Commun. Math. Phys. 188, 69 – 88 (1997)

Communications in
Mathematical Physics

The Thermal Equilibrium Solution of a Generic Bipolar Quantum Hydrodynamic Model

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Received: 26 March 1996 / Accepted: 13 January 1997

Dedicated to Helmut Neunzert at his 60th birthday

Abstract: The thermal equilibrium state of a bipolar, isothermic quantum fluid confined to a bounded domain $\Omega \subset \mathbb{R}^d, d = 1, 2$ or $d = 3$ is entirely described by the particle densities n, p, minimizing the energy

$$\varepsilon^2 \int |\nabla\sqrt{n}|^2 + \varepsilon^2 \int |\nabla\sqrt{p}|^2 + \int G_1(n) + \int G_2(p) + \frac{\lambda^2}{2} \int |\nabla V[n - p - C]|^2,$$

where $G_{1,2}$ are strictly convex real valued functions, $-\lambda^2 \Delta V = n - p - C$, with $\int(n - p - C) = \int V = 0$. It is shown that this variational problem has a unique minimizer in

$$\left\{(n, p) \in L^1(\Omega) \times L^1(\Omega) : n, p \geq 0, \sqrt{n}, \sqrt{p} \in H^1(\Omega), \int n = N, \int p = P\right\}$$

and some regularity results are proven. The semi-classical limit $\varepsilon \to 0$ is carried out recovering the minimizer of the limiting functional. The subsequent zero space charge limit $\lambda \to 0$ leads to extensions of the classical boundary conditions. Due to the lack of regularity the asymptotics $\lambda \to 0$ can not be settled on Sobolev embedding arguments. The limit is carried out by means of a compactness-by-convexity principle.

1. Introduction

Quantum hydrodynamic models (QHDs) give a fairly accurate account of the macroscopic behavior of ultra small semiconductor devices in terms of only macroscopic quantities such as particle densities, current densities and electric fields.

Within semiconductor device modeling QHDs are located between microscopic quantum models (Schrödinger-Poisson systems [16, 15], Bloch's equation [3, 13] or kinetic-type quantum transport equations [14]) and macroscopic semi-classical hydrodynamic models [14]. Presently the interplay between these different approaches is a field of intensive research. Actual research deal with the derivation of QHDs from microscopic quantum models (essentially based on Madelung's transformation, see [6] for a review) and investigations of the semi-classical limit $\hbar \to 0$.

All quantum models of semiconductor devices investigated so far are *unipolar*, i.e. these models involve *only one* particle type, namely electrons. Hence a consistency problem arises. Whenever quantum effects are negligible, solutions of QHDs should recover the qualitative behavior of solutions of semi-classical models. However most of the established semi-classical approaches involve in a crucial way *two* particle types, namely electrons and holes. Therefore the analysis of unipolar QHDs has to be extended to bipolar QHDs.

Unipolar QHDs reduce in thermal equilibrium to generic unipolar constitutive laws [2]. The (scaled) bipolar extension of the constitutive laws reads

$$\begin{cases} n\nabla V + T_1 \nabla R_1(n) - \varepsilon^2 n \nabla \dfrac{\Delta\sqrt{n}}{\sqrt{n}} = \mathbf{0}, \\ -p\nabla V + T_2 \nabla R_2(p) - \xi\varepsilon^2 p \nabla \dfrac{\Delta\sqrt{p}}{\sqrt{p}} = \mathbf{0}, \\ -\lambda^2 \Delta V = n - p - C, \\ \displaystyle\int n = N, \quad \int p = P, \quad \int V = 0 \end{cases} \tag{1}$$

In (1) the functions n, p, V are unknown, where $n = n(x) \geq 0$ is the particle density of electrons (negatively charged) in the conduction band, $p = p(x) \geq 0$ is the particle density of holes (positively charged) in the valence band, $V = V(x)$ is the (negative) electrostatic potential and x ranges over Ω, a bounded domain in $\mathbb{R}^d$, where $d = 1, 2$ or $d = 3$. ε is the scaled Planck's constant and ξ is the ratio of the effective masses of electrons and holes. The device dependent parameters T_1, T_2 (electron and hole reference temperature, respectively) and the minimal Debye length λ are assumed to be constant. $R_{1,2} : [0, \infty) \to [0, \infty)$ are the respective pressure functions. (Typically, the pressure function is continuously differentiable and increasing.) C is the doping profile. It is assumed that the impurity atoms are fully ionized, i.e. $C = N_D - N_A$, where $N_D = N_D(x), N_A = N_A(x) \geq 0$ are the space densities of donator and acceptor atoms, respectively. N is the total number of electrons in the conductivity band and P is the total number of holes in the valence band. N, P are related to the densities of donator and acceptor atoms via

$$N = n_i + \int N_D, P = n_i + \int N_A,$$

where $n_i > 0$ is an intrinsic constant taking into account that the number of electrons in the conduction band (as well as the number of holes in the valence band) is not only determined by the doping but also by intrinsic thermal excitation processes. The relation between N, P and C implies total charge neutrality. Hence Poisson's equation has (at least for $n - p - C \in L^2(\Omega)$) exactly one solution V satisfying $\int V = 0$.

Since our main conclusions will not depend on the particular values of the positive parameters T_1, T_2, ξ we simply set

$$T_1 = T_2 = \xi = 1.$$

Equations (1) provide only a necessary condition for the thermal equilibrium state. Equations (1) do not take into account that the thermal equilibrium solution minimizes the system's total energy $\mathcal{E}_{\varepsilon\lambda}$. If (1) has more than one solution – this happens in some semi-classical settings [18] – the physically relevant solution of (1) is distinguished as a minimizer of $\mathcal{E}_{\varepsilon\lambda}$. One is therefore compelled to minimize

$$\begin{aligned}\mathcal{E}_{\varepsilon\lambda}(\nu,\pi) = \quad & \varepsilon^2 \int |\nabla\sqrt{\nu}|^2 + \varepsilon^2 \int |\nabla\sqrt{\pi}|^2 + \int G_1(\nu) + \int G_2(\pi) \\ & + \frac{\lambda^2}{2} \int |\nabla V[\nu-\pi-C]|^2\end{aligned}$$

in

$$\Gamma_\varepsilon \equiv \left\{(\nu,\pi) \in L^1(\Omega)\times L^1(\Omega) : \nu,\pi \geq 0, \sqrt{\nu}, \sqrt{\pi} \in H^1(\Omega), \int \nu = N, \int \pi = P\right\},$$

where $G_{1,2}$ is a primitive of $g_{1,2}(t) \equiv \frac{1}{t}\frac{dR_{1,2}(t)}{dt}$ and

$$-\lambda^2 \Delta V[\nu-\pi-C] = \nu-\pi-C, \int V[\nu-\pi-C] = 0.$$

A straightforward *formal* computation shows that the Euler-Lagrange equations of the functional $\mathcal{E}_{\varepsilon\lambda}$ are

$$\begin{cases} \varepsilon^2\Delta\sqrt{n} = \sqrt{n}(V + \ g_1(n) - \alpha_1) \\ \varepsilon^2\Delta\sqrt{p} = \sqrt{p}(-V + \ g_2(p) - \alpha_2) \\ -\lambda^2\Delta V = n - p - C \\ \int n = N, \quad \int p = P, \quad \int V = 0, \end{cases} \tag{2}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ are the Lagrange-multipliers associated with the constraints $\int \nu = N, \int \pi = P$. If the minimizer (n,p) of $\mathcal{E}_{\varepsilon\lambda}$ in Γ_ε satisfies (2) *and* $n, p > 0$ one gets (1) from (2) by simple algebraic manipulations and taking gradients.

The formulation of (1) as a variational problem provides a natural justification of the normalizing condition $\int V = 0$. For fixed $(\nu,\pi) \in \Gamma_\epsilon$ the potential $V[\nu-\pi-C]$ minimizes the electric field energy [9]

$$\mathcal{F}_{el}[W] = \frac{\lambda^2}{2} \int_\Omega |\nabla W|^2 - \int_\Omega (\nu-\pi-C)W,$$

where W ranges in a set Γ_W such that $\inf_{W\in\Gamma_W} \mathcal{F}_{el}[W] = \mathcal{F}_{el}[V[\nu-\pi-C]]$. To make $\inf_{W\in\Gamma_W} \mathcal{F}_{el}[W]$ as small as possible one has to choose Γ_W as large as possible:

$$\Gamma_W \equiv \left\{W \in H^1(\Omega) : \int W = 0\right\}.$$

Due to the assumed total charge neutrality $\mathcal{F}_{el}$ is well defined on Γ_W and attains its unique minimizer in Γ_W. The normalizing condition $\int_\Omega W = 0$ eliminates physically irrelevant additive constants. It is readily seen that $V[\nu-\pi-C]$ satisfies homogeneous Neumann conditions.

Remark 1. a) Replacing formally the terms $\sqrt{n}$ by Ψ_1, $\sqrt{p}$ by Ψ_2, n by $|\Psi_1|^2$ and p by $|\Psi_2|^2$ Eqs. (2) can be written as an scaled, stationary, nonlinear Schrödinger-Poisson system.

$$\begin{cases} -\varepsilon^2 \Delta\Psi_1 + V\Psi_1 + \Psi_1\ g_1\left(|\Psi_1|^2\right) = \alpha_1\Psi_1 \\ -\varepsilon^2 \Delta\Psi_2 - V\Psi_2 + \Psi_2\ g_2\left(|\Psi_2|^2\right) = \alpha_2\Psi_2 \\ \qquad -\lambda^2\Delta V = |\Psi_1|^2 - |\Psi_2|^2 - C \\ \displaystyle\int |\Psi_1|^2 = N,\ \int |\Psi_2|^2 = P,\ \int V = 0. \end{cases}$$

In this formulation α_1, α_2 are energy eigenvalues. The corresponding variational problem is to minimize the functional

$$\mathcal{E}^*_{\varepsilon\lambda}(\Psi_1, \Psi_2) = \varepsilon^2 \int |\nabla\Psi_1|^2 + \varepsilon^2 \int |\nabla\Psi_2|^2 + \int G_1(|\Psi_1|^2) + \int G_2(|\Psi_2|^2 + \frac{\lambda^2}{2}\int |\nabla V[|\Psi_1|^2 - |\Psi_2|^2 - C]|^2$$

in the set

$$\Gamma^*_\epsilon = \left\{(\Psi_1, \Psi_2) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{C}) : \int |\Psi_1|^2 = N,\ \int |\Psi_2|^2 = P\right\}.$$

It is not very difficult to check that the minimizer of $\mathcal{E}^*_{\varepsilon\lambda}$ equals up to a physically irrelevant constant phase factor $(\sqrt{n}, \sqrt{p})$.

b) The normalizing condition $\int V = 0$ implies that V satisfies homogeneous Neumann boundary conditions. This means that no external voltage is present. In voltage-driven applications however the thermal equilibrium state is influenced by external electric potentials. In this case Dirichlet (or mixed Dirichlet-Neumann) boundary data for V are prescribed. In [17] the analysis of a unipolar QHD with these boundary data is carried out. The extension to bipolar models of the investigations in [17] as well as the modifications of the results of Subsect. 2.2 and 2.3 are rather straightforward and can be left to the reader. Essential for the treatment of the electric energy are the estimates

$$\|V[f]\|_{L^\infty}, \|V[f]\|_{H^1} \le K\|f\|_{L^2},$$

where $\Delta V[f] = f$. Such estimates hold for reasonable Dirichlet (or mixed Dirichlet-Neumann) boundary data for V.

Equations (1) involve the dimensionless parameters ε, λ. Due to the presence of quantum effects ε is of not negligible order of magnitude for ultra small semiconductor devices. For "standard" devices however quantum effects play no major role. In these settings one has

$$\varepsilon^2 \ll \lambda^2 \ll 1,$$

and one is therefore compelled to study the consecutive limits $\varepsilon \to 0$ and $\lambda \to 0$.

The smallness of ε^2 is a high temperature effect as well as due to the smallness of Planck's constant. The terms involving ε^2 represent corrections to an otherwise classical model. Carrying out the limit $\varepsilon \to 0$ means to go back from quantum mechanics to classical physics.

It is the aim of this paper to analyze the variational problem of minimizing $\mathcal{E}_{\varepsilon\lambda}$ in Γ_ϵ, to give a rigorous derivation of the associated Euler-Lagrange equations (2), to prove that the minimizer of $\mathcal{E}_{\varepsilon\lambda}$ in Γ_ϵ solves (1), to carry out the semi-classical limit $\epsilon \to 0$ and to justify the employment of semi-classical boundary conditions whenever quantum effects are negligible and the scaled minimal Debye length is small.

All subsequent investigations are based on (mild) assumptions given at the beginning of Sect. 2. Subsections 2.1, 2.2, 2.3 are concerned with the statements of the results. The proofs are given in Subsects. 3.1, 3.2, 3.3. The core of the analysis of the semi-classical limit $\varepsilon \to 0$ (subsection 3.2) are properties of the functional $\mathcal{E}_{\circ\lambda}$ obtained from $\mathcal{E}_{\varepsilon\lambda}$ by setting formally $\varepsilon = 0$. This functional $\mathcal{E}_{\circ\lambda}$ possesses a unique minimizer in a set $\Gamma_\circ$ with $\Gamma_\circ \supset \Gamma_\epsilon, \Gamma_\circ \not\subset \Gamma_\epsilon$. Although the comparison functions of $\Gamma_\circ$ are less regular than those of Γ_ϵ, the minimizer of $\mathcal{E}_{\circ\lambda}$ in $\Gamma_\circ$ is actually an element of Γ_ϵ. This regularity result allows in connection with ε-independent estimates to pass to the limit $\varepsilon \to 0$ strongly in $H^1(\Omega)$. Subsection 3.3 is concerned with the justification of semi-classical boundary conditions for QHDs. The minimizer of $\mathcal{E}_{\circ\lambda}$ in $\Gamma_\circ$ does not recover the usual semi-classical boundary conditions [12, 14]. This is not to be expected because the semi-classical boundary conditions are derived from the zero space charge assumption $\lambda = 0$. Setting $\lambda = 0$ in $\mathcal{E}_{\circ\lambda}$ gives a functional $\mathcal{E}_{\circ\circ}$ to be minimized in a set $\Gamma_{\circ\circ} \subset \Gamma_\circ, \Gamma_{\circ\circ} \not\supset \Gamma_\circ$. $\mathcal{E}_{\circ\circ}$ possesses a unique minimizer (n_c, p_c) in $\Gamma_{\circ\circ}$ satisfying the semi-classical boundary conditions. However the investigation of $\lambda \to 0$ requires some effort. The main difficulty to pass to the limit $\lambda \to 0$ is the lack of regularity of (n_c, p_c). In fact the limiting densities n_c, p_c are in general *not* continuous while for all $\lambda > 0$ the minimizers of $\mathcal{E}_{\circ\lambda}$ belong to $C(\Omega)$. Hence compactness arguments based on embeddings of $H^1(\Omega)$ in some L^p-space (as used to perform the semi-classical limit) are not applicable. However a compactness-by-convexity principle (Lemma 3) allows to carry out the limit $\lambda \to 0$.

2. Statement of the Results

The subsequent investigations are based on the following assumptions:

$$
\textbf{(A)} \quad \begin{cases}
a)\ \Omega \subset I\!R^d, d = 1, 2 \text{ or } d = 3 \text{ is a bounded domain with } \partial\Omega \in C^{0,1}. \\
b) \text{ There exists a } K > 0 \text{ only depending on } \Omega \text{ such that} \\
\qquad \|V[f]\|_{L^\infty} \le K \|f\|_{L^2}. \\
c)\ C \in L^\infty(\Omega). \\
d)\ N - P = \int C\ ,\ N > \int C^+\ ,\ P > \int C^-. \\
e)\ g_{1,2} \in C(0,\infty) \cap L^1_{loc}([0,\infty)) \text{ is strictly increasing,} \\
\qquad \lim_{t\to\infty} g_{1,2}(t) = \infty \quad \text{and} \quad \underline{g_{1,2}} \equiv \lim_{t\to 0+} g_{1,2}(t) \in [-\infty, \infty).
\end{cases}
$$

Remark 2. a) Assumption **(A)**b) is essentially a requirement on the smoothness of $\partial\Omega$. For instance it is well known, see e.g. [5], that for $\partial\Omega \in C^\infty$ the estimate

$$\|V[f]\|_{H^2} \le K\|f\|_{L^2}$$

holds. This estimate implies in dimensions $d \leq 3$ assumption b), because due to $\partial\Omega \in C^{0,1}$ the embedding $H^2(\Omega) \to C_B(\Omega)$ is continuous [1].
b) The assumptions **(A)**e) are satisfied for functions $g_{1,2}$ deduced from the most frequently employed pressure functions of the form $R_{1,2}(t) = t^a, a \in [1, \infty)$.

2.1. Existence and uniqueness of a minimizer. The main result of this subsection is

Theorem 1. *Assume* **(A)**. *Then for all* $\varepsilon, \lambda > 0$ *the functional* $\mathcal{E}_{\varepsilon\lambda}$ *has a unique minimizer* (n, p) *in* Γ_ϵ *which solves the associated Euler-Lagrange equations (2) as well as (1). Furthermore,*

- n, p, V *satisfy homogeneous Neumann boundary conditions,*
- *for all* $t \in (0, 1)$*, the functions* $\sqrt{n}, \sqrt{p}, n, p, V$ *belong to* $C^{1,t}_{loc}(\Omega) \cap C_B(\Omega) \cap H^1(\Omega)$*,*
- n, p *are strictly positive in* Ω*, i.e.* $n(x), p(x) > 0$ *for all* $x \in \Omega$*,*
- *if* $\underline{g_1} = -\infty$*, then there exists a constant* $K > 1$ *such that* $1/K \leq n \leq K$*.*
- *if* $\underline{g_2} = -\infty$*, then there exists a constant* $K > 1$ *such that* $1/K \leq p \leq K$*.*

2.2. The semi-classical limit $\varepsilon \to 0$*.* Keeping $\lambda > 0$ fixed and given $\varepsilon \in (0, \infty)$ let $(n_\varepsilon, p_\varepsilon)$ be the unique minimizer of $\mathcal{E}_{\varepsilon\lambda}$ in Γ_ϵ and let $V_\varepsilon = V[n_\varepsilon - p_\varepsilon - C]$. By setting $\varepsilon = 0$ and formal manipulations Eqs. (1) become

$$\begin{cases} n_\circ \nabla V_\circ + \nabla R_1(n_\circ) = \mathbf{0}, \\ -p_\circ \nabla V_\circ + \nabla R_2(p_\circ) = \mathbf{0}, \\ -\lambda^2 \Delta V_\circ = n_\circ - p_\circ - C, \\ \int n_\circ = N, \quad \int p_\circ = P, \quad \int V_\circ = 0, \end{cases} \tag{3}$$

the energy functional $\mathcal{E}_{\varepsilon\lambda}$ becomes

$$\mathcal{E}_{\circ\lambda}(\nu, \pi) = \int G_1(\nu) + \int G_2(\pi) + \frac{\lambda^2}{2} \int |\nabla V[\nu - \pi - C]|^2,$$

i.e. $\sqrt{\nu}, \sqrt{\pi} \in H^1(\Omega)$ is not required anymore and $\mathcal{E}_{\circ\lambda}$ should be minimized in

$$\Gamma_\circ = \left\{ (\nu, \pi) \in L^1(\Omega) \times L^1(\Omega) : \ \nu, \pi \geq 0 \ , \ \int \nu = N, \ \int \pi = P \right\}.$$

The limit $\varepsilon = 0$ of the Euler-Lagrange equations (2) is less straightforward. In contrast to the quantum case the appearance of "vacuum-sets" (subsets of Ω where $n_\circ$ or $p_\circ$ vanishes) is possible. Hence by a simple canceling the differential operators in (2) some information is lost on vacuum-sets. A rigorous analysis shows that the Euler-Lagrange equations become in the limit $\varepsilon = 0$ variational *in*equalities

$$\begin{cases} 0 = V_\circ + g_1(n_\circ) - \alpha_{1\circ} & \text{if } n_\circ > 0, \\ 0 \leq V_\circ + g_1(n_\circ) - \alpha_{1\circ} & \text{if } n_\circ = 0, \\ 0 = -V_\circ + g_2(p_\circ) - \alpha_{2\circ} & \text{if } p_\circ > 0, \\ 0 \leq -V_\circ + g_2(p_\circ) - \alpha_{2\circ} & \text{if } p_\circ = 0, \\ -\lambda^2 \Delta V_\circ = n_\circ - p_\circ - C, & \\ \int n_\circ = N, \int p_\circ = P, \quad \int V_\circ = 0, & \end{cases} \tag{4}$$

where $\alpha_{1\circ}, \alpha_{2\circ} \in \mathbb{R}$. Some more information about $n_\circ, p_\circ$ is available by introducing the generalized inverse $h_{1,2}$ of $g_{1,2}$:

$$\begin{cases} h_{1,2} : \mathbb{R} \to [0, \infty) \\ \qquad t \ \mapsto \begin{cases} 0 & \text{if } t \leq \underline{g_{1,2}} \\ g_{1,2}^{-1}(t) & \text{if } t > \underline{g_{1,2}}. \end{cases} \end{cases}$$

Lemma 1. *Assume* **(A)** *and let* $\lambda > 0$. *Then the functional* $\mathcal{E}_{\circ\lambda}$ *has a unique minimizer* $(n_\circ, p_\circ)$ *in* $\Gamma_\circ$ *solving the associated variational inequalities (4). Furthermore,*

- *for all* $t \in (0, 1)$, *the electric potential* $V_\circ$ *belongs to* $C^{1,t}_{loc}(\Omega) \cap C_B(\Omega) \cap H^1(\Omega)$,
- $n_\circ, p_\circ \in C_B(\Omega)$, $n_\circ \leq \sup_\Omega C + P/\text{meas}(\Omega)$, $p_\circ \leq -\inf_\Omega C + N/\text{meas}(\Omega)$,
- *for all* $t \in (0, 1)$, $g_1(n_\circ) \in C^{1,t}_{loc}(\{n_\circ > 0\}) \cap H^1(\{n_\circ > 0\})$,
- *for all* $t \in (0, 1)$, $g_2(p_\circ) \in C^{1,t}_{loc}(\{p_\circ > 0\}) \cap H^1(\{p_\circ > 0\})$,
- *if* $\underline{g_1} = -\infty$, *then there exists a* $K > 1$ *such that* $1/K \leq n_\circ \leq K$,
- *if* $\underline{g_2} = -\infty$, *then there exists a* $K > 1$ *such that* $1/K \leq p_\circ \leq K$,
- $n_\circ = h_1(\alpha_{1\circ} - V_\circ), p_\circ = h_2(\alpha_{2\circ} + V_\circ)$ *and the electric potential* $V_\circ$ *solves the semi-linear elliptic equation*

$$-\lambda^2 \Delta V_\circ = h_1(\alpha_{1\circ} - V_\circ) - h_2(\alpha_{2\circ} + V_\circ) - C \quad , \quad \int V_\circ = 0.$$

The following convergence result of $(n_\varepsilon, p_\varepsilon, V_\varepsilon)$ to $(n_\circ, p_\circ, V_\circ)$ as $\varepsilon \to 0$ requires $\sqrt{n_\circ}, \sqrt{p_\circ} \in H^1(\Omega)$. Sufficient conditions for $\sqrt{n_\circ}, \sqrt{p_\circ} \in H^1(\Omega)$ can be most easily formulated in terms of $h_{1,2}$ and $g_{1,2}$ [19]:

Corollary 1. *Assume* **(A)** *and let* $\lambda > 0$. *Then* $\sqrt{n_\circ}, \sqrt{p_\circ}$ *belong to* $H^1(\Omega)$ *if* $g_j, h_j, j = 1, 2$ *satisfy one of the following conditions:*

a) $\sqrt{h_j} \in C^{0,1}_{loc}(\mathbb{R})$.
b) $\underline{g_j} = -\infty$ *and* $h_j \in C^{0,1}_{loc}(\mathbb{R})$.
c) $g_j \in C^1_{loc}(0, \infty)$, $\underline{g_j} = -\infty$ *and* $\frac{d\, g_j(t)}{d\, t} > 0$ *for* $t \in (0, \infty)$.

Remark 3. In applications $g_{1,2}(t)$ usually equals to $\log(t)$ for small t so b) applies.

Theorem 2. *Assume* **(A)** *and* $\sqrt{n_\circ}, \sqrt{p_\circ} \in H^1(\Omega)$. *Then*

- $V_\varepsilon \to V_\circ$ *strongly in* $H^1(\Omega)$ *and strongly in* $L^\infty(\Omega)$ *as* $\varepsilon \to 0$,
- $n_\varepsilon \to n_\circ$ *and* $p_\varepsilon \to p_\circ$ *strongly in* $L^r(\Omega), r \in [1, \infty)$ *and weak* in* $L^\infty(\Omega)$ *as* $\varepsilon \to 0$,
- $\sqrt{n_\varepsilon} \to \sqrt{n_\circ}$ *and* $\sqrt{p_\varepsilon} \to \sqrt{p_\circ}$ *strongly in* $H^1(\Omega)$ *as* $\varepsilon \to 0$,
- *if* $\underline{g_1} = -\infty$ *then there exists an* $\varepsilon^* > 0$ *and a* $K > 1$ *which is independent of* $\varepsilon \in (0, \varepsilon^*)$ *such that* $1/K \leq n_\varepsilon, n_\circ \leq K$ *and* $n_\varepsilon \to n_\circ$ *strongly in* $H^1(\Omega)$ *as* $\varepsilon \to 0$,
- *if* $\underline{g_2} = -\infty$ *then there exists an* $\varepsilon^* > 0$ *and a* $K > 1$ *which is independent of* $\varepsilon \in (0, \varepsilon^*)$ *such that* $1/K \leq p_\varepsilon, p_\circ \leq K$ *and* $p_\varepsilon \to p_\circ$ *strongly in* $H^1(\Omega)$ *as* $\varepsilon \to 0$.

2.3. The limit $\lambda \to 0$. Throughout this section let (n_λ, p_λ) be the unique minimizer of $\mathcal{E}_{\circ\lambda}$ in $\Gamma_\circ$ and let $V_\lambda = V[n_\lambda - p_\lambda - C]$. Equations (3) are known as semi-classical hydrodynamic semiconductor device model in thermal equilibrium. For this model the definition of the built-in potential is based on the zero space charge assumption which means that λ is set to zero in Poisson's equation [12].

To analyze the limit $\lambda \to 0$ set formally $\lambda = 0$ in (3):

$$\begin{cases} n_c \nabla V_c + \nabla R_1(n_c) = \mathbf{0}, \\ -p_c \nabla V_c + \nabla R_2(p_c) = \mathbf{0}, \\ 0 = n_c - p_c - C, \\ \int n_c = N, \quad \int p_c = P, \quad \int V_c = 0 \end{cases} \tag{5}$$

The functional $\mathcal{E}_{\circ\lambda}$ becomes formally

$$\mathcal{E}_{\circ\circ}(\nu, \pi) = \int G_1(\nu) + \int G_2(\pi)$$

to be minimized in

$$\Gamma_{\circ\circ} = \left\{ (\nu, \pi) \in L^1(\Omega) \times L^1(\Omega) : \nu, \pi \geq 0, \int \nu = N, \int \pi = P, \; \nu - \pi - C = 0 \right\}.$$

The associated Euler-Lagrange equations are

$$\begin{cases} \gamma = g_1(n_c) + g_2(p_c) \text{ if } n_c p_c > 0 \\ \gamma \leq g_1(n_c) + g_2(p_c) \text{ if } n_c p_c = 0, \end{cases} \tag{6}$$

where $\gamma \in \mathbb{R}$.

The solvability of this minimization problem is the content of

Lemma 2. *Assume* **(A)**. *Then $\mathcal{E}_{\circ\circ}$ has a unique minimizer (n_c, p_c) in $\Gamma_{\circ\circ}$ and*

- $n_c, p_c \in L^\infty(\Omega)$,
- n_c, p_c *satisfy (6),*
- $\text{meas}(\{n_c = 0\} \cap \{p_c = 0\}) = 0$,
- $n_c p_c$ *does not vanish identically on* Ω, *i.e.* $\int n_c p_c > 0$,
- $\{n_c = 0\} = \{p_c = C^-\}$ *and* $\{p_c = 0\} = \{n_c = C^+\}$,
- *if* $\underline{g_1} = -\infty$ *then there exists a* $K > 1$ *such that* $1/K \leq n_c \leq K$,
- *if* $\underline{g_1} = -\infty$ *then there exists a* $K > 1$ *such that* $1/K \leq p_c \leq K$,
- $g_1(n_c), g_2(p_c) \in L^\infty(\Omega)$,
- *defining*

$$\beta_1 \equiv \frac{1}{\text{meas}(\Omega)} \left(\gamma \, \text{meas}(\{n_c = 0\}) + \int_{\{n_c > 0\}} g_1(n_c) - \int_{\{n_c = 0\}} g_2(p_c) \right),$$

$\beta_2 \equiv \gamma - \beta_1$, *and setting*

$$V_c \equiv \begin{cases} \beta_1 - g_1(n_c) \text{ if } n_c > 0 \\ g_2(p_c) - \beta_2 \text{ if } n_c = 0 \end{cases}$$

the quintuple $(\beta_1, \beta_2, n_c, p_c, V_c)$ *is a solution of (5).*

The main problem when passing to the limit $\lambda \to 0$ is that the limit solution (n_c, p_c) is less regular than the minimizers $n_\lambda, p_\lambda \in H^1(\Omega)$. Hence there are no uniform $H^1(\Omega)$-estimates on $\sqrt{n_\lambda}, \sqrt{p_\lambda}$. Available estimates concern $\int G_1(n_\lambda)$ and $\int G_2(p_\lambda)$ so the subsequent Lemma and its Corollary are fundamental.

Lemma 3. *(Compactness-by-Convexity) Let $\Omega \subset \mathbb{R}^d,\ d \in \mathbb{N}$, be a bounded domain and let $G : [0, \infty) \to \mathbb{R}$ be strictly convex and continuous. For $n \in \mathbb{N}$ let $f_n, f \in L^1(\Omega)$ with $f_n, f \geq 0$ a.e. on Ω. Assume that $||f_n||_{L^1} \to ||f||_{L^1}$ as $n \to \infty$ and suppose that there exists a $\vartheta \in (0, 1)$ such that*

$$\int G(f) = \lim_{n\to\infty} \int G(f_n) = \lim_{n\to\infty} \int G(\vartheta f + (1 - \vartheta) f_n) \equiv L \in \mathbb{R}.$$

Then $f_n \to f$ strongly in $L^1(\Omega)$ as $n \to \infty$.

Corollary 2. *Let Ω and G as in Lemma 3. For $n \in \mathbb{N}$ let $f_n, f \in L^1(\Omega)$ with $f_n, f \geq 0$ a.e. on Ω and assume that $f_n \to f$ weakly in $L^1(\Omega)$ as well as*

$$\int G(f) = \lim_{n\to\infty} \int G(f_n) \equiv L < \infty,$$

as $n \to \infty$. Then $f_n \to f$ strongly in $L^1(\Omega)$ as $n \to \infty$.

Remark 4. a) In Lemma 3 it is assumed that ϑ is constant. By obvious modifications this assumption can be a bit weakened to require that there exists a sequence $(\vartheta_n)_{n\in\mathbb{N}}$ with $\vartheta_n \in (0, 1)$ and $\lim_{n\to\infty} \vartheta_n = \vartheta \in (0, 1)$ such that

$$\int G(f) = \lim_{n\to\infty} \int G(f_n) = \lim_{n\to\infty} \int G(\vartheta_n f + (1 - \vartheta_n) f_n) \equiv L \in \mathbb{R}.$$

Setting $\underline{\Theta} = \inf \{\vartheta_n : n \in \mathbb{N}\}$, $\overline{\Theta} = \sup \{\vartheta_n : n \in \mathbb{N}\}$, both in Lemma 3 and Corollary 2 the assumption $G \in C([0, \infty))$, G strictly convex, can be replaced by

$$\forall k > 1,\ \forall \vartheta \in [\underline{\Theta}, \overline{\Theta}]\ :\ \exists c > 0\ :$$

$$\forall u, v \in [\tfrac{1}{k}, k],\ u \leq v\ :\ \vartheta G(v) + (1 - \vartheta)G(v - u) - G(v - (1 - \vartheta)u) \geq C\ u.$$

b) There are many sufficient conditions known which allow to pass from weak L^1-convergence (or convergence in the sense of distributions) to strong L^1-convergence, see e.g. Brézis [4] and the references given there. In Lemma 3 however no convergence of the sequence (f_n) is assumed.

The main result of this subsection is

Theorem 3. *Assume* **(A)**. *Then*

- $n_\lambda \to n_c, p_\lambda \to p_c, V_\lambda \to V_c$ *strongly in* $L^r(\Omega), r \in [1, \infty)$ *and weak* in* $L^\infty(\Omega)$ *as* $\lambda \to 0$,
- $\|V_\lambda\|_{H^1} = o(1/\lambda)$ *as* $\lambda \to 0$,
- *if* $\underline{g_1} = -\infty$ *then there exists a* $\lambda^* > 0$ *and a constant* $K > 1$ *which is independent of* $\lambda \in (0, \lambda^*)$ *such that* $1/K \leq n_\lambda, n_c \leq K$,
- *if* $\underline{g_2} = -\infty$ *then there exists a* $\lambda^* > 0$ *and a constant* $K > 1$ *which is independent of* $\lambda \in (0, \lambda^*)$ *such that* $1/K \leq p_\lambda, p_c \leq K$.

Remark 5. a) Convergence in the $L^\infty(\Omega)$-norm can in general not be expected because $n_\lambda, p_\lambda, V_\lambda \in C(\Omega)$ for all $\lambda > 0$ while for not continuous C one has $n_c, p_c, V_c \notin C(\Omega)$.
b) If $g_{1,2}(t) = \log(t)$, see [12], then the functions n_c, p_c, V_c are given by

$$\begin{cases} n_c = (C/2) + \sqrt{(C/2)^2 + \delta^2} \\ p_c = -(C/2) + \sqrt{(C/2)^2 + \delta^2} \\ V_c = \beta_1 - \log\left((C/2) + \sqrt{(C/2)^2 + \delta^2}\,\right) = \\ \quad = \log\left(-(C/2) + \sqrt{(C/2)^2 + \delta^2}\,\right) - \beta_2 \end{cases} \tag{7}$$

where $\delta^2 = e^{\beta_1 + \beta_2}$ is uniquely determined by $\int \left((C/2) + \sqrt{(C/2)^2 + \delta^2}\right) = N$, or equivalently by $\int \left(-(C/2) + \sqrt{(C/2)^2 + \delta^2}\right) = P$. Equations (7) recover the classical expressions for the thermal equilibrium distributions of of n_c, p_c, V_c, see [12]. The parameter δ^2 (as well as β_1, β_2) is uniquely determined by N and P.

3. Proofs

3.1. Proofs of Subsection 2.1.

Proof of Theorem 1. The proof extends a similar argumentation of [17] to bipolar models. Some modifications are however necessary to handle the operator $V[f]$ whose corresponding operator in [17] is positive. For the sake of simplicity assume that $g_1 = g_2 = g$.

Step 1. For $i \in (1, \infty], t \in [0, \infty)$ let $g_i(t) \equiv \min\{it, \max\{-i, g(t)\}\}$ and $G_i(t) = \int_1^t g_i(\sigma)\, d\sigma$. We shall minimize

$$E_i^+(r, s) = \varepsilon^2 \int |\nabla r|^2 + \varepsilon^2 \int |\nabla s|^2 + \int G_i\left((r^+)^2\right) + \int G_i\left((s^+)^2\right) + \frac{\lambda^2}{2} \int \left|\nabla V[(r^+)^2 - (s^+)^2 - C]\right|^2$$

in

$$\Gamma^+ \equiv \left\{(r, s) \in H^1(\Omega) \times H^1(\Omega) : \int (r^+)^2 = N, \int (s^+)^2 = P\right\},$$

where r^+, s^+ are the positive parts of r, s. The aim of the subsequent analysis is to carry out the limit $i \to \infty$. Various i-independent positive constants are denoted by K.

Lemma 4. *Assume* **(A)**. *Then, for all* $i \in (1, \infty]$, *the functional* E_i^+ *possesses a unique minimizer* (R_i, S_i) *in* Γ^+ *and* $R_i, S_i \geq 0$.

Proof of Lemma 4. The existence of a minimizer $(R_i, S_i) \in \Gamma^+$ follows from standard theory, see e.g. [7, 11]. One easily checks that $(R_i^+, S_i^+) \in \Gamma^+$ (cutting maps $H^1(\Omega)$ into $H^1(\Omega)$, see e.g. [10]) and $E_i^+(R_i^+, S_i^+) \leq E_i^+(R_i, S_i)$, where equality holds iff $R_i^- = S_i^- = 0$. Therefore $R_i, S_i \geq 0$. Assume that (R_i, S_i) and (R^1, S^1) are distinct non-negative minimizers of E_i^+ in Γ^+. Then a straightforward calculation shows that for all $\vartheta \in (0,1)$ the pair $(R_\vartheta, S_\vartheta)$,

$$R_\vartheta \equiv \sqrt{\vartheta(R_i)^2 + (1-\vartheta)(R^1)^2} \geq 0,$$
$$S_\vartheta \equiv \sqrt{\vartheta(S_i)^2 + (1-\vartheta)(S^1)^2} \geq 0$$

belongs to Γ^+ with $E_i^+(R_\vartheta, S_\vartheta) < \vartheta E_i^+(R_i, S_i) + (1-\vartheta)E_i^+(R^1, S^1)$ which contradicts the assumed minimality of $E_i^+(R_i, S_i)$ and $E_i^+(R^1, S^1)$ in Γ^+. $\square$

Step 2. Similar to [17] it can be easily seen that for all $i \in (1, \infty)$ (the case $i = \infty$ has to be excluded here because of the possible lack of differentiability of $G_i(t)$ at $t = 0$) the pair (R_i, S_i) satisfies the respective Euler-Lagrange equations

$$\begin{cases} \varepsilon^2 \Delta R_i = R_i \left(V_i + g_i(R_i^2) - \alpha_{i1}\right), \\ \varepsilon^2 \Delta S_i = S_i \left(-V_i + g_i(S_i^2) - \alpha_{i2}\right), \\ -\lambda^2 \Delta V_i = R_i^2 - S_i^2 - C, \\ \int R_i^2 = N, \int S_i^2 = P, \int V_i = 0, \end{cases} \tag{8}$$

where it is taken into account that $R_i, S_i \geq 0$. The space of test functions of (8) is $H^1(\Omega)$. Hence R_i, S_i satisfy homogeneous Neumann boundary conditions.
Step 3. The limit $i \to \infty$ is prepared by deriving i-independent estimates on R_i, S_i. Here some modifications of the proof of [17] are necessary. Due to the fact that E_i^+ is uniformly (with respect to i) bounded from below and $\|R_i\|_{L^2}, \|S_i\|_{L^2} \leq K$, one gets $\|R_i\|_{H^1}, \|S_i\|_{H^1} \leq K$ which gives $\|R_i\|_{L^6}, \|S_i\|_{L^6} \leq K$. Due to assumption **(A)**b),c) it follows $\|V_i\|_{L^\infty} \leq K$. Combining these estimates we get $\int R_i^2 |V_i|, \int R_i |V_i| \leq K$ and we can proceed along the lines of Sect. 3.3 of [17] to establish the estimates $\int R_i^2 g_i\left(R_i^2\right), \int S_i^2 g_i\left(S_i^2\right) \leq K$ and $|\alpha_{i1}|, |\alpha_{i2}| \leq K$.

Lemma 5. *Assume* **(A)**. *Then* $0 \leq R_i, S_i \leq K$.

Proof of Lemma 5. Given $a > 1$ we use $\dfrac{[R_i - a]^+}{R_i}$ as test function in the first equation of (8). This gives

$$\varepsilon^2 a \int \frac{|\nabla [R_i - a]^+|^2}{R_i^2} + \int [R_i - a]^+ \left(V_i + g_i\left(R_i^2\right) - \alpha_{i1}\right) = 0$$

such that previous estimates imply

$$\left(K - g_i\left(a^2\right)\right) \int [R_i - a]^+ \geq \varepsilon a \int \frac{|\nabla [R_i - a]^+|^2}{R_i^2} \geq 0,$$

and due to $\lim_{t\to\infty} g_i(t) = \infty$ we have $R_i \leq K$. $S_i \leq K$ follows in analogy. $\square$

Step 4. The estimates derived so far allow to choose a sequence $(R_i, S_i)_{i\in\mathbb{N}}$ such that $R_i \to R, S_i \to S$ weakly in $H^1(\Omega)$ and weak* in $L^\infty(\Omega)$ as $i \to \infty$. It remains to show that the pair (R, S) is actually the minimizer of E_∞^+ in Γ^+ solving the corresponding

Euler-Lagrange equations. It can be seen as in [17] that (R, S) is the minimizer of E^+_∞ in Γ^+ . To pass to the limit in the Euler-Lagrange equations (8) we distinguish between two cases. If $\underline{g} = -\infty$ it follows from the maximum principle and previous estimates that $K \leq R_i, S_i$, see [17] for the details. If $\underline{g} \in I\!R$, then the map $t \mapsto \sqrt{t}g(t)$ is continuous on $[0, \infty)$. In both cases we can pass to the limit $i \to \infty$ in the weak formulation of (8) with arbitrary test functions in $H^1(\Omega)$. This settles the boundary conditions and the limiting equations. The regularity of R, S follows from the fact that $\Delta R, \Delta S$ are both in $L^\infty(\Omega)$. If $\underline{g} = -\infty$ then the lower estimate for R, S follows from $R_i, S_i \geq K$, if $\underline{g} \in I\!R$, the strict positivity of R, S follows from Harnack's inequality.

Identifying n with R^2 and p with S^2 settles the proof of Theorem 1. □

3.2. Proofs of Subsection 2.2.

Proof of Lemma 1. Lemma 1 modifies a result in [18] where mixed Dirichlet-Neumann boundary conditions are concerned. For the sake of a smoother presentation assume $g = g_1 = g_2$.

Step 1. For $i \in (1, \infty], t \in [0, \infty)$ let

$$g_i(t) = \begin{cases} t - (1/i) + g(1/i) \ , 0 \leq t \leq (1/i) \\ \qquad g(t) \qquad\quad , (1/i) < t < i \\ \quad t - i + g(i) \qquad , t \geq i \end{cases} ,$$

and set $G_i(t) \equiv \int_1^t g_i(\sigma)\, d\sigma$. g_i is strictly monotone increasing. Let

$$\begin{aligned} h_i : I\!R &\to [0, \infty) \\ t \ &\mapsto \begin{cases} 0 & \text{if } t \leq g(1/i) - (1/i) \\ g_i^{-1}(t) & \text{if } t > g(1/i) - (1/i) \end{cases} . \end{aligned}$$

It is readily seen that for $i \in (1, \infty)$ the function G_i is strictly convex and belongs to $C^1[0, \infty)$. Furthermore $G_i(t) = O(t^2)$ as $t \to \infty$. We shall minimize the functional

$$\mathcal{E}^i_{\circ\lambda}(\nu, \pi) \equiv \int G_1(\nu) + \int G_2(\pi) + \frac{\lambda^2}{2} \int |\nabla V[\nu - \pi - C]|^2$$

in the set

$$\Gamma_\circ = \left\{ (\nu, \pi) \in L^1(\Omega) \times L^1(\Omega) : \nu, \pi \geq 0, \int \nu = N, \int \pi = P \right\},$$

where the last term of $\mathcal{E}^i_{\circ\lambda}$ is set to $+\infty$ whenever the problem $-\lambda^2 \Delta V = \nu - \pi - C, \int V = 0$ admits no solution in $H^1(\Omega)$. (ν, π belong only to $L^1(\Omega)$.) It follows from standard theory that $\mathcal{E}^i_{\circ\lambda}$ possesses for all $i \in (1, \infty)$ a unique minimizer $(n_i, p_i) \in \Gamma_\circ$. The case $i = \infty$ has to be excluded here because of the possible lack of coercivity of the functional $\mathcal{E}_{\circ\lambda}$ in $L^1(\Omega)$ (or any other $L^r(\Omega)$ space as well). Furthermore the standard theory also provides that (n_i, p_i) solves the corresponding variational inequalities

$$\begin{cases} 0 \ = \ V_i + g_i(n_i) - \alpha_{i1} & \text{if } n_i > 0, \\ 0 \ \leq \ V_i + g_i(n_i) - \alpha_{i1} & \text{if } n_i = 0, \\ 0 \ = \ -V_i + g_i(p_i) - \alpha_{i2} & \text{if } p_i > 0, \\ 0 \ \leq \ -V_i + g_i(p_i) - \alpha_{i2} & \text{if } p_i = 0, \\ -\lambda^2 \Delta V_i \ = \ n_i - p_i - C, \\ \int n_i \ = N, \int p_i \ = \ P, \quad \int V_i \ = \ 0. \end{cases} \tag{9}$$

This system can be written as a single semi-linear equation in terms of the electrostatic potential V_i:

$$-\lambda^2 \Delta V_i = h_i\left(\alpha_{i1} - V_i\right) - h_i\left(\alpha_{i2} + V_i\right) - C \quad , \quad \int V_i = 0,$$

where

$$n_i = h_i\left(\alpha_{i1} - V_i\right) \quad , \quad p_i = h_i\left(\alpha_{i2} + V_i\right). \tag{10}$$

It follows by the strict monotonicity of h_i via the maximum principle that $V_i \in L^\infty(\Omega)$ with $\underline{V_i} \leq V \leq \overline{V_i}$, where $\underline{V_i}, \overline{V_i}$ satisfy the inequalities

$$h_i(\alpha_{i1} - \underline{V_i}) - h_i(\alpha_{i2} + \underline{V_i}) \leq \sup_\Omega C, h_i(\alpha_{i1} - \overline{V_i}) - h_i(\alpha_{i2} + \overline{V_i}) \geq \inf_\Omega C. \tag{11}$$

Furthermore, the normalizing conditions $\int h_i\left(\alpha_{i1} - V_i\right) = N$ and $\int h_i\left(\alpha_{i2} + V_i\right) = P$ imply

$$\begin{cases} h_i(\alpha_{i1} - \overline{V_i}) \leq N' \equiv N/\text{meas}(\Omega) \leq h_i(\alpha_{i1} - \underline{V_i}), \\ h_i(\alpha_{i2} + \underline{V_i}) \leq P' \equiv P/\text{meas}(\Omega) \leq h_i(\alpha_{i2} + \overline{V_i}). \end{cases} \tag{12}$$

Step 2. We carry out the limit $i \to \infty$ by deriving i-independent estimates. Various i-independent positive constants are denoted by K. It follows from (10),(11), (12) and the non negativity of n_i, p_i that $n_i \leq \sup_\Omega C + P'$, $p_i \leq -\inf_\Omega C + N'$. Hence $\|\Delta V_i\|_{L^\infty} \leq K$ which gives by **(A)**b) the estimate $\|V_i\|_{L^\infty} \leq K$. It follows from (9) that $\alpha_{i1}, \alpha_{i2} \leq K$. To establish lower estimates for α_{i1} assume that $\liminf_{i\to\infty} \alpha_{i1} = -\infty$. Passing if necessary to a subsequence we have due to (12),

$$N' \leq h(\alpha_{i1} + K).$$

Choose i large enough such that $\alpha_{i1} + K < g_i(N') = g(N')$. Then, if $\alpha_{i1} + K > g(1/i) - (1/i)$ the contradiction

$$N' \leq h_i(\alpha_{i1} + K) = g_i^{-1}(\alpha_{i1} + K) = g^{-1}(\alpha_{i1} + K), \text{ i.e. } g(N') \leq \alpha_{i1} + K$$

follows. If however $\alpha_{i1} + K \leq g(1/i) - (1/i)$, then $N' \leq h_i(\alpha_{i1} + K) = 0$, which is a contradiction. This proves that $\liminf_{i\to\infty} \alpha_{i1} \in I\!R$ and a similar argumentation for α_{i2} settles $|\alpha_{i1}|, |\alpha_{i2}| \leq K$.

Step 3: The estimates of Step 2 ensure that - possibly after passing to a subsequence - $\lim_{i\to\infty} \alpha_{i1} = \alpha_{1\circ}$, $\lim_{i\to\infty} \alpha_{i2} = \alpha_{2\circ}$ as well as

$$n_i \to n_\circ \quad , \quad p_i \to p_\circ \quad \text{weak* in } L^\infty(\Omega), \quad \text{as } i \to \infty.$$

Hence $V_i \to V_\circ$ weak* in $L^\infty(\Omega)$ and strongly in $H^1(\Omega)$, as $i \to \infty$, where $V_\circ = V[n_\circ - p_\circ - C]$. Passing if necessary to a subsequence gives

$$V_i \to V_\circ \quad \text{almost everywhere in } \Omega, \quad \text{as } i \to \infty.$$

We proceed by a case distinction.
a) If $\underline{g} = -\infty$ then by means of $g_i(n_i) \geq \alpha_{i1} - V_i \geq -K$, the estimate $n_i \geq K$ follows. Hence $g_i(n_i) = g(n_i)$ as well as $n_i = h(\alpha_{i1} - V_i)$ for all sufficiently large i and by continuity of h we have $n_i \to n_\circ = h(\alpha_{1\circ} - V_\circ)$ almost everywhere in Ω as $i \to \infty$ which gives via $\|n_i\|_{L^\infty} \leq K$,

$$n_i \to n_\circ = h(\alpha_{1\circ} - V) \quad \text{strongly in } L^r(\Omega)\ ,\ r \in [1, \infty), \quad \text{as } i \to \infty.$$

b) If $\underline{g} \in I\!\!R$ then $h_i \to h$ uniformly on compact subsets of $I\!\!R$ as $i \to \infty$ which gives via $\|\alpha_{i1} - V_i\|_{L^\infty} \leq K$ and convergence almost everywhere of $\alpha_{i1} - V_i$,

$$n_i \to n_\circ = h(\alpha_{1\circ} - V_\circ) \quad \text{strongly in } L^r(\Omega)\ ,\ r \in [1, \infty), \quad \text{as } i \to \infty.$$

In analogy we get in both cases

$$p_i \to p_\circ = h(\alpha_{2\circ} + V_\circ) \quad \text{strongly in } L^r(\Omega)\ ,\ r \in [1, \infty), \quad \text{as } i \to \infty.$$

Step 4. It remains to prove that $(n_\circ, p_\circ)$ is the minimizer of $\mathcal{E}_{\circ\lambda}$ in $\Gamma_\circ$. (By strict convexity of $\mathcal{E}_{\circ\lambda}$ there is at most one minimizer.) As shown in Step 3 the triple $(n_\circ, p_\circ, V_\circ)$ satisfies the variational inequalities (4). Now it is an easy exercise to verify for all $(\nu, \pi) \in \Gamma_\circ$,

$$\lim\inf_{\vartheta \to 0} \frac{\mathcal{E}_{\circ\lambda}(n_\circ + \vartheta(\nu - n_\circ), p_\circ + \vartheta(\pi - p_\circ)) - \mathcal{E}_{\circ\lambda}(n_\circ, p_\circ)}{\vartheta} \geq 0.$$

The convexity of $\mathcal{E}_{\circ\lambda}$ implies that $(n_\circ, p_\circ)$ is a minimizer of $\mathcal{E}_{\circ\lambda}$ in $\Gamma_\circ$. The regularity results stated in Lemma 1 follow from standard theory [8]. □

Proof of Theorem 2. The proof is divided into two steps. In the first step strong convergence of $n_\varepsilon, p_\varepsilon$ in $H^1(\Omega)$ as $\varepsilon \to 0$ is proven. Then uniform L^∞-estimates are established.
Step 1. Various ε-independent positive constants are denoted by K. We note that

$$\mathcal{E}_{\varepsilon\lambda}(n_\varepsilon, p_\varepsilon) - \mathcal{E}_{\circ\lambda}(n_\varepsilon, p_\varepsilon) = \varepsilon^2 \int |\nabla\sqrt{n_\varepsilon}|^2 + \varepsilon^2 \int |\nabla\sqrt{p_\varepsilon}|^2 \geq 0$$

for all $\varepsilon > 0$. Due to $\sqrt{n_\circ}, \sqrt{p_\circ} \in H^1(\Omega)$, for all $\varepsilon > 0$,

$$\mathcal{E}_{\varepsilon\lambda}(n_\varepsilon, p_\varepsilon) \leq \mathcal{E}_{\varepsilon\lambda}(n_\circ, p_\circ) = \varepsilon^2 \int |\nabla\sqrt{n_\circ}|^2 + \varepsilon^2 \int |\nabla\sqrt{p_\circ}|^2 + \mathcal{E}_{\circ\lambda}(n_\circ, p_\circ),$$

as well as $\mathcal{E}_{\circ\lambda}(n_\circ, p_\circ) \leq \mathcal{E}_{\circ\lambda}(n_\varepsilon, p_\varepsilon)$. Combining these estimates we get for all $\varepsilon \geq 0$,

$$\int |\nabla\sqrt{n_\varepsilon}|^2 + \int |\nabla\sqrt{p_\varepsilon}|^2 \leq \int |\nabla\sqrt{n_\circ}|^2 + \int |\nabla\sqrt{p_\circ}|^2,$$

and due to $\|\sqrt{n_\varepsilon}\|_{L^2} = N,\ \|\sqrt{p_\varepsilon}\|_{L^2} = P$ this implies $\|\sqrt{n_\varepsilon}\|_{H^1},\ \|\sqrt{p_\varepsilon}\|_{H^1} \leq K$. Passing to a subnet one has

$$\sqrt{n_\varepsilon} \to \sqrt{n_*}\ ,\ \sqrt{p_\varepsilon} \to \sqrt{p_*} \quad \text{weakly in } H^1(\Omega), \quad \text{as } \epsilon \to 0.$$

The compactness of the embedding $H^1(\Omega) \to L^6(\Omega)$ gives

$$n_\varepsilon \to n_*\ ,\ p_\varepsilon \to p_* \quad \text{strongly in } L^3(\Omega), \quad \text{as } \epsilon \to 0.$$

This convergence implies by (**A**)b) that

$$V_\varepsilon \to V_* \quad \text{strongly in } L^\infty(\Omega), H^1(\Omega), \quad \text{as } \epsilon \to 0,$$

where $V_* = V[n_* - p_* - C]$. To prove $n_* = n_\circ,\ p_* = p_\circ$ note that

$$\begin{aligned}\mathcal{E}_{\circ\lambda}(n_\circ, p_\circ) &\leq \liminf_{\varepsilon\to 0} \mathcal{E}_{\circ\lambda}(n_\varepsilon, p_\varepsilon) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_{\varepsilon\lambda}(n_\varepsilon, p_\varepsilon) \leq \limsup_{\varepsilon\to 0} \mathcal{E}_{\varepsilon\lambda}(n_\varepsilon, p_\varepsilon)\\ &\leq \liminf_{\varepsilon\to 0} \mathcal{E}_{\varepsilon\lambda}(n_\circ, p_\circ) = \mathcal{E}_{\circ\lambda}(n_\circ, p_\circ).\end{aligned}$$

Hence $\mathcal{E}_{\circ\lambda}(n_\circ, p_\circ) = \lim_{\varepsilon\to 0} \mathcal{E}_{\varepsilon\lambda}(n_\varepsilon, p_\varepsilon)$. On the other hand by the weakly sequential $L^2(\Omega)$-continuity of the functional $\mathcal{E}_{\circ\lambda}$,

$$\mathcal{E}_{\circ\lambda}(n_*, p_*) \leq \liminf_{\varepsilon\to 0} \mathcal{E}_{\circ\lambda}(n_\varepsilon, p_\varepsilon) \leq \lim_{\varepsilon\to 0} \mathcal{E}_{\varepsilon\lambda}(n_\varepsilon, p_\varepsilon) = \mathcal{E}_{\circ\lambda}(n_\circ, p_\circ),$$

so (n_*, p_*) is minimizer of $\mathcal{E}_{\circ\lambda}$ in $\Gamma_\circ$. (Obviously, $(n_*, p_*) \in \Gamma_\circ$.) By uniqueness of the minimizer of $\mathcal{E}_{\circ\lambda}$ in $\Gamma_\circ$ one has $n_* = n_\circ,\ p_* = p_\circ$.
Step 2. As shown in Step 1 we have $\|V_\varepsilon\|_{L^\infty} \leq K$. We observe by strong convergence of $\sqrt{n_\varepsilon}$ to $\sqrt{n_\circ}$ in $L^1(\Omega)$ and $\int \sqrt{n_\circ} > 0$ that there exists an $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ the estimate $\int \sqrt{n_\varepsilon} \geq K$ holds. For $\varepsilon < \varepsilon^*$ set $m_\varepsilon \equiv N / \int \sqrt{n_\varepsilon}$. We observe that $m_\varepsilon \leq K$ for all $\varepsilon < \varepsilon^*$. This allows us to proceed as in the proof of Lemma 4 in [17] to get the estimate $\left|\int n_\varepsilon g_1(n_\varepsilon)\right| \leq K$ for all $\varepsilon < \varepsilon^*$. Using $\sqrt{n_\varepsilon}$ as test function in the first equation of (2) we get

$$\alpha_{1\varepsilon} N = \varepsilon^2 \int |\nabla\sqrt{n_\varepsilon}|^2 + \int n_\varepsilon V_\varepsilon + \int n_\varepsilon g_1(n_\varepsilon),$$

and therefore by previous estimates $|\alpha_{1\varepsilon}| \leq K$ for all $\varepsilon < \varepsilon^*$. Using the maximum principle and the monotonicity of g_1 in the first equation of (2) it follows that $\overline{n_\varepsilon} \equiv \sup_\Omega n_\varepsilon > 0$ satisfies the inequality $g_1(\overline{n_\varepsilon}) \leq \alpha_{1\varepsilon} - \underline{V_\varepsilon}$, where $\underline{V_\varepsilon} \equiv \inf_\Omega V_\varepsilon$. Hence $n_\varepsilon \leq h_1(\alpha_{1\varepsilon} - \underline{V_\varepsilon}) \leq K$ for all $\varepsilon < \varepsilon^*$, because $\|V_\varepsilon\|_{L^\infty} \leq K$. This settles by non-negativity $\|n_\varepsilon\|_{L^\infty} \leq K$. If $\underline{g_1} = -\infty$ we can again apply the maximum principle in the first equation of (2) to get for $\underline{n_\varepsilon} \equiv \inf_\Omega n_\varepsilon > 0$ in analogy for all $\varepsilon < \varepsilon^*$ the estimate $n_\varepsilon \geq h_1(\alpha_{1\varepsilon} - \overline{V_\varepsilon}) \geq K$, where $\overline{V_\varepsilon} \equiv \sup_\Omega V_\varepsilon$. The L^∞-estimates concerning $p_\varepsilon, p_\circ$ follow in analogy. Finally the regularity results are consequences of standard theory [8]. $\square$

3.3. Proofs of Subsection 2.3.

Proof of Lemma 2:. We rewrite the minimization problem as follows. The functional

$$\mathcal{E}(\rho) \equiv \int G_1(C^+ + \rho) + \int G_2(C^- + \rho)$$

is to be minimized in

$$\Gamma \equiv \left\{\rho \in L^1(\Omega) : \rho \geq 0, \int \rho = N - \int C^+\right\}.$$

Due to **(A)**d) we have $\int C^+ < N$ and therefore $\Gamma \neq \{0\}$. As a strictly convex functional $\mathcal{E}$ possesses at most one minimizer. We introduce the function

$$\begin{aligned} g : \Omega \times [0,\infty) &\to [-\infty, +\infty) \\ (x,s) \quad &\mapsto g_1(C^+(x)+s) + g_2(C^-(x)+s). \end{aligned}$$

It is readily seen that for fixed $x \in \Omega$ the function $g(x,.)$ is strictly monotone increasing and continuous. Furthermore, for fixed $x \in \Omega$ we have $\lim_{s\to\infty} g(x,s) = \infty$. This allows to define for fixed $x \in \Omega$ the function

$$\begin{aligned} r(x,.) : I\!R &\to [0,+\infty) \\ \gamma &\mapsto \begin{cases} 0 & \text{if } \gamma \le g(x,0) \\ [g(x,.)]^{-1}(\gamma) & \text{if } \gamma > g(x,0) \end{cases}. \end{aligned}$$

For fixed $x \in \Omega$ the function $r(x,.)$ is continuous and monotone increasing. Given $\gamma \in I\!R$ we note that $r(x,\gamma) \in L^\infty(\Omega)$ as well as

$$\lim_{\gamma\to-\infty} \sup_{x\in\Omega} r(x,\gamma) = 0 \quad , \quad \lim_{\gamma\to\infty} \inf_{x\in\Omega} r(x,\gamma) = \infty,$$

which gives

$$\lim_{\gamma\to-\infty} \int r(x,\gamma) = 0 \quad , \quad \lim_{\gamma\to\infty} \int r(x,\gamma) = \infty.$$

Furthermore the map $\gamma \mapsto \int r(x,\gamma)$ is continuous. Hence there exists a $\gamma^* \in I\!R$ such that $\int r(x,\gamma^*) = N - \int C^+$. Set $r^*(x) = r(x,\gamma^*)$ and $n_c = g_1(C^+ + r^*)$, $p_c = g_2(C^- + r^*)$. Then

$$g_1(C^+ + r^*) + g_2(C^- + r^*) \ge \gamma^*,$$

where equality holds whenever $r^* > 0$. Since r^* does not vanish identically we have by strict monotonicity of g the estimate $\gamma^* > g_1(0) + g_2(0)$, which proves meas$(\{n_c = 0\} \cap \{p_c = 0\}) = 0$. If the function $n_c p_c$ vanishes identically on Ω then by $n_c = C^+ + r^*$ and $p_c = C^- + r^*$ the identity $(C^+ + r^*)(C^- + r^*) = 0$ will follow which gives due to $C^+C^- = 0$ the contradiction $r^*(|C| + r^*) = 0$, i.e. $r^* = 0$. We have

$$\lim\inf_{\vartheta\to 0} \frac{\mathcal{E}(r^* + \vartheta(\rho - r^*)) - \mathcal{E}(r^*)}{\vartheta} \ge 0$$

for all $\rho \in \Gamma$. Hence r^* is a minimizer of $\mathcal{E}$ in Γ. The remaining assertions of Lemma 2 follow by straightforward verifications. $\square$

Proof of Lemma 3. If $\|f\|_{L^1} = 0 = \lim_{n\to\infty} \|f_n\|_{L^1}$, then $f_n \to 0 = f$ strongly in $L^1(\Omega)$ and there is nothing to do. If $\|f\|_{L^1} \equiv K > 0$, suppose by contradiction that there exists an $\varepsilon \in (0, 8K)$ such that $\|f_n - f\|_{L^1} > \varepsilon$ for a subsequence n. Set $g_n \equiv f_n - f$. Then $f_n - f = g_n^+ - g_n^-$ and $f_n + g_n^- = f + g_n^+$. By non-negativity of f_n, f and $\int f_n \to \int f$ as $n \to \infty$ one gets $\lim_{n\to\infty} \int g_n^+ = \lim_{n\to\infty} \int g_n^-$. On the other hand $\varepsilon < \int |f_n - f| = \int g_n^+ + \int g_n^-$ for all $n \in I\!N$. Hence $\lim_{n\to\infty} \int g_n^- \ge \frac{\varepsilon}{2}$ and therefore $\int g_n^- \ge \frac{\varepsilon}{4}$ for a subsequence n. Choose $M_\varepsilon > 0$ such that $\int_{\{f > M_\varepsilon\}} f < \frac{1}{8}\varepsilon$ and put $\Omega_\varepsilon \equiv \{f \le M_\varepsilon\}$ which has nonzero measure:

$$0 < K - \frac{1}{8}\varepsilon < \int_\Omega f - \int_{\{f>M_\varepsilon\}} f = \int_{\Omega_\varepsilon} f.$$

On the other hand $0 \leq \int_{\{f>M_\varepsilon\}} g_n^- \leq \int_{\{f>M_\varepsilon\}} f < \frac{1}{8}\varepsilon$, because of $0 \leq f_n = f + g_n^+ - g_n^-$, and either $g_n^+ = 0$ which implies $f \geq g_n^-$, or $g_n^+ > 0$ which gives $0 = g_n^- \leq f$. But then

$$\frac{1}{8}\varepsilon < \int_\Omega g_n^- - \int_{\{f>M_\varepsilon\}} g_n^- = \int_{\Omega_\varepsilon} g_n^-.$$

Now set $\delta_\varepsilon \equiv \varepsilon/(16 \text{ meas}(\Omega_\varepsilon))$ and define $C_n \equiv \{g_n^- \geq \delta_\varepsilon\} \cap \Omega_\varepsilon$ which has non zero measure:

$$\frac{1}{8}\varepsilon < \int_{\Omega_\varepsilon} g_n^- = \int_{\Omega_\varepsilon \setminus C_n} g_n^- + \int_{C_n} g_n^- \leq \frac{\varepsilon}{16} + \int_{C_n} g_n^-.$$

Since $0 \leq f_n = f + g_n^+ - g_n^-$ and $g_n^+ \equiv 0$ on C_n, one has $0 < \delta_\varepsilon \leq g_n^- \leq f \leq M_\varepsilon$ almost everywhere on C_n. Set $R_\varepsilon \equiv \{(u,v) \in \mathbb{R}^2 : \delta_\varepsilon \leq u \leq v \leq M_\varepsilon\}$ and define

$$\begin{aligned} F: \quad & R_\varepsilon \to \mathbb{R} \\ & (u,v) \mapsto (\vartheta G(v) + (1-\vartheta)G(v-u) - G(v-(1-\vartheta)u))/u. \end{aligned}$$

Since G is strictly convex and $0 < \delta_\varepsilon \leq u \leq v \leq M_\varepsilon$, it follows that $F > 0$ on R_ε. Furthermore G is continuous and so is F on the compact set R_ε. Hence there exists a $C_\varepsilon > 0$ such that $F \geq C_\varepsilon \quad$ on R_ε. But then

$$\begin{aligned} &\int_\Omega \vartheta G(f) + (1-\vartheta)G(f_n) - G(\vartheta f + (1-\vartheta)f_n) \\ &\geq \int_{C_n} \vartheta G(f) + (1-\vartheta)G(f + g_n^+ - g_n^-) - G(\vartheta f + (1-\vartheta)f + (1-\vartheta)(g_n^+ - g_n^-)) \\ &= \int_{C_n} F(\vartheta, g_n^-, f)g_n^- \geq C_\varepsilon \int_{C_n} g_n^- \geq \varepsilon C_\varepsilon/16 > 0, \end{aligned}$$

because $g_n^+ \equiv 0$ on C_n. Hence we get the contradiction

$$\begin{aligned} L &= \lim_{n\to\infty} \int_\Omega G(\vartheta f + (1-\vartheta)f_n) \\ &\leq -\varepsilon C_\varepsilon/16 + \lim_{n\to\infty} \vartheta \int_\Omega G(f) + \lim_{n\to\infty} (1-\vartheta) \int_\Omega G(f_n) = -\varepsilon C_\varepsilon/16 + L. \quad \Box \end{aligned}$$

Proof of Corollary 2. Set $\vartheta_n \equiv \frac{1}{2}$. Then by convexity

$$\limsup_{n\to\infty} \int G\left(\frac{1}{2}f + \frac{1}{2}f_n\right) \leq \frac{1}{2}G(f) + \frac{1}{2}\lim_{n\to\infty} \int G(f_n) = L,$$

while by weak lower semi-continuity $L = \int G(f) \leq \liminf_{n\to\infty} \int G\left(\frac{1}{2}f + \frac{1}{2}f_n\right)$. Hence $L = \lim_{n\to\infty} \int G(\frac{1}{2}f + \frac{1}{2}f_n)$. On the other hand the weak L^1-convergence implies that $\|f_n\|_{L^1} = \int f_n \longrightarrow \int f = \|f\|_{L^1}$ as $n \to \infty$. The result follows from Lemma 3. $\Box$

Proof of Theorem 3. Various λ-independent positive constants are denoted by K. We note that

$$\mathcal{E}_{\circ\circ}(n_c,p_c) \le \mathcal{E}_{\circ\circ}(n_\lambda,p_\lambda) \le \mathcal{E}_{\circ\lambda}(n_\lambda,p_\lambda) \le \mathcal{E}_{\circ\lambda}(n_c,p_c),$$

which gives

$$\begin{aligned}\mathcal{E}_{\circ\circ}(n_c,p_c) &\le \limsup_{\lambda\to0}\mathcal{E}_{\circ\circ}(n_\lambda,p_\lambda) \liminf_{\lambda\to0}\mathcal{E}_{\circ\lambda}(n_\lambda,p_\lambda)\\ &\le \limsup_{\lambda\to0}\mathcal{E}_{\circ\lambda}(n_\lambda,p_\lambda) \le \limsup_{\lambda\to0}\mathcal{E}_{\circ\lambda}(n_c,p_c) = \mathcal{E}_{\circ\circ}(n_c,p_c),\end{aligned}$$

and therefore

$$\mathcal{E}_{\circ\circ}(n_c,p_c) = \lim_{\lambda\to0}\mathcal{E}_{\circ\lambda}(n_\lambda,p_\lambda),$$

as well as $\frac{\lambda^2}{2}\int|\nabla V_\lambda|^2 \le K$. As $\|n_\lambda\|_{L^\infty}, \|p_\lambda\|_{L^\infty} \le K$, see Lemma 1, one has by passing to a subnet $n_\lambda \to n_*,\ p_\lambda \to p_*$ weak* in $L^\infty(\Omega)$ as well as $\lambda V_\lambda \to W_\circ$ weakly in $H^1(\Omega)$. It follows for all test functions $\varphi \in H^1(\Omega)$,

$$0 = \lim_{\lambda\to0}\lambda^2\int\nabla V_\lambda\nabla\varphi = \lim_{\lambda\to0}\int(n_\lambda - p_\lambda - C)\varphi,$$

which implies $n_\lambda - p_\lambda - C \to 0$ weakly in $H^1(\Omega)$ as $\lambda \to 0$ and therefore $n_*, p_* \in \Gamma_{\circ\circ}$. Thanks to weak sequential lower semi-continuity in $L^2(\Omega)$ one has $\int G_1(n_*) \le \liminf_{\lambda\to0}\int G_1(n_\lambda)$, $\int G_2(p_*) \le \liminf_{\lambda\to0}\int G_2(p_\lambda)$, and therefore

$$\mathcal{E}_{\circ\circ}(n_*,p_*) \le \limsup_{\lambda\to0}\mathcal{E}_{\circ\circ}(n_\lambda,p_\lambda) \le \mathcal{E}_{\circ\circ}(n_c,p_c).$$

But (n_c,p_c) is the unique minimizer of $\mathcal{E}_{\circ\circ}$ in $\Gamma_{\circ\circ}$. Hence $n_* = n_c,\ p_* = p_c$, and as a consequence of $\mathcal{E}_{\circ\circ}(n_*,p_*) = \mathcal{E}_{\circ\circ}(n_c,p_c) = \lim_{\lambda\to0}\mathcal{E}_{\circ\lambda}(n_\lambda,p_\lambda)$ one gets

$$\lim_{\lambda\to0}\frac{\lambda^2}{2}\int|\nabla V_\lambda|^2 = 0,$$

as well as $\int G_1(n_\lambda) \to \int G_1(n_c)$, $\int G_2(p_\lambda) \to \int G_2(p_c)$, as $\lambda \to 0$. Now it follows from Corollary 2 that

$$n_\lambda \to n_c\ ,\ p_\lambda \to p_c \quad \text{strongly in } L^1(\Omega), \quad \text{as } \lambda \to 0,$$

and therefore $n_\lambda \to n_c,\ p_\lambda \to p_c$ a.e. on Ω for a subnet λ. Due to convergence almost everywhere and convergence weak* in $L^\infty(\Omega)$ we have

$$n_\lambda \to n_c\ ,\ p_\lambda \to p_c \quad \text{strongly in } L^r(\Omega)\ ,\ r\in[1,\infty), \quad \text{as } \lambda \to 0.$$

The uniform L^∞-estimates on n_λ, p_λ imply $g_1(n_\lambda), g_2(p_\lambda) \le K$. Hence by integration of (4) and $\int V_\lambda = 0$ we get upper estimates for the Lagrange multipliers: $\alpha_{1\lambda}, a_{2\lambda} \le K$. Due to convergence almost everywhere and due to the continuity of $g_{1,2}$ we have

$$g_1(n_\lambda) + g_2(p_\lambda) \to g_1(n_c) + g_2(p_c) \ge \gamma \quad \text{a.e. on } \Omega, \quad \text{as } \lambda \to 0.$$

Hence there exists a $\lambda^* \in (0,\infty)$ such that for all $\lambda \in (0,\lambda^*)$ the estimate $g_1(n_\lambda) + g_2(p_\lambda) \ge -K$ holds a.e. on Ω. Hence, if $\underline{g_1} = -\infty$, then there exists a $K > 1$ such that $1/K \le n_\lambda \le K$ for all $\lambda < \lambda^*$ and an equivalent estimate follows for p_λ whenever $\lim_{u\to0} g_2(u) = -\infty$. To establish lower estimates for $\alpha_{1\lambda}, a_{2\lambda}$, assume by contradiction that for a subnet $\lim_{\lambda\to 0}\alpha_{1\lambda} = -\infty$. Then on the set $\{n_\lambda > 0\}$ - whose measure is at least $N/(P' + \overline{C})$ - the equality $V_\lambda = \alpha_{1\lambda} - g_1(n_\lambda)$ holds which gives $V_\lambda \to -\infty$ uniformly on $\{n_\lambda > 0\}$. Hence due to $\int V_\lambda = 0$ we have $\int_{\{n_\lambda=0\}} V_\lambda \to \infty$ leading to $\lim_{\lambda\to 0}\overline{V_\lambda} = \infty$. We have due to (4) the inequality $\alpha_{2\lambda} \le -V_\lambda + g_2(p_\lambda)$, and therefore $\lim_{\lambda\to 0}\alpha_{2\lambda} = -\infty$ which settles in analogy $V_\lambda \to +\infty$ uniformly on $\{p_\lambda > 0\}$. Hence by continuity of $n_\lambda, p_\lambda, V_\lambda$ we have $\{n_\lambda > 0\}\cap\{p_\lambda > 0\} = \emptyset$, and therefore $n_\lambda p_\lambda = 0$ for all sufficiently small λ. Due to convergence almost everywhere it follows that $n_c p_c = 0$, which contradicts Lemma 2. This and an equivalent investigation of $\alpha_{2\lambda}$ settles $|\alpha_{1\lambda}|, |\alpha_{2\lambda}| \le K$, and we conclude from (4) that

$$\alpha_{1\lambda} - g_1(n_\lambda) \le V_\lambda \le g_2(p_\lambda) - \alpha_{2\lambda},$$

which gives $\|V_\lambda\|_{L^\infty} \le K$ for all $\lambda \le \lambda^*$ which settles by passing to a subnet

$$V_\lambda \to V_* \quad \text{weak* in } L^\infty(\Omega), \quad \text{as } \lambda \to 0$$

as well as $\int V_* = 0$. Passing to another subnet we have, due to the uniform estimates on $\alpha_{1\lambda}, a_{2\lambda}$, the existence of $\beta_{1*}, \beta_{2*} \in I\!R$ such that $\alpha_{1\lambda} \to \beta_{1*}$ and $\alpha_{2\lambda} \to \beta_{2*}$ as $\lambda \to 0$. Due to strong convergence in $L^1(\Omega)$ and due to Egorov's, Theorem there exists for each $\delta > 0$ an $\Omega_\delta \subset \Omega$ with $\text{meas}(\Omega \setminus \Omega_\delta) \le \delta$ such that

$$g_1(n_\lambda) - \alpha_{1\lambda} \to g_1(n_c) - \beta_{1*} \quad \text{uniformly on } \Omega_\delta, \quad \text{as } \lambda \to 0.$$

Hence

$$V_\lambda = g_1(n_\lambda) - \alpha_{1\lambda} \to V_* \quad \text{uniformly on } \Omega_\delta \cap \{n > 0\}, \quad \text{as } \lambda \to 0,$$

which settles $V_* = V_c + \beta_{1*} - \beta_1$ almost everywhere on $\{n_c > 0\}$. A similar argumentation gives $V_* = V_c - \beta_{2*} + \beta_2$ almost everywhere on $\{p_c > 0\}$. As shown in Lemma 2 the function $n_c p_c$ does not vanish identically on Ω, which settles $\beta_{1*} - \beta_1 = -\beta_{2*} + \beta_2$, and therefore $\beta_{2*} = \gamma - \beta_{1*}$. As $\{n_c = 0\} \subset \{p_c > 0\}$, see Lemma 2, we conclude via $\int V_* = \int V_c = 0$ that

$$0 = (\beta_{1*} - \beta_1)\,\text{meas}(\{n_c > 0\} + (\beta_2 - \beta_{2*})\,\text{meas}(\{n_c = 0\},$$

and therefore $\beta_{1*} = \beta_1$ and $\beta_{2*} = \beta_2$, and therefore $V_* = V_c$ on $\{n_c > 0\}\cup\{p_c > 0\} = \Omega$. Furthermore, as seen above, we have

$$V_\lambda \to V_c \quad \text{almost everywhere on } \{n_c > 0\} \cup \{p_c > 0\}, \quad \text{as } \lambda \to 0.$$

This settles in connection with weak* convergence in $L^\infty(\Omega)$,

$$V_\lambda \to V_c \quad \text{strongly in } L^r(\Omega)\ r \in [1,\infty), \quad \text{as } \lambda \to 0,$$

and finishes the proof of Theorem 3. $\square$

Acknowledgement. The author acknowledges support from EC-network, contract # ERBCHRXCT 930413 and support from the Deutsche Forschungsgemeinschaft, project MA 1662/2-1 entitled "Mathematische Analysis und Numerik von Quantenhydrodynamischen Modellen der Halbleiterphysik (QHD)". The author is indebted to the ENS, Departement de Mathématique, Cachan, and the Université Paul Sabatier, Toulouse, where parts of this research were carried out.

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Communicated by J.L. Lebowitz