b)\|F\|_{L^2(\mathcal{Q}/_{\mathcal{L}})} \tag{A.10}$$

with $C(1+b)$ independent of F and σ. Thus, we may define $K : f \mapsto \mathcal{D}[B(\cdot)^{-1}(\mathcal{D}^{-1}f)(\cdot)]$ as a bounded operator from $L^2(\mathcal{Q})$ into $D(\widehat{A}) \subset H^{2m}(\mathcal{Q})$. The boundedness in $H^{2m}(\mathcal{Q})$ is a consequence of (A.10) and the criterion (A.7). The fact that K maps into the closed subspace $D(\widehat{A})$ of $H^{2m}(\mathcal{Q})$ follows since $B(\sigma)^{-1}$ maps into $D(B)$. Obviously, K is the desired inverse of $\widehat{A}$ and the 'if' part is proved.

The norm identity follows easily as $B(\sigma)^{-1}$ as the operator from $L^2(\mathcal{Q}/_{\mathcal{L}})$ into itself depends continuously on σ. ☐

The main result of this appendix reads as follows.

Theorem A.4. *Let $A(\partial_x)$ be an elliptic operator on $\mathcal{Q}$ with $\mathcal{L}$–periodic coefficients and $\mathcal{B}$ a boundary operator on $\partial\mathcal{Q}$ satisfying the conditions from above. Then, we have*

$$\mathit{spec}(\widetilde{A}) = \mathit{spec}(\widehat{A}) = \mathit{closure}\Big(\bigcup_{\sigma\in\mathcal{T}^*} \mathit{spec}(B(\sigma))\Big), \tag{A.11}$$

where $B(\sigma)$ are the associated Bloch operators, cf. (A.8).

Proof. The first identity was already proved in Theorem A.2.

Denote by $\mathcal{C}$ the set on the right–hand side of (A.11) and take $\lambda_0 \notin \text{spec}(\widehat{A})$. Then, by Lemma A.3 we know that $B(\sigma) - \lambda_0 I$, $\sigma \in \mathcal{T}^*$ is invertible with the inverse having a uniform bound b. Hence, for each $\sigma \in \mathcal{T}^*$ the set $\{ \lambda \in \mathbb{C} : |\lambda - \lambda_0| < 1/b \}$ is in the resolvent set of $B(\sigma)$ which implies $\lambda_0 \notin \mathcal{C}$. This proves $\mathcal{C} \subset \text{spec}(\widehat{A})$.

Now take $\lambda_1 \notin \mathcal{C}$. Then, the function $q : \sigma \mapsto \|(B(\sigma) - \lambda_1 I)^{-1}\|_{L^2(\mathcal{Q}/_\mathcal{L}) \to L^2(\mathcal{Q}/_\mathcal{L})}$ is well–defined and maps $\mathcal{T}^*$ into $(0, \infty)$. It is continuous, since $B(\sigma)$ as the bounded operator from $D(B)$ into $L^2(\mathcal{Q}/_\mathcal{L})$ is continuous in σ. Moreover, q decays like $(1 + |\sigma|^2)^{-m}$ for large σ because of (A.9). Thus, $b = \sup\{ q(\sigma) \ : \ \sigma \in \mathcal{T}^* \}$ is finite and Lemma A.3 provides a bounded inverse of $\widehat{A} - \lambda_1 I$. This shows $\text{spec}(\widehat{A}) \subset \mathcal{C}$ and the theorem is proved. □

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