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Distribution Functions for Random Variables for Ensembles of Positive Hermitian Matrices

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Abstract: Distribution functions for random variables that depend on a parameter are computed asymptotically for ensembles of positive Hermitian matrices. The inverse Fourier transform of the distribution is shown to be a Fredholm determinant of a certain operator that is an analogue of a Wiener-Hopf operator. The asymptotic formula shows that, up to the terms of order $o(1)$, the distributions are Gaussian.

1. Introduction

In the theory of random matrices one is led naturally to consider the probability distribution on the set of eigenvalues of the matrices. For $N \times N$ random Hermitian matrices one can show that under reasonable assumptions, the probability density that the eigenvalues $\lambda_1, \dots, \lambda_N$ lie in the intervals

$$(x_1, x_1 + dx_1), \dots, (x_N, x_N + dx_N)$$

is given by the formula

$$P_N(x_1, \dots, x_N) = \frac{1}{N!} \det K(x_i, x_j) \,\Big|_{i,j=1}^{N}, \tag{1}$$

where

$$K_N(x, y) = \sum_{i=0}^{N-1} \phi_i(x)\phi_i(y), \tag{2}$$

and ϕ_i is obtained by orthonormalizing the sequence $\left\{x^i e^{-x^2/2}\right\}$ over $\mathbf{R}$.

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For $N \times N$ positive Hermitian matrices the probability density has the same form except that ϕ_i is replaced by the functions obtained by orthonormalizing the sequence $\left\{x^{\nu/2}e^{-x/2}x^i\right\}$ over $\mathbf{R}^+$. We will not describe here exactly how these particular densities arise but instead refer the reader to [8].

We can define a random variable on the space of eigenvalues by considering $f(x_1, \ldots, x_N)$ where f is any symmetric function of the x_i's. A particular case of interest is a random variable of the form $\sum_{i=1}^N f(x_i)$, where f is a function of a real variable. Such a random variable is generally called a linear statistic.

In previous work [8, 3, 1, 6], the variance of the random variable was computed in the large N limit. More precisely, the function f and the kernel $K_N(x, y)$ were suitably rescaled so that the limit as $N \to \infty$ of the variance could be computed. The precise details of this are in the next section.

Our goal in this paper is to compute the distribution function for a class of the linear statistics that depend on a parameter α. We now describe the sections of the paper and main results. In the next section we outline the random matrix theory and show how the distribution functions can be computed using Fredholm determinants. In Sect. 3 we replace the function $f(x)$ in the linear statistic by $f_\alpha(x) = f(x/\alpha)$. For random variables of this type we show that the inverse Fourier transform of the distribution function $\check{\phi}(k)$ has an asymptotic expansion of the form

$$\check{\phi}(k) \sim e^{ak^2+bk} \tag{3}$$

as $\alpha \to \infty$. This of course implies that the actual distribution is asymptotically Gaussian. Here a and b depend on f and α. This is proved for both the Hermitian matrices and positive Hermitian matrices. In the latter case with $\nu = -1/2$, a very simple proof is given in Sect. 3. For $\nu > -1/2$, a completely different proof is obtained in Sect. 4.

Most of the results are obtained by using simple operator theory identities in the theory of Wiener-Hopf operators. The central idea is that the various quantities which yield information about random variables can all be computed in terms of traces or determinants of integral operators. Some of the computations lead directly to a familiar problem in the theory of Wiener-Hopf operators, while others require modifications and generalizations of these results.

2. Preliminaries

In this section we show how to compute the mean, variance, and inverse Fourier transform of the distribution of the random variable. Computations for the mean and variance have been given before in many places. However, we reproduce all of these here for completeness sake and also to highlight the use of operator theory ideas.

We begin by considering P_N for $N \times N$ random Hermitian matrices. We want to consider large matrices and thus we let $N \to \infty$, but this leads to a trivial result unless we rescale K_N in a particular way. We replace $K_N(x, y)$ with

$$\frac{1}{\sqrt{2N}} K_N\left(\frac{x}{\sqrt{2N}}, \frac{y}{\sqrt{2N}}\right). \tag{4}$$

Rescaling K_N is equivalent to rescaling the mean spacing of the eigenvalues. (See [12] for details.)

From the theory of Hermite polynomials it is easy to see that as $N \to \infty$,

$$\frac{1}{\sqrt{2N}} K_N \left(\frac{x}{\sqrt{2N}}, \frac{y}{\sqrt{2N}} \right) \to \frac{\sin(x-y)}{\pi(x-y)}. \tag{5}$$

This last function is known as the sine kernel. Now consider a random variable of the form

$$\sum_{i=1}^{N} f(x_i \sqrt{2N}),$$

where in all that follows f is a continuous real-valued function belonging to $L_1(\mathbf{R})$ and which vanishes at $\pm\infty$. The appearance of the $\sqrt{2N}$ should not be surprising here since the above rescaling spreads out the eigenvalues and hence should be reflected in the random variable. The mean μ_N is

$$\int \cdots \int \sum_{i=1}^{N} f(x_i \sqrt{2N}) P_N(x_1, \dots, x_N) dx_1 \cdots dx_N. \tag{6}$$

Now the function P_N has the important property [8]

$$\begin{aligned} &\frac{N!}{(N-n)!} \int \cdots \int P_N(x_1, \dots, x_n, x_{n+1}, \dots, x_N) dx_{n+1} \cdots dx_N \\ &= \det K(x_i, x_j) \big|_{i,j=1}^{n} . \end{aligned} \tag{7}$$

Thus, (6) is easily seen to be

$$\int_{-\infty}^{\infty} f(x\sqrt{2N}) K_N(x,x)\, dx \tag{8}$$

which, after changing x to $x/\sqrt{2N}$, becomes

$$\int_{-\infty}^{\infty} f(x) \frac{1}{\sqrt{2N}} K_N \left(\frac{x}{\sqrt{2N}}, \frac{y}{\sqrt{2N}} \right) dx.$$

Thus, as $N \to \infty$,

$$\mu_N \to \mu = \int_{-\infty}^{\infty} f(x) K(x,x)\, dx, \tag{9}$$

where $K(x,y)$ is the sine kernel.

A very similar computation for the variance $\mathrm{var}_N f$, again using (7), yields

$$\mathrm{var}\, f := \lim_{N\to\infty} \mathrm{var}_N f = -\int\int f(x) f(y) K^2(x,y)\, dx\, dy + \int f^2(x) K(x,x)\, dx. \tag{10}$$

Both the mean and the variance can be interpreted as traces of certain Wiener-Hopf operators. To see this, consider the operator $A(f)$ on $L_2(-1,1)$ with kernel

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-it(x-y)}\, dt. \tag{11}$$

This operator can easily be seen to be the product $\mathcal{F}\mathcal{M}_f\mathcal{F}^{-1}P$, where $Pg = \chi_{(-1,1)}g$, $\mathcal{M}_f g = fg$ and $\mathcal{F}$ is the Fourier transform. A moment's thought shows that $\mu = \mathrm{tr}\,\{A(f)\}$ and $\mathrm{var}\, f = \mathrm{tr}\,\{A(f^2) - (A(f))^2\}$.

A more difficult, yet also straightforward problem, is to find an expression for the distribution function of a random variable of this type. A fundamental formula from probability theory shows that if we call the probability distribution function ϕ_N, then

$$\check{\phi}_N(k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{ik \sum_{j=1}^{N} f(x_j \sqrt{2N})} P_N(x_1, \ldots, x_N) dx_1 \cdots dx_N. \tag{12}$$

Thus,

$$\begin{aligned}
\check{\phi}_N(k) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} e^{ikf(x_j\sqrt{2N})} P_N(x_1, \ldots, x_N)\, dx_1 \cdots dx_N \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} ((e^{ikf(x_j\sqrt{2N})} - 1) + 1) P_N(x_1, \ldots, x_N)\, dx_1 \cdots dx_N \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \{1 + \sum_{j=1}^{N} (e^{ikf(x_j\sqrt{2N})} - 1) \\
&\quad + \sum_{j<l}^{N} (e^{ikf(x_j\sqrt{2N})} - 1)(e^{ikf(x_l\sqrt{2N})} - 1) + \ldots\} \\
&\quad \times P_N(x_1, \ldots, x_N)\, dx_1 \cdots dx_N \\
&= 1 + \frac{1}{1!} \int_{-\infty}^{\infty} (e^{ikf(x\sqrt{2N})} - 1) K_N(x, x)\, dx \\
&\quad + \frac{1}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{ikf(x_1\sqrt{2N})} - 1)(e^{ikf(x_2\sqrt{2N})} - 1) \\
&\quad \det(K_N(x_j, x_l)) \,|_{1 \le j,l \le 2}\ dx_1\, dx_2 \\
&\quad + \cdots + \frac{1}{N!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} (e^{ikf(x_j\sqrt{2N})} - 1) P_N(x_1, \ldots, x_N)\, dx_1 \cdots dx_N.
\end{aligned}$$

In each integral we rescale to obtain

$$\begin{aligned}
\check{\phi}_N(k) = 1 &+ \frac{1}{1!} \int_{-\infty}^{\infty} K'(x_1, x_1)\, dx_1 + \frac{1}{2!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K'(x_1, x_2)\, dx_1\, dx_2 \\
&+ \cdots + \frac{1}{N!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} K'(x_1, \ldots, x_N)\, dx_1 \cdots dx_N,
\end{aligned} \tag{13}$$

where

$$K'(x_1, \ldots, x_n) = \det \left((e^{ikf(x_j)} - 1) K_N\left(\frac{x_j}{\sqrt{2N}}, \frac{x_l}{\sqrt{2N}}\right) \frac{1}{\sqrt{2N}} \right)_{1 \le j,l \le n}. \tag{14}$$

Letting $N \to \infty$ we see this is the formula for the Fredholm determinant $\det(I + K)$, where K has kernel

$$K(x, y) = (e^{ikf(x)} - 1) \frac{\sin(x - y)}{\pi(x - y)}. \tag{15}$$

As before we can express this last quantity in terms of the operator $A(\sigma)$

$$\check{\phi}(k) = \lim_{N\to\infty} \check{\phi}_N(k) = \det(I + A(\sigma)), \tag{16}$$

where $\sigma(x) = e^{ikf(x)} - 1$.

The preceding computations can all be carried out in the case of positive Hermitian matrices. In this case we replace $K_N(x, y)$ with

$$\frac{1}{4N} K_N(\frac{x}{4N}, \frac{y}{4N})$$

and from the theory of Laguerre polynomials we see that as $N \to \infty$,

$$\frac{1}{4N} K_N(\frac{x}{4N}, \frac{y}{4N}) \to \frac{J_\nu(\sqrt{x})\sqrt{y}J_\nu{}'(\sqrt{y}) - \sqrt{x}J_\nu{}'(\sqrt{x})J_\nu(\sqrt{y})}{2(x-y)}, \tag{17}$$

where J_ν is the Bessel function of order ν. The details of this are found in [13]. The rescaling here forces the eigenvalue density to be bounded near zero and is called "scaling at the hard edge." The kernel (17) is known as the Bessel kernel.

We can again write the mean, the variance, and the Fourier transform of the distribution in terms of operators. This time the relevant operator $B(f)$ is defined on $L_2(0, 1)$ with kernel given by

$$K(x, y) = \int_0^\infty t\sqrt{xy} f(t) J_\nu(tx) J_\nu(ty)\, dt. \tag{18}$$

If we begin with the linear statistic (the $\sqrt{x}$ is merely for convenience, and we again assume that f is continuous, in $L_1(\mathbf{R}^+)$ and vanishes at $+\infty$)

$$\sum_{i=1}^{N} f(\sqrt{x_i 4N}), \tag{19}$$

then nearly identical computations show that

$$\begin{aligned} \mu &= \text{tr}\, B(f), \\ \text{var}\, f &= \text{tr}\, \{B(f^2) - (B(f))^2\}, \\ \check{\phi}(k) &= \det(I + B(\sigma)), \end{aligned}$$

where $\sigma = e^{ikf(x)} - 1$. We summarize these results in the following:

Theorem 1. *(a) Given a random variable of the form* $\sum_{i=1}^N f(x_i\sqrt{2N})$ *defined on the space of eigenvalues of* $N \times N$ *Hermitian matrices with probability distribution given in (1), we have*

$$\begin{aligned} \mu &:= \lim_{N\to\infty} \mu_N = \text{tr}\,(A(f)), \\ \text{var}\, f &:= \lim_{N\to\infty} \text{var}_N f = \text{tr}\, \{A(f^2) - (A(f))^2\}, \\ \check{\phi}(k) &:= \lim_{N\to\infty} \check{\phi}_N(k) = \det(I + A(\sigma)), \end{aligned}$$

where $\sigma(x) = e^{ikf(x)} - 1$.

(b) Given a random variable of the form $\sum_{i=1}^N f(\sqrt{x_i 4N})$ *defined on the space of eigenvalues of positive* $N \times N$ *Hermitian matrices, we have*

$$\begin{aligned} \mu &:= \lim_{N\to\infty} \mu_N = \text{tr}\,(B(f)), \\ \text{var}\, f &:= \lim_{N\to\infty} \text{var}_N f = \text{tr}\, \{B(f^2) - (B(f))^2\}, \\ \check{\phi}(k) &:= \lim_{N\to\infty} \check{\phi}_N(k) = \det(I + B(\sigma)), \end{aligned}$$

where $\sigma(x) = e^{ikf(x)} - 1$.

When linear statistics are considered [3, 10], one is often concerned with a statistic of the form $\sum_{i=1}^{N} f(x_i/\alpha)$, where α is a real parameter approaching infinity. This is the case, for example, in the study of disordered conductors where large α corresponds to a high density metallic regime. The above formulas still hold, of course, but now they depend on the parameter. We will call the operators that depend on the parameter α by $A_\alpha(f)$ and $B_\alpha(f)$, respectively. In the next sections we will compute the mean, variance, and distribution function asymptotically as $\alpha \to \infty$.

3. The Mean, Variance, and Distribution Function as $\alpha \to \infty$

For random Hermitian matrices, computing the various limits are applications of the continuous analogues of the Strong Szegö Limit Theorem. For then, $A_\alpha(f)$ is just the classical Wiener-Hopf operator defined on the interval $(-\alpha, \alpha)$, and all of the quantities are known asymptotically as $\alpha \to \infty$. We provide the answers here for completeness.

Theorem 2. *Assume that $f \in L_1(\mathbf{R})$ is continuous, and vanishes at $\pm\infty$ and that in addition its Fourier transform $\hat{f}$ satisfies*

$$\int_{-\infty}^{\infty} |x||\hat{f}(x)|^2\, dx \;<\; \infty.$$

Then

$$\mu = \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} f(x)\, dx,$$
$$\operatorname{var} f = 2 \int_{0}^{\infty} x \hat{f}(x)\hat{f}(-x)\, dx \;+\; o(1)$$

and

$$\check{\phi}(k) \sim \exp\left\{ \frac{\alpha}{2\pi} \int_{-\infty}^{\infty} ik f(x)\, dx - k^2 \int_{0}^{\infty} x \hat{f}(x)\hat{f}(-x)\, dx \right\}.$$

The Bessel case is significantly more complicated. There is no corresponding Szegö type theorem. We begin by computing the mean. The operator $B_\alpha(\sigma)$ has kernel

$$\int_0^\infty \sqrt{xy}\, t f(t/\alpha) J_\nu(tx) J_\nu(ty)\, dt.$$

Thus the mean μ is given by

$$\begin{aligned} \mu &= \int_0^1 \int_0^\infty xt f(t/\alpha) J_\nu^2(tx)\, dt\, dx \\ &= \alpha^2 \int_0^\infty \int_0^1 xt f(t) J_\nu^2(\alpha t x)\, dx\, dt \\ &= \alpha^2 \int_0^\infty f(t) \int_0^1 xt J_\nu^2(\alpha t x)\, dx\, dt. \end{aligned} \tag{20}$$

Now

$$\int_0^1 x J_\nu^2(\alpha t x)\, dx = \frac{1}{2} \left\{ J_\nu^2(\alpha t) - J_{\nu+1}(\alpha t) J_{\nu-1}(\alpha t) \right\}$$

and

$$J_{\nu-1}(\alpha t) = -J_{\nu+1}(\alpha t) + \frac{2\nu}{\alpha t} J_\nu(\alpha t).$$

Therefore the integral (20) becomes

$$\alpha \int_0^\infty f(t) \frac{\alpha t}{2} \left\{ J_\nu^2(\alpha t) + J_{\nu+1}^2(\alpha t) - \frac{2\nu}{\alpha t} J_{\nu+1}(\alpha t) J_\nu(\alpha t) \right\} dt$$

or

$$\alpha \int_0^\infty f(t) \frac{\alpha t}{2} \left\{ J_\nu^2(\alpha t) + J_{\nu+1}^2(\alpha t) \right\} dt - \alpha\nu \int_0^\infty f(t) J_{\nu+1}(\alpha t) J_\nu(\alpha t)\, dt. \qquad (21)$$

The first integral equals

$$\frac{\alpha}{\pi} \int_0^\infty f(t)\, dt \;\; + o(1), \qquad (22)$$

which can be easily seen by using the asymptotic properties of Bessel functions. The second integral is asymptotically

$$\frac{\nu}{2} f(0) + o(1).$$

This uses the identity $\int_0^\infty J_{\nu+1}(x) J_\nu(x)\, dx = \frac{1}{2}$.

Thus we have

$$\mu = \frac{\alpha}{\pi} \int_0^\infty f(t)\, dt - \frac{\nu}{2} f(0) + o(1). \qquad (23)$$

For the variance we refer to [1] where the calculation was already done. There it was shown that

$$\text{var} f \sim \frac{1}{\pi^2} \int_{-\infty}^\infty |M(f)(2iy)|^2 y \tanh(\pi y) dy. \qquad (24)$$

We note however, that this can also be written as

$$\text{var} f \sim \frac{1}{\pi^2} \int_0^\infty x (C(f)^2)\, dx, \qquad (25)$$

where $C(f)(x) = \int_0^\infty f(y) \cos(xy) dy$ denotes the cosine transform of f. This is an exercise involving the properties of the Mellin transform, and we leave it to the reader.

To compute the distribution function, we first turn our attention to the case where $\nu = -1/2$. Our operator $B_\alpha(\sigma)$ has kernel

$$\begin{aligned} &\frac{2}{\pi} \int_0^\infty \sigma(t/\alpha) \cos xt \cos yt dt \\ &= \frac{1}{\pi} \int_0^\infty \sigma(t/\alpha)(\cos((x-y)t) + \cos((x+y)t))\, dt \\ &= \frac{\alpha}{\pi}(C(\sigma)((x-y)\alpha) + C(\sigma)((x+y)\alpha)). \end{aligned}$$

This is unitarily equivalent to the operator on $L_2(0, \alpha)$ with kernel

$$\frac{1}{\pi}(C(\sigma)(x-y) + C(\sigma)(x+y)). \tag{26}$$

The operator with kernel $\frac{1}{\pi}(C(\sigma)(x-y))$ is the finite Wiener-Hopf operator, usually denoted as $W_\alpha(\sigma)$, and the operator with kernel $\frac{1}{\pi}(C(\sigma)(x+y))$ is the Hankel operator $H_\alpha(\sigma)$. (The only difference between this definition of a finite Wiener-Hopf operator and the one given earlier for A_α is the difference in the domain. The two are unitarily equivalent.) If we consider the operators on $L_2(0,\infty)$ in what follows, we will denote them by $W(\sigma)$ and $H(\sigma)$ respectively. Also, whenever it is necessary to consider the extension of σ to the entire real axis, it will always be the even extension.

Thus the problem of finding the distribution function asymptotically becomes the same as computing the Fredholm determinant $\det(I + B_\alpha(\sigma)) = \det(I + W_\alpha(\sigma) + H_\alpha(\sigma))$ asymptotically. To do this we need some basic facts about Wiener-Hopf operators and we collect them in the following theorem. These are well-known and can all be found in [4].

Theorem 3. *a) Suppose ϕ and ψ are even bounded functions in $L_1(\mathbf{R})$. Then*

$$W(\phi)H(\psi) + H(\phi)W(\psi) = H(\phi\psi)$$

and

$$W(\phi)W(\psi) = W(\phi\psi) - H(\phi)H(\psi).$$

b) Suppose ϕ and ψ are bounded functions in $L_1(\mathbf{R})$. If the Fourier transform $\hat{\phi}(x)$ vanishes for x negative, then $W(\psi)W(\phi) = W(\phi\psi)$ and if $\hat{\phi}(x)$ vanishes for x positive, then $W(\phi)W(\psi) = W(\phi\psi)$.

We define $W(\sigma)$ and $H(\sigma)$ with $\sigma = 1 + f$ and f in L_1 by $W(\sigma) = I + W(f)$ and $H(\sigma) = H(f)$. Both of these definitions are natural when thought of in a distributional setting, and the above theorem holds with these definitions as well.

The next theorem is of primary importance in the computations that follow.

Theorem 4. *Suppose $\phi = 1+f$, $\phi^{-1} = 1+g$, where f and g are bounded even functions. Then the inverse of $W(\phi) + H(\phi)$ is $W(\phi^{-1}) + H(\phi^{-1})$.*

Proof. Using Theorem 3 parts a) and b) we have,

$$\begin{aligned}
&(W(\phi) + H(\phi))(W(\phi^{-1}) + H(\phi^{-1}))\\
&= W(\phi)W(\phi^{-1}) + H(\phi)W(\phi^{-1}) + W(\phi)H(\phi^{-1}) + H(\phi)H(\phi^{-1})\\
&= I - H(\phi)H(\phi^{-1}) + H(\phi\phi^{-1}) + H(\phi)H(\phi^{-1})\\
&= I + H(1) = I.
\end{aligned}$$

The same computation holds for $(W(\phi^{-1}) + H(\phi^{-1}))(W(\phi) + H(\phi))$, and so we have shown that these operators are inverses of each other. □

It is well known from the theory of Wiener-Hopf operators that under appropriate conditions $\det(I + W_\alpha(\sigma))$ has the asymptotic expansion $G(\sigma)^\alpha E(\sigma)$, where

$$G(\sigma) = \exp \frac{1}{2\pi}\int_{-\infty}^{\infty} \log(1 + \sigma(\xi))d\xi$$

and $E(\sigma) = \det(W(\phi)W(\phi^{-1}))$ with $\phi = 1 + \sigma$. This is simply another version of Theorem 2. With additional assumptions on ϕ, it is very easy to adapt this proof to the Bessel case $\nu = -1/2$ to show that

$$\det(I + W_\alpha(\sigma) + H_\alpha(\sigma)) \sim G(\sigma)^\alpha E'(\sigma) \tag{27}$$

and $E'(\sigma) = \det((W(\phi) + H(\phi))W(\phi^{-1}))$. Thus to compute the distribution, we need to know the form of the above determinant. This is contained in the next theorem.

Theorem 5. *Suppose $\sigma = e^{ikf} - 1$, where f is even, continuous, piecewise C^2 and vanishes at infinity. Suppose also that $f \in L_1$ and the function*

$$\xi \to (1 + \xi^2)(|f''(\xi)| + |f'(\xi)|^2) \in L_2.$$

Then as $\alpha \to \infty$, we have

$$\det(I+W_\alpha(\sigma)+H_\alpha(\sigma)) \sim \exp\{\frac{\alpha}{\pi}\int_0^\infty ikf(x)\,dx+\frac{ik}{4}f(0)-\frac{k^2}{2\pi^2}\int_0^\infty x|C(f)(x)|^2\,dx\}. \tag{28}$$

Proof. The conditions on σ ensure that the above integrals converge, and that the operators $H(\phi)$ and $H(f)$ are trace class. The reader is referred to [2] for details. These assumptions also guarantee that (27) holds. It is also easy to see that $G(\phi) = \exp\{\frac{\alpha}{\pi}\int_0^\infty ikf(x)\,dx\}$. To complete the proof we need a concrete representation for $\det((W(\phi) + H(\phi))W(\phi^{-1}))$. Define

$$h(k) = \log\det((W(\phi) + H(\phi))W(\phi^{-1})),$$

where $\phi = e^{ikf}$. Let $h(k) = \log\det((W(\phi) + H(\phi))W(\phi_{-1}))$. We need to show the second derivative of h is constant in k. A standard formula [5] yields

$$\begin{aligned}
h'(k) &= \mathrm{tr}((W(\phi^{-1}))^{-1}(W(\phi) + H(\phi))^{-1} \times \frac{d(W(\phi) + H(\phi))W(\phi^{-1})}{dk}) \\
&= \mathrm{tr}((W(\phi^{-1}))^{-1}(W(\phi) + H(\phi))^{-1}) \\
&\quad \times\{(W(\phi) + H(\phi))W(\phi^{-1}(-if)) + W(\phi if)W(\phi^{-1}) + H(\phi if)W(\phi^{-1})\} \\
&= \mathrm{tr}\{(W(\phi^{-1}))^{-1}W(\phi^{-1}(-if)) + (W(\phi^{-1}))^{-1}W(\phi^{-1})W(\phi if)W(\phi^{-1}) \\
&\quad +(W(\phi^{-1}))^{-1}W(\phi^{-1})H(\phi if)W(\phi^{-1}) + (W(\phi^{-1}))^{-1}H(\phi^{-1})W(\phi if)W(\phi^{-1}) \\
&\quad +(W(\phi^{-1}))^{-1}H(\phi^{-1})H(\phi if)W(\phi^{-1})\}.
\end{aligned}$$

This uses Theorem 4. Simplifying further and using the fact that $H(\phi^{-1})$ is trace class we have

$$\begin{aligned}
h'(k) = \mathrm{tr}\{&(W(\phi^{-1}))^{-1}W(\phi^{-1}(-if)) + W(\phi if)W(\phi^{-1}) \\
&+H(\phi if)W(\phi^{-1}) + H(\phi^{-1})W(\phi if) + H(\phi^{-1})H(\phi if)\}.
\end{aligned}$$

Now apply Theorem 3, part a) and the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ to find

$$\begin{aligned}
h''(k) = \mathrm{tr}\{&(W(\phi^{-1}))^{-1}W((\phi^{-1})(if)^2) \\
&-(W(\phi^{-1}))^{-1}W(\phi^{-1}(-if))(W(\phi^{-1}))^{-1}W(\phi^{-1}(-if))\}.
\end{aligned}$$

The conditions on ϕ guarantee that the function ϕ has a factorization $\phi = (g_- + 1)(g_+ + 1)$ such that the Fourier transforms of g_+ and g_- vanish for positive and negative real values respectively. Then using Theorem 3, part b), it is easy to see that we can write

$$W(\phi) = W(g_- + 1)W(g_+ + 1), W(\phi^{-1})^{-1} = W(g_+ + 1)W(g_- + 1).$$

A repeated application of these identities allows us to write $h''(k) = \mathrm{tr}H(if)H(if)$, and $h''(k)$ is independent of k. Thus at this point we have $h(k) = ak^2 + bk + c$, where $2a = -\mathrm{tr}((H(f))^2$. A direct computation shows that $a = -\frac{1}{2\pi^2}\int_0^\infty x|C(f)(x)|^2\,dx$. To compute b, notice that $h'(0)$ is $\mathrm{tr}H(if) = \frac{i}{2\pi}\int_0^\infty C(f(x))\,dx$. Also $h(0) = \mathrm{tr}\log(I) = 0$. Thus the last theorem holds. $\square$

4. The General Case

In this section we show that under certain conditions, the distribution function for general ν has the same form as in the case of $\nu = -1/2$. The only difference is in the mean which was computed in the last section. The attack on the problem is entirely different here. Instead of computing determinants asymptotically, we compute the traces of the operators $(B_\alpha(\sigma))^n$ and then piece together the answers to get an answer for the trace of $\log(I + B_\alpha(\sigma))$ and from that to the desired determinant.

To begin we need to show that $\mathrm{tr}\, f(B_\alpha(\sigma))$ makes sense for a class of analytic functions f. Just as we can associate the Wiener-Hopf operator with the Fourier transform and a multiplication operator, we can also write

$$B_\alpha(\sigma) = P\mathbf{H}\mathcal{M}_\sigma\mathbf{H},$$

where **H** is the Hankel transform and P is the projection on $L_2(0,1)$. Since the Hankel transform is unitary on $L_2(0,\infty)$ ([11]), the operator norm $||B_\alpha(\sigma)||$ is less than the infinity norm $||\sigma||_\infty$ of σ. Thus $f(B_\alpha(\sigma))$ is defined for f analytic on a disk centered at the origin with radius $||\sigma||_\infty + \delta, \delta > 0$. The operator $B_\alpha(\sigma)$ is also trace class for σ in L_1 by Mercer's Theorem ([5] Ch.III) as is $f(B_\alpha(\sigma))$ for f satisfying the above and $f(1) = 0$.

We need some lemmas that will prove to be useful. These may be known already, but we include them for completeness.

Lemma 6. *Suppose* $-1 < p < 1,\ 0 < \lambda, \delta < 1, \mu < 0, p + \mu + \delta < 0$ *and* $0 < t < 1$. *Then*

$$\int_0^\infty s^p(1+s)^\mu|1-s|^{-1+\lambda}|1-ts|^{-1+\delta}\,ds$$
$$\leq A\max(|1-t|^{-1+\lambda}, |1-t|^{-1+\delta})\max(t^{-\lambda}, t^{-p-\lambda}), \tag{29}$$

where A *is some constant independent of* t.

Proof. We have

$$\int_0^\infty s^p(1+s)^\mu|1-s|^{-1+\lambda}|1-ts|^{-1+\delta}\,ds$$
$$= t^{-1+\delta}\int_0^1 s^p(1+s)^\mu|1-s|^{-1+\lambda}|1/t-s|^{-1+\delta}\,ds$$

$$+t^{-1+\delta}\int_1^{1/t} s^p(1+s)^\mu|1-s|^{-1+\lambda}|1/t-s|^{-1+\delta}\ ds$$
$$+t^{-1+\delta}\int_{1/t}^{\infty} s^p(1+s)^\mu|1-s|^{-1+\lambda}|1/t-s|^{-1+\delta}\ ds.$$

We consider each of the above integrals. In each, A is a possibly different constant independent of t but can depend on the other parameters. First,

$$\begin{aligned}&\int_0^1 s^p(1+s)^\mu|1-s|^{-1+\lambda}|1/t-s|^{-1+\delta}\ ds\\ &\leq |1/t-1|^{-1+\delta}\int_0^1 s^p(1+s)^\mu|1-s|^{-1+\lambda}ds\\ &\leq A|t-1|^{-1+\delta}t^{1-\delta}.\end{aligned}$$

Next,

$$\begin{aligned}&\int_1^{1/t} s^p(1+s)^\mu|1-s|^{-1+\lambda}|1/t-s|^{-1+\delta}\ ds\\ &\leq A\max(1,t^{-p})\int_1^{1/t}|1-s|^{-1+\lambda}|1/t-s|^{-1+\delta}ds\\ &= A\max(1,t^{-p})|1/t-1|^{-1+\lambda+\delta}\\ &= A\max(1,t^{-p})t^{-\lambda-\delta+1}|1-t|^{-1+\lambda+\delta}\\ &\leq A\max(1,t^{-p})t^{-\lambda-\delta+1}|1-t|^{-1+\lambda}.\end{aligned}$$

Finally,

$$\begin{aligned}&\int_{1/t}^{\infty} s^p(1+s)^\mu|1-s|^{-1+\lambda}|1/t-s|^{-1+\delta}\ ds.\\ &\leq |1-1/t|^{-1+\lambda}\int_{1/t}^{\infty} s^{p+\mu}|1/t-s|^{-1+\delta}\ ds\\ &\leq |1-1/t|^{-1+\lambda}t^{-p-\mu-\delta}A\\ &= A|t-1|^{-1+\lambda}t^{-p-\mu-\delta-\lambda+1}.\end{aligned}$$

Putting this together we have that the original integral is bounded by

$$A\max(|1-t|^{-1+\lambda},|1-t|^{-1+\delta})\max(1,t^{-\lambda},t^{-p-\lambda}).\quad\square$$

Lemma 7. *Suppose* $-1<p<1,\ 0<\lambda,\delta<1,\mu<0,p+\mu+\lambda<0$ *and* $t>1$. *Then*

$$\int_0^{\infty} s^p(1+s)^\mu|1-s|^{-1+\lambda}|1-ts|^{-1+\delta}\ ds\leq A\max(|1-t|^{-1+\lambda},|1-t|^{-1+\delta})\max(1,t^{-p}),\tag{30}$$

where A *is some constant independent of* t.

Proof. The proof of this is almost identical to the previous lemma, and we leave the details to the reader.

Lemma 8. *Suppose* $|x| < 1, \mathrm{Re}\, c > 0, \mathrm{Re}\,(c-b) > 0,$ *and* $\mathrm{Re}\,(c-a-b) < 0$. *Then the hypergeometric function* $F(a,b,c,x)$ *satisfies the estimate*

$$|F(a,b,c,x)| \leq A|1-x|^{\mathrm{Re}\,(c-a-b)}$$

with A *independent of* x.

Proof. The hypergeometric function satisfies the identity $F(a,b,c,x) = (1-x)^{c-a-b}$ $F(c-a,c-b,c,x)$. Using Euler's integral formula for F, we have

$$F(c-a,c-b,c,x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{c-b-1}(1-t)^{b-1}(1-tx)^{-c+a}\, dx. \tag{31}$$

The last integral is bounded by $\int_0^1 t^{\mathrm{Re}\,(c-b-1)}(1-t)^{\mathrm{Re}\,(a+b-c-1)}dx$ or $\frac{\Gamma(\mathrm{Re}\,(c-b))\Gamma(\mathrm{Re}\,(a+b-c))}{\Gamma(\mathrm{Re}\,a)}$.

We next find an integral expression for the trace of $(B_\alpha(\sigma))^n$. We proceed informally at first and later state things rigorously. Using (18) we can write this trace as

$$\int_0^\infty \cdots \int_0^\infty \int_0^1 \cdots \int_0^1 \prod_{i=1}^n s_i x_i \sigma(x_i/\alpha) J_\nu(x_i s_i) J_\nu(x_i s_{i+1})\, ds_1 \ldots ds_n dx_1 \ldots dx_n,$$

where $s_{1+n} = s_1$. Let $\hat{\sigma}$ be the Mellin transform of σ, where $c > 0$. Then the above becomes

$$\frac{1}{(2\pi i)^n} \int_{c-i\infty}^{c+i\infty} \cdots \int_{c-i\infty}^{c+i\infty} \int_0^\infty \cdots \int_0^\infty \int_0^1 \cdots \int_0^1 \prod_{i=1}^n \{s_i x_i^{1-z_i} J_\nu(x_i s_i) J_\nu(x_i s_{i+1}) \hat{\sigma}(z_i)\}$$
$$\times \alpha^{z_1 + \ldots z_n}\, ds_1 \ldots ds_n dx_1 \ldots dx_n dz_1 \ldots dz_n.$$

Now use the formula

$$\int_0^\infty x^{-\lambda} J_\nu(ax) J_\nu(bx) dx$$
$$= \frac{(ab)^\nu \Gamma(\nu + \frac{1-\lambda}{2})}{2^\lambda (a+b)^{2\nu-\lambda+1}\Gamma(1+\nu)\Gamma(1/2+\frac{\lambda}{2})} F(\nu + \frac{1-\lambda}{2}, \nu + \frac{1}{2}; 2\nu+1; \frac{4ab}{(a+b)^2}),$$

where $F(a,b;c;z)$ is the hypergeometric function ${}_2F_1$, n times in the integral to get the expression

$$\frac{1}{(2\pi i)^n} \int_{c-i\infty}^{c+i\infty} \cdots \int_{c-i\infty}^{c+i\infty} \int_0^1 \cdots \int_0^1 \alpha^{z_1 + \ldots + z_n}$$
$$\times \prod_{i=1}^n \hat{\sigma}(z_i) \frac{s_i^{2\nu+1}\Gamma(\nu+1-z_i/2) F(\nu+1-z_i/2, \nu+\frac{1}{2}; 2\nu+1; \frac{4s_i s_{i+1}}{(s_i+s_{i+1})^2})}{2^{z_i-1}\Gamma(1+\nu)\Gamma(z_i/2)(s_i+s_{i+1})^{2\nu-z_i+2}}$$
$$\times ds_1 \ldots ds_n dz_1 \ldots dz_n.$$

Next we make the change of variables

$$\begin{aligned} s_1 &= s_1' \\ s_2 &= s_2' s_1' \\ &\vdots \\ s_n &= s_n' \ldots s_1', \end{aligned}$$

and the integral becomes

$$\frac{1}{(2\pi i)^n}\int_{c-i\infty}^{c+i\infty}\cdots\int_{c-i\infty}^{c+i\infty}\int_0^1\int_0^{\frac{1}{s_1}}\cdots\int_0^{\frac{1}{s_1\cdots s_{n-1}}}(\alpha/2)^{z_1+\ldots+z_n}2^n\hat{\sigma}(z_n)\frac{\Gamma(\nu+1-z_n/2)}{\Gamma(1+\nu)\Gamma(z_n/2)}$$

$$\times s_1^{z_1+\ldots+z_n-1}(1+s_n\ldots s_2)^{-2\nu+z_n-2}F(\nu+1-z_n/2,\nu+\frac{1}{2};2\nu+1;\frac{4s_n\ldots s_2}{(1+s_n\ldots s_2)^2})$$

$$\times\prod_{i=1}^{n-1}\{\frac{\hat{\sigma}(z_i)\Gamma(\nu+1-z_i/2)}{\Gamma(1+\nu)\Gamma(z_i/2)}s_{i+1}^{2\nu+1+z_{i+1}+\ldots z_{n-1}}(1+s_{i+1})^{-2\nu+z_i-2}$$

$$\times F(\nu+1-z_i/2,\nu+\frac{1}{2};2\nu+1;\frac{4s_{i+1}}{(1+s_{i+1})^2})\}ds_n\ldots ds_1dz_1\ldots dz_n.$$

Write the inside integral as

$$\int_0^1\int_0^{\frac{1}{s_1}}\cdots\int_0^{\frac{1}{s_1\cdots s_{n-1}}}\ldots ds_n\ldots d_1-\int_0^1\int_0^{\infty}\cdots\int_0^{\infty}\ldots ds_n\ldots ds_1$$

$$+\int_0^1\int_0^{\infty}\cdots\int_0^{\infty}\ldots ds_n\ldots ds_1.$$

The last integral in the above sum, inserted in the main integral, is the same as $\operatorname{tr}(B_\alpha(\sigma^n))$. After reversing the order of integration to $ds_1\ldots ds_n$, the first two terms combine to yield limits of integration

$$-\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\int_{\min(1,\frac{1}{s_2},\ldots,\frac{1}{s_2\cdots s_n})}^1,$$

and then the first integration can be done. The result is that

$$\operatorname{tr}(B_\alpha(\sigma))^n=\operatorname{tr}B_\alpha(\sigma^n)+C(\sigma),$$

where $C(\sigma)$ is given by the expression

$$\frac{-1}{(2\pi i)^n}\int_{c-i\infty}^{c+i\infty}\cdots\int_{c-i\infty}^{c+i\infty}(\alpha/2)^{z_1+\ldots+z_n}\prod_{i=1}^{n}\frac{\hat{\sigma}(z_i)2\Gamma(\nu+1-z_i/2)}{\Gamma(1+\nu)\Gamma(z_i/2)}$$

$$\times\int_0^{\infty}\cdots\int_0^{\infty}\frac{1-(\min(1,\frac{1}{s_2},\ldots,\frac{1}{s_2\ldots s_n}))^{z_1+\ldots+z_n}}{z_1+z_2+\ldots+z_n}$$

$$\times(1+s_n\ldots s_2)^{-2\nu+z_n-2}F(\nu+1-z_n/2,\nu+\frac{1}{2};2\nu+1;\frac{4s_n\ldots s_2}{(1+s_n\ldots s_2)^2})$$

$$\times\{\prod_{i=2}^{n}s_i^{2\nu+1+z_i+\ldots z_{n-1}}(1+s_i)^{-2\nu+z_{i-1}-2}\times F(\nu+1-z_{i-1}/2,\nu+\frac{1}{2};2\nu+1;\frac{4s_i}{(1+s_i)^2})\}$$

$$\times ds_n\ldots ds_2dz_1\ldots dz_n.$$

We next write this integral as

$$\frac{-1}{(2\pi i)^n}\int_{c-i\infty}^{c+i\infty}\cdots\int_{c-i\infty}^{c+i\infty}G(z_i)\int_0^{\infty}\cdots\int_0^{\infty}H(z_i;s_i)ds_n\dots ds_2dz_1\dots dz_n.$$

The idea from here on out is to evaluate this integral asymptotically using complex analysis. This will be done in several stages and by breaking the integral into several parts. To begin we first consider the interior integration

$$\int_0^{\infty}\cdots\int_0^{\infty}H(z_i;s_i)ds_2\dots ds_n.$$

Consider this as an integral over $R_1 \cup R_2$, where R_1 is a union of disjoint sets, $R_1 = \cup_{i=2}^n U_i$ such that on U_i, s_i is bounded away from 1 and where R_2 is the complement of R_1. □

Lemma 9. *Suppose that* $-2\nu - 1 < 0$. *The integral of* $H(z_i;s_i)$ *over* U_i *is bounded and the* z_i *variables can be changed in such a way so that the integrated function is analytic in a particular* z *variable to the left of the imaginary axis.*

Proof. For convenience let $i = 2$ (although the proof is the same for any i) and let $z_1 + \dots + z_n = z_1^{'}$ with the other variables remaining the same. Suppose that Re $z_i = c$ for $i = 3,\dots,n$ and that $c > 0$. Suppose also that Re $z_1^{'} = b$ with $|b| < c$. We now refer to z_1' as z. Our goal is to show that this integral is bounded and that as a function of z is analytic to the left of the imaginary axis. By repeated application of Lemma 8, we can say that the integral is bounded by a constant times

$$\int_{|s_2-1|\geq B}\int_0^{\infty}\cdots\int_0^{\infty}\left|\frac{1-(\min(1,\frac{1}{s_2},\dots,\frac{1}{s_2\dots s_n}))^{z_1+\dots+z_n}}{z_1+z_2+\dots+z_n}\right|$$

$$\times\prod_{i=3}^{n}(1+s_i)^{-2\nu-1}s_i^{(n-i)c}|1-s_i|^{c-1}$$

$$\times s_2^{b-2c}(1+s_2)^{-2\nu-1}|1-s_2|^{b-1}|1-s_2\dots s_n|^{c-1}$$

$$\times ds_2\dots ds_n.$$

This is valid as long as $2\nu + 1 > 0$ and Re $z_i - 1/2 < 0$, which is the case here if we assume that c is small enough. Next, we estimate

$$\left|\frac{1-(\min(1,\frac{1}{s_2},\dots,\frac{1}{s_2\dots s_n}))^{z_1+\dots+z_n}}{z_1+z_2+\dots+z_n}\right|$$

by using the fact that $|1 - x^z| \leq \max |z|x^{\mathrm{Re}\, z'}| \ln x|$, where the max is taken over the z' values on a line connecting 0 and z, x between 0 and 1. Thus, $|1 - x^z| \leq K|z|x^{\mathrm{Re}\, z}x^{-\epsilon}$ for some positive ϵ chosen shortly. Inserting this in the integral we have that the integral is bounded by a constant times

$$\int_{|s_2-1|\geq B}\int_0^{\infty}\cdots\int_0^{\infty}\sum_{j=2}^{n}\{\max(1,(s_2\ldots s_j)^{\epsilon}(s_2\ldots s_j)^{-b+\epsilon})\}$$
$$\times\prod_{i=3}^{n}(1+s_i)^{-2\nu-1}s_i^{(n-i)c}|1-s_i|^{c-1}$$
$$\times s_2^{b-2c}(1+s_2)^{-2\nu-1}|1-s_2|^{b-1}|1-s_2\ldots s_n|^{c-1}ds_2\ldots ds_n.$$

The reason for both terms in the "max" part of the integral is that b could be either positive or negative. Now let's begin with the s_n integration. Then the first interior integral has the form

$$\int_0^{\infty} s_n^p(1+s_n)^{-2\nu-1}|1-s_n|^{c-1}|1-s_2\ldots s_n|^{-1+c}ds_n.$$

The value for p is either $\pm a$, where $a=|b-\epsilon|<c$. The next step is to apply Lemmas 6 and 7. We use $\lambda=\delta=c$ and p as above. The result is that this integral is bounded by a constant times

$$|1-s_2\ldots s_{n-1}|^{c-1}\times\max(1,(s_2\ldots s_{n-1})^{-c},(s_2\ldots s_{n-1})^{-c-p}(s_2\ldots s_{n-1})^{-p}).$$

We collect powers and use the lemmas twice with respect to the s_{n-1} integration and powers of $p=\pm(2c)$. At the next integration step the powers of $p=\pm 3c$ and so on until we arrive at the s_2 integration. Here we will have

$$\int_{|s_2-1|\geq B} s_2^p|1-s_2|^q|1+s_2|^{-2\nu-1}ds_2,$$

where p and q are appropriate powers. These integrals satisfy all the conditions necessary for the lemmas as long as c and b are small enough. We will have at most 2^n integrals in this process. Hence the integral of H over U_i is analytic in the z variable in a strip $|\mathrm{Re}\, z|<c$ by the application of Morera's Theorem and Fubini's Theorem. □

We remark here that this proof also is easily modified to show that the interchange of integrals done at the beginning of the section are valid and the expression $C(\sigma)$ is the one of interest.

Lemma 10. *Suppose that σ has $[\nu]+2$ derivatives all in L_1 and that $-2\nu-1<0$. Then the integral*

$$\int_{c-i\infty}^{c+i\infty}\cdots\int_{c-i\infty}^{c+i\infty}G(z_i)\int\cdots\int_{R_1}H(z_i;s_i)ds_2\ldots ds_n dz_1\ldots dz_n$$

is $O(\alpha^{-\delta})$, where $\delta>0$.

Proof. Note that the condition in the hypothesis implies that

$$\int_{c-i\infty}^{c+\infty}|\hat{\sigma}(z)||z|^{\nu+1/2}<\infty. \tag{32}$$

We first replace the inside integral with a sum of integrals over U_i. For each of these we change variables as in the last lemma. We can then perform the integration over the

z variable by moving it to a line to the left of the imaginary axis. Thus we have that each of these integrals is bounded by a constant times

$$\frac{\alpha^b}{(\pi)^n 2^b}\int_{b-i\infty}^{b+i\infty}\int_{c-i\infty}^{c+i\infty}\cdots\int_{c-i\infty}^{c+i\infty}\left|\prod_{i=2}^{n}\frac{\hat{\sigma}(z_i)\Gamma(\nu+1-z_i/2)}{\Gamma(1+\nu)\Gamma(z_i/2)}\right|$$

$$\times\left|\frac{\hat{\sigma}(z-\sum_{z_j\neq z}z_j)\Gamma(\nu+1-(z-\sum_{z_j\neq z}z_j)/2)}{\Gamma(1+\nu)\Gamma((z-\sum_{z_j\neq z}z_j)/2)}\right|dzdz_2\ldots dz_n.$$

This last integral is bounded by a product of integrals all of the form

$$\int_{c-i\infty}^{c+i\infty}|\hat{\sigma}(z)|\left|\frac{\Gamma(\nu+1-z/2)}{\Gamma(z/2)}\right|dz,$$

and these in turn are bounded by (32) using the basic asymptotics properties of the Gamma function. □

We now turn our attention to the region R_2. To begin we make another change of variables,

$$\frac{1}{s_2} = 1 - s_2', \tag{33}$$

$$\frac{1}{s_2 s_3} = 1 - s_2' - s_3', \tag{34}$$

$$\vdots \tag{35}$$

$$\frac{1}{s_2\ldots s_n} = 1 - s_2' - s_3' - \ldots - s_n'. \tag{36}$$

Under the change of variables, the region R_2 is transformed to a region R_3 which can be assumed to be a symmetric region containing the origin, and where the sum

$$|s_2 + \ldots + s_j| \le a < 1$$

(we drop the "primes" again) for some a. Notice that the exact form of R_1 was unnecessary in the previous computation. Thus the integral over R_2 is transformed to

$$\int_{R_3}\cdots\int I(z_i; s_i)ds_2\ldots ds_n,$$

where

$$I(z_i; s_i) = \frac{1-(1-\max(0, s_2, \ldots, s_2+\ldots+s_n))^{z_1+\ldots+z_n}}{z_1+\ldots+z_n}$$

$$\times|s_2|^{z_2-1}\ldots|s_n|^{z_{n-1}-1}|s_2+\ldots+s_n|^{z_n-1}$$

$$\times f(s_2, \ldots, s_n, z_1, \ldots, z_n),$$

where the function f is smooth in the s variables.

The following lemmas will help keep track of the contribution of the R_3 integral.

Lemma 11. *Suppose* Re $z_i = c, 0 < c < 1$, *for* $i \geq 3$. *Then the integral*

$$\int_{R_3} \cdots \int |s_2|^{z_1+1}|s_3|^{z_2-1} \ldots |s_n|^{z_{n-1}-1}|s_2 + \ldots s_n|^{z_n-1} ds_2 \ldots ds_n$$

can be thought of as an analytic function in the z_1 *variable that can be extended to a strip containing the imaginary axis.*

Proof. First note that the following integral with z and w real and between zero and one satisfies

$$\int_a^b |x|^{z-1}|x+y|^{w-1} dx \leq A|y|^{z+w-1},$$

where the constant only depends on the z and w variable. A repeated application of this estimate in the above integral yields a final integration of

$$\int_a^b |s_2|^{\text{Re}\, z_1+(n-2)c} ds_2.$$

Thus, once again the analytic continuation argument holds. □

Lemma 12. *Suppose* Re $z_i = c, 0 < c < 1$, *for* $i \geq 2$ *and* Re $z_1 = d$. *Then the integral*

$$\int_{R_3} \cdots \int |s_2|^{z_1}|s_3|^{z_2}|s_4|^{z_3-1} \ldots |s_n|^{z_{n-1}-1}|s_2 + \ldots s_n|^{z_n-1} ds_2 \ldots ds_n$$

can be thought of as an analytic function in the z_1 *variable that can be extended to a strip containing the imaginary axis.*

Proof. We begin the integration just as in the previous integral. After $n-3$ integrations we arrive at an integral with an estimate of the form

$$\int_a^b \int_a^b |s_2|^d |s_3|^c |s_3 + s_2|^{(n-3)c-1} ds_2 ds_3.$$

We can estimate this by looking at three integrals

$$\int_a^b \int_{-1}^1 |s_2|^{d+(n-2)c}|s_3|^c|s_3+1|^{(n-3)c-1} ds_3 ds_2,$$

$$\int_a^b \int_1^{b/s_2} |s_2|^{d+(n-2)c}|s_3|^c|s_3+1|^{(n-3)c-1} ds_3 ds_2,$$

and

$$\int_a^b \int_{a/s_2}^{-1} |s_2|^{d+(n-2)c}|s_3|^c|s_3+1|^{(n-3)c-1} ds_3 ds_2.$$

We can say, for example, that the last integral is less than a constant times

$$\int_a^b |s_2|^{d-(n-4)c} ds_2,$$

and thus is finite for Re z_1 in a strip about the imaginary axis. The other two integrals are handled in the same manner. So by our standard argument the analytic extension is defined. □

Now let us return to our function $I(z_i; s_i)$. We can write the expression

$$\frac{1-(1-\max(0,s_2,\ldots,s_2+\ldots+s_n))^{z_1+\ldots z_n}}{z_1+\ldots+z_n}$$

as

$$\max(0,s_2,\ldots,s_2+\ldots s_n)+(\max(0,s_2,\ldots,s_2+\ldots s_n))^2\times g(z_1+\ldots+z_n,s_2,\ldots s_n),$$

where the last function is a bounded continuous function in the variables.

Lemma 13. *The contribution of*

$$\frac{-1}{(2\pi i)^n}\int_{c-i\infty}^{c+i\infty}\ldots\int_{c-i\infty}^{c+i\infty}G(z_i)\int\ldots\int_{R_2}(\max(0,s_2,\ldots,s_2+\ldots s_n))^2$$
$$\times|s_2|^{z_1-1}\ldots|s_n|^{z_{n-1}-1}|s_2+\ldots+s_n|^{z_n-1}$$
$$\times f(s_2,\ldots,s_n,z_1,\ldots,z_n)g(z_1+\ldots+z_n,s_2,\ldots s_n)$$
$$\times ds_n\ldots ds_2dz_1\ldots dz_n$$

is $O(\alpha^{-\delta})$.

Proof. We simply consider the set where say $s_1+s_2+\ldots+s_j$ is the maximum of the terms. We then expand the square so that we have a term of the form s_is_k. We then apply the above lemmas after an appropriate re-ordering of the variables and the lemma holds.

The next step is to replace the function f in the expression for $I(z_i; s_i)$ with the first term of its Taylor expansion. This expansion gives an "extra" s_i (combined with the ones from the $\max(0,s_2,\ldots,s_2+\ldots+s_n)$) term in the estimates which, as the above lemmas show, is all we need to show that this part of the integral does not contribute in the asymptotic expansion.

So we are finally at the one critical term that gives a contribution in the expansion. This term is

$$\frac{-1}{(2\pi i)^n}\int_{c-i\infty}^{c+i\infty}\ldots\int_{c-i\infty}^{c+i\infty}G(z_i)\int\ldots\int_{R_3}\max(0,s_2,\ldots,s_2+\ldots+s_n)$$
$$\times|s_2|^{z_1-1}\ldots|s_n|^{z_{n-1}-1}|s_2+\ldots+s_n|^{z_n-1}$$
$$\times f(0,0,\ldots,0,z_1,\ldots,z_n)ds_n\ldots ds_2dz_1\ldots d_n.$$

We can easily compute $f(0,0,\ldots,0,z_1,\ldots,z_n)$ to see that it equals

$$\prod_1^n\frac{2^{-2\nu-1}\Gamma(2\nu+1)\Gamma(-z_i/2+1/2)}{\Gamma(\nu+1/2)\Gamma(\nu+1-z_i/2)}.$$

We can simplify further using the formula for $G(z_i)$ and the duplication formula for the Gamma function to arrive at

$$C(\sigma)=\frac{-1}{(2\pi i)^n}\int_{c-i\infty}^{c+i\infty}\ldots\int_{c-i\infty}^{c+i\infty}(\alpha/2)^{z_1+\ldots+z_n}\pi^{-n/2}\prod_1^n\frac{\hat{\sigma}(z_i)\Gamma(-z_i/2+1/2)}{\Gamma(z_i/2)}$$
$$\times\int\ldots\int_{R_3}\max(0,s_2,\ldots,s_2+\ldots+s_n)$$

$$\times |s_2|^{z_1-1}\dots|s_n|^{z_{n-1}-1}|s_2+\dots+s_n|^{z_n-1}ds_2\dots ds_n dz_1\dots dz_n + O(\alpha^{-\delta}). \quad (37)$$

Notice that this expression is now independent of ν. Our final steps are to compute the contribution from the above integral and we, by the way, finally have an integral which will yield a contribution. We begin with a well-known identity due to Mark Kac, which was used originally to prove the continuous analogue of the Strong Szegö Limit Theorem. It reads

$$\sum_\sigma \max(0, a_{\sigma_1}, a_{\sigma_1}+a_{\sigma_2}, \dots, a_{\sigma_1}+\dots+a_{\sigma_n}) = \sum_\sigma \sum_{k=1}^n a_{\sigma_1}\theta(a_{\sigma_1}+\dots+a_{\sigma_k}),$$

where $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ otherwise and the sums are taken over all permutations in n variables.

Because of this identity we can rewrite the integral in (37) as

$$\sum_{j=2}^n \frac{-1}{(2\pi i)^n}\int_{c-i\infty}^{c+i\infty}\dots\int_{c-i\infty}^{c+i\infty}(\alpha/2)^{z_1+\dots+z_n}\pi^{-n/2}\prod_1^n \frac{\hat{\sigma}(z_i)\Gamma(-z_i/2+1/2)}{\Gamma(z_i/2)}$$

$$\times\int\dots\int_{R_3\cap\{s_2+\dots+s_j>0\}} s_2|s_2|^{z_1-1}\dots|s_n|^{z_{n-1}-1}|s_2+\dots+s_n|^{z_n-1}ds_2\dots ds_n dz_1\dots dz_n. \quad (38)$$

It is straightforward to see how this identity can be used if the integrand is symmetric in the variables. In our case, the integrand is not obviously symmetric in the variables, but can always be made so by changing the z variables. Thus we can apply the identity.

We once again consider the inner integral and call $z = z_1+\dots+z_n$ leaving the other variables as is, and show how this inner integral can be thought of as analytic in z in a strip containing the imaginary axis. The difference is that in this case there will be a pole at $z = 0$. □

Now we suppose that $j > 2$. For $j = 2$ the following computation is almost identical and the conclusion is the same. Let us rewrite the inner integral in (38) as two integrals

$$\int_0^b \int_B\dots\int s_2|s_2|^{z-z_2-z_3-\dots z_n-1}\dots|s_n|^{z_{n-1}-1}|s_2+\dots+s_n|^{z_n-1}ds_n\dots ds_2$$

$$+\int_{-b}^0 \int_B\dots\int s_2|s_2|^{z-z_2-z_3-\dots z_n-1}\dots|s_n|^{z_{n-1}-1}|s_2+\dots+s_n|^{z_n-1}ds_n\dots ds_2,$$

where B is some $n-2$ dimensional set. In the first (the computations for the second integral being almost identical) of these we make yet another change of variables:

$$s_3 = s_3' s_2'$$
$$\vdots$$
$$s_n = s_n' s_2'$$

to arrive at

$$\int_0^b s_2^{z-1}\int_{B/s_2}\dots\int |s_3|^{z_2-1}\dots|s_n|^{z_{n-1}-1}|1+s_3+\dots+s_n|^{z_n-1}ds_n\dots ds_3 ds_2.$$

The original set R_3 was chosen to be symmetric and contain the origin. So here we chose it to be something convenient, say a cube C with size length l. With this choice we can write B/s_2 as $C/s_2 \cap \{s_3 + \ldots + s_n + 1 > 0\}$. Next integrate by parts with respect to the s_2 variable. The result is that the above integral becomes:

$$s_2^z k(s_2) - \int_0^b s_2^z d/ds_2(k(s_2)) ds_2,$$

where

$$k(s_2) = \int_{B/s2} \ldots \int |s_3|^{z_2 - 1} \ldots |s_n|^{z_{n-1} - 1} |1 + s_3 + \ldots + s_n|^{z_n - 1} ds_n \ldots ds_3.$$

The function $k(s_2)$ has a derivative given by the formula

$$k'(s_2) = -s_2^{-1} \int_D f(s_3, \ldots, s_n)\, (\mathbf{n} \cdot s_2^{-1}(s_3, \ldots, s_n)) dS,$$

where D is the boundary of the set C/s_2 which lies in the half-space defined by $\{s_3 + \ldots + s_n + 1 > 0\}$, the vector **n** is the outward normal to the surface, the function f is simply the one given in the above integral restricted to the surface, and dS is surface measure. We can estimate the derivative of $k(s_2)$ on any boundary edge to be at most a constant times $s_2^{(n-2)c}$ for Re $z_i = c$. Thus we have proved the following:

Lemma 14. *The function of z defined by*

$$\int_0^b \int_B \ldots \int s_2 |s_2|^{z - z_2 - z_3 - \ldots z_n - 1} \ldots |s_n|^{z_{n-1} - 1} |s_2 + \ldots + s_n|^{z_n - 1} ds_n \ldots ds_2$$

$$+ \int_{-b}^0 \int_B \ldots \int s_2 |s_2|^{z - z_2 - z_3 - \ldots z_n - 1} \ldots |s_n|^{z_{n-1} - 1} |s_2 + \ldots + s_n|^{z_n - 1} ds_n \ldots ds_2$$

is analytic in a strip containing the imaginary axis except at the point $z = 0$. Further, the contribution of this integral with the z integration moved to a line to the left of the axis is given by the residue at $z = 0$ plus $O(\alpha^{-\delta})$.

We note here that there are no other poles given our conditions on σ, (32) and the formula for $G(z_i)$.

For $j > 2$, the above computation also shows exactly what the residue is, namely:

$$\int_{\mathbf{R}^{n-2} \cap \{s_3 + \ldots + s_j > -1\}} \ldots \int |s_3|^{z_2 - 1} \ldots |s_n|^{z_{n-1} - 1} |1 + s_3 + \ldots + s_n|^{z_n - 1} ds_n \ldots ds_3$$

$$- \int_{\mathbf{R}^{n-2} \cap \{s_3 + \ldots + s_j > -1\}} \ldots \int |s_3|^{z_2 - 1} \ldots |s_n|^{z_{n-1} - 1} | - 1 + s_3 + \ldots + s_n|^{z_n - 1} ds_n \ldots ds_3.$$

To find an explicit formula for this integral we start with the following formula that can be easily proved using formulas for the Beta function.

For

$$0 < \mathrm{Re}\, p, \mathrm{Re}\, q < 1, \mathrm{Re}\, (p + q) < 1,$$

$$\int_{-\infty}^{\infty} |x|^{p-1}|x+y|^{q-1}dx = |y|^{p+q-1}\frac{2\Gamma(p)\Gamma(q)\cos(\pi p/2)\cos(\pi q/2)}{\Gamma(p+q)\cos((p+q)\pi/2)}. \tag{39}$$

Define $t(p,q)$ to be

$$\frac{2\Gamma(p)\Gamma(q)\cos(\pi p/2)\cos(\pi q/2)}{\Gamma(p+q)\cos((p+q)\pi/2)}.$$

The residue is then (B is the Beta function)

$$B(z_2+\ldots+z_{j-1}, z_j+\ldots+z_n)\prod_{k=j}^{n-1} t(z_k, z_n+\ldots+z_{k+1})\prod_{k=2}^{j-2} t(z_k, z_{k+1}+\ldots+z_{j-1}).$$

We leave this as an exercise to the reader. For $j = 2$ the residue can also be easily computed using the definition of $t(p,q)$ and it is seen to be

$$\prod_{k=2}^{n-1} t(z_k, z_n+\ldots+z_{k+1}).$$

Combining all of the above results we are left with the following theorem.

Theorem 15. *Suppose σ has $[\nu]+2$ continuous derivatives in L_1. Then*

$$\operatorname{tr}(B_\alpha(\sigma))^n = \operatorname{tr} B_\alpha(\sigma^n) + C(\sigma),$$

where

$$C(\sigma) = \frac{-1}{\pi^2}\sum_{j=1}^{n-1}\frac{1}{j}\int_0^\infty xC(\sigma^j)(x)C(\sigma^{n-j})(x)dx + o(1).$$

Proof. Recall we were computing the integral

$$\sum_{j=2}^{n}\frac{-1}{(2\pi i)^n}\int_{c-i\infty}^{c+i\infty}\cdots\int_{c-i\infty}^{c+i\infty}(\alpha/2)^{z_1+\ldots+z_n}\pi^{-n/2}\prod_1^n\frac{\hat\sigma(z_i)\Gamma(z_i/2+1/2)}{\Gamma(z_i/2)}$$

$$\times\int\cdots\int_{R_3\cap\{s_2+\ldots+s_j>0\}} s_2|s_2|^{z_1-1}\ldots|s_n|^{z_{n-1}-1}|s_2+\ldots+s_n|^{z_n-1}ds_2\ldots ds_n dz_1\ldots. dz_n. \tag{40}$$

For each j we rename the variables and compute the residue as above. For $j > 2$ the residue is

$$\frac{-1}{(2\pi i)^{n-1}}\int_{c-i\infty}^{c+i\infty}\cdots\int_{c-i\infty}^{c+i\infty}\pi^{-n/2}\prod_2^n\frac{\hat\sigma(z_i)\Gamma(-z_i/2+1/2)}{\Gamma(z_i/2)}$$

$$\times\frac{\hat\sigma(-z_2-\ldots-z_n)\Gamma((z_2+\ldots+z_n)/2+1/2)}{\Gamma((-z_2-\ldots-z_n)/2)}B(z_2+\ldots+z_{j-1}, z_j+\ldots+z_n)$$

$$\times\prod_{k=j}^{n-1} t(z_k, z_n+\ldots+z_{k+1})\prod_{k=2}^{j-2} t(z_k, z_{k+1}+\ldots+z_{j-1})dz_2\ldots dz_n. \tag{41}$$

Notice that

$$t(p,q)t(p+q,r) = 2^2 \frac{\Gamma(p)\Gamma(q)\Gamma(r)\cos(p)\cos(q)\cos(r)}{\Gamma(p+q+r)\cos((p+q+r)\pi/2)}.$$

Using this identity in (4) we have that the above integral is

$$\frac{-1}{(2\pi i)^{n-1}} \int_{c-i\infty}^{c+i\infty} \cdots \int_{c-i\infty}^{c+i\infty} 2^{n-3}\pi^{-n/2} \prod_{i=2}^{n} \frac{\hat{\sigma}(z_i)\Gamma(z_i)\cos(z_i\pi/2)\Gamma(-z_i/2+1/2)}{\Gamma(z_i/2)}$$

$$\times \frac{\hat{\sigma}(-z_2-\ldots-z_n)\Gamma((z_2+\ldots+z_n)/2+1/2)}{\Gamma\frac{(-z_2-\ldots-z_n)}{2}\Gamma(z_2+\ldots+z_n)\cos(z_2+\ldots+z_{j-1})\pi/2\cos(z_j+\ldots+z_n)\pi/2} dz_2\ldots dz_n. \tag{42}$$

From the duplication formula for the Gamma function, this can be simplified to

$$\frac{-1}{(2\pi i)^{n-1}} \int_{c-i\infty}^{c+i\infty} \cdots \int_{c-i\infty}^{c+i\infty} 2^{-2}\pi^{-1} \prod_{i=2}^{n} \hat{\sigma}(z_i)$$

$$\times \frac{\hat{\sigma}(-z_2-\ldots-z_n)(z_2+\ldots+z_n)\sin((z_2+\ldots+z_n)\pi/2)}{\cos((z_2+\ldots+z_{j-1})\pi/2)\cos((z_j+\ldots+z_n)\pi/2)} dz_2\ldots dz_n. \tag{43}$$

Now we change variables with

$$z_{j-1} = z_2+\ldots+z_{j-1},\, z_n = z_j+\ldots+z_n,$$

and the above integral becomes

$$\frac{-1}{(2\pi i)^{n-1}} \int_{c-i\infty}^{c+i\infty} \cdots \int_{c-i\infty}^{c+i\infty} 2^{-2}\pi^{-1}(\prod_{i=2}^{j-2} \hat{\sigma}(z_i))\hat{\sigma}(z_{j-1}-\ldots-z_2) \prod_{i=j}^{n-1} \hat{\sigma}(z_i)$$

$$\times \hat{\sigma}(z_n-\ldots-z_j)\hat{\sigma}(-z_{j-1}-z_n)(z_{j-1}+z_n) \frac{\sin((z_{j-1}+z_n)\pi/2)}{\cos(z_{j-1}\pi/2)\cos(z_n\pi/2)} dz_2\ldots dz_n. \tag{44}$$

The convolution theorem for the Mellin transform shows that this can be reduced to the integral

$$\frac{-1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} 2^{-2}\pi^{-1} \widehat{\sigma^{j-2}}(z_{j-1})\widehat{\sigma^{n-j}}(z_n)$$

$$\times \hat{\sigma}(-z_{j-1}-z_n)(z_{j-1}+z_n) \frac{\sin((z_{j-1}+z_n)\pi/2)}{\cos(z_{j-1}\pi/2)\cos(z_n\pi/2)} dz_{j-1}dz_n. \tag{45}$$

Notice this can also be written as

$$\frac{-1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} 2^{-2}\pi^{-1} \widehat{\sigma^{j-2}}(z_{j-1})\widehat{\sigma^{n-j+1}}(z_n)$$

$$\times \hat{\sigma}(-z_{j-1}-z_n)(z_{j-1}+z_n) \frac{\sin(z_{j-1}\pi/2)}{\cos(z_{j-1}\pi/2)} dz_{j-1}dz_n \tag{46}$$

$$-\frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{c-i\infty}^{c+i\infty} 2^{-2}\pi^{-1} \widehat{\sigma^{j-2}}(z_{j-1})\widehat{\sigma^{n-j+1}}(z_n)$$

$$\times \hat{\sigma}(-z_{j-1}-z_n)(z_{j-1}+z_n) \frac{\sin(z_n\pi/2)}{\cos(z_n\pi/2)} dz_{j-1}dz_n. \tag{47}$$

Before we proceed further we need three formulas from the theory of Mellin transforms. These are

$$\text{the Mellin transform of } \int_x^\infty \phi(x)dx = z^{-1}\Phi(z+1),$$

where Φ is the transform of ϕ,

$$\text{the Mellin transform of} \quad x\phi'(x) = -z\Phi(z),$$

where Φ is the transform of ϕ, and finally

$$\frac{2}{\pi}\int_0^\infty xC(\phi)(x)C(\psi)(x)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} \Phi(z)\Psi(-z)z\tan(z\pi/2)dz.$$

These can be found in any standard table of transforms, although the third requires a straightforward computation combined with the convolution theorem.

So now we apply the second formula along with convolution with respect to the z_n variable and we have for each $2 < j < n$,

$$\frac{1}{8\pi^2 i}\int_{c-i\infty}^{c+i\infty} \widehat{\sigma^{j-2}}(z_{j-1})x\widehat{\sigma^{n-j+1}}\sigma'(-z_{j-1})\frac{\sin(z_{j-1}\pi/2)}{\cos(z_{j-1}\pi/2)}dz_{j-1} \tag{48}$$

$$+\frac{1}{8\pi^2 i}\int_{c-i\infty}^{c+i\infty} \widehat{\sigma^{n-j+1}}(z_{j-1})x\widehat{\sigma^{j-2}}\sigma'(-z_{j-1})\frac{\sin(z_{j-1}\pi/2)}{\cos(z_{j-1}\pi/2)}dz_{j-1}. \tag{49}$$

Next apply the first formula after inserting a factor of z_{j-1}/z_{j-1} to write the above as

$$\frac{1}{2\pi^2}\int_0^\infty xC(\sigma^{j-2})(x)C(\int_x^\infty \sigma^{n-j+1}\sigma')(x)dx \tag{50}$$

$$+\frac{1}{2\pi^2}\int_0^\infty xC(\sigma^{n-j+1})(x)C(\int_x^\infty \sigma^{j-2}\sigma')(x)dx \tag{51}$$

or

$$\frac{-1}{2\pi^2}\frac{1}{n-j+2}\int_0^\infty xC(\sigma^{j-2})(x)C(\sigma^{n-j+2})(x)\,dx \tag{52}$$

$$+\frac{-1}{2\pi^2}\frac{1}{j-1}\int_0^\infty xC(\sigma^{j-1})(x)C(\sigma^{n-j+1})(x)\,dx. \tag{53}$$

We can do the $j = 2, j = n$ cases separately just as easily (the above formulas are not even all required in that case) and putting the two cases together and reindexing when necessary we arrive at the conclusion of the theorem.

Our final step is to extend this to functions other than powers. The standard uniformity arguments used in the Wiener-Hopf theory apply here if we can show that

$$||\text{tr}\, f(B_\alpha(\sigma)) - \text{tr}\, B_\alpha(f(\sigma))||_1 = O(1)$$

uniformly for σ replaced by $1 - \lambda + \lambda\sigma$ and λ in some complex neighborhood of $[0, 1]$. The details of this are found in [14]. The norm above is the trace norm. Given sufficient analyticity conditions on f, it is only necessary to prove $||B_\alpha(\sigma_1)B_\alpha(\sigma_2) - B_\alpha(\sigma_1\sigma_2)||_1 = O(1)$, where the $O(1)$ here depends on properites of σ_i. A trace norm of a product can always be estimated by the product of two Hilbert-Schmidt norms and in this case we need to estimate the Hilbert Schmidt norm of the operator with kernel

$$X_{(1,\infty)}(z)\int_0^\infty \sigma_i(t/\alpha)\sqrt{xz}tJ_\nu(xt)J_\nu(tz)dt.$$

Using integration by parts, and integration formulas for Bessel functions this is easily estimated to be bounded. For analogous details see [14]. Thus for suitably defined f we can extend our previous theorem to the more general case. The f of interest is $\log(1+z)$. This will satisfy the necessary analyticity conditions if we consider small enough k. The necessary conditions are collected in the following:

Theorem 16. *Suppose f is a real-valued function with $[\nu]+2$ derivatives all contained in L_1. Then for sufficiently small k (say $k < ||\sigma||_\infty^{-1}$)*

$$\check{\phi}(k) \sim \exp\left\{\frac{\alpha}{\pi}\int_0^\infty ikf(x)dx - \frac{ik\nu}{2}f(0) - \frac{k^2}{2\pi^2}\int_0^\infty xC(f)^2(x)dx\right\}.$$

Proof. The form of the answer follows from the computation of the mean given earlier and from the fact that the constant term in the previous theorem is exactly half of the answer in Szegö's Theorem. Thus the above answer for the log function must be half as well. □

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