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Generic Metrics and Connections on Spin- and Spin^c-Manifolds

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Abstract: We study the dependence of the dimension $h^0(g, A)$ of the kernel of the Atyiah-Singer Dirac operator $\mathcal{D}_{g,A}$ on a spin^c-manifold M on the metric g and the connection A. The main result is that in the case of spin-structures the value of $h^0(g)$ for the generic metric is given by the absolute value of the index provided dim$M \in \{3, 4\}$. In dimension 2 the mod-2 index theorems have to be taken into a account and we obtain an extension of a classical result in the theory of Riemann surfaces. In the spin^c-case we also discuss upper bounds on $h^0(g, A)$ for generic metrics, and we obtain a complete result in dimension 2. The much simpler dependence on the connection A and applications to Seiberg–Witten theory are also discussed.

Contents

1. Introduction

Given a spin-(spin^c-)manifold M much effort has been invested in the study of the Dirac operator $\mathcal{D}_g$ on spinors for particular metrics. Alternatively, metrics have been used as an auxiliary tool to derive topological information about the underlying spin-manifold. Via the index formula, knowledge of dimKer$\mathcal{D}_g$ may lead to topological obstructions. Thus if for instance Ker$\mathcal{D}_g = \{0\}$ on a closed $4k$-manifold M then the $\hat{A}$-genus of M vanishes. In particular, this is true if the metric g has positive scalar curvature [L].

Thus the dependence of dimKer$\mathcal{D}_g$ on the metric has been studied from the very beginning of the subject [L, Hi]. In general, one expects a spin-manifold to have arbitrarily many harmonic spinors for a suitable metric. The existence of harmonic spinors for suitable metrics in dimensions $0, \pm 1 \mod 8$ is proved in [Hi,Th.4.5]. The same holds in dimensions $3 \mod 4$ [Bär1, Bär2], and in fact the proof of loc.cit. can probably be extended to show that in dimensions $3 \mod 4$ there are indeed metrics with $h^0(g)$ arbitrarily large [BärP].

However, it has been conjectured that for the generic metric (a term to be made precise) the dimension of the space of harmonic spinors is equal to the absolute value of the index [BG, K2, Bär1]. More precisely,

Problem. Is it true that for a given spin^c-structure on a closed m-dimensional manifold M with fixed connection A on the canonical bundle we have

$$h^0(g, A) := \text{dimKer}\mathcal{D}_{g,A} = \begin{cases} |\text{Index}\mathcal{D}^+_{g,A}| & , m \text{ even} \\ 0 & , m \text{ odd} \end{cases}$$

for the generic metric g on M?

It is the purpose of this article to study this problem. Note that the index of the Dirac operator does not depend on the choice of metric and connection.

It is known that in dimensions 1 and 2 mod 8 the theorem is not true for spin-manifolds in the form stated because by the mod 2-Index theorems $h^0(g)$ and $h^+(g) :=$ dimKer$\mathcal{D}^+_g$ respectively are constant modulo 2 [AtSi] (see Remark 3.3 below). Thus in these dimensions the problem must be rephrased as follows:

Problem. Is it true that for a given spin^c-structure on a closed manifold M of dimension $1, 2 \mod 8$ with fixed connection A on the canonical bundle L the functions $h^0(g, A)$ and $h^+(g, A)$ respectively are constant on a generic set of metrics and are either 0 or 1 on this set?

The corresponding problem for variations of complex Kählerian structures has also been formulated [Hi,p.24] and has been conclusively answered in [K2] by exhibiting a counterexample.

Some of the motivation for studying the dependence of the Dirac on the metric comes from Seiberg–Witten theory. Here, it would be desirable to have a priori knowledge about dimKer$\mathcal{D}^+_{g,A}$ for a suitable metric. However, this is not possible because the connection A is part of a solution of the Seiberg–Witten equations. We shall discuss this issue below in Sect. 9.

The point of view adopted in this paper is the following: The Dirac operator will be viewed as a map $\mathcal{D}_{.,A} : \mathcal{M} \to \mathcal{B}(\mathcal{H}^1(\Sigma), \mathcal{H}^0(\Sigma))$, i.e. as a map from the space of metrics to the space of bounded linear operators between suitable Sobolev spaces. The Dirac operator $\mathcal{D}_{g,A}$ is a Fredholm operator. Note that the space of Fredholm operators $\mathcal{F} := \mathcal{F}(\mathcal{H}^1(\Sigma), \mathcal{H}^0(\Sigma))$ is stratified by the sets $\mathcal{F}_{n,k} := \{f \in \mathcal{F}, \text{dimKer}(f) = n, \text{dimCoker}(f) = k\}$. Each such set is a locally closed analytic submanifold of the Banach space $\mathcal{B}(\mathcal{H}^1(\Sigma), \mathcal{H}^0(\Sigma))$ of bounded linear maps [Kos]. As the Dirac operator is a formally self-adjoint operator, $\mathcal{D}_{g,A} \in \mathcal{F}_{n,n}$ for $n = h^0(g, A)$. We shall show that unless g is subject to certain restrictions, the Fréchet derivative $D\mathcal{D}_g$ at g has image not tangential to $\mathcal{F}_{n,n}$. Thus we may slightly perturb g to get a metric g' with $\mathcal{D}_{g'} \in \mathcal{F}_{n',n'}$, where $h^0(g', A) = n' < n$. If however Im$D\mathcal{D}_g$ is tangential to $\mathcal{F}_{n,n}$ (in which case we call g critical) this argument fails. The property of g being critical is a geometric condition which can be expressed in terms of a simple formula the analysis of which yields severe restrictions on the geometry of the Riemannian manifold (M, g).

As there are signifiant differences between the situation in dimension 2 in comparison to the situation in dimensions 3 or 4 we state the results separately for each dimension:

Theorem 1.1. *Let M be a closed oriented 2-dimensional manifold. For a fixed spin^c-structure and a fixed connection A on the canonical bundle P_{U_1} with $c_1(P_{U_1}) \neq 0$ the generic metric satisfies dimKer$\mathcal{D}_{g,A}$ = $|\frac{1}{2}c_1(P_{U_1})|$.*

If a given spin-structure is twisted by a connection B on the trivial bundle, thought of as a 1-form $B \in i\Omega^1(M)$, such that B is closed and defines an element in $H^1(M, 2\pi i\mathbb{Z})$, the generic metric satisfies dimKer$\mathcal{D}_g$ = 0 or 2 depending only on the spin-structure. For other B the generic metric has no nontrivial harmonic spinors.

This theorem provides a complete answer to the problem. The theorem can be reformulated in the language of the theory of Riemann surfaces, compare Theorem 7.1 below.

Theorem 1.2. *Let M be a closed oriented 3-manifold.*

(i) For a fixed spin-structure the generic metric has no nontrivial harmonic spinors.

(ii) For a fixed spin^c-structure and a fixed connection A on the canonical bundle the dimension of the space of harmonic spinors is at most 2 for a generic metric.

Theorem 1.3. *Let M be an oriented closed 4-manifold.*

(i) For a fixed spin-structure there are no nontrivial harmonic spinors of negative (positive) chirality for the generic metric if Index$\mathcal{D}^+_g \geq 0$ (≤ 0).

(ii) For Spin^c-structures, the same conclusion as in (i) holds if Index$\mathcal{D}^+_{g,A} \notin \{0, \pm 1\}$.

(iii) If Index$\mathcal{D}^+_{g,A}$ = ± 1 then $h^+ + h^- \leq 3$ for the generic metric.

(iv) If Index$\mathcal{D}^+_{g,A}$ = 0 then $h^+ = h^- \leq 2$ for the generic metric.

In the spin^c-case we may not only vary the metric but also the connection on the canonical bundle. One obtains the following:

Theorem 1.4. *Let (M, g) be a Riemannian spin^c-manifold with fixed spin^c-structure.*

(i) If dim$M$ = 2 or 4 then for the generic connection on the canonical bundle there are no nontrivial negative (positive) harmonic spinor provided Index$\mathcal{D}^+_{g,A} \geq 0$ (≤ 0).

(ii) If dim$M$ = 1 or 3 there are no nontrivial harmonic spinors for the generic connection.

(iii) The same conclusions hold if both metric and connection are varied.

This result has been proved independently N.Anghel [Ang,Th.1.5], and the four-dimensional case is contained in [Mor,Lem.6.9.3].

It is natural to consider not only variations of the 0-eigenvalue but of other eigenvalues, too. In fact, we shall formulate the more general results for arbitrary eigenvalues. The main difference in the discussion of zero- and nonzero eigenvalues stems from the fact that only the dimension of the 0-eigenspace is a conformal invariant whereas the dimension of the other eigenspaces varies with the metric in a conformal class. Thus in the discussion of the 0-eigenvalue (in dimensions > 2) the main difficulty will be to fix a suitable metric in the given conformal class. As the results for nonzero eigenvalues seem to be of lesser importance we refer the reader to Sect. 8 for a statement of results.

This paper is organized as follows: We shall first discuss the dependence of the Dirac operator on both the metric on the base-manifold and the connection on the canonical bundle. Our discussion is essentially an extension of the corresponding discussion in [BG], but we prefer to alter their definitions in order to better take into account conformal rescaling.

We shall then define and discuss the term "generic" before describing formulas which describe a first-order obstruction to the existence of deformations of the metric and/or the connection on the canonical bundle which reduce the dimension of the space of harmonic spinors. In fact, we shall prove the obstruction formula for all eigenvalues, not only for the 0-eigenvalue.

Restricting the discussion to harmonic spinors, the aim is then to show that in dimensions 2 to 4 this obstruction is indeed only a first-order obstruction, i.e. that unless the metric and/or connection is minimal there are deformations which do indeed reduce the dimension of the space of harmonic spinors.

As an immediate application we first prove the rather simple Theorem 1.4 and we make preliminary remarks on dimensions 3 and 4. Then we discuss the case dim$M = 2$ where the main feature is Serre-duality whereas conformal invariance plays no role. As indicated above, Theorem 1.1 has a translation into the language of the theory of Riemann surfaces. This translation is carried out in Sect. 7.

In dimensions 3 and 4 conformal invariance is the key-feature and most effort has to be put into the conformal fixing of the metric. It might be tempting to choose the metric within the conformal class such that the scalar-curvature is constant, but that approach seems to lead nowhere. Instead, we will locally rescale the metric such that harmonic spinors will have constant length.

We shall then consider nonzero eigenvalues. Here, the main feature is the appearance of Killing spinors which allows us to prove that for the generic metric in dimension 2 or 3 there are no λ-eigenspinors for a fixed number $\lambda \neq 0$.

Finally, we shall briefly discuss the Seiberg–Witten moduli spaces. The upshot of the discussion is the observation that for any connection A on the canonical bundle which comes from a solution to the Seiberg–Witten equations with parameter a metric g, the pair (g, A) in general is non-generic in our sense if Index$\mathcal{D}^+_{g,A} \leq 0$.

In an appendix we prove a result for analytic families of elliptic operators which is implicit in the literature but for which no general statement and proof seems to be known. We make use of the theorem in our discussion of generic metrics and connections.

2. The Dependence of $\mathcal{D}_{g,A}$ on the Metric and Connection

This section contains an exposition of the results of Bourguignon and Gauduchon [BG] with the aim of extending their discussion to variations of the Dirac operator with respect to variations of connections on twisting bundles. In addition, we shall redefine the identification of spinor bundles for different metrics so as to take into account the L^2-Hilbert space structure induced on the spinor bundles by the corresponding volume forms.

2.1. Preliminaries. First, let us briefly review the terminology which we shall employ. For a thorough exposition see for example [LM]. Given an m-dimensional Riemannian manifold (M, g) we shall by $P_{SO}(M)$ denote the bundle of orthonormal frames.

The manifold M is spin, if and only if there is a 2-fold connected cover of $P_{SO}(M)$ such that on each fibre the covering map reduces to the standard two-fold cover $\rho : Spin_m \to SO_m$. Such a covering is a principal $Spin_m$ bundle and we denote this bundle by $P_{Spin}(M, g)$.

Similarly, M is spin^c, if and only if there is a S^1-bundle P and a connected double cover of the fibre product $P_{SO}(M) \times_M P$ which on each fibre is the two-fold covering map $\tilde{\rho} : Spin^c_m \to SO_m \times S^1$. Such a cover is a principal $Spin^c_m$-bundle which we shall denote by $P_{Spin^c}(M, g, P)$. We shall refer to P as the canonical bundle of the Spin^c-structure.

If m is even let Σ_m be the irreducible module for the Clifford-algebra $\mathbb{C}l_m$, and if m is odd let Σ_m be the irreducible module for $\mathbb{C}l_m$ on which the volume element $i^{[(m+1)/2]} e_1 \ldots e_m$ acts as $+Id$. Given a spin- or spin^c-structure, we form the spinor bundles $\Sigma_g := P_{Spin}(M, g) \times_{rep} \Sigma_m$ and $\Sigma_g := P_{Spin^c}(M, g, P) \times_{rep} \Sigma_m$ respectively (where rep denotes the representation of $Spin_m$ and $Spin^c_m$ respectively which come from the standard embedding $Spin_m \subset Spin^c_m \subset \mathbb{C}l_m$). Note that in even dimensions Σ splits into the $\pm$-eigenbundles for the (fibrewise) action of the volume element.

In the spin-case the Atiyah–Singer Dirac operator $\mathcal{D}_g$ acting on sections of Σ is defined by

$$\mathcal{D}_g : C^\infty(\Sigma) \xrightarrow{\tilde{\nabla}^g} \Omega^1(M) \otimes C^\infty(\Sigma) \xrightarrow{\cong} C^\infty(TM) \otimes C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma)\,,$$

where the last arrow is Clifford-multiplication, and where $\tilde{\nabla}^g$ denotes the connection on Σ induced by the Levi–Cività-connection on $P_{SO}(M, g)$. In the spin^c-case, given a connection on the canonical bundle P, we get an induced connection $\tilde{\nabla}^{g,A}$ on Σ and thus the Atiyah–Singer Dirac operator $\mathcal{D}_{g,A}$ acting on sections of Σ.

2.2. The Identification. In order to compare the Dirac operator on a fixed manifold M with fixed spin-(spin^c) structure for different metrics (and connections on the canonical bundle) we need a canonical way of identifying the spinor bundles Σ_g and Σ_h for different metrics g and h. We shall briefly review how this is done [BG].

Consider for the moment a real m-dimensional vector space V. Given two metrics $g, h \in \mathrm{Sym}(V^* \otimes V^*)$ there is a unique positive endomorphism H of V such that $h(.,.) = g(H.,.)$. Let $b := H^{-1/2}$. If E is a g-orthonormal frame then $b(E)$ is a h-orthonormal frame. Thus b defines a smooth SO_m-equivariant map of the manifold of g-orthonormal frames $P(g)$ to the manifold of h-orthonormal frames $P(h)$.

Let $g_t := (1-t)g + th$, and let $b_t : P(g) \to P(g_t)$ be the associated map. Let $\pi : \tilde{P}(g_t) \to P(g_t)$ be the connected 2-fold covering which (after a choice of basepoint) we may identify with the connected 2-fold covering $\rho : Spin_m \to SO_m$. Given $E \in P(g)$ choose $\tilde{E} \in \tilde{P}(g)$ such that $\pi(\tilde{E}) = E$. Then the path $(t, b_t) \subset \bigcup_{t\in[0,1]} P(g_t)$ lifts uniquely to a path β_t in $\bigcup_{t\in[0,1]} \tilde{P}(g_t)$ such that $\beta_0(\tilde{E}) = \tilde{E}$. Clearly, we have $\beta_t(E.q) = \beta_t(E).q$ for $q \in Spin_m$.

We thus get a $Spin_m$-equivariant map $\beta_{h,g} = \beta_1 : \tilde{P}(g) \to \tilde{P}(h)$. Of course, in the preceding discussion we may replace the path g_t of metrics by any smooth path of metrics connecting g and h. The resulting map β_1 is independent of the path chosen because the space of metrics is contractible.

Note that because of the invariant description we may extend $b_{h,g}$ and $\beta_{h,g}$ to bundles to obtain SO_m- resp. $Spin_m$-equivariant smooth bundle maps $b_{h,g} : P_{SO}(M, g) \to P_{SO}(M, h)$ and $\beta_{h,g} : P_{Spin}(M, g) \to P_{Spin}(M, h)$ (provided M is spin), such that $\beta_{h,g}$ covers $b_{h,g}$. Of course, we have $\beta_{h,g} = \beta^{-1}_{g,h}$.

Similarly, if M is spin^c, fix a spin^c-structure with canonical bundle P. The $SO_m \times S^1$-equivariant bundle map $b_{h,g} \times Id$ lifts to a $Spin^c_m$-equivariant bundle map $\beta_{h,g}$: $P_{Spin^c}(M,g,P) \to P_{Spin^c}(M,h,P)$.

The map $\beta_{h,g}$ extends to an isometry $\beta_{h,g} : \Sigma_g \to \Sigma_h$ of Hermitian bundles. For any pair (g,A) and (h,B) of metrics and connections on P denote by $\tilde{\nabla}^{g,A}$ and $\tilde{\nabla}^{h,B}$ respectively the connections induced on Σ_g resp. Σ_h by the Levi–Cività-connections on TM and A resp. B on P. Then $\beta^{-1}_{h,g} \circ \tilde{\nabla}^{h,B} \circ \beta_{h,g}$ is a connection on Σ_g, and in fact it is the connection induced by the pair $(b^{-1}_{h,g} \circ \tilde{\nabla}^{h,B} \circ b_{h,g},\ B)$.

Note that g is $b^{-1}_{h,g} \circ \nabla^{h,B} \circ b_{h,g}$-parallel but that $b^{-1}_{h,g} \circ \nabla^{h,B} \circ b_{h,g}$ is usually not torsion-free. Also note that we have the following identity:

$$\beta_{h,g}(X.s) = b_{h,g}(X).\beta_{h,g}(s)\,.$$

It may now be tempting to use $\beta_{h,g}$ to pull back the Dirac operator on sections of Σ_h to a differential operator on sections of Σ_g [BG]. However, even though $\beta_{h,g}$ induces an isometry of Hermitian bundles it does not induce an isometry of Hilbert spaces $L^2(\Sigma_g, \mathrm{dvol}_g)$ and $L^2(\Sigma_h, \mathrm{dvol}_h)$, where dvol_g and dvol_h denote the volume forms. Instead, let a positive function $f_{h,g}$ be defined by $\mathrm{dvol}_h = f^2_{h,g}\mathrm{dvol}_g$ and set

$$\hat{\beta}_{h,g} := \frac{1}{f_{h,g}}\,\beta_{h,g}\,.$$

This $\hat{\beta}_{h,g}$ induces an isometry of Hilbert spaces $L^2(\Sigma_g, \mathrm{dvol}_g)$ and $L^2(\Sigma_h, \mathrm{dvol}_h)$. The pull-back

$$\bar{\mathcal{D}}_{h,B} := \hat{\beta}^{-1}_{h,g} \circ \mathcal{D}_{h,B} \circ \hat{\beta}_{h,g}$$

then has the same properties (symmetry, self-adjoint closure etc.) as $\mathcal{D}_{h,B}$. We have

$$\begin{aligned}\bar{\mathcal{D}}_{h,B} &= f_{h,g}\beta^{-1}_{h,g} \circ \mathcal{D}_{h,B} \circ f^{-1}_{h,g}\beta_{h,g}\\ &= \beta^{-1}_{h,g} \circ \mathcal{D}_{h,B} \circ \beta_{h,g} - f^{-1}_{h,g} b_{g,h}(\mathrm{grad}_h f_{h,g})\,,\end{aligned}$$

where $b_{h,g}(\mathrm{grad}_h f_{h,g})$ operates via Clifford multiplication. For any smooth function f we have $g(b_{g,h}(\mathrm{grad}_h f), .) = g(b_{h,g}(\mathrm{grad}_g f), .)$. We thus obtain

$$\bar{\mathcal{D}}_{h,B} = \beta^{-1}_{h,g} \circ \mathcal{D}_{h,B} \circ \beta_{h,g} - f^{-1}_{h,g} b_{h,g}(\mathrm{grad}_g f_{h,g})\,.$$

2.3. Computing the derivative of the Dirac Operator. We shall have to compute the derivative of $\mathcal{D}_{h,B}$ with respect to h and B. First, note that the second summand does not depend on B. We shall compute this term first:

Pick $k \in C^\infty\mathrm{Sym}(T^*M \otimes T^*M)$ and let $g_t := g + tk$ for small t. Then $b_{g_t,g} = (Id + tK)^{-\frac{1}{2}}$ where $K \in C^\infty\mathrm{Sym}_g(TM)$ is defined by $g(K., .) = k(., .)$. Thus $\frac{d}{dt}\big|_{t=0} b_{g_t,g} = -\frac{1}{2}K$. Now $\mathrm{dvol}_{g_t} = \sqrt{\det(I + tK)}\mathrm{dvol}_g$. Hence $f_{g_t,g} = (\det(I + tK))^{1/4}$ and $\frac{d}{dt}\big|_{t=0} f_{g_t,g} = \frac{1}{4}\mathrm{Tr}_g k$. Note that $f_{g,g} \equiv 1$ and thus

$$\frac{d}{dt}\bigg|_{t=0}\left(\frac{1}{f_{g_t,g}} b_{g_t,g}(\mathrm{grad}_g f_{g_t,g})\right) = \frac{1}{4}\mathrm{grad}_g(\mathrm{Tr}_g k)\,.$$

To deal with the first summand we shall write it in terms of a local frame: If $\{e_1, \ldots, e_m\}$ is a local g-orthonormal frame on some open contractible set $U \subset M$ one may compute

$$\beta_{h,g}^{-1} \circ \mathcal{D}_{h,B} \circ \beta_{h,g} = \sum_{i=1}^{m} e_i.\tilde{\nabla}^{g,A}_{b_{h,g}(e_i)} + \sum_{i=1}^{m} e_i.\left(\beta_{h,g}^{-1} \circ \tilde{\nabla}^{h,B}_{b_{h,g}(e_i)} \circ \beta_{h,g} - \tilde{\nabla}^{g,A}_{b_{h,g}(e_i)}\right),$$

see [BG]. We may think of Σ_g over U as coming from a spin-structure tensor product $P_{Spin}(U,g)$. Given a Hermitian connection A on $U \times \mathbb{C}$ write A as $A = d + \phi_A$, $\phi_A \in i\Omega^1(U)$. Let $\tilde{\nabla}^g$ be the connection on $P_{Spin}(U,g) \times_\rho \Sigma_m$ induced by the Levi–Cività-connection. Then $\tilde{\nabla}^{g,A} = \tilde{\nabla}^g + \frac{1}{2}\phi_A$ over U. It is then immediate that

$$\left.\frac{d}{dt}\right|_{t=0} \bar{\mathcal{D}}_{g,A+ta} = \frac{1}{2}a, \qquad a \in i\Omega^1(M),$$

where a acts via Clifford multiplication.

Finally, we are left with computing $\left.\frac{d}{dt}\right|_{t=0} \beta_{g_t,g}^{-1} \circ \mathcal{D}_{g_t,A} \circ \beta_{g_t,g}$ for $g_t := g + tk$. This has been done in [BG], where the following formula is obtained:

$$\left.\frac{d}{dt}\right|_{t=0} \beta_{g_t,g}^{-1} \circ \mathcal{D}_{g_t,A} \circ \beta_{g_t,g} = -\frac{1}{2}\sum_i e_i \tilde{\nabla}^{g,A}_{K(e_i)} + \frac{1}{4}(d(\mathrm{Tr}_g k) - \mathrm{div}_g k)$$

Note that in comparison to [BG] we prefer to use the opposite sign convention for the divergence operator.

We obtain the following formula which is an immediate consequence of the preceding discussion:

Proposition 2.4. *The derivative of $\bar{\mathcal{D}}_{g,A}$ at (g,A) in the direction (k,a), $k \in C^\infty Sym$ $(T^*M \otimes T^*M)$ and $a \in i\Omega^1(M)$, is given by*

$$(D\bar{\mathcal{D}})_{(g,A)}(k,a) = -\frac{1}{2}\sum_i e_i \tilde{\nabla}^{g,A}_{K(e_i)} - \frac{1}{4}\mathrm{div}_g k + \frac{1}{2}a,$$

where in the last two terms the 1-forms act via Clifford-multiplication.

Remark 2.5. More generally, if E is a complex vector bundle with connection ∇^E we may compute the Fréchet derivative of $\bar{\mathcal{D}}_{g,A,\nabla^E}$ on the twisted spinor bundle $\Sigma_g \otimes E$. The same computation as above then yields:

$$(D\bar{\mathcal{D}})_{(g,A,\nabla^E)}(k,a,\Phi) = -\frac{1}{2}\sum_i e_i \tilde{\nabla}^{g,A,\nabla^E}_{K(e_i)} - \frac{1}{4}\mathrm{div}_g k + \frac{1}{2}a + \sum_i e_i. \otimes \Phi(e_i),$$

where $\Phi \in \Omega^1(M) \otimes \mathrm{End}(E)$.

Remark 2.6. It should be remarked that the conformal invariance of the dimension of the space of harmonic spinors is not only a feature of the Atiyah–Singer operator but is a quite general phenomenon. More precisely, let M be a spin^c-manifold with fixed spin^c-structure, a metric g and a connection A on the canonical bundle. Let $\rho : \mathbb{C}l_m \to \mathrm{End}(W)$ be any hermitian representation and form the bundle $\Sigma = P_{spin^c}(M,g,P_{U_1}) \times_\rho W$, and let E be any complex vector bundle with connection. Then the dimension of the space of harmonic spinors of the twisted Dirac operator on $\Sigma \otimes E$ is a conformal invariant. The proof (which involves computations similar to the ones above) proceeds precisely as in [Hi;BFGK,Th.13;Hij1,Prop.4.3.1]. In fact, if $h = e^{2f}g$ set $\bar{\beta}_{h,g} := e^{-\frac{m-1}{2}f}\beta_{h,g}$. Then $\mathcal{D}_{h,A,E} = e^{-f}\bar{\beta}_{h,g} \circ \mathcal{D}_{g,A,E} \circ \bar{\beta}_{h,g}^{-1}$.

3. Generic Metrics and Connections

Definition. *Let $E \to M$ be a smooth (real or complex) vector bundle over the closed manifold M, and let $\mathcal{E} \subset C^\infty(E)$ be a C^0-open subset of smooth sections of E. We shall call a subset $\mathcal{E}' \subset \mathcal{E}$ C^k-generic in $\mathcal{E}$ if $\mathcal{E}'$ is C^∞-dense and C^k-open in $\mathcal{E}$.*

Note that if $\mathcal{E}'$ is C^k-generic in $\mathcal{E}$ then it is also C^l-generic for any $l > k$.

In our applications, $\mathcal{E}' = \mathcal{M} \subset C^\infty \mathrm{Sym}(T^*M \otimes T^*M)$, $\mathcal{E}' = \mathcal{M} \times \mathcal{A}$ and $\mathcal{E}' = \mathcal{A}$ according to context, where $\mathcal{M}$ denotes the set of smooth metrics on M and $\mathcal{A} = i\Omega^1(M)$.

In the sequel consider the Dirac operator defined on a bundle Σ obtained from $P_{Spin}(M, g)$ and $P_{Spin^c}(M, g, P_{U_1})$ respectively by a hermitian representation $\rho : \mathbb{C}l_m \to \mathrm{End}(W)$. Let $\mathcal{M}^\lambda_{min} \subset \mathcal{M}$ (alternatively $(\mathcal{M} \times \mathcal{A})^\lambda_{min} \subset \mathcal{M} \times \mathcal{A}$, or $\mathcal{M}^\lambda_{min}(A) = \mathcal{M} \times \{A\}$, or indeed $\mathcal{A}^\lambda_{min}(g) = \{g\} \times \mathcal{A}$) denote the set of metrics (of metrics and connections on the canonical bundle, of metrics, of connections on the canonical bundle) for which $\mathrm{dimKer}(\mathcal{D}_{g,A} - \lambda)$ is minimal among all possible choices (in the third case we assume the connection to be fixed, in the fourth case we assume a metric g to be fixed).

Proposition 3.1. *The sets $\mathcal{M}^\lambda_{min} \subset \mathcal{M}$, $(\mathcal{M} \times \mathcal{A})^\lambda_{min} \subset \mathcal{M} \times \mathcal{A}$, and $\mathcal{M}^\lambda_{min}(A) \subset \mathcal{M} \times \{A\}$ are C^1-generic. The set $\mathcal{A}^\lambda_{min}(g) \subset \mathcal{A}$ is C^0-generic.*

Proof. Suppose M is spin. We shall argue the first case: Fix a connection ∇ on Σ. Then $\mathcal{D}_g = S_1 \circ \nabla + S_2$, where $S_1 \in C^\infty \mathrm{Hom}(\Omega^1(M) \otimes \Sigma, \Sigma)$ and $S_2 \in C^\infty \mathrm{End}(\Sigma)$. Then

$$\|\mathcal{D}_g s\|_{L^2} \leq \max|S_1|.\|\nabla s\|_{L^2} + \max|S_2|.\|s\|_{L^2} \leq \mathrm{const.}(\max|S_1| + \max|S_2|)\|s\|_{\mathcal{H}^1}\,.$$

S_1 and S_2 depend only on g and its first derivatives. Thus $g \mapsto \mathcal{D}_g \in \mathcal{B}(\mathcal{H}^1(\Sigma), \mathcal{H}^0(\Sigma))$ is continuous in the C^1-topology on $\mathcal{M}$. If $\mathrm{dimKer}\mathcal{D}_g$ is minimal then so is $\mathrm{dimKer}\mathcal{D}_{g'}$ for $\mathcal{D}_{g'}$ in a neighbourhood of $\mathcal{D}_g$ in the norm topology on $\mathcal{B}(\mathcal{H}^1(\Sigma), \mathcal{H}^0(\Sigma))$. This shows that $\mathcal{M}^\lambda_{min}$ is C^1-open.

Let $g \in \mathcal{M}^\lambda_{min}$ and $h \in \mathcal{M}$. Set $g_t := (1 - t)g + th$. The family of operators $\overline{\mathcal{D}}_{g_t}$ is self-adjoint and analytic in t in the sense of the appendix. Proposition 11.4 of this appendix shows that for all but finitely many $t \in [0, 1]$ we have $g_t \in \mathcal{M}^\lambda_{min}$. It follows that $\mathcal{M}^\lambda_{min}$ is C^∞-dense in $\mathcal{M}$.

The case $(\mathcal{M} \times \mathcal{A})^\lambda_{min} \subset \mathcal{M} \times \mathcal{A}$ is argued similarly. In the case $\mathcal{A}^\lambda_{min}(g) \subset \mathcal{A}$ note that with $\mathcal{D}_{g,A} = S_1 \circ \nabla + S_2$ the sections S_1 and S_2 depend continuously on A. The argument now proceeds as before. □

Example 3.2. Suppose M is spin. If M has a metric g of positive scalar curvature, then by the preceding proposition we know that for each metric h in the C^1-generic set $\mathcal{M}^\lambda_{min}$ of metrics on M there are no harmonic spinors, because the Dirac-operator for the metric g has none by [L;LM,Cor.8.9]. Thus because for simply connected closed manifolds of dimension $m \geq 5$ the existence of positive scalar curvature metrics is equivalent to the vanishing of certain topological obstructions [GL, Sto] we find a rich class of spin-manifolds for which the answer to the problem in the introduction is affirmative.

Remark 3.3. As stated in the introduction, the problem in its original form does not hold in dimensions $1, 2$ mod 8. To see this let $\overline{K3}$ be a $K3$-surface with the opposite orientation and define $M := \overline{K3}\#(S^1 \times S^3)$. Then M has signature $\sigma(M) = 16$. Let $Y_1 := M \times M \times S^1$ and $Y_2 := M \times M \times F$, where F is a closed 2-manifold of genus ≥ 2. Choose any spin-structure on M and take the spin-structure on S^1 which does not extend to the disc, and then furnish Y_1 with the product spin-structure. By multiplicativity of the spin-number we see that $h^0(g) \equiv 1 \bmod 2$ [AtSi,Th.3.1]. Similarly, by Remark 3 of [At,p.60] we see that Y_2 has a spin-structure with $h^0(g) \equiv 1 \bmod 2$.

Convention. We shall refer to metrics in $\mathcal{M}^\lambda_{min}$ as either *minimal* or *generic*. Similarly, we shall call metrics in $\mathcal{M}^\lambda_{min}(A)$ respectively connections in $\mathcal{A}^\lambda_{min}(g)$ minimal or generic, and pairs in $(\mathcal{M} \times \mathcal{A})^\lambda_{min}$ are referred to as either minimal or generic, too.

4. The Obstruction Formula

Let M be closed spin^c-manifold and fix a metric g and a connection A on the canonical bundle. The formula of the first section shows that $\bar{\mathcal{D}} : \mathcal{M} \to \mathcal{DO}_1$ as map from the Fréchet space of smooth metrics to the Fréchet space of differential operators of order 1 is at least C^1. Thus so is $\bar{\mathcal{D}} : \mathcal{M} \to \mathcal{B}(\mathcal{H}^1(\Sigma), \mathcal{H}^0(\Sigma))$, where $\mathcal{H}^1(\Sigma)$ is the Sobolev space of order 1 and $\mathcal{H}^0(\Sigma) = L^2(\Sigma)$.

Let $\mathcal{F}$ denote the set of Fredholm operators in $\mathcal{B}(\mathcal{H}^1(\Sigma), \mathcal{H}^0(\Sigma))$, and let $\mathcal{F}_{n,k}$ denote the stratum $\mathcal{F}_{n,k} := \{f \in \mathcal{F}, \mathrm{dimKer}(f) = n, \mathrm{dimCoker}(f) = k\}$. By [Kos] each $\mathcal{F}_{n,k}$ is a locally closed analytic submanifold of $\mathcal{B}$ and the fibre of the analytic normal bundle of $\mathcal{F}_{n,k}$ at f is given by Hom(Ker(f), Coker(f)).

Suppose $\bar{\mathcal{D}}_{g,A} - \lambda \in \mathcal{F}_{n,n}$ for a fixed $\lambda \in \mathbb{R}$ (recall that because $\bar{\mathcal{D}}_{g,A} - \lambda$ is formally self-adjoint we have Ker$(\bar{\mathcal{D}}_{g,A} - \lambda)$ = Coker$(\bar{\mathcal{D}}_{g,A} - \lambda) \subset C^\infty(\Sigma)$). If there is $(k, a) \in \mathrm{Sym}(T^*M \otimes T^*M) \times i\Omega^1(M)$ such that $(D\bar{\mathcal{D}})_{g,A}(k, a)$ is not tangential to $\mathcal{F}_{n,n}$ then for small t the operator $\bar{\mathcal{D}}_{g_t,A_t} - \lambda$ will not be in $\mathcal{F}_{n,n}$. Here, as before, $g_t = g + tk$ and $A_t = A + ta$. By upper semicontinuity of the dimension of the kernel of $\bar{\mathcal{D}}_{g_t,A_t}$ for some sufficiently small t we have $\bar{\mathcal{D}}_{g_t,A_t} \in \mathcal{F}_{n',n'}$ with $n' < n$.

Note that if we rescale the metric by a constant factor μ^2, $\mu > 0$, we have $\bar{\mathcal{D}}_{\mu^2 g,A} = \frac{1}{\mu}\bar{\mathcal{D}}_{g,A}$. Thus for no eigenvalue $\lambda \neq 0$ can the image of the differential $D\bar{\mathcal{D}}_{g,A}$ at $\bar{\mathcal{D}}_{g,A} - \lambda$ be tangential to $\mathcal{F}_{n,n}$ for variations of the metric unless we restrict to such variations which preserve the total volume. Hence

Convention. For brevity's sake we shall call a pair (g, A) *critical at the eigenvalue* λ if the image of $D(\bar{\mathcal{D}} - \lambda)_{g,A}$ restricted to elements $(k, a) \in \mathrm{Sym}(T^*M \otimes T^*M) \times i\Omega^1(M)$ with $\int \mathrm{Tr}_g k \,\mathrm{dvol}_g = 0$ is tangential to $\mathcal{F}_{n,n}$. Similarly, we call a metric (connection) critical at the eigenvalue λ if for a fixed connection (metric) the image of $D(\bar{\mathcal{D}} - \lambda)_{g,A}$ is tangential to $\mathcal{F}_{n,n}$, where the derivative is computed with respect to variations in the metric (connection) only.

A good criterion with which to decide whether $\mathrm{Im}D\bar{\mathcal{D}}_{g,A}$ is tangential to $\mathcal{F}_{n,n}$ is the following:

Proposition 4.1. *The pair (g, A) is critical at the eigenvalue λ if and only if*

(i) $\langle X.\Psi_1, \Psi_2\rangle = 0\,,$

(ii) $\langle X.\tilde{\nabla}^{g,A}_X \Psi_1, \Psi_2\rangle + \langle \Psi_1, X.\tilde{\nabla}^{g,A}_X \Psi_2\rangle = \frac{2\lambda}{m}\langle \Psi_1, \Psi_2\rangle g(X, X)\,,$

(iii) $\langle \Psi_1, \Psi_2\rangle = const$ *if* $\lambda \neq 0\,,$

for all $X \in C^\infty(TM)$ and $\Psi_i \in \mathrm{Ker}(\mathcal{D}_{g,A} - \lambda)$.

In case we vary the connection only, the condition for A being critical is equivalent to (i), and if we vary the metric only, (ii) and (iii) are equivalent to the metric being critical.

Proof. The image of $D\bar{\mathcal{D}}_{g,A}$ is tangential to $\mathcal{F}_{n,n}$ at $\mathcal{D}_{g,A}$ if and only if

$$\big(D\bar{\mathcal{D}}_{g,A}(k,a)\Psi_1,\Psi_2\big)_{L^2}=0$$

for all $\Psi_i\in\mathrm{Ker}(\mathcal{D}_{g,A}-\lambda)$ and $(k,a)\in\mathrm{Sym}(T^*M\otimes T^*M)\times i\Omega^1(M)$ with $\int\mathrm{Tr}_g k\,\mathrm{dvol}_g=0$.
Define $Q^{g,A}_{\Psi_1,\Psi_2}(X,Y):=\frac{1}{2}\langle X.\tilde{\nabla}^{g,A}_Y\Psi_1,\Psi_2\rangle+\frac{1}{2}\langle Y.\tilde{\nabla}^{g,A}_X\Psi_1,\Psi_2\rangle$. Then

$$\langle\sum_i e_i.\tilde{\nabla}^{g,A}_{Ke_i}\Psi_1,\Psi_2\rangle=\sum_{i,j}k(e_i,e_j)Q^{g,A}_{\Psi_1,\Psi_2}(e_i,e_j)=\langle k,Q^{g,A}_{\Psi_1,\Psi_2}\rangle\,,$$

where the term on the right-hand side means the usual pointwise $\mathbb{C}$-bilinear product of $\mathbb{C}$-valued symmetric bilinear forms. With this notation the condition that the image of $D\mathcal{D}_{g,A}$ be tangential to $\mathcal{F}_{n,n}$ is equivalent to:

$$0=\int_M\left(-\frac{1}{2}\langle k,Q^{g,A}_{\Psi_1,\Psi_2}\rangle-\frac{1}{4}\langle(\mathrm{div}_g k).\Psi_1,\Psi_2\rangle+\frac{1}{2}\langle a.\Psi_1,\Psi_2\rangle\right)\mathrm{dvol}_g$$

for all λ-eigenspinors Ψ_1 and Ψ_2.

If we set $k=0$ then we immediately obtain the first condition of the proposition. This also implies that the integral over the third term vanishes identically.

We may repeat the above argument with Ψ_1 and Ψ_2 interchanged. Denote by $\bar{Q}^{g,A}_{\Psi_2,\Psi_1}$ the complex conjugate of $Q^{g,A}_{\Psi_2,\Psi_1}$. Then adding the corresponding equations we get

$$0=\int_M\langle k,Q^{g,A}_{\Psi_1,\Psi_2}+\bar{Q}^{g,A}_{\Psi_2,\Psi_1}\rangle\mathrm{dvol}_g \tag{4.1.1}$$

for all $k\in C^\infty\mathrm{Sym}(T^*M\otimes T^*M)$ with $\int\mathrm{Tr}_g k\,\mathrm{dvol}_g=0$. This implies that the section in the bundle of symmetric bilinear forms $Q^{g,A}_{\Psi_1,\Psi_2}+\bar{Q}^{g,A}_{\Psi_2,\Psi_1}$ is equal to its trace part and that its trace is constant. For $\lambda\neq 0$ the latter condition is equivalent to (iii) of the proposition, whereas the former is just (ii).

Now let $Z:=\sum\langle e_i.\Psi_1,\Psi_2\rangle e_i$ with respect to a local g-orthonormal frame. Z is globally defined, and computing at a point $x\in M$, where we may assume the local g-orthonormal frame to satisfy $\nabla^g e_i|_x=0$ we find:

$$\begin{aligned}L_Zg(X,X)&=2g(\nabla^g_X Z,X)|_x\\&=2X\langle e_i.\Psi_1,\Psi_2\rangle|_x\,g(e_i,X)|_x\\&=2\langle X.\tilde{\nabla}^{g,A}_X\Psi_1,\Psi_2\rangle|_x-2\langle\Psi_1,X.\tilde{\nabla}^{g,A}_X\Psi_2\rangle|_x\\&=2Q^{g,A}_{\Psi_1,\Psi_2}(X,X)|_x-2\bar{Q}^{g,A}_{\Psi_2,\Psi_1}(X,X)|_x\,.\end{aligned}$$

Adding (ii) (multiplied by a factor of 2) to the last equation yields

$$\text{(ii)'}\quad\frac{1}{4}L_Zg(X,X)=\langle X.\tilde{\nabla}^{g,A}_X\Psi_1,\Psi_2\rangle+\frac{\lambda}{m}\langle\Psi_1,\Psi_2\rangle g(X,X)\,,$$

which is of course equivalent to (ii). To prove that (i), (ii) and (iii) imply that $\mathrm{Im}D\bar{\mathcal{D}}_{g,A}$ is tangential to $\mathcal{F}_{n,n}$, observe that $(L.g)^*(k)=-2\sum(\mathrm{div}_g k)(e_i)e_i$ [Be,1.60]. Thus (ii)' implies

$$-\frac{1}{2}\int_M\langle k,Q^{g,A}_{\Psi_1,\Psi_2}\rangle\mathrm{dvol}_g=-\frac{1}{8}\int_M\langle k,L_Zg\rangle\mathrm{dvol}_g+\frac{\lambda}{2m}\int\langle\Psi_1,\Psi_2\rangle\mathrm{Tr}_g k\,\mathrm{dvol}_g$$

$$= \frac{1}{4} \int_M \langle \mathrm{div}_g k.\Psi_1, \Psi_2 \rangle \mathrm{dvol}_g \, .$$

The last equality is clear for $\lambda = 0$. In case $\lambda \neq 0$ recall that $\langle \Psi_1, \Psi_2 \rangle$ is constant and $\int \mathrm{Tr}_g k \, \mathrm{dvol}_g = 0$. But this equation precisely states that (g, A) is critical. Inspection of the proof shows that if we restrict to variations of the metric, (ii) and (iii) are equivalent to the metric being critical. And in case we vary only the connection, the property of A being critical is equivalent to (i) only. □

The following is an immediate corollary of the definitions and the preceding proposition:

Corollary 4.2. *For generic metrics conditions (ii) and (iii) of the proposition are satisfied. For generic connections (i) is satisfied. For generic pairs of metrics and connections (i), (ii) and (iii) are satisfied.*

Remark 4.3. Note that an eigenvalue λ which admits a Killing spinor, i.e. a spinor Ψ which satisfies $\tilde{\nabla}_X \Psi = -\frac{\lambda}{m} X.\Psi$, is a critical eigenvalue for variations of the metric which preserve the total volume [BG,Prop.28]. In Proposition 9.1 below we shall prove a partial converse to this.

Remark 4.4. Consider only the eigenvalue 0: It is clear from Remark 2.6 above that (i) is conformally invariant. Some straightforward but tedious computation shows that the vanishing of $Q^{g,A}_{\Psi_1,\Psi_2} + \overline{Q}^{g,A}_{\Psi_2,\Psi_1}$ is a conformally invariant statement, too. More precisely, if $h = e^{2f} g$ we have $\mathcal{D}_{h,A,E} = e^{-f} \bar{\beta}_{h,g} \circ \mathcal{D}_{g,A,E} \circ \bar{\beta}^{-1}_{h,g}$ by Remark 2.6 above. Furthermore, $\tilde{\nabla}^{h,A}_X = \beta_{h,g} \{ \tilde{\nabla}^{g,A}_X + \frac{1}{4}(X.\nabla f. + \nabla f.X.) \} \beta^{-1}_{h,g}$ [LM,p.134]. We may compute $Q^{h,A}_{\bar{\beta}\Psi_1,\bar{\beta}\Psi_2} + \overline{Q}^{h,A}_{\bar{\beta}\Psi_2,\bar{\beta}\Psi_1}$, where we write $\beta := \beta_{h,g}$ and $\bar{\beta} := \bar{\beta}_{h,g}$ to simplify notation:

$$\begin{aligned} Q^{h,A}_{\bar{\beta}\Psi_1,\bar{\beta}\Psi_2} &= \langle X.\tilde{\nabla}^{h,A}_X \bar{\beta}\Psi_1, \bar{\beta}\Psi_2 \rangle \\ &= e^{-\frac{m-3}{2} f} \langle X.(\beta^{-1} \tilde{\nabla}^{h,A}_X) \bar{\beta}\Psi_1, \Psi_2 \rangle \\ &= e^{(2-m)f} \left\{ Q^{g,A}_{\Psi_1,\Psi_2} + \frac{1}{4} \langle (X.X.\nabla f - X.\nabla f.X).\Psi_1, \Psi_2 \rangle \right. \\ &\quad \left. - \frac{m-1}{2} (Xf) \langle X.\Psi_1, \Psi_2 \rangle \right\} \\ &= e^{(2-m)f} \left\{ Q^{g,A}_{\Psi_1,\Psi_2} - \frac{m}{2} (Xf) \langle X.\Psi_1, \Psi_2 \rangle \right\} . \end{aligned}$$

Adding this to the corresponding result for $\overline{Q}^{h,A}_{\bar{\beta}\Psi_2,\bar{\beta}\Psi_1}$ yields:

$$Q^{h,A}_{\bar{\beta}\Psi_1,\bar{\beta}\Psi_2} + \overline{Q}^{h,A}_{\bar{\beta}\Psi_2,\bar{\beta}\Psi_1} = e^{(2-m)f} \left(Q^{g,A}_{\Psi_1,\Psi_2} + \overline{Q}^{g,A}_{\Psi_2,\Psi_1} \right) .$$

This shows that (ii) is a conformally invariant equation.

Note that if $m = 2$ the form $Q := Q^{g,A}_{\Psi_1,\Psi_2} + \overline{Q}^{g,A}_{\Psi_2,\Psi_1}$ is independent of the choice of metric in a given conformal class. It depends only on the connection A (and thus on the holomorphic structure on the line bundle Σ^+) and the choice of harmonic spinors. To better understand the meaning of this consider the case of flat connections A only. Let $q(X, Y) := \mathrm{Re}\langle X.\tilde{\nabla}^g_Y \Psi_+, \Psi_- \rangle + \mathrm{Re}\langle \Psi_+, X.\tilde{\nabla}^g_Y \Psi_- \rangle$. Note that $\mathrm{Tr}_g q = 0$ and thus q is anti-J-invariant, i.e. $q(J., J.) = -q(., .)$, where J denotes the complex structure induced by the metric g. As $\Lambda^{2,0}(M, J)$ is trivial, q is in fact a symmetric form. Note that we recover Q from q by the identity $Q = q - iq^J$ with $q^J(., .) := q(J., .)$.

Fix $p \in M$ and choose an ON-frame $\{e_1, e_2\}$ around p and a vector field X with $\nabla e_1|_p = \nabla X|_p = 0$. Compute at the point p:

$$\begin{aligned}
\mathrm{div}_g q &= \sum_i e_i q(e_i, X) \\
&= \sum_i e_i \left(\mathrm{Re}\langle X.\tilde{\nabla}^g_{e_i}\Psi_+, \Psi_-\rangle + \mathrm{Re}\langle \Psi_+, X.\tilde{\nabla}^g_{e_i}\Psi_-\rangle\right) \\
&= \mathrm{Re}\langle X.\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\Psi_+, \Psi_-\rangle + \mathrm{Re}\langle \Psi_+, X.\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\Psi_-\rangle \\
&= -\mathrm{Re}\langle X.\tilde{\nabla}^*\tilde{\nabla}\Psi_+, \Psi_-\rangle - \mathrm{Re}\langle \Psi_+, X.\tilde{\nabla}^*\tilde{\nabla}\Psi_-\rangle\,.
\end{aligned}$$

Using the Weitzenböck formula $\mathcal{D}_g^2 = \tilde{\nabla}^*\tilde{\nabla} + s/4$ we see that $\mathrm{div}_g q = 0$. The condition that a symmetric bilinear form be trace-free is invariant under conformal changes. In dimension 2 the property of a symmetric bilinear form being divergence-free is a conformally invariant property, too. Thus we see that q defines an element in the tangent space $T_{[g]}\mathcal{T}$ to Teichmüller space $\mathcal{T}$ at the point defined by the conformal class $[g]$ of g, see for example [Tr]. The image of the map which assigns to each pair (Ψ_+, Ψ_-) of harmonic spinors the form $q(X,Y) := \mathrm{Re}\langle X.\tilde{\nabla}^g_Y\Psi_+, \Psi_-\rangle + \mathrm{Re}\langle \Psi_+, X.\tilde{\nabla}^g_Y\Psi_-\rangle$ is thus the subspace of the tangent space $T_{[g]}\mathcal{T}$ which contains those infinitesimal deformations which reduce the dimension of the space of harmonic spinors. Conformal invariance of q thus reflects the fact that spin-geometry on 2-manifolds is essentially equivalent to the study of holomorphic square roots of the canonical bundle $K = \Omega^{1,0}(M)$ on Riemann surfaces. For this point of view see Sect. 7 below.

Remark 4.5. Define a gauge-transformation to be a smooth map $u : M \to U_1$. Such a u acts on (g, A) by the rule $u.(g, A) := (g, u(A) = A + 2udu^{-1})$. It is immediate that $\tilde{\nabla}^{g,u(A)} = u \circ \tilde{\nabla}^{g,A} \circ u^{-1}$. It follows that $\mathcal{D}_{g,u(A)} = u \circ \mathcal{D}_{g,A} \circ u^{-1}$, which in particular implies that $\tilde{\nabla}^{g,u(A)}$ and $\tilde{\nabla}^{g,A}$ have the same spectrum, and it is also immediate that if (i), (ii) and (iii) of the proposition hold for (g, A) and some λ in the spectrum of $\tilde{\nabla}^{g,A}$, then they also hold for $(g, u(A))$. Thus the condition that λ be critical is invariant under gauge-transformations.

Remark 4.6. Let M be even dimensional. The complex volume element $i^{\frac{m}{2}} e_1 \dots e_m \in \mathbb{C}l_m(TM)$ acts on Σ and splits it into the $\pm$-eigenbundles Σ_g^+ and Σ_g^-. $\mathcal{D}_{g,A}$ intertwines Σ_g^+ and Σ_g^-. It is clear that $\beta_{h,g}$ respects this splitting, i.e. $\beta_{h,g} : \Sigma_g^\pm \to \Sigma_h^\pm$. Thus we may consider $\mathcal{D}_{h,B} = \mathcal{D}^+_{h,B} + \mathcal{D}^-_{h,B}$ as operator on $\Sigma_g^\pm$. We may thus ask under what conditions on (g, A) is $\mathrm{Im} D\mathcal{D}^+_{g,A}$ tangential to $\mathcal{F}_{n,k}(\mathcal{H}^1(\Sigma_g^+), \mathcal{H}^0(\Sigma_g^-))$. Because $\mathrm{Ker}\mathcal{D}_{g,A} = \mathrm{Ker}\mathcal{D}^+_{g,A} \oplus \mathrm{Coker}\mathcal{D}^+_{g,A}$ we do not get any new information. In fact, what one would get if one proceeded as in the above proof are equations (i) and (ii) with Ψ_1 replaced by Ψ_+ and Ψ_2 replaced by Ψ_-. But these equations are contained in the above proposition, and conversely if these equations are known for Ψ_+ and Ψ_- we retrieve (i) and (ii) above because these equations are symmetric in Ψ_+ and Ψ_-.

Remark 4.7. Equations (ii) and (iii) are essentially contained in [BG]: If the analytic functions $\lambda_1(t), \dots, \lambda_n(t)$ (pairwise different) with $\lambda_i(0) = \lambda$ are eigenvalues for $\mathcal{D}_{g_t} - \lambda$, where the dimension of the λ-eigenspace is n then the equations $\frac{d}{dt}\big|_{t=0}\lambda_i = 0$ are implied by (ii) replacing harmonic spinors by eigenspinors with eigenvalue λ for $\mathcal{D}_g$ [BG,Th.24]. Conversely, the proof of [BG,Th.24] may easily be modified to prove that if $\frac{d}{dt}\big|_{t=0}\lambda_i = 0$ for all i then the metric g is critical at the eigenvalue λ. Thus the bifurcation-theoretic approach of [BG] is equivalent to our approach.

5. Partial Proofs

In this section we will prove Theorem 1.4 and the statements of Theorems 1.2 and 1.3 concerning spin^c-manifolds.

The following proof is essentially the proof of [Hij2] which however is applied rather differently in this reference.

Proof of Theorem 1.4. Consider dimensions 2 and 4 first. By Proposition 4.1 we know that for a fixed metric the image of $\mathcal{D}^+_{g,A}$ is tangential to some $\mathcal{F}_{n,k}$ with $n, k > 0$ if and only if $\langle X.\Psi_+, \Psi_-\rangle = 0$ for all harmonic spinors Ψ_+ and Ψ_-. Suppose that neither spinor vanishes. Then by the unique continuation principle [BW] there is an open dense subset of M on which neither vanishes. In dimension 2 the complex fibre dimension of $\Sigma^\pm$ is 1 and in dimension 4 it is 2. Thus there is always a vector field X such that $\langle X.\Psi_+, \Psi_-\rangle \not\equiv 0$. This shows that unless here are no nontrivial harmonic spinors of either positive or negative type we may deform the connection so as to reduce the dimension of the space of harmonic spinors. In dimensions 1 and 3 one may argue similarly: In these dimensions the complex fibre dimensions of Σ are 1 and 2 respectively. Thus given a nontrivial harmonic spinor Ψ we may always find a vector field X with $\langle X.\Psi, \Psi\rangle \not\equiv 0$. Thus unless there are no nontrivial harmonic spinors we may deform the connection so as to reduce the dimension of the space of harmonic spinors. □

In dimensions ≥ 5 it might happen that for a given metric and connection $T_pM.\mathcal{H}_p \cap \mathcal{H}_p = \{0\}$ for every $p \in M$, where $\mathcal{H} \neq \{0\}$ is the space of harmonic spinors and $\mathcal{H}_p$ is the subspace in the fibre Σ_p spanned by harmonic spinors. In this case (i) of Proposition 4.1 is satisfied but we have no means of deforming the connection so as to reduce the dimension of $\mathcal{H}$.

Also note that we are not able to extend our arguments to dimensions 7 and 8 as in [Hij2] because in dimensions 7 and 8 there is in general no parallel real structure on the spinor bundle Σ for a given spin^c-structure.

The following lemma contains parts of the statements of Theorems 1.2 and 1.3 concerning spin^c-manifolds:

Lemma 5.1. *Let M be a closed oriented 3- or 4-manifold with fixed spin^c-struture and fixed metric g and connection A on the canonical bundle. Suppose there are nontrivial harmonic spinors (of both chiralities in dimension 4) and that condition (ii) of Proposition 4.1 is satisfied. Then*

- *(i) All nontrivial harmonic spinors vanish on the same set N and on any connected set in the complement of N we have $|\Psi_1|/|\Psi_2|$ = const for nontrivial harmonic spinors Ψ_i.*
- *(ii) If dim$M = 3$ the dimension of the space of harmonic spinors is at most 2.*
- *(iii) If dim$M = 4$ and Index$\mathcal{D}_{g,A} \neq 0$, a generic metric has $h^+ + h^- \leq 3$ and Index$\mathcal{D}_{g,A} \in \{\pm 1\}$ unless either of $h^\pm$ is zero.*
- *(iv) If dim$M = 4$ and Index$\mathcal{D}_{g,A} = 0$ then $h^+ = h^- \leq 2$ for the generic metric.*

Proof. Consider first the 4-dimensional case: Suppose Im$\mathcal{D}^+_{g,A}$ is tangential to $\mathcal{F}_{n,k}$, $n, k \geq 1$ such that there are linearly independent harmonic spinors Ψ^1_+, Ψ^2_+, and let Ψ_- be a nontrivial negative harmonic spinor. Then by Proposition 4.1,

$$\langle X.\tilde{\nabla}^g_X \Psi^i_+, \Psi_-\rangle + \langle \Psi^i_+, X.\tilde{\nabla}^g_X \Psi_-\rangle = 0$$

for $i \in \{1, 2\}$. Suppose for the moment that there is an open connected set $U \subset M$ on which Ψ_+^1 does not vanish and where $\Psi_+^2 = f\Psi_+^1$ for a smooth function f. Plugging into the equation yields $(Xf)\langle X.\Psi_+^1, \Psi_-\rangle = 0$. At a fixed point $p \in U$ we may choose a basis $\{X_1, \dots, X_4\}$ for T_pM such that $\langle X_k.\Psi_+^1, \Psi_-\rangle \neq 0$. Thus $df|_p = 0$, and hence f is constant on U. By the unique continuation principle Ψ_+^2 is a constant multiple of Ψ_+^1 in contradiction to the assumption. Thus the set of points p at which $\Psi_+^1|_p$ and $\Psi_+^2|_p$ are linearly independent is open and dense. Fix a connected open subset U such that neither $\Psi_+^1|_p$ and $\Psi_+^2|_p$ vanish or are linearly dependent at any point p in U and such that Ψ_- vanishes nowhere on U.

Given another harmonic spinor Ψ_+' we may write $\Psi_+' = f_1\Psi_+^1 + f_2\Psi_+^2$ (where $f_i \in C^\infty(M, \mathbb{C})$) over U. Replacing Ψ_+^i in the equation by Ψ_+' we obtain

$$(Xf_1)\langle X.\Psi_+^1, \Psi_-\rangle + (Xf_2)\langle X.\Psi_+^2, \Psi_-\rangle = 0\,.$$

Let $F_i \subset TU$ be the subbundle Ker($X \in TU|_p \to \langle X.\Psi_+^i, \Psi_-\rangle|_p$). Both F_i have 2-dimensional real fibres and $F_1 \cap F_2 = \{0\}$. By the previous equation, a section $X_1 \in C^\infty(F_2)$ satisfies $X_1f_2 = 0$, and a section $X_2 \in C^\infty(F_1)$ satisfies $X_2f_1 = 0$.

Fix a point $p \in U$ and $0 \neq X_2 \in F_2|_p$ with $X_2f_2|_p = 0$, and choose $X_1 \in F_1|_p$. Set $X = X_1 + X_2$ and plug into the above equation. Then $0 = (X_1f_1)\langle X_2.\Psi_+^1, \Psi_-\rangle$. By fibrewise linear independence of Ψ_+^1 and Ψ_+^2 on U we find $X_1f_1|_p = 0$. Hence f_1 is constant on each component of U, and similarly f_2 is constant on each component, too. By the unique continuation principle Ψ_+' is a linear combination of Ψ_+^1 and Ψ_+^2.

Thus if $h^+, h^- \geq 2$ we find (by applying the above argument to positive and negative harmonic spinors) $h^+ = h^- = 2$. Thus if Index$\mathcal{D}_{g,A} \neq 0$ and both h^+ and h^- are positive we find that either h^+ or h^- are ≤ 1, and h^+ and h^- differ by one.

In dimension 3 it suffices to note that if $h \geq 2$ then two linearly independent harmonic spinors Ψ_1 and Ψ_2 have $\Psi_1|_p$ and $\Psi_2|_p$ linearly independent for p in some open dense set. This is proved as the corresponding statement in dimension 4. Then arguing as before we see that $h \leq 2$.

If the spin^c-structure is in fact a spin-structure we have a quaternion-structure on the spinor-bundle. Thus a critical metric on M which has both positive and negative harmonic spinors satisfies $h^+ = h^- = 2$. □

6. Dimension 2

In order to prove Theorem 1.1 we find it convenient to view the Picard-torus of a smooth line bundle L on a Riemann surface (M, g, J) in terms of connections on L. We shall always assume that M carries a metric which induces the given complex structure.

Given a line bundle L over a Riemann surface (M, J) and a partial connection $\nabla^{0,1}$ on L, this partial connection induces a holomorphic structure on L. This follows from the usual integrability theorems [Do,Th.2.1.53] because $\Omega^{2,0} \oplus \Omega^{0,2} = \{0\}$. When we want to emphasize that L is considered as a holomorphic bundle with the structure induced by $\nabla^{0,1}$ we write $(L, \nabla^{0,1})$.

Given an isomorphism f of L (which we think of as a smooth map $f : M \mapsto \mathbb{C}^*$) we may pull back a given partial connection $\nabla^{0,1}$ along f to obtain the partial connection $f \circ \nabla^{0,1} \circ f^{-1} = \nabla^{0,1} + f(\bar\partial f^{-1})$. Then $(L, \nabla^{0,1})$ and $(L, \nabla^{0,1} + f(\bar\partial f^{-1}))$ are holomorphically equivalent. And if $(L, \nabla^{0,1})$ and $(L, \nabla^{0,1} + \phi)$, $\phi \in \Omega^{0,1}(M)$, are holomorphically equivalent then there is a smooth function $f : M \to \mathbb{C}^*$ such that $f \circ \nabla^{0,1} \circ f^{-1} = \nabla^{0,1} + f(\bar\partial f^{-1}) = \nabla^{0,1} + \phi$. Thus $\phi = f(\bar\partial f^{-1})$.

Note that the additive group $\{f(\bar\partial f^{-1}) : f \in C^\infty(M,\mathbb{C}^*)\}$ splits as $\bar\partial(C^\infty(M,\mathbb{C}))\oplus H^{0,1}(M,2\pi i\mathbb{Z})$, by writing $f = ue^h$ with $u : M \to S^1$ a harmonic map and $h : M \to \mathbb{C}$. Here $H^{0,1}(M,2\pi i\mathbb{Z})$ is the projection of $H^1(M,2\pi i\mathbb{Z}) \subset H^1(M,\mathbb{C})$ into the $(0,1)$-component.

Note that any holomorphic structure on L is defined by $\nabla^{0,1}$ for a suitable connection. Thus the moduli-space of holomorphic structures on L is the quotient

$$\Omega^{0,1}/(\bar\partial(C^\infty(M,\mathbb{C}))\oplus H^{0,1}(M,2\pi i\mathbb{Z})) = H^{0,1}(M,\mathbb{C})/H^{0,1}(M,2\pi i\mathbb{Z})\,.$$

This quotient is a complex torus, called the Picard torus of L.

Now fix a Hermitian metric h on L. For any holomorphic structure there is a connection which induces the given holomorphic structure and preserves h, i.e. $\nabla h = 0$.

Given two h-preserving connections ∇_1 and ∇_2 on (L,h) which induce the same holomorphic structure on L, we have $\nabla_2^{0,1} - \nabla_1^{0,1} = \phi \in \bar\partial(C^\infty(M,\mathbb{C}))\oplus H^{0,1}(M,2\pi i\mathbb{Z})$. Because $\nabla_2 - \nabla_1 \in i\Omega^1(M,\mathbb{R})$, we find that $\nabla_2 - \nabla_1 = \phi - \bar\phi$.

We now return to spin-structures on M: Let $K = \Omega^{1,0}(M)$ be the canonical bundle of (M,J). Spin-strucures on M correspond to holomorphic square-roots of K by [Hi,Th.2.2]. Fix some such square-root. Given a metric on M which induces the given complex structure, K and L inherit hermitian metrics. Let $\tilde\nabla$ be the hermitian connection on L, and ∇ the hermitian connection on K. Note that $\tilde\nabla\otimes\tilde\nabla = \nabla$. As all square-roots of K are isomorphic as unitary bundles we may think of them as being of the form $(L,\tilde\nabla+\omega)$ with $\omega \in i\Omega^1(M,\mathbb{R})$. Taking the square we get a connection $\nabla' := \tilde\nabla\otimes\tilde\nabla + 2\omega$ on $K = L\otimes L$. Observe now:

Lemma 6.1. *∇' induces the same holomorphic structure on K as does ∇ if and only if $2\omega^{0,1} \in \bar\partial(C^\infty(M,\mathbb{C}))\oplus H^{0,1}(M,2\pi i\mathbb{Z})$, that is if and only if the cohomology class $[\omega^{0,1}]$ is contained in the lattice obtained by projecting $\frac{1}{2}H^1(M,2\pi i\mathbb{Z})$ into $H^{0,1}(M,\mathbb{C})$.*

Observe that $\dim H^1(M,\mathbb{R}) = \mathrm{rank} H^1(M,\mathbb{Z}) = 2\,\mathrm{genus}(M)$. We thus retrieve the well known fact that there are $2^{2genus(M)}$ spin-structures on M.

Armed with these preliminary remarks we can now embark upon a proof of Theorem 1.1. The following lemma is the analogue of Lemma 5.1 for dimensions 3 and 4 above.

Lemma 6.2. *Let M be a closed 2-dimensional manifold and fix a spin^c-structure, a metric g on M and a connection A on the auxiliary bundle P_{U_1}, and let $\mathcal{D}_{g,A} : \Sigma^+ \to \Sigma^-$ be the Atiyah–Singer Dirac operator. Suppose that at g, Im$D\mathcal{D}^+_{g,A}$ is tangential to $\mathcal{F}_{n,k}$, $n,k > 0$. Then either $c_1(P_{U_1}) = 0$ and dimKer$\mathcal{D}^+_{g,A}$ = dimKer$\mathcal{D}^-_{g,A} \leq 1$, or $c_1(P_{U_1}) \neq 0$ and there are no harmonic spinors of either positive or negative chirality. In the first case, given two nontrivial harmonic spinors Ψ_+ and Ψ_- of positive and negative chirality respectively, we have $|\Psi_+| = \lambda|\Psi_-|$ for some $\lambda > 0$. In this case Ψ_+ and Ψ_- vanish on the same finite set of points.*

Proof. Pick two harmonic spinors Ψ_+ and Ψ_-. Then by Proposition 4.1,

$$\langle X.\tilde\nabla^g_X\Psi_+, \Psi_-\rangle + \langle\Psi_+, X.\tilde\nabla^g_X\Psi_-\rangle = 0\,.$$

Suppose that neither spinor vanishes identically. By the unique continuation principle [BW] we may choose an open connected subset $U \subset M$, where neither Ψ_+ nor Ψ_- vanish. Note that the fibre dimension of each $\Sigma^\pm$ is 1. Let Ψ_+' be another harmonic spinor, and over U write $\Psi_+' = f\Psi_+$ for some $f \in C^\infty(U,\mathbb{C})$. Replace Ψ_+ in the previous equation by Ψ_+' to obtain $(Xf)\langle X.\Psi_+, \Psi_-\rangle = 0$. As this holds for every vector field over U we see that f is constant on U. By the unique continuation principle Ψ'_+ is a constant multiple of Ψ_+, and hence $\dim\mathrm{Ker}\mathcal{D}^+_{g,A} = 1$. Repeating the argument with Ψ_- shows that $\dim\mathrm{Ker}\mathcal{D}^-_{g,A} = 1$, too, and thus $\mathrm{Index}\mathcal{D}^+_{g,A} = \frac{1}{2}c_1(P_{U_1}) = 0$.

Let X be a vector field on U with $|X| = 1$. Then there is the identity $|\Psi_-|^2\Psi_+ = X.\Psi_-\langle\Psi_+, X.\Psi_-\rangle$. Then

$$\begin{aligned} |\Psi_-|^2\, X|\Psi_+|^2 &= |\Psi_-|^2\langle\tilde\nabla_X\Psi_+,\Psi_+\rangle + |\Psi_-|^2\langle\Psi_+,\tilde\nabla_X\Psi_+\rangle \\ &= \langle\tilde\nabla_X\Psi_+, X.\Psi_-\rangle\langle X.\Psi_-,\Psi_+\rangle + \langle X.\Psi_-,\tilde\nabla_X\Psi_+\rangle\langle\Psi_+, X.\Psi_-\rangle \\ &= \langle\Psi_+, X.\tilde\nabla_X\Psi_-\rangle\langle X.\Psi_-,\Psi_+\rangle + \langle X.\tilde\nabla_X\Psi_-,\Psi_+\rangle\langle\Psi_+, X.\Psi_-\rangle \\ &= 2\mathrm{Re}\left(\langle\Psi_+, X.\tilde\nabla_X\Psi_-\rangle\overline{\langle\Psi_+, X.\Psi_-\rangle}\right)\,. \end{aligned}$$

Note that the last expression is symmetric in Ψ_+ and Ψ_-. Thus we obtain the equation $|\Psi_-|^2\, X|\Psi_+|^2 = |\Psi_+|^2\, X|\Psi_-|^2$. Given a point p in M with $\Psi_+|_p \not\equiv 0$ we conclude that $|\Psi_+| = \lambda|\Psi_-|$ in a neighbourhood of p. Because Ψ_- does not vanish on a dense open set by the unique continuation principle [BW] it follows that $\lambda > 0$. Thus if $\Psi_+|_p = 0$ at some $p \in M$ then also $\Psi_-|_p = 0$. By symmetry, Ψ_+ and Ψ_- vanish on the same set, and because Ψ_+ is a holomorphic section of Σ^+ with respect to the holomorphic structure induced by $\tilde\nabla^{0,1}$ [Hi], we have $|\Psi+| = \lambda|\Psi_-|$ for some $\lambda > 0$ on all of M. □

Lemma 6.3. *Let M be a closed 2-manifold. Let g be a metric on M and $a \in i\Omega^1(M)$, and fix a spin-structure on M. Denote the positive spinor bundle by L and let $\tilde\nabla$ be the connection on L induced by g. Let $\mathcal{D}_{g,A}$ be the Dirac operator obtained from the connection $\tilde\nabla+a$. Suppose that Im$D\mathcal{D}_{g,A}$ is tangential to $\mathcal{F}_{n,n}$ for $n > 0$. Then $(L,\tilde\nabla+a)$ is a holomorphic square-root of K and thus a spin-structure, possibly different from L. There is a smooth function $f : M \to S^1$ with $a = df/2f$. The form a is closed and defines an element $[a] \in H^1(M, 2\pi i\mathbb{Z})$.*

Proof. Let K denote the canonical bundle, and let L be the square root of K defined by the spin-structure. Given a metric on M there is an antilinear isomorphism $h : \overline{K}\otimes L \to L$ given on smooth sections v and w of either bundle by $\langle v, h(w)\rangle = \int vw$ [At]. Let ϕ be a local section of K of unit length over some open set $U \subset M$, and let $\overline{\phi}$ be the corresponding section of $\overline{K}$. Let σ be a section of L with $\sigma\otimes\sigma = \sigma^2 = \phi$. Then necessarily $|\sigma| = 1$. For $f \in C^\infty(U,\mathbb{C})$:

$$\langle\sigma, h(f\overline{\phi}\wedge\sigma)\rangle_{L^2} = \int f\,\overline{\phi}\wedge\phi = i\int f\ \mathrm{dvol}_g\,.$$

Using a Dirac-sequence for f we find that $h(\overline{\phi}\wedge\sigma) = i\sigma$. Now pick any point $p \in U$ and choose a g-orthonormal frame $\{e_1, e_2\}$ in a neighbourhood of p such that $\nabla e_1|_p = 0$. Thus $\nabla\sigma|_p = 0$ and $\nabla\overline{\phi}|_p = 0$. It is immediate that $\nabla h|_p = 0$. As p was arbitrary it follows that h is parallel, and h is unitary with respect to the hermitian metrics on both bundles. Pull back the connection $\hat\nabla + a$ on $\overline{K}\otimes L$ to L along h. We compute $h\circ(\hat\nabla + a)\circ h^{-1} = \tilde\nabla - a$, because h is parallel and antilinear and $a \in i\Omega^1(M)$. Let X be a smooth vector field and denote by J the complex structure on M induced by the metric. Compute:

$$\begin{aligned} \tilde\nabla^{0,1}_X - a^{0,1}(X) &= \frac{1}{2}\left(\hat\nabla_X - a(X) + i\hat\nabla_{JX} - ia(JX)\right) \\ &= \frac{1}{2}h\circ\left(\hat\nabla_X + a(X) - i\hat\nabla_{JX} - ia(JX)\right)\circ h^{-1} \end{aligned}$$

$$= h \circ \left(\hat{\nabla}^{1,0} + a^{1,0}\right) \circ h^{-1} .$$

Let $\Delta_a^{0,1}$ be the Laplace-operator on $L \oplus (\Omega^{0,1}(M) \otimes L)$ associated with the operator $\hat{\nabla}^{0,1} + a^{0,1}$, and let similarly $\Delta_a^{1,0}$ be the Laplace-operator associated with the operator $\hat{\nabla}^{1,0} + a^{1,0}$. Let Ψ_- be a section of $\overline{K} \otimes L$. Then by [Hi] Ψ_- is harmonic if and only if $\Delta_a^{0,1}\Psi_- = 0$. This is equivalent to demanding $\Delta_a^{1,0}\Psi_- = 0$, because $\Delta_a^{1,0} = \Delta_a^{0,1}$. The latter condition translates into $(\hat{\nabla}^{1,0} + a^{1,0})\Psi_- = 0$. By the previous computation we see that Ψ_- is harmonic if and only if $h(\Psi_-)$ is a holomorphic section of L with respect to the holomorphic structure induced by $\tilde{\nabla} - a$.

Now suppose the metric on M is critical and there are harmonic spinors Ψ_+ and Ψ_-. By the previous lemma we may assume that $|\Psi^+| = |h(\Psi_-)|$. Let P be the finite set of points where both sections vanish, and choose a function $f \in C^\infty(M \setminus P)$ such that $h(\Psi_-) = f\Psi_+$. Harmonicity of Ψ_+ is equivalent to $(\tilde{\nabla}^{0,1} + a^{0,1})\Psi_+ = 0$, which together with $(\tilde{\nabla}^{0,1} - a^{0,1})h(\Psi_-) = 0$ implies $\overline{\partial} f - 2fa^{0,1} = 0$ by substitution. Thus $a^{0,1} = \overline{\partial} f/2f$. Because $|f| = 1$ we have $b := df/2f \in i\Omega^1(M)$. As $a \in i\Omega^1(M)$, too, both a and b satisfy $a^{1,0} = -\overline{a^{0,1}}$ and $b^{1,0} = -\overline{b^{0,1}}$ which implies $a = b = df/2f$. By continuity we see that a is a closed form. Pick any $p \in P$ and a neighbourhood D of p, which we may assume diffeomorphic to a disc. There $a = idh$ for some smooth real valued function h on D. Cutting out a radial line of D yields a contractible set D', where we may assume $f = e^{ig}$ for some smooth real valued function g. Thus on D' we have $dh = \frac{1}{2}dg$. Thus $g = 2h + \text{const}$ on D'. It follows that f may be smoothly continued into p.

In total we have found a function $f \in C^\infty(M, \mathbb{C})$ with $|f| = 1$ such that $a^{0,1} = \overline{\partial} f/2f$. But by Lemma 6.1 of this section and the discussion preceding it this implies that $(L, \tilde{\nabla}^{0,1} + a^{0,1})$ is a holomorphic square-root of K. □

We may now proceed to prove the main theorem of this section:

Proof of Theorem 1.1. By Lemma 6.2 of this section we see that all there remains to do is to study the case of twisted spin-structures. Let g be a metric on M and $a \in i\Omega^1(M)$, and fix a spin-structure on M. Denote the positive spinor bundle by L and let $\tilde{\nabla}$ be the connection on L induced by g. Let $\mathcal{D}_{g,A}$ be the Dirac operator obtained from the connection $\tilde{\nabla} + a$. If a is closed and represents a class in $H^1(M, 2\pi i\mathbb{Z})$ then $(L, \tilde{\nabla} + a)$ is again a square root of K, for an arbitrary metric g.

Suppose there is a metric g with nontrivial harmonic spinors of both chiralities which is critical. By Lemma 6.2 we have $h^+ = h^- = 1$, and Lemma 6.3 implies that $(L, \tilde{\nabla} + a)$ is in fact another spin-structure and $a = df/2f$ for some smooth function $f : M \to S^1$. Hence a defines an element in $H^1(M, 2\pi i\mathbb{Z})$. Thus unless the twisted spin-structure is a spin structure itself, no metric can be critical and hence any metric with nontrivial harmonic spinors may be deformed to one without. In the spin-case the value $h^+ = \text{dimKer}\mathcal{D}_g^+ \mod 2$ is independent of the choice of metric and depends only on the spin-structure [At, Mum, ACGH]. As a critical metric has $h^+ \in \{0, 1\}$, we see that metrics with $h^+ > 1$ cannot be critical and can thus be perturbed to a new metric with less harmonic spinors. Thus if $\text{dimKer}\mathcal{D}_g^+ = 0 \mod 2$ the generic metric will have $\text{dimKer}\mathcal{D}_g^+ = 0$. In the other case $\text{dimKer}\mathcal{D}_g^+ = 1$ for the generic metric. □

Remark 6.4. The initiated will have noticed that $h : \overline{K} \otimes L \to L$ is essentially the Serre-duality map, possibly up to sign. In fact, if $\overline{*}_L$ denotes the complex conjugate of the Hodge-$*$-operator with coefficients in L [W, p. 166] we have a map

$$\bar{K} \otimes L \xrightarrow{\bar{*}_L} K \otimes L^{-1} \cong L\,.$$

We find $\bar{*}(\bar{\phi} \otimes \sigma) = -i\phi$, and thus the above map is identified as the following map: $\bar{\phi} \otimes \sigma \overset{\bar{*}_L}{\longmapsto} -i\phi \otimes \langle ., \sigma\rangle = -i\sigma \otimes \sigma \otimes \langle ., \sigma\rangle \overset{\cong}{\longmapsto} -i\sigma$. Hence h is the negative of the Serre-duality map.

Remark 6.5. We should mention that there are spin-structures on M for which there are always harmonic spinors. In fact, their number can be computed and it turns out to be $2^{g-1}(2^g - 1)$ [At,Th.3] where g = genus(M).

7. Applications to Riemann Surfaces

Theorem 1.1 is really a theorem in the theory of Riemann surfaces, their moduli and the moduli of holomorphic line bundles.

Let L_c denote the positive spinor bundle for a fixed spin-structure on a Riemann surface. Here, the index c is the parameter of the complex structures on the underlying closed 2-manifold M. Let F be a Hermitian line bundle with connection A with respect to which the Hermitian metric is parallel. Then given a complex c on M, the connection induces a holomorphic structure on F. Denote this holomorphic bundle by $F_{A,c}$.

We may now restate Theorem 1.1 as follows: If $c_1(F) \neq 0$ then for a dense open subset of Teichmüller space $h^0(L_c \otimes F_{A,c}) = 0$ in case $c_1(F) < 0$, and $h^0(L_c \otimes F_{A,c}) = c_1(F)$ in case $c_1(F) > 0$. If $c_1(F) = 0$, write $A = d + a$, $a \in i\Omega^1(M)$, with respect to some trivialization of F. Then unless $a \in H^1(M, 2\pi i\mathbb{Z})$, $h^0(L_c \otimes F_{A,c}) = 0$ for generic c. Otherwise, $L_c \otimes F_{A,c}$ is a holomorphic square root of K_c and for the generic complex structure $h^0(L_c \otimes F_{A,c}) = 0$ or 1 depending only on the spin-structure. Thus in particular we have the following extension of the classical results of [At, Mum]:

Theorem 7.1. *The function $h^0 : c \in \mathcal{T}(M) \mapsto h^0(L_c)$ is constant on a generic (i.e. dense and open) subset $\mathcal{C}$ of Teichmüller space $\mathcal{T}(M)$. On $\mathcal{C}$ the image of h^0 is contained in $\{0, 1\}$, and the actual value depends only on which spin-structure is chosen.*

In the theory of Riemann surfaces spin-structures are often called Theta-characteristics. Note that if (M, c) is hyperelliptic then $h^0(L_c) = [(g + 1)/2]$ for at least one square root of K [BaS,Th.3,Th.4]. Thus for genus(M) ≥ 3 the generic set $\mathcal{C}$ of the proposition is not all of Teichmüller-space for at least one square-root of K.

Remark 7.2. One may ask what kind of subset the set $\mathcal{D}(L) := \{\, c \in \mathcal{T}(M) |\, h^0(L_c) > 1 \,\}$ is. First, by [Gro,Th.3.1 & Rem.3.2.2] there is a smooth analytic space V and an analytic submersion $\pi : V \to \mathcal{T}(M)$ such that $\pi^{-1}(c) = (M, c)$, i.e. M furnished with the complex structure c. Using Grauert's upper-semicontinuity theorem [Gra,Satz 3;GR,5.10.4] we find that $\mathcal{D}(L)$ is an analytic subset of $\mathcal{T}(M)$. I.e. $\mathcal{D}(L)$ is a locally finite union of irreducible analytic subsets of $\mathcal{T}(M)$. Compare also [Far].

Remark 7.3. An obvious question is whether the sets $\mathcal{D}(L) := \{\, c \in \mathcal{T}(M) |\, h^0(L_c) > 1 \,\}$ do depend on the square root L of K. First, note that for genus(M) < 3 the value of $h^0(L_c)$ is independent of the complex structure c and $h^0(L_c) \in \{0, 1\}$ [Hi,Prop.2.3]. Thus $\mathcal{D}(L) = \emptyset$ for genus(M) < 3. Thus consider the case genus(M) ≥ 3: By [BaS,Th.3,Th.4], on any hyperelliptic surface (M, c) there is always a square-root L of K for which $h^0(L_c) = 0$ and a square root L'_c for which $h^0(L'_c) = [(g + 1)/2]$. This shows that the sets $\mathcal{D}(L)$ do indeed depend upon the spin-structure chosen. Needless to say, we may take the union $\mathcal{D} := \bigcup_{L^2=K} \mathcal{D}(L)$ to find an analytic subset such that on the complement the function $c \mapsto h^0(L_c) \in \{0, 1\}$ is constant for every square root of K.

Remark 7.4. Theorem 1.4 may be read as follows: For generic holomorphic structures h on a line bundle F over a fixed Riemann surface, $h^0(F_h) = c_1(F) + 1 - \text{genus}(M)$ if $c_1(F) \geq \text{genus}(M) - 1$, and $h^0(F_h) = 0$ if $c_1(F) < \text{genus}(M) - 1$. Here, a generic set is a dense open subset of the Picard torus for F. This is of course a basic result of Brill–Noether theory [Gu,p.51]. The fact that $h^0(F_h) \geq c_1(F) + 1 - \text{genus}(M)$ for every h is a trivial consequence of the Riemann–Roch theorem. Brill–Noether theory also shows that the set of holomorphic structures for which $h^0(L_h)$ is greater than the minimal value is a union of analytic subsets.

8. Dimensions 3 and 4

In this section we will prove the statement on spin-manifolds in Theorem 1.3 first and then indicate the necessary changes in dimension 3. *Thus assume for the time being that M is a closed spin-4-manifold with a fixed spin-structure and a metric which is critical and has both nontrivial positive and negative harmonic spinors* (if g was not critical we could deform the metric so as to reduce the dimension of the space of harmonic spinors). The aim is to show that (M, g) is conformally flat. We shall even show that in this situation (M, g) is conformally equivalent to a flat torus, see Proposition 8.12 below. This shows that only in this particular situation a metric may be critical without being minimal. Otherwise critical metrics are precisely the minimal metrics.

Fix two nontrivial harmonic spinors Ψ_+ and Ψ_-. The proof of Theorem 1.3 will extend over a rather long list of lemmas.

Lemma 8.1. *On each connected set on which Ψ_+ does not vanish there is $\lambda > 0$ with $|\Psi_+| = \lambda|\Psi_-|$. In particular, Ψ_+ and Ψ_- vanish on the same set. Moreover, linear combinations of Ψ_+ and $J\Psi_+$ (respectively Ψ_- and $J\Psi_-$) are the only positive (negative) harmonic spinors on M, i.e. $h^+ = h^- = 2$.*

Proof. The last statement is proved above in Lemma 5.1. Let X be a vector field on some open connected with $|X| = 1$. Then $|\Psi_-|^2\Psi_+ = \langle\Psi_+, X.\Psi_-\rangle X.\Psi_- + \langle\Psi_+, X.J\Psi_-\rangle X.J\Psi_-$. With this we may compute

$$|\Psi_-|^2 X|\Psi_+|^2 = 2\text{Re}\langle\Psi_+, X\tilde{\nabla}_X\Psi_-\rangle\langle X.\Psi_-, \Psi_+\rangle$$
$$+2\text{Re}\langle\Psi_+, X\tilde{\nabla}_X J\Psi_-\rangle\langle X.J\Psi_-, \Psi_+\rangle\,.$$

By (ii) of Proposition 4.1 this is symmetric in Ψ_+ and Ψ_-, and arguing as before in the proof of Lemma 6.2 we may deduce the lemma. □

Fix a connected open set U on which neither Ψ_+ nor Ψ_- vanish. We may assume that $|\Psi_+| = |\Psi_-|$ by the preceding lemma. We may fix an ON-frame by the rule:

$$e_1.\Psi_+ = \Psi_- \quad e_2.\Psi_+ = i\Psi_- \quad e_3.\Psi_+ = J\Psi_- \quad e_4.\Psi_+ = -iJ\Psi_-$$

(to see that this is oriented compute $e_1e_2e_3e_4\Psi_+ = -\Psi_+$). In the sequel we shall always let X and Y be vector fields on U with $|X| = |Y| = 1$ and $X \perp Y$ such that they map Ψ_+ to a harmonic spinor under Clifford multiplication.

Lemma 8.2. *Let $\omega_X(e_i, e_j) := \langle\nabla_{e_i}X, e_j\rangle - \langle\nabla_{e_j}X, e_i\rangle$. Then the following holds:*

$$div(X)\Psi_+ + 2\tilde{\nabla}_X\Psi_+ - \omega_X\Psi_+ = 0\,,$$

where $\omega_X\Psi_+ := \sum_{i<j}\omega_X(e_i, e_j)e_ie_j\Psi_+$.

Proof. By definition of X, $X.\Psi_+$ is harmonic. Thus:

$$\begin{aligned} 0 &= \sum_i e_i \tilde{\nabla}_{e_i}(X.\Psi_+) \\ &= \sum_i e_i(\nabla_{e_i}X)\Psi_+ + e_i X \tilde{\nabla}_{e_i}\Psi_+ \\ &= \sum_{i,k} \langle \nabla_{e_i}X, e_k\rangle e_i e_k \Psi_+ - 2\tilde{\nabla}_X \Psi_+ \\ &= -\sum_i \langle \nabla_{e_i}X, e_i\rangle \Psi_+ - 2\tilde{\nabla}_X\Psi_+ + \sum_{i<k} \omega_X(e_i, e_k) e_i e_k \Psi_+ \\ &= -\mathrm{div}(X)\Psi_+ - 2\tilde{\nabla}_X \Psi_+ + \omega_X \Psi_+ . \end{aligned}$$

Of course, the particular choice of ON-frame plays no role here. □

Lemma 8.3. *Let X be as above and Z any smooth vector field which is everywhere orthogonal to X. Then*

$$2Re\langle \tilde{\nabla}_Z \Psi_+, ZX\Psi_+\rangle + \langle \nabla_Z X, Z\rangle |\Psi_+|^2 = 0 .$$

Proof. We may assume that $|Z| = 1$. Equation (ii) of Proposition 4.1 yields

$$\begin{aligned} 0 &= \langle Z\tilde{\nabla}_Z\Psi_+, X\Psi_-\rangle + \langle \Psi_+, Z\tilde{\nabla}_Z(X\Psi_+)\rangle \\ &= -\langle \tilde{\nabla}_Z \Psi_+, ZX\Psi_-\rangle + \langle \Psi_+, Z(\nabla_Z X)\Psi_+\rangle + \langle \Psi_+, ZX\tilde{\nabla}_Z\Psi_+\rangle \\ &= -2\mathrm{Re}\langle \tilde{\nabla}_Z\Psi_+, ZX\Psi_-\rangle - \langle \nabla_Z X, Z\rangle |\Psi_+|^2 + \langle \Psi_+, ZW\Psi_+\rangle , \end{aligned}$$

where $W := \nabla_Z X - \langle \nabla_Z X, Z\rangle Z$ is orthogonal to both Z and X. Thus

$$\mathrm{Re}\langle \Psi_+, ZW\Psi_+\rangle = -\mathrm{Re}\langle \Psi_+, ZW\Psi_+\rangle ,$$

and hence $\langle \Psi_+, ZW\Psi_+\rangle$ is imaginary-valued. The lemma now follows. □

It is useful to observe that $\langle \Psi_+, e_i e_j \Psi_+\rangle$ is always imaginary-valued if $i \neq j$. This follows as in the preceding proof.

Lemma 8.4. *$\nabla_{e_i} e_j$ is a multiple of e_i in each fibre provided $i \neq j$.*

Proof. Let $i < j$. By Lemma 8.2:

$$\mathrm{div}(e_i)\langle \Psi_+, e_i e_j \Psi_+\rangle + 2\langle \tilde{\nabla}_{e_i}\Psi_+, e_i e_j \Psi_+\rangle - \langle \omega_{e_i}\Psi_+, e_i e_j \Psi_+\rangle = 0 .$$

Taking real parts we obtain:

$$2\mathrm{Re}\langle \tilde{\nabla}_{e_i}\Psi_+, e_i e_j \Psi_+\rangle - \mathrm{Re}\langle \omega_{e_i}\Psi_+, e_i e_j\Psi_+\rangle = 0 .$$

Now $\mathrm{Re}\langle \omega_{e_i}\Psi_+, e_i e_j \Psi_+\rangle = \omega_{e_i}(e_i, e_j)|\Psi_+|^2 \pm \omega_{e_i}(e_k, e_l)|\Psi_+|^2$, where $k < l$ are different from i, j, and the sign is taken to be + if e_k, e_l, e_i, e_j is oriented and − otherwise. Plugging in the definition of ω_{e_i} we obtain

$$2\mathrm{Re}\langle \tilde{\nabla}_{e_i}\Psi_+, e_i e_j \Psi_+\rangle - \langle \nabla_{e_i} e_i, e_j\rangle |\Psi_+|^2 \pm \left(\langle \nabla_{e_k} e_i, e_l\rangle - \langle \nabla_{e_l} e_i, e_k\rangle\right) |\Psi_+|^2 = 0 .$$

By the preceding lemma the sum of first two terms vanishes. Thus

$$\langle\nabla_{e_k} e_i, e_l\rangle - \langle\nabla_{e_l} e_i, e_k\rangle = -\langle e_i, [e_k, e_l]\rangle = 0$$

for arbitrary choices of k, l and $i \neq k$ and $i \neq l$. By the Koszul formula [O'N,3.11]

$$2\langle\nabla_{e_i} e_j, e_k\rangle = -\langle e_i, [e_j, e_k]\rangle + \langle e_j, [e_k, e_i]\rangle + \langle e_k, [e_i, e_j]\rangle$$

we see that $\nabla_{e_i} e_j \perp e_k$ for $i \neq j$ and k different from both i, j. The lemma follows. □

Lemma 8.5. *The value of* $\langle\nabla_{e_k} e_i, e_k\rangle$ *is independent of the choice of* $k \neq i$.

Proof. To this end let $j \neq k$ and $j \neq i$ and compute

$$\begin{aligned} 0 &= \langle e_j \tilde{\nabla}_{e_k} \Psi_+, e_i \Psi_+\rangle + \langle \Psi_+, e_j \tilde{\nabla}_{e_k}(e_i \Psi_+)\rangle \\ &\quad + \langle e_k \tilde{\nabla}_{e_j} \Psi_+, e_i \Psi_+\rangle + \langle \Psi_+, e_k \tilde{\nabla}_{e_j}(e_i \Psi_+)\rangle \\ &= 2\mathrm{Re}\langle \tilde{\nabla}_{e_k} \Psi_+, e_i e_j \Psi_+\rangle + 2\mathrm{Re}\langle \tilde{\nabla}_{e_j} \Psi_+, e_i e_k \Psi_+\rangle \\ &\quad + \left(\langle\nabla_{e_k} e_i, e_k\rangle - \langle\nabla_{e_j} e_i, e_j\rangle\right) \langle \Psi_+, e_j e_k \Psi_+\rangle \,, \end{aligned}$$

where we have used the preceding lemma. The first two terms are real-valued and the last term is imaginary-valued. Thus

$$\left(\langle\nabla_{e_k} e_i, e_k\rangle - \langle\nabla_{e_j} e_i, e_j\rangle\right) \langle \Psi_+, e_j e_k \Psi_+\rangle = 0\,.$$

If $e_j e_k \Psi_+ \in \{\pm i \Psi_+\}$ then $\langle\nabla_{e_k} e_i, e_k\rangle = \langle\nabla_{e_j} e_i, e_j\rangle$. Otherwise replace the first Ψ_+ in each bracket by $J\Psi_+$ and compute:

$$\begin{aligned} 0 &= \langle e_j \tilde{\nabla}_{e_k} J\Psi_+, e_i \Psi_+\rangle + \langle J\Psi_+, e_j \tilde{\nabla}_{e_k}(e_i \Psi_+)\rangle \\ &\quad + \langle e_k \tilde{\nabla}_{e_j} J\Psi_+, e_i \Psi_+\rangle + \langle J\Psi_+, e_k \tilde{\nabla}_{e_j}(e_i \Psi_+)\rangle \\ &= \left(\langle\nabla_{e_k} e_i, e_k\rangle - \langle\nabla_{e_j} e_i, e_j\rangle\right) \langle J\Psi_+, e_j e_k \Psi_+\rangle \,, \end{aligned}$$

and if now $e_j e_k \Psi_+ \in \{\pm J\Psi_+, \pm iJ\Psi_+\}$, then again $\langle\nabla_{e_k} e_i, e_k\rangle = \langle\nabla_{e_j} e_i, e_j\rangle$. □

Lemma 8.6. *If* $|\Psi_+|$ *is constant on* U *then the* e_i *and all harmonic spinors are parallel over* U.

Proof. If $|\Psi_+|$ is constant on U, $\langle\tilde{\nabla}\Psi_+, \Psi_+\rangle$ is imaginary-valued. Thus $0 = \mathrm{div}(e_i)|\Psi_+|^2$ by Lemma 8.2. Hence $\mathrm{div}(e_i) = 0$. But

$$\mathrm{div}(e_i) = \sum_{k \neq i} \langle\nabla_{e_k} e_i, e_k\rangle\,,$$

and by the preceding lemma we see that $\langle\nabla_{e_k} e_i, e_k\rangle = 0$ for all i, k, and thus by Lemma 8.4 each e_i is parallel on U. Lemma 8.2 then implies that $\Psi_\pm$ are parallel. □

Lemma 8.7. (M, g) *is conformally flat.*

Proof. Given $p \in M$ with $\Psi_+|_p \neq 0$ choose a smooth function f with $f = \frac{1}{3} \ln |\Psi_+|^2$ in a neighbourhood of p. Let $g' := e^{2f} g$. By Remark 2.6 the spinor $\overline{\beta}_{g',g}\Psi_+$ has constant norm in a neighbourhood of p. Thus g' is flat in a neighbourhood of p by the last lemma. Thus the Weyl-tensor for g vanishes on the set of points where Ψ_+ does not vanish. But this set in dense in M, and hence the Weyl-tensor for g vanishes identically, which means that (M, g) is conformally flat. □

Remark 8.8. In dimension 5 the situation is considerably more complicated: There are 5-dimensional non-conformally flat Riemannian spin-manifolds the nontrivial spinors of which are all parallel [BFGK, p.150]. Thus this metric is critical but not minimal.

Proof of Theorem 1.3. Note that we may perturb g by a C^1-small deformation to get a metric with $h^+(g) \leq 2$ and nonvanishing Weyl-tensor $W(g)$. Hence this metric cannot be critical by the preceding discussion, and we may thus find a small perturbation of g to get a new metric without harmonic spinors. □

Proof of Theorem 1.2. We now indicate the changes in the above proof necessary to prove the corresponding result in dimension 3. Thus let (M, g) be an oriented Riemannian 3-Manifold with fixed spin-structure such that there is a nontrivial harmonic spinor Ψ for the metric g which is critical in the sense of the remark following Proposition 4.1. Define a local ON-frame $\{e_1, e_2, e_3\}$ such that

$$e_1\Psi = i\Psi \quad e_2\Psi = iJ\Psi \quad e_3\Psi = J\Psi.$$

Lemma 8.2 holds, and because $e_1e_2e_3\Psi = -\Psi$ we obtain

$$\text{div}(X)\Psi + 2\tilde{\nabla}_X\Psi - \omega_X(e_1, e_2)e_3\Psi + \omega_X(e_1, e_3)e_2\Psi - \omega_X(e_2, e_3)e_1\Psi = 0$$

This equation immediately implies

$$2\text{Re}\langle\tilde{\nabla}_{e_k}\Psi, e_k\Psi\rangle \pm \omega_{e_k}(e_i, e_j)|\Psi|^2 = 0\,,$$

where $i < j$ are both different from k, and the sign depends on k. Note that by (ii) of Proposition 4.1,

$$\text{Re}\langle\tilde{\nabla}_{e_k}\Psi, e_k\Psi\rangle = -\text{Re}\langle e_k\tilde{\nabla}_{e_k}\Psi, \Psi\rangle = \text{Re}\langle\Psi, e_k\tilde{\nabla}_{e_k}\Psi\rangle = -\text{Re}\langle e_k\Psi, \tilde{\nabla}_{e_k}\Psi\rangle\,,$$

and hence $\text{Re}\langle\tilde{\nabla}_{e_k}\Psi, e_k\Psi\rangle = 0$. Thus we find $\omega_{e_k}(e_i, e_j) = 0$. Hence $\langle e_k, [e_i, e_j]\rangle = 0$, and as before we conclude that $\nabla_{e_i}e_j$ is a multiple of e_i. Lemmata 8.5 to 8.7 remain valid (where we have to replace Ψ_+ by Ψ, of course). In the proof of Lemma 8.7 we have to replace the Weyl-tensor by the anti-symmetrisation of the Schouten-tensor. We can now argue as before to conclude the proof of Theorem 1.2. □

Remark 8.9. In dimension 5 the situation is more complicated: There are 5-dimensional non-conformally flat Riemannian spin-manifolds the nontrivial harmonic spinors of which are all parallel [BFGK,p.150]. Thus these metrics are critical but not minimal and we cannot reproduce the above arguments. Of course, one might try to prove that 5-dimensional closed spin-manifolds with critical but not minimal metric must be isometric to the examples of [BFGK,p.150].

A little more work will yield all the conformal closed oriented 3-manifolds and spin-4-manifolds which admit a critical metric. We first need the following lemma:

Lemma 8.10. *Let $U' \subset U \subset \mathbb{R}^n$, $n \geq 3$, be open and connected, and denote by g the Euclidean metric on $\mathbb{R}^n$. Let $f : U' \to \mathbb{R}$ be a function such that $e^{2f}g$ is flat. Then f can be uniquely continued to a function ϕ on U with the possible exception of one point such that $e^{2\phi}g$ is a flat metric on U.*

Proof. By possibly shrinking U' we may find an open set $V' \subset \mathbb{R}^n$ and an isometry $\sigma' : (V', g) \to (U', e^{2f}g)$. σ' extends uniquely to a global conformal diffeomorphism $\sigma : S^n \to S^n$ [KuPi2,p.12]. Let $V := \sigma^{-1}U - \{\infty\}$, where we identify $\mathbb{R}^n$ with the sphere minus the North pole ∞ via stereographic projection. Thus $\sigma : (V, g) \to (U - \{\sigma(\infty)\}, g)$ is a conformal diffeomorphism, and thus there is a function $\phi : U - \{\sigma(\infty)\} \to \mathbb{R}$ such that $(\sigma^{-1})^* g = e^{2\phi} g$ and $\phi = f$ on U'. This was under the assumption of a possibly shrunk U', in order to prove the lemma it thus suffices to show that ϕ is unique.

Given now two extensions ϕ_1 and ϕ_2 of f (defined on all of U with the possible exception of a point for each ϕ_i) such that $e^{2\phi_i} g$ is flat, choose a connected open set $V \subset U$ on which both ϕ_i are defined such that there are open sets $V_i \subset \mathbb{R}^n$ and isometries $\sigma_i : (V_i, g) \to (V, e^{2\phi_i} g)$. If $\phi_1 = \phi_2$ on a connected open subset V_0 of V, the map $\sigma_2^{-1} \circ \sigma_1 : (\sigma_1^{-1} V_0, g) \to (\sigma_2^{-1} V_0, g)$ is an isometry. Then $\sigma_2^{-1} \circ \sigma_1$ extends uniquely to an isometry of $\mathbb{R}^n$, in particular $\sigma_2^{-1} \circ \sigma_1 | \sigma_1^{-1} V$ is an isometry. It follows that $\phi_1 = \phi_2$ on all of V. By connectedness of $U - \{\sigma(\infty)\}$ the result follows. □

Lemma 8.11. *The set of points on which Ψ_+ (respectively Ψ in dimension 3) vanishes is discrete.*

Proof. Let U be a connected open subset of M which (after possibly conformally rescaling the metric g first) is isometric to some open subset of Euclidean space. Let $U' \subset U$ an open connected subset on which Ψ_+ (respectively Ψ) does not vanish. Let $f := \frac{1}{3} \ln |\Psi_+|^2 : U' \to \mathbb{R}$. Then $e^{2f} g$ is a flat metric on U'. By the preceding lemma, f may be continued to a function to all of U with the possible exception of a single point. Thus Ψ_+ cannot vanish on U minus that point. □

We can now prove the following proposition:

Proposition 8.12. *Let (M, g) be a closed Riemannian spin-manifold of either dimension 3 or 4 with fixed spin-structure with harmonic spinors (of both chiralities in dimension 4) such that the metric g is critical for the eigenvalue 0. Then M is a torus and g is conformally equivalent to a flat metric.*

Proof. Let $\tilde{M}$ be the universal cover of M and F the discrete set of points on which Ψ_+ (Ψ in the case of dimension 3) vanishes, and let $\tilde{F}$ be its preimage in $\tilde{M}$. A standard monodromy argument shows that there is a local isometry $\delta : \tilde{M} \setminus \tilde{F} :\to \mathbb{R}^n$, $n = \dim M$, where $\tilde{M} \setminus \tilde{F}$ is furnished with the metric obtained by pulling back the flat metric $e^{2f} g$ from $M \setminus F$ with f being defined as in the previous lemma. This map uniquely extends to a conformal map $\delta : \tilde{M} \to S^n$ [KuPi2]. It follows that the holonomy $\Gamma \subset Conf(S^n)$ of M fixes ∞. By Theorem C of [Kam], (M, g) is conformally covered by either S^n, $S^{n-1} \times S^1$ or a torus T^n with the natural conformal structures, and the conformal class of g contains a metric of positive scalar curvature in the first two cases and a flat metric in the third. Thus by the standard Weitzenböck formula [L], harmonic spinors can occur only when M is covered by the torus. Replacing g by a conformally equivalent flat metric shows that Ψ_+ and Ψ_- (respectively Ψ) are parallel. Thus $\Sigma^{\pm}$ (respectively Σ) is trivialized by parallel sections, and thus so is TM, and hence (M, g) has trivial holonomy. By [Wo,Cor.3.4.6] we conclude that (M, g) is a flat torus. □

Remark 8.13. Observe the following fact: Given a metric which is not critical (for the eigenvalue 0), the quadratic form $Q = Q^{g,A}_{\Psi_1,\Psi_2} + \overline{Q}^{g,A}_{\Psi_2,\Psi_1}$ (in dimension 4) does not vanish identically for any choice of harmonic spinors $\Psi_\pm$. Otherwise, the above arguments go through to show that (M, g) is conformally flat and is in fact a torus with the flat conformal structure as we shall see in a moment. In particular the metric would be critical. We may thus assume that Q does not vanish identically.

Fix some open subset $U \subset M$. If $Q|U$ did vanish identically we could again conclude that $(U, g|U)$ is conformally flat. By a slight perturbation of g within U we may assume that this is not the case and that Q does not vanish identically on U.

Let $\phi \geq 0$ be a smooth function supported in U such that ϕQ does not vanish identically. By Equation 4.1.1 the quadratic form $\mathrm{Re}(\phi Q)$ defines a deformation direction along which $\dim\mathrm{Ker}\mathcal{D}_g$ decreases.

The same arguments go through in dimensions 2 and 3, and inspection of the proof of Theorem 1.4 shows that we may argue similarly for deformations of connections on the canonical bundle. Thus in total we have:

Proposition 8.14. *Let* $\dim M = 2,3$ *or 4 and* U *an open set of* M. *Given a metric* g *which is not minimal we may find a minimal metric* g' *which is* C^1*-close to* g *and is equal to* g *outside* U.

Let $\dim M = 1, \ldots, 4$. *Given a connection* A *on the canonical bundle which is not minimal we may find a minimal connection* A' *which is* C^0*-close to* A *and is equal to* A *outside* U.

This extends an observation of [Hi,p.45].

9. Critical Eigenvalues $\neq 0$

In this section we shall prove a partial converse to [BG,Prop.29] which asserts that eigenvalues which admit a Killing spinor are critical for all variations of the metric which preserve the total volume.

Proposition 9.1. *Let* M *be a closed oriented 2- or 3-manifold with a fixed spin-structure. If for some metric* g *on* M *some eigenvalue* $\lambda \neq 0$ *is critical for variations of the metric which preserve the total volume, then* (M, g) *is covered by the round sphere (up to rescaling by a constant factor). In dimension 2,* (M, g) *is isometric to the round sphere.*

Corollary 9.2. *Let* M *be a closed oriented 2- or 3-manifold with fixed spin-structure. Fix* $\lambda \neq 0$. *The set of metrics with given total volume for which* λ *is not an eigenvalue of* $\mathcal{D}_g$ *is* C^1*-generic.*

Proof of the Proposition. Let Ψ be a nontrivial eigenspinor for the eigenvalue λ. By Proposition 4.1 the norm of Ψ is constant and we may assume it to be = 1. Let

$$\omega(X, Y) := \mathrm{Re}\langle X.\tilde{\nabla}^g_Y \Psi, \Psi\rangle - \frac{\lambda}{m} g(X, Y)$$

for arbitrary vector fields X and Y. By (ii) of Proposition 4.1 ω is a 2-form. By $T \subset \Sigma$ denote the image of TM under Clifford multiplication with Ψ, i.e $T = TM.\Psi$. Let H be the orthogonal complement of $\mathbb{R}\Psi \oplus T$ with respect to the metric $\mathrm{Re}\langle .,. \rangle$. Set $\tilde{\nabla}^H \Psi := \mathrm{pr}_H \tilde{\nabla}\Psi$, where pr_H denotes orthogonal projection onto H. For a local ON-frame $\{e_1, \ldots, e_m\}$

$$\tilde{\nabla}^g_{e_i} \Psi = \sum_{i \neq j} \omega(e_i, e_j) e_j.\Psi - \frac{\lambda}{m} e_i.\Psi + \tilde{\nabla}^H_{e_i} \Psi\,.$$

Multiply this with e_i and sum over i to obtain

$$0 = 2\sum_{i<j}\omega(e_i,e_j)e_i.e_j.\Psi + \sum_i e_i.\tilde\nabla^H_{e_i}\Psi\,.$$

Consider the case $m = 2$: Then $e_1e_2.\Psi$ is a local section of H. Now the previous formula reads

$$0 = 2\omega(e_1,e_2)e_1.e_2.\Psi + \mathrm{Re}\langle\tilde\nabla^H_{e_2}\Psi, e_1e_2.\Psi\rangle e_1.\Psi - \mathrm{Re}\langle\tilde\nabla^H_{e_1}\Psi, e_1e_2.\Psi\rangle e_2.\Psi\,.$$

Thus all coefficients vanish and hence $\tilde\nabla^g_X\Psi = -\frac{\lambda}{2}X.\Psi$.

Now consider $m = 3$: Here $H = \{0\}$ because the real fibre dimension of Σ is 4. Using $e_1e_2e_3.\Psi = -\Psi$ we obtain

$$0 = \omega(e_1,e_2)e_3.\Psi + \omega(e_2,e_3)e_1.\Psi + \omega(e_3,e_1)e_2.\Psi$$

and thus $\omega = 0$, whence $\tilde\nabla^g_X\Psi = -\frac{\lambda}{3}X.\Psi$.

Thus if $m = 2$ or 3 then Ψ is a Killing-spinor. The proposition now follows from [BFGK,Th.8,p.31] because in dimensions 2 and 3 Einstein metrics of constant scalar curvature are in fact constant curvature metrics. $\square$

Remark 9.3. In dimension 3, one might be tempted into believing that M must in fact be the sphere. This, however, is not the case: Identify $S^3 \cong SU_2$ and let $E = (e_1, e_2, e_3)$ be a left-invariant ON-frame on SU_2, where the e_i satisfy the relations

$$[e_i, e_j] = 2\mu e_k, \quad \mu\in\mathbb{R}^*$$

for cyclic permutations (i, j, k) of $(1, 2, 3)$. Then $\nabla_{e_i}e_j = \mu e_k$. Let Γ be a discrete subgroup of SU_2 and $M := \Gamma\setminus SU_2$ the quotient. The metric on SU_2 and E descend to M. View E as a section $E : M \to P_{SO}M$. Lift E to a section $\tilde E$ of the $Spin_3$-bundle $P_{Spin}M$ associated with the trivial spin-structure. Fix $v\in\mathbb{C}^2$ and let Ψ be the section given by

$$\Psi(m) = [\tilde E(m), v] \in (P_{Spin}M\times\mathbb{C}^2)/SU_2 = P_{Spin}(M)\times_{rep}\mathbb{C}^2\,,$$

where $SU_2 = Spin_3$ acts via $u.(g, v) := (gu, u^{-1}v)$. Then

$$\tilde\nabla\Psi = \frac{1}{2}\sum_{i<j}\omega_{ji}e_ie_j\Psi, \quad \omega_{ji} = \langle\nabla e_i, e_j\rangle = \mu e_k^*\,.$$

Recalling that $e_1e_2e_3.\Psi = -\Psi$ it is now immediate that Ψ is a Killing spinor.

10. A Remark on Seiberg–Witten Moduli Spaces

One motivation for studying generic metrics and connections on spin^c-manifolds comes from Seiberg–Witten theory. Let M be a closed oriented 4-manifold with $b^+\geq 1$, and fix a spin^c-structure on M with canonical bundle L. For a given metric g on M and a self-dual 2-form η the Seiberg–Witten equations are equations for a connection A on L and a section Ψ of Σ^+:

$$\begin{aligned}\mathcal{D}_{g,A}\Psi &= 0\,,\\ \rho(F_A^+) &= \sigma(\Psi,\Psi) + \rho(i\eta)\,,\end{aligned}$$

where F_A^+ is the self-dual part of the curvature. The map $\rho : i\Omega^+(M) \to \mathrm{End}_0(\Sigma^+)$ is given by Clifford multiplication and the bilinear map $\sigma : \Sigma_+ \otimes \overline{\Sigma_+} \to \mathrm{End}_0(\Sigma^+)$ is defined as follows:

$$\sigma(\Psi_1, \Psi_2) := \Psi_1 \otimes \Psi_2^* - \frac{1}{2}\mathrm{Tr}(\Psi_1 \otimes \Psi_2^*) Id \,.$$

For (g, η) in a dense open subset $\mathcal{D} \subset \mathcal{M} \times \Omega^+(M)$ the space of solutions is a smooth manifold which contains only irreducible solutions, i.e. solutions (Ψ, A) with $\Psi \not\equiv 0$.

One may ask whether the set consisting of pairs (g, A), where A comes from a solution of the Seiberg–Witten equations for a fixed pair (g, η) is disjoint from the set of pairs $(g, B) \in \mathcal{M} \times \mathcal{A}$ of metrics and connections for which the space of harmonic spinors is larger than required by the index of the Dirac operator. Here (g, η) are parameters which we will choose in $\mathcal{D}$, i.e. the corresponding Seiberg–Witten moduli space contains irreducible solutions only and is smooth. By Theorem 1.4 we know that for a generic pair (g, B) the dimension of the space of harmonic spinors is indeed equal to the absolute value of the index. In particular, if the index is negative there are no nontrivial positive harmonic spinors for the generic pair (g, B).

Now let the index be nonpositive, i.e $c_1(L)^2 - \sigma(M) \leq 0$. Suppose there is a pair (g, η) of parameters with at least one nontrivial solution (Ψ, A) to the Seiberg–Witten equations such that (g, A) is generic. Because the index is nonpositive Theorem 1.4 implies that $\Psi \equiv 0$, and hence $(g, \eta) \notin \mathcal{D}$.

Suppose we are given a spin^c-structure $\mathfrak{c}$ with $c_1(L)^2 - \sigma(M) \leq 0$. If the Seiberg–Witten invariant $SW_{g,\eta}(\mathfrak{c})$ is nontrivial (for parameters $(g, \eta) \in \mathcal{D}$) then for any nontrivial solution (Ψ, A) of the Seiberg–Witten equations the pair (g, A) is not generic by the preceding argument. Thus if we could show that whenever $SW_{g,\eta}(\mathfrak{c})$ is nontrivial we can find at least one solution (Ψ, A) for which (g, A) is generic then $c_1(L)^2 - \sigma(M) > 0$.

Problem. Is it true that if for a given spin^c-structure $\mathfrak{c}$ the Seiberg–Witten invariant $SW_{g,\eta}(\mathfrak{c})$ is nontrivial, the index of the Dirac operator is positive, i.e $c_1(L)^2 - \sigma(M) > 0$?

That this is not true in general will be shown in the following proposition. This proposition may also be interpreted as saying that if the answer to the problem should be affirmative for $b^+ > 1$ then there is no proof which relies on infinitesimal arguments, i.e there is no proof which tries to argue that a C^∞-small deformation of both g and η may be found such that (g, A) is generic for some solution A of the equations. For such an argument would also apply to the case $b^+ = 1$.

Proposition 10.1. *Let M be a geometrically ruled surface over the curve C. There is a connected open set $U \in \mathcal{M} \times \Omega^+(M)$ such that for $(g, \eta) \in U$ the moduli space of solutions for the anti-canonical spin^c-structure $\mathfrak{c}_{can}$ contains no reducible solution and $SW_{g,\eta}(\mathfrak{c}_{can}) \neq 0$. Furthermore $b^+(M) = 1$ and the signature of the Dirac operator is negative provided genus$(C) > 1$. In particular, if the connection A comes from a solution to the Seiberg–Witten equations for the parameters (g, η) then (g, A) is not contained in the generic set.*

Note that the anti-canonical spin^c-structure has as canonical bundle $L = K := \Omega^{2,0}$ [LM,App.D].

Proof. Because M is Kähler and $p_g(M) = 0$, $b^+(M) = 1$. The set of pairs $(g, \eta) \in \hat{U} \subset \mathcal{M} \times \Omega^+(M)$ for which the Seiberg–Witten moduli space contains no reducible solution is open. The dimension of the moduli space is 0. Let F be a fibre. Then because F has trivial normal bundle, $c_1(M)|F = c_1(F)$ and $c_1(K)|F = c_1(K_F) = -c_1(F)$. Now F is $\mathbb{C}P^1$, thus $c_1(F) = 2$. It follows that

$$\left(\frac{c_1(M) + 2c_1(K)}{2}[F]\right)^{\text{genus}(C)} = \pm 1\,.$$

Corollary 1.4 of [LL] now implies that there is one component $U \subset \hat{U}$ such that for $(g,\eta) \in U$ the Seiberg–Witten invariants satisfy $SW_{g,\eta}(\mathfrak{c}_{\text{can}}) \neq 0$. Note that the canonical bundle of the spin^c-structure $\mathfrak{c}$ is $L = K = \Omega^{2,0}(M)$. Furthermore, by [Beau,Prop.III.21] $c_1(K)^2 = 8(1 - \text{genus}(C))$ and the signature of M is $\sigma = 0$. Hence the index of the Dirac operator is $(c_1(K)^2 - \sigma)/8 < 0$ for genus$(C) > 1$. For a solution (Ψ, A) of the Seiberg–Witten equations for parameters $(g,\eta) \in U$ the pair (g, A) cannot be in the generic set because $\Psi \not\equiv 0$. □

11. Appendix: Analytic Families of Differential Operators

This section is technical in nature and serves to prove a very simple analyticity theorem for differential operators. This theorem is a formalization of the proof of [Ber,Lemme 3.15]. Equivalent statements have also been proven independently in [Ang,Th.1.1,Th.1.2] with similar applications as in this paper.

In the sequel let E and F always denote smooth $\mathbb{C}$-vectorbundles over a closed manifold M with dim$M = m$. On M a smooth measure shall be fixed once and for all. Let α always denote a multiindex in $\mathbb{N}^k$. By $\Gamma(E)$ we denote the space of (possibly discontinuous) sections of E.

Definition. *We say that $s_y \in \Gamma(E) = \Gamma(M,E)$ depends analytically on $y \in Y \subset \mathbb{R}^n$ if for fixed $p \in M$ the map $y \mapsto s_y(p) \in E_p$ is analytic in a uniform manner, i.e. for every $y_0 \in Y$ there are $s_\alpha \in \Gamma(E)$ and $R > 0$ with $B_R(y_0) \subset Y$ and*

$$s_y = \sum_\alpha \frac{1}{\alpha!} s_\alpha (y - y_0)^\alpha \qquad |y - y_0| < R\,.$$

If $\phi \in \Gamma(\text{Hom}(E,E))$ then the analyticity of s_y implies the analyticity of $\phi(s_y)$. The definition is local in nature, i.e if $V_1 \cup V_2 = M$ and s_y is analytic over both V_i then s_y is also analytic over M.

If M is compact the definition of analyticity of s_y is equivalent to demanding that the coordinate functions in any local trivialization be analytic in a uniform manner. Thus for most arguments it suffices to consider functions which depend analytically on a parameter.

Given $s_y \in \Gamma(M \times \mathbb{C})$, where $y \in D_R := \{(x_1,\ldots,x_n) \in \mathbb{R}^n,\ |x_i| < R\}$, and $s_\alpha \in \Gamma(M \times \mathbb{C})$ with $s_y = \sum_\alpha \frac{1}{\alpha!} s_\alpha y^\alpha$ we have for fixed $p \in M$ and $0 < R' < R$ the Cauchy integral formula:

$$s_y(p) = s(p,y) = \frac{1}{(2\pi i)^n} \int_C \frac{s(p,\zeta_1,\ldots,\zeta_n)}{(\zeta_1 - y_1)\cdots(\zeta_n - y_n)} d\zeta_1 \ldots \zeta_n\,,$$

$$s_\alpha(p) = \frac{1}{(2\pi i)^n} \int_C \frac{s(p,\zeta_1,\ldots,\zeta_n)}{\zeta_1^{\alpha_1+1}\cdots\zeta_n^{\alpha_n+1}} d\zeta_1 \ldots \zeta_n\,,$$

where $C := C_1 \times \ldots \times C_n$ with $C_j(t) = R' e^{s\pi i t_j}$.

For the moment we shall work in a fixed coordinate system on some open subset U of M. Suppose $s_y(p) = s(p, y)$ is differentiable in the p-coordinate for each fixed y and assume furthermore that $D_1 s : U \times Y \to \mathrm{Hom}_{\mathbb{R}}(TU, \mathbb{C})$ is continuous jointly in both variables. Suppose inductively that this holds for all $D_1^\alpha s(p, y)$ with $|\alpha| < j$ and $1 \le j \le k$. Then the Cauchy integral formula shows that for $|\gamma| \le k$, $D_1^\gamma s(p, y)$ is analytic in y, the s_α are in C^k and

$$D_1^\gamma s(p, y) = \sum_\alpha \frac{1}{\alpha!} D^\gamma s_\alpha \, y^\alpha \, .$$

Thus for fixed $\epsilon > 0$ we may find $q \in \mathbb{N}$ such that for $|y_i| < R' < R$,

$$\|s_y(p) - \sum_{|\alpha| \le q} \frac{1}{\alpha!} s_\alpha y^\alpha\|_{C^k} < \epsilon \, .$$

If M is compact we thus have the following

Lemma 11.1. *Let $s_y \in \Gamma(E)$ depend analyticaly on $y \in Y \subset \mathbb{R}^n$ such that s is in $C^k(M \times Y, E)$. Given $\epsilon > 0$ and $y_0 \in Y$ we may choose $R > 0$ and $q \in \mathbb{N}$ such that $B_R(y_0) \subset Y$ and*

$$\|s_y(p) - \sum_{|\alpha| \le q} \frac{1}{\alpha!} s_\alpha (y - y_0)^\alpha\|_{C^k} < \epsilon \qquad |y - y_0| < R.$$

Lemma 11.2. *Let E and F be smooth complex vector bundles over M, $\phi_y \in C^\infty(M \times Y, Hom(E, F))$ be a section which depends analytically on y. Then ϕ_y defines a continuous linear map $\mathcal{H}^s(E) \to \mathcal{H}^s(F)$ which depends analytically on y, where $\mathcal{H}^s$ denotes the Sobolev space of order $s \in \mathbb{R}$.*

Proof. Choose connections ∇_E and ∇_F in E and F respectively and (hermitian) metrics on both vector bundles. Then for $S \in C^\infty(E)$ and $k \in \mathbb{N}$:

$$\|\nabla_E^k(\phi_y S)\|_{L^2(F)} \le \text{const.} \|\phi_y\|_{C^k} \left(\sum_{l=0}^k \|\nabla_E^l S\|_{L^2(E)} \right) .$$

Hence $\|\phi_y S\| \le \text{const.} \|\phi_y\|_{C^k} \|S\|_{\mathcal{H}^k}$. Thus $\phi_y : \mathcal{H}^k(E) \to \mathcal{H}^k(F)$ is continuous and its operator norm is bounded by a constant multiple of $\|\phi_y\|_{C^k}$. For $k \in -\mathbb{N}$ this follows by duality, and for $s \in \mathbb{R}$, ϕ_y is continuous with operator norm bounded by a constant multiple of $\|\phi_y\|_{C^k}$ for $k \in \mathbb{N}$ with $k \ge |s|$, by the interpolation argument of [Fo,3.21]. By the previous lemma, given $k \in \mathbb{N}$ and $y_0 \in U$ we may find $R > 0$ and $q \in \mathbb{N}$ such that

$$\|\phi_y(p) - \sum_{|\alpha| \le q} \frac{1}{\alpha!} \phi_\alpha (y - y_0)^\alpha\|_{C^k} < \epsilon \qquad |y - y_0| < R$$

for given $\epsilon > 0$. Thus $\sum_{|\alpha| \le q} \frac{1}{\alpha!} \phi_\alpha (y - y_0)^\alpha$ converges uniformly to ϕ_y in the operator norm. This proves the lemma. $\square$

Definition. *Let $D_y : C^\infty(E) \to C^\infty(F)$ be a differential operator of order q on a compact manifold M which depends upon a variable $y \in U$, where $U \subset \mathbb{R}^n$ is open. We say that D_y depends analytically on y if in every local trivialization of E and F over $V \subset M$*

$$D_y = \sum_{|\alpha| \leq q} A_{\alpha,y} \frac{\partial}{\partial x^\alpha} ,$$

where the $A_{\alpha,y}$ are analytic uniformly in y and are smooth jointly in both variables.

This definition clearly is independent of the particular trivializations chosen.

Proposition 11.3. *An analytic differential operator $D_y : C^\infty(E) \to C^\infty(F)$ of order q extends to a bounded linear operator $D_y : \mathcal{H}^{s+q}(E) \to \mathcal{H}^s(F)$ which is analytic in y.*

Proof. We may think of D_y as a section d_y of Hom(J^qE, F), where J^qE denotes the q^{th} jet-bundle of E. This section clearly is analytic in y. If $j_q : C^\infty(E) \hookrightarrow C^\infty(J^qE)$ denotes the standard inclusion then $D_y = d_y \circ j_q$. The map d_y extends to an analytic bounded linear map $\mathcal{H}^s(J^qE) \to \mathcal{H}^s(F)$ and j_q extends to a bounded linear map $\mathcal{H}^{s+q}(E) \to \mathcal{H}^s(J^qE)$. Thus their composition is an analytic bounded linear map $\mathcal{H}^{s+q}(E) \to \mathcal{H}^s(F)$. □

The following proposition is the upshot of the preceding discussion. This proposition has also been proved in [Ang,Th.1.1] for perturbations of order smaller than the order of D.

Proposition 11.4. *Let D_t, $t \in (a, b)$, be an analytic family of differential operators of order q acting on the smooth sections of a complex vectorbundle E over M such that D_t is elliptic for each t. Let μ := min$\{$dimKer(D_t), $t \in (a, b)\}$. Then the set $T := \{t \in (a, b)$, dimKer$(D_t) > \mu\}$ is discrete.*

Proof. If $s \notin T$ then for all t in a neighbourhood of s, we have $t \notin T$, by upper semicontinuity. This in particular implies that the set T is closed. Fix any $s \in (a, b)$ in the boundary of T. Split $\mathcal{H}^q(E)$ orthogonally as $K \oplus H$, where K := Ker(D_s), and split $\mathcal{H}^0(E)$ orthogonally as $C \oplus D$, where C := Coker(D_s). We may decompose D_t as

$$D_t = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}$$

with respect to this splitting, where $d_t : H \to D$ is invertible for t near s. Let k := dimKer$D_s - \mu > 0$. Set $R(t) := b_t \circ d_t^{-1} \circ c_t - a_t$ and note that dimKer(D_t) = dimKer(D_s) if and only if $R(t) = 0$ [Kos]. Because s is in the boundary of T, there is a $(k \times k)$-minor of $R(t)$ with nonvanishing determinant for a set of points with s as an accumulation point. But $R(t)$ depends analytically on t, and thus this minor has nonvanishing determinant at $t \neq s$ in a neighbourhood of s. Thus there is an open interval (t_1, t_2) with $a < t_1 < s < t_2 < b$ such that for $t \in (t_1, t_2) \setminus \{s\}$ we have $t \notin T$. Thus T is discrete. □

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