Commun. Math. Phys. 187, 523 – 566 (1997)

Communications in
Mathematical Physics

Harmonic Crystal on the Wall: A Microscopic Approach

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Received: 28 August 1996 / Accepted: 23 December 1996

Abstract: A three dimensional Winterbottom type construction in the regime of partial wetting is derived in a scaling limit of a gas of microscopic Gaussian SOS droplets under the fixed volume constraint. The proof is based on a coarse graining of the random microscopic region "wetted" by the crystal, random walk representations of various quantities related to free massless fields and a stability analysis of the torsional rigidity problem.

1. Introduction

1.1. Macroscopic Winterbottom construction. The shape of a small crystal in the equilibrium with its vapour is assumed, disregarding gravitation, to minimize the anisotropic surface energy. The corresponding construction was obtained at the turn of the century by Wulff [36], and much work since then was devoted to a rigorous mathematical treatment and further generalizations of the underlying variational problem ([8, 15, 19, 33] to mention a few) on one hand, and to extensions of the construction to other physical situations of interest, e.g. to the case of a particle on a solid substrate [35], see [19] for its mathematical counterpart.

From the purely statistical mechanical point of view, though, the problem of a rigorous derivation of these optimal macroscopic shapes directly from the structure of local microscopic interactions and an analysis of the corresponding Gibbs measures in an appropriate scaling limit remained open and long pending until the late eighties, when, almost simultaneously, it was solved for several two dimensional models. The simplest one was the gas of generalized SOS droplets in 1 + 1 dimensions [9], which gave rise

⋆ Partially support by Swiss NF grant 20-41925.94
⋆⋆ Partially supported by Swiss NF grant 20-41925.94, by NSF grant DMS 9504513 and by the Commission of the EU under the contract CHRX-CT93-0411

to a Winterbottom-like shape in the scaling limit. A two dimensional Wulff construction was derived in the context of the supercritical phase of Bernoulli bond percolation in [1]. Finally, the $2D$ Ising model at very low temperatures was solved in the ground breaking monograph [12], which accomplished the program initiated in early works on phase separation [23, 24]. The approach of [12] was simplified in [27], using duality methods with mastery, the latter article being of a fundamental interest in its own right. Most of the results in [17 and 27] were formulated on the level of very precise local limit theorems. Their weaker integral versions were pushed all the way up to the critical temperature in [17, 18].

Further remarkable results on complete analyticity and phase separation were obtained in [29 and 30].

All the above results, however, are two dimensional, the higher dimensional problems being so far considered at least as much formidable as interesting. In this work we obtain a three dimensional droplet shape in the scaling limit of a 2 + 1 Gaussian counterpart of the model considered in [9]. To be more precise, we consider the free lattice field $(X_i)_{i \in S_N}$, in a square box $S_N \subset \mathbb{Z}^2$ of side length $2N$. This is the centered Gaussian random field whose covariance matrix is given by $(-\Delta)^{-1}$, where Δ is the discrete Laplacian on S_N with Dirichlet boundary conditions. We interpret this field as two dimensional random surface in the $2+1$-dimensional space $\mathbb{Z}^2 \times \mathbb{R}$. This random field is then equipped with three additional ingredients which govern the relation between this surface and the "wall" $\mathbb{Z}^2 \times \{0\}$:

1. An attractive surface to wall interaction,
2. A hard wall condition, meaning that the surface has to stay on the positive side of the wall,
3. A macroscopic restriction on the volume between the surface and the wall.

A formal description will be given in 1.3.

This is the microscopic model. The macroscopic picture is obtained by scaling the lengths by a factor $1/N$. The main aim of this paper is to prove a law of large numbers for this macroscopic shape.

Our limit macroscopic shape is reminiscent of the one provided by the general Winterbottom construction, and, because of the underlying Gaussian field, we call it a harmonic crystal. Compared, for example, with the Ising model or even with the supercritical Bernoulli bond percolation the model itself provides a rather poor approximation to the phenomena of phase separation. In this respect our intrusion into three dimensions, though, perhaps, being not without physical and mathematical appeal, is of a quite restricted nature, and many of the core problems for higher dimensional interfaces remain unsolved. An interesting aspect of our results and the method to prove them is that in three dimensions a nontrivial coarse graining procedure becomes imperative for the proof. This could be relevant for studying more complicated 3D models in the phase separation regime. Indeed, probably one of the most formidable problems on the way to a rigorous justification of a genuine Wulff construction directly from the microscopic local interactions, e.g. in the context of the 3D Ising model, is to define a natural scaling, which would substitute the 2D skeleton computations of [12 or 27].

A simplifying feature of the Gaussian interactions is the possibility to use random walk representations to compute many quantities exactly. This is lost if we substitute the quadratic interaction by a general convex one, whatever growth, smoothness and strict convexity properties are assumed. Furthermore, the geometry of the anharmonic crystal becomes more complicated as well – instead of a Poisson problem for the Laplacian one has to solve a semilinear elliptic equation. Besides the fact that the corresponding solution in the latter case cannot be explicitly computed, one also loses the scaling relation enjoyed by the torsional rigidity in the Gaussian case. Recently, however, there has been considerable progress in the study of anharmonic models with convex potentials [16, 26]. In particular, it was shown that such models admit a useful random walk representation, and, moreover, many computations for these random walks can be reduced to the corresponding computations for the simple random walk using the Brascamp-Lieb inequalities [8]. Based on these works, one can derive a droplet construction also in the non Gaussian setting. The corresponding results are under way.

Finally, we would like to remark that the concentration results here are obtained in the L_1 norm. It would certainly be possible to upgrade them to L_2 or even to L_p. The real issue, however, would be to obtain concentration in the L_∞ norm. Apart from being a stronger and geometrically nicer result, such an assertion would confirm a heuristic belief that an intrinsic statistical stability of shapes is better than an impartial stability of the isoperimetric problems involved (see a brief discussion about the corresponding problem for the 3D Wulff problem in [12]). One result of this type was obtained for the membrane problem in [4] and [32].

We conclude this subsection by giving a brief description of the Winterbottom construction (cf. [35, 19]):

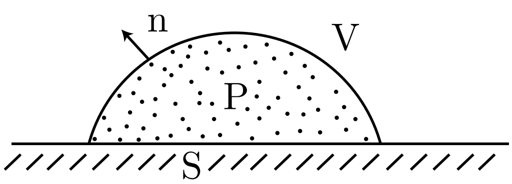

Fig. 1

Consider a small particle P placed on a solid foreign substrate S and in the equilibrium with its vapour V. (Fig. 1).

If the gravitation is disregarded, then the energy of the particle is given by

$$E(P) = \int_{PV} \tau_{PV}(n_s)\, ds + |PS|(\tau^f_{PS} - \tau^f_{VS}),$$

where $\tau_{PV} : S^2 \to \mathbb{R}_+$ is the anisotropic particle-vapour surface tension, and τ^f_{PS} and τ^f_{PV} are surface tensions of the particle-solid and vapour-solid flat interfaces respectively, and PV, PS are the corresponding interfaces. n_s is the normal vector to the particle-vapour interface at the point s and $|PS|$ denotes the area of the particle-solid interface, to which we will refer as to the "wetted" region.

The equilibrium shape of the particle is assumed to minimize the energy $E(P)$ at a fixed volume v. The solution to this variational problem was formulated in [35] and is, in fact, a version of the Wulff construction. The particle-vapour equilibrium Wulff shape K_{PV} centered at the origin, is defined by

$$K_{PV} = \bigcap_{n \in S^2} \{x \in \mathbb{R}^3 \ : \ (x, n) \le \tau_{PV}(n)\},$$

where $(\bullet, \bullet)$ is the scalar product in $\mathbb{R}^3$. K^S_{PV} is its intersection with the half space $H = \{\, x \in \mathbb{R}^3 \ : \ (x, e_3) \ge \tau^f_{VS} - \tau^f_{PS} \,\}$, where $e_3 = (0, 0, 1)$.

If $\tau^f_{VS} - \tau^f_{PS} \geq \tau^f_{PV}$, where $\tau^f_{PV} \triangleq \tau_{PV}(e_3)$, we are in the situation of complete wetting, i.e. the particle will spread out to form a thin layer separating V from S. Otherwise the equilibrium shape is obtained by an appropriate dilatation of K^S_{PV} in order to adjust its volume. In the latter case there are still three possibilities:

Complete drying: $\tau^f_{VS} - \tau^f_{PS} \leq -\tau^f_{PV}$,
Repelling wall: $-\tau^f_{PV} < \tau^f_{VS} - \tau^f_{PS} \leq 0$,
and
Attracting wall: $0 < \tau^f_{VS} - \tau^f_{PS} < \tau^f_{PV}$.

In the first case the shape K^S_{PV} coincides with the "free" Wulff shape K_{PV}. In the case of a repelling wall S the optimal shape K^S_{PV} is depicted in Fig. 2.

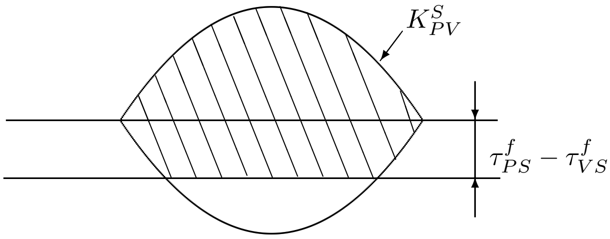

Fig. 2

Finally, in the case of an attracting wall, the optimal shape K^S_{PV} is presented in Fig. 3.

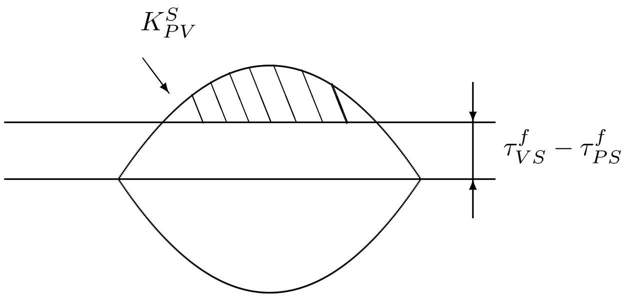

Fig. 3

Note that in the latter case the optimal PV interface can be represented as a function over the "wetted" region PS. Our model tacitly assumes the attractiveness of the wall:

$$\tau^f_{PV} > \Delta_f \triangleq \tau^f_{PV} - \tau^f_{VS} + \tau^f_{PS} > 0, \tag{1.1.1}$$

i.e. our results pertain to this case only. Strict positivity of Δ_f, which emerges in the macroscopic limit for the model we consider here is discussed in Subsect. 8.2.

We proceed by specifying the exact expression for the energy in the harmonic case.

1.2. The macroscopic description of the harmonic crystal. For the Gaussian model we consider here, the angle dependent surface tension is defined as follows (see [22] for general definitions and related properties) :

Let $\xi \in \mathbb{R}^2$, and consider the Gaussian random field over $S_N \triangleq NS(1) \cap \mathbb{Z}^2 \triangleq N[-1,1]^2 \cap \mathbb{Z}^2$ with the Hamiltonian

$$\mathcal{H}_{N,\xi}(x) = \frac{1}{2}\sum_{\langle k,l\rangle}(x_k - x_l)^2, \quad x \in \mathbb{R}^{S_N}$$

with ξ-tilted boundary conditions on ∂S_N:

$$x_k = (\xi, k),$$

for $k \in \partial S_N$, where $(\bullet,\bullet)$ is the scalar product in $\mathbb{R}^2$, and ∂S_N is the outer boundary of S_N, i.e. the set of points in $\mathbb{Z}^2 \setminus S_N$ which have a neighbor in S_N. The sum in the above definition of the Hamiltonian is over unordered pairs of nearest neighbor points in $S_N \cup \partial S_N$. Then the Gaussian surface tension σ_G in the direction of the unit vector $n \in \mathbb{S}^2$; $n = \frac{1}{\sqrt{1+\xi^2}}(\xi, 1)$, is defined by

$$\sigma_G(n) = -\frac{1}{\sqrt{1+|\xi|^2}} \lim_{N\to\infty} \frac{1}{N^2} \log \frac{Z_{N,\xi}}{Z_{N,0}},$$

where the partition function $Z_{N,\xi}$ is given by:

$$Z_{N,\xi} = \int_{\mathbb{R}^{S_N}} \mathrm{e}^{-\mathcal{H}_{N,\xi}(x)} dx.$$

In the Gaussian case one can easily compute σ_G,

$$\sigma_G(n) = \frac{1}{2}\frac{|\xi|^2}{\sqrt{1+|\xi|^2}}.$$

Consequently, the integrated Gaussian surface tension over an interface, parametrized by a function $u = u(x)$ is given by the integral

$$\frac{1}{2}\int |\nabla u|^2 dx.$$

With this computation in mind we proceed to define the macroscopic model in more precise terms. Let $H^{1,2}$ be the usual Sobolev space of functions with one square integrable weak derivative, and $H_0^{1,2}$ be the ones with compact support. If D is an open set, we denote by $H_0^{1,2}(D)$ the $H^{1,2}$ functions which have a compact support in D. A nonnegative function $u \in H_0^{1,2}$ is called a profile, and supp (u) (which is uniquely defined by u, up to Lebesgue measure 0) the wetted region. Then the energy of the particle with the profile u is given by

$$E(u) = \frac{1}{2}\int_D |\nabla u|^2\, dx + \Delta_f |\text{supp }(u)|, \tag{1.2.1}$$

where Δ_f is assumed to be positive, $\Delta_f > 0$. We define the *harmonic crystal* of volume v to be the $H_0^{1,2}$ solution to the variational problem

$$E(u) \mapsto \min, \quad \text{given} \quad V(u) \stackrel{\Delta}{=} \int u dx = v. \tag{1.2.2}$$

Note that if u solves (1.2.2), then so do all the shifts of u; $u(x + \bullet)$, for any $x \in \mathbb{R}^2$. Below we will see that actually all the solutions of (1.2.2) are shifts of some function $h_v(\bullet)$.

In order to determine h_v, we remark that the minimum in (1.2.2) equals

$$\inf_a \left(\inf_{\substack{D \text{ open} \\ |D|=a}} \inf_{\substack{u \in H_0^{1,2}(D) \\ V(u)=v}} \frac{1}{2} \int_D |\nabla u|^2 \, dx + \Delta_f a \right), \tag{1.2.3}$$

and thus, (1.2.2) is split into three minimization problems which we can all solve. Indeed,

$$\inf_{\substack{u \in H_0^{1,2}(D) \\ V(u)=v}} \int_D |\nabla u|^2 \, dx = \frac{v^2}{\chi(D)}, \tag{1.2.4}$$

where $\chi(D)$ is the torsional rigidity of D [28], given by

$$\chi(D) = \int_D u_D(x) \, dx, \tag{1.2.5}$$

where, u_D is the solution of the Poisson equation

$$\begin{aligned} \Delta u_D &= -1 \quad \text{in } D, \\ u_D|_{\partial D} &= 0. \end{aligned}$$

Moreover, the infimum in (1.2.4) is attained at $u_D^v \stackrel{\Delta}{=} (v/\chi(D))u_D$. Next, it is well known [28], that the maximal torsional rigidity over domains of a fixed area a is the one for the circle B_a,

$$\max_{|D|=a} \chi(D) = \chi(B_a) = \frac{a^2}{8\pi}. \tag{1.2.6}$$

Substituting this into (1.2.3) we find that the optimal area $\bar{a} = \bar{a}(v, \Delta_f)$ is found by minimizing the convex function

$$\frac{4\pi v^2}{a^2} + \Delta_f a$$

and the optimal profile h_v is given by

$$h_v(x) \;=\; \frac{2v}{\bar{a}} \left(1 - \frac{\pi |x|^2}{\bar{a}} \right) \vee 0, \tag{1.2.7}$$

Obviously, any shift of h_v is also optimal, and these are all the solutions of (1.2.2).

1.3. The microscopic model and the result. The first result on the droplet shape in the scaling limit of a Gaussian higher dimensional model is contained in a recent article [3], based on results on entropic repulsion for Gaussian lattice fields. In this model however, the wetted region played no role, more precisely, all the microscopic droplets under consideration were bound to wet the square $S(1) = [-1,1]^2$ (actually in [3] all the computations were done in the general case of $d > 1$ dimensional cubes).

In order to obtain the limit shapes given by (1.2.7) one has to consider a “gas of droplets” with varying microscopic wetted regions as, for example, the one studied in [9] in the case of 1 + 1-dimensional models. To set up notations, let $S_N = NS(1) \cap \mathbb{Z}^2$ and $\Omega_N = \mathbb{R}^{S_N}$. Our random field $X(\bullet) \in \Omega_N$ represents then the heights of droplets at lattice points $k \in S_N$ and we assume that all the mass of the particle is confined to the box S_N, i.e. $X(\bullet) \equiv 0$ on $\mathbb{Z}^2 \setminus S_N$. We define

$$\widehat{\mathbb{P}}_N(X(\bullet) \in dx) = \frac{1}{\widehat{Z}_N} \exp\left\{ -\frac{1}{2} \sum_{|k-l|=1} (x_k - x_l)^2 \right\} \times \prod_{k \in S_N} (\mathrm{e}^{-J} dx_k + \delta_0(dx_k)) \prod_{k \in \mathbb{Z}^2 \setminus S_N} \delta_0(dx_k), \qquad (1.3.1)$$

where $|\bullet|$ denotes the L^1-norm on $\mathbb{Z}^2$. The $\delta_0(dx_k)$ part is responsible for the attraction between the surface and the wall. If it is absent, we have the purely Gaussian model which had been the starting point of [3]. Our model becomes more transparent, if we rewrite it in a different form after opening all the brackets on the right-hand side of (1.3.1):

$$\widehat{\mathbb{P}}_N(X(\bullet) \in dx) = \frac{1}{\widehat{Z}_N} \sum_{A \subseteq S_N} e^{-J|A|} \exp\left\{ \frac{1}{2} \langle \Delta_d x, x \rangle \right\} \times \prod_{k \in A} dx_k \prod_{k \in \mathbb{Z}^2 \setminus A} \delta_0(dx_k),$$

where $|A|$ is the cardinality of A, Δ_d is the lattice Laplacian and $\langle \bullet, \bullet \rangle$ is the scalar product in $\mathbb{R}^{\mathbb{Z}^2}$. Indeed, the expression above gives the joint distribution of the microscopic wetted region $A \subseteq S_N$ and the microscopic droplet profiles $X(\bullet)$ over A. Our scaled microscopic profile $\xi_N \in L^1(\mathbb{R}^2)$ is given by

$$\xi_N(x) = \frac{1}{N} \sum_{k \in \mathbb{Z}^2} X(k) \mathbf{1}_{\{\|k - Nx\| < 1/2\}}, \qquad (1.3.2)$$

where $\|(x_1, x_2)\| \triangleq \max(|x_1|, |x_2|)$. Thus, ξ_N is just a scaled plaquette reconstruction of the microscopic particle profile over $\mathbb{R}^2$ from the field $X(\bullet)$. Note that

$$\mathrm{supp}(\xi_N) \subseteq S(1).$$

Finally, define the volume V_N of the gas of droplets as

$$V_N \triangleq \sum_{k \in S_N} X(k) = N^3 \int_{S(1)} \xi_N(x)\, dx.$$

We are going to prove a result about convergence to the optimal harmonic shape under the hard wall condition

$$X(\bullet) \in \Omega_+ \triangleq \{x(\bullet) \in \mathbb{R}^{\mathbb{Z}^2} : x(k) \geq 0 \ \forall\, k \in \mathbb{Z}^2\}.$$

Define:

$$\widehat{\mathbb{P}}_{N,+} = \widehat{\mathbb{P}}_N(\bullet \mid \Omega_+),$$

Theorem A. *For each $J \in \mathbb{R}$ define*

$$\Delta_{f,N} = \Delta_{f,N}(J) = J + \frac{1}{|S_N|} \log \frac{\widehat{Z}_{N,+}}{Z_N},$$

where $\widehat{Z}_{N,+} \triangleq \widehat{Z}_N \widehat{\mathbb{P}}_N(\Omega_+)$, $\widehat{Z}_N$ is the normalizing constant in (1.3.1) *and*

$$Z_N \triangleq \int_{\Omega_N} \exp(1/2\langle \Delta_N x, x\rangle)\, dx, \tag{1.3.3}$$

Δ_N being the lattice Laplacian with zero boundary conditions on S_N. Then the limit

$$\Delta_f = \lim_{N\to\infty} \Delta_{f,N} \tag{1.3.4}$$

exists and is a nondecreasing convex nonnegative function of J. Moreover,

$$\Delta_f(J) > 0 \tag{1.3.5}$$

for J large enough.

Assume now that J is such that (1.3.5) *holds, and let $v > 0$ be fixed (and small enough to enable $B_{\overline{a}} \subset S(1)$) and let $h_v(\bullet)$ be given by* (1.2.7) *with $a = \overline{a}(v, \Delta_f)$. Then there exists a sequence ν_N, $\lim_{N\to\infty} \nu_N = 0$, such that*

$$\widehat{\mathbb{P}}_{N,+}\left(\min_{x\in\mathbb{R}^2} \|h_v(x+\bullet) - \xi_N(\bullet)\|_{L^1(\mathbb{R}^2)} > \nu_N \,\middle|\, V_N \geq N^3 v \right) \leq \nu_N. \tag{1.3.6}$$

The theorem above implies a sharp concentration of microscopic profiles around the optimal harmonic crystal shape (1.2.7) under the measures (1.3.1) and the hard wall condition Ω_+.

Remark 1.3.1. In contrast with the situation in [3], the box S_N is playing here a very minor rôle and could be replaced by any region $NV \cap \mathbb{Z}^2$, $V \subset \mathbb{R}^2$, where V satisfies the condition that some translate of $B_{\overline{a}}$ is contained in V, and still the same limiting shape would appear. It is in fact true, although we don't need this, that a thermodynamic limit $\widehat{P}_\infty$ of $\widehat{P}_N$ as $N \to \infty$ exists and defines a random field on $\mathbb{Z}^2$ (see [11]). Of course, we cannot start with $\widehat{P}_\infty$ as then, due to translation invariance, the droplet does not "know" where to emerge, but it should be obvious that the only rôle of the finite box S_N is to keep the droplet confined.

In the next section we sketch the scheme of the proof and describe the principal results and estimates involved. Subsequent sections are devoted to rigorous proofs of these results: Sect. 3 deals with the coarse graining, Sect. 4 with the estimates on various partition functions, Sect. 5 with the stability of the related torsional rigidity problem, Sect. 6 with the concentration estimates over fixed wetted regions, Sect. 7 with the approximation of relevant macroscopic quantities by their mesoscopic counterparts, and, finally, Sect. 8 contains the proof of the main Theorem A.

Remark 1.3.2. In what follows we shall use two types of constants: fixed constants related to coarse graining or symbols like π, and two varying constants c and δ. The exact values of the latter constants are of no importance for us, except that they should belong to $(0, \infty)$. Moreover, they will always enter the estimates below in such a way, that if a certain estimate is true with (c, δ), it will also be true with (c', δ'), where $c' \geq c$ and $\delta' \leq \delta$. Thus, whenever we write c (δ) we actually mean the maximum (minimum) of the corresponding constant over all the estimates involved. Luckily, we get by with only a finite number of them, so, all the results remain valid under this convention.

2. Outline of the Proof

2.1. Strategy. For $A \subseteq S_N$ let Δ_A denote the lattice Laplacian with zero boundary conditions on A. Define

$$\mathbb{P}_A(X(\bullet) \in dx) = \frac{1}{Z_A} \exp\left(\frac{1}{2}\langle \Delta_A x, x\rangle\right) \prod_{k\in A} dx_k \prod_{k\in\mathbb{Z}^2\setminus A} \delta_0(dx_k), \tag{2.1.1}$$

where the partition function Z_A is given by

$$Z_A = \int_{\mathbb{R}^A} \exp\left(\frac{1}{2}\langle \Delta_A x, x\rangle\right) dx. \tag{2.1.2}$$

Then $\widehat{\mathbb{P}}_{N,+}$ is a convex combination;

$$\widehat{\mathbb{P}}_{N,+}(\bullet) = \sum_{A\subseteq S_N} e^{-J|A|} \frac{Z_A}{\widehat{Z}_{N,+}} \mathbb{P}_A(\bullet\ ;\ \Omega_+). \tag{2.1.3}$$

Under each $\mathbb{P}_A$ the volume V_N has a Gaussian distribution, and it is not difficult to compute that

$$\mathbb{P}_A(V_N \geq N^3 v) = \exp\left(-N^2 \frac{v^2}{2\chi_N(A)}(1 + o(1))\right), \tag{2.1.4}$$

where $\chi_N(A)$ is the approximate torsional rigidity of A, given by

$$\chi_N(A) = \frac{1}{N^2} \sum_{k\in A} u_{A,N}(k), \tag{2.1.5}$$

where $u_{A,N} \in \mathbb{R}^A$ solves

$$N^2 \Delta_A u_{A,N} \ = \ -1 \quad \text{in } A. \tag{2.1.6}$$

One might naively think that the main contribution to $\widehat{\mathbb{P}}_N(V_N \geq N^3 v)$ in the representation (2.1.3) comes from those A-s which are close in shape to some optimal microscopic wetted region A_{opt}, which minimizes

$$J|A| - \log Z_A + N^2 \frac{v^2}{\chi_N(A)}. \tag{2.1.7}$$

This, however, is not the case. It turns out that microscopically the wetted region under $\widehat{\mathbb{P}}_{N,+}$ is given by an almost optimal shape, which supports most of the droplet volume and a non negligible "noisy" shallow region. One already sees the problem, when remarking that the logarithm of the number of terms in the right-hand side of (2.1.3) is of the same order of magnitude as (2.1.7). In other words, on the microscopic scale the entropy competes with probabilistic weights. Note also that the macroscopic quantity Δ_f in (1.3.4) is not produced in (2.1.7).

As usual, in order to cancel the entropy and to generate all the relevant macroscopic quantities, one has to introduce an intermediate (mesoscopic) scale. We describe this scale in the next subsection, it enables us to restrict attention to mesoscopic wetted regions $B \subseteq S_N$, which are composed of blocks of the size N^b, $b \in (0,1)$. We then decompose $\widehat{\mathbb{P}}_{N,+}$ according to the value of the mesoscopic wetted region $\mathcal{M}$;

$$\widehat{\mathbb{P}}_{N,+}(\bullet) = \sum_{\substack{B \subseteq S_N \\ \text{mesoscopic}}} \widehat{\mathbb{P}}_{N,+}(\bullet \,;\, \mathcal{M} = B)$$

In its turn, due to our basic expansion (2.1.3) of $\widehat{\mathbb{P}}_{N,+}$ in microscopic wetted regions,

$$\widehat{\mathbb{P}}_{N,+}(\bullet \,;\, \mathcal{M} = B) = \sum_{A \subseteq S_N} e^{-J|A|} \frac{Z_A}{\widehat{Z}_{N,+}} \mathbb{P}_A(\bullet \,;\, \mathcal{M} = B \,;\, \Omega_+). \tag{2.1.8}$$

It happens that the essential contribution to the above sum comes only from those microscopic $A \subseteq S_N$, which cover B sufficiently well. A precise formulation of the latter statement is given in Subsect. 2.2. Thus, given a mesoscopic $B \subseteq S_N$, the sum on the right-hand side of (2.1.8) is effectively only over A-s, satisfying $B \subseteq A$. In order to make estimates on such a sum one should be able to decouple both Z_A and $\widehat{Z}_{N,+}$ over the boundary of B. The corresponding estimates on various partition functions are stated in Subsect. 2.3. Roughly all this leads to the representation of $\widehat{\mathbb{P}}_{N,+}$ as

$$\widehat{\mathbb{P}}_{N,+}(\bullet) \approx \sum_{\substack{B \subseteq S_N \\ \text{mesoscopic}}} e^{-J|B|} \frac{Z_B}{\widehat{Z}_{B,+}} \mathbb{P}_B(\bullet \,;\, \Omega_+), \tag{2.1.9}$$

where $\widehat{Z}_{B,+}$ is defined analogously to $\widehat{Z}_{N,+}$, with B playing the role of S_N. Note that there are only $O(e^{cN^{2(1-b)}})$ terms in the right-hand side of (2.1.10). Thus the mesoscopic wetted regions should concentrate around minimizers of

$$E^v_{N,f}(B) \triangleq J\frac{|B|}{N^2} + \frac{1}{N^2}\log\frac{\widehat{Z}_{B,+}}{Z_B} + \frac{1}{2}\frac{v^2}{\chi_N(B)}. \tag{2.1.10}$$

Provided that $E^v_{N,f}(B)$ is a good approximation to

$$\Delta_f \frac{|B|}{N^2} + \frac{1}{2}\frac{v^2}{\chi(B)}$$

and that the shape of the infimum in (1.2.3) is stable, one obtains a concentration of the mesoscopic wetted regions around the shifts of macroscopic optimal $B_{\bar{a}}$, and the problem is reduced to concentration estimates on

$$\mathbb{P}_{B,+}(\bullet \,|\, V_N \geq N^3 v)$$

for almost optimal B. The latter task can be accomplished by means of Gaussian computations, which however yield concentration around

$$u^v_{B,N} \triangleq \frac{v}{\chi_N(B)} u_{B,N}$$

instead of h_v. Thus, the last step should be to estimate the $L^1(\mathbb{R}^2)$ deviation of $u^v_{B,N}$ from h_v for almost optimal B.

To summarize we have the following tasks to perform:

1. Coarse graining, i.e. introduction of an intermediate mesoscopic scale and derivation of the corresponding control estimates,
2. Estimates on the partition functions Z_A and $\widehat{Z}_{B,+}$,
3. Stability estimates on the torsional rigidity,
4. Concentration estimates on $\mathbb{P}_{B,+}(\bullet|V_N \geq N^3 v)$ for B close to $B_{\bar{a}}$,
5. Approximation of χ by χ_N and of h_v by $u^v_{B,N}$.

We proceed by stating all the relevant results along these lines. The proofs are relegated to subsequent sections.

2.2. Coarse graining. Our coarse graining procedure is based on ideas introduced by Donsker and Varadhan in their treatment of the Wiener sausage. There are two scales involved:

1. The coarse graining scale $M = N^b$; $b \in (0, 1)$, and
2. The cutting level $H = N^\gamma$, $\gamma \in (0, 1)$.

The choice of b and γ is specified in Subsect. 3.1 below, but we always assume for notational convenience that $2M+1$ divides $2N+1$, but this is, of course, of no importance. Recall that $S(1) = [-1, 1]^2$ and $S_N = NS(1) \cap \mathbb{Z}^2$. We define the smoothing kernel Γ_M, supported in S_M, as follows:

Let $D_k = \{ i \in \mathbb{Z}^2 : \|i\| = k \} \stackrel{\Delta}{=} \partial S_k$ and set $D_k(i) = i + D_k$ denote the boundary of the k-square $S_k(i)$ centered at i, and let $\{\eta_n\}_{n\in\mathbb{N}}$ denote the simple random walk on $\mathbb{Z}^2$. For any $i \in \mathbb{Z}^2$ and $j \in D_k(i)$ define

$$\gamma_k(i, j) = \mathbb{P}^{RW}_i (\eta_{\tau_{D_k(i)}} = j), \tag{2.2.1}$$

where $\mathbb{P}^{RW}_i$ is the law of $\eta_\bullet$ starting at i, and $\tau_{D_k(i)}$ is the first hitting time of $D_k(i)$ by $\eta_\bullet$. Then we define

$$\Gamma_M(i, j) = \frac{2}{M(M+1)} \sum_{k=1}^{M} k\gamma_k(i, j). \tag{2.2.2}$$

Note that $\Gamma_M(i, \bullet)$ is a probability measure on $S_M(i)$. Also, γ_k and Γ_M are shift invariant; $\gamma_k(i, j) = \gamma_k(i - j)$ and $\Gamma_M(i, j) = \Gamma_M(i - j)$. The smoothened field $X_M \in \mathbb{R}^{\mathbb{Z}^2}$ is defined by

$$X_M(i) = \sum_j \Gamma_M(i - j)X(j), \quad i \in \mathbb{Z}^2. \tag{2.2.3}$$

Note that $X_M \equiv 0$ outside S_{N+M} under $\widehat{P}_N$. Define the coarse grained lattice

$$\mathbb{Z}^2_M = (2M + 1)\mathbb{Z}^2.$$

The next step is to split $\mathbb{Z}^2$ into the blocks of the size M:

$$\mathbb{Z}^2 = \bigcup_{i\in\mathbb{Z}^2_M} S_M(i).$$

Our coarse grained field $\widetilde{X}_M(\bullet)$ is defined to be constant on each of these blocks, namely

$$\widetilde{X}_M(j) = \sum_{i\in\mathbb{Z}^2_M} X_M(i)\mathbf{1}_{\{\|j-i\|\leq M\}}. \tag{2.2.4}$$

Obviously, the support of $\widetilde{X}_M$ is contained in S_N.

We shall call a finite union of M-blocks $S_M(i)$; $i \in \mathbb{Z}^2_M$, a *mesoscopic region*. Remark that the number of mesoscopic subsets of S_N equals

$$2^{(2N+1)^2/(2M+1)^2} \;\leq\; 2^{(N/M)^2} \;=\; \exp\bigl(N^{2(1-b)}\log 2\bigr).$$

Given our cutting level $H = N^\gamma$, we define the *mesoscopic wetted region*

$$\mathcal{M} \;=\; \mathcal{M}(\widetilde{X}_M) \;=\; \{\, i\in\mathbb{Z}^2 \,:\, \widetilde{X}_M \;\geq\; H\} \;\subseteq\; S_N.$$

The mesoscopic M-scale above leads to an "entropic reduction" in the representation of $\widehat{\mathbb{P}}_N$ given in (2.1.3). In fact, this representation has just too many summands to be immediately useful. The small heights cutoff given by H is necessary to get rid of the "shallow" part of the wetted region and, simultaneously, to produce the macroscopic quantity Δ_f. It happens that as far as questions of concentrations are considered, one can restrict attention to mesoscopic profiles:

Theorem 2.2.1. *For all N and all $A\subseteq S_N$,*

$$\mathbb{P}_A\left(\sum_{i\in\mathbb{Z}^2}|X(i)-\widetilde{X}_M(i)|\geq N^{3-\delta}\right)\leq \exp\left(-\frac{1}{c}N^{2+\delta}\right). \tag{2.2.5}$$

Consequently, if $\xi_{N,M}$ is the scaled plaquette reconstruction from $\widetilde{X}_M$, i.e.

$$\xi_{N,M}(x) = \frac{1}{N}\sum_{k\in\mathbb{Z}^2}\widetilde{X}_M(k)\mathbf{1}_{\{\|k-Nx\|<1/2\}},$$

then

$$\widehat{\mathbb{P}}_N(\|\xi_N-\xi_{N,M}\|_{L^1(\mathbb{R}^2)}\geq N^{-\delta})\leq \exp\left(-\frac{1}{c}N^{2+\delta}\right). \tag{2.2.6}$$

This super exponential estimate lies at the heart of our coarse graining approach.

We need also another super exponential estimate, which we call the volume filling lemma. It asserts that a mesoscopic wetted region B cannot be effectively produced by $\mathbb{P}_A$ in the decomposition (2.1.3) of $\widehat{\mathbb{P}}_{N,+}$, unless B is sufficiently well covered by A.

Lemma 2.2.2. *For all N and $A,B\subseteq S_N$,*

$$\mathbb{P}_A(\mathcal{M}(\widetilde{X}_M)=B)\leq \exp\left(-\frac{1}{c}N^{2+\delta}\right) \tag{2.2.7}$$

as soon as $|B\setminus A|\geq N^{2-\delta}$.

2.3. Estimates on partition functions. Let $A\subseteq S_N$. We use the following notations:

$$\partial A \;=\; \Bigl\{k\in A \,:\, \exists l\in\mathbb{Z}^2\setminus A \text{ with } \|k-l\|=1\Bigr\}, \tag{2.3.1}$$

$$A_t \;=\; \Bigl\{k\in A \,:\, \min_{l\in\mathbb{Z}^2\setminus A}\|l-k\| \;\geq\; t\Bigr\}. \tag{2.3.2}$$

Lemma 2.3.1. *a) There exist constants $q, r > 0$, such that for any $A \Subset \mathbb{Z}^2$,*

$$q|A| - r\max_t |\partial A_t| \ \le\ \log Z_A \ \le\ q|A|. \tag{2.3.3}$$

b) There exists a constant $\hat{q} > 0$, for any $t > 0$, a constant $c = c(t) > 0$, such that for any mesoscopic wetted region $B \Subset \mathbb{Z}^2$, satisfying $|B| \ge tN^2$,

$$\hat{q} - cN^{-b}\log N \ \le\ \frac{\log \widehat{Z}_B}{|B|} \ \le\ \hat{q}. \tag{2.3.4}$$

c) For any mesoscopic region $B \subseteq S_N$ and any sets $A \subseteq S_N \setminus B$ and $C \subseteq B$,

$$0 \ \le\ \frac{1}{N^2}\log\frac{Z_{A\vee C}}{Z_A Z_C} \ \le\ cN^{-b}\log N. \tag{2.3.5}$$

d) For any sets B, A and C as above

$$-cN^{-\delta} \ \le\ \frac{1}{N^2}\log\frac{\mathbb{P}_{A\vee C}\big(\,\Omega_+\big)}{\mathbb{P}_A\big(\,\Omega_+\big)\mathbb{P}_C\big(\,\Omega_+\big)} \ \le\ cN^{-\delta}. \tag{2.3.6}$$

Remark 2.3.2. The condition of $|B| \ge tN^2$ is not essential, but it simplifies the proof of (2.3.4). If $t > 0$ is chosen small enough, the restriction to mesoscopic regions satisfying $|B| \ge tN^2$ will be seen to be harmless, as regions where this fails have a negligible contribution in (2.1.9) for the events we are interested in. The notion of $t > 0$ being small enough is quantified in Subsect. 8.1 (See the proof of Proposition 8.1.2).

2.4. Stability results for the torsional rigidity. Let $D \subseteq \mathbb{R}^2$ be a bounded domain with a piecewise $\mathbf{C}^2$ boundary, u_D the solution of the Poisson equation

$$\begin{aligned} \Delta u_D &= -1 \text{ in } D,\\ u|_{\mathbb{R}^2\setminus D} &\equiv 0, \end{aligned}$$

and let $\text{lev}_\mu u_D$ be the level sets of u_D,

$$\text{lev}_\mu u_D = \{\, x \in D \,:\, u_D(x) \ge \mu \,\}.$$

Define $a(\mu) = |\text{lev}_\mu u_D|$. Then, $a(\mu)$ is a strictly decreasing continuous function and let $\mu = \mu(a) \,:\, [0, |D|] \to \mathbb{R}_+$, be the inverse of $a(\bullet)$. Finally, set

$$D_a = \text{lev}_{\mu(a)} u_D \quad \text{and} \quad l_D(a) = |\partial D_a|.$$

Note, that $|D_a| = a$ and $l_D(a) \ge |\partial B_a| \overset{\Delta}{=} s(a)$, where as in Subsect. 1.2 B_a is the circle of the area a.

Theorem 2.4.1.

$$\chi(D) \le \int_0^{|D|} \frac{a^2}{l_D(a)^2}\, da, \tag{2.4.1}$$

and

$$\max_{B:|B|=|D|} \chi(B) - \chi(D) = \chi(B_{|D|}) - \chi(D) \ge \frac{1}{4\pi}\int_0^{|D|} a\left(1 - \frac{s(a)}{l_D(a)}\right) da. \tag{2.4.2}$$

The right-hand side of (2.4.2) is a measure of deviation of D from the shape of the circle $B_{|D|}$, and the claim itself asserts that the torsional rigidity is stable with respect to this measure. From the representation

$$\chi(D) = \frac{1}{2}\int_D \mathbb{E}_x^{BM}\tau_D\,dx,$$

where τ_D is the first exit time from D of the two dimensional Brownian motion, it is clear the $\chi(\bullet)$ cannot be stable with respect to the Hausdorff distance, indeed adding a thin long hair does not change substantially both the area and the torsional rigidity. We shall see, however, that Theorem 2.4.1 above already implies stability with respect to another "natural" measure of deviation – the area of symmetric difference,

$$d_\Delta(D) = \inf_{x\in\mathbb{R}^d} |D\Delta(x + B_{|D|})|. \tag{2.4.3}$$

An even more important consequence for us here is the stability with respect to the inradius of D: Let D be simply connected and let $\varrho = \varrho(D)$ be the inradius (i.e. the radius of the largest inscribed circle) of D. Note that

$$\max_{|B|=|D|} \varrho(B) = \varrho(B_{|D|}) = \sqrt{\frac{|D|}{\pi}}.$$

Lemma 2.4.2.

$$\varrho(B_{|D|})^2 - \varrho(D)^2 = \frac{|D|}{\pi} - \varrho(D)^2 \le c\sqrt[3]{\chi(B_{|D|}) - \chi(D)}. \tag{2.4.4}$$

As a consequence we obtain the following result on the $L^1(\mathbb{R}^2)$ stability of the crystal shape: Let

$$E_f^v(D) = \frac{v^2}{2\chi(D)} + \Delta_f|D|$$

and $u_D^v = \sqrt{v/\chi(D)}u_D$, i.e. u_D^v is the shape of the minimal energy harmonic drop of the volume v bound to wet D. Then,

Lemma 2.4.3. *Let $D \subseteq S(1)$ with a piecewise smooth boundary ∂D, and let v be a fixed number; $v > 0$ Then,*

$$\inf_{x\in\mathbb{R}^2} \|u_D^v - h_v(\bullet + x)\|_{L^1(R)} \le c\sqrt[6]{E_f^v(D) - E_f^v(B_{\bar{a}})}, \tag{2.4.5}$$

where h_v is the harmonic crystal shape defined in (1.2.7).

Remark 2.4.4. The power $1/6$ in (2.4.5) is by no means optimal, but is adequate for our purpose. Note that in Lemma 2.4.3 the region D is not required to be simply connected.

2.5. Concentration of $\mathbb{P}_B(\bullet \,|\, V_N \ge N^3 v)$. Let $A \subseteq S_N$, and $u_{A,N}$ is the solution of the approximate Poisson equation (2.1.6) on A. Define $\overline{u}_{A,N}^v : S(1) \to \mathbb{R}$ as

$$\overline{u}_{A,N}^v(x) = \frac{v}{\chi_N(A)} \sum_{k\in S_N} u_{A,N}(k)\mathbf{1}_{\{\|k-Nx\|<1/2\}}. \tag{2.5.1}$$

Then the following estimate on the concentration of the scaled profile $\xi_N(\bullet)$, defined in (1.3.2), around $\overline{u}_{A,N}^v$ is valid:

Lemma 2.5.1. *For each $A \subseteq S_N$ and $a \in \mathbb{R}_+$,*

$$\mathbb{P}_A\big(\|\xi_N - \overline{u}^v_{A,N}\|_{L^1(\mathbb{R}^2)} \geq a \,|\, V_N \geq N^3 v\big) \leq \exp\bigg(-\frac{a^2}{c}N^2\bigg).$$

2.6. Approximation by discrete quantities. For a mesoscopic region $B \subseteq S_N$ we define $\bar{B} \subseteq S(1)$ by

$$\bar{B} = \frac{1}{N}\bigcup_{k\in B}\bigg(k + \frac{1}{2}S(1)\bigg). \tag{2.6.1}$$

Lemma 2.6.1.

$$|\chi_N(B) - \chi(\bar{B})| \leq cN^{-b}, \tag{2.6.2}$$

uniformly in N and mesoscopic $B \subseteq S_N$.

This, combined with the estimates on the partition functions stated in Lemma 2.3.1, leads to the following approximation result:

Lemma 2.6.2. *For any $t > 0$ there exist $\delta = \delta(t) > 0$ and $c = c(t) > 0$, such that any mesoscopic $B \subseteq S_N$, satisfying $|B| \geq tN^2$ and $\chi_N(B) \geq t$, also satisfies*

$$|E^v_f(\bar{B}) - E^v_{N,f}(B)| \leq cN^{-\delta}, \tag{2.6.3}$$

where $E_{N,f}(B)$ is given by (2.1.10)*, and, as before, $E^v_f(\bar{B}) \triangleq \Delta_f|\bar{B}| + \frac{1}{2}v^2/\chi(\bar{B})$.*

Finally, we get the following stability estimate for mesoscopic wetted regions B:

Lemma 2.6.3. *For any mesoscopic $B \subseteq S_N$,*

$$\inf_x \|\overline{u}^v_{B,N} - h_v(x + \bullet)\|_{L^1(\mathbb{R}^1)} \leq c\sqrt[6]{|\mathbb{E}^v_{N,f}(B) - E^v_f(B_{\overline{a})}|}, \tag{2.6.4}$$

where $\overline{u}^v_{B,N}$ is the approximated profile defined in (2.5.1)*, and $h_v, \overline{a}$ are respectively the optimal profile and the area of the optimal wetted region, which were defined in* Subsect. 1.2.

3. Coarse Graining

3.1. Scaling parameters. We start by fixing a small (say $b \leq 0.1$) but positive value of b. The exact condition on the "smallness" of b will be made precise at the end of Subsect. 3.2. We choose γ satisfying

$$\gamma + 2b < 1 < 1 + 2\delta < \gamma + 4b.$$

The first inequality enables to make the following reduction, which paves the way to the proof of Theorem A in Subsect. 8.1:

Let $i \in \mathbb{Z}^2_M$ and assume that $\widetilde{X}_M(i) \leq N^\gamma$, that is assume that $S_M(i) \cap \mathcal{M}(\widetilde{X}_M) = \emptyset$. Then, for each $\gamma' \in (\gamma + 2b, 1)$,

$$\{i \notin \mathcal{M}(\widetilde{X}_M)\} \cap \Omega_+ \implies \{X(k) \leq cN^{\gamma'}\ \forall\, k \in S_M(i)\}.$$

This implication is explained in Subsect. 8.1. Finally, the inequality $1 + 2\delta < \gamma + 4b$ is used to prove the volume filling estimate of Subsect. 3.3.

3.2. Proof of Theorem 2.2.1.

Proposition 3.2.1. *For all $A \subseteq S_N$ and for any $t \in \mathbb{R}_+$, the following estimate holds:*

$$\mathbb{P}_A\left(\sum_{k\in S_N} |X(k) - \widetilde{X}_M(k)| \geq tN^3\right) \leq \exp\left(cN^2 - \frac{1}{c}N^{2+8b}t^2\right). \tag{3.2.1}$$

Remark 3.2.2. The claim of Theorem 2.2.1 follows from the proposition above, if we take $t = N^{-\delta}$ for δ small enough.

Proof. We follow [13], to estimate

$$\mathbb{P}_A\left(\sum_{k\in S_N} |X(k) - \widetilde{X}_M(k)| \geq tN^3\right) \leq 2^{(2N+1)^2} \max_{\sigma\in\{-1,1\}^{S_N}} \mathbb{P}_A(Y_M(\sigma) \geq N^3 t), \tag{3.2.2}$$

where

$$Y_M(\sigma) \triangleq \sum_{k\in S_N} \sigma(k)(X(k) - \widetilde{X}_M(k)).$$

Now, $Y_M(\sigma)$ is zero mean Gaussian under each $\mathbb{P}_A$ with the variance $V(A,\sigma) \triangleq \mathbb{E}_A(Y_M(\sigma))^2$ given by

$$V(A,\sigma) = \sum_{k,k'\in\mathbb{Z}_M^2} \sum_{\substack{i\in S_M(k)\\ i'\in S_M(k')}} \sigma_i\sigma_{i'}\overline{\lambda}_A(i,i'), \tag{3.2.3}$$

where $\overline{\lambda}_A(\bullet,\bullet)$ is given by

$$\begin{aligned}\overline{\lambda}_A(i,i') \triangleq G_A(i,i') &- \sum_j \Gamma_M(k-j)G_A(j,i')\\ &- \sum_{j'} \Gamma_M(k'-j')G_A(i,j')\\ &+ \sum_{j,j'} \Gamma_M(k-j)\Gamma_M(k'-j')G_A(j,j').\end{aligned}$$

G_A above stays for the Green function of the simple random walk on A with zero boundary conditions, and the smoothing kernel Γ_M was defined in (2.2.2).

Pick now $a \in (0,1)$, $a > b$, (the exact value is to be specified later), and set $L = N^a$. In order to split the right-hand side in (3.2.3) define

$$\partial_L A = \{\, k \in A \,:\, \min_{l\in A^c} \|k - l\| \leq L \,\}.$$

We introduce, then, the following families of pairs of subindices $(k,k') \in \mathbb{Z}_M^2 \times \mathbb{Z}_M^2$:

$$\mathcal{A}_1 = \{\, (k,k') \,:\, \|k-k'\| \geq 2L+1 \text{ and } \{k,k'\}\cap(A\setminus\partial_L A) \neq \emptyset \,\},$$

$$\mathcal{A}_2 = \{\, (k,k') \,:\, \|k-k'\| \leq 2L \,\},$$

$$\mathcal{A}_3 = \{\, (k,k') \,:\, \{k,k'\} \subseteq A^c \cup \partial_L A \,\}.$$

Setting

$$\Psi(k,k') \triangleq \max_{\sigma\in\{-1,1\}^{S_N}} \left| \sum_{i\in S_M(k)} \sum_{i'\in S_M(k')} \sigma_i \sigma_i \overline{\lambda}_A(i,i') \right|,$$

we obtain:

$$V(A,\sigma) \le \sum_{\mathcal{A}_1} \Psi_A(k,k') + \sum_{\mathcal{A}_2} \Psi_A(k,k') + \sum_{\mathcal{A}_3} \Psi_A(k,k'). \tag{3.2.4}$$

Estimate on $\sum_{\mathcal{A}_1}$. Assume that $k \in A \setminus \partial_L A$ and $\|k-k'\| > 2L+1$. This means that $S_L(k) \subseteq A \setminus S_L(k')$. Consequently, for each $l \in S_L(k')$, the function

$$i \mapsto G_A(i,l)$$

is harmonic on $S_L(k)$. Therefore, for $i \in S_M(k)$ and $i' \in S_M(k')$,

$$\overline{\lambda}_A(i,i') = [G_A(i,i') - G_A(k,i')] + \sum_{j'\in S_N(k')} \Gamma_M(k'-j')[G_A(k,j') - G_A(i,j')].$$

Similarly, for each $l \in S_M(k')$, the function

$$i \mapsto G_A(i,l) - G_A(k,l)$$

is again harmonic on $S_L(k)$ and equals zero at $i=k$. Also, by Theorem 1.6.6 in [20],

$$G_A(i,l) \le G_{S_N}(0,0) \le c\log N. \tag{3.2.5}$$

Consequently, using Theorem 1.7.1. a) of [20], we infer that there exists a constant $c>0$, such that

$$\max_{(k,k')\in A_1} \max_{\substack{i\in S_M(k)\\ i'\in S_M(k')}} |\overline{\lambda}_A(i,i')| \le c\frac{M}{L}\log N.$$

Therefore,

$$\sum_{\mathcal{A}_1} \Psi_A(k,k') \le cN^4 N^{b-a}\log N \tag{3.2.6}$$

for some $c>0$.

Estimate on $\sum_{\mathcal{A}_2}$. From (3.2.5) and a trivial estimate

$$|\mathcal{A}_2| \le N^{2+2a-4b},$$

it follows that

$$\sum_{\mathcal{A}_2} \Psi_A(k,k') \le cN^{2+2a}\log N. \tag{3.2.7}$$

Estimate on $\sum_{\mathcal{A}_3}$. Note, first of all, that

$$\sum_{i\in S_M(k)} \sum_{i'\in S_M(k')} \sigma(i)\sigma(i')\overline{\lambda}_A(i,i') = \sum_{i,i'} G_A(i,i')\overline{\Gamma}_M(\sigma)(i,i'),$$

where

$$\bar{\Gamma}_M(\sigma)(i,i) = \sigma(i)\sigma(i') - \sigma(i)\Gamma_M(k,i')\sum_{l'\in S_M(k')}\sigma(l) - \sigma(i')\Gamma_M(k-i)\sum_{l\in S_M(k)}\sigma(l)$$
$$+ \Gamma_M(k-i)\Gamma_M(k'-i')\sum_{l\in S_M(k)}\sigma(l)\sum_{l'\in S_M(k')}\sigma(l').$$

However, due to our definition of the kernel $\Gamma_M(\bullet)$ and by the well known results on the exit distribution of the simple random walk (see e.g. Lemma 1.7.4 in [20]), there exists a constant $c > 1$, such that $\frac{1}{cM^2} \le \Gamma_M(l) \le \frac{c}{M^2}$ for all $l \in S_M$. As a result, all $|\bar{\Gamma}_M|$ are, independently of M and σ, bounded above by some finite constant $c > 0$, and

$$\sum_{\mathcal{A}_3} \Psi_A(k,k') \le c \sum_{i,j\in\partial_L A} G_A(i,j). \tag{3.2.8}$$

Thus, it remains to estimate the right-hand side of (3.2.8).

Let τ_A be the exit time of a simple random walk (RW) from A. Define the following sequence of stopping times:

$$\tau_1 = \inf\{\, n \ge 0 \,:\, \eta_n \in \partial_L A, n < \tau_A \,\},$$

and for $m \ge 1$,

$$\tau_{m+1} = \inf\{\, n \ge L^2 + \tau_m \,:\, \eta_n \in \partial_L A, n < \tau_A \,\}$$

(with the usual convention $\inf\{\emptyset\} = \infty$). Then, for each $i \in \partial_L A$,

$$\sum_{j\in\partial_L A} G_A(i,j) = \mathbb{E}_i^{RW} \sum_{n=0}^{\tau_A} \mathbf{1}_{\partial_L A}(\eta_n) \le L^2 \sum_{m=1}^{\infty} \mathbb{P}_i^{RW}(\tau_m < \infty).$$

Now, for each $i \in \partial_L A$,

$$\mathbb{P}_i^{RW}(\tau_A > L^2) \le 1 - \min_{\|k\|\le L} \mathbb{P}_0^{RW}(\tau_{\{k\}} \le L^2).$$

But,

$$\mathbb{P}_0^{RW}(\tau_{\{k\}} \le L^2) \ge \varrho/\log L \;\; \forall\, k : \|k\| \le L \tag{3.2.9}$$

for some $\varrho > 0$. In fact, we have by the last exit decomposition

$$\mathbb{P}_0^{RW}(\tau_{\{k\}} \le L^2) \ge \sum_{m=1}^{L^2} \mathbb{P}_0^{RW}(\xi_m = k)\,\mathbb{P}_0^{RW}(\tau_{\{0\}} > L^2).$$

From the standard local central limit theorem, we have $\mathbb{P}_0^{RW}(\xi_m = k) \ge c/L^2$, if $L^2/2 \le k \le L^2$, and m has the same parity as k. Therefore

$$\sum_{m=1}^{L^2} \mathbb{P}_0^{RW}(\xi_k = k) \ge c > 0.$$

On the other hand, it is known that $\mathbb{P}_0^{RW}(\tau_{\{0\}} > L^2) \sim \pi/2\log L$ (see [31], Sect. 16, Theorem E.1). Therefore, (3.2.9) follows. Consequently,

$$\mathbb{P}_i^{RW}(\tau_m < \infty) \le (1 - \varrho/\log L)^m,$$

and

$$\sum_{\mathcal{A}_3} \Psi_A(k,k') \leq cN^{2+2a} \log N. \tag{3.2.10}$$

Combining (3.2.6), (3.2.7) and (3.2.10), we obtain that there exists $c > 0$, such that $\forall\, A \subseteq S_N$,

$$\max_{\sigma\in\{-1,1\}^{S_N}} V(A,\sigma) \leq c(N^{2+2a} \log N \;+\; N^{4+b-a}). \tag{3.2.11}$$

Therefore with $b \in (0,1)$ fixed the optimal a to yield the best possible estimate along these lines is given by $2 + 2a = 4 + b - a < 4$. For our purposes, however, it would be sufficient to remark that for a choice of $b \in (0,1)$ small enough, (3.2.11) implies that

$$\max_{\sigma\in\{-1,1\}^{S_N}} V(A,\sigma) \leq cN^{4-8b}.$$

In a view of (3.2.2) this leads to the claim of the proposition.

3.3. Volume filling estimate. The volume filling lemma (Lemma 2.2.2) is a direct consequence of (3.2.1). Indeed if $|B \setminus A| \geq N^{2-\delta}$, then $\mathbb{P}_A$-a.s. on $\{\mathcal{M}(\widetilde{X}_M) = B\}$,

$$\sum_{k\in S_N} |X(k) - \widetilde{X}_M(k)| \geq N^{2-\delta+\gamma}.$$

Therefore, by virtue of (3.2.1), for any such A,

$$\mathbb{P}_A\big(\mathcal{M}(\widetilde{X}_M) = B\big) \leq \exp\left(cN^2 - \frac{1}{c}N^{2(\gamma+4b-\delta)}\right).$$

Thus, (2.2.7) follows, as soon as

$$\gamma \;+\; 4b \;-\; \delta \;>\; 1 \;+\; \delta,$$

which is one of the two scaling conditions, specified in Subsect. 3.1.

4. Estimates on Partition Functions

4.1. Random walk representation. Recall that for $A \Subset \mathbb{Z}^2$, we have defined Z_A as

$$Z_A = \int_{\mathbb{R}^A} \exp\left(\frac{1}{2}\langle \Delta_A x, x\rangle\right) \prod_{k\in A} dx_k,$$

where Δ_A is the discrete Dirichlet Laplacian on $A \subseteq \mathbb{Z}^2$;

$$\Delta_A = 4(P_A - I),$$

where I is the identity operator, and P_A the transition matrix of the simple random walk, killed at exiting from A. Let λ_k^A; $k = 1,\ldots,|A|$, be the eigenvalues of $-\Delta_A$, and μ_k^A; $k = 1,\ldots,|A|$, the corresponding eigenvalues of P_A;

$$\lambda_k^A = 4(1 - \mu_k^A).$$

Then,

$$\log Z_A = \frac{|A|}{2}\log 2\pi - \frac{1}{2}\sum_{k=1}^{|A|}\log\lambda_k^A = \frac{|A|}{2}\log\frac{\pi}{2} - \frac{1}{2}\sum_{k=1}^{|A|}\log(1-\mu_k^A). \tag{4.1.1}$$

We follow [14] in our approach to the right-hand side of (4.1.1): Note, first of all, that

$$\sum_{k=1}^{|A|}\log(1-\mu_k^A) = -\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{|A|}(\mu_k^A)^n = -\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}(P_A^n).$$

Furthermore, since obviously $\mathrm{Tr}(P_A^n) = 0$ for any odd n, we obtain:

$$\sum_{k=1}^{|A|}\log(1-\mu_k^A) = -\sum_{n=1}^{\infty}\frac{1}{2n}\mathrm{Tr}(P_A^{2n}). \tag{4.1.2}$$

To investigate the right-hand side of (4.1.2) we use the following random walk representation:

$$\mathrm{Tr}(P_A^{2n}) = \sum_{k\in A}\mathbb{P}_k^{RW}(\eta_{2n} = k; \tau_A > 2n),$$

and, consequently,

$$-\sum_{k=1}^{|A|}\log(1-\mu_k^A) = \sum_{k\in A}\sum_{n=1}^{\infty}\frac{1}{2n}\mathbb{P}_k^{RW}(\eta_{2n} = k, \tau_A > 2n), \tag{4.1.3}$$

where τ_A is the first exit time from A.

4.2. Estimates of Z_A. It is easy to see what the volume term of (4.1.3) is. Let $T_1, T_2, \dots$ be the hitting times of $0 \subset \mathbb{Z}^2$ by our random walk. Set

$$q = \mathbb{E}_0^{RW}\sum_{k=1}^{\infty}\frac{1}{T_k} = \sum_{n=1}^{\infty}\frac{1}{2n}\mathbb{P}_0^{RW}(\eta_{2n} = 0). \tag{4.2.1}$$

Then, as it follows from (4.1.3),

$$-\sum_{k=1}^{|A|}\log(1-\mu_k^A) \le q|A|. \tag{4.2.2}$$

Remark 4.2.1. Note that the right-hand side of (4.2.1) is summable, since by the local CLT (see e.g. [20], Theorem 1.2.1), $\mathbb{P}_0^{RW}(\eta_{2n} = 0) \sim 1/n$.

Proposition 4.2.2. *Define*

$$r = \sum_{n=1}^{\infty}\frac{1}{2n}\mathbb{E}_0^{RW}\left(\max_{1\le m\le 2n}|\eta_m|\right)\mathbf{1}_{\{\eta_{2n}=0\}}. \tag{4.2.3}$$

Then, $r < \infty$, and for each $A \Subset \mathbb{Z}^2$,

$$-\sum_{k=1}^{|A|}\log(1-\mu_k^A) \ge q|A| - r\max_t|\partial A_t|, \tag{4.2.4}$$

where A_t was defined in Subsect. 2.3.

Proof. Consider

$$g(n) \triangleq \mathbb{E}_0^{RW}\Big(\max_{1\le m\le 2n} |\eta_m| \mid \eta_{2n} = 0\Big).$$

We claim that

$$g(n) \le c\sqrt{n} \tag{4.2.5}$$

for some $c > 0$. Since, as mentioned before, $\mathbb{P}_0^{RW}(\eta_{2n} = 0) \sim 1/n$, the n^{th} term in the sum on the right-hand side of (4.2.3) is, thereby, of order $n^{-3/2}$, and $r < \infty$ as claimed. To show (4.2.5) set $Y_n = \max_{1\le m\le 2n} |\eta_m|$. Then, for each $K \in \mathbb{Z}_+$,

$$g(n) = \mathbb{E}_0^{RW}(Y_n \mid \eta_{2n} = 0) \le K + \sum_{k\ge K} \mathbb{P}_0^{RW}(Y_n \ge k \mid \eta_{2n} = 0). \tag{4.2.6}$$

However,

$$\mathbb{P}_0^{RW}(Y \ge k \mid \eta_{2n} = 0) = \frac{1}{\mathbb{P}_0^{RW}(\eta_{2n} = 0)} \mathbb{E}_0^{RW} \mathbf{1}_{\{\tau_{S_k}\le 2n\}} \mathbf{1}_{\{\eta_{2n}=0\}}, \tag{4.2.7}$$

where τ_{S_k} is the exit time from the box S_k. Decomposing the expectation in the right-hand side of (4.2.7) we obtain

$$\mathbb{E}_0^{RW} \mathbf{1}_{\{\tau_{S_k}\le 2n\}} \mathbf{1}_{\{\eta_{2n}=0\}} = \sum_{m=1}^{2n} \mathbb{E}_0^{RW} \mathbf{1}_{\{\tau_{S_k}=m\}} \mathbb{P}_{\eta_m}^{RW}(\eta_{2n-m} = 0).$$

By the local CLT, $\forall\, y \,:\, \|y\| = k \in [K, 2n]$,

$$\mathbb{P}_y^{RW}(\eta_{2n-m} = 0) \le (1 + o(1)) \frac{2}{\pi(2n-m)} \exp\Big(-\frac{k^2}{2n-m}\Big).$$

Therefore, optimizing in the right-hand side above and substituting the result into (4.2.7), we obtain that

$$\mathbb{P}_0^{RW}(Y \ge k \mid \eta_{2n} = 0) \le c\frac{n}{k^2} \mathbb{P}_0^{RW}(Y_n \ge k)$$

for some $c > 0$. Thus, choosing $K = \sqrt{n}$, we infer from (4.2.6) that

$$g(n) \le \sqrt{n} + c \sum_{k\ge\sqrt{n}} \mathbb{P}_0^{RW}(Y_n \ge k) \le \sqrt{n} + c\mathbb{E}_0^{RW} Y_n.$$

Finally, $\mathbb{E}_0^{RW} Y_n$ is of order $\sqrt{n}$ by the usual submartingale argument.

We turn now to the proof of (4.2.4): By (4.1.3),

$$-\sum_{k=1}^{|A|} \log(1 - \mu_k^A) = q|A| - \sum_{k\in A} \sum_{n=1}^{\infty} \frac{1}{2n} \mathbb{P}_k^{RW}(\eta_{2n} = k, \tau_A \le 2n). \tag{4.2.8}$$

Recall that for $t \in \mathbb{N}$ we defined $A_t = \{\, k \in A \,:\, \min_{l\in\mathbb{Z}^2\setminus A} \|l - k\| \ge t \,\}$. Now, if $k \in \partial A_t$,

$$\sum_{n=1}^{\infty} \frac{1}{2n} \mathbb{P}_k^{RW}(\eta_{2n} = k, \tau_A \le 2n) \le \sum_{n=1}^{\infty} \frac{1}{2n} \mathbb{P}_0^{RW}(\eta_{2n} = 0; \tau_{S_t} \le 2n).$$

Therefore,

$$\sum_{k\in A}\sum_{n=1}^{\infty}\frac{1}{2n}\mathbb{P}_k^{RW}(\eta_{2n}=k,\tau_A\le 2n)$$
$$\le \max_t|\partial A_t|\sum_{t=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2n}\mathbb{P}_0^{RW}(\eta_{2n}=0,\tau_{S_t}\le 2n). \tag{4.2.9}$$

However,

$$\sum_{t=1}^{\infty}\mathbb{P}_0^{RW}(\eta_{2n}=0;\tau_{S_t}\le 2n)=\mathbb{E}_0^{RW}Y_n\mathbf{1}_{\{\eta_{2n}=0\}},$$

where, as before, $Y_n=\max_{1\le m\le 2n}\|\eta_m\|$. Consequently, the right-hand side of (4.2.9) equals $r\max_t|\partial A_t|$, and, substituting the latter estimate into (4.2.8), we arrive at the claim of the proposition.

Our next task is to prove the decoupling estimate (2.3.5). Let A, B, C be as in the conditions of Lemma 2.3.1 c), i.e. $B\subseteq S_N$ is a mesoscopic region, $A\subseteq S_N\setminus B$ and $C\subseteq B$. Then it follows from (4.1.1) and the representation (4.1.3),

$$0\le\ 2\log\frac{Z_{A\vee C}}{Z_AZ_C}=\sum_{k\in A}\sum_{n=1}^{\infty}\frac{1}{2n}\mathbb{P}_k^{RW}(\eta_{2n}=k;\tau_A\le 2n<\tau_{A\vee C})$$
$$+\sum_{k\in C}\sum_{n=1}^{\infty}\frac{1}{2n}\mathbb{P}_k^{RW}(\eta_{2n}=k,\tau_C\le 2n<\tau_{A\vee C}). \tag{4.2.10}$$

The contribution to the right-hand side of (4.2.10) comes only from those random walks, which start in A (respectively C), and in $2n$ steps visit C (respectively A) without leaving $A\vee C$. Any such random walk has to cross $\partial_BC\stackrel{\Delta}{=}\partial B\cap C$. Consequently

$$2\log\frac{Z_{A\vee C}}{Z_AZ_C}\ \le\ \sum_{k\in\partial_BC}\sum_{n=1}^{\infty}\mathbb{P}_k^{RW}(\eta_{2n}=k,\tau_{A\vee C}>2n)$$
$$\le\ |\partial_BC|\max_{k\in\partial_BC}\sum_{n=1}^{\infty}\mathbb{P}_k^{RW}(\eta_{2n}=k;\tau_{S_N}>2n)$$
$$=\ |\partial_BC|\max_{k\in\partial_BC}G_{S_N}(k,k),$$

where G_{S_N} is the Green's function of the simple random walk, killed upon an exist from S_N. However, by Theorem 1.6.6 of [20], $\max_{k\in S_N}G_{S_N}(k,k)\le c\log N$ for some $c>0$. Also, by the very definition of the mesoscopic region, $|\partial_BC|\le|\partial B|\le cN^{2-b}$. Therefore,

$$\frac{1}{2}\log\frac{Z_{A\vee C}}{Z_AZ_C}\le c\log N|\partial_BC|\le cN^{2-b}\log N$$

as was asserted in (2.3.5).

4.3. Estimates on $\widehat{Z}_B$. The partition function $\widehat{Z}_\bullet$ obviously possesses the following superadditive property:

$$B\cap B'=\emptyset\Rightarrow\log\widehat{Z}_{B\vee B'}\ge\log\widehat{Z}_B+\log\widehat{Z}_{B'}. \tag{4.3.1}$$

Because of the results of the previous subsection, one can supplement (4.3.1) with an appropriate lower bound:

$$\widehat{Z}_{B\vee B'} = \sum_{A\subseteq B}\sum_{C\subseteq B'} e^{-J(|A|+|C|)} Z_{A\vee C} \le \exp(c|\partial B'|\log N)\widehat{Z}_B\widehat{Z}_C, \tag{4.3.2}$$

where,

$$|\partial_B B'| = |\{\, k\in B' \,:\, \exists\, l\in B \text{ with } \|k-l\| = 1\,\}| \le cN^{2-b}.$$

For each $k \in \mathbb{Z}_+$, define $Q(k) = \frac{1}{|S_k|}\log\widehat{Z}_{S_k}$. Then, $\{Q(2^m k)\}_{m=1}^{\infty}$ is an increasing sequence, and, by (4.3.2),

$$Q(2^m k) \le Q(2^{m+1}k) \le Q(2^m k) + \frac{c}{2^m k}\log 2^m k.$$

Thus, if we define,

$$\hat{q} = \lim_{m\to\infty} Q(2^m k), \tag{4.3.3}$$

then, for $k < N$,

$$Q(k) \le \hat{q} \le c\frac{\log N}{k} + Q(k).$$

Of course, we have to justify the tacit assumption that $\hat{q}$ in (4.3.3) doesn't depend on the base k chosen, but this again follows from (4.3.2), since for all $k, l \in \mathbb{Z}_+$,

$$Q(k) \le Q(kl) \le Q(k) + c\frac{\log k}{k},$$

and, in a completely symmetric way,

$$Q(l) \le Q(kl) \le Q(l) + c\frac{\log l}{l}.$$

In particular, for $M = N^b$,

$$Q(M) \le \hat{q} \le Q(M) + cN^{-b}\log N. \tag{4.3.4}$$

Therefore, for a mesoscopic region $B = \bigvee_{k\in\mathcal{B}\subseteq\mathbb{Z}^2_M} S_M(k)$,

$$\sum_{k\in\mathcal{B}}\log\widehat{Z}_{S_M(k)} \le \log\widehat{Z}_B \le \sum_{k\in\mathcal{B}}\log\widehat{Z}_{S_M(k)} + cN^{2(1-b)}\log N.$$

Since $\log\widehat{Z}_{S_M(k)} = (2N^b+1)^2 Q(M)$, $|B| = (2N^b+1)^2|\mathcal{B}|$, and also due to our assumption $|B| \ge tN^2$, we conclude that for some $c = c(t) > 0$,

$$Q(M) \le \frac{1}{|B|}\log\widehat{Z}_B \le Q(M) + cN^{-b}\log N,$$

and (2.3.4) now follows from (4.3.4).

4.4. The hard wall condition. If $D \subset S_N$, we denote by $\partial^+ D$ the outer boundary of D, i.e. the points which are not in D but have a neighbor point in D. If $x \in (\mathbb{R}^+)^{\partial^+ D}$, we write $\mathbb{P}_{D,x}$ for the law of the free field on $\mathbb{R}^D$ with boundary condition x on $\partial^+ D$. With this notation, we have $\mathbb{P}_{D,0} = \mathbb{P}_D$, where the latter is restricted to configurations on D. We will need some properties of FKG type.

Lemma 4.4.1. *a) For all* $x \in (\mathbb{R}^+)^{\partial^+ D}$*, we have*

$$\mathbb{P}_{D,x}(\Omega_+) \geq \mathbb{P}_D(\Omega_+).$$

b) Let $D_1 \subset D_2$ *and* $f : \mathbb{R}^{D_1} \to \mathbb{R}$ *be bounded, measurable and increasing in all arguments. Then*

$$\mathbb{E}_{D_1}(f(X) \,|\, \Omega_+) \leq \mathbb{E}_{D_2}(f(X) \,|\, \Omega_+).$$

Proof. a) Let h_x be the solution of the discrete Dirichlet problem in D, with boundary condition x on $\partial^+ D$. If $X(i)$, $i \in D$, is distributed according to $\mathbb{P}_D$, then $X(i) + h_x(i)$ is distributed according to $\mathbb{P}_{D,x}$. As $h_x \geq 0$, the statement follows.

b) If $A \subset S_N$, let Ω_+^A be the event $\{X(i) \geq 0, i \in A\}$. It was proved in [10], Lemma 3.1 that for $A \subset B$ the law $\mathbb{P}_B(\bullet \,|\, \Omega_+^A)$ on $\mathbb{R}^B$ is associated, i.e. for any bounded measurable functions f_1, $f_2 : \mathbb{R}^B \to \mathbb{R}$ which are increasing in all arguments, one has

$$\mathbb{E}_B(f_1 f_2 \,|\, \Omega_+^A) \geq \mathbb{E}_B(f_1 \,|\, \Omega_+^A)\mathbb{E}_B(f_2 \,|\, \Omega_A^+).$$

(See the proof of Lemma 3.1 of [10].) We apply this to $B \triangleq D_2$ and $A \triangleq (D_1 \cup \partial^+ D_1) \cap D_2$. Setting $\mathbb{P}_{B,A}^+ \equiv \mathbb{P}_B(\bullet \,|\, \Omega_+^A)$, we obtain for any $t > 0$,

$$\mathbb{E}_{B,A}^+(f(X) \,|\, X(i) \leq t, i \in \partial^+ D_1 \cap D_2) \leq \mathbb{E}_{B,A}^+(f(X)).$$

Letting $t \downarrow 0$, the l.h.s. converges to $\mathbb{E}_{D_1}(f(X) \,|\, \Omega_+)$, and so we have

$$\mathbb{E}_{D_1}(f(X) \,|\, \Omega_+) \leq \mathbb{E}_{D_2}(f(X) \,|\, \Omega_A^+).$$

Using the fact that f and $1_{\Omega_+^{D_2 \setminus A}}$ are increasing, the r.h.s. is

$$\leq \mathbb{E}_{D_2}(f(X) \,|\, \Omega_+),$$

which proves the claim. □

Lemma 4.4.2. *Let* $\varepsilon > 0$*. Then there exists* $N_\varepsilon \in \mathbb{N}$ *such that for* $N \geq N_\varepsilon$ *and all* $D \subset S_N$*, we have*

$$\mathbb{P}_D(\Omega_+) \leq 2\,\mathbb{P}_D\Big(\Omega_+, \max_i X_i \leq N^\varepsilon\Big).$$

Proof.

$$\mathbb{P}_D(\Omega_+) = \mathbb{P}_D\Big(\Omega_+, \max_i X_i \leq N^\varepsilon\Big) + \mathbb{P}_D\Big(\max_i X_i > N^\varepsilon \,|\, \Omega_+\Big)\mathbb{P}_D(\Omega_+). \qquad (4.4.1)$$

By Lemma 4.4.1b), we have

$$\begin{aligned}\mathbb{P}_D\Big(\max_i X_i > N^\varepsilon \,|\, \Omega_+\Big) &\leq \mathbb{P}_{S_N}\Big(\max_{i \in S_N} X_i > N^\varepsilon \,|\, \Omega_+\Big) \\ &\leq \mathbb{P}_{S_N}\Big(\max_{i \in S_N} X_i > N^\varepsilon\Big)/\mathbb{P}_{S_N}(\Omega_+). \qquad (4.4.2)\end{aligned}$$

The numerator is estimated in a rough way by

$$\mathbb{P}_{S_N}\Big(\max_{i \in S_N} X_i > N^\varepsilon\Big) \leq 5N^2 \max_i \mathbb{P}_{S_N}(X_i > N^\varepsilon) \leq 5N^2 \exp\Big(-c\frac{N^{2\varepsilon}}{\log N}\Big),$$

as the maximal variance of X_i under $\mathbb{P}_{S_N}$ is of order $\log N$. $P_{S_N}(\Omega_+)$ is of order $\exp(-c(\log N)^2)$ ([5]). □

Let now A, C, B be as in the statement of Lemma 2.3.1, i.e. B is a mesoscopic region, and $C \subset B$, $A \subset S_N \setminus B$. Let $\partial^- B$ be the set of points in B which are at distance 1 from ∂B, i.e. those points in B which can be joined by two bonds of the lattice with B^c but not with one. Let $D^- \triangleq \partial^- B \cap C$, $D \triangleq \partial B \cap C$, $D^+ \triangleq \partial^+ B \cap A$. We denote by Y^+, Y^-, Y the restriction of a configuration X to D^+, D^-, and D respectively. If $y^+ \in \mathbb{R}^{D^+}$, $y^- \in \mathbb{R}^{D^-}$, $y \in \mathbb{R}^D$, we denote by $f(y^+, y^- \mid y)$ the conditional density of the $\mathbb{P}_{A\vee C}$-law of (Y^+, Y^-) given $Y = y$.

Lemma 4.4.3. *If $\varepsilon > 0$, then there exists $c > 0$ such that*

$$|\log f(y^+, y^- \mid y) - \log f(y^+, y^- \mid 0)| \leq c|D|N^{2\varepsilon},$$

for $0 \leq y, y^+, y^- \leq N^\varepsilon$. Here we write $0 \leq y \leq N^\varepsilon$ if all the components of y satisfy this condition.

Proof. With an abuse of notation, we write $f(y^+, y^-)$ for the density of (Y^+, Y^-) under $\mathbb{P}_{A\vee C}$, $f(y)$ for the density of Y, and $f(y \mid y^+, y^-)$ for the conditional density of Y given (Y^+, Y^-). Writing

$$f(y^+, y^- \mid y) = \frac{f(y \mid y^+, y^-) f(y^+, y^-)}{f(y)},$$

we see that it suffices to prove

$$|\log f(y \mid y^+, y^-) - \log f(0 \mid y^+, y^-)| \leq c|D|N^{2\varepsilon}, \tag{4.4.3}$$

and

$$|\log f(y) - \log f(0)| \leq c|D|N^{2\varepsilon}, \tag{4.4.4}$$

uniformly in $0 \leq y, y^+, y^- \leq N^\varepsilon$. For some positive function $\varphi : \mathbb{R}^{D^+ \cup D^-} \to \mathbb{R}$, we have

$$\begin{aligned} &f(y \mid y^+, y^-) \\ &\quad = \varphi(y^+, y^-) \exp\Bigg\{ -\frac{1}{2} \sum_{\substack{i\in D \\ j\in D^+, |i-j|=1}} (y(i) - y^+(j))^2 - \frac{1}{2} \sum_{\substack{i\in D \\ j\in D^-, |i-j|=1}} (y(i) - y^-(j))^2 \Bigg\}. \end{aligned}$$

Using this, (4.4.3) clearly follows. To prove (4.4.4), we introduce $\overline{f}(y)$ as the density of

$$P_{A\vee C}\{Y \in \bullet, |Y^+| \leq 2N^\varepsilon, |Y^-| \leq 2N^\varepsilon\}.$$

Clearly

$$f(y) = \overline{f}(y) + f(y)\, \mathbb{P}_{A\vee C}\Big(\max_{i\in D^+\cup D^-} |X(i)| > 2N^\varepsilon \mid Y = y \Big). \tag{4.4.5}$$

By a similar argument as in the proof of Lemma 4.4.2, we have

$$\lim_{N\to\infty} \mathbb{P}_{A\vee C}\Big(\max_{i\in D^+\cup D^-} |X(i)| > 2N^\varepsilon \mid Y = y \Big) = 0,$$

uniformly in $A, C \subset S_N$, and $|y| \leq N^\varepsilon$. Using this, we get from (4.4.5):

$$\overline{f}(y) \leq f(y) \leq 2\overline{f}(y), \tag{4.4.6}$$

uniformly in $A, C \subset S_N$, $|y| \leq N^\varepsilon$, provided N is large enough. Now

$$\bar{f}(y) = \int_{\{y^+, |y^+| \le 2N^\varepsilon\}} \int_{\{y^- : |y^-| \le 2N^\varepsilon\}} f(y \mid y^+, y^-) \, \mathbb{P}_{A \vee C}(Y^+, Y^-)^{-1} (dy^+, dy^-).$$

By an obvious slight modification of the argument leading to (4.4.3), this proves (4.4.4). □

Proof of Lemma 2.3.1 d) For two expressions $\varphi(A, C, N)$, $\psi(A, C, N) > 0$, where $C \subset B$, $A \subset S_N \setminus B$, B mesoscopic, we write $\varphi \sim \psi$ if

$$|\log \varphi(A, C, N) - \log \psi(A, C, N)| \le c|D|N^{2\varepsilon},$$

for any ε, uniformly in A, B, C, if N is large enough. Let

$$C' \triangleq C \setminus (D \vee D^-), \qquad A' \triangleq A \setminus D^+.$$

If $y^+ \in \mathbb{R}^{D^+}$, $y^- \in \mathbb{R}^{D^-}$, we write $\tilde{y}(y^+, y^-)$ for the boundary condition on $\partial^+(A' \vee C')$ which is y^+ on D^+, y^- on D^- and 0 otherwise.

Using Lemma 4.4.2, we get

$$\begin{aligned}
&\mathbb{P}_{A \vee C}(\Omega_+) \sim \mathbb{P}_{A \vee C}(\Omega_+ \text{ and } Y, Y^+, Y^- \le N^\varepsilon) \\
&= \int_{\{0 \le y \le N^\varepsilon\}} \mathbb{P}_{A \vee C} Y^{-1}(dy) \int_{\{0 \le y^+, y^- \le N^\varepsilon\}} dy^+ \, dy^- \, f(y^+, y^- \mid y) \, \mathbb{P}_{A' \vee C', \tilde{y}}(\Omega_+) \\
&\sim \int_{\{0 \le y \le N^\varepsilon\}} \mathbb{P}_{A \vee C} Y^{-1}(dy) \int_{\{0 \le y^+, y^- \le N^\varepsilon\}} dy^+ \, dy^- \, f(y^+, y^- \mid 0) \, \mathbb{P}_{A' \vee C', \tilde{y}}(\Omega_+) \\
&= P_{A \vee C}(0 \le Y \le N^\varepsilon) \, \mathbb{P}_{(A \vee C) \setminus D}(\Omega_+, 0 \le Y^+, Y^- \le N^\varepsilon) \\
&\sim \mathbb{P}_{(A \vee C) \setminus D}(\Omega_+) \\
&= \mathbb{P}_A(\Omega_+) \, \mathbb{P}_{C'}(\Omega_+),
\end{aligned}$$

where we have used Lemma 4.4.3 and Lemma 4.4.2. Applying that to $A = \emptyset$, we get

$$\mathbb{P}_C(\Omega_+) \sim \mathbb{P}_{C'}(\Omega_+),$$

and therefore

$$\mathbb{P}_{A \vee C}(\Omega_+) \sim \mathbb{P}_A(\Omega_+) \, \mathbb{P}_C(\Omega_+). \qquad \square$$

5. Stability Results for the Torsional Rigidity

5.1. Main estimate. We use the notations introduced in Subsect. 2.4. Since u_D is analytic in the interior of D, $\nabla u_D = 0$ at most at finitely many points inside D, and $\Gamma_\mu \triangleq \partial \text{lev}_\mu u_D$ is an analytic curve for all, except finitely many $\mu \in [0, \max u_D]$. Following [2],

$$a'(\mu) = -\int_{\Gamma_\mu} \frac{ds}{|\nabla u_D|} = -|\Gamma_\mu| \frac{1}{|\Gamma_\mu|} \int_{\Gamma_\mu} \frac{ds}{|\nabla u_D|},$$

where we use $|\Gamma_\mu|$ to denote the length of Γ_μ. Therefore, by Jensen's inequality,

$$a'(\mu) \le -|\Gamma_\mu| \left(\frac{1}{|\Gamma_\mu|} \int_{\Gamma_\mu} |\nabla u_D| \, ds \right)^{-1} = -\frac{|\Gamma_\mu|^2}{a(\mu)}.$$

Consequently,

$$\mu'(a) = \frac{1}{a'(\mu)}\bigg|_{\mu=\mu(a)} \geq -\frac{a}{l_D^2(a)} \;\geq\; -\frac{1}{4\pi},$$

for all but at most finite number of $a \in [0, |D|]$. Now note that

$$\begin{aligned}\chi(D) \;&=\; \int_D u_D(x)\,dx = \int_0^{|D|} \mu(a)\,da \\ &= -\int_0^{|D|} a\,\mu'(a)\,da \leq \int_0^{|D|} \frac{a^2}{l_D^2(a)}\,da,\end{aligned} \tag{5.1.1}$$

and (2.4.1) follows.

An explicit calculation reveals that

$$\chi(B_{|D|}) = \int_0^{|D|} \frac{a^2}{s(a)^2}\,da = \frac{|D|^2}{8\pi}. \tag{5.1.2}$$

Subtracting (5.1.1) from (5.1.2) (recall $s(a) = 2\sqrt{\pi a}$), we obtain

$$\chi(B_{|D|}) - \chi(D) \geq \int_0^{|D|} a^2\left(\frac{1}{s(a)^2} - \frac{1}{l_D^2(a)}\right)da \geq \frac{1}{4\pi}\int_0^{|D|} a\left(1 - \frac{s(a)}{l_D(a)}\right)da, \tag{5.1.3}$$

and the proof of Theorem 2.4.1 is, thereby, concluded.

5.2. Stability of the inradius. Let $D \subseteq S(1)$ be a simply connected (but not necessarily connected) domain with a piecewise smooth boundary. Since for each $a \in [0, |D|]$, the inradius $\varrho(D_a)$ of D_a satisfies $\varrho(D_a) \leq \varrho(D)$, the Bonnensen inequality (see e.g [25], (4.7)) implies that

$$l_D(a)^2 - s(a)^2 \geq \pi^2\left(\sqrt{\frac{a}{\pi}} - \varrho(D)\right)^2$$

for each $a \in [\pi\varrho(D)^2, |D|]$. Therefore,

$$\frac{l_D(a)}{s(a)} \geq \left(1 + \frac{\pi}{4a}\left(\sqrt{\frac{a}{\pi}} - \varrho(D)\right)^2\right)^{1/2},$$

and, consequently,

$$1 - \frac{s(a)}{l_D(a)} \geq 1 - \left(1 + \frac{1}{4a}(\sqrt{a} - \sqrt{\pi}\varrho(D))^2\right)^{-1/2}.$$

At this point we stop pushing for precise constants, and simply observe that due to a trivial estimate; $\forall\alpha \geq 0,\ 1 - (1+\alpha)^{-1/2} \geq \alpha/2(1+\alpha)$,

$$1 - \frac{s(a)}{l_D(a)} \geq \frac{1}{c}(a - \pi\varrho(D)^2),$$

for all $a \in [\pi\varrho(D)^2, |D|]$ and c large enough. Substituting this into (5.1.3), and performing the integration over the interval $a \in [\pi\varrho(D)^2, |D|]$, we infer that,

$$\chi(B_{|D|}) - \chi(D) \geq \frac{1}{c}(|D| - \pi\varrho(D)^2)^3, \tag{5.2.1}$$

and the claim of Lemma 2.4.2 follows.

Remark 5.2.1. As mentioned in the introduction, our stability estimate (5.2.1) readily implies stability in terms of the area of the symmetric difference function $d_\Delta(\bullet)$, introduced in (2.4.3). Indeed, for a simply connected D,

$$d_\Delta(D) \leq 2(|D| - \pi\varrho(D)^2) \leq c\sqrt[3]{(\chi(B_{|D|}) - \chi(D))}. \tag{5.2.2}$$

We shall see in Subsect. 5.4 that such an estimate can be easily extended to the case of not simply connected domains as well.

5.3. Stability of crystal shapes. Let $D \subseteq S(1)$ be as in the previous subsection, and assume without loss of generality that the largest inscribed circle of D is centered at the origin, i.e. that

$$B_{\pi\varrho(D)^2} \subseteq D.$$

We are going to estimate $\|u_D^v - h_v\|_{L^1(\mathbb{R}^2)}$. Let $h_v^r(\bullet)$ denote the shape of the harmonic droplet of the volume v, which is bound to wet the circle of radius r, centered at the origin (see (1.2.6) and (1.2.7),

$$h_v^r(x) = \frac{2v}{\pi r^2}(r^2 - |x|^2) \vee 0.$$

Set $r(D) = \sqrt{\frac{|D|}{\pi}}$. Then, for any $\alpha > 0$,

$$\|u_D^v - h_v\|_{L^1(\mathbb{R}^2)} \leq \|u_D^v - h_{\alpha v}^{\varrho(D)}\|_{L^1} + \|h_v^{r(D)} - h_{\alpha v}^{\varrho(D)}\|_{L^1} + \|h_v - h_v^{r(D)}\|_{L^1}. \tag{5.3.1}$$

We choose $\alpha = \chi(B_{\pi\varrho(D)^2})/\chi(D)$, so that

$$h_{\alpha v}^{\varrho(D)} = \frac{v}{\chi(D)} u_{B_{\pi\varrho(D)^2}}.$$

Then, using monotonicity in domain of the solution of the Poisson equation with Dirichlet boundary condition,

$$\|u_D^v - h_{\alpha v}^{\varrho(D)}\|_{L^1} = v\left(1 - \frac{\chi(B_{\pi\varrho(D)^2})}{\chi(D)}\right) \tag{5.3.2}$$

and

$$\|h_v^{r(D)} - h_{\alpha v}^{\varrho(D)}\|_{L^1} = v\left(1 - \frac{\chi(B_{\pi\varrho(D)^2})}{\chi(B_{|D|})}\right).$$

On the other hand, a straightforward computation reveals that

$$\|h_v^{r(D)} - h_v\|_{L^1} = \frac{v}{2\pi}\frac{|\,|D| - \overline{a}|}{|D| + \overline{a}}. \tag{5.3.3}$$

To facilitate notations set $\Delta(D) = E_f^v(D) - E_f^v(B_{\bar{a}})$. Then,

$$\Delta(D) \geq \max\{E_f^v(D) - E_f^v(B_{|D|})\,,\ E_f^v(B_{|D|}) - E_f^v(B_{\bar{a}})\}.$$

Since $E_f^v(B_{|D|})$ can be computed exactly, we infer that for some $c > 0$,

$$E_f^v(B_{|D|}) - E_f^v(B_{\bar{a}}) \geq \frac{1}{c}(|D| - \bar{a})^2,$$

or, substituting the above estimate into (5.3.3),

$$\|h_v^{r(D)} - h_v\|_{L^1} \leq c\sqrt{\Delta(D)}. \tag{5.3.4}$$

On the other hand,

$$\Delta(D) \geq E_f^v(D) - E_f^v(B_{|D|}) = \frac{v^2(\chi(B_{|D|}) - \chi(D))}{2\chi(B_{|D|})\chi(D)} \geq \frac{(\pi v)^2}{8}(\chi(B_{|D|}) - \chi(D)), \tag{5.3.5}$$

the last inequality follows from (1.2.6 and the fact that $D \subseteq S(1)$ (and hence, by (1.2.6), both $\chi(B_{|D|})$) and $\chi(D)$ are bounded above by $2/\pi$).

By (5.2.1) this means that one can choose $c = c(v)$, such that

$$|D| - \pi\varrho(D)^2 \leq c(v)\sqrt[3]{\Delta(D)}.$$

Consequently, we can use (1.2.6) to derive,

$$\begin{aligned}\chi(D) - \chi(B_{\pi\varrho(D)^2}) \;\leq\; \chi(B_{|D|}) - \chi(B_{\pi\varrho(D)^2}) &= \frac{|D|^2 - (\pi\varrho(D)^2)^2}{8\pi}\\ &\leq \frac{1}{\pi}(|D| - \pi\varrho(D)^2) \leq c\sqrt[3]{\Delta(D)}.\end{aligned}$$

Substituting the latter inequalities into (5.3.2) and (5.3.3), we finally obtain

$$\|u_D^v - h_{\alpha v}^{\varrho(D)}\|_{L^1} \leq \frac{c}{\chi(D)}\sqrt[3]{\Delta(D)} \tag{5.3.6}$$

and

$$\|h_v^{\sqrt{|D|/\pi}} - h_v\|_{L^1} \leq \frac{c}{|D|^2}\sqrt[3]{\Delta(D)} \tag{5.3.7}$$

Since for any $k > 0$, $E_f^v(D) \leq k \Rightarrow \chi(D) \geq v^2/2k$, (5.3.1) and (5.3.4)–(5.3.7) imply that there exists $c > 0$, such that

$$\min_{x \in S(1)} \|u_D^v - h_v(\bullet + x)\|_{L^1} \;\leq\; c\sqrt[3]{\Delta(D)}, \tag{5.3.8}$$

uniformly in simply connected domains $D \subseteq S(1)$.

5.4. Estimates for general domains $D \subseteq S(1)$. It remains to prove Lemma 2.4.3 in the case when $D \subseteq S(1)$ is not necessarily simply connected. For such a domain D let $\{D_a\}_{a\in[0,|D|]}$ be the rearrangement of the level sets of u_D, defined in Subsect. 2.4. Recall that

$$\Delta(D) \geq\; E_f^v(D) - E_f^v(B_{|D|}) \;=\; \frac{v^2}{2}\left(\frac{1}{\chi(D)} \;-\; \frac{1}{\chi(B_{|D|})}\right).$$

Since we are interested only in the case of $\Delta(D)$ being small, we may assume that $\chi(D)$ is bounded away from zero uniformly in all domains D in question. Then, by virtue of (2.4.2),

$$\Delta(D) \;\geq\; \frac{1}{c}\int_0^{|D|} a\left(1 - \frac{s(a)}{l_D(a)}\right)da.$$

Setting $a^* = \max\{a \,:\, l_D(a) \leq s(a) + \sqrt{\Delta(D)}\} \vee |D|/2$, we therefore obtain (modifying the constant c according to the convention of Remark 1.3.2):

$$\sqrt{\Delta(D)} \;\geq\; \frac{|D| - a^*}{c} \;=\; \frac{|D \setminus D^*|}{c},$$

where $D^* \stackrel{\Delta}{=} D_{a^*}$. Since D^* is a level set of u_D, and using the estimate on μ' derived at the beginning of Subsect. 5.1, we conclude that

$$\max_{x \in D\setminus D^*} u_D(x) \;\leq\; \frac{1}{4\pi}|D \setminus D^*|.$$

Consequently,

$$\begin{gathered}\|u_D - u_{D^*}\|_{L^1} \;=\; \chi(D) - \chi(D^*) \;\leq\; c\sqrt{\Delta(D)} \\ \text{and} \\ \Delta(D^*) \;\leq\; c\sqrt{\Delta(D)}.\end{gathered} \tag{5.4.1}$$

Thus everything boils down to the following problem:

Given $D \subseteq S(1)$ with a piecewise smooth boundary ∂D satisfying

$$|\partial D| \;\leq\; s(|D|) \;+\; \Delta(D),$$

prove that:

$$\inf_{x \in S(1)} \|u_D^v - h_v(x + \bullet)\|_{L^1} \;\leq\; c\sqrt[3]{\Delta(D)}. \tag{5.4.2}$$

Again, since we are interested only in the case of $\Delta(D)$ being small, it can be assumed from the beginning that $|D| \geq \overline{a}/2$. First of all, notice, that if D contains two disjoint components, $D = D_1 \vee D_2$, then

$$\begin{aligned}\chi(D) \;=\; \chi(D_1) + \chi(D_2) \;&\leq\; \chi(B_{|D|})\left(\frac{|D_1|^2 + |D_2|^2}{|D|^2}\right) \\ &=\; \chi(B_{|D|})\left(1 \;-\; \frac{2|D_1||D_2|}{|D|^2}\right),\end{aligned}$$

and, consequently,

$$E_f^v(D) \;\geq\; \Delta_f|D| + \frac{v^2}{2\chi(B_{|D|})}\left(1 \;-\; \frac{2|D_1||D_2|}{|D|^2}\right)^{-1} \;\geq\; E_f^v(B_{|D|}) + \frac{v^2}{2\pi}\frac{2|D_1||D_2|}{|D|^2}.$$

Therefore,

$$\frac{2|D_1||D_2|}{|D|^2} \;\leq\; \frac{2\pi}{v^2}\Delta(D),$$

or, $\min\{|D_1|, |D_2|\} \leq c\Delta(D)$. Thus, in order to prove (2.5.1), one can restrict attention to the case where D is connected.

So assume that D satisfies (5), is connected, but possibly not simply connected, i.e. that $D = G \setminus \overline{R}$, where $G \subseteq S(1)$ is connected and simply connected, and $R \subset G$ is open; both domains having piecewise smooth boundaries. Since, $|D| \;\leq\; |G|$ and

$|\partial D| = |\partial G| + |\partial R|$, we immediately infer that $|\partial R| \le \Delta(D)$, and, consequently, that $|R| \le \Delta(D)^2/4\pi$. Furthermore,

$$\chi(D) \le \chi(G) \le \chi(B_{|D|+|R|}) = \frac{(|D| + |R|)^2}{8\pi} \le \chi(D) + c\Delta(D), \qquad (5.4.3)$$

where the last inequality follows by the estimate on $|R|$ above and by (5.3.5) of the previous subsection. Equation (5.4.3) and the above estimate on $|R| = |G \setminus D|$ already contain all the information we need to prove (5.4.2). Indeed, we readily obtain that

$$\Delta(G) \le c\Delta(D) \text{ and } \|u_G - u_D\|_{L^1} = \chi(G) - \chi(D) \le c\Delta(D),$$

and it remains, thereby, to apply (5.3.8) to the function u_G^v over the simply connected domain G.

6. Concentration Under $\mathbb{P}_B\big(\bullet \mid V_N \ge N^3 v\big)$

6.1. Gaussian concentration estimates. We give a proof of Lemma 2.5.1. Using the representation of the approximate torsional rigidity

$$N^4 \chi_N(A) = \sum_i \mathbb{E}_i(\tau_{A^c}),$$

where $\mathbb{E}_i$ is the expectation of an ordinary symmetric random walk on $\mathbb{Z}^2$ starting at i, and τ_{A^c} is the first hitting time of A^c, we see that

$$\chi_N(A) \le \chi_N(S_N),$$

and obviously (see also Lemma 2.6.1

$$\lim_{N\to\infty} \chi_N(S_N) = \chi(S(1)) < \infty.$$

Therefore, we have

$$K := \sup_N \sup_{A \subset S_N} \chi_N(A) < \infty. \qquad (6.1.1)$$

Now, we have

$$\mathrm{var}_{\mathbb{P}_A}(V_N) = N^4 \chi_N(A). \qquad (6.1.2)$$

We write

$$\begin{aligned}
&\mathbb{P}_A(\|\xi_N - \overline{u}^v_{A,N}\|_1 \ge a \mid V_N \ge N^3 v) \\
&\quad \le \int_v^{v+a/2} \mathbb{P}_A(\|\xi_N - u^v_{A,N}\|_1 \ge a \mid V_N = N^3 x)\, \mathbb{P}_A\left(\frac{V_N}{N^3} \in dx \,\middle|\, V_N \ge N^3 v\right) \\
&\qquad + \mathbb{P}_A\left(V_N \ge N^3\left(v + \frac{a}{2}\right) \,\middle|\, V_N \ge N^3 v\right). \qquad (6.1.3)
\end{aligned}$$

Using (6.1.1) and (6.1.2), we get

$$\begin{aligned}
&\mathbb{P}_A\left(V_N \ge N^3\left(v + \frac{a}{2}\right) \,\middle|\, V_N \ge N^3 v\right) \\
&\quad \le \exp\left\{-\frac{N^2(v + \frac{a}{2})^2}{2\chi_N(A)} - \frac{N^2 v^2}{2\chi_N(A)}\right\} \le \exp\left(-\frac{N^2 a^2}{8K}\right).
\end{aligned}$$

Using this, (6.1.3), and the obvious fact that

$$\|u^v_{A,N} - u^x_{A,N}\|_1 \le |x - v|,$$

it suffices to prove

$$\sup_{x\in[v,v+\frac{a}{2}]} \mathbb{P}_A\left(\|\xi_N - \overline{u}^x_{A,N}\|_1 \ge \frac{a}{2} \,\middle|\, V_N = N^3 x\right) \le e^{-\frac{a^2N^2}{c}}. \tag{6.1.4}$$

The random field $(X(i))_{i\in A}$ under the conditioned law $\mathbb{P}_A(\ \ | V_N = N^3 v)$ is Gaussian with mean

$$\mathbb{E}_A(X(i) \,|\, V_N = N^3 x) = N x u_{A,N}(i)$$

and covariances

$$\mathrm{cov}_A(X(i), X(j) \,|\, V_N = N^3 x) = g_A(i,j) - \frac{u_{A,N}(i)u_{A,N}(j)}{\chi_N(A)},$$

where $g_A(i,j) \triangleq \mathbb{E}_A(X(i)X(j))$. Remark that

$$u_{A,N}(i) = \frac{1}{N^2}\sum_{j\in A} g_A(i,j),$$

and as $\chi_N(A) = \sum_{i\in A} u_{A,N}(i)/N^2$, we see that

$$\sigma_A^2 \triangleq \sum_{i,j\in A} |\mathrm{cov}_A(X(i), X(j) \,|\, V_N = N^3 x)| \le 2N^4\chi_N(A) \le 2N^4\chi_N(S_N). \tag{6.1.5}$$

We apply now one of the standard isoperimetric inequalities for Gaussian measures (see e.g. [21], (4.4)). First remark that

$$\begin{aligned}\mu &\triangleq \mathbb{E}_A\left(\sum_{j\in A} |X(j) - xNu_{A,N}(j)| \,\middle|\, V_N = N^3 x\right)\\ &\le \sum_{j\in A}\sqrt{g_A(j,j)} \le cN^2\sqrt{\log N} \le \frac{aN^{3-\delta}}{4}\end{aligned}$$

if N is large enough. Therefore, using (4.4) of [21], we get

$$\begin{aligned}&\mathbb{P}_A\left(\|\xi_N - \overline{u}^x_{A,N}\|_1 \ge \frac{a}{2} \,\middle|\, V_N = N^3 x\right)\\ &\quad = \mathbb{P}_A\left(\sum_{j\in A} |X(j) - xNu_{A,N}| \ge \frac{aN^3}{2} \,\middle|\, V_N = N^3 x\right)\\ &\quad \le \mathbb{P}_A\left(\sum_{j\in A} |X(j) - xNu_{A,N}(j)| \ge \mu + \frac{aN^3}{4} \,\middle|\, V_N = N^3 x\right)\\ &\quad \le \exp\left(-\frac{a^2N^6}{32\sigma^2}\right),\end{aligned} \tag{6.1.6}$$

where

$$\sigma^2 = \sup\left\{ \mathrm{var}_A\left(\sum_{j\in A} X(j)g(j) \,\middle|\, V_N = N^3 v \right) \,:\, |g(j)| \le 1 \text{ for all } j \right\} \le \sigma_A^2 .$$

Using (6.1.5), we see that the r.h.s. of (6.1.6) is bounded by $\exp(-a^2N^2/64\chi_N(S_N))$. As

$$\sup_N \chi_N(S_N) < \infty,$$

we have proved Lemma 2.5.1.

7. Approximation by Discrete Quantities

7.1. Estimates on discrete rigidities. Recall that the discrete rigidity $\chi_N(A)$ of a lattice domain $A \Subset \mathbb{Z}^2$ was defined in (2.1.5). We follow [34] to take advantage of the variational characterization of χ_N:

$$\frac{1}{\chi_N(A)} = N^4 \inf_{\substack{u\ge 0 \text{ in } A \\ u=0 \text{ on } \mathbb{Z}^2\setminus A}} \frac{\sum_{\langle k,l\rangle}(u(k)-u(l))^2}{(\sum_k u(k))^2}, \tag{7.1.1}$$

where the sum in the numerator is over all unoriented pairs of nearest neighbours in $\mathbb{Z}^2$.

Note, by the way, that for a domain $D \Subset \mathbb{R}^2$ with a piecewise $\mathbf{C}^2$ boundary, the torsional rigidity $\chi(D)$ is given by a similar formula,

$$\frac{1}{\chi(D)} = \inf_{\substack{u>0 \text{ on } D \\ u=0 \text{ on } D^c}} \frac{\int_D |\nabla u|^2\, dx}{(\int_D u\, dx)^2}. \tag{7.1.2}$$

Proposition 7.1.1. *Let $D \Subset \mathbb{R}^2$ have a piecewise $\mathbf{C}^2$ boundary, and assume that a finite $A \subset \mathbb{Z}^2$ is such that*

$$\min\{\, \|x-k\| \,:\, x \in ND, k \in \mathbb{Z}^2 \setminus A \,\} \ge 1/2. \tag{7.1.3}$$

Then,

$$\chi(D) \le \chi_N(A). \tag{7.1.4}$$

Proof. The proof follows Sect. 2 of [34], where a similar inequality for the membrane problem was established. We adopt it here for the sake of completeness.

Let $u \in H_0^1(D)$. For each $(\alpha,\beta) \in \frac{1}{2}S(1) \stackrel{\Delta}{=} \frac{1}{2}[-1,1]^2$, define

$$V_{\alpha,\beta}(k) = u\left(\frac{k+(\alpha,\beta)}{N}\right).$$

Because of the condition (7.1.3), $V_{\alpha,\beta} \equiv 0$ on $\mathbb{Z}^2 \setminus A$. Moreover if $u \ge 0$ on D, then $V_{\alpha,\beta} \ge 0$ as well. By (7.1.1), $\forall\,(\alpha,\beta)$,

$$\left(\sum_k V_{\alpha,\beta}(k)\right)^2 \le N^4 \chi_N(A) \sum_{\langle k,l\rangle} (V_{\alpha,\beta}(k) - V_{\alpha,\beta}(l))^2.$$

However, by Jensen's inequality

$$\int_{\frac{1}{2}S(1)} \Big(\sum_k V_{\alpha,\beta}(k) \Big)^2 \, d\alpha \, d\beta \geq N^2 \Big(\int_D u(x) \, dx \Big)^2,$$

and

$$\int_{\frac{1}{2}S(1)} \Big(\sum_{\langle k,l \rangle} (V_{\alpha,\beta}(k) - V_{\alpha,\beta}(l))^2 \Big) \, d\alpha \, d\beta \leq \frac{1}{N^2} \int_D |\nabla u|^2 \, dx.$$

The claim follows now by (7.1.2).

The estimate (7.1.4) controls the approximation from above by discrete rigidities. A possibility to control it below as well is provided by the following

Proposition 7.1.2. *Let A be a finite subset of $\mathbb{Z}^2$ and define*

$$A_2 = \Big\{ k \in \mathbb{Z}^2 \, : \, \min_{l \in A} \|k - 2l\| \leq 1 \Big\}.$$

Then,

$$\chi_N(A) \leq \chi_{2N}(A_2). \tag{7.1.5}$$

Proof. Given a function $u \, : \, \mathbb{Z}^2 \to \mathbb{R}_+$, such that $u|_{\mathbb{Z}^2 \setminus A} \equiv 0$ and u is not identically zero, let us define $\widetilde{u} \, : \, \mathbb{Z}^2 \to \mathbb{R}_+$ via

$$\widetilde{u}(k) = \frac{1}{|A(k)|} \sum_{m \in A(k)} u(m), \tag{7.1.6}$$

where $A(k) = \{ \, m \in \mathbb{Z}^2 \, : \, \|k - 2m\| < 1 \, \}$. Then, by the direct substitution of (7.1.6),

$$\sum_{k \in \mathbb{Z}^2} \widetilde{u}(k) = 4 \sum_{m \in \mathbb{Z}^2} u(m),$$

and

$$\sum_{\langle k,l \rangle} (\widetilde{u}(k) - \widetilde{u}(l))^2 \leq \sum_{\langle m,n \rangle} (u(m) - u(n))^2.$$

Since by the very construction $\widetilde{u} \equiv 0$ outside A_2, (7.1.5) follows from the variational characterization (7.1.1).

We are in a position now to prove Lemma 2.6.1. Let $B \subseteq S_N$ be a mesoscopic region, and define $\bar{B} \subseteq S(1)$ as in (2.6.1),

$$\bar{B} = \frac{1}{N} \bigcup_{k \in B} (k + (1/2)S(1)).$$

By Propositions 7.1.1 and 7.1.2,

$$\chi(\bar{B}_{-,N}) \leq \chi_N(B) \leq \lim_{m \to \infty} \chi_{2^m N}(B_{2^m}),$$

where $\bar{B}_{-,N} \stackrel{\Delta}{=} \{ \, x \in \bar{B} \, : \, \min_{y \in \partial \bar{B}} \|x - y\| > 3/N \, \}$, and $B_{2^m} = (\ldots(B_2)_2)_2 \ldots)_2$ (m times). Using results of [6] and the monotonicity of χ in a domain, we conclude that

$$\chi_N(B) \leq \chi(\bar{B}_{+,N}),$$

where $\bar{B}_{+,N} \stackrel{\Delta}{=} \{ x \in \mathbb{R}^2 : \min_{y\in B}\|x-y\| \le 3/N \}$. Consequently, for any $B \subseteq S_N$ mesoscopic,

$$|\chi_N(B) - \chi(\bar{B})| \le \chi(\bar{B}_{+,N}) - \chi(\bar{B}_{-,N}), \tag{7.1.7}$$

and it remains to estimate the right-hand side of (7.1.7) uniformly in $B \subseteq S_N$.

Let $u_{B,+}$ and $u_{B,-}$ be the solutions of the Poisson equation on $\bar{B}_{+,N}$ and $\bar{B}_{-,N}$ respectively. Set

$$a_N = \max_{x\in \bar{B}_{+,N}\setminus \bar{B}_{-,N}} u_{B,+}(x). \tag{7.1.8}$$

Then,

$$\begin{aligned}\chi(\bar{B}_{+,N}) - \chi(\bar{B}_{-,N}) &= \int_{\bar{B}_{+,N}} u_{B,+}(x)\,dx - \int_{\bar{B}_{-,N}} u_{B,-}(x)\,dx \\ &\le a_N|\bar{B}_{+,N}| \le 4a_N. \end{aligned}\tag{7.1.9}$$

Indeed, for $x \in \bar{B}_{-,N}$,

$$u_{B,+}(x) = u_{B,-}(x) + \int_{\partial \bar{B}_{-,N}} u_{B,+}(\xi)\varrho(x, d\xi),$$

where $\varrho(x,\bullet)$ is the exit distribution (harmonic measure) on $\partial\bar{B}_{-,N}$ for the Brownian motion starting at x.

In order to estimate a_N in (7.1.8), let $G_N = 3S(1) \setminus N^{b-1}S(1)$, and let u_{G_N} be the solution of the Poisson equation with Dirichlet boundary conditions on ∂G_N. Set

$$a'_N = \max_{\|x\|\le N^{b-1}+9/N} u_{G_N}(x).$$

By the monotonicity considerations one infers that $a'_N \ge a_N$ for all mesoscopic $\bar{B} \subseteq S(1)$. In order to estimate a'_N define a new domain

$$\widetilde{G}_N = N^{b-1}(2S(1) \setminus S(1)) \subseteq G_N.$$

Then u_{G_N} is majorized by the solution of the Poisson equation on $\widetilde{G}_N$ subject to the boundary conditions 0 on $\partial(N^{b-1}S(1))$ and m_N on $\partial(2N^{b-1}S(1))$, where

$$m_N = \max_{\|x\|=2N^{b-1}} u_{G_N}(x). \tag{7.1.10}$$

Then, of course, $m_N \le c$ independently of N. Since the blowup of $\widetilde{G}_N$ by the factor N^{1-b} is simply the square annulus $2S(1)\setminus S(1)$, we can use Brownian scaling to conclude that

$$a'_N \;\le\; cN^{-b},$$

and the claim of Lemma 2.6.1 follows.

Let us turn to the proof of Lemma 2.6.2. By Lemma 2.3.1, for each box $S_M(k) = (k + N^b S(1)) \cap \mathbb{Z}^2$,

$$q|S_M| \ge \log Z_{S_M(k)} \ge q|S_M| - 8rN^b.$$

Consequently, for a mesoscopic $B = \bigcup_{k\in\mathcal{B}\subseteq\mathbb{Z}^2_M} S_M(k)$,

$$q|B| \geq \log Z_B \geq \sum_{k\in\mathcal{B}} \log Z_{S_M(k)} \geq q|B| - 8rN^{2-b}.$$

On the other hand, by the estimates (2.3.5) and (2.3.6),

$$-\,cN^{-\delta} \;\leq\; \frac{1}{N^2}\log\frac{\widehat{Z}_{B\vee B',+}}{\widehat{Z}_{B,+}\widehat{Z}_{B',+}} \;\leq\; cN^{-\delta}, \tag{7.1.11}$$

for any two disjoint mesoscopic regions B and B'. Consequently, a rerun of the subadditivity argument of Subsect. 4.3 reveals that the limit

$$\hat{q}_+ \;\stackrel{\Delta}{=}\; \lim_{N\to\infty}\frac{1}{N^2}\log\widehat{Z}_{N,+}$$

exists, and, moreover, any mesoscopic B with $|B| \geq tN^2$ satisfies

$$\left|\,\hat{q}_+ \,-\, \frac{1}{|B|}\log\widehat{Z}_{B,+}\right| \;\leq\; cN^{-\delta}. \tag{7.1.12}$$

Since by the definition $\Delta_f = J + \hat{q}_+ - q$, we conclude that any mesoscopic region $B \subseteq S_N$ with $|B| \geq tN^2$ satisfies,

$$\frac{1}{N^2}\left|\Delta_f|B| - \log\frac{Z_B}{\widehat{Z}_{B,+}} - J|B|\right| \leq cN^{-\delta}.$$

But, by the assumption, $\chi_N(B) \geq t$, so the latter inequality combined with Lemma 2.6.1 implies the desired estimate (2.6.3).

7.2. Approximation near the optimal shape. Let us assume that

$$\Delta_N(B) \stackrel{\Delta}{=} |E^v_{N,f}(B) - E^v_f(B_{\bar{a}})|$$

is small enough. Such an assumption clearly imposes restrictions on $\chi_N(B)$ from below, and all the results from the previous subsections apply. The proof of Lemma 2.6.3 follows closely the scheme developed in Sect. 5, and we shall use some of the notations introduced therein. In particular, we can restrict our attention to the case of simply connected domains. So, let

$$\overline{B} = \frac{1}{N}\bigcup_{k\in B}\left(k + \frac{1}{2}S(1)\right)$$

be connected and simply connected. Also let $\bar{\varrho} = \varrho(\overline{B})$ be the inradius of $\overline{B}$, and, to facilitate the notations, let us assume that $B_{\pi\bar{\varrho}^2}$ is the corresponding incircle. Note that due to the results of the previous subsection and the stability estimate (5.2.1), we may assume that

$$\bar{\varrho} \geq \frac{1}{2}\sqrt{\frac{\bar{a}}{\pi}} > 0. \tag{7.2.1}$$

Finally, let B_a^N be the discretization of B_a,

$$B_a^N \stackrel{\Delta}{=} NB_a \cap \mathbb{Z}^2,$$

and define

$$h^r_{v,N} = \overline{u}^v_{B^N_{\pi r^2},N}.$$

Set $\alpha = \chi_N(B^N_{\pi\overline{\varrho}^2})/\chi_N(B)$. Then,

$$\|\overline{u}^v_{B,N} - h_v\|_{L^1} \leq \|\overline{u}^v_{B,N} - h^{\overline{\varrho}}_{\alpha v,N}\|_{L^1} + \|h^{\overline{\varrho}}_{\alpha v,N} - h^{\overline{\varrho}}_{\alpha v}\|_{L^1} + \|h^{\overline{\varrho}}_{\alpha v} - h_v\|_{L^1}. \quad (7.2.2)$$

Since, $B^N_{\pi\overline{\varrho}^2} \subseteq B$, the first term on the right-hand side of (7.2.2) equals $(1 - \chi_N(B^N_{\pi\overline{\varrho}^2})/\chi_N(B))v$. However, in a view of Lemma 2.6.1 and (7.2.1),

$$\chi_N(B) - \chi_N(B^N_{\pi\overline{\varrho}^2}) \leq 2(\chi(\overline{B}) - \chi(B_{\pi\varrho^2})).$$

Proceeding as at the end of Subsect. 5.3, and using (2.6.3), we obtain that

$$\chi(\overline{B}) - \chi(B_{\pi\overline{\varrho}^2}) \leq c\sqrt[3]{\Delta_N(B)}.$$

By (7.2.1) both $\chi_N(B)$ and $\chi_N(B^N_{\pi\overline{\varrho}^2})$ are bounded below by a positive constant which does not depend on B and $\overline{\varrho}$. Consequently, there exists $c > 0$, such that

$$\|\overline{u}^v_{B,N} - h^{\overline{\rho}}_{\alpha v,N}\|_{L^1} \leq c\sqrt[3]{\Delta_N(B)}. \quad (7.2.3)$$

For the second term on the right-hand side of (7.2.2) we can simply use results on discretization errors [6]] for the Poisson equation on a regular domain $B_{\pi\overline{\varrho}^2}$, which assert that

$$\|h^{\overline{\varrho}}_{\alpha v,N} - h^{\overline{\varrho}}_{\alpha v}\|_{L^1} \leq c/N. \quad (7.2.4)$$

Finally, the remaining term on the right-hand side of (7.2.2) can be estimated exactly as it was done in Subsect. 5.3. Indeed, because of (7.2.1) and Lemma 2.6.2,

$$\Delta(\overline{B}) \leq 2\Delta_N(B),$$

and all the estimates of Subsect. 5.3 apply. Thus

$$\|h^{\overline{\varrho}}_{\alpha v} - h_v\|_{L^1} \leq c\sqrt[3]{\Delta_N(B)} \quad (7.2.5)$$

for some $c = c(v) > 0$. Substituting (7.2.3)–(7.2.5) into (7.2.2), and following the pattern laid down in Subsect. 5.4 to incorporate the not simply connected case, we arrive at the conclusion of Lemma 2.6.3.

8. Proof of Theorem A

8.1. The proof. In order to facilitate notations let us define

$$\Xi_N = \{X(\bullet) : \min_{x\in D} \|\xi_N - h_v(\bullet + x)\|_{L^1} \geq \nu_N\},$$

where, as before, ξ_N is the plaquette reconstruction from the random field $X(\bullet)$, h_v is the optimal harmonic shape given by (1.2.7) and the sequence $\{\nu_N\}$; $\lim \nu_N = 0$, is to be appropriately selected in the course of the proof. Our derivation of the asserted rate of convergence of $\widehat{\mathbb{P}}_{N,+}(\Xi_N \mid V_N \geq N^3 v)$ to zero is based on the disjoint decomposition of the event Ξ_N with respect to mesoscopic wetted regions:

$$\widehat{\mathbb{P}}_{N,+}\big(\, \Xi_N \,|\, V_N \geq N^3 v\big) \;=\; \frac{1}{\widehat{\mathbb{P}}_{N,+}\big(V_N \geq N^3 v\big)} \sum_{\substack{B \subseteq S_N \\ \text{mesoscopic}}} \widehat{\mathbb{P}}_{N,+}\big(\, \Xi_N \,;\, \mathcal{E}(v,B)\big), \tag{8.1.1}$$

where $\mathcal{E}(v,B) \stackrel{\Delta}{=} \{V_N \geq N^3 v\} \cap \{\mathcal{M}(\widetilde{X}_M) = B\}$.
The proof of the theorem comprises two estimates:

1. A lower bound on $\widehat{\mathbb{P}}_{N,+}\big(V_N \geq N^3 v\big)$
   and
2. Uniformly in mesoscopic regions $B \subseteq S_N$ an upper bound on
   $\widehat{\mathbb{P}}_{N,+}\big(\, \Xi_N \,;\, \mathcal{E}(B,v)\big)$.

Proposition 8.1.1.

$$\widehat{\mathbb{P}}_{N,+}\big(\, V_N \geq N^3 v\big) \;\geq\; \exp\big(\, -N^2(m_{N,f} + cN^{-\delta})\big), \tag{8.1.2}$$

where $m_{N,f} \stackrel{\Delta}{=} \min\{E^v_{N,f}(B)\,;\, B \subseteq S_N$ *mesoscopic*$\}$.

Proposition 8.1.2. *Let* $c > 0$ *and* $\delta > 0$ *be fixed to satisfy (8.1.2) above. There exists a sequence* $\{\nu_N\}$; $\lim_{N\to\infty} \nu_N = 0$, *such that for any* $B \subseteq S_N$ *mesoscopic,*

$$\widehat{\mathbb{P}}_{N,+}\big(\, \Xi_N \,;\, \mathcal{E}(B,v)\big) \;\leq\; \exp\big(\, -N^2(m_{N,f} + cN^{-\delta}) - cN^{2-b}\big). \tag{8.1.3}$$

Since, as we have seen before, the number of all mesoscopic subregions of S_N is bounded above by $\exp(N^{2(1-b)})$, the conclusion of Theorem A follows.

Proof of Proposition 8.1.1 Let B_N be an optimal mesoscopic region, i.e let

$$E^v_{N,f}(B_N) \;=\; m_{N,f}.$$

Then,

$$\begin{aligned} \widehat{\mathbb{P}}_{N,+}\big(\, V_N \geq N^3 v\big) \;&\geq\; \sum_{A \supseteq B_N} \mathrm{e}^{-J|A|} \frac{Z_A}{\widehat{Z}_{N,+}} \mathbb{P}_A\big(\, V_N \geq N^3 v \,;\, \Omega_+\big) \\ &\geq\; \sum_{A \supseteq B_N} \mathrm{e}^{-J|A|} \frac{Z_A}{\widehat{Z}_{N,+}} \mathbb{P}_A\big(\, V_N \geq N^3 v\,\big) \mathbb{P}_A\big(\, \Omega_+\big), \end{aligned} \tag{8.1.4}$$

where the second inequality follows from the FKG properties of $\mathbb{P}_A$. Moreover, for each $A \supseteq B_N$,

$$\mathbb{P}_A\big(\, V_N \geq N^3 v\,\big) \;\geq\; \mathbb{P}_{B_N}\big(\, V_N \geq N^3 v\,\big),$$

and, by Lemma 2.3.1 d),

$$\mathbb{P}_A\big(\, \Omega_+\big) \;\geq\; \mathbb{P}_{A\setminus B_N}\big(\, \Omega_+\big)\mathbb{P}_{B_N}\big(\, \Omega_+\big)\mathrm{e}^{-cN^{2-\delta}}.$$

Finally, as it was remarked in Sect. 4, $\mathbb{P}_{B_N}\big(\, \Omega_+\,\big) \geq \mathrm{e}^{-cN^{2-b}}$ uniformly in mesoscopic regions B. Consequently,

$$\begin{aligned} &\mathbb{P}_{N,+}\big(\, V_N \geq N^3 v\big) \\ &\geq\; \mathrm{e}^{-J|B_N| - cN^{2-\delta}} \mathbb{P}_{B_N}\big(\, V_N \geq\; N^3 v\big) \times \sum_{D \supseteq B_N^c} \mathrm{e}^{-J|D|} \frac{Z_{B_N \vee D}}{\widehat{Z}_{N,+}} \mathbb{P}_D\big(\, \Omega_+\big). \end{aligned}$$

By (7.1.11),

$$\sum_{D\supseteq S_N\setminus B_N} \mathrm{e}^{-J|D|}\, Z_D \mathbb{P}_D\big(\,\Omega_+\big) \;=\; \widehat{Z}_{B_N^c,+} \;\geq\; \frac{\widehat{Z}_{N,+}}{\widehat{Z}_{B_N,+}}\mathrm{e}^{-cN^{2-\delta}}.$$

Since, V_N is Gaussian under $\mathbb{P}_{B_N}$,

$$\mathbb{P}_{B_N}\big(\,V_N \;\geq\; N^3 v\big) \;\geq\; \exp\bigg(\,-N^2\frac{v^2}{2\chi_N(B_N)} - c\log N\bigg),$$

and the substitution of all the above reductions into (8.1.4) leads to

$$\begin{aligned}&\widehat{\mathbb{P}}_{N,+}\big(\,V_N \;\geq\; N^3 v\big)\\ &\geq\; \exp\bigg(\,-N^2\bigg(J\frac{|B_N|}{N^2} + \frac{1}{N^2}\log\frac{\widehat{Z}_{B_N,+}}{Z_{B_N}} + \frac{v^2}{2\chi_N(B_N)}\bigg) - cN^{2-\delta}\bigg).\end{aligned}$$

Equation (8.1.2) follows now by the definition (2.1.10) of $E^v_{N,f}$ and the optimal choice of B_N; $E^v_{N,f}(B_N) = m_{N,f}$

Proof of Proposition 8.1.2 We split all the mesoscopic wetted regions into two families:

$$\mathcal{F}_1 \;=\; \{B - \text{mesoscopic} \;:\; E^v_{N,f}(B) \;>\; m_{N,f} + \epsilon_N\}$$

and

$$\mathcal{F}_2 \;=\; \{B - \text{mesoscopic} \;:\; E^v_{N,f}(B) \;\leq\; m_{N,f} + \epsilon_N\},$$

where the sequence $\{\epsilon_N\}$ is to be specified later.

Estimates for $B \in \mathcal{F}_1$. From now on we pick a number γ';

$$\gamma + 2b \;<\; \gamma' \;<\; 1,$$

which is possible due to the choice of the scaling parameters b and γ in Subsect. 3.1.

Now,

$$\Omega_+ \cap \{\,\mathcal{M}(\widetilde{X}_M) \;=\; B\,\} \Longrightarrow \{X(k) \leq N^{\gamma'},\ \forall\, k \notin B\,\}. \tag{8.1.5}$$

Indeed, by Lemma 1.7.4 in [20], $\Gamma(M) \geq \delta/M = \delta N^{-2b}$. Therefore,

$$\{X(k) \geq N^{\gamma'}\} \;\Rightarrow\; \{\widetilde{X}_M(k) \geq \delta N^{\gamma'-2b}\} \Rightarrow \{\widetilde{X}_M(k) \gg N^{\gamma}\},$$

which contradicts the assumption $\{k \notin \mathcal{M}(\widetilde{X}_M)\}$.

Next notice that one can disregard mesoscopic B-s which are too small. For fix a small positive number t, and assume that $|B| \leq tN^2$. By (8.1.5),

$$\Omega_+ \cap \mathcal{E}(v, B) \Longrightarrow \{V_B \geq N^3 v - cN^{2+\gamma'} \geq N^3 v(1 - cN^{-\delta})\},$$

where $V_B \triangleq \sum_{k\in B} X(k)$. However, for each $A \subseteq S_N$, V_B is a zero mean Gaussian under $\mathbb{P}_A$ with the variance bounded above by

$$\sum_{k,l\in B} G_{S_N}(k,l) \;\leq\; 2N^2|B| \;\leq\; 2tN^4.$$

Therefore,

$$\widehat{\mathbb{P}}_N\big(\mathcal{E}(v,B)\,;\,\Omega_+\big)\;\le\;\exp\Big(-\frac{v^2N^2}{4t}\Big).$$

Thus, for t small enough (8.1.3) is automatically satisfied.

With such a small positive t fixed we can proceed to consider only those $B\in\mathcal{F}_1$, for which $|B|\ge tN^2$. We have:

$$\begin{aligned}\widehat{\mathbb{P}}_{N,+}\big(\,\Xi_N\,;\,\mathcal{E}(v,B)\big)\;&\le\widehat{\mathbb{P}}_{N,+}\big(\,\mathcal{E}(v,B)\big)\\ &=\sum_A \mathrm{e}^{-J|A|}\frac{Z_A}{\widehat{Z}_{N,+}}\mathbb{P}_A\big(\,\mathcal{M}(\widetilde{X}_M)=B\,;\,V_N\ge N^3v\,;\,\Omega_+\big).\end{aligned}\tag{8.1.6}$$

Due to the volume filling estimate (2.2.7), any substantial contribution to the sum above can come only from those A-s, which satisfy $|B\setminus A|\le cN^{2-\delta}$. Also, in a view of (8.1.5), we can further develop the right-hand side of (8.1.6) using:

$$\begin{aligned}&\mathbb{P}_A\big(\,\mathcal{M}(\widetilde{X}_M)=B\,;V_N\ge N^3v\,;\,\Omega_+\big)\\ \le\;& P_A\big(\,X|_{A\setminus B}\le N^{\gamma'}\,;\,\Omega_+^{A\setminus B}\,;\,V_B\ge N^3(v-cN^{-\delta})\big).\end{aligned}$$

We want to condition on the values of spins at $\partial_{A\setminus B}B$, which are known to stay below $N^{\gamma'}$, to decouple between events over $A\cap B$ and $A\setminus B$. As in Subsect. 4.4 let $\mathbb{P}_{A\cap B,x}$ denote the Gaussian measure on $A\cap B$ subject to boundary conditions x on $S_N\setminus(A\cap B)$. Clearly, for each number $a\in\mathbb{R}$,

$$\max_{x\in[0,N^{\gamma'}]^{A\setminus B}}\mathbb{P}_{A\cap B,x}\big(\,V_B\ge a\,\big)\;\le\;\mathbb{P}_{A\cap B}\big(\,V_N\ge a-cN^{2+\gamma'}\,\big).$$

Therefore,

$$\mathbb{P}_A\big(\,\mathcal{E}(v,B)\,;\,\Omega_+\big)\;\le\;\mathbb{P}_{A\cap B}\big(\,V_N\ge vN^3(1-cN^{-\delta})\big)\mathbb{P}_A\big(\,\Omega_+^{A\setminus B}\big).$$

Finally, as it becomes apparent from the proof of Lemma 2.3.1 d) in Subsect. 4.4,

$$\mathbb{P}_A\big(\,\Omega_+^{A\setminus B}\big)\;\le\;\mathrm{e}^{cN^{2-\delta}}\mathbb{P}_{A\setminus B}\big(\,\Omega_+\big),$$

and, of course, $\mathbb{P}_{A\cap B}\big(\,V_N\ge a\big)\le\mathbb{P}_B\big(\,V_N\ge a\big)$ for each number $a\in\mathbb{R}$.

Proceeding as in the proof of Proposition 8.1.1 we, therefore, obtain:

$$\widehat{\mathbb{P}}_{N,+}\big(\,\Xi_N\,;\,\mathcal{E}(v,B)\big)\;\le\;\exp\big(-N^2E^v_{N,f}(B)\;+\;cN^{2-\delta}\big).\tag{8.1.7}$$

Since it was assumed, that $E^v_{N,f}(B)\ge m_{N,f}+\epsilon_N$, we deduce from (8.1.2) and (8.1.7) that

$$\widehat{\mathbb{P}}_{N,+}\big(\,\Xi_N\,;\,\mathcal{E}(v,B)\big)\;\le\;\exp\big(\,-N^2(\epsilon_N-cN^{-\delta})\big)\widehat{\mathbb{P}}_{N,+}\big(V_N\ge N^3v\big).$$

Then, the choice $\epsilon_N=cN^{-\delta}\;+\;cN^{-b}$ does the job.

Estimates for $B\in\mathcal{F}_2$ So let $B\subseteq S_N$ be such, that

$$E^v_{N,f}(B)\;\le\;m_{N,f}\;+\;\epsilon_N.\tag{8.1.8}$$

This, of course, imposes a restriction on $|B|$ from below; for example $|B|\ge N^2\overline{a}/2$ for N large enough. We proceed exactly as in the $\mathcal{F}_1$ case to conclude, that uniformly in B-s satisfying (8.1.8) and in $A\subseteq S_N$; $|B\setminus A|\le cN^{2-\delta}$,

$$\mathbb{P}_A\big(\Xi_N;\mathcal{E}(v,B);\Omega_+\big) \\ \leq \mathrm{e}^{cN^{2-\delta}}\mathbb{P}_{A\setminus B}\big(\Omega_+\big)\max_{x\in[0,N^{\gamma'}]^{S_N\setminus(A\cap B)}}\mathbb{P}_{A\cap B,x}\big(\Xi_N;V_N\geq N^3v(1-cN^{-\delta})\big).$$

If, $E^v_{N,f}(A\cap B)\geq m_{N,f}+\epsilon_N$, then the corresponding term in the expansion

$$\widehat{\mathbb{P}}_{N,+}\big(\,\Xi_N\,;\,\mathcal{E}(v,B)\big) \;=\; \sum_A \mathrm{e}^{-J|A|}\frac{Z_A}{\widehat{Z}_{N,+}}\mathbb{P}_A\big(\,\Xi_N\,;\,\mathcal{E}(v,B)\,;\,\Omega_+\big)$$

can be treated as in the $\mathcal{F}_1$ case. Thus, it remains to consider only those A-s, for which

$$E^v_{N,f}(A\cap B)\;<\;m_{N,f}\;+\;\epsilon_N. \tag{8.1.9}$$

Only at this stage the event Ξ_N at last comes into play. Notice that for such an almost optimal $A\cap B$ the profile $\overline{u}^v_{A\cap B,N}$ is already very close to the profile $\overline{u}^v_{B,N}$, and hence, by the approximation and stability results of Sects. 5 and 7, to some shift of h_v itself. More precisely,

$$\|\overline{u}^v_{B,N}\;-\;\overline{u}^v_{A\cap B,N}\|_{L^1} \\ \leq\;\left\|\overline{u}^v_{B,N}-\frac{\chi_N(A\cap B)}{\chi_N(B)}\overline{u}^v_{A\cap B,N}\right\|_{L^1}+\left\|\left(1-\frac{\chi_N(A\cap B)}{\chi_N(B)}\right)\overline{u}^v_{A\cap B,N}\right\|_{L^1}.$$

However, since $\chi_N(A\cap B)\leq\chi_N(B)$, and the integral of a positive function $\overline{u}^v_{D,N}$ is v regardless of the region $D\subseteq S_N$, one infers that

$$\|\overline{u}^v_{B,N}-\overline{u}^v_{N,A\cap B}\|_{L^1}\leq 2v\left(1-\frac{\chi_N(A\cap B)}{\chi_N(B)}\right).$$

On the other hand, the fact that $|B\setminus A|\leq cN^{2-\delta}$ in conjunction with the random walk representation of Sect. 4 and withthe inequalities (8.1.8) and (8.1.9) implies that

$$|E^v_{N,f}(B)\;-\;E_{N,f}(A\cap B)|\;\leq\;2\epsilon_N,$$

and that $\chi_N(B)\geq v^2/4m_{N,f}$. Consequently,

$$\|\overline{u}^v_{A\cap B,N}\;-\;\overline{u}^v_{B,N}\|_{L^1}\;\leq\;c\epsilon_N,$$

and, by virtue of the stability result (2.6.4),

$$\min_{x\in S(1)}\|\overline{u}^v_{A\cap B,N}\;-\;h_v(\bullet+x)\|_{L^1}\;\leq\;c\sqrt[6]{\epsilon_N}. \tag{8.1.10}$$

Thus, it remains to give an estimate on

$$\mathbb{P}_{A\cap B,x}\big(\|\xi_N\;-\;\overline{u}_{A\cap B,N}\|_{L^1}\;\geq\;\nu_N\;;\;V_N\geq N^3v(1-cN^{-\delta})\big),$$

uniformly in A,B and boundary conditions $x\in[0,N^{\gamma'}]^{S_N\setminus(A\cap B)}$, and then to choose the sequence $\{\nu_N\}$ in accordance with all the restrictions imposed by different estimates involved. In fact we can reduce the bounds for different x-s to a single estimate at $x\equiv 0$. Indeed, fix an $x\in[0,N^{\gamma'}]^{S_N\setminus(A\cap B)}$ and define $\overline{u}^x$ to be the plaquette reconstruction of the solution to the (discrete) harmonic equation on $A\cap B$ with boundary conditions x. Then, under $\mathbb{P}_{A\cap B,x}$, the field $\tilde{X}(\bullet)\stackrel{\Delta}{=}X(\bullet)-\overline{u}^x(\bullet)$ is Gaussian with zero boundary conditions on $S_N\setminus(A\cap B)$. Therefore, since by the maximum principle $0\leq\overline{u}^x\leq N^{\gamma'}$,

$$\begin{aligned}&\mathbb{P}_{A\cap B,x}\big(\|\xi_N - \bar{u}_{A\cap B,N}\|_{L^1} \geq \nu_N\ ;\ V_N \geq N^3 v(1-cN^{-\delta})\big)\\&\quad\leq \mathbb{P}_{A\cap B}\big(\|\xi_N - \bar{u}_{A\cap B,N}\|_{L^1} \geq \nu_N - N^{\gamma'-1}\ ;\ V_N \geq N^3 v(1-cN^{-\delta}) - N^{\gamma'-1}\big).\end{aligned}$$

Combining the latter estimate with (8.1.10), we see that the sequence $\{\nu_N\}$ should satisfy

$$N^{\gamma'-1} \vee c\sqrt[6]{\epsilon_N} \ \ll\ \nu_N \ \ll\ 1. \tag{8.1.11}$$

However, once the choice of $\{\nu_N\}$ complies with (8.1.11), we are entitled to use the concentration estimates of Sect. 6 to assert that for each $B \in \mathcal{F}_2$ and each $A \subseteq S_N$, such that $|B \setminus A| \leq cN^{2-\delta}$,

$$\widehat{\mathbb{P}}_N\big(\ \varXi_N\ ;\ \mathcal{E}(v,B)\ \big) \leq\ \exp\Big(-N^2(m_{N,f} - cN^{-\delta} + \epsilon_N \wedge \frac{\nu_N^2}{c})\Big).$$

Recall that we have already chosen $\epsilon_N = c(N^{-b} + N^{-\delta})$. Then,

$$\nu_N\ =\ \sqrt[12]{\epsilon_N} \vee N^{(\gamma'-1)/2},$$

both satisfies the requirement (8.1.11) and leads to the desired estimate (8.1.3).

8.2. Positivity of Δ_f. The fact that the limit in (1.3.4) is well defined was established in the end of Subsect. 7.1.

One can rewrite $\Delta_{f,N}$ as

$$\Delta_{f,N}\ =\ \Delta_{f,N}(J)\ =\ \frac{1}{|S_N|}\log\frac{\widehat{Z}_{N,+}}{\mathrm{e}^{-J|S_N|}Z_N\mathbb{P}_{S_N}(\Omega_+)}\ +\ \frac{1}{|S_N|}\log\mathbb{P}_{S_N}(\Omega_+),$$

where

$$\widehat{Z}_{N,+}\ =\ \widehat{Z}_{N,+}(J)\ =\ \sum_{A\subseteq S_N}\mathrm{e}^{-J|A|}Z_A\mathbb{P}_A(\Omega_+).$$

However, by the results of [10],

$$\lim_{N\to\infty}\frac{1}{|S_N|}\log\mathbb{P}_{S_N}(\Omega_+)\ =\ 0.$$

Consequently, Δ_f is nonnegative.

Differentiating $\Delta_{f,N}$ with respect to J, we obtain:

$$\frac{d}{dJ}\Delta_{f,N}\ =\ 1\ -\ \frac{1}{|S_N|}\widehat{\mathbb{E}}_{N,+}|\mathcal{D}|, \tag{8.2.1}$$

and

$$\frac{d^2}{d^2J}\Delta_{f,N}\ =\ \frac{1}{|S_N|}\mathbb{V}\mathrm{ar}_{N,+}(|\mathcal{D}|),$$

where $\mathcal{D}$ is the random microscopic wetted region. Since $\mathcal{D} \subseteq S_N$, $\Delta_{f,N}$ is nondecreasing and convex. Moreover, (8.2.1) above clearly indicates that the question of whether $\Delta_{f,N} > 0$ for all $J \in \mathbb{R}$ or not is essentially the question of the wetting transition in our model. We do not attempt to solve it here - such a computation would involve a rather delicate analysis of the entropic repulsion phenomena for two-dimensional Gaussian fields with 0-boundary conditions, which would be closer in spirit to [5] than to the problems we are addressing in this article. Instead we shall give a rather crude and straightforward proof of the positivity of Δ_f for large enough values of J. Namely, we claim that

$$\Delta_f(J) \;>\; \frac{J - \log\sqrt{2} - q}{2}, \tag{8.2.2}$$

where q is defined in (4.2.1). Indeed, by the results of Subsect. 4.2,

$$\log Z_N \;\leq\; |S_N| \left(\frac{1}{2} \log \frac{\pi}{2} \;+\; q \right).$$

On the other hand, a trivial computation shows that for every $A \subseteq S_N$,

$$Z_A \mathbb{P}_A(\Omega_+) \;\geq\; \left(\int_0^\infty \mathrm{e}^{-\frac{x^2}{2}} dx \right)^{|A|} \;=\; \exp\left\{ \frac{|A|}{2} \log \frac{\pi}{4} \right\}.$$

In particular, for any $A \subseteq S_N$ such that $|A| = \frac{|S_N|}{2}$,

$$\log \frac{\mathrm{e}^{-J|A|} Z_A \mathbb{P}_A(\Omega_+)}{\mathrm{e}^{-J|S_N|} Z_N} \;\geq\; \frac{|S_N|}{2} \big(J \;-\; \log\sqrt{2} \;-\; q \big),$$

and (8.2.2) follows.

Acknowledgement. D.I. wants to thank Michiel van den Berg for providing important references about torsional rigidity and spectral properties of discrete Laplacians, and Anton Bovier for several useful conversations they had.

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Communicated by J.L. Lebowitz