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Analyticity in Time and Smoothing Effect of Solutions to Nonlinear Schrödinger Equations

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Dedicated to Professor Haruo Shizuka on his sixtieth birthday

Abstract: In this paper we consider analyticity in time and smoothing effect of solutions to nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{2p}u, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = \phi, & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $\lambda \in \mathbb{C}$, $p \in \mathbb{N}$. We prove that if ϕ satisfies

$$\left\| e^{|x|^2}\phi \right\|_{H^{[n/2]+1}} < \infty, \tag{2}$$

then there exists a unique solution $u(t,x)$ of (1) and positive constants T, C_0, C_1 such that $u(t,x)$ is analytic in time and space variables for $t \in [-T,T]\setminus\{0\}$ and $x \in \Omega = \{x; |x| < R\}$ and has an analytic continuation $U(z_0,z)$ on

$$\left\{ z_0 = t + i\tau; -C_0t^2 < \tau < C_0t^2, t \in [-T,T]\setminus\{0\} \right\}$$

and

$$\{z = x + iy; -C_1|t| < y < C_1(t), (t,x) \in [-T,T]\setminus\{0\}\times\Omega\}.$$

In the case $n = 1,2,3$ the condition (2) can be relaxed as follows:

$$\left\| e^{|x|^2}\phi \right\|_{H^m} < \infty,$$

where $m = 0$ if $n = 1$, $p = 1$, $m = 1$ if $n = 2$, $p \in \mathbb{N}$ and $m = 1$ if $n = 3$, $p = 1$.

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1. Introduction

In this paper we consider analyticity in time and smoothing effect of solutions to nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{2p}u, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{C}$, $p \in \mathbb{N}$. Equation in (1.1) appears in various physical applications, such as plasma physics, nonlinear optics, and nonrelativistic quantum physics. There have been many works on global existence of solutions, and on asymptotic behavior of solutions (see [Ca] and references cited therein).

The analyticity in space and smoothing effect of solutions to (1.1) were studied in [H-Sai 1]. In particular, [H-Sai 1] showed that the solution u of (1.1) with $p = 1$ has an analytic continuation on the strip

$$S(|t|) = \{z = x + iy; -|t| < y_j < |t|, j = 1, \dots, n\}$$

for any time provided that $\left\| \prod_{j=1}^n (\cosh x_j)\phi \right\|_{H^m}$ is sufficiently small with $m > n$ and $n \geq 2$. If we restrict our attention to the local existence of solutions, the method used in [H-Sai 1] is applicable to (1.1) when $n = 1$.

The analyticity in time of solutions to (1.1) replacing $\lambda|u|^{2p}u$ by $F(u, \overline{u})$ was proved in [H-K.K] first under the analytical condition on the data, where $F(u, \overline{u})$ is a polynomial with respect to u and $\overline{u}$ (see [H-K.K, Theorem 1.1]).

Our purpose in this paper is to prove analyticity in time of solutions to (1.1) without regularity assumption on the data. Our main tool is the operator $K = |x|^2 + nit + 2itx \cdot \nabla + 2it^2\partial_t$, which almost commutes with $L = i\partial_t + \frac{1}{2}\Delta$. Indeed we have $[L, K] = 4itL$ which yields $LK^l u = (K+4it)^l Lu$. Theorems are obtained through Propositions 3.1–3.3 which state the existence of solutions in analytic function spaces involving the operator K. In order to prove Propositions 3.1–3.3 we need the multiplication lemmas (Lemma 2.7, Lemmas 2.12–2.14). Lemma 2.7 is used to prove Proposition 3.2, Lemma 2.12 is used to prove Proposition 3.1 and Lemmas 2.13–2.14 are used to prove Proposition 3.3, respectively. The main tool in the previous work [H-K.K] was the operator $P = x\cdot\nabla+2t\partial_t$ which has the commutation relation $[L, P] = 2L$.

Differences between the proof in this paper and the previous one follow from the facts that the operator K does not commute with the time variable t and the operator $x \cdot \nabla$, and K is not the first order differential operator. The fact that K does not commute with the time variable t means that we can not use the Reibniz rule in $(K + 4it)^l$ and so we need to prepare Lemma 2.3 which prevents us from considering the analyticity in time of solutions in large time. On the other hand the fact that the operator P commutes with the constant 2 appears in $[L, P] = 2L$ enables us to use the Reibniz rule in $(P+2)^l$. Furthermore P is the first order differential operator which commutes with the operator $x \cdot \nabla$. Since K is not the first order differential operator, we introduce the multiplication term $e^{-\frac{i|x|^2}{2t}}$ to prove the multiplication lemmas. By making use of $e^{-\frac{i|x|^2}{2t}}$, we easily see that

$$\tilde{K} = |x|^2 + 2itx \cdot \nabla + 2it^2\partial_t = 2ite^{\frac{i|x|^2}{2t}}(x \cdot \nabla + t\partial_t)e^{-\frac{i|x|^2}{2t}}.$$

The operator $\tilde{K}$ is considered as the first order differential operator for nonlinear terms satisfying the gauge condition and we see that $(i/2it)\tilde{K}$ commutes with the operator $e^{\frac{i|x|^2}{2t}} x \cdot \nabla e^{-\frac{i|x|^2}{2t}}$ although $\tilde{K}$ does not commute with $x \cdot \nabla$. We also know $\tilde{K}$ is almost equivalent to K through Lemma 2.3. We give the strategy of the proofs of Theorems 1–2. The desired results are established by showing

$$\left\| e^{\frac{i|x|^2}{2t}} u \right\|_{G^{b_1}(t\partial; G^{b_2 t^2}(\partial_t; L^2(\Omega)))} < \infty$$

for some constants b_1, b_2 and sufficiently small t, where the norm is defined below. We show the above estimate by combining Propositions 3.1–3.3 and Lemmas 4.1–4.2. We note that Lemma 4.1 also prevents us to prove the analyticity in time in large time t. Lemma 4.1 says that the relation between $\tilde{K}^l$ and $t^l(\frac{1}{2it}\tilde{K})^l = t^l e^{\frac{i|x|^2}{2t}}(x\cdot\nabla + t\partial_t)^l e^{-\frac{i|x|^2}{2t}}$.

The operator P was also used to prove the Gevrey smoothing effect in space variable in [B-H-K.K]. Roughly speaking it was shown that the data ϕ belongs to a Gevrey class of order 2, then solutions of some nonlinear Schrödinger equations become analytic in the space variable for $t \neq 0$. Korteweg-de Vries equation's version of the operator P written as $x\partial_x + 3t\partial_t$ is also useful to study analyticity in time and Gevrey regularizing effect in space variables for solutions (see [B-H-K.K] for details).

Local smoothing effect of solutions to linear Schrödinger equations were studied by [Co-Sau, Sj and V] for the homogeneous case and by [Ke-P-V] for the inhomogeneous case. Kenig, Ponce and Vega [Ke-P-V] applied them to the proof of local existence results of nonlinear Schrödinger equations with nonlinearities having the derivatives of unknown function.

This paper is organized as follows. In Sect. 2 we prove multiplication lemmas (Lemma 2.7, Lemmas 2.12–2.14) which are needed to prove local existence of analytic solutions of (1.1) which are established in Sect. 3. Section 4 is devoted to prove Theorems 1.1 and 1.2. In Sect. 5 we give some applications.

Theorem 1.1. *We let Ω be a ball in $\mathbb{R}^n$ with radius R center at the origin and assume that*

$$\left\| e^{|x|^2}\phi \right\|_{H^{[n/2]+1}} < \infty.$$

Then for any R, there exists a unique solution u of (1.1) *and positive constants T, C_0, C_1 such that u is analytic in time and space variables for $(t,x) \in [-T,T] \setminus \{0\} \times \Omega$ and has an analytic continuation $U(z_0, z)$ on*

$$\left\{ z_0 = t + i\tau : -C_0 t^2 < \tau < C_0 t^2, t \in [-T,T] \setminus \{0\} \right\}$$

and

$$\begin{aligned} \{z &= x + iy; -C_1|t| < y_j \\ &< C_1|t| < y_i < C_1|t|, (t,x) \in [-T,T] \setminus \{0\} \times \Omega, j = 1, \ldots, n\}. \end{aligned}$$

Theorem 1.2. *We assume that*

$$\left\| e^{|x|^2}\phi \right\|_{L^2} < \infty \quad \text{for} \quad p = 1, n = 1,$$

$$\left\| e^{|x|^2}\phi \right\|_{H^1} < \infty \quad \text{for} \quad \begin{cases} p \in \mathbb{N}, n = 2, \\ p = 1, n = 3. \end{cases}$$

Then the same results as in Theorem 1.1 *holds.*

In the case of the linear Schrödinger equations

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

we have the same as in the proof of [H-Sai2, Theorem 1].

Proposition 1.1. *We assume that*

$$\left\| e^{|x|^2} \phi \right\|_{L^2} < \infty.$$

Then there exists a unique solution $u(t,x)$ *of* (1.2) *such that* $u(t,x)$ *has an anylytic continuation* $U(t,z)$ *to* $S(\infty) = \{z; z \in \mathbb{C}^n\}$ *and*

$$e^{-\frac{iz^2}{2t}} U(t,z) \in A(\infty, 2t^2),$$

where

$$A(\infty,\alpha) = \Big\{ \text{the set of all analytic functions } f(z) \text{ on } S(\infty) \text{ such that}$$

$$\frac{1}{(\alpha\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{y^2}{\alpha}} |f(z)|^2 \, dx\, dy < \infty \quad \text{for each } \alpha \Big\}.$$

Furthermore $U(t,z)$ *satisfies*

$$\frac{1}{(2t^2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\left(\frac{u}{t}+2x\right)^2 + 2|x^2|} |U(t,x+iy)|^2 \, dx\, dy = \int_{\mathbb{R}^n} e^{2|x|^2} |\phi(x)|^2 \, dx. \tag{1.3}$$

From Proposition 1.1 and the same argument as in Sect. 4 the result about analyticity in time follows. However we can not expect the estimate (1.3) in the case of nonlinear Schrödinger equations because the function space $A(\infty, 2t^2)$ does not work for (1.1). We notice that Proposition 1.1 follows from the use of the operator $J = x + it\nabla$.

Notatation and function spaces. Let X be a Banach space with norm $\|\cdot\|_X$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, where $|\alpha| = \sum_{j=1}^n \alpha_j$, $\partial_j = \partial/\partial x_j$ and $\alpha_j \in \mathbb{N} \cup \{0\}$. We define analytic function spaces as follows:

$$G^a(\partial; X) = \left\{ f \in X; ||f||_{G^a(\partial;X)} = \sum_{\alpha \in (\mathbb{N}\cup 0)^n} \frac{a^{|\alpha|}}{\alpha!} ||\partial^\alpha f||_X < \infty \right\},$$

where

$$\sum_{\alpha \in (\mathbb{N}\cup 0)^n} \frac{a^{|\alpha|}}{\alpha!} ||\partial^\alpha f||_X = \sum_{\alpha \in (\mathbb{N}\cup 0)^n} \frac{a^{|\alpha|}}{(\alpha_1!)\cdots(\alpha_n!)} ||\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f||_X.$$

In the following we denote the infinite sum $\sum_{\alpha \in (\mathbb{R}\cup 0)^n}$ by $\sum_\alpha$.

We also define

$$G^a(A; X) = \left\{ f \in X; ||f||_{G^a(A;X)} = \sum_N \frac{a^N}{N!} ||A^N f||_X < \infty \right\},$$

where

$$A = K = |x|^2 + nit + 2itx \cdot \nabla + 2it^2\partial_t,$$

or

$$A = \tilde{K} = |x|^2 + 2itx \cdot \nabla + 2it^2\partial_t \quad \text{with} \quad x \cdot \nabla = \sum_{j=0}^{n} x_j\partial_j.$$

We note that

$$\tilde{K} = 2ite^{\frac{i|x|^2}{2t}}(x \cdot \nabla + t\partial_t)e^{-\frac{i|x|^2}{2t}}.$$

The usual Sobolev space $H^{m,p}$ is defined by

$$H^{m,p} = \left\{ f \in L^p; ||f||_{H^{m,p}} = \sum_{|\alpha|\leq m} ||\partial^\alpha f||_{L^p} < \infty \right\},$$

and we let $H^m = H^{m,2}$. We let with $J = (J_j)_{1\leq j\leq n}, J_j = x_j + it\partial_j$,

$$R^{m,p}(t) = \left\{ f \in L^p; ||f||_{R^{m,p}(t)} = \sum_{|\alpha|+|\beta|\leq m} ||J^\alpha \partial^\beta f||_{L^p} < \infty \right\}$$

and $R^m(t) = R^{m,2}(t)$. For simplicity we write $G^a(A, B; X) = G^a(B; G^a(A; X))$.

2. Preliminary Estimates

Lemma 2.1. *We assume that operators A and B satisfy the commutation relations*

$$[A, B] = -\beta A^2, \quad [A, \gamma] = [B, \gamma] = 0,$$

where $\beta, \gamma \in \mathbb{C}$. Then we have

$$(A + B)^l = \sum_{1\leq k\leq l} \binom{l}{k} \prod_{j=0}^{k-1}(1 + \beta j)A^k B^{l-k} + B^l. \tag{2.1}$$

Proof. We prove (2.1) by induction. When $l = 1$, it is clear that (2.1) holds. We assume that (2.1) holds for any l. Then we have by the assumption

$$(A + B)^{l+1} = \sum_{1\leq k\leq l} \binom{l}{k} \prod_{j=0}^{k-1}(1 + \beta j)(A + B)A^k B^{l-k} + (A + B)B^l. \tag{2.2}$$

We next prove by induction

$$[B, A^k] = \beta k A^{k+1}. \tag{2.3}$$

The case $k = 1$ follows from the assumtion. Assume (2.3) for any k, then

$$\begin{aligned} [B, A^{k+1}] &= BA^{k+1} - A^{k+1}B \\ &= (BA^k - A^k B)A + A^k(BA - AB) \\ &= [B, A^k]A + A^k[B, A] = \beta(k + 1)A^{k+2}. \end{aligned}$$

This implies (2.3). We apply (2.3) to (2.2) to obtain

$$
\begin{aligned}
&(A+B)^{l+1} \\
&= \sum_{1\le k\le l} \binom{l}{k} \prod_{j=0}^{k-1}(1+\beta j)((A^{k+1}+A^k B+\beta k A^{k+1})B^{l-k}) + (A+B)B^l \\
&= \sum_{1\le k\le l} \binom{l}{k} \prod_{j=0}^{k-1}(1+\beta j)((1+\beta k)A^{k+1}B^{l-k}+A^k B^{l+1-k}) + (A+B)B^l \\
&= \sum_{2\le k'\le l-1} \binom{l}{k'-1} \prod_{j=0}^{k'-2}(1+\beta j)(1+\beta(k'-1))A^{k'}B^{l+1-k'} \\
&+ \sum_{1\le k\le l} \binom{l}{k} \prod_{j=0}^{k-1}(1+\beta j)A^k B^{l+1-k} + (A+B)B^l \\
&= \sum_{2\le k\le l} \left(\binom{l}{k-1}+\binom{l}{k}\right) \prod_{j=0}^{k-1}(1+\beta j)A^k B^{l+1-k} \\
&+ \prod_{j=0}^{l}(l+\beta j)A^{l+1} + \binom{l}{1} AB^l + (A+B)B^l \\
&= \sum_{2\le k\le l} \binom{l+1}{k} \prod_{j=0}^{k-1}(1+\beta j)A^k B^{l+1-k} \\
&+ \binom{l+1}{l+1} \prod_{j=0}^{l}(1+\beta j)A^{l+1} + \binom{l+1}{1} AB^l + B^{l+1} \\
&= \sum_{1\le k\le l+1} \binom{l+1}{k} \prod_{j=0}^{l}(1+\beta j)A^k B^{l+1-k} + B^{l+1}. \qquad (2.4)
\end{aligned}
$$

This implies (2.1). $\square$

Lemma 2.2. *We have for* $\tilde{K}_d = \tilde{K} + idt$,

$$||f(t)||_{G^a(\tilde{K}_d+\alpha t;X(t))} \le \frac{C}{1-ab|t|}||f(t)||_{G^a(\tilde{K}_d;X(t))},$$

provided that $ab|t| < 1, ||t\cdot||_{X(t)} \le |t|||\cdot||_{X(t)}$, *where* $b > 2$.

Proof. Since $[\alpha t, \tilde{K}_d] = -\frac{2i}{\alpha}(\alpha t)^2$, we have by Lemma 2.1 with $A = \alpha t$, $B = \tilde{K}_d$, $\beta = 2i/\alpha$

$$(\tilde{K}_d + \alpha t)^l = \sum_{l\le k\le l} \binom{l}{k} \prod_{j=0}^{k-1}(1+\frac{2i}{\alpha}j)(\alpha t)^k \tilde{K}_d^{l-k} + \tilde{K}_d^l. \qquad (2.5)$$

By (2.5)

$$||f(t)||_{G^a(\tilde{K}_d+\alpha t;X(t))} = \sum_l \frac{a^l}{l!}||(\tilde{K}_d+\alpha t)^l f(t)||_{X(t)}$$

$$\leq \sum_{l\geq 1} \frac{a^l}{l!} \left(\sum_{1\leq k\leq l} \binom{l}{k} \prod_{j=0}^{k-1} (1 + \frac{2}{|\alpha|} j) |\alpha|^k |t|^k ||\tilde{K}_d^{l-k} f(t)||_{X(t)} \right) + \sum_{l} \frac{a^l}{l!} ||\tilde{K}_d^l f(t)||_{X(t)}$$

$$\leq \sum_{l\geq 1} \sum_{1\leq k\leq l} \frac{a^{l-k}}{(l-k)!} \frac{(2a|t|)^k}{k!} \prod_{j=0}^{k-1} (\frac{|\alpha|}{2} + j) ||\tilde{K}_d^{l-k} f(t)||_{X(t)} + ||f(t)||_{G^a(\tilde{K}_d;X(t))}$$

$$\leq \left(\sum_{k\geq 1} (2a|t|)^k \frac{1}{k!} \prod_{j=0}^{k-1} (\frac{|\alpha|}{2} + j) \right) ||f(t)||_{G^a(\tilde{K}_d;X(t))} + ||f(t)||_{G^a(\tilde{K}_d;X(t))}. \qquad (2.6)$$

It is clear that

$$\frac{1}{k!} \prod_{j=0}^{k-1} (\frac{|\alpha|}{2} + j) \leq C\tilde{a}^k \quad \text{for} \quad \tilde{a} > 1.$$

Hence we have by (2.6),

$$||f(t)||_{G^a(\tilde{K}_d+\alpha t;X(t))} \leq C(\sum_k (ab|t|)^k) ||f(t)||_{G^a(\tilde{K}_d;X(t))},$$

which implies the lemma.

Lemma 2.3. *We have for* $\tilde{K}_d = \tilde{K} + idt$,

$$||f||_{G^a(\tilde{K}_d+\alpha t;X(T))} \leq \frac{C}{1-abT} ||f||_{G^a(\tilde{K}_d;X(T))},$$

provided that $abT < 1, ||t \cdot ||_{X(T)} \leq |T| || \cdot ||_{X(T)}$, *where* $b > 2$.

Proof. In the same way as in the proof of Lemma 2.2 we have the lemma. In what follows we use the notation

$$M(t) = e^{-\frac{i|x|^2}{2t}}, \qquad \hat{f} = M(t)f$$

and the relation

$$J^\alpha = M(-t)(it\partial)^\alpha M(t).$$

Lemma 2.4. *We have for* $p_1 \geq 1$,

$$||f_1 f_2 \overline{f}_3||_{G^a(J;L^{p_1})} \leq \prod_{j=1}^{3} ||f_j||_{G^a(J;L^{p_{j+1}})},$$

where $1/p_1 = \sum_{j=1}^3 (1/p_{j+1})$.

Proof. We have by Reibniz' rule,

$$\|f_1f_2\overline{f}_3\|_{G^a(J;L^{P_1})} = \sum_\alpha \frac{a^{|\alpha|}}{\alpha!}\left\|J^\alpha(f_1f_2\overline{f}_3)\right\|_{L^{P_1}}$$

$$= \sum_\alpha \frac{a^{|\alpha|}}{\alpha!}\left\|(it\partial)^\alpha(\hat{f}_1\hat{f}_2\overline{\hat{f}_3})\right\|_{L^{P_1}}$$

$$\sum_\alpha \frac{(a|t|)^{|\alpha|}}{\alpha!}\left\|\sum_{\substack{\beta\le\alpha\\ \gamma\le\beta}}\binom{\alpha}{\beta}\binom{\beta}{\gamma}(\partial^{\alpha-\beta}\hat{f}_1)(\partial^{\beta-\gamma}\hat{f}_2)(\partial^\gamma\overline{\hat{f}_3})\right\|_{L^{P_1}}$$

$$\le \sum_\alpha \sum_{\substack{\beta\le\alpha\\ \le\beta}} \frac{(a|t|)^{|\alpha|}}{(\alpha-\beta)!(\beta-\gamma)!\gamma!}\left\|(\partial^{\alpha-\beta}\hat{f}_2)(\partial^{\beta-\gamma}\overline{\hat{f}_3})\right\|_{L^{P_1}}$$

$$\le \sum_\alpha \sum_{\substack{\alpha\le\alpha\\ \gamma\le\beta}} \frac{(a|t|)^{|\alpha|}}{(\alpha-\beta)!(\beta-\gamma)!\gamma!}\left\|(\partial^{\alpha-\beta}\hat{f}_1)\right\|_{L^{P_2}}\left\|(\partial^{\beta-\gamma}\hat{f}_2)\right\|_{L^{P_3}}\left\|(\partial^\gamma\overline{\hat{f}_3})\right\|_{L^{P_4}}$$

(by Hölder's inequality)

$$\le \prod_{j=1}^3 \|f_j\|_{G^a(J;L^{p_{j+1}})}.$$

□

Lemma 2.5. *We have for* $p_1, q_1 \ge 1$,

$$||f_1f_2\overline{f}_3||_{G^\alpha(J;L^{q_1}(-T,T;L^{p_1}))} \le \prod_{j=1}^3 ||f_j||_{G^a(J;L^{q_{j+1}}(-T,T;L^{p_{j+1}}))},$$

where $1/p_1 = \sum_{j=1}^3(p_{j+1})$ *and* $1/q_1 = \sum_{j=1}^3(1/q_{j+1})$.

Proof. By the definition

$$||g||_{G^a(J;L^{q_1}(-T,T;L^{p_1}))} = \sum_\alpha \frac{a^{|\alpha|}}{\alpha!}\left(\int_{-T}^T ||J^\alpha g(t)||^{q_1}_{L^{p_1}}dt\right)^{1/q_1}.$$

In the same way as in the proof of Lemma 2.4,

$$||f_1f_2\overline{f}_3||_{G^a(J;L^{q_1}(-T,T;L^{p_1}))}$$
$$\le \sum_\alpha \sum_{\substack{\beta\le\alpha\\ \gamma\le\beta}} \frac{a^{|\alpha|}}{(\alpha-\beta)!(\beta-\gamma)!\gamma!}\left(\int_{-T}^T ||(J^{\alpha-\beta}f_1)(J^{\beta-\gamma}f_2)(\overline{J^\gamma f_3})||^{q_1}_{L^{p_1}}dt\right)^{1/q_1}$$
$$\le \sum_\alpha \sum_{\substack{\beta\le\alpha\\ \gamma\le\beta}} \frac{a^{|\alpha|}}{(\alpha-\beta)!(\beta-\gamma)!\gamma!}||J^{\alpha-\beta}f_1||_{L^{q_2}(-T,T;L^{p_2})}$$
$$\times||J^{\beta-\gamma}f_2||_{L^{q_3}(-T,T;L^{p_3})}||J^\gamma f_3||_{L^{q_4}(-T,T;L^{p_4})},$$

which gives the lemma.

Lemma 2.6. *We have for* $p_1 \geq 1$,

$$||f_1 f_2 \overline{f}_3||_{G^a(J,\tilde{K};L^{p_1})} \leq \prod_{j=1}^{3} ||f_j||_{G^a(J,\tilde{K};L^{p_{j+1}})},$$

where $1/p_1 = \sum_{j=1}^{3}(1/p_{j+1})$.

Proof. We have

$$\|f_1 f_2 \overline{f_3}\|_{G^a(J,\tilde{K};L^{p_1})} = \sum_{l} \frac{a^l}{l!} \|\tilde{K}^l (f_1 f_2 \overline{f_3})\|_{G^a(J;L^{p_1})}. \tag{2.7}$$

Since

$$\tilde{K}^l = M(-t)(2itx \cdot \nabla + 2it^2 \partial_t)^l M(t),$$

we easily see that

$$\tilde{K}^l (f_1 f_2 \overline{f_3}) = M(-t)(2itx \cdot \nabla + 2it^2 \partial_t)^l (\hat{f}_1 \hat{f}_2 \overline{\hat{f}_3}).$$

The operator

$$2it\tilde{P} = 2it(x \cdot \nabla + t\partial_t)$$

is the first order differential operator and so in the same way in the proof of Lemma 2.4 we find that the right-hand side of (2.7) is bounded from above by

$$\sum_{l} \sum_{\substack{k \leq l \\ j \leq k}} \frac{a^l}{(l-k)!(k-j)!j!} \left\| \left((2it\tilde{P})^{l-k} \hat{f}_1\right) \left((2it\tilde{P})^{k-j} \hat{f}_2\right) \left((2it\tilde{P})^{j} \overline{\hat{f}_3}\right) \right\|_{G^a(J;L^{p_1})}. \tag{2.8}$$

We apply Lemma 2.4 to (2.8) to get the lemma.

Lemma 2.7. *We have for* $p_1, q_1 \geq 1$,

$$\|f_1 f_2 \overline{f_3}\|_{G^a\left(J;\tilde{K};L^{q_1}(-T,T;L^{p_1})\right)} \leq \prod_{j=1}^{3} \|f_j\|_{G^a\left(J,\tilde{K};L^{q_{j+1}}(-T,T;L^{p_{j+1}})\right)},$$

where $1/p_1 = \sum_{j=1}^{3}(1/p_{j+1})$ *and* $1/q_1 = \sum_{j=1}^{3}(1/q_{j+1})$.

Proof. In the same way as in the proof of Lemma 2.6 we have the lemma by using Lemma 2.5.

Lemma 2.8. *We have*

$$\|f_1 f_2 \overline{f_3}\|_{Y(T)} \leq C \prod_{j=1}^{3} \|f_j\|_{Y(T)},$$

where $Y(T) = L^\infty(-T,T;R^m(t))$ *and* $m \geq [n/2]+1$.

Proof. By integration by parts and the commutation relation

$$[\partial_k, J_k] = \delta_{jk} \tag{2.9}$$

we obtain

$$\|f_1 f_2 \overline{f_3}\|_{R^m(t)} \le C \sum_{|\alpha|\le m} \left(\left\| J^\alpha (f_1 f_2 \overline{f_3}) \right\|_{L^2} + \left\| \partial^\alpha (f_1 f_2 \overline{f_3}) \right\|_{L^2} \right). \tag{2.10}$$

Hence we have with $\hat{f} = M(t) f$,

$$\begin{aligned}
&\sum_{|\alpha|\le m} \left\| J^\alpha (f_1 f_2 \overline{f_3}) \right\|_{L^2} = \sum_{|\alpha|\le m} \left\| (it\partial)^\alpha (\hat{f}_1 \hat{f}_2 \overline{\hat{f}_3}) \right\|_{L^2} \\
&\le \sum_{|\alpha|\le m} \sum_{\substack{\alpha\le\alpha \\ \gamma\le\beta}} \binom{\alpha}{\beta}\binom{\beta}{\gamma} |t|^{|\alpha|} \left\| (\partial^{\alpha-\beta}\hat{f}_1)(\partial^{\beta-\gamma}\hat{f}_2)(\overline{\partial^\gamma \hat{f}_3}) \right\|_{L^2} \\
&\le C \sum_{\substack{|\alpha|=|\alpha_1|+|\alpha_2|+|\alpha_3| \\ |\alpha|\le m}} \left\| (it\partial)^{\alpha_1} \hat{f}_1 \right\|_{L^{p_1}} \left\| (it\partial)^{\alpha_2} \hat{f}_2 \right\|_{L}^{p_2} \left\| (it\partial)^{\alpha_3} \hat{f}_3 \right\|_{L^{p_3}} \\
&\hspace{20em} \text{(by Hölder's inequality)} \\
&\le C \sum_{|\alpha|\le m} \prod_{j=1}^{3} \left\| (it\partial)^\alpha \hat{f}_j \right\|_{L^2}^{\alpha_j} \|f_j\|_{L^\infty}^{1-\alpha_j} \quad \text{(by Sobolev's inequality)},
\end{aligned}$$

where

$$\frac{1}{p_j} = \frac{|\alpha_j|}{n} + a_j \left(\frac{1}{2} - \frac{m}{n} \right), \quad \sum_{j=1}^{3} \frac{1}{p_j}.$$

Hence we have

$$\sum_{|\alpha|\le m} \left\| J^\alpha (f_1 f_2 \overline{f_3}) \right\|_{L^2} \le C \sum_{|\alpha|\le m} \prod_{j=1}^{3} \|J^\alpha f_j\|_{L^2}^{a_j} \|f_j\|_{L^\infty}^{1-a_j}.$$

We again apply Sobolev's inequality to get

$$\sum_{|\alpha|\le m} \left\| J^\alpha (f_1 f_2 \overline{f_3}) \right\|_{L^2} \le C \prod_{j=1}^{3} \|f_j\|_{R^m(t)}. \tag{2.11}$$

In the same way as in the proof of (2.11) we have

$$\sum_{|\alpha|\le m} \left\| \partial^\alpha (f_1 f_2 \overline{f_3}) \right\|_{L^2} \le C \prod_{j=1}^{3} \|f_j|_{R^m(t)}. \tag{2.12}$$

From (2.10)–(2.12) the lemma follows. □

Lemma 2.9. *We have*

$$\|f_1 f_2 \overline{f_3}\|_{G^a(J;Y(T))} \le C \prod_{j=1}^{3} \|f_j\|_{G^a(J;Y(T))},$$

where $Y(T) = L^\infty(-T, T; R^m(t))$ *and* $m \ge [n/2] + 1$.

Proof. We have by Lemma 2.8,

$$\|f_1 f_2 \overline{f_3}\|_{G^a(J;Y(T))} = \sum_\alpha \frac{a^{|\alpha|}}{\alpha!} \left\|J^\alpha (f_1 f_2 \overline{f_3})\right\|_{Y(T)}$$
$$\leq \sum_\alpha \sum_{\substack{\beta \leq \alpha \\ \gamma \leq \beta}} \frac{a^{|\alpha|}}{(\alpha-\beta)!(\beta-\gamma)!\gamma!} \left\|(J^{\alpha-\beta} f_1)(J^{\beta-\gamma} f_2)(\overline{J^\gamma f_3})\right\|_{Y(T)}$$
$$\leq C \sum_\alpha \sum_{\substack{\beta \leq \alpha \\ \gamma \leq \beta}} \frac{a^{|\alpha|}}{(\alpha-\beta)!(\beta-\gamma)!\gamma!} \left\|J^{\alpha-\beta} f_1\right\|_{Y(T)} \left\|J^{\beta-\gamma} f_2\right\|_{Y(T)} \|J^\gamma f_3\|_{Y(T)},$$

which gives the lemma. □

Lemma 2.10. *We have*

$$\|f_1 f_2 \overline{f_3}\|_{G^a(J,\tilde{K};Y(T))} \leq C \prod_{j=1}^{3} \|f_j\|_{G^a(J,\tilde{K};Y(T))},$$

where $Y(T) = L^\infty(-T,T;R^m(t))$ *and* $m \geq [n/2]+1$.

Proof. We have by the definition

$$\|f_1 f_2 \overline{f_3}\|_{G^a(J,\tilde{K};Y(T))} = \sum_l \frac{a^l}{l!} \left\|\tilde{K}^l (f_1 f_2 \overline{f_3})\right\|_{G^a(J;Y(T))}.$$

In the same way as in the proof of (2.8) we obtain

$$\|f_1 f_2 \overline{f_3}\|_{G^a(J,\tilde{K};Y(T))}$$
$$\leq \sum_l \sum_{\substack{k \leq l \\ j \leq k}} \frac{a^l}{(l-k)!(k-j)!j!} \left\|(\tilde{K}^{l-k} f_1)(\tilde{K}^{k-j} f_2)\left(\overline{\tilde{K}^j f_3}\right)\right\|_{G^a(J;Y(T))}.$$

We apply Lemma 2.9 to the right-hand side of the above to get lemma. □

Lemma 2.11. *We have*

$$\left\||f|^{2p} f - |g|^{2p} g\right\|_{G^a(J,\tilde{K};Y(T))}$$
$$\leq C \left(\|f\|^{2p}_{G^a(J,\tilde{K};Y(T))} + \|g\|^{2p}_{G^a(J,\tilde{K};Y(T))}\right) \|f-g\|_{G^a(J,\tilde{K};Y(T))},$$

where $Y(T) = L^\infty(-T,T;R^m(t))$ *and* $m \geq [n/2]+1$.

Proof. We prove by induction with respect to p . When $p = 1$ we have the lemma by Lemma 2.10. We assume that the lemma holds for any p. From the equality

$$|f|^{2p+2} f - |g|^{2p+2} g = |f|^2 \left(|f|^{2p} f - |g|^{2p} g\right) + |g|^{2p} g \left(\overline{f}(f-g) + g(\overline{f} - \overline{g})\right),$$

and Lemma 2.10 it follows that

$$\begin{aligned}
&\left\||f|^{2p+2}f-|g|^{2p+2}g\right\|_{G^a(J,\tilde{K};Y(T))}\\
&\le C\left(\|f\|^2_{G^a(J,\tilde{K};Y(T))}\left\||f|^{2p}f-|g|^{2p}g\right\|_{G^a(J,\tilde{K};Y(T))}\right.\\
&+\left\||g|^{2p}g\right\|_{G^a(J,\tilde{K};Y(T))}\\
&\left.\left(\|f\|_{G^a(J,\tilde{K};Y(T))}+\|g\|_{G^a(J,\tilde{K};Y(T))}\right)\|f-g\|_{G^a(J,\tilde{K};Y(T))}\right).
\end{aligned}$$

Thus the lemma for the case $p+1$ follows from the assumption. This completes the proof of Lemma 2.11.

Lemma 2.12. *We have*

$$\begin{aligned}
&\||f|^{2p}f-|g|^{2p}g\|_{G^a(J,K+4it,Y(T))}\\
&\le\frac{C}{(1-abT)^{2p+2}}\left(\|f\|^{2p}_{G^a(J,K;Y(T))}+\|g\|^{2p}_{G^a(J,K;Y(T))}\right)\|f-g\|_{G^a(J,K;Y(T))},
\end{aligned}$$

provided that $abT>1$, *where* $b<2$, $Y(T)=L^\infty(-T,T;R^m(t))$ *and* $m\ge[n/2]+1$.

Proof. Lemma 2.3 with $d=0$, $\alpha=(4+n)i$ and $X(T)=G^a(J;Y(T))$ gives

$$\|\cdot\|_{G^a(J,K+4it;Y(T))}\le\frac{C}{1-abT}\|\cdot\|_{G^a(J,\tilde{K};Y(T))},\tag{2.13}$$

since

$$\|t\cdot\|_{X(T)}\le T\|\cdot\|_{X(T)}.$$

We again use Lemma 2.3 with $d=n$, $\alpha=-ni$ to obtain

$$\|\cdot\|_{G^a(J,\tilde{K};Y(T))}\le\frac{C}{1-abT}\|\cdot\|_{G^a(J,K;Y(T))}.\tag{2.14}$$

The lemma follows from (2.13) and Lemma 2.11. □

Lemma 2.13. *We assume that* $n=2$ *or* 3. *Then we have*

$$\begin{aligned}
&\left\|\left(\prod_{j=1}^{p}f_j\overline{f}_{j+p}\right)f_{2p+1}\right\|_{G^a\left(J,\tilde{K};L^r(-T,T;R^{1,r'}(t))\right)}\\
&\le C\sum_{k=1}^{2p+1}\prod_{\substack{j=1\\ j\ne k}}^{2p+1}\|f_j\|_{G^a\left(J,\tilde{K};L^\infty(-T,T;R^1(t))\right)}\|f_k\|_{G^a\left(J,\tilde{K};L^r(-T,T;R^{1,r}(t))\right)},
\end{aligned}$$

where $r=2+(4/n)$, $(1/r)+(1/r')=1$ *and* $p=1$ *if* $n=3$, $p\in\mathbb{N}$ *if* $n=2$.

Proof. We have by Hölder inequality

$$\begin{aligned}
&\left\|\left(\prod_{j=1}^{p}f_j\overline{f}_{j+p}\right)f_{2p+1}\right\|_{L^r(-T,T;R^{1,r'}(t))}\\
&=\sum_{|\alpha|+|\beta|\le1}\left\|J^\alpha\partial^\beta\left(\prod_{j=1}^{p}f_j\overline{f}_{j+p}\right)f_{2p+1}\right\|_{L^r(-T,T;L^{r'})}\\
&\le C\sum_{k=1}^{2p+1}\prod_{\substack{j=1\\ j\ne k}}^{2p+1}\|f_j\|_{L^\infty(-T,T;L^{p(n+2)})}\|f_k\|_{L^r(-T,T;R^{1,r}(t))}.
\end{aligned}\tag{2.15}$$

By Sobolev's inequality,

$$\left\| \left(\prod_{j=1}^{p} f_j \overline{f}_{j+p} \right) f_{2p+1} \right\|_{L^r(-T,T;R^{1,r'}(t))} \leq C \sum_{k=1}^{2p+1} \prod_{\substack{j=1 \\ j\neq k}}^{2p+1} \|f_j\|_{L^\infty(-T,T;R^1(t))} \|f_k\|_{L^r(-T,T;R^{1,r}(t))}. \tag{2.16}$$

In the same way as in the proof of Lemma 2.9 we obtain by (2.16),

$$\left\| \left(\prod_{j=1}^{p} f_j \overline{f}_{j+p} \right) f_{2p+1} \right\|_{G^a(J,L^r(-T,T;R^{1,r'}(t)))} \leq C \sum_{k=1}^{2p+1} \prod_{\substack{j=1 \\ j\neq k}}^{2p+1} \|f_j\|_{G^a(J;L^\infty(-T,T;R^1(t)))} \|f_k\|_{G^a(J;L^r(-T,T;R^{1,r}(t)))}. \tag{2.17}$$

From (2.17) and the similar argument as in the proof of Lemma 2.10 the lemma follows. □

Lemma 2.14. *We assume that* $n = 2$ *or* 3*. Then we have*

$$\left\| \left(\prod_{j=1}^{p} f_j \overline{f}_{j+p} \right) f_{2p+1} \right\|_{G^a(J,\tilde{K};L^1(-T,T;R^1(t)))} \leq CT^{\frac{3}{5}} \sum_{\substack{j,k,l=1 \\ j\neq k\neq l}} \|f_j\|_{G^a(J,\tilde{K};L^\infty(-T,T;R^1(t)))} \|f_k\|_{G^a(J,\tilde{K};L^r(-T,T;R^{1,r}(t)))} \times \|f_l\|_{G^a(J,\tilde{K};L^r(-T,T;R^{1,r}(t)))} \quad \text{for } n = 3,$$

$$\left\| \left(\prod_{j=1}^{p} f_j \overline{f}_{j+p} \right) f_{2p+1} \right\|_{G^a(J,\tilde{K};L^1(-T,T;R^1(t)))} \leq CT^{\frac{3}{4}} \sum_{k=1}^{2p+1} \prod_{\substack{j=1 \\ j\neq k}}^{2p+1} \|f_j\|_{G^a(J,\tilde{K};L^\infty(-T,T;R^1(t)))} \|f_k\|_{G^a(J,\tilde{K};L^r(-T,T;R^{1,r}(t)))} \quad \text{for} \quad n = 2,$$

where $r = 2 + (4/n)$, $p = 1$ *if* $n = 3$ *and* $p \in \mathbb{N}$ *if* $n = 2$.

Proof. We have by Hölder's inequality

$$\left\| \left(\prod_{j=1}^{p} f_j \overline{f}_{j+p} \right) f_{2p+1} \right\|_{L^1(-T,T;R^1(t))} = \sum_{|\alpha|+|\beta|\leq 1} \left\| J^\alpha \partial^\beta \left(\left(\prod_{j=1}^{p} f_j \overline{f}_{j+p} \right) f_{2p+1} \right) \right\|_{L^1(-T,T;L^2)}$$

$$\leq C \sum_{k=1}^{2p+1} \prod_{\substack{j=1 \\ j \neq k}}^{2p+1} \|f_j\|_{L^{q_j}(-T,T;L^{2p(n+2)})} \|f_k\|_{L^{q_k}(-T,T;R^{1,r}(t))}, \tag{2.18}$$

where $\sum_{j=1}^{2p+1} 1/q_j = 1$. On the other hand by Sobolev's inequality we have for $n = 3$,

$$\|f\|_{L^{2p(n+2)}} = \|f\|_{L^{10}} \leq C\|f\|_{H^1}^{1/2} \|f\|_{H^{1,r}}^{1/2}, \tag{2.19}$$

and for $n = 2$

$$\|f\|_{L^{2p(n+2)}} \leq C\|f\|_{H^1}. \tag{2.20}$$

We use (2.19) and (2.20) in the right-hand side of (2.18) to get

$$\begin{aligned}
&\left\| \left(\prod_{j=1}^{p} f_j \overline{f}_{j+p} \right) f_{2p+1} \right\|_{L^1(-T,T;R^1(t))} \\
&\leq C \sum_{\substack{j,k,l=1 \\ j \neq k \neq l}}^{3} \|f_j\|_{L^{\frac{n+2}{2}}(-T,T;R^1(t))} \|f_k\|_{L^r(-T,T;R^{1,r}(t))} \|f_l\|_{L^r(-T,T;R^{1,r}(t))} \\
&\leq CT^{\frac{n}{n+2}} \sum_{\substack{j,k,l=1 \\ j \neq k \neq l}} \|f_j\|_{L^\infty(-T,T;R^1(t))} \|f_k\|_{L^r(-T,T;R^{1,r}(t))} \\
&\qquad \|f_l\|_{L^r(-T,T;R^{1,r}(t))} \quad \text{for n=3}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \left(\prod_{j=1}^{p} f_j \overline{f}_{j+p} \right) f_{2p+1} \right\|_{L^1(-T,T:R^1(t))} \\
&\leq C \sum_{k=1}^{2p+1} \prod_{\substack{j=1 \\ j \neq k}}^{2p+1} \|f_j\|_{L^{\frac{8p}{3}}(-T,T;R^1(t))} \|f_k\|_{L^r(-T,T;R^{1,r}(t))} . \\
&\leq CT^{\frac{3}{4}} \sum_{k=1}^{2p+1} \prod_{\substack{j=1 \\ j \neq k}}^{2p+1} \|f_j\|_{L^\infty(-T,T;R^1(t))} \|f_k\|_{L^r(-T,T;R^{1,r}(t))} \quad \text{for } n = 2.
\end{aligned}$$

The rest of the proof is done in the same way as in the proof of Lemma 2.13 and so we leave it to the reader.

Define $U(t)$ by

$$U(t)(\phi) = \mathcal{F}^{-1} e^{-i|\xi|^2 t/2} \hat{\phi},$$

which is the fundamental solution of linear Schrödinger equation

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n. \end{cases}$$

For $U(t)$ we have

Lemma 2.15. *(1) We let* $2 \leq p \leq \infty$ *and* $(1/p) + (1/p') = 1$. *Then for any* $\phi \in L^{p'}$,

$$\|U(t)\phi\|_{L^p} \leq C|t|^{-\frac{n}{2}(1-\frac{2}{p})}\|\phi\|_{L^{p'}}. \tag{2.21}$$

(2) We let $r = 2 + (4/n)$*. for any* $\phi \in L^2$,

$$\|U(\cdot)\phi\|_{L^r(\mathbb{R};L^r)} \leq C\|\phi\|_{L^2}. \tag{2.22}$$

Lemma 2.15 (2.21) is well known. Lemma 2.15 (2.22) is due to Stricharz [St].

3. Existence of Analytic Solutions

Proposition 3.1. *We assume that*

$$\phi \in G^a\left(x, |x|^2; R^m(0)\right) \quad \text{with} \quad m \geq [\frac{n}{2}] + 1.$$

Then there exists a unique solution $u(t,x)$ *of* (1.1) *and a positive constant* T *such that*

$$u(t,x) \in G^\alpha\left(J, K; L^\infty(-T,T; R^m(t))\right) \quad \text{for} \quad t \in [-T,T].$$

Proof. We only treat the case of positive time, since the negative time is treated similarly. To prove Proposition 3.1 we introduce the function space

$$X_T = \{f \in L^\infty(0,T;L^2); ||f||_{X_T} = \|f\|_{G^a(J,K;Y(T))} < \infty\},$$

where

$$Y(T) = L^\infty(0,T;R^m(t)).$$

We consider the linearized equation of (1.1),

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda|v|^{2p}v, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = \phi(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $v \in X_T$. We define M by $u = Mv$. It is sufficient to prove M is a contraction mapping from a closed ball

$$X_{T,\rho} = \{f \in X_T; ||f||_{X_T} \leq \rho\}$$

into itself for some time T. Applying both sides of (3.1) by $J^\beta\partial^\gamma J^\alpha K^l$, we obtain

$$i\partial_t J^\beta\partial^\gamma J^\alpha K^l u + \frac{1}{2}\Delta J^\beta\partial^\gamma J^\alpha K^l u = \lambda J^\beta\partial^\gamma J^\alpha (K+4it)^l |v|^{2p}v, \tag{3.2}$$

where we have used the commutation relations

$$[L,J] = 0, \quad [L,K] = 4itL,$$

with

$$L = i\partial_t + \frac{1}{2}\Delta.$$

From (3.2) it follows that

$$\|J^\beta \partial^\gamma J^\alpha K^l u\|_{L^2} \leq \|x^\beta \partial^\gamma x^\alpha |x|^{2l} \phi\|_{L^2}$$
$$+ C \int_0^t \|J^\beta \partial^\gamma J^\alpha (K + 4i\tau)^l |v|^{2p} v\|_{L^2} d\tau. \tag{3.3}$$

Multiplying both sides of (3.3) by $a^{|\alpha|+l}/\alpha! l!$, making a summation with respect to α, l, β, γ, we get

$$||u(t)||_{G^a(J,K;Y(T))} \leq ||\phi||_{G^a}(x, |x|^2; R^m(0)) + C \int_0^T |||v|^{2p} v(\tau)||_{G^a(J,K+4i\tau;Y(T))} d\tau. \tag{3.4}$$

By Lemma 2.3 with $d = n, \alpha = (4 + n)i, X(T) = G^a(J; Y(T))$ we see that

$$|| \cdot ||_{G^a(J,K+4it;Y(T))} \leq \frac{C}{1 - abT} || \cdot ||_{G^a(J,K;Y(T))}. \tag{3.5}$$

We apply (3.5) and Lemma 2.12 to the second term of the right-hand side of (3.4) to obtain

$$||u(t)||_{G^a(J,K;Y(T))} \leq ||\phi||_{G^a(x,|x|^2;R^m(0))} + C \int_0^T \frac{1}{(1 - abT)^{2p+3}} ||v||^{2p+1}_{g^a(J,K;Y(T))} d\tau$$

from which it follows that

$$||u||_{X_T} \leq ||\phi||_{G^a(x,|x|^2;R^m(0))} + CT\rho^{2p+1} \tag{3.6}$$

provided that

$$T < \frac{1}{2ab}.$$

We take

$$||\phi||_{G^a(x,|x|^2;R^m(0))} \leq \frac{\rho}{2} \quad \text{and } C_\rho^{2p+1} T \leq \frac{\rho}{2}.$$

Then (3.6) gives us

$$||u||_{X_T} \leq \rho. \tag{3.7}$$

In the same way as in the proof of (3.7) we have by Lemma 2.12,

$$||Mv_1 - Mv_2||_{X_T} \leq CT\rho^{2p} ||v_1 - v_2||_{x_T} \leq \frac{1}{2} ||v_1 - v_2||_{X_T}, \tag{3.8}$$

provided that $cT\rho^{2p} \leq \frac{1}{2}$. From (3.7) and (3.8) we see that there exists a T such that M is a contraction mapping from $X_{T,\rho}$ into itself. This completes the proof of Proposition 3.1.

Proposition 3.2. *We assume that* $p = 1, n = 1$ *and*

$$\phi \in G^a(x, |x|^2; L^2).$$

Then there exists a unique solution $u(t, x)$ *of (1.1) and a positive constant* T *such that*

$$u(t, x) \in G^a(J, K; L^2) \quad for \quad t \in [-T, T].$$

Proof. To prove Proposition 3.2 we introduce the function space

$$X_T = \{f \in L^\infty(0,T;L^2); ||f||_{X_T} < \infty\},$$

where

$$||f||_{X_T} = ||f||_{G^a(J,K;L^\infty(0,T;L^2))} + ||f||_{G^a(J,K;L^6(0,T;L^6))}.$$

We also define a closed ball

$$X_{T,\rho} = \{f \in X_T; ||f||_{X_T} \le \rho\}.$$

We now prove that there exists a T such that M defined by $u = Mv$ is a contraction mapping from X_T, ρ into itself. In the same way as in the proof of (3.2) we have by (3.1),

$$i\partial_t J^\alpha K^l u + \frac{1}{2}\Delta J^\alpha K^l u = \lambda J^\alpha (K + 4it)^l |v|^2 v,$$

which can be written as

$$J^\alpha K^l u(t) = U(t)x^\alpha |x|^{2l}\phi - i\int_0^t U(t-\tau)\lambda J^\alpha (K+4i\tau)^l |v|^2 v(\tau)d\tau. \tag{3.9}$$

By virtue of Lemma 2.13 we get

$$\begin{aligned}&||J^\alpha K^l u(t)||_{L^6} \le ||U(t)x^\alpha |x|^{2l}\phi||_{L^6}\\ &+C\int_0^t (t-\tau)^{-\frac{1}{3}}||J^\alpha (K+4i\tau)^l|v|^2 v(\tau)||_{L^{6/5}}d\tau.\end{aligned}$$

Taking L^6 norm in time, using Lemma 2.15, we obtain

$$\begin{aligned}&||J^\alpha K^l u||_{L^6(0,T;L^6)} \le C||x^\alpha |x|^{2l}\phi||_{L^2}\\ &+C||\int_0^t (t-\tau)^{-\frac{1}{3}}||J^\alpha (K+4i\tau)^l|v|^2 v(\tau)||_{L^{6/5}}d\tau||_{L^6(0,T)}.\end{aligned} \tag{3.10}$$

By Hölder's inequality

$$\int_0^t (t-\tau)^{\frac{1}{3}} g(\tau)d\tau \le \left(\int_0^t (t-\tau)^{-\frac{2}{3}}d\tau\right)^{\frac{1}{2}}\left(\int_0^t |g(\tau)|^2 d\tau\right)^{\frac{1}{2}} \le Ct^{\frac{1}{6}}||g||_{L^2(0,T)}. \tag{3.11}$$

We use (3.11) with

$$g(\tau) = ||J^\alpha (K+4i\tau)^l |v|^2 v||_{L^{6/5}}$$

to (3.10) to have

$$||J^\alpha K^l u||_{L^6(0,T;L^6)} \le C||x^\alpha |x|^{2l}\phi||_{L^2} + CT^{\frac{1}{3}}||J^\alpha (K+4it)^l|v|^2 v||_{L^2(0,T;L^{6/5})}. \tag{3.12}$$

Multiplying both sides of (3.12) by $a^{|\alpha|+l}/a!l!$, making a summation with respect to α, l, we get

$$\begin{aligned}&\|u\|_{G^\alpha(J,K;L^6(0,T;L^6))}\\ &\le C\left(\|\phi\|_{G^\alpha(x,|x|^2;L^2)} + T^{\frac{1}{3}}\left\||v|^2 v\right\|_{G^\alpha(J,K+4it;L^2(0,T;L^{6/5}))}\right).\end{aligned} \tag{3.13}$$

We have by Lemma 2.3 with $X(T) = G^{\alpha}\left(J; L^2(0,T;L^{6/5})\right)$, $d = 0$, $\alpha = (n+4)i$,

$$
\begin{aligned}
&\left\| |v|^2 v \right\|_{G^{\alpha}\left(J,K+4it;L^2(0,T;L^{6/5})\right)} = \left\| |v|^2 v \right\|_{G^{\alpha}\left(K+4it;G^{\alpha}(J;L^2(0,T;L^{6/5}))\right)} \\
&\le \frac{C}{1-abT} \left\| |v|^2 v \right\|_{G^{\alpha}\left(J,\tilde{K};L^2(0,T;L^{6/5})\right)} \\
&\le \frac{C}{1-abT} \|v\|^2_{G^{\alpha}\left(J,\tilde{K};L^6(0,T;L^6)\right)} \|v\|_{G^{\alpha}\left(J,\tilde{K};L^6(0,T;L^2)\right)} \quad \text{(by Lemma 2.7)}.
\end{aligned}
$$

We again use Lemma 2.3 to obtain

$$
\||v|^2 v\|_{G^{\alpha}\left(J,K+4it;L^2(0,T;L^{6/5})\right)} \le \frac{C}{(1-abT)^4} T^{\frac{1}{6}} \rho^3. \tag{3.14}
$$

Hence by (3.13) and (3.14),

$$
\|u\|_{G^{\alpha}\left(J,K;L^6(0,T;L^6)\right)} \le C \left(\|\phi\|_{G^{\alpha}\left(x,|x|^2;L^2\right)} + T^{\frac{1}{2}} \rho^3 \right), \tag{3.15}
$$

provided that

$$
T \le \frac{1}{2ab}.
$$

In the same way as (3.4) we have

$$
\begin{aligned}
&\|u\|_{G^{\alpha}\left(J,K;L^{\infty}(0,T;L^2)\right)} \le \|\phi\|_{G^{\alpha}\left(x,|x|^2;L^2\right)} + C \int_0^T \||v|^2 v\|_{G^{\alpha}\left(J,K+4i\tau;L^2\right)} d\tau \\
&\le \|\phi\|_{G^{\alpha}\left(x,|x|^2;L^2\right)} + C \left\| |v|^2 v \right\|_{G^{\alpha}\left(J,K+4it;L^1(0,T;L^2)\right)}.
\end{aligned} \tag{3.16}
$$

We have

$$
\begin{aligned}
&\left\| |v|^2 v \right\|_{G^{\alpha}\left(J,K+4it;L^2(0,T;L^2)\right)} \\
&\le \frac{C}{1-abT} \left\| |v|^2 v \right\|_{G^{\alpha}\left(J,\tilde{K};L^1(0,T;L^2)\right)} \quad \text{(by Lemma 2.3)} \\
&\le \frac{C}{1-abT} \|v\|^3_{G^{\alpha}\left(J,\tilde{K};L^3(0,T;L^6)\right)} \quad \text{(by Lemma 2.7)} \\
&\le \frac{C}{(1-abT)^4} \|v\|^3_{G^{\alpha}\left(J,K;L^3(0,T;L^6)\right)} \quad \text{(by Lemma 2.3)}.
\end{aligned} \tag{3.17}
$$

We use (3.17) in the right-hand side of (3.16) to get

$$
\|u(t)\|_{G^{\alpha}(J,K;L^2)} \le \|\phi\|_{G^{\alpha}(x,|x|^2;L^2)} + CT^{\frac{1}{2}} \rho^3. \tag{3.18}
$$

From (3.15) and (3.18) it follows that

$$
\|u\|_{X_T} \le C \left(\|\phi\|_{G^{\alpha}\left(x,|x|^2;L^2\right)} + T^{\frac{1}{2}} \rho^3 \right). \tag{3.19}
$$

In the same way as in the proof of (3.19) we have by Lemma 2.6 and Lemma 2.7,

$$
\|Mv_1 - M_{v_2}\|_{X_T} \le CT^{\frac{1}{2}} \rho^2 \|v_1 - v_2\|_{X_T}. \tag{3.20}
$$

We take

$$
C\|\phi\|_{G^{\alpha}\left(x,|x|^2;L^2\right)} \le \frac{\rho}{2}, \quad CT^{\frac{1}{2}} \rho^2 \le \frac{1}{2}.
$$

Then (3.19) and (3.20) show that there exists a T such that M is a contraction mapping from $X_{T,\rho}$ into itself. This completes the proof of Proposition 3.2.

Proposition 3.3. *We assume that* $n = 2$ *or* 3 *and* $p = 1$ *when* $n = 3$, $p \in \mathbb{N}$ *when* $n = 2$, *and*

$$\phi \in G^{\alpha}\left(x, |x|^2; R^1(0)\right).$$

Then there exists a unique solution $u(t,x)$ *of* (1.1) *and a positive constant* T *such that*

$$u(t,x) \in G^{\alpha}\left(J, K; R^1(t)\right) \quad \text{for} \quad t \in [-T, T].$$

Proof. We define the function space as follows:

$$X_T = \left\{ f \in L^{\infty}(0,T;L^2); \|f\|_{X_T} < \infty \right\},$$

where

$$\|f\|_{X_T} = \|f\|_{G^{\alpha}\left(J,K;L^{\infty}(0,T;R^1(t))\right)} + \|f\|_{G^{\alpha}\left(J,K;L^{r}(0,T;R^{1,r}(t))\right)},$$

and $r = 2 + 4/n$. We also define a closed ball

$$X_{T,\rho} = \{ f \in X_T; \|f\|_{X_T} \le \rho \}.$$

We let M be defined by $u = Mv$, where $v \in X_{T,\rho}$. In the same way as in the proof of (3.10),

$$\begin{aligned} &\|J^{\alpha}K^{l}u\|_{L^r\left(0,T:R^{1,r}(t)\right)} \le C\|x^{\alpha}|x|^{2l}\phi\|_{R^1(0)} \\ &+ C\left\|\int_0^t (t-\tau)^{-\frac{n}{2}(1-\frac{2}{r})}\left\|J^{\alpha}(K+4i\tau)^l|v|^{2p}v(\tau)\right\|_{R^{1,r'}(\tau)}d\tau\right\|_{L^r(0,T)}. \end{aligned} \tag{3.21}$$

The similar arguments as in (3.13) and (3.21) give

$$\begin{aligned} &\|u\|_{G^{a}\left(J,K;L^r(0,T;R^{1,r}(t))\right)} \le C\Big(\|\phi\|_{G^{a}\left(x,|x|^2;R^1(0)\right)} \\ &+ T^{\frac{2n}{2n+4}}\left\||v|^{2p}v\right\|_{G^{a}\left(J,K+4it;L^r(0,T;R^{1,r'}(t))\right)}\Big). \end{aligned} \tag{3.22}$$

We have

$$\begin{aligned} &\left\||v|^{2p}v\right\|_{G^{a}\left(J,K+4it;L^r(0,T;R^{1,r'}(t))\right)} \\ &\le \frac{C}{1-abT}\left\||v|^{2p}v\right\|_{G^{a}\left(J,\tilde{K};L^r(0,T;R^{1,r'}(t))\right)} \quad \text{(by Lemma 2.3)} \\ &\frac{C}{1-abT}\|v\|^{2p}_{G^{a}\left(J,\tilde{K};L^{\infty}(0,T;R^1(t))\right)}\|v\|_{G^{a}\left(J,\tilde{K};L^r(0,T;R^{1,r}(t))\right)} \quad \text{(by Lemma 2.13)} \\ &\le \frac{C}{(1-abT)^{2p+2}} \\ &\times\|v\|^{2p}_{G^{a}\left(J,K;L^{\infty}(0,T;R^1(t))\right)}\|v\|_{G^{a}\left(J,K;L^(0,T;R^1(t))\right)} \quad \text{(by Lemma 2.3).} \end{aligned} \tag{3.23}$$

From (3.22) and (3.23) it follows that

$$\|u\|_{G^{\alpha}\left(J,K;L^r(0,T;R^{1,r}(t))\right)} \le C\left(\|\phi\|_{G^{\alpha}\left(x,|x|^2;R^1(t)\right)} + T^{\frac{2n}{2n+4}}\rho^{2p+1}\right), \tag{3.24}$$

provided that

$$T \le \frac{1}{2ab}.$$

In the same way as in the proof of (3.16)

$$\begin{aligned}&\|u\|_{G^a\left(J,K;L^\infty(0,T;R^1(t))\right)}\\&\le \|\phi\|_{G^\alpha\left(x,|x|^2;R^1(0)\right)} + C\left\||v|^{2p}v\right\|_{G^\alpha\left(J,K+4it;L^1(0,T;R^1(t))\right)}.\end{aligned}\tag{3.25}$$

We have by Lemma 2.3,

$$\left\||v|^{2p}v\right\|_{G^\alpha\left(J,K+4it;L^1(0,T;R^1(t))\right)} \le \frac{C}{1-abT}\left\||v|^{2p}v\right\|_{G^\alpha\left(J,\tilde{K};L^1(0,T;R^1(t))\right)}.\tag{3.26}$$

We apply Lemma 2.14 to (3.26) to see that the right-hand side of (3.26) is bounded from above by

$$\frac{CT^{3/5}}{1-abT}\|v\|_{G^\alpha\left(J,\tilde{K};L^\infty(0,T;R^1(t))\right)}\|v\|^2_{G^\alpha\left(J,\tilde{K};L^r(0,T;R^{1,r}(t))\right)} \quad \text{for } n = 3 \tag{3.27}$$

and

$$\frac{CT^{3/4}}{1-abT}\|v\|^{2p}_{G^\alpha\left(J,\tilde{K};L^\infty(0,T;R^1(t))\right)}\|v\|_{G\alpha\left(J,\tilde{K};L^r(0,T;R^{1,r}(t))\right)} \quad \text{for } n = 2. \tag{3.28}$$

We use Lemma 2.3 in (3.27) and (3.28) to get

$$\left\||v|^{2p}v\right\|_{G^\alpha\left(J,K+4it;L^1(0,T;R^1(t))\right)} \le C\begin{cases} T^{3/5}\rho^3 & \text{for } n = 3,\\ T^{3/4}\rho^{2p+1} & \text{for } n = 2,\end{cases}\tag{3.29}$$

provided that

$$T \le \frac{1}{2ab}.$$

Hence by (3.25) and (3.29),

$$\|u\|_{G^\alpha\left(J,K;L^\infty(0,T;R^1(t))\right)} \le \|\phi\|_{G^\alpha\left(x,|x|^2;R^1(0)\right)} + C\max\left(T^{\frac{3}{5}},T^{\frac{3}{4}}\right)\rho^{2p+1}.\tag{3.30}$$

From (3.24) and (3.30) we see that there exists a T such that

$$\|u\|_{X_T} \le \rho.\tag{3.31}$$

In the same way as in the proof of (3.31) we have by Lemma 2.3, Lemma 2.13 and Lemma 2.14

$$\|Mv_1 - Mv_2\|_{X_T} \le \frac{\rho}{2}\|v_1 - v_2\|_{X_T}\tag{3.32}$$

for a sufficiently small T. Proposition 3.3 follows from (3.31) and (3.32).

4. Proofs of Theorems

We first prove

Lemma 4.1. *We let* $\tilde{P} = x \cdot \nabla + t\partial_t$*. Then we have for any* $k \in \mathbb{N}$*,*

$$\sum_{0 \le l \le m} \frac{a^{l+k}}{(l+k)!} \|t^l \tilde{P}^l (t\tilde{P})^k f\|_{X_T} \le \sum_{0 \le l \le m} \frac{a^k}{(l+k)!} \left(\frac{a}{2 - e^{a|t|}} \right) \left\|(t\tilde{P})^{l+k} f\right\|_{X(t)},$$

provided that $\|t \cdot \|_{X_t} \le |t| \| \cdot \|_{X(t)}$ *and* $2 - e^{a|t|} > 0$.

Proof. We prove the lemma by induction. It is clear that the lemma holds for $m = 0$ and any k. We assume that the lemma holds for m and any k. We have

$$\begin{aligned}
&\sum_{0 \le l \le m+1} \frac{a^{l+k}}{(l+k)!} \left\|t^l \tilde{P}^l (t\tilde{P})^k f\right\|_{X(t)} \\
&= \frac{a^k}{k!} \left\|(t\tilde{P})^k f\right\|_{X(t)} \\
&+ \sum_{1 \le l \le m+1} \frac{a^{l+k}}{(l+k)!} \left\| \left(t^{l-1}[t, \tilde{P}^{l-1}]\tilde{P} + t^{l-1}\tilde{P}^{l-1}(t\tilde{P})\right) (t\tilde{P})^k f\right\|_{X(t)} \\
&\le \frac{a^k}{k!} \left\|(t\tilde{P})^k f\right\|_{X(t)} + \sum_{1 \le l \le m+1} \frac{a^{l+k}}{(l+k)!} \left\|t^{l-1}\tilde{P}^{l-1}(t\tilde{P})^{k+1} f\right\|_{X(t)} \\
&\quad + \sum_{2 \le l \le m+1} \frac{a^{l+k}}{(l+k)!} \left\|t^{l-1}[t, \tilde{P}^{l-1}]\tilde{P}(t\tilde{P})^k f\right\|_{X(t)}.
\end{aligned} \tag{4.1}$$

We prove by induction

$$\tilde{P}^l t = t(\tilde{P} + 1)^l. \tag{4.2}$$

In case $l = 0$, (4.2) is valid. We assume that (4.2) holds for l. Then we have

$$\begin{aligned}
\tilde{P}^{l+1} t &= \tilde{P} t (\tilde{P} + 1)^l \quad \text{(by assumption)} \\
&= \left(t\tilde{P} + [\tilde{P}, t]\right) (\tilde{P} + 1)^l = t(\tilde{P} + 1)^{l+1}.
\end{aligned} \tag{4.3}$$

This completes the proof of (4.2). From (4.2) we have

$$\tilde{P}^l t = t \sum_{1 \le j \le l} \binom{l}{j} \tilde{P}^{l-j} + t\tilde{P}^l.$$

Hence

$$[t, \tilde{P}^l] = -t \sum_{1 \le j \le l} \binom{l}{j} \tilde{P}^{l-j}. \tag{4.4}$$

From (4.4) it follows that

$$\begin{aligned}
&\sum_{2\le l\le m+1} \frac{a^{l+k}}{(l+k)!} \left\| t^{l-1}[t, \tilde{P}^{l-1}]\tilde{P}(t\tilde{P})^k f \right\|_{X(t)} \\
&\le \sum_{2\le l\le m+1} \frac{a^{l+k}}{(l+k)!} \sum_{1\le j\le l-1} \binom{l-1}{j} |t|^l \left\| \tilde{P}^{l-j}(t\tilde{P})^k f \right\|_{X(t)} \\
&= \sum_{2\le l\le m+1} \left(\sum_{1\le j\le l-1} \frac{a^{l+k}(l-1)!}{(l+k)!(l-1-j)!j!} |t|^l \left\| \tilde{P}^{l-j}(t\tilde{P})^k f \right\|_{X(t)} \right). \qquad (4.5)
\end{aligned}$$

We have for $k \in \mathbb{N} \cup \{0\}$,

$$\frac{(l-1)!}{(l+k)!(l-1-j)!} \le \frac{1}{(l+k-j)!}.$$

Hence the right-hand side of (4.5) is bounded from above by

$$\sum_{2\le l\le m+1} \left(\sum_{1\le j\le l-1} \frac{a^{l+k}|t|^j}{(l+k-j)!j!} \left\| t^{l-j}\tilde{P}^{l-j}(t\tilde{P})^k f \right\|_{X(t)} \right). \qquad (4.6)$$

Since

$$\sum_{2\le l\le m+1} \left(\sum_{1\le j\le l-1} a_{l-j} b_j \right) \le \left(\sum_{1\le l\le m} a_l \right) \left(\sum_{1\le l\le m} b_l \right),$$

we obtain by (4.6) if we put

$$\begin{aligned}
a_l &= \frac{a^{l+k}}{(l+k)!} \left\| t^l \tilde{P}^l(t\tilde{P})^k f \right\|_{X(t)}, \quad b_l = \frac{(a|t|)^t}{l!}, \\
&\le \left(\sum_{1\le l\le m} \frac{(a|t|)^l}{l!} \right) \left(\sum_{1\le l\le m} \frac{a^{l+k}}{(l+k)!} \left\| t^l \tilde{P}^l(t\tilde{P})^k f \right\|_{X(t)} \right) \\
&\le \left(e^{a|t|} - 1 \right) \sum_{1\le l\le m} \frac{a^{l+k}}{(l+k)!} \left\| t^l \tilde{P}^l(t\tilde{P})^k f \right\|_{X(t)}. \qquad (4.7)
\end{aligned}$$

From (4.1) and (4.7) it follows that

$$\begin{aligned}
&\textstyle\sum_{0\le l\le m+1} \frac{a^{l+k}}{(l+k)!} \left\| t^l \tilde{P}^l(t\tilde{P})^k f \right\|_{X(t)} \\
&\le \tfrac{a^k}{k!} \left\| (t\tilde{P})^k f \right\|_{X(t)} + \textstyle\sum_{0\le l\le m} \frac{a^{l+1+k}}{(l+1+k)!} \left\| t^l \tilde{P}^l(t\tilde{P})^{k+1} f \right\|_{X(t)} \\
&\quad + \left(e^{a|t|} - 1 \right) \textstyle\sum_{1\le l\le m} \frac{a^{l+k}}{(l+k)!} \left\| t^l \tilde{P}^l(t\tilde{P})^k f \right\|_{X(t)} \\
&\le \tfrac{a^k}{k!} \left\| (t\tilde{P})^k f \right\|_{X(t)} + \textstyle\sum_{0\le l\le m} \frac{a^{1+k}}{(l+1+k)!} \left(\frac{a}{2-e^{a|t|}} \right)^l \left\| (t\tilde{P})^{l+k+1} f \right\|_{X(t)} \\
&\quad + \left(e^{a|t|} - 1 \right) \textstyle\sum_{1\le l\le m} \frac{a^{l+k}}{(l+k)!} \left\| t^l \tilde{P}^l(t\tilde{P})^k f \right\|_{X(t)} \quad \text{(by assumption)} \\
&= \tfrac{a^k}{k!} \left\| (t\tilde{P})^k f \right\|_{X(t)} + a \textstyle\sum_{1\le l\le m+1} \frac{a^k}{(l+k)!} \left(\frac{a}{2-e^{a|t|}} \right)^{l-1} \left\| (t\tilde{P})^{l+k} f \right\|_{X(t)} \\
&\quad + \left(e^{a|t|} - 1 \right) \textstyle\sum_{1\le l\le m} \frac{a^{l+k}}{(l+k)!} \left\| t^l \tilde{P}^l(t\tilde{P})^k f \right\|_{X(t)}.
\end{aligned}$$

Therefore

$$\sum_{1\le l\le m+1} \frac{a^{l+k}}{(l+k)!} \left\| t^l \tilde{P}^l (t\tilde{P})^k f \right\|_{X(t)}$$
$$\le \sum_{1\le l\le m+1} \frac{a^k}{(l+k)!} \left(\frac{a}{2-e^{a|t|}} \right)^l \left\| (t\tilde{P})^{l+k} f \right\|_{X(t)},$$

which means the lemma holds for $m+1$. This completes the proof of the lemma.

We next prove the multi-dimensional version of [H-K.K, Lemma 2.4 (2.2)].

Lemma 4.2. *We have for any* $k \in \mathbb{N}$,

$$\sum_{0\le l\le m} \frac{a^{l+k}}{(l+k!)} \left\| (x\cdot\nabla)^{l+k} f \right\|_{X(t)}$$
$$\le \sum_{|\alpha|\le m} \frac{a^k}{(a+k)!} \left(\frac{a}{1-a} \right)^{|\alpha|} \left\| (x\cdot\nabla)^k x^\alpha \partial^\alpha f \right\|_{X(t)},$$

provided that $0 < a < 1$.

Proof. We prove by induction. It is clear that the lemma holds for $m = 0$ and any k. We assume that the lemma holds for m and any k. Then we have by the assumption

$$\begin{aligned}
&\sum_{0\le l\le m+1} \frac{a^{l+k}}{(l+k)!} \left\| (x\cdot\nabla)^{k+1} f \right\|_{X(t)} \\
&= \frac{a^k}{k!} \left\| (x\cdot\nabla)^k f \right\|_{X(t)} + \sum_{0\le l\le m} \frac{a^{l+1+k}}{(l+1+k)!} \left\| (x\cdot\nabla)^{k+1} (x\cdot\nabla)^l j \right\|_{X(t)} \qquad (4.8)\\
&\le \frac{a^k}{k!} \left\| (x\cdot\nabla)^k f \right\|_{X(t)} + \sum_{|\alpha|\le m} \frac{a^{1+k}}{(\alpha+1+k)!} \left(\frac{a}{1-a} \right)^{|\alpha|} \left\| (x\cdot\nabla)^{k+1} x^\alpha \partial^\alpha f \right\|_{X(t)}.
\end{aligned}$$

Since

$$(x\cdot\nabla) x^\alpha \partial^\alpha = \sum_{1\le j\le n} (a_j x^\alpha \partial^\alpha + x_j x^\alpha \partial_j \partial^\alpha),$$

we have by (4.8),

$$\begin{aligned}
&\sum_{0\le l\le m+1} \frac{a^{l+k}}{(l+k)!} \left\| (x\cdot\nabla)^{k+1} f \right\|_{X(t)} \le \frac{a^k}{k!} \left\| (x\cdot\nabla)^k f \right\|_{X(t)} \\
&+ \sum_{|\alpha|\le m} \frac{a^{k+1}}{(\alpha+k+1)!} \left(\frac{a}{1-a} \right)^{|\alpha|} \left(\sum_{1\le j\le n} \Big(a_j \left\| (x\cdot\nabla)^k x^\alpha \partial^\alpha f \right\|_{X(t)} \right. \\
&\qquad \left. + \left\| (x\cdot\nabla)^k x_j x^\alpha \partial_j \partial^\alpha f \right\|_{X(t)} \Big) \right). \qquad (4.9)
\end{aligned}$$

By a simple calculation

$$\sum_{1\le j\le n} \frac{a_j}{(\alpha+1)!} = \left(\frac{\alpha_1+\alpha_2+\cdots \alpha_n}{(\alpha+1)!(\alpha+2)!\cdots(\alpha_n+1)!} \right) \le \frac{1}{\alpha!}.$$

Hence by (4.9),

$$\begin{aligned}
&\sum_{0\le l\le m+1} \frac{a^{l+k}}{(l+k)!} \left\| (x\cdot\nabla)^{k+1} f \right\|_{X(t)} \\
&\le \frac{a^k}{k!} \left\| (x\cdot\nabla)^k f \right\|_{X(t)} + a \sum_{1\le|\alpha|\le m} \frac{a^k}{(\alpha+k)!} \left(\frac{a}{1-a} \right)^{|\alpha|} \left\| (x\cdot\nabla)^k x^\alpha \partial^\alpha f \right\|_{X(t)} \\
&\quad + (1-a) \sum_{1\le|\alpha|\le m+1} \frac{a^k}{(\alpha+k)!} \left(\frac{a}{1-a} \right)^{|\alpha|} \left\| (x\cdot\nabla)^k x^\alpha \partial^\alpha f \right\|_{X(t)} \\
&\le \sum_{|\alpha|\le m+1} \frac{a^k}{(\alpha+k)!} \left(\frac{a}{1-a} \right)^{|\alpha|} \left\| (x\cdot\nabla)^k x^\alpha \partial^\alpha f \right\|_{X(t)}.
\end{aligned}$$

This completes the proof of the lemma.

Proof of Theorems 1.1 and 1.2. Theorems 1.1 and 1.2 are obtained if we prove that there exist constants b_1, b_2, and T such that

$$\|M(t)u\|_{G^{b_1}\left(t\partial;G^{b_2t^2}(\partial_t;L^2(\Omega))\right)} < \infty, \quad M(t) = e^{-\frac{i|x|^2}{2t}} \tag{4.10}$$

for $t \in (-T,T)\setminus\{0\}$. For simplicity we assume that $t > 0$ since the negative time is treated similarly. By [H-K.K Lemma 2.4 (2.1)]

$$\begin{aligned}
&\|M(t)u\|_{G^{b_1}\left(t\partial;G^{b_2t^2}(\partial t;L^2(\Omega))\right)} \\
&= \sum_\alpha \frac{(b_1)^{|\alpha|}}{\alpha!} \sum_l \frac{(b_2t^2)^t}{l!} \left\| \partial_t^l (it\partial)^\alpha M(t)u \right\|_{L^2(\Omega)} \\
&\le \sum_\alpha \frac{(b_1)^{|\alpha|}}{\alpha!} \sum_l \frac{1}{l!} \left(\frac{b_2t}{1-b_2t} \right)^l \|(t\partial_t)^t v_\alpha\|_{L^2(\Omega)}
\end{aligned} \tag{4.11}$$

where $v_\alpha = (it\partial)^\alpha M(t)u$, and positive constants b_1 and b_2 are determined later. By Reibniz' rule we see that for $\tilde{P} = x\cdot\nabla + t\partial_t$,

$$(t\partial_t)^l = (\tilde{P} - x\cdot\nabla)^l = \sum_{0\le k\le l} \binom{l}{k} \tilde{P}^{l-k} (x\cdot\nabla)^k,$$

since $[x\cdot\nabla, \tilde{P}] = 0$. We use the above equation in the right-hand side of (4.11) to obtain

$$\begin{aligned}
&\|M(t)u\|_{G^{b_1}\left(t\partial;G^{b_2t^2}(\partial_t;L^2(\Omega))\right)} \\
&\le \sum_{\alpha,l} \frac{(b_1)^{|\alpha|}}{\alpha!} \sum_{k\le l} \left(\frac{b_2t}{1-b_2t} \right)^l \frac{1}{(l-k)!k!} \left\| (x\cdot\nabla)^k \tilde{P}^{l-k} v_\alpha \right\|_{L^2(\Omega)} \\
&\le \sum_{\alpha,l_1,l_2} \frac{(b_1)^{|\alpha|}}{\alpha!} \left(\frac{b_2t}{1-b_2t} \right)^{l_1+l_2} \frac{1}{l_1!l_2!} \left\| (x\cdot\nabla)^{l_1} \tilde{P}^{l_2} v_\alpha \right\|_{L^2(\Omega)}.
\end{aligned}$$

We use Lemma 4.2 to the above to get

$$
\begin{aligned}
&\|M(t)u\|_{G^{b_1}\left(t\partial;G^{b_2t^2}(\partial_t;L^2(\Omega))\right)} \\
&\le \sum_{\alpha} \frac{(b_1)^{|\alpha|}}{\alpha!} \sum_{l_1,\beta} \left(\frac{b_2t}{1-b_2t}\right)^{l_1} \left(\frac{b_2t}{1-2b_2t}\right)^{|\beta|} \frac{1}{l_1!\beta!} \|x^\beta \partial^\beta \tilde{P}^{l_1} v_\alpha\|_{L^2(\Omega)} \\
&\le \sum_{\alpha} \frac{(b_1)^{|\alpha|}}{\alpha!} \sum_{l_1,\beta} \frac{(2b_2t)^{t_1}(2Rb_2t)^{|\beta|}}{l_1!\beta!} \|\partial^\beta \tilde{P}^{l_1} v_\alpha\|_{L^2(\Omega)} \\
&= \sum_{\alpha!} \frac{(b_1)^{|\alpha|}}{\alpha!} \sum_{l_1,\beta} \frac{(2b_2t)^{l_1}(2Rb_2)^{|\beta|}}{l_1!\beta!} \left\| J^\beta \left(M(-t)\tilde{P}M(t)\right)^{l_1} J^\alpha u \right\|_{L^2(\Omega)},
\end{aligned}
\tag{4.12}
$$

provided that

$$
t < \frac{1}{2b_2}.
$$

By Lemma 4.1 and (4.12)

$$
\begin{aligned}
&\|M(t)u\|_{G^{b_1}\left(t\partial;G^{b_2t^2}(\partial_t;L^2(\Omega))\right)} \\
&\le \sum_{\alpha,l_1,\beta} \frac{1}{\alpha!l_1!\beta!}(b_1)^{|\alpha|} \left(\frac{b_2}{2-e^{2b_2|t|}}\right)^{l_1} (2Rb_2)^{|\beta|} \|J^\alpha \tilde{K}^{l_1} J^\alpha u\|_{L^2} \\
&\le \sum_{\alpha,l_1,\beta} \frac{1}{\alpha!l_1!\beta!}(b_1)^{|\alpha|}(2b_2)^{l_1}(2Rb_2)^{|\beta|} \|\tilde{(}K)^{l_1} J^{\alpha+\beta} u\|_{L^2},
\end{aligned}
\tag{4.13}
$$

provided that

$$
t < \frac{1}{2b_2} \log \frac{3}{2},
$$

where we have used the commutation relation

$$
[\tilde{K}, J] = 0. \tag{4.14}
$$

Hence we have by (4.13),

$$
\begin{aligned}
&\|M(t)u\|_{G^{b_1}\left(t\partial;G^{b_2t^2}(\partial_t;L^2(\Omega))\right)} \\
&\le \sum_{\alpha,\beta} \frac{b_1^{|\alpha|}(2Rb_2)^{|\beta|}}{\alpha!\beta!} \|J^{\alpha+\beta} u\|_{G^{2b_2}(\tilde{K};L^2(\Omega))} \\
&\le C\|u\|_{G^{(b_1+2Rb_2)}\left(J;G^{2b_2}(\tilde{K};L^2(\Omega))\right)}.
\end{aligned}
\tag{4.15}
$$

We put

$$
a = \max\{b_1 + 2Rb_2, 2b_2\} \quad \text{with} \quad b_2 \le \frac{1}{2t} \log \frac{3}{2}.
$$

Then (4.14) and (4.15) imply that

$$
\begin{aligned}
&\|M(t)u\|_{G^{b_1}\left(t\partial;G^{b_2t^2}(\partial_t;L^2(\Omega))\right)} \\
&\le C\|u\|_{G^a\left(\tilde{K},J;L^2(\Omega)\right)} \le C\|u\|_{G^a(J,\tilde{K};L^2)}.
\end{aligned}
\tag{4.16}
$$

From Propositions 3.1–3.3 we see that the right-hand side of (4.16) is bounded if

$$\|\phi\|_{G^a\left(x,\|x|^2;R^m(0)\right)} \le C \left\| e^{|x|^2}\phi \right\|_{H^m}. \tag{4.17}$$

We have

$$\begin{aligned}\|\phi\|_{G^a\left(x,|x|^2;R^m(0)\right)} &= \sum_{\alpha,k} \frac{a^{|\alpha|+k}}{\alpha!k!} \sum_{|\beta|+|\gamma|\le m} \left\| x^\beta \partial^\gamma x^\alpha |x|^{2k}\phi \right\|_{L^2} \\ &\le \sum_{|\beta|+|\gamma|\le m} \sum_{\alpha,k} \frac{a^{|\alpha|+k}}{\alpha!k!} \left\| x^\alpha |x|^{2k} x^\beta \partial^\gamma \phi \right\|_{L^2},\end{aligned}$$

from which it follows that

$$\begin{aligned}\|\phi\|^2_{G^a\left(x,|x|^2;R^m(0)\right)} &\le \sum_{|\beta|+|\gamma|\le m} \sum_{\alpha,k} \frac{\left(\sqrt{2}a\right)^{2(|\alpha|+k)}}{(\alpha!)^2(k!)^2} \left\| x^\alpha |x|^{2k} x^\beta \partial^\gamma \phi \right\|^2_{L^2} \\ &\le \sum_{|\beta|+|\gamma|\le m} \sum_{\alpha,k} \frac{\left(2\sqrt{2}a\right)^{2(|\alpha|+k)}}{(2a)!(2k)!} \left\| x^\alpha |x|^{2k} x^\beta \partial^\gamma \phi \right\|^2_{L^2} \quad \left(\text{by } \frac{1}{(a!)^2} \le \frac{2^{2|\alpha|}}{(2\alpha)!}\right) \\ &\le C \sum_{|\beta|+|\gamma|\le m} \left\| \prod_{j=1}^{b} \cosh\left(2\sqrt{2}a(x_j + |x|^2)\right) x^\beta \partial^\gamma \phi \right\|^2_{L^2} \\ &\le C \sum_{|\beta|+|\gamma|\le m} \left\| e^{b|x|^2} x^\beta \partial^\gamma \phi \right\|^2_{L^2} \quad \text{for } b > 2\sqrt{2}a.\end{aligned} \tag{4.18}$$

From (4.18) it follows that

$$\|\phi\|_{G^a\left(x,|x|^2;R^m(0)\right)} \le C \left\| e^{|x|^2}\phi \right\|_{H^m} \quad \text{for } a < \frac{1}{2\sqrt{2}}.$$

This completes the proof of Theorem 1.1–1.2.

5. Applications

Our proof of the theorems can be applicable to

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda|u|^{2p}u + V(x)u, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, p \in \mathbb{N}, \lambda \in \mathbb{C}, \\ u(0,x) = \phi(x) \quad x \in \mathbb{R}^n \end{cases} \tag{5.1}$$

and

$$\begin{cases} i\partial_t u + \partial_x^2 u + 2i\delta\partial_x(|u|^2u) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}, \delta \in \mathbb{R}, \\ u(0,x) = \phi(x), \quad x \in \mathbb{R}. \end{cases} \tag{5.2}$$

We have for (5.1)

Proposition 5.1. *In addition to the assumption on Theorems 1.1–1.2 we assume that*

$$\sum_{\alpha} \frac{a^{|\alpha|}}{\alpha!} \|x^\alpha \partial^\alpha V\|_{G^b(\partial;H^m)} < \infty,$$

where $0 < a < 1$, $0 < b$. *Then the same result as in Theorems 1.1–1.2 are valid for the solutions of (5.1).*

Proof. In the same way as in the proofs of Theorems 1.1–1.2 we have the result if we prove that there exists a positive constant a_1 such that

$$V \in G^{a_1}\left(x \cdot \nabla; G^b(\partial; H^m)\right).$$

By Lemma 4.2 with $X = G^b(\partial; H^m)$ and $k = 0$ we find that

$$\|V\|_{G^{a_1}(x\cdot\nabla;X)} \le \sum_{\alpha} \left(\frac{a_1}{1-a_1}\right)^{|\alpha|} \frac{1}{\alpha!} \|x^\alpha \partial^\alpha V\|_X < \infty,$$

provided that $\frac{a_1}{1-a_1} \le a$. This completes the proof of Proposition 5.1.

The condition on V given in Proposition 5.1 is satisfied if $V(x)$ has an analytic continuation $V(z)$ on the complex domain

$$\begin{aligned}&\Gamma_{\sqrt{2}a,\sqrt{2}b} = \{z \in \mathbb{C}; z_j = x_j + iy_j; -\infty < x_j < +\infty,\\ &-\sqrt{2}b - (\tan\alpha)|x_j| < y_i < \sqrt{2}b + (\tan\alpha)|x_j|,\\ &j = 1, 2, \ldots, n, 0 < \alpha = \sin^{-1}\sqrt{2}a < \pi/2\}\end{aligned}$$

and

$$\int_{\Gamma_{\sqrt{2}a,\sqrt{2}b}} |V(z)|^2 dxdy < \infty,$$

(see the proof of Theorem 1.1 in([H-K.K]). Hence if $\sqrt{2}b < 1$,

$$V(x) = \frac{1}{(1+|x|^2)^m}, \quad \frac{1}{2}\left[\frac{n}{2}\right] < m \in \mathbb{N}$$

can be considered as the typical example satisfying the condition in Proposition 5.1.

The derivative nonlinear Schrödinger equation (5.2) can be translated into the system of nonlinear Schrödinger equations without nonlinear terms having derivatives of unknown function by using a gauge transformation. Indeed putting $u_1 = E^2 u$ and $u_2 = E\partial_x(Eu)$ with

$$E(t,x) = \exp(i\delta \int_{-\infty}^{x} |u(t,y)|^2 dy),$$

we have

$$\left\{\begin{array}{l} i\partial_t u_1 + \partial_x^2 u_1 = 2i\delta u_1^2 \overline{u}_2, \\ i\partial_t u_2 + \partial_x^2 u_2 = -2i\delta u_2^2 \overline{u}_1, \end{array}\right\}$$

(see [H] for details). Hence in the same way as in the proof of Theorem 1.2 we obtain for (5.2)

Proposition 5.2. *We assume that*

$$||e^{|x|^2}\phi||_{H^1} < \infty.$$

Then the same result as in Theorem 1.1 is valid for the solutions of (5.2).

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