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A Characterization of Affine Kac-Moody Lie Algebras

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Abstract: We give a new characterization of the affine Kac-Moody algebras in terms of extended affine Lie algebras. We also present new realizations of the twisted affine Kac-Moody algebras.

Introduction

The purpose of this paper is twofold. We present a new characterization of the affine Kac-Moody Lie algebras and then go on to give some new realizations for the twisted affine algebras.

This work grew out of our study, in [AABGP], of extended affine Lie algebras (EALA's, for short) and their root systems. EALA's were first introduced in [H-KT] by R. Høegh-Krohn and B. Torresani (under the name irreducible quasisimple Lie algebras) as a generalization of finite dimensional simple Lie algebras and affine Kac-Moody Lie algebras. Thereafter classifications of tame EALA's of simply-laced type (except A_1) were carried out in [BGK] and [BGKN]. In [AABGP], we developed the basic structure theory of EALA's; gave a satisfying picture and many classification results for their root systems; and introduced many new examples of these algebras.

The characterization of affine Lie algebras proved in this paper says that a Lie algebra $\mathcal{L}$ over the complex field is an affine Kac-Moody Lie algebra if and only if it is a tame EALA with nullity $\nu = 1$. Moreover the realizations we give for the twisted algebras show how they can be viewed as Lie algebras arising from some of the usual and well-known constructions of finite dimensional simple Lie algebras in characteristic 0 studied by N. Jacobson, G. Seligman, and J. Tits, among others. Thus, for example, we are able to show that the twisted affine algebra $F_4^{(2)}$ (using the notation from [MP]) has structure tied up with a Jordan algebra while that of $G_2^{(3)}$ is connected with a Cayley algebra.

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The algebras $B_l^{(2)}$ ($l \geq 2$), $C_l^{(2)}$ ($l \geq 3$) and $BC_l^{(2)}$ ($l \geq 1$) are dealt with using matrix constructions.

To describe our main result in more detail, we briefly recall some terminology from [AABGP]. We begin with a complex Lie algebra $\mathcal{L}$ which has a non-degenerate invariant symmetric bilinear form and a self-centralizing finite dimensional diagonalizable abelian subalgebra $\mathcal{H}$. One then gets a root space decomposition of $\mathcal{L}$ and assumes that if x is an element in a root space of a non-isotropic root then ad x acts locally nilpotently on $\mathcal{L}$. One also assumes that the set R of roots of $\mathcal{L}$ is a discrete subset of $\mathcal{H}^*$, that the set $R^\times$ of non-isotropic roots is indecomposable and that there are no isolated isotropic roots. Such an algebra is called an *extended affine Lie algebra*. Let $\mathcal{V}$ be the real span of the roots. This is a positive semi-definite space and the *nullity* ν of R is by definition the dimension of the radical of this space. Finally let $\mathcal{L}_c$ be the subalgebra of $\mathcal{L}$ generated by the non-isotropic root spaces of $\mathcal{L}$ and call this the *core* of $\mathcal{L}$. Then $\mathcal{L}_c$ is in fact an ideal of $\mathcal{L}$, and $\mathcal{L}$ is called *tame* if the kernel of the representation $\rho : \mathcal{L} \to \mathrm{End}(\mathcal{L}_c)$ given by $\rho(x) = \mathrm{ad}\, x|_{\mathcal{L}_c}$ is just the center of $\mathcal{L}_c$. Our main result then says that such an algebra (i.e., a tame EALA of nullity one) is an affine Kac-Moody Lie algebra. (The converse of this result follows from well known properties of affine Lie algebras.) Our proof relies on some of the general results in [AABGP] dealing with EALA's as well as some results about the root systems involved. We think this result is interesting in its own right but note here that it also clearly identifies the affine Kac-Moody algebras within the class of tame EALA's and hence shows how the latter algebras can be considered as natural generalizations of the former.

Other characterizations of the affine Kac-Moody Lie algebras are known. For example, from the deep and beautiful work of O. Mathieu in [Ma1-3] which extends earlier work of Kac in [K3] one finds the following result. If $L = \oplus_{n\in\mathbb{Z}} L_n$ is a simple $\mathbb{Z}$-graded Lie algebra which is infinite dimensional in both directions and if the dimensions of L_n are uniformly bounded then either L is a Witt algebra or an affine Lie algebra. Moreover, the Witt algebra does not admit a non-degenerate invariant form. Of course, here, in this characterization, one means by affine Lie algebra the loop version, so there is no center and no degree derivation added. There are some other interesting characterizations within Kac-Moody Lie algebras. One, given in [BC], is in terms of universal enveloping algebras: If $\mathfrak{g}$ is a symmetrizable Kac-Moody Lie algebra then its universal enveloping algebra is an Ore domain which is not Noetherian if and only if $\mathfrak{g}$ is an affine Kac-Moody Lie algebra. Another characterization, given in [G], is in terms of the second homology group of Lie algebras: If $\mathfrak{g}$ is a Kac-Moody Lie algebra (not necessarily symmetrizable) then the second homology group $H_2(\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}])$ of the Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ is infinite dimensional if and only if $\mathfrak{g}$ is affine.

The paper is organized as follows. In Sect. 1 we will recall the terminology, notation and results that we need from [AABGP]. Our main result, the characterization of affine Lie algebras, is contained in Sect. 2. This depends on numerous results about root systems as well as some results about sl_2-triples which appear in our algebras. Finally in Sect. 3 we present our realizations following along the lines in [AABGP].

Throughout the paper, *all algebras will be over the field* $\mathbb{C}$ *of complex numbers.* By an *affine Kac-Moody Lie algebra* we will mean a minimally realized affine Kac-Moody Lie algebra as defined in [MP, Sects. 4.1–4.3] or, equivalently, an affine Kac-Moody Lie algebra as defined in [K2, Chapter 6]. Such a Lie algebra then has a 1-dimensional center and a derived algebra of codimension 1.

1. Extended Affine Lie Algebras and their Root Systems

In this section, we recall the definitions and facts that we will need from [AABGP].

An *extended affine Lie algebra* (EALA, for short) is a triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ consisting of two complex Lie algebras $\mathcal{H} \subseteq \mathcal{L}$ and a symmetric bilinear form $(\cdot,\cdot) : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ such that the axioms (EA1) through (EA5) described below hold. The first two axioms are

(EA1) *The form* $(\cdot,\cdot)$ *is non-degenerate and invariant.*

(EA2) $\mathcal{H} \neq (0)$ *is finite dimensional, abelian, and self-centralizing. Moreover* ad $h \in$ *End*$(\mathcal{L})$ *is diagonalizable for all* $h \in \mathcal{H}$.

From (EA2) we obtain the usual root space decomposition

$$\mathcal{L} = \oplus_{\alpha \in \mathcal{H}^*} \mathcal{L}_\alpha, \tag{1.1}$$

where $\mathcal{L}_\alpha = \{x \in \mathcal{L} | [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}$. The *root system*

$$R = \{\alpha \in \mathcal{H}^* | \mathcal{L}_\alpha \neq (0)\} \tag{1.2}$$

is divided into *isotropic* and *non-isotropic* roots,

$$R = R^0 \uplus R^\times$$

where $R^0 = \{\alpha \in R | (\alpha, \alpha) = 0\}$ and $R^\times = \{\alpha \in R | (\alpha, \alpha) \neq 0\}$. Note that $0 \in R^0$ and that $\mathcal{L}_0 = \mathcal{H}$.

Reasoning along standard lines we obtain that

$$(\mathcal{L}_\alpha, \mathcal{L}_\beta) = (0) \quad \text{unless } \alpha + \beta = 0. \tag{1.3}$$

In particular the restriction of $(\cdot,\cdot)$ to $\mathcal{H} \times \mathcal{H}$ is non-degenerate. This allows us to identify $\mathcal{H}$ with $\mathcal{H}^*$. For $\alpha \in \mathcal{H}^*$ let $t_\alpha \in \mathcal{H}$ be given by

$$\alpha(h) = (t_\alpha, h) \quad \text{for all } h \in \mathcal{H}. \tag{1.4}$$

The map $\alpha \to t_\alpha$ is an isomorphism and allows us to transfer the form to $\mathcal{H}^*$ by setting

$$(\alpha, \beta) = (t_\alpha, t_\beta) \quad \text{for all } \alpha, \beta \in \mathcal{H}^*. \tag{1.5}$$

It is easy to see that

$$[x_\alpha, x_{-\alpha}] = (x_\alpha, x_{-\alpha}) t_\alpha$$

for $x_\alpha \in \mathcal{L}_\alpha, x_{-\alpha} \in \mathcal{L}_{-\alpha}$. Thus

$$[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}] = \mathbb{C} t_\alpha \quad \text{for all } \alpha \in R. \tag{1.6}$$

The next axiom

(EA3) $\text{ad}_{\mathcal{L}}(x)$ *is locally nilpotent for all* $x \in \mathcal{L}_\alpha$ *and* $\alpha \in R^\times$

allows us to construct automorphisms of $\mathcal{L}$ of the form

$$\exp \text{ad}\, t e_\alpha \exp \text{ad}(-t^{-1} f_\alpha) \exp \text{ad}\, t e_\alpha \tag{1.7}$$

whenever $e_\alpha \in \mathcal{L}_\alpha, f_\alpha \in \mathcal{L}_{-\alpha}, \alpha \in R^\times$, and $t \in \mathbb{C}^\times$. Armed with this we obtain the following (see Theorem I.1.29 in [AABGP]):

Theorem 1.8. *Suppose that $\mathcal{L}$ satisfies (EA1)–(EA3). Let $\alpha \in R^\times$ be non-isotropic. Then*

- $\mathbb{Z}$ *for all $\beta \in R$.* (1.9)
- *The map*

$$w_\alpha : x \to x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha \tag{1.10}$$

is a reflection in α of $\mathcal{H}^*$ stabilizing R.

- $\mathbb{C}\alpha \cap R = \{\alpha, 0, -\alpha\}$. (1.11)
- $\dim_{\mathbb{C}} \mathcal{L}_\alpha = 1$ *and $\mathcal{L}_{-\alpha} \oplus \mathbb{C}t_\alpha \oplus \mathcal{L}_\alpha$ is a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.* (1.12)
- *For any $\beta \in R$ there exist two non-negative integers u and d such that for $n \in \mathbb{Z}$ we have*

$$\beta + n\alpha \in R \quad \Leftrightarrow \quad -d \leq n \leq u. \tag{1.13}$$

Moreover, $d - u = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

Let $\mathcal{W} = \mathcal{W}_{\mathcal{L}}$ be the subgroup of $GL(\mathcal{H}^*)$ generated by the reflections w_α for $\alpha \in R^\times$. $\mathcal{W}$ is called the *Weyl group* of our algebra $\mathcal{L}$.

The remaining axioms relate the geometry associated with the form $(\cdot,\cdot)$ to the topology of Euclidean spaces.

(EA4) *R is discrete subset of $\mathcal{H}^*$.*

(EA5) *$R^\times$ is indecomposable (i.e., $R^\times$ cannot be decomposed as a disjoint union $R_1 \uplus R_2$, where R_1 and R_2 are nonempty subsets of $R^\times$ satisfying $(R_1, R_2) = \{0\}$) and R^0 has no isolated roots (i.e., given $\sigma \in R^0$ there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$).*

Assume now that *$\mathcal{L}$ is an EALA*, or in other words that $\mathcal{L}$ satisfies the axioms (EA1)–(EA5). From these axioms we obtain the following crucial fact about R^0 (see Proposition I.2.1 in [AABGP]):

$$(R, R^0) = (0). \tag{1.14}$$

Further if we let

$$\mathcal{V} = \sum_{\alpha \in R} \mathbb{R}\alpha. \tag{1.15}$$

then

Proposition 1.16. *The form $(\cdot,\cdot)$ can be scaled so that its restriction to $\mathcal{V}$ is real valued and positive semidefinite.*

This fact was an assumption about $(\cdot,\cdot)$ made in [H-KT]. Therein the authors reported that V. Kac had conjectured that it follows from (EA1)–(EA5). This was proved in Theorem I.2.14 of [AABGP]. From now on *we assume that our form $(\cdot,\cdot)$ is scaled so that (1.16) holds.*

At the outset we thus have

(R0) *$\mathcal{V}$ is a finite dimensional real space, $(\cdot,\cdot)$ is a positive semidefinite symmetric bilinear form on $\mathcal{V}$, and $R = R^\times \uplus R^0$, where*

$$R^\times = \{\alpha \in R : (\alpha,\alpha) \neq 0\} \quad \textit{and} \quad R^0 = \{\alpha \in R : (\alpha,\alpha) = 0\}.$$

Moreover it follows from the axioms (EA1)–(EA5) and Theorem 1.8 that we have:

(R1) $0 \in R$.
(R2) $-R = R$.

(R3) *R spans $\mathcal{V}$.*
(R4) $\alpha \in R^\times \Rightarrow 2\alpha \notin R$.
(R5) *R is discrete in $\mathcal{V}$.*
(R6) *If $\alpha \in R^\times$ and $\beta \in R$, then there exist non-negative integers d and u so that*

$$\{\beta + n\alpha : n \in \mathbb{Z}\} \cap R = \{\beta - d\alpha, ..., \beta + u\alpha\} \quad \text{and} \quad d - u = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}.$$

(R7) *$R^\times$ cannot be decomposed as a disjoint union $R_1 \uplus R_2$, where R_1 and R_2 are nonempty subsets of $R^\times$ satisfying $(R_1, R_2) = \{0\}$.*
(R8) *For any $\sigma \in R^0$, there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R$.*

The properties just listed suggest the following definition. An *extended affine root system* (EARS for short) is defined to be a subset R of a real vector space $\mathcal{V}$ with a form $(\cdot,\cdot)$ so that the axioms (R0)–(R8) hold. Thus, we have just seen that the root system of an EALA is an EARS.

Assume now that R is an arbitrary EARS in a real vector space $\mathcal{V}$. So in particular our discussion will apply to the root system of an EALA $\mathcal{L}$. We now describe how R can be constructed from a finite root system and semilattices [AABGP, Chapter II].

Let

$$\mathcal{V}^0 = \{x \in \mathcal{V} | (x, \mathcal{V}) = (0)\}.$$

Set $\overline{\mathcal{V}} = \mathcal{V}/\mathcal{V}^0$ and let $^- : \mathcal{V} \to \overline{\mathcal{V}}$ be the canonical map. Since our form is positive semidefinite we have

$$\mathcal{V}^0 = \{\alpha \in \mathcal{V} | (\alpha,\alpha) = 0\} \quad \text{and} \quad R^0 = R \cap \mathcal{V}^0.$$

We define the *nullity* of R to be the (real) dimension ν of $\mathcal{V}^0$. If R is the root system of an EALA $\mathcal{L}$, ν is also called the *nullity* of $\mathcal{L}$.

Next one has that $\overline{R} = \{\overline{\alpha} | \alpha \in R\}$ is a finite irreducible (not necessarily reduced) root system [AABGP, Prop. II.2.9]. (Here we depart from [Bou] in assuming that an irreducible finite root system contains 0.) We define the *finite type* of R, which we usually refer to simply as the *type* of R, to be the type X_l of the finite root system $\overline{R}$. If R is the root system of an EALA $\mathcal{L}$, X_l is also called the *type* of $\mathcal{L}$.

We now want to lift $\overline{R}$ to a root system in $\mathcal{V}$. Fix a base $\overline{\Pi} = \{\overline{\alpha}_1, \ldots, \overline{\alpha}_l\}$ for $\overline{R}$ and choose $\dot\alpha_i$ in R so that $\overline{\dot\alpha_i} = \overline{\alpha}_i$. Let $\dot{\mathcal{V}}$ be the real span of the $\dot\alpha_i$'s. Then,

$$\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^\circ.$$

and $^-$ restricts to an isometry of $\dot{\mathcal{V}}$ onto $\overline{\mathcal{V}}$. If we let

$$\dot R = \{\dot\alpha \in \dot{\mathcal{V}} : \dot\alpha + \sigma \in R \text{ for some } \sigma \in \mathcal{V}^\circ\},$$

then $\dot R \simeq \overline{R}$ (isomorphism of root systems).

For $\dot\alpha \in \dot R^\times = \dot R \setminus \{0\}$, we define

$$S_{\dot\alpha} = \{\sigma \in \mathcal{V}' : \dot\alpha + \sigma \in \mathcal{R}\}.$$

Then,

$$R = R^0 \cup \Big(\bigcup_{\dot\alpha \in \dot R^\times} (\dot\alpha + S_{\dot\alpha})\Big).$$

By means of the Weyl group one can see that $S_{\dot\alpha}$ depends only on the length of $\dot\alpha$ (Prop. II.2.15 in [AABGP]). Set S (respectively L, E) to be $S_{\dot\alpha}$ whenever $\dot\alpha$ is short (respectively long, extra-long). (Here, $\dot\alpha \in \dot R^\times$ is said to be *short* if it has minimal length in $\dot{R}^\times$, *extra long* if it is twice a short root of $\dot{R}^\times$, and *long* if it is neither short nor extra long.) Then

$$R = (S + S) \cup \Big(\bigcup_{\dot\alpha \in \dot{R}_{sh}} (\dot\alpha + S)\Big) \cup \Big(\bigcup_{\dot\alpha \in \dot{R}_{lg}} (\dot\alpha + L)\Big) \cup \Big(\bigcup_{\dot\alpha \in \dot{R}_{ex}} (\dot\alpha + E)\Big), \tag{1.17}$$

where $\dot{R}^\times = \dot{R}_{sh} \uplus \dot{R}_{lg} \uplus \dot{R}_{ex}$ is the decomposition of $\dot{R}^\times$ according to length. Note that $R^0 = S + S$. One also knows that $L \subseteq S$ and $E \subseteq S$ and so S, L and E consist of isotropic roots.

The key feature here is that S and L are *semilattices*. That is, they are subsets $\mathcal{S}$ of $\mathcal{V}^0$ such that

(S1) $0 \in \mathcal{S}$.
(S2) $-\mathcal{S} = \mathcal{S}$.
(S3) $\mathcal{S} + 2\mathcal{S} \subseteq \mathcal{S}$.
(S4) $\mathcal{S}$ spans $\mathcal{V}^0$.
(S5) $\mathcal{S}$ is discrete in $\mathcal{V}^0$.

As for E it is a *translated semilattice* (i.e., E is nonempty and (S2)–(S5) hold) which satisfies $E \cap 2S = \emptyset$.

Up to this point, we have shown how to decompose any EARS as in (1.17) using a finite root system and up to 3 semilattices or translated semilattices of isotropic roots. Conversely, we can use a finite root system and semilattices to construct EARS:

Construction 1.18. Suppose that $\dot{R}$ is an irreducible finite root system of type X_l in a finite dimensional real vector space $\dot{\mathcal{V}}$ with positive definite symmetric bilinear form $(\cdot,\cdot)$. We decompose the set $\dot{R}^\times$ of nonzero elements of $\dot{R}$ according to length as $\dot{R}^\times = \dot{R}_{sh} \uplus \dot{R}_{lg} \uplus \dot{R}_{ex}$. Let $\mathcal{V}^0$ be a finite dimensional real vector space, let $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$, and extend $(\cdot,\cdot)$ to $\mathcal{V}$ in such a way that $(\mathcal{V}, \mathcal{V}^0) = \{0\}$.

(a) (*The simply laced construction*). Suppose that X_l is simply laced, i.e. $X_l = A_l$ ($l \geq 1$), D_l ($l \geq 4$), E_6, E_7 or E_8. Suppose that S is a semilattice in $\mathcal{V}^0$. If $X_l \neq A_1$ suppose further that S is a lattice in $\mathcal{V}^0$. Put

$$R = R(X_l, S) := (S + S) \cup \Big(\bigcup_{\dot\alpha \in \dot{R}^\times} (\dot\alpha + S)\Big).$$

(b) (*The reduced nonsimply laced construction*). Suppose that X_l is reduced and non-simply laced, i.e. $X_l = B_l$ ($l \geq 2$), C_l ($l \geq 3$), F_4 or G_2. Suppose that S and L are semilattices in $\mathcal{V}^0$ so that

$$L + kS \subseteq L \quad \text{and} \quad S + L \subseteq S,$$

where $k = 2$ if $X_l = B_l$ ($l \geq 2$), C_l ($l \geq 3$) or F_4, and $k = 3$ if $X_l = G_2$. Further, if $X_l = B_l$ ($l \geq 3$) suppose that L is a lattice, if $X_l = C_l$ ($l \geq 3$) suppose that S is a lattice, and if $X_l = F_4$ or G_2 suppose that both S and L are lattices. Put

$$R = R(X_l, S, L) := (S + S) \cup \Big(\bigcup_{\dot\alpha \in \dot{R}_{sh}} (\dot\alpha + S)\Big) \cup \Big(\bigcup_{\dot\alpha \in \dot{R}_{lg}} (\dot\alpha + L)\Big).$$

(c) (*The BC_l construction*, $l \geq 2$). Suppose that $X_l = BC_l$ ($l \geq 2$). Suppose that S and L are semilattices in $\mathcal{V}^0$ and E is a translated semilattice in $\mathcal{V}^0$ such that $E \cap 2S = \emptyset$, and

$$L + 2S \subseteq L,\ S + L \subseteq S,\ E + 2L \subseteq E \ \text{ and } \ L + E \subseteq L.$$

If $l \geq 3$, suppose further that L is a lattice. Put

$$\begin{aligned} R &= R(BC_l, S, L, E) \\ &:= (S + S) \cup (\textstyle\bigcup_{\dot\alpha \in \dot R_{sh}} (\dot\alpha + S)) \cup (\bigcup_{\dot\alpha \in \dot R_{lg}} (\dot\alpha + L)) \cup (\bigcup_{\dot\alpha \in \dot R_{ex}} (\dot\alpha + E)). \end{aligned}$$

(d) (*The BC_1 construction*). Suppose that $X_l = BC_1$. Suppose that S is a semilattice in $\mathcal{V}^0$ and E is a translated semilattice in $\mathcal{V}^0$ such that $E \cap 2S = \emptyset$ and

$$E + 4S \subseteq E \ \text{ and } \ S + E \subseteq S.$$

Put

$$R = R(BC_1, S, E) := (S + S) \cup (\bigcup_{\dot\alpha \in \dot R_{sh}} (\dot\alpha + S)) \cup (\bigcup_{\dot\alpha \in \dot R_{ex}} (\dot\alpha + E)).$$

We can now state the main result on the structure of EARS (Theorem II.2.37 in [AABGP]).

Theorem 1.19 *Let X_l be one of the types for a finite root system. Starting from a finite root system $\dot R$ of type X_l and up to three semilattices or translated semilattices (as indicated in the construction), Construction 1.18 produces an extended affine root system of type X_l. Conversely, any extended affine root system of type X_l is isomorphic to a root system obtained from the part of Construction 1.18 corresponding to type X_l.*

Using Theorem 1.19, it is easy to classify EARS of nullity 1. (This classification can also be deduced from the more general arguments in Sect. II.4 of [AABGP]. However, these more general arguments are not necessary in nullity 1.) Indeed, suppose that R is an EARS of nullity 1. Then, R is obtained as in Construction 1.18 from some $\dot R$, S, L and E satisfying the assumptions in the construction. Since $\dim(\mathcal{V}^0) = 1$, any semilattice in $\mathcal{V}^0$ is a lattice in $\mathcal{V}^0$ (see Corollary II.1.7 of [AABGP]). So

$$S = \mathbb{Z}\delta \tag{1.20}$$

for some $0 \neq \delta \in S$. Next, if $R_{ex} \neq \emptyset$, we have

$$E = \delta + 2\mathbb{Z}\delta. \tag{1.21}$$

Indeed, we have $E+4S \subseteq E$ and $S+E \subseteq S$ and so E is the union of cosets of $4S$ in S. But then since $E \cap 2S = \emptyset$ and $E+2E \subseteq E$, it follows that $E = (\delta+4S) \cup (3\delta+4S) = \delta+2\mathbb{Z}\delta$ as claimed. Also, if $R_{lg} \neq \emptyset$, we have

$$L = \mathbb{Z}p\delta \tag{1.22}$$

for some integer $p > 0$ (since L is a lattice contained in S). Moreover, if R has type B_l ($l \geq 2$), C_l ($l \geq 3$), F_4 or G_2, we have $kS \subseteq L \subseteq S$ and so $p = 1$ or k (where k is as in Construction 1.18 (b)). On the other hand if R has type BC_l ($l \geq 2$), we have $2S \subseteq L \subseteq S$ and $L + E \subseteq L$ which forces $p = 1$. Summarizing: If $R_{lg} \neq \emptyset$,

$$p = \begin{cases} 1 \text{ or } 2 & \text{if } R \text{ has type } B_l (l \geq 2), C_l (l \geq 3) \text{ or } F_4 \\ 1 \text{ or } 3 & \text{if } R \text{ has type } G_2 \\ 1 & \text{if } R \text{ has type } BC_l (l \geq 2) \end{cases}. \tag{1.23}$$

So R is one of the EARS in the following table:

Table 1.24: EARS when $\nu = 1$

EARS	Finite type X_l	Affine label
$R(X_l, \mathbb{Z}\delta)$	$A_l(l \geq 1),\ D_l(l \geq 4),\ E_6,\ E_7,\ E_8$	$X_l^{(1)}$
$R(X_l, \mathbb{Z}\delta, \mathbb{Z}\delta)$	$B_l(l \geq 2),\ C_l(l \geq 3),\ F_4,\ G_2$	$X_l^{(1)}$
$R(X_l, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$	$B_l(l \geq 2),\ C_l(l \geq 3),\ F_4$	$X_l^{(2)}$
$R(G_2, \mathbb{Z}\delta, 3\mathbb{Z}\delta)$	G_2	$G_2^{(3)}$
$R(BC_1, \mathbb{Z}\delta, \delta + 2\mathbb{Z}\delta)$	BC_1	$BC_1^{(2)}$
$R(BC_l, \mathbb{Z}\delta, \mathbb{Z}\delta, \delta + 2\mathbb{Z}\delta)$	$BC_l(l \geq 2)$	$BC_l^{(2)}$

We note that in the last column of Table 1.24 we have attached to each root system R an *affine label* (or *affine type*) of the form $X_l^{(t)}$, where X_l is the type of R and t is an integer between 1 and 3 called the *tier number* of R. (The affine labels are those used in [MP]. See the note below.) We will see in the next section that if $\mathcal{L}$ is a tame EALA of nullity 1 with root system R then $\mathcal{L}$ is isomorphic to an affine Kac-Moody Lie algebra constructed from an affine matrix A of type $X_l^{(t)}$.

For the convenience of the reader, we note that the correspondence between the affine labels in [MP] and the affine labels from V. Kac' book [K2] is as follows (with the affine labels from [K2] listed second):

$$X_l^{(1)} \leftrightarrow X_l^{(1)} \text{ if } X_l \text{ is reduced },$$

$$B_l^{(2)} \leftrightarrow D_{l+1}^{(2)}\ (l \geq 2), \quad C_l^{(2)} \leftrightarrow A_{2l-1}^{(2)}\ (l \geq 3),$$

$$F_4^{(2)} \leftrightarrow E_6^{(2)}, \quad G_2^{(3)} \leftrightarrow D_4^{(3)} \quad \text{and} \quad BC_l^{(2)} \leftrightarrow A_{2l}^{(2)}\ (l \geq 1).$$

2. Tame EALA's of Nullity One

In this section, we obtain the characterization of affine Kac-Moody Lie algebras as tame EALA's of nullity 1.

We begin by considering EALA's with arbitrary nullity. So *let $\mathcal{L}$, or more precisely $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$, be an EALA with root system R and nullity ν.* The subalgebra $\mathcal{L}_c$ of $\mathcal{L}$ generated by the non-isotropic subspaces $\mathcal{L}_\alpha$, $\alpha \in R^\times$, is called the *core* of $\mathcal{L}$. It is easy to see that $\mathcal{L}_c$ is an ideal of $\mathcal{L}$. Its orthogonal complement

$$\mathcal{L}_c^\perp = \{x \in \mathcal{L} | (x, \mathcal{L}_c) = (0)\}$$

is nothing but the centralizer of $\mathcal{L}_c$. Thus

$$\mathcal{L}_c^\perp = \{x \in \mathcal{L} | [x, \mathcal{L}_c] = (0)\}. \tag{2.1}$$

$\mathcal{L}$ is said to be *tame* if $\mathcal{L}_c^\perp$ equals the center $\mathbf{Z}(\mathcal{L}_c)$ of $\mathcal{L}_c$. Equivalently

$$\mathcal{L} \text{ is tame if and only if } \mathcal{L}_c^\perp \subseteq \mathcal{L}_c. \tag{2.2}$$

We assume for the rest of the section that the EALA $\mathcal{L}$ is tame.

We use the notation of Sect. 1 for the root system R. So R is decomposed as in (1.17) using $\dot{R}$, S, L and E. By Prop. II.1.11 of [AABGP] the group $\langle S \rangle$ generated by S is a lattice in $\mathcal{V}^0$. In fact

$$\langle S \rangle = \mathbb{Z}\delta_1 \oplus \cdots \oplus \mathbb{Z}\delta_\nu, \tag{2.3}$$

where the elements $\delta_1, \ldots, \delta_\nu \in S$ form a basis for $\mathcal{V}^0$ over $\mathbb{R}$. Also, thanks to (II.2.34) and (II.2.35) of [AABGP] we have $S, L, E \subseteq \langle S\rangle$. Furthermore,

$$\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0 \tag{2.4}$$

with $\dot{\mathcal{V}} = \oplus_{i=1}^{l}\mathbb{R}\dot{\alpha}_i$ and $\mathcal{V}^0 = \oplus_{i=1}^{\nu}\mathbb{R}\delta_i$. Let

$$\dot{Q}_{\mathbb{C}} = \sum_{i=1}^{l}\mathbb{C}t_{\dot{\alpha}_i} \quad \text{and} \quad Q^0_{\mathbb{C}} = \sum_{i=1}^{\nu}\mathbb{C}t_{\delta_i}.$$

Then by (1.6) and (1.17) and the fact that $\mathcal{L}_c \cap \mathcal{H} = \sum_{\alpha\in R^\times}[\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$, one can easily show that

$$\mathcal{L}_c \cap \mathcal{H} = \dot{Q}_{\mathbb{C}} \oplus Q^0_{\mathbb{C}}, \tag{2.5}$$

Our next objective is to show that

$$\dim_{\mathbb{C}} \mathcal{H} = l + 2\dim_{\mathbb{C}} Q^0_{\mathbb{C}}. \tag{2.6}$$

Since $(\dot{Q}_{\mathbb{C}} + Q^0_{\mathbb{C}}, Q^0_{\mathbb{C}}) = (0)$ and the restriction of the form to both $\mathcal{H}$ and $\dot{Q}_{\mathbb{C}}$ is non-degenerate, it follows that there exists a subspace $\mathcal{D}$ of $\mathcal{H}$ with the following properties:

- $\mathcal{H} = \dot{Q}_{\mathbb{C}} \oplus Q^0_{\mathbb{C}} \oplus \mathcal{D}$;
- $\dim \mathcal{D} \geq \dim Q^0_{\mathbb{C}}$;
- $(\dot{Q}_{\mathbb{C}}, \mathcal{D}) = (0)$.

If $\dim \mathcal{D} > \dim Q^0_{\mathbb{C}}$ then we can find $x \in \mathcal{D} \setminus \{0\}$ such that $(\dot{Q}_{\mathbb{C}} \oplus Q^0_{\mathbb{C}}, x) = (0)$. By (1.3) and (2.5) we get $x \in \mathcal{L}_c^{\perp}$ and $x \notin \mathcal{L}_c$. This contradicts tameness. Hence we have $\dim \mathcal{D} = \dim Q^0_{\mathbb{C}}$ and (2.6) follows.

Assume now that the nullity ν of $\mathcal{L}$ is 1. (We are still assuming that $\mathcal{L}$ is tame.) Then, we have

$$S = \mathbb{Z}\delta,\ E = \delta + 2\mathbb{Z}\delta \quad \text{and} \quad L = \mathbb{Z}p\delta \tag{2.7}$$

as in (1.20)–(1.23). Moreover, R is one of the root systems in Table 1.24. Let X_l be the (finite) type of R and let $X_l^{(t)}$ be the affine label of R from Table 1.24.

For convenience, from now on we write α_i instead of $\dot{\alpha}_i$, $i = 1, \ldots, l$. Then, with respect to the fixed base $\{\alpha_1, \ldots, \alpha_l\}$ for $\dot{R}$ there exists a unique root ξ_l of maximal height in $\dot{R}$. If $\dot{R}$ has roots of different lengths then in addition to ξ_l (which is either long or extra-long) there exists a unique short root ξ_s of maximal height. Note that if $\dot{R}$ has type BC_l ($l \geq 1$), we have $\xi_l = 2\xi_s$. We use these maximal roots ξ_l and ξ_s to define

$$\alpha_0 = -\xi + \delta, \tag{2.8}$$

where $\xi \in \dot{R}$ is defined according to the affine label of R as follows:

Table 2.9

Affine label	ξ
$X_l^{(1)}$ or $BC_l^{(2)}$	ξ_l = the highest root of $\dot{R}$
$G_2^{(3)}$ or $X_l^{(2)}, X_l \neq BC_l$	ξ_s = the highest short root of $\dot{R}$

The maximality of height, (1.9) and (2.7) yield

$$\frac{2(\alpha_0, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z}_{\leq 0} \quad \text{for all } 1 \leq i \leq l. \tag{2.10}$$

Next we establish two results (2.11) and (2.12) which will eventually allow us to prove the main theorem of this section without having to appeal to the realizations of the different affine Kac-Moody Lie algebras.

First of all:

Lemma 2.11 *Let $\alpha \in R$. Then $\alpha = \sum_{i=0}^{l} a_i \alpha_i$ for some unique $a_0, a_1, \ldots, a_l \in \mathbb{Z}$. Furthermore the a_i's are all non-negative or all non-positive.*

Proof. Indeed if the a_i's exist they are unique as the α_i's are linearly independent. To establish existence, let $\alpha \in R$. By (1.17) we can write $\alpha = \dot{\alpha} + n\delta$ for some $\dot{\alpha} \in \dot{R}$ and $n \in \mathbb{Z}$. Substituting α by $-\alpha$ if necessary we may assume that $n \geq 0$. Now (2.11) follows from (1.17) and the following observation about finite root systems:

$$\dot{\alpha} + \xi_l \in \sum_{i=1}^{l} \mathbb{Z}_{\geq 0}\alpha_i, \quad \dot{\alpha} + 2\xi_s \in \sum_{i=1}^{l} \mathbb{Z}_{\geq 0}\alpha_i \text{ for } \dot{\alpha} \in \dot{R} \qquad \text{and}$$

$$\dot{\alpha} + \xi_s \in \sum_{i=1}^{l} \mathbb{Z}_{\geq 0}\alpha_i \text{ for } \dot{\alpha} \in \dot{R}_{sh}.$$

Lemma 2.12 *If $\alpha \in R^\times$ then*

$$\alpha = w\alpha_i,$$

for some i, $0 \leq i \leq l$, and some w in the subgroup of $\mathcal{W}$ generated by the fundamental reflections $w_{\alpha_0}, \ldots, w_{\alpha_l}$.

Proof. We reason by induction on $ht(\alpha)$ where $ht(\cdot)$, is the height function defined by (2.11). We may assume that $ht(\alpha) > 1$. By (1.17) we see that $(\alpha, \alpha_j) > 0$ for some $0 \leq j \leq l$. Thus $1 \leq ht(w_{\alpha_j}\alpha) < ht(\alpha)$. By induction $w_{\alpha_j}\alpha = w\alpha_i$, and therefore $\alpha = w_{\alpha_j} w\alpha_i$ is as desired.

Next, since the α_i's are non-isotropic, each of the 3-dimensional spaces

$$\mathfrak{sl}_2^{(i)} = \mathcal{L}_{-\alpha_i} \oplus [\mathcal{L}_{-\alpha_i}, \mathcal{L}_{\alpha_i}] \oplus \mathcal{L}_{\alpha_i}$$

is a Lie algebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ (see (1.21)). We can thus find $3(l+1)$ elements e_i, h_i, f_i, $0 \leq i \leq l$ of $\mathcal{L}$ such that for all i,

$$e_i \in \mathcal{L}_{\alpha_i}, f_i \in \mathcal{L}_{-\alpha_i}, [h_i, e_i] = 2e_i,\ [e_i, f_i] = h \quad \text{and} \quad [h_i, f_i] = -2f_i. \tag{2.13}$$

The elements h_i above are unique since h_i is the unique element of $[\mathcal{L}_{\alpha_i}, \mathcal{L}_{-\alpha_i}] = \mathbb{C}t_{\alpha_i}$ satisfying $\alpha_i(h_i) = 2$. By (1.4) we have

$$h_j = \frac{2t_{\alpha_j}}{(\alpha_j, \alpha_j)}. \tag{2.14}$$

So by (1.9) we have

$$\alpha_i(h_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \in \mathbb{Z}. \tag{2.15}$$

Using some standard facts about finite root systems, (2.10) and (2.15), one can show that the $(l+1) \times (l+1)$ matrix A given by

$$A = (A_{ij}), \quad A_{ij} = \alpha_i(h_j)$$

is an indecomposable (generalized) Cartan matrix. If we write (see Table 2.9)

$$\xi = \sum_{i=1}^{l} n_i \alpha_i,$$

and set $n_0 = 1$, then for $\mathbf{n} = (n_0, n_1, \cdots, n_l)$ we have $\mathbf{n}A = \mathbf{0}$. Indeed, the j^{th} entry of $\mathbf{nA}$ is

$$\sum_{i=0}^{l} n_i A_{ij} = A_{0j} + \sum_{i=1}^{l} n_i \alpha_i(h_j) = \alpha_0(h_j) + \xi(h_j) = \delta(h_j) = 0.$$

We have established that A is an affine Cartan matrix with null root $\mathbf{n}$. It is easy to verify that A is of type $X_l^{(t)}$ according to the list in Chapter 3.5 of [MP]. We call A the *affine Cartan matrix associated to* $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$. We will see that $\mathcal{L}$ is isomorphic to an affine Kac-Moody algebra with Cartan matrix A.

Observe that in $\mathcal{L}$ the following familiar looking relations hold:

$$\begin{array}{ll} (r1) & [h_i, e_j] = A_{ji} e_j, \;\; [h_i, f_j] = -A_{ji} f_j \text{ for } 0 \le i, j \le l; \\ (r2) & [e_i, f_j] = \delta_{ij} h_i \text{ for } 0 \le i, j \le l; \\ (r3) & (\text{ad } e_j)^{-A_{ij}+1} e_i = 0, \;\; (\text{ad } f_j)^{-A_{ij}+1} f_i = 0 \text{ for } 0 \le i \ne j \le l. \end{array} \tag{2.16}$$

Indeed, (r1) is a consequence of (2.13) and so is (r2) in the case $i = j$. The case $i \neq j$ of (r2) follows from (2.11). To prove (r3), assume $j \neq i$. We have $[f_i, e_j] = 0$, since $-\alpha_i + \alpha_j \notin R$ by (2.11). Then by Lemma I.1.21 in [AABGP] we get (r3) (or one can use the standard $\mathfrak{sl}_2$ theory to easily show this fact).

Now let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Kac-Moody Lie algebra constructed from the matrix A and a minimal realization of A. In the notation of [MP, Sect. 4.2], $\mathfrak{g}$ is the Lie algebra $\mathfrak{g}(A, \mathcal{R})$, where $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ is a minimal realization of A. (See also [K2, Chapter 6], where $\mathfrak{g}$ is denoted by $\mathfrak{g}(A)$.) So $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$, $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_l\} \subseteq \mathfrak{h}^*$ is a base for the root system Δ of $(\mathfrak{g}, \mathfrak{h})$ and $\Pi^\vee = \{\alpha_0^\vee, \ldots, \alpha_l^\vee\} \subseteq \mathfrak{h}$ is a corresponding co-base. Observe that we are using the same notation α_i for a root of $(\mathfrak{g}, \mathfrak{h})$ and for a root of $(\mathcal{G}, \mathcal{H})$, $i = 0, \ldots, l$. We regard this as an identification which we extend to an identification of the root lattice $\sum_{i=0}^{l} \mathbb{Z}\alpha_i$ of $\mathfrak{g}$ in $\mathfrak{h}^*$ with $\sum_{i=0}^{l} \mathbb{Z}\alpha_i$ in $\mathcal{H}^*$.

For later use we note the following consequence of (2.12):

Corollary 2.17. *Let $\alpha = \sum_{i=0}^{l} a_i \alpha_i \in \mathcal{H}^*$. If $\alpha \in R^\times$, then α, when viewed as an element of $\mathfrak{h}^*$, is a real root of $\mathfrak{g}$.*

If $\mathbf{D}\mathfrak{g}$ is the derived algebra of $\mathfrak{g}$ (i.e., $\mathbf{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$) we have the following information (see Proposition 4.1.12 in [MP]):

$$(\mathbf{D}\mathfrak{g})^\alpha = \mathfrak{g}^\alpha \quad \text{for all } \alpha \in \mathfrak{h}^* \setminus \{\mathbf{0}\}, \tag{2.18}$$

$$(\mathbf{D}\mathfrak{g})^{\mathbf{0}} = \mathbf{D}\mathfrak{g} \cap \mathfrak{h} = \oplus_{\mathbf{i=0}}^{\mathbf{l}} \mathbb{C}\alpha_{\mathbf{i}}^\vee. \tag{2.19}$$

Since $\dim \mathfrak{h} = l + 2$ (the realization being minimal) it follows from (2.18)-(2.19) that there exists $d \in \mathfrak{h}$ such that

$$\mathfrak{h} = (\oplus_{i=0}^{l} \mathbb{C}\alpha_i^\vee) \oplus \mathbb{C}d, \tag{2.20}$$

$$\delta(d) \neq 0, \quad \text{where } \delta \in \Delta \text{ is null}, \tag{2.21}$$

$$\mathfrak{g} = \mathbf{D}\mathfrak{g} \oplus \mathbb{C}d. \tag{2.22}$$

By (2.16), [MP, Proposition 4.3.3] and the Gaber-Kac theorem (see [MP, Theorem 4.6.4]) there exists a natural Lie algebra homomorphism

$$\rho : \mathbf{D}\mathfrak{g} \to \mathcal{L}$$

satisfying

$$\rho((\mathbf{D}\mathfrak{g})^{\alpha_i}) = \mathcal{L}_{\alpha_i}, \tag{2.23}$$

$$\rho(\alpha_i^\vee) = h_i, \tag{2.24}$$

for all $0 \leq i \leq l$. Choose $D \in \mathcal{H}$ so that

$$\alpha_i(D) = \alpha_i(d)$$

for all $0 \leq i \leq l$. By (1.14) and (2.21) it follows that

$$D \notin \dot{Q}_{\mathbb{C}} \oplus Q^0_{\mathbb{C}} = \oplus_{i=0}^l \mathbb{C}h_i.$$

Now (2.6) yields

$$\mathcal{H} = (\oplus_{i=0}^l \mathbb{C}h_i) \oplus \mathbb{C}D.$$

We can now in view of (2.22) extend ρ to a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathcal{L},$$

so that $\rho(d) = D$. Since $\mathfrak{h} \cong \mathcal{H}$ under ρ we conclude from (2.23) and [MP, Prop. 4.3.9] that ρ is injective. Note that ρ is graded:

$$\rho(\mathfrak{g}^\alpha) \subseteq \mathcal{L}_\alpha \quad \text{for all } \alpha \in \sum_{i=0}^l \mathbb{Z}\alpha_i. \tag{2.25}$$

Next we show that

$$\rho(\mathbf{D}\mathfrak{g}) = \mathcal{L}_{\mathbf{c}} \quad \text{and} \quad \rho(\mathfrak{g}) = \mathcal{L}_{\mathbf{c}} + \mathbb{C}\mathbf{D}. \tag{2.26}$$

(The sum $\mathcal{L}_c + \mathbb{C}D$ is actually direct by (2.5)). For this we must show that $\mathcal{L}_\alpha \subseteq \rho(\mathbf{D}\mathfrak{g})$ whenever $\alpha \in R^\times$. By (2.17) α is a real root of $\mathfrak{g}$ so $\mathfrak{g}^\alpha \neq (0)$. Now (2.26) follows from the fact that ρ is injective, graded and dim $\mathcal{L}_\alpha = 1$.

Let $\mathcal{M} = \mathcal{L}_c + \mathbb{C}D$. This is a subalgebra of $\mathcal{L}$ and we know that

$$\mathcal{H} \subseteq \mathcal{M}. \tag{2.27}$$

Consequently, by [MP, Prop. 2.1.1], $\mathcal{M}$ is a graded subalgebra:

$$\mathcal{M} = \oplus_{\alpha \in \mathcal{H}^*} \mathcal{M}_\alpha \quad \text{where } \mathcal{M}_\alpha = \mathcal{M} \cap \mathcal{L}_\alpha. \tag{2.28}$$

Let us now show that

$$\mathcal{M}^\perp := \{x \in \mathcal{L} | (x, \mathcal{M}) = (0)\} = (0). \tag{2.29}$$

Indeed, we have

$$\begin{aligned} \mathcal{M}^\perp &\subseteq \mathcal{L}_c^\perp \quad (\text{as } \mathcal{L}_c \subseteq \mathcal{M}) \\ &= \mathbf{Z}(\mathcal{L}_c) \quad (\text{by tameness}) \\ &\subseteq \mathcal{H} \quad (\text{use } \rho \text{ and } \mathbf{Z}(\mathbf{D}\mathfrak{g}) \subseteq \mathfrak{h} \text{ [MP, Prop. 4.3.4]}). \end{aligned}$$

Now (1.3) and (2.27) yield $\mathcal{M}^\perp = \{0\}$.

Finally we can show that

$$\mathcal{L} = \mathcal{M}. \tag{2.30}$$

By (2.28) it suffices to show that $\mathcal{M}_\alpha = \mathcal{L}_\alpha$ for all $\alpha \in \mathcal{H}^*$. Let $\alpha \in \mathcal{H}^*$, and set

$$\mathcal{L}(\alpha) = \mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha} \quad \text{and} \quad \mathcal{M}(\alpha) = \mathcal{M}_\alpha \oplus \mathcal{M}_{-\alpha}.$$

Consider the canonical map

$$\chi : \mathcal{L}(\alpha) \to \mathcal{M}(\alpha)^*$$

given by

$$\chi(x)(y) = (x, y) \quad \text{for all } x \in \mathcal{L}(\alpha),\ y \in \mathcal{M}(\alpha).$$

By (1.3) and (2.29), χ is injective. On the other hand $\mathcal{M}(\alpha)$ is finite dimensional because of (2.25) and (2.26). Thus (2.30) holds true.

We have therefore proved

Theorem 2.31 *Let $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ be a tame extended affine Lie algebra of nullity one. Then there is a graded isomorphism*

$$\mathcal{L} \cong \mathfrak{g}(A),$$

where A is the affine Cartan matrix associated to $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$ and $\mathfrak{g}(A)$ is a (minimally realized) Kac-Moody Lie algebra constructed from A.

Thus we have proved the implication "$\Longleftarrow$" in the following characterization of affine Kac-Moody Lie algebras. The reverse implication "$\Longrightarrow$" follows from well known properties of affine algebras.

Theorem 2.32 *A Lie algebra $\mathcal{L}$ over $\mathbb{C}$ is isomorphic to an affine Kac-Moody Lie algebra if and only if $\mathcal{L}$ is isomorphic to a tame extended affine Lie algebra of nullity 1.*

3. Constructions

As we have seen in Sect. 1, the EARS of nullity 1 are:

(a) $R(X_l, \mathbb{Z}\delta)$ and $R(X_l, \mathbb{Z}\delta, \mathbb{Z}\delta)$, where X_l is a reduced type,
(b) $R(B_l, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$ ($l \geq 2$),
(c) $R(C_l, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$ ($l \geq 3$),
(d) $R(BC_1, \mathbb{Z}\delta, \delta + 2\mathbb{Z}\delta)$ and $R(BC_l, \mathbb{Z}\delta, \mathbb{Z}\delta, \delta + 2\mathbb{Z}\delta)$ ($l \geq 2$),
(e) $R(F_4, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$ and
(f) $R(G_2, \mathbb{Z}\delta, 3\mathbb{Z}\delta)$.

In this section, for each of the above root systems R, we describe a construction of a tame EALA with root system R. It follows from Theorem 2.31 that these Lie algebras are affine Kac-Moody Lie algebras. Also, all affine types are obtained, and so the Lie algebras that we construct are precisely the affine Kac-Moody Lie algebras. (This gives another proof of the implication "$\Longrightarrow$" in Theorem 2.32.)

The constructions that we describe are special cases of the constructions given in Chapter III of [AABGP]. We present them here since they take on a much simpler form than in [AABGP], where EALA's of arbitrary nullity were considered. We do not give proofs of any of the facts that we describe. The interested reader can either directly check the assertions or read the more general proofs in Chapter III of [AABGP].

We begin by recalling the nullity 1 case of the general construction of EALA's described in [AABGP, Chapter III, §1]. Assume that $\mathcal{G} = \sum_{n\in\mathbb{Z}} \mathcal{G}^n$ is a $\mathbb{Z}$-graded Lie algebra over $\mathbb{C}$ which possesses a nondegenerate invariant symmetric bilinear form $(\cdot,\cdot)$ and a nontrivial finite dimensional ad-diagonalizable abelian subalgebra $\dot{\mathcal{H}}$ such that the restriction of $(\cdot,\cdot)$ to $\dot{\mathcal{H}}$ is nondegenerate. Then as is usual we can transfer $(\cdot,\cdot)$ to a form on the dual space $\dot{\mathcal{H}}^*$ of $\dot{\mathcal{H}}$. Let

$$\mathcal{G} = \sum_{\dot\alpha\in\dot{\mathcal{H}}^*} \mathcal{G}_{\dot\alpha}, \text{ where } \mathcal{G}_{\dot\alpha} = \{x \in \mathcal{G} : [h,x] = \dot\alpha(h)x \text{ for all } h \in \dot{\mathcal{H}}\},$$

be the root space decomposition of $\mathcal{G}$ relative to $\dot{\mathcal{H}}$, and put $\dot R = \{\dot\alpha \in \dot{\mathcal{H}}^* : \mathcal{G}_{\dot\alpha} \neq \{0\}\}$. We suppose further that the following conditions hold:

- $\mathcal{G}$ is generated as a Lie algebra by $\sum_{\dot\alpha\in\dot R\setminus\{0\}} \mathcal{G}_{\dot\alpha}$.
- The restriction of the form $(\cdot,\cdot)$ to the real space $\dot{\mathcal{V}}$ spanned by $\dot R$ is a positive definite real valued form such that $\dot R$ is an irreducible finite root system (including 0) in the Euclidean space $(\dot{\mathcal{V}}, (\cdot,\cdot))$.
- $\mathcal{G}_{\dot\alpha} = \sum_{n\in\mathbb{Z}^n} (\mathcal{G}^n \cap \mathcal{G}_{\dot\alpha})$ for $\dot\alpha \in \dot R$.
- $\mathcal{G}^0 \cap \mathcal{G}_{\dot\alpha} \neq \{0\}$ for each $\dot\alpha \in \dot R \setminus \{0\}$ such that $\frac{1}{2}\dot\alpha \notin \dot R$.
- $\dot{\mathcal{H}} = \mathcal{G}^0 \cap \mathcal{G}_0$.
- $\mathcal{G}^n \neq \{0\}$ for at least one nonzero $n \in \mathbb{Z}$, and
- $m, n \in \mathbb{Z}, m + n \neq 0 \Longrightarrow (\mathcal{G}^m, \mathcal{G}^n) = \{0\}$.

Using this data, we can construct a tame EALA $\mathcal{L}$ of nullity 1. To do this, let

$$\mathcal{L} = \mathcal{G} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with anti-commutative product $[\cdot,\cdot]'$ defined by

$$\begin{aligned}&[\mathcal{L}, c]' = \{0\},\\ &[d, x]' = nx \text{ for all } x \in \mathcal{G}^n, \text{ and}\\ &[x, y]' = [x, y] + \delta_{m,-n}(x, y)c \text{ for } x \in \mathcal{G}^m, y \in \mathcal{G}^n.\end{aligned}$$

Next we define a form $(\cdot,\cdot)$ on $\mathcal{L}$ such that $(\cdot,\cdot)$ extends the form $(\cdot,\cdot)$ on $\mathcal{G}$ and

$$(c,c) = (d,d) = 0, \quad (c,d) = 1 \quad \text{and} \quad (c,\mathcal{G}) = (d,\mathcal{G}) = 0.$$

Finally, put $\mathcal{H} = \dot{\mathcal{H}} \oplus \mathbb{C}c \oplus \mathbb{C}d$. Then, it follows from [AABGP, Prop. III.1.20] that $\mathcal{L}$ is a tame EALA of nullity 1. Hence, by Theorem 2.31, $\mathcal{L}$ is an affine Kac-Moody Lie algebra.

We are now ready to present, for each EARS R in the list at the beginning of this section, a construction of a tame EALA $\mathcal{L}$ of nullity 1 with root system R. In each case we will specify a Lie algebra $\mathcal{G}$ with a $\mathbb{Z}$-grading, a form $(\cdot,\cdot)$ and a subalgebra $\dot{\mathcal{H}}$ as above; and use the general construction just described to construct the EALA $\mathcal{L} = \mathcal{G} \oplus \mathbb{C}c \oplus \mathbb{C}d$. In each of the constructions, we will use the ring

$$\mathcal{S} = \mathbb{C}[t, t^{-1}]$$

of Laurent polynomials over $\mathbb{C}$. Note that $\mathcal{S}$ has a natural $\mathbb{Z}$-grading $\mathcal{S} = \sum_{n\in\mathbb{Z}} \mathcal{S}^n$, where $\mathcal{S}^n = \mathbb{C}t^n$, $n \in \mathbb{Z}$. Also we will use the linear map $\epsilon : \mathcal{S} \to \mathcal{S}$ defined by linear extension of

$$\epsilon(t^n) = \begin{cases} 1 & \text{if } n = 0\\ 0 & \text{if } n \neq 0.\end{cases}$$

Finally, in each case, at the beginning of the construction we will list the EARS R being considered, the affine type of the resulting Kac-Moody Lie algebra $\mathcal{L}$ and the relevant section of [AABGP, Chapter III] which contains the more general construction.

(a) $R = R(X_l, \mathbb{Z}\delta)$ **or** $R(X_l, \mathbb{Z}\delta, \mathbb{Z}\delta)$**, where** X_l **is a reduced type. Affine type** $= X_l^{(1)}$**.** [AABGP, Sect. III.1]

This is the classical construction of the nontwisted affine Kac-Moody Lie algebras ([K1] and [M]). Let X_l be a reduced type, let $\dot{\mathcal{G}}$ be a finite dimensional simple Lie algebra of type X_l over $\mathbb{C}$ and let $\dot{\mathcal{H}}$ be a Cartan subalgebra of $\dot{\mathcal{G}}$. Let $\mathcal{G} = \mathcal{S}\otimes_{\mathbb{C}}\dot{\mathcal{G}}$. We define a $\mathbb{Z}$-grading on $\mathcal{G}$ by putting $\mathcal{G}^n = \mathcal{S}^n\otimes_{\mathbb{C}}\dot{\mathcal{G}}$ for $n \in \mathbb{Z}$. The form $(\cdot,\cdot)$ on $\mathcal{G}$ is defined by

$$(a\otimes_{\mathbb{C}}x, b\otimes_{\mathbb{C}}y) = \epsilon(ab)\kappa(x,y)$$

for $a, b \in \mathcal{S}$ and $x, y \in \dot{\mathcal{G}}$, where κ is the Killing form on $\dot{\mathcal{G}}$. Finally, we identify $\dot{\mathcal{H}} = 1\otimes_{\mathbb{C}}\dot{\mathcal{H}}$ as a subalgebra of $\mathcal{G}$. Then, applying the general construction, $\mathcal{L} = \mathcal{G}\oplus\mathbb{C}c\oplus\mathbb{C}d$ is a tame EALA with root system $R(X_l, \mathbb{Z}\delta)$ if X_l is simply laced and $R(X_l, \mathbb{Z}\delta, \mathbb{Z}\delta)$ otherwise. So, by Theorem 2.31, $\mathcal{L}$ is an affine Kac-Moody Lie algebra of affine type $X_l^{(1)}$.

(b) $R = R(B_l, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$**,** $l \geq 2$**. Affine type** $= B_l^{(2)}$**.** [AABGP, Sect. III.3]

Let $l \geq 2$. We begin by letting

$$\mathcal{G} = \{X \in M_{2l+2}(\mathcal{S}) : G^{-1}X^tG = -X\}, \text{ where } G = \begin{bmatrix} 0 & I_l & 0 & 0 \\ I_l & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & t \end{bmatrix} \in M_{2l+2}(\mathcal{S}).$$

Then, $\mathcal{G}$ is a Lie algebra over $\mathbb{C}$ under the commutator product, and $\mathcal{G}$ is the set of all $(2l+2)\times(2l+2)$-matrices over $\mathcal{S}$ of the form

$$\begin{bmatrix} A & S & -C^t & -tE^t \\ T & -A^t & -B^t & -tD^t \\ B & C & 0 & -ta \\ D & E & a & 0 \end{bmatrix},$$

where $A, S, T \in M_l(\mathcal{S}) = M_{l\times l}(\mathcal{S})$, $B, C, D, E \in M_{1\times l}(\mathcal{S})$, $a \in \mathcal{S}$, $S^t = -S$ and $T^t = -T$. We define a $\mathbb{Z}$-grading on $M_{2l+2}(\mathcal{S})$, and hence by restriction on $\mathcal{G}$, by putting

$$\deg(t^n e_{pq}) = 2n + \delta_{p,2l+2} - \delta_{q,2l+2}$$

for $n \in \mathbb{Z}$ and $1 \leq p, q \leq 2l+2$. (Here of course the elements e_{pq} are the matrix units.) The form $(\cdot,\cdot)$ on $\mathcal{G}$ is defined by

$$(X, Y) = \epsilon(\mathrm{tr}(XY))$$

for $X, Y \in \mathcal{G}$. Finally the subalgebra $\dot{\mathcal{H}}$ of $\mathcal{G}$ is defined by

$$\dot{\mathcal{H}} = \{\sum_{i=1}^{l} \alpha_i(e_{ii} - e_{l+i,l+i}) : \alpha_i \in \mathbb{C}\}.$$

With this input, the general construction produces a tame EALA $\mathcal{L} = \mathcal{G}\oplus\mathbb{C}c\oplus\mathbb{C}d$ with root system $R(B_l, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$. Hence, again by Theorem 2.31, $\mathcal{L}$ is an affine Kac-Moody Lie algebra of affine type $B_l^{(2)}$.

(d) $R = R(C_l, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$**,** $l \geq 3$**. Affine type** $= C_l^{(2)}$**.** [AABGP, Sect. III.4].

Let $l \geq 3$. We first let $^-$ be the involution of $\mathcal{S}$ such that $\bar{t} = -t$. We then define

$$\mathcal{G} = \{X \in M_{2l}(\mathcal{S}) : G^{-1}\bar{X}^t G = -X, \text{tr}(X) = 0\}, \quad \text{where } G = \begin{bmatrix} 0 & I_l \\ -I_l & 0 \end{bmatrix} \in M_{2l}(\mathcal{S}).$$

$\mathcal{G}$ is a Lie algebra over $\mathbb{C}$ under the commutator product, and $\mathcal{G}$ is the set of all $2l \times 2l$-matrices over $\mathcal{S}$ of the form

$$\begin{bmatrix} A & S \\ T & -\bar{A}^t \end{bmatrix}$$

where $A, S, T \in M_l(\mathcal{S})$, $\text{tr}(A) = \overline{\text{tr}(A)}$, $S^t = S$ and $T^t = T$. We define a $\mathbb{Z}$-grading on $M_{2l}(\mathcal{S})$, and hence by restriction on $\mathcal{G}$, by putting

$$\deg(t^n e_{pq}) = n$$

for $n \in \mathbb{Z}$ and $1 \leq p, q \leq 2l$. The form $(\cdot,\cdot)$ and the subalgebra $\dot{\mathcal{H}}$ are defined exactly as in (b) above. This time the general construction produces a tame EALA $\mathcal{L} = \mathcal{G} \oplus \mathbb{C}c \oplus \mathbb{C}d$ with root system $R(C_l, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$, and therefore $\mathcal{L}$ is an affine Kac-Moody Lie algebra of affine type $C_l^{(2)}$.

(d) $R = R(BC_l, \mathbb{Z}\delta, \delta + 2\mathbb{Z}\delta)$**,** $l = 1$**, and** $R = R(BC_l, \mathbb{Z}\delta, \mathbb{Z}\delta, \delta + 2\mathbb{Z}\delta)$**,** $l \geq 2$**. Affine type =** $BC_l^{(2)}$**.** [AABGP, Sect. III.3]

Let $l \geq 1$. Let $^-$ be the involution of $\mathcal{S}$ defined in (c) above, and put

$$\mathcal{G} = \{X \in M_{2l+1}(\mathcal{S}) : G^{-1}\bar{X}^t G = -X, \text{tr}(X) = 0\}, \quad \text{where}$$

$$G = \begin{bmatrix} 0 & I_l & 0 \\ I_l & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in M_{2l+1}(\mathcal{S}).$$

Again $\mathcal{G}$ is a Lie algebra over $\mathbb{C}$ under the commutator product, and this time $\mathcal{G}$ is the set of all $(2l+1) \times (2l+1)$-matrices over $\mathcal{S}$ of the form

$$\begin{bmatrix} A & S & -\bar{C}^t \\ T & -\bar{A}^t & -\bar{B}^t \\ B & C & a \end{bmatrix},$$

where $A, S, T \in M_l(\mathcal{S})$, $B, C \in M_{1\times l}(\mathcal{S})$, $a \in \mathcal{S}$, $\text{tr}(A) - \overline{\text{tr}(A)} + a = 0$, $\bar{S}^t = -S$ and $\bar{T}^t = -T$. We define a $\mathbb{Z}$-grading on $M_{2l}(\mathcal{S})$, and hence by restriction on $\mathcal{G}$, by putting

$$\deg(t^n e_{pq}) = n$$

for $n \in \mathbb{Z}$ and $1 \leq p, q \leq 2l+1$. The form $(\cdot,\cdot)$ and the subalgebra $\dot{\mathcal{H}}$ are defined exactly as in (b) above. Then the general construction produces a tame EALA $\mathcal{L} = \mathcal{G} \oplus \mathbb{C}c \oplus \mathbb{C}d$ with root system $R(BC_l, \mathbb{Z}\delta, \delta + 2\mathbb{Z}\delta)$ if $l = 1$ and $R(BC_l, \mathbb{Z}\delta, \mathbb{Z}\delta, \delta + 2\mathbb{Z}\delta)$ if $l \geq 2$. Therefore $\mathcal{L}$ is an affine Kac-Moody Lie algebra of affine type $BC_l^{(2)}$.

(e) $R = R(F_4, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$**. Affine type =** $F_4^{(2)}$**.** [AABGP, Sect. III.5]

Let $\mathbb{J}$ be the 27–dimensional exceptional simple Jordan algebra over $\mathbb{C}$ with product denoted by $\cdot$ (see [S, Chapter IV]). Let $T : \mathbb{J} \to \mathbb{C}$ be the normalized trace on $\mathbb{J}$; that is T is the usual trace normalized so that $T(1) = 1$. Then $\mathbb{J} = \mathbb{C}1 \oplus \mathbb{J}_0$, where $\mathbb{J}_0 = \{x \in \mathbb{J} : T(x) = 0\}$. Further, let $\dot{\mathcal{G}} = [L_{\mathbb{J}}, L_{\mathbb{J}}]$, where L_x is the left multiplication operator by $x \in \mathbb{J}$. Then, $\dot{\mathcal{G}}$ is the Lie algebra of all derivations of $\mathbb{J}$ and $\dot{\mathcal{G}}$ is the simple Lie algebra of type F_4 over $\mathbb{C}$. Put

$$\mathcal{G} = (\mathcal{R}t \otimes_{\mathbb{C}} \mathbb{J}_0) \oplus (\mathcal{R} \otimes_{\mathbb{C}} \dot{\mathcal{G}}),$$

where $\mathcal{R} = \mathbb{C}[t^2, t^{-2}]$ in $\mathcal{S}$. We define an anticommutative multiplication [,] on $\mathcal{G}$ by

$$[at \otimes_{\mathbb{C}} x, bt \otimes_{\mathbb{C}} y] = abt^2 \otimes_{\mathbb{C}} [L_x, L_y], \quad [a \otimes_{\mathbb{C}} D, bt \otimes_{\mathbb{C}} x] = (ab)t \otimes_{\mathbb{C}} Dy \quad \text{and}$$

$$[a \otimes_{\mathbb{C}} D, b \otimes_{\mathbb{C}} E] = ab \otimes_{\mathbb{C}} [D, E]$$

for $a, b \in \mathcal{R}$, $x, y \in \mathbb{J}_0$ and $D, E \in \dot{\mathcal{G}}$. Then, $\mathcal{G}$ is a Lie algebra over $\mathbb{C}$. The $\mathbb{Z}$-grading on $\mathcal{G}$ is defined by

$$\deg(t^{2n+1} \otimes_{\mathbb{C}} x) = 2n + 1 \quad \text{and} \quad \deg(t^{2n} \otimes_{\mathbb{C}} D) = 2n$$

for $x \in \mathbb{J}_0$ and $D \in \dot{\mathcal{G}}$. The form $(\cdot, \cdot)$ on $\mathcal{G}$ is the unique symmetric bilinear form such that $\mathcal{R}t \otimes_{\mathbb{C}} \mathbb{J}_0$ is orthogonal to $\mathcal{R} \otimes_{\mathbb{C}} \dot{\mathcal{G}}$,

$$(at \otimes_{\mathbb{C}} x, bt \otimes_{\mathbb{C}} y) = \epsilon(abt^2)T(x \cdot y) \quad \text{and} \quad (a \otimes_{\mathbb{C}} D, b \otimes_{\mathbb{C}} [L_u, L_v]) = \epsilon(ab)T((Du) \cdot v)$$

for $a, b \in \mathcal{R}$, $D \in \dot{\mathcal{G}}$ and $x, y \in \mathbb{J}_0$ and $u, v \in \mathbb{J}$. Finally, we obtain a subalgebra $\dot{\mathcal{H}}$ of $\mathcal{G}$ by identifying a Cartan subalgebra $\dot{\mathcal{H}}$ of $\dot{\mathcal{G}}$ with $1 \otimes_{\mathbb{C}} \dot{\mathcal{H}}$. Then the general construction produces a tame EALA $\mathcal{L} = \mathcal{G} \oplus \mathbb{C}c \oplus \mathbb{C}d$ with root system $R(F_4, \mathbb{Z}\delta, 2\mathbb{Z}\delta)$, and therefore $\mathcal{L}$ is an affine Kac-Moody Lie algebra of affine type $F_4^{(2)}$.

(f) $R = R(G_2, \mathbb{Z}\delta, 3\mathbb{Z}\delta)$**. Affine type =** $G_2^{(3)}$**.** [AABGP, Sect. III.5]

Let $\mathbb{A}$ be the 8-dimensional Cayley algebra over $\mathbb{C}$ (see [S, Chapter III]). Let $T : \mathbb{A} \to \mathbb{C}$ be the normalized trace on $\mathbb{A}$, in which case we have $\mathbb{A} = \mathbb{C}1 \oplus \mathbb{A}_0$, where $\mathbb{A}_0 = \{x \in \mathbb{A} : T(x) = 0\}$. Moreover, if $x, y \in \mathbb{A}$, we have

$$xy = T(xy)1 + x * y$$

for some unique $x * y \in \mathbb{A}_0$. Next, let $\dot{\mathcal{G}} = D_{\mathbb{A},\mathbb{A}}$, where $D_{x,y} = \frac{1}{4}(L_{[x,y]} - R_{[x,y]} - 3[L_x, R_y])$ for $x, y \in \mathbb{J}$. (Here L_x and R_x denote the left and right multiplication operators by x in $\mathbb{A}$.) Then $\dot{\mathcal{G}}$ is the Lie algebra of all derivations of $\mathbb{A}$ and $\dot{\mathcal{G}}$ is the simple Lie algebra of type G_2 over $\mathbb{C}$. Put

$$\mathcal{G} = (\mathcal{R}t \otimes_{\mathbb{C}} \mathbb{A}_0) \oplus (\mathcal{R}t^2 \otimes_{\mathbb{C}} \mathbb{A}_0) \oplus (\mathcal{R} \otimes_{\mathbb{C}} \dot{\mathcal{G}}),$$

where $\mathcal{R} = \mathbb{C}[t^3, t^{-3}]$ in $\mathcal{S}$. We define an anticommutative multiplication [,] on $\mathcal{G}$ by

$$[at \otimes_{\mathbb{C}} x, bt \otimes_{\mathbb{C}} y] = (ab)t^2 \otimes_{\mathbb{C}} x * y, \quad [at^2 \otimes_{\mathbb{C}} x, bt^2 \otimes_{\mathbb{C}} y] = (abt^3)t \otimes_{\mathbb{C}} x * y,$$

$$[a \otimes_{\mathbb{C}} D, bt \otimes_{\mathbb{C}} x] = (ab)t \otimes_{\mathbb{C}} Dx, \quad \left[a \otimes_{\mathbb{C}} D, bt^2 \otimes_{\mathbb{C}} x\right] = (ab)t^2 \otimes_{\mathbb{C}} Dx,$$

$$\left[at \otimes_{\mathbb{C}} x, bt^2 \otimes_{\mathbb{C}} y\right] = abt^3 \otimes_{\mathbb{C}} D_{x,y}, \quad [a \otimes_{\mathbb{C}} D, b \otimes_{\mathbb{C}} E] = ab \otimes_{\mathbb{C}} [D, E]$$

for $a, b \in \mathcal{R}$, $x, y \in \mathbb{A}_0$ and $D, E \in \dot{\mathcal{G}}$. Then, $\mathcal{G}$ is a Lie algebra over $\mathbb{C}$. The $\mathbb{Z}$-grading on $\mathcal{G}$ is defined by

$$\deg(t^{3n+1} \otimes_{\mathbb{C}} x) = 3n + 1, \quad \deg(t^{3n+2} \otimes_{\mathbb{C}} x) = 3n + 2 \quad \text{and} \quad \deg(t^{3n} \otimes_{\mathbb{C}} D) = 3n$$

for $x \in \mathbb{A}_0$ and $D \in \dot{\mathcal{G}}$. Next the form $(\cdot, \cdot)$ on $\mathcal{G}$ is the unique symmetric bilinear form such that $\mathcal{R}t \otimes_{\mathbb{C}} \mathbb{A}_0$ is orthogonal to $\mathcal{R}t \otimes_{\mathbb{C}} \mathbb{A}_0 + \mathcal{R} \otimes_{\mathbb{C}} \dot{\mathcal{G}}$, $\mathcal{R}t^2 \otimes_{\mathbb{C}} \mathbb{A}_0$ is orthogonal to $\mathcal{R}t^2 \otimes_{\mathbb{C}} \mathbb{A}_0 + \mathcal{R} \otimes_{\mathbb{C}} \dot{\mathcal{G}}$,

$$(at\otimes_{\mathbb{C}}x, bt^2\otimes_{\mathbb{C}}y) = \epsilon(abt^3)T(xy) \quad \text{and} \quad (a\otimes_{\mathbb{C}}D, b\otimes_{\mathbb{C}}D_{u,v}) = \epsilon(ab)T((Du)v)$$

for $a, b \in \mathcal{R}$, $D \in \dot{\mathcal{G}}$ and $x, y \in \mathbb{A}_0$ and $u, v \in \mathbb{A}$. Again, we obtain a subalgebra $\dot{\mathcal{H}}$ of $\mathcal{G}$ by identifying a Cartan subalgebra $\dot{\mathcal{H}}$ of $\dot{\mathcal{G}}$ with $1\otimes_{\mathbb{C}}\dot{\mathcal{H}}$. This time the general construction produces a tame EALA $\mathcal{L} = \mathcal{G}\oplus\mathbb{C}c\oplus\mathbb{C}d$ with root system $R(G_2, \mathbb{Z}\delta, 3\mathbb{Z}\delta)$, and therefore $\mathcal{L}$ is an affine Kac-Moody Lie algebra of affine type $G_2^{(3)}$.

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