Automorphism Group of $k((t))$:
Applications to the Bosonic String[★]

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Abstract: This paper is concerned with the formulation of a non-pertubative theory of the bosonic string. We introduce a formal group G which we propose as the "universal moduli space" for such a formulation. This is motivated because G establishes a natural link between representations of the Virasoro algebra and the moduli space of curves.

Contents

1. Introduction

On the moduli space of smooth algebraic curves of genus g, $\mathcal{M}_g$, one can define a family of determinant invertible sheaves $\{\lambda_n | n \in \mathbb{Z}\}$. In a remarkable paper, Mumford ([Mu]) proved the existence of canonical isomorphisms:

$$\lambda_n \overset{\sim}{\to} \lambda_1^{(6n^2 - 6n + 1)} \quad \forall n \in \mathbb{Z}$$

which have been studied in depth from different approaches.

For instance, within the frame of string theory, these isomorphisms are one of the main tools in the explicit computation of the Polyakov measure for bosonic strings in genus g ([BK, MM]). Proposals for developing a genus-independent (or "non-pertubative") formulation of the theory of bosonic strings have been made by several authors (e.g. [BR, Mo, BNS]).

In this paper we propose a "universal moduli space" as the main ingredient for a non-perturbative string theory which is different from those introduced by the above authors.

Following the spirit of previous papers ([AMP, MP]), where a "formal geometry" of curves and Jacobians was developed (see [BF, P] for other applications of these ideas), we introduce a formal group scheme G representing the functor of automorphisms of $k((t))$ (see Sect. 3); more precisely, the points of G with values in a k-scheme S are:

$$G(S) = \mathrm{Aut}_{H^0(S, \mathcal{O}_S) - \mathrm{alg}} H^0(S, \mathcal{O}_S)((t)).$$

The formal group scheme G might be interpreted as a formal moduli scheme for parametrized formal curves. The canonical action of G on the infinite Grassmannian $\mathrm{Gr}(k((t))dt^{\otimes n})$ allows us to construct an invertible sheaf, Λ_n, on G (for every $n \in \mathbb{Z}$) endowed with a bitorsor structure. Using a generalization of the Lie Theory for certain non commutative groups (given in Appendix B), we prove that these sheaves satisfy an analogous formula of the Mumford Theorem; that is, there exist canonical isomorphisms (see Theorem 4.5):

$$\Lambda_n \xrightarrow{\sim} \Lambda_1^{(6n^2 - 6n + 1)} \quad \forall n \in \mathbb{Z}.$$

To show that our formula is a local version of Mumford's, rather than a mere "coincidence", we relate G and the moduli of curves by means of infinite Grassmannians (see subsection 4.4 for precise statements). Let $\mathcal{M}_g^\infty$ be the moduli space of pointed curves of genus g with a given parameter at the point (see Definition 4.7). Then, the action of G on $\mathrm{Gr}(k((t)))$ induces an action, ϕ, on $\mathcal{M}_g^\infty$. Moreover, given a rational point $X \in \mathcal{M}_g^\infty$, the action induces a morphism of schemes:

$$G \xrightarrow{\phi_X} \mathcal{M}_g^\infty.$$

Let $\widehat{\phi}_X$ be the composite of the immersion of $\widehat{G}$ (the formal completion of G at the identity) into G, ϕ_X, and the projection $\mathcal{M}_g^\infty \to \mathcal{M}_g$. Let $(\mathcal{M}_G^\infty)_X^{\widehat{\ }}$ be the formal completion of $\mathcal{M}_G^\infty$ at X. Then, from the surjectivity of the map $\widehat{G} \to (\mathcal{M}_G^\infty)_X^{\widehat{\ }}$ induced by ϕ_X (see Theorem 4.11), it follows easily that there exist isomorphisms:

$$\widehat{\phi}_X^*(\lambda_n) \xrightarrow{\sim} \Lambda_n \quad \forall n \in \mathbb{Z}.$$

Finally, the last section offers a proposal on how to apply these results to a non-perturbative formulation of the bosonic string. The explicit development of these ideas and the geometric interpretation of partition functions in terms of the geometry of the group G will be performed elsewhere.

2. Background on Grassmannians

2.1. The Grassmannian $\mathrm{Gr}(k((t)))$. This section summarizes results on infinite Grassmannians as given in [AMP] in order to set notations and to recall the facts we will need.

Below, V will always denote the k-vector space $k((t))$ and V^+ the subspace $k[[t]]$. Let $\mathcal{B}^f$ be the set of subspaces generated by $\{t^{s_0}, t^{s_1}, \dots\}$ for every strictly increasing sequence of integers $s_0 < s_1 < \dots$ such that $s_{i+1} = s_i + 1$ for $i >> 0$. Let $\mathcal{B}$ denote the set of subspaces of V given by the t-adic completion of the elements of $\mathcal{B}^f$. We can now interpret $\mathcal{B}$ as a basis of a topology on V. It is easy to characterize the neighborhoods of 0 as the set of subspaces A of V such that there exists an integer $n >> 0$ with $t^n k[[t]] \subseteq A$ and it is of finite codimension.

Now the pair $(V, \mathcal{B})$ satisfies the following properties:

- the topology is separated and V is complete,
- for every $A, B \in \mathcal{B}$, it holds that $(A + B)/(A \cap B)$ is finite dimensional,
- if $A, B \in \mathcal{B}$, then $A + B, A \cap B \in \mathcal{B}$,
- $V/A = \varinjlim_{B \in \mathcal{B}} (B + A/A)$ for every $A \in \mathcal{B}$,

and hence there exists a k-scheme, called the Grassmannian of $(V, \mathcal{B})$ and denoted by $\mathrm{Gr}^\bullet(V)$, whose S-valued points is the set:

$$\left\{ \begin{array}{l} \text{quasi-coherent sub-}\mathcal{O}_S\text{-modules } L \subseteq \hat{V}_S \text{ such that for every point } s \in S, \\ L_{k(s)} \subseteq \hat{V}_{k(s)} \text{ and there exists an open neighborhood } U \text{ of } s \text{ and } A \in \mathcal{B} \\ \text{such that } \hat{V}_U/L_U + \hat{A}_U = (0) \text{ and } L_U \cap \hat{A}_U \text{ is free of finite type} \end{array} \right\}$$

($k(s)$ is the residual field of s) where $\hat{L}_T := \varprojlim(L/L \cap A_S) \otimes_{\mathcal{O}_S} \mathcal{O}_T$ for a submodule L of V_S and a morphism of k-schemes $T \to S$.

The very construction of $\mathrm{Gr}^\bullet(V)$ shows that $\{F_A \mid A \in \mathcal{B}\}$ is an open covering by affine subschemes where F_A is the k-scheme whose S-valued points are:

$$\left\{ \text{locally free sub-}\mathcal{O}_S\text{-modules } L \subseteq \hat{V}_S \text{ such that } L_S \oplus \hat{A}_S \simeq \hat{V}_S \right\}.$$

From this fact one deduces (see [AMP]) that the complexes of $\mathcal{O}_{\mathrm{Gr}^\bullet(V)}$-modules $\mathcal{L} \oplus \hat{A}_{\mathrm{Gr}^\bullet(V)} \to \hat{V}_{\mathrm{Gr}^\bullet(V)}$ are perfect ($\mathcal{L}$ being the universal object of $\mathrm{Gr}^\bullet(V)$) for every $A \in \mathcal{B}$. Moreover, the Euler–Poincaré characteristic of the complex $\mathcal{L} \oplus \hat{A}_{\mathrm{Gr}^\bullet(V)} \to \hat{V}_{\mathrm{Gr}^\bullet(V)}$:

$$L \longmapsto \dim(L \cap V^+) - \dim(V/L + V^+)$$

gives the decomposition of $\mathrm{Gr}^\bullet(V)$ into connected components. The connected component of characteristic 0 will be denoted by $\mathrm{Gr}(V)$. It is easy to show that these complexes are all quasi-isomorphic.

From the theory of [KM] on determinants, it follows that their determinants are well defined and that they are isomorphic. The choice of $V^+ \in \mathcal{B}$ now enables us to construct a line bundle on the Grassmannian as follows: on the connected component of characteristic n consider the determinant of $\mathrm{Det}(\mathcal{L} \oplus t^n \hat{V}^+_{\mathrm{Gr}^n(V)} \to \hat{V}_{\mathrm{Gr}^n(V)})$. The resulting bundle will be called "the determinant bundle" and will be denoted simply by Det_V.

It is also known that given a complex $\mathcal{L} \oplus \hat{A}_{\mathrm{Gr}^{\bullet}(V)} \xrightarrow{\delta_A} \hat{V}_{\mathrm{Gr}^{\bullet}(V)}$ ($A \in \mathcal{B}$), the morphism δ_A gives a section of $\mathrm{Det}(\mathcal{L} \oplus \hat{A}_{\mathrm{Gr}^{\bullet}(V)} \to \hat{V}_{\mathrm{Gr}^{\bullet}(V)})^*$. By fixing the basis $\{t^n | n \in \mathbb{Z}\}$ of V one checks that the induced isomorphisms among determinants of these complexes are compatible (see [AMP]). Using such isomorphisms the above-defined section gives a section Ω_A of Det_V^*. The section defined on the connected component of characteristic n by the determinant of the addition homomorphism $\mathcal{L} \oplus t^n \hat{V}_{\mathrm{Gr}^n(V)}^+ \to \hat{V}_{\mathrm{Gr}^n(V)}$ will be denoted by Ω_+.

2.2. The linear group $\mathrm{Gl}(V)$.

For each k-scheme S, let us denote by $\mathrm{Aut}_{\mathcal{O}_S}(\hat{V}_S)$ the group of automorphisms of the $\mathcal{O}_S$-module $\hat{V}_S$.

Definition 2.1. • *A sub-$\mathcal{O}_S$-module $L \subseteq \hat{V}_S$ is said to be a $\mathcal{B}$-neighborhood if there exists a vector subspace $A \in \mathcal{B}$ such that $\hat{A}_S \subset L$ and $L/\hat{A}_S$ is locally free of finite type.*
• *An automorphism $g \in \mathrm{Aut}_{\mathcal{O}_S}(\hat{V}_S)$ is called $\mathcal{B}$-bicontinuous if $g(\hat{A}_S)$ and $g^{-1}(\hat{A}_S)$ are $\mathcal{B}$-neighborhoods for all $A \in \mathcal{B}$.*
• *The linear group, $\mathrm{Gl}(V)$, of $(V, \mathcal{B})$ is the contravariant functor over the category of k-schemes defined by:*

$$S \rightsquigarrow \mathrm{Gl}(V)(S) := \{g \in \mathrm{Aut}_{\mathcal{O}_S}(\hat{V}_S) \text{ such that } g \text{ is } \mathcal{B}\text{-bicontinuous}\}.$$

Theorem 2.2. *There exists a natural action, μ, of $\mathrm{Gl}(V)$ on the Grassmannian, preserving the determinant bundle.*

Proof. The first part is easy to show. It suffices to prove that $g(L)$ belongs to $\mathrm{Gr}^{\bullet}(V)(S)$ for an S-valued point $L \in \mathrm{Gr}^{\bullet}(V)(S)$ and an arbitrary $g \in \mathrm{Gl}(V)(S)$ using that g is $\mathcal{B}$-bicontinuous.

Note that given $g \in \mathrm{Gl}(V)(S)$ and an S-scheme, T, one has an induced isomorphism $\hat{V}_S/\hat{A}_S \to \hat{V}_S/g(\hat{A}_S)$ for each $A \in \mathcal{B}$. Twisting by $\mathcal{O}_T$, and taking the inverse limit over $A \in \mathcal{B}$, one obtains an $\mathcal{O}_T$-automorphism g_T of $\hat{V}_T$, which due to the very construction is $\mathcal{B}$-bicontinuous. Moreover, the map:

$$\mathrm{Gl}(V)(S) \to \mathrm{Gl}(V)(T),$$
$$g \mapsto g_T$$

is functorial. So, for an element $g \in \mathrm{Gl}(V)(S)$ we have constructed $g_T \in \mathrm{Gl}(V)(T)$ for every S-scheme T; hence, g yields an S-automorphism of $\mathrm{Gr}^{\bullet}(V)_S := \mathrm{Gr}^{\bullet}(V) \times_k S$. We have then constructed a functor homomorphism:

$$\mathrm{Gl}(V) \to \underline{\mathrm{Aut}}(\mathrm{Gr}^{\bullet}(V)),$$
$$g \mapsto g_{\bullet},$$

where $\underline{\mathrm{Aut}}(\mathrm{Gr}^{\bullet}(V))(S) := \mathrm{Aut}_{S\text{-sch}}(\mathrm{Gr}^{\bullet}(V)_S)$.

With the expression "preserving the determinant bundle" we mean that $g_{\bullet}^* p_1^* \mathrm{Det} \simeq p_1^* \mathrm{Det} \otimes p_2^* N$ (where p_i denotes the projection onto the ith factor of $\mathrm{Gr}^{\bullet}(V) \times_k S$) for a line bundle N over S. It is therefore enough to prove the statement when S is a local affine scheme.

Recall that:

$$g_\bullet^* p_1^* \operatorname{Det}_V \simeq \operatorname{Det}\left(g_\bullet^* p_1^* \mathcal{L} \oplus g_\bullet^* p_1^* \hat{A}_{\operatorname{Gr}^\bullet(V)} \to g_\bullet^* p_1^* \hat{V}_{\operatorname{Gr}^\bullet(V)}\right)$$

for $A \in \mathcal{B}$. Take $A \in \mathcal{B}$ such that $\hat{A}_S \subseteq g^{-1}(\hat{V}_S^+)$ and $g^{-1}(\hat{V}_S^+)/\hat{A}_S$ are free of finite type. Then, g induces an isomorphism:

$$g_\bullet^* p_1^* \operatorname{Det}_V \simeq p_1^* \operatorname{Det}_V \otimes \operatorname{Det}\left(p_1^* \hat{V}_{\operatorname{Gr}^\bullet(V)}^+ / g_\bullet^*(p_1^* \hat{A}_{\operatorname{Gr}^\bullet(V)})\right)^*.$$

From the very construction of $g_\bullet$ it follows that there is an isomorphism:

$$p_1^* \hat{V}_{\operatorname{Gr}^\bullet(V)}^+ / g_\bullet^*(p_1^* \hat{A}_{\operatorname{Gr}^\bullet(V)}) \simeq p_2^*\left(\hat{V}_S^+ / g(\hat{A}_S)\right)$$

and the claim follows. □

Theorem 2.3. *There exists a canonical central extension of functors of groups over the category of k-schemes:*

$$0 \to \mathbb{G}_m \to \widetilde{\operatorname{Gl}}(V) \to \operatorname{Gl}(V) \to 0$$

and a natural action, $\tilde{\mu}$, of $\widetilde{\operatorname{Gl}}(V)$ over the vector bundle, $\mathbb{V}(\operatorname{Det}_V)$, defined by the determinant bundle lifting the action μ.

Proof. For an affine k-scheme S, define $\mathcal{G}(S)$ as the set of commutative diagrams (in the category of S-schemes):

$$
\begin{array}{ccc}
\mathbb{V}(\operatorname{Det}_V^*)_S & \xrightarrow{\ \bar{g}\ } & \mathbb{V}(\operatorname{Det}_V^*)_S \\
\downarrow & & \downarrow \\
\operatorname{Gr}^\bullet(V)_S & \xrightarrow{\ g\ } & \operatorname{Gr}^\bullet(V)_S,
\end{array}
$$

where $\bar{g}$ is an isomorphism and $g \in \operatorname{Gl}(V)(S)$ and the homomorphism $\mathcal{G} \to \operatorname{Gl}(V)$ by $\bar{g} \mapsto g$. For an arbitrary scheme S define $\mathcal{G}(S)$ by sheafication; that is, consider a covering $\{U_i\}$ by open affine subschemes of S and $\mathcal{G}(S)$ the kernel of the restriction homomorphisms:

$$\prod_i \mathcal{G}(U_i) \rightrightarrows \prod_{i,j} \mathcal{G}(U_i \cap U_j).$$

We have then obtained an extension:

$$0 \to \prod_{\mathbb{Z}} \mathbb{G}_m \to \mathcal{G} \to \operatorname{Gl}(V) \to 0,$$

since $H^0(\operatorname{Gr}^\bullet(V)_S, \mathcal{O}_{\operatorname{Gr}^\bullet(V)_S}) = \prod_{\mathbb{Z}} H^0(S, \mathcal{O}_S)$ ([AMP]).

Finally, define $\widetilde{\operatorname{Gl}}(V)(S)$ as the direct image of this extension by the morphism $\prod_{\mathbb{Z}} \mathbb{G}_m \to \mathbb{G}_m$ which maps $\{a_i\}$ to a_0. Observe that for any projection $\{a_i\} \mapsto a_n$ the resulting extensions are isomorphic. □

Let us compute the cocycle associated with this central extension. For the sake of clarity we shall begin with the finite dimensional situation: V finite dimensional, $\{v_1, \ldots, v_d\}$ a basis, $\mathcal{B}$ consists of all finite dimensional subspaces and $V^+ :=\ <v_{n+1}, \ldots, v_d>$ (for an integer $0 \leq n \leq d$). Then, $\mathrm{Gr}(V)$ parametrizes the n-dimensional subspaces of V. Let $\bar{g}$ denote the morphism $<v_1, \ldots, v_n> \hookrightarrow V \overset{g}{\to} V \to V/V^+$ for an element $g \in \mathrm{Gl}(V)$ (observe that $\bar{g}$ consists of the first n columns and rows of the matrix associated with g).

We now have the following exact sequence:

$$0 \to \mathbb{G}_m \to \widetilde{\mathrm{Gl}}(V) \overset{p}{\to} \mathrm{Gl}(V) \simeq \mathrm{Aut}(\wedge^n V) \to 0.$$

Let us consider the subgroup $\mathrm{Gl}^+(V)$ consisting of those automorphisms $g \in \mathrm{Gl}(V)$ such that $\bar{g}$ is an isomorphism. It is easy to check that:

$$g \longmapsto \big(g, \det(\bar{g})\big)$$

is a section of p over $\mathrm{Gl}^+(V)$. The cocycle associated to the central extension is given by:

$$c(g_1, g_2) = \det\big(\bar{g}_1 \circ (\overline{g_1 \circ g_2})^{-1} \circ \bar{g}_2\big).$$

The cocyle corresponding to the Lie algebra level follows from a straightforward computation. Let $\mathrm{Id} + \epsilon_i D_i$ be a $k[\epsilon_i]/\epsilon_i^2$-valued point of $\mathrm{Gl}(V)$ ($i = 1, 2$). The very definition of the cocycle:

$$c_{\mathrm{Lie}}(D_1, D_2)\epsilon_1\epsilon_2 = c(\mathrm{Id} + \epsilon_1 D_1, \mathrm{Id} + \epsilon_2 D_2) - c(\mathrm{Id} + \epsilon_2 D_2, \mathrm{Id} + \epsilon_1 D_1)$$

yields the expression:

$$c_{\mathrm{Lie}}(D_1, D_2) = \mathrm{Tr}(D_1^{+-} D_2^{-+} - D_2^{+-} D_1^{-+}), \tag{2.1}$$

where $D_i^{+-} : V^+ \to V^- :=\ <v_1, \ldots, v_n>$ is induced by $\mathrm{Id} + \epsilon_i D_i \in \mathrm{Gl}(V)$ with respect to the decomposition $V \simeq V^- \oplus V^+$ (and, analogously, $D_i^{-+} : V^- \to V^+$).

The case of $(V = k((t)), \mathcal{B}, V^+ = k[[t]])$ and $V^- = t^{-1}k[t^{-1}]$ is very similar and the same formulae remain valid.

3. The Automorphism Group of $k((t))$: G

This section aims at studying the functor (on groups) over the category of k-schemes defined by:

$$S \rightsquigarrow G(S) := \mathrm{Aut}_{H^0(S, \mathcal{O}_S)\text{-alg}} H^0(S, \mathcal{O}_S)((t)),$$

where the group law in G is given by the composition of automorphisms (here $R((t))$ stands for $R[[t]][t^{-1}]$ for a commutative ring R with identity; or, what amounts to the same, the Laurent developments in t with coefficients in R).

3.1. Elements of G. Let us consider the following functor over the category of k-schemes:

$$S \leadsto \underline{k((t))}^*(S) := \left\{ \text{invertibles of } H^0(S, \mathcal{O}_S)((t)) \right\}.$$

The first result is quite easy to show:

Lemma 3.1. *The functor homomorphism:*

$$\psi_R : \text{Aut}_{R\text{-}alg}\, R((t)) \to \underline{k((t))}^*(R)$$
$$g \mapsto g(t)$$

induces an injection of G into the connected component of t, $\underline{k((t))}^*_1$. Moreover, $G(R) \to \underline{k((t))}^*_1(R)$ is a semigroup homomorphism with respect to the following composition law on $\underline{k((t))}^*_1$:

$$m : \underline{k((t))}^*_1(R) \times \underline{k((t))}^*_1(R) \to \underline{k((t))}^*_1(R), \tag{3.1}$$
$$(g(t), h(t)) \mapsto h(g(t)).$$

Theorem 3.2. *The morphism ψ_R induces a natural isomorphism of functors:*

$$G \xrightarrow{\sim} \underline{k((t))}^*_1.$$

Proof. The only delicate part of the proof is the surjectivity of ψ_R. The idea is to relate $G(R)$ with the group of automorphisms of $R[[x]][y]$.

Let I be the ideal of $R[[x]][y]$ generated by $(x \cdot y - 1)$, and let $\text{Aut}_I\, R[[x]][y]$ be the group:

$$\left\{ g \in \text{Aut}_{R\text{-}alg}\, R[[x]][y] \text{ such that } g(I) = I \right\}.$$

Since there is an isomorphism $R[[x]][y]/I \xrightarrow{\sim} R((t))$ (which maps x to t and y to t^{-1}), one has a morphism $\text{Aut}_I\, R[[x]][y] \to \text{Aut}_{R\text{-}alg}\, R((t))$, and a commutative diagram:

$$
\begin{array}{ccccc}
\text{Aut}_I\, R[[x]][y] & \xrightarrow{\bar{\psi}_R} & R[[x]][y] & \xrightarrow{\pi} & R[[x]][y]/I \\
\downarrow & & & & \simeq\downarrow \\
\text{Aut}_{R\text{-}alg}\, R((t)) & \xrightarrow{\psi_R} & R((t))^*_1 & \longrightarrow & R((t))^*,
\end{array}
$$

where $\bar{\psi}_R(f) := f(x)$.

Observe that the induced morphism:

$$\pi \left(\left\{ \begin{array}{l} \text{series } f(x, y) \in x \cdot R[[x]] \oplus \text{Rad}(R)[y] \\ \text{such that the coefficient of } x \text{ is invertible} \end{array} \right\} \right) \longrightarrow R((t))^*_1$$

is surjective. The claim being equivalent to the surjectivity of ψ_R, it is then enough to show that:

$$\left\{ \begin{array}{l} \text{series } f(x, y) \in x \cdot R[[x]] \oplus \text{Rad}(R)[y] \\ \text{such that the coefficient of } x \text{ is invertible} \end{array} \right\} \subseteq \text{Im}(\bar{\psi}_R).$$

Given an element $x \cdot f(x) + n(y) \in x \cdot R[[x]] \oplus \mathrm{Rad}(R)[y]$, where $f(0)$ is invertible, consider the following R-endomorphism:

$$\phi : R[[x]][y] \to R[[x]][y]$$
$$x \mapsto x \cdot f(x) + n(y)$$
$$y \mapsto \frac{y}{f(x)} \cdot \left(1 + \frac{y \cdot n(y)}{f(x)}\right)^{-1}$$

(which is well defined since $f(x) \in R[[x]]^*$ and $n(y)$ is nilpotent).

Provided that ϕ is an isomorphism, it holds that $\phi(I) = I$ and that $\bar{\psi}_R(\phi) = x \cdot f(x) + n(y)$. To show that ϕ is actually an R-isomorphism of $R[[x]][y]$, observe that $\phi = \phi_3 \circ \phi_2 \circ \phi_1$, where ϕ_1, ϕ_2, ϕ_3 are R-isomorphisms of $R[[x]][y]$ defined by:

$$\begin{cases} \phi_2(x) = x \cdot f(x) \\ \phi_2(y) = y \end{cases} \qquad \begin{cases} \phi_3(x) = x \\ \phi_3(y) = \frac{y}{f(x)} \cdot \left(1 + \frac{y \cdot n(y)}{f(x)}\right)^{-1} \end{cases} \qquad \begin{cases} \phi_1(x) = x + (\phi_3 \circ \phi_2)^{-1}(n(y)) \\ \phi_1(y) = y \end{cases}$$

$\square$

3.2. Formal scheme structure of G.

Set an k-scheme S and an element $f \in \underline{k((t))^*}(S)$. From [AMP] we know that the function:

$$S \longrightarrow \mathbb{Z},$$
$$s \mapsto v_s(f) := \text{ order of } f_s \in k(s)((t))$$

is locally constant and that the connected component of t, $\underline{k((t))^*_1}$, is identified with the set of S-valued points of a formal k-scheme, $k((t))^*_1$. One therefore obtains an isomorphism between the functor G and the functor of points of the formal scheme $k((t))^*_1$:

$$G(S) \xrightarrow{\sim} k((t))^*_1(S) = \left\{ \begin{array}{c} \text{series } (a_r t^r + \cdots + a_0 + a_1 t + \ldots)t \text{ such that} \\ a_r, \ldots, a_{-1} \in \mathrm{Rad}(R), \ a_0 \in R^* \text{ and } r < 0 \end{array} \right\}$$

(where $R = H^0(S, \mathcal{O}_S)$).

3.3. Subgroups of G.

Two important subgroups of $G \xrightarrow{\sim} k((t))^*_1$ are the subschemes G_+ and G_- defined by:

$$G_+(S) := \left\{ t \cdot (1 + \sum_{i>0} a_i t^i) \text{ where } a_i \in R \right\},$$

$$G_-(S) := \left\{ \begin{array}{c} \text{polynomials } t \cdot (a_r t^r + \cdots + a_1 t^{-1} + 1) \text{ such} \\ \text{that } a_i \in R \text{ are nilpotent and } r \text{ arbitrary} \end{array} \right\}$$

respectively.

Let $\widehat{G}$ (respectively $\widehat{G}_-$, $\widehat{\mathbb{G}}_m$ and $\widehat{G}_+$) be the completion of the formal scheme G (G_-, $\mathbb{G}_m$ and G_+) at the point $\{\mathrm{Id}\}$.

Lemma 3.3. *The subgroups $\widehat{G}_-$, $\widehat{\mathbb{G}}_m$ and $\widehat{G}_+$ commute with each other and*

$$\widehat{G}_- \cdot \widehat{\mathbb{G}}_m \cdot \widehat{G}_+ = \widehat{G}.$$

Proof. Recall that $\mathrm{Hom}(\mathrm{Spec}(A), \widehat{G})$ is the union of $\mathrm{Hom}(\mathcal{O}/\mathfrak{m}_{\mathcal{O}}^n, A)$, where $\mathcal{O}$ is the ring of G and $\mathfrak{m}_{\mathcal{O}}$ is the maximal ideal corresponding to the identity. It therefore suffices to show that:

1. $\widehat{G}_-(A)$, $\widehat{\mathbb{G}}_m(A)$ and $\widehat{G}_+(A)$ commute with each other,
2. $\widehat{G}_-(A) \cdot \widehat{\mathbb{G}}_m(A) \cdot \widehat{G}_+(A) = \widehat{G}(A)$,

for each local and rational k-algebra A such that $\mathfrak{m}_A^{n+1} = 0$ for $n >> 0$.

Let us proceed by induction on n. The case $n = 1$ is a simple computation.

1. Let us prove that $\widehat{G}_-(A)$ and $\widehat{\mathbb{G}}_m(A)$ commute with each other. Consider the following subgroup of $G(A)$:

$$H(A) := \left\{ a_n t^{-n} + \ldots + a_0 \text{ with } a_i \in \mathfrak{m}_A \text{ for } i < 0 \text{ and } a_0 \in A^* \right\}$$

and note that we have the group exact sequence:

$$0 \to \widehat{H}(k[\mathfrak{m}_A^n]) \to \widehat{H}(A) \xrightarrow{\rho} \widehat{H}(B) \to 0,$$

where $B = A/\mathfrak{m}_A^n$.

For an element $h \in \widehat{H}(A)$ there exist $h_- \in \widehat{G}_-(B)$ and $h_0 \in \widehat{\mathbb{G}}_m(B)$ such that $\rho(h) = \rho(h_- \circ h_0)$; or what amounts to the same:

$$h_-^{-1} \circ h \circ h_0^{-1} \in \widehat{H}(k[\mathfrak{m}_A^n]).$$

The induction hypothesis implies that $\widehat{H}(k[\mathfrak{m}_A^n]) = \widehat{G}_-(k[\mathfrak{m}_A^n]) \cdot \widehat{\mathbb{G}}_m(k[\mathfrak{m}_A^n])$ and hence there exist $h'_- \in \widehat{G}_-(k[\mathfrak{m}_A^n])$ and $h'_0 \in \widehat{\mathbb{G}}_m(k[\mathfrak{m}_A^n])$ such that:

$$h_-^{-1} \circ h \circ h_0^{-1} = h'_- \circ h'_0$$

and therefore:

$$\widehat{H} = \widehat{G}_- \cdot \widehat{\mathbb{G}}_m.$$

Analogously, one proves that $\widehat{H} = \widehat{\mathbb{G}}_m \cdot \widehat{G}_-$.

The proofs of the other commutation relations are similar.

2. Note that $g_0 \circ g_- = g_- \circ g_0$ for $g_0 \in \widehat{\mathbb{G}}_m(A)$ and $g_- \in \widehat{G}_-(k[\mathfrak{m}_A^n])$ and proceed similarly. $\square$

Theorem 3.4. *The functor G is canonically a subgroup of* $\mathrm{Gl}(V)$.

Proof. Note that it suffices to show that $G_-(S)$, $\mathbb{G}_m(S)$ and $G_+(S)$ are canonically subgroups of $\mathrm{Gl}(V)(S)$ for each k-scheme S, since:

- $\widehat{G} = \widehat{G}_- \cdot \widehat{\mathbb{G}}_m \cdot \widehat{G}_+$,
- $\widehat{G}_- = G_-$ and $\widehat{G}_+ \subseteq G_+$,
- $G = \widehat{G} \cdot G_+$.

By the very definition of $\mathrm{Gl}(V)$, it is enough to prove the case when S is a local affine scheme, $\mathrm{Spec}(R)$.

The cases of $\mathbb{G}_m$ and G_+ are straightforward since:

$$\phi(t^n R[[t]]) = t^n R[[t]] \quad \forall n$$

for $\phi \in \mathbb{G}_m(S)$ or $\phi \in G_+(S)$.

Let us now consider $\phi \in G_-(S)$. Let $u(t)$ be such that $\phi^{-1}(t) = t(1 + u(t))$. It then holds that:

$$\phi^{-1}(t^r) = t^r(1 + u(t))^r = t^r \cdot \sum_{i=0}^{r} \binom{r}{i} u(t)^i.$$

Since $u(t)$ is nilpotent, there exists s such that:

$$\phi^{-1}(t^r R[[t]]) \subseteq t^s R[[t]],$$

in other words:

$$t^r R[[t]] \subseteq \phi(t^s R[[t]]).$$

The Nakayama lemma implies that the family $\{\phi(t^s), \ldots, \phi(t^{r-1})\}$ generates $\phi(t^s R[[t]])/t^r R[[t]]$. Using the fact that $\phi \in G_-$ one proves that they are linearly independent; summing up, $\phi(t^s R[[t]])/t^r R[[t]]$ is free of finite type. □

3.4. The Lie algebra of G, $\mathfrak{Lie}(G)$.

Theorem 3.5. *There is a natural isomorphism of Lie algebras:*

$$\mathfrak{Lie}(G) \xrightarrow{\sim} k((t))\partial_t$$

compatible with their natural actions on the tangent space to the Grassmannian, $T\mathrm{Gr}(V)$. (From now on $\mathrm{Der}_k k((t))$ will denote $k((t))\partial_t$.)

Proof. Take an element $g(t) = t(1 + \epsilon g_0(t)) \in \mathfrak{Lie}(G)$ (recall that by definition $\mathfrak{Lie}(G) = G(k[\epsilon]/\epsilon^2) \times_{G(k)} \{Id\}$). Let us compute $\mu(g)(t^m)$ for $m \in \mathbb{Z}$:

$$\mu(g)(t^m) = g(t)^m = t^m(1 + \epsilon g_0(t))^m =$$
$$= t^m(1 + m\epsilon g_0(t)) = (Id + \epsilon \cdot g_0(t)t\partial_t)(t^m).$$

It is now natural to define the following map:

$$\mathfrak{Lie}(G) \to \mathrm{Der}_k k((t)),$$
$$t(1 + \epsilon g_0(t)) \mapsto g_0(t)t \cdot \partial_t,$$

and this turns out to be an isomorphism of k-vector spaces.

In order to check that this map is actually an isomorphism of Lie algebras, let us compute explicitly the Lie algebra structure of $\mathfrak{Lie}(G)$.

Given two elements $g_n(t) = t(1 + \epsilon_1 t^n)$ and $g_m(t) = t(1 + \epsilon_2 t^m)$ (where $\epsilon_i^2 = 0$), we have:

$$g_n(g_m(t)) = g_m(g_n(t))(1 + (m - n)\epsilon_1\epsilon_2 t^{m+n})$$

that is:

$$[g_m, g_n] = (m - n)g_{m+n}.$$

Since $[t^{m+1}\partial_t, t^{n+1}\partial_t] = (m - n) \cdot t^{m+n+1}\partial_t$, one concludes that the map is in fact an isomorphism of Lie algebras.

Let us check that the actions of these Lie algebras on $T\mathrm{Gr}(V)$ coincide. Fix a rational point $U \in \mathrm{Gr}(V)$ and take an element $g(t) = t(1 + \epsilon g_0(t)) \in \mathfrak{Lie}(G)$. Clearly, the image of (g, U) by μ lies on:

$$T_U\mathrm{Gr}(V) = \mathrm{Gr}(V)(k[\epsilon]/\epsilon^2) \underset{\mathrm{Gr}(V)(k)}{\times} \{U\} \simeq \mathrm{Hom}_k(U, V/U),$$

which is associated with the morphism:

$$U \hookrightarrow V \xrightarrow{tg_0(t)} V \to V/U.$$

Consider an element $D \in \mathrm{Der}_k\, k((t))$. Then the image of (D, U) under the action of $\mathrm{Der}_k\, k((t))$ on $T\mathrm{Gr}(V)$ is:

$$U \hookrightarrow V \xrightarrow{D} V \to V/U$$

and the conclusion follows.   □

Let Vir denote the Virasoro algebra; that is, the Lie algebra with a basis $\{\{d_m | m \in \mathbb{Z}\}, c\}$ and Lie brackets given by:

$$[d_m, c] = 0,$$

$$[d_m, d_n] = (m - n)d_{m+n} + \delta_{n,-m}\frac{(m^3 - m)}{12}c.$$

By abuse of notation Vir and Virasoro will also denote the Lie algebra given by $\varprojlim_n$ Vir $/\{d_m | m > n\}$. Both algebras have a "universal" central extension:

$$\mathrm{Ext}^1(k((t))\partial_t, \mathbb{C}) = \mathbb{C} \cdot \mathrm{Vir}$$

and this is the important feature for our approach (see [KR] Lecture 1, [ACKP, 2.1], [LW]).

Definition 3.6. *The central extension of G given by Theorem 2.3, $\widetilde{G}$, will be called the Virasoro Group.*

Proposition 3.7. *The Lie algebra of $\widetilde{G}$, $\mathfrak{Lie}(\widetilde{G})$, is isomorphic to the Virasoro algebra, Vir.*

Remark 1. Let us compute the cocycle associated with $\mathfrak{Lie}(\widetilde{G})$. Let $\mathrm{Gl}^+(V)$ be the subgroup of $\mathrm{Gl}(V)$ consisting of elements g such that $g(F_{V+}) = F_{V+}$. Since $\widehat{G}$ is contained in $\mathrm{Gl}^+(V)$, one can use the formula 2.1. Recall that a basis of $\mathfrak{Lie}(\widehat{G})$ is given by the set $\{g_n(t) := t(1 + \epsilon t^n) | n \in \mathbb{Z}\}$ since $\mathfrak{Lie}(\widehat{G}) = \widehat{G}(k[\epsilon]/\epsilon^2)$.

The element of $\mathrm{Gl}^+(V)$ (a $\mathbb{Z} \times \mathbb{Z}$ matrix) corresponding to g_m is:

$$(g_m)_{ij} = \begin{cases} 1 & \text{if } i = j \\ \epsilon \cdot j & \text{if } i = j + m \\ 0 & \text{otherwise} \end{cases}$$

and the cocycle is therefore:

$$c(g_m, g_n) = \delta_{n,-m} \cdot \sum_{j=0}^{n-1} j(j - n) = \delta_{n,-m} \cdot \frac{m^3 - m}{6}.$$

3.5. Central extensions of G. We begin with an explicit construction of an important family of central extensions of G.

Fix two integer numbers α, β and consider the k-vector space

$$V_{\alpha,\beta} := t^\alpha k((t))(dt)^{\otimes\beta}.$$

The natural isomorphism:

$$d_{\alpha,\beta} : V \longrightarrow V_{\alpha,\beta},$$
$$f(t) \mapsto t^\alpha f(t)(dt)^{\otimes\beta}$$

allows us to define a triplet $(V_{\alpha,\beta}, \mathcal{B}_{\alpha,\beta} := d_{\alpha,\beta}(\mathcal{B}), V_{\alpha,\beta}^+ := d_{\alpha,\beta}(V^+))$. One has therefore an isomorphism:

$$\mathrm{Gr}(V) \xrightarrow{\sim} \mathrm{Gr}(V_{\alpha,\beta}).$$

Observe that the action of G on $V_{\alpha,\beta}$ defined by:

$$\big(g(t), t^\alpha f(t)(dt)^{\otimes\beta}\big) \mapsto g(t)^\alpha f(g(t))(dg(t))^{\otimes\beta}$$
$$= t^\alpha \left(\frac{g(t)}{t}\right)^\alpha f(g(t))g'(t)^\beta (dt)^{\otimes\beta}$$

induces an action on $\mathrm{Gr}(V_{\alpha,\beta})$ (by a straightforward generalization of Theorem 3.4), and also in $\mathrm{Gr}(V)$:

$$\mu_{\alpha,\beta} : G \times \mathrm{Gr}(V) \to \mathrm{Gr}(V).$$

Note that $\mu_{0,0}$ is the action of G on $\mathrm{Gr}(V)$ defined in the previous section. Moreover, these actions are related by:

$$\mu_{\alpha,\beta}(g(t)) = \left(\left(\frac{g(t)}{t}\right)^\alpha \cdot g'(t)^\beta\right) \circ \mu_{0,0}(g(t)),$$

where the first factor is the homothety defined by itself.

Theorem 2.3 implies that there exists a central extension:

$$0 \to \mathbb{G}_m \to \widetilde{G}_{\alpha,\beta} \to G \to 0$$

corresponding to the action $\mu_{\alpha,\beta}$. Moreover, it follows from its proof that $\widetilde{G}_{\alpha,\beta}$ consists of commutative diagrams:

$$
\begin{array}{ccc}
\mathbb{V}(\mathrm{Det}_V^*) & \xrightarrow{\;\bar{g}\;} & \mathbb{V}(\mathrm{Det}_V^*) \\
\downarrow & & \downarrow \\
\mathrm{Gr}(V) & \xrightarrow{\;\mu_{\alpha,\beta}(g)\;} & \mathrm{Gr}(V)
\end{array}
$$

or equivalently:

$$\widetilde{G}_{\alpha,\beta} = \{(g, \tilde{g}) \text{ where } g \in G \text{ and } \tilde{g} : \mu_{\alpha,\beta}(g)^* \mathrm{Det}_V \xrightarrow{\sim} \mathrm{Det}_V\}$$

since $\mu_{\alpha,\beta}(g)^* \mathrm{Det}_V \simeq \mathrm{Det}_V$ for all $g \in G$. It is not difficult to show that the extensions $\widetilde{G}_{\alpha,\beta}$ and $\widetilde{G}_{\alpha',\beta}$ are isomorphic for every $\alpha, \alpha' \in \mathbb{Z}$. Then, $\widetilde{G}_{0,\beta}$ (respectively $\mu_{0,\beta}$) will be denoted by $\widetilde{G}_\beta$ (μ_β). The group law of $\widetilde{G}_\beta$ is:

$$(h, \tilde{h}) \cdot (g, \tilde{g}) = (h \cdot g, \tilde{g} \circ \mu_\beta(g)^*(\tilde{h}))$$

since we have:

$$\mu_\beta(h \cdot g)^* \operatorname{Det}_V = (\mu_\beta(g)^* \circ \mu_\beta(h)^*) \operatorname{Det}_V \xrightarrow{\mu_\beta(g)^*(\tilde{h})} \mu_\beta(g)^* \operatorname{Det}_V \xrightarrow{\tilde{g}} \operatorname{Det}_V .$$

These central extensions induce extensions of the Lie algebra $\mathfrak{Lie}(G)$ whose corresponding cocycles are:

$$c_\beta(m, n) = \delta_{n,-m} \cdot \sum_{j=0}^{n-1} (j + (m+1)\beta)(j - n + (n+1)\beta)$$

$$= \delta_{n,-m} \cdot \left(\frac{m^3 - m}{6} \right)(1 - 6\beta + 6\beta^2). \tag{3.2}$$

To obtain such a formula, one only has to check that the matrix corresponding to $\mu_\beta(g_m)$ is:

$$(\mu_\beta(g_m))_{ij} = \begin{cases} 1 & \text{if } i = j \\ \epsilon \cdot (j + (m+1)\beta) & \text{if } i = j + m \, . \\ 0 & \text{otherwise} \end{cases}$$

Remark 2. It is worth pointing out that one can continue with this geometric point of view for studying the representations of $\mathfrak{Lie}(G)$ since it acts on the space of global sections of the Determinant line bundle which contains the "standard" Fock space. (For an explicit construction of sections of Det_V^*, see [AMP]). An algebraic study of the representations of Vir induced by $\mu_{\alpha,\beta}$ has been done in [KR].

3.6. Line bundles on G. Formula (3.2) may be stated in terms of line bundles. For this goal, let us first recall from [SGA] the relationships among line bundles, bitorsors and extensions.

Recall that a central extension of the group G by $\mathbb{G}_m$:

$$0 \to \mathbb{G}_m \to \mathcal{E} \to G \to 0$$

($\mathcal{E}$ being a group) determines a bitorsor over $\big((\mathbb{G}_m)_G, (\mathbb{G}_m)_G\big)$, which will be denoted by $\mathcal{E}$ again.

Moreover, given two bitorsors $\mathcal{E}$ and $\mathcal{E}'$, one defines their product by $\mathcal{E} \overset{\mathbb{G}_m}{\times} \mathcal{E}'$, which is the quotient of $\mathcal{E} \times \mathcal{E}'$ by the action of $\mathbb{G}_m$:

$$\mathbb{G}_m \times \big(\mathcal{E} \times \mathcal{E}'\big) \to \mathcal{E} \times \mathcal{E}',$$
$$(g, (e, e')) \mapsto (e \cdot g, g \cdot e')$$

(where the dot denotes the actions on $\mathcal{E}$ and $\mathcal{E}'$).

From [SGA] §1.3.4 we know that the group law of $\mathcal{E}$ induces a canonical isomorphism:

$$p_1^* \mathcal{E} \overset{\mathbb{G}_m}{\times} p_2^* \mathcal{E} \overset{\sim}{\to} m^* \mathcal{E} \tag{3.3}$$

of $\big((\mathbb{G}_m)_{G \times G}, (\mathbb{G}_m)_{G \times G}\big)$-bitorsors (where $p_i : G \times G \to G$ is the projection in the ith component and m the group law of G).

Conversely, a bitorsor $\mathcal{E}$ satisfying (3.3) and an associative type property (see [SGA] for the precise statement) determines an extension of G.

Observe that one can associate a line bundle to such an extension. Given:

$$0 \to \mathbb{G}_m \to \mathcal{E} \to G \to 0$$

consider the line bundle:

$$\mathcal{L} := \mathcal{E} \overset{\mathbb{G}_m}{\times} \mathbb{A}^1_k,$$

where $\mathcal{E}$ is interpreted as a principal fiber bundle of group $\mathbb{G}_m$ and $\mathbb{G}_m$ acts on $\mathbb{A}^1_k$ by the trivial character and on $\mathcal{E}$ via the inclusion $\mathbb{G}_m \subset \mathcal{E}$. Further, the structure of $\mathcal{E}$ implies that there exists a canonical isomorphism:

$$p_1^* \mathcal{L} \otimes p_2^* \mathcal{L} \overset{\sim}{\to} m^* \mathcal{L}. \tag{3.4}$$

One proves that the product of bitorsors corresponds to the tensor product of line bundles; that is, for two extensions $\mathcal{E}$ and $\mathcal{E}'$ there exists a canonical isomorphism:

$$L_{\mathcal{E} \overset{\mathbb{G}_m}{\times} \mathcal{E}'} \overset{\sim}{\to} L_{\mathcal{E}} \otimes L_{\mathcal{E}'}.$$

Conversely, if $\mathcal{L}$ is a line bundle satisfying (3.4) and an associative type property, then the principal fibre bundle $\underline{\mathrm{Isom}}(\mathcal{O}_G, \mathcal{L})$ is a principal fibre bundle of group $\mathbb{G}_m$ which can be endowed with the structure of central extension such that the associated line bundle is $\mathcal{L}$.

Definition 3.8. *The invertible sheaf on G associated with $\widetilde{G}_\beta$ will be denoted by Λ_β.*

4. Main Results

4.1. Modular properties of the τ-function. Let us fix a point $X \in \mathrm{Gr}(V)$ and a non-negative integer β. From Theorem 2.2 we know that there exists L_β, a line bundle over G, such that:

$$\mu_{\beta}^*{}_\bullet p_2^* \mathrm{Det}_V \simeq p_2^* \mathrm{Det}_V \otimes p_1^* L_\beta, \tag{4.1}$$

where

$$G \times \mathrm{Gr}(V) \overset{\mu_\beta}{\longrightarrow} G \times \mathrm{Gr}(V) \overset{p_2}{\to} \mathrm{Gr}(V).$$

Then, restricting to $G \times X$ and looking at sections we have:

$$\Omega_+(\mu_\beta(g)(X)) = l_\beta(g) \cdot \Omega_+(X)$$

for a certain section $l_\beta(g)$ of L_β (we assume here that $\Omega_+(X) \neq 0$, so that it generates $(\mathrm{Det}_V^*)_X$).

The above identity is the cornerstone of the modular properties of the τ-functions. However, let us give a more precise statement. Assume that the orbit of X under Γ (consisting of an invertible Laurent series acting by multiplication, see [AMP]) is contained in F_{V+}. Note, further, that L_β may be trivialized. Then, with the above premises, the following theorem holds:

Theorem 4.1. *There exists a function $\bar{l}_\beta(g)$ on G, such that*

$$\tau_{\mu_\beta(g)(X)} = \bar{l}_\beta(g) \cdot \tau_X.$$

To finish this section let us offer a few hints on the explicit computation of l_β. The previous statement is to be understood as an equality of S-valued functions (for a fixed k-scheme S and $g \in G(S)$).

However, in order to describe this isomorphism explicitly it suffices to deal with the case of the universal automorphism, $\mathbf{g}$, corresponding to the identity point of $G(G)$. Note that the following relation holds:

$$\mu_\beta(\mathbf{g}) = \mathbf{g}' \circ \mu_{\beta-1}(\mathbf{g})$$

(where $\mathbf{g}'$ acts as a homothety) and observe that the proof of Theorem 2.2 implies that the existence of canonical isomorphisms:

$$\mu_\beta(\mathbf{g})_\bullet^* p_2^* \operatorname{Det}_V \simeq \mu_{\beta-1}(\mathbf{g})_\bullet^* p_2^* \operatorname{Det}_V \otimes p_1^*(N), \quad \beta \geq 1,$$
$$\mu_0(\mathbf{g})_\bullet^* p_2^* \operatorname{Det}_V \simeq p_2^* \operatorname{Det}_V \otimes p_1^*(M),$$

where

- $M = (\wedge \hat{V}_G^+ / \mathbf{g}(\hat{A}_G)) \otimes (\wedge \hat{V}_G^+ / \hat{A}_G)^*$,
- $N = (\wedge \hat{V}_G^+ / \mathbf{g}' \cdot \hat{A}_G) \otimes (\wedge \hat{V}_G^+ / \hat{A}_G)^*$,

($A \in \mathcal{B}$ is locally chosen such that $A \subset V^+$, $\mathbf{g}' \cdot \hat{A}_G \subset \hat{V}_G^+$ and $\mathbf{g}(\hat{A}_G) \subset \hat{V}_G^+$). Thus, we obtain:

$$L_\beta^* = M \otimes N^\beta \tag{4.2}$$

and the computation of $\bar{l}_\beta(g) := l_\beta(g)/l_\beta(1)$ ($g \in G(S)$) is now straightforward.

Remark 3. The above theorem can be interpreted as the formal version of Theorems 5.10 and 5.11 of [KNTY].

4.2. Central extensions of G and $\mathfrak{Lie}(G)$.

Along the rest of this section it will be assumed that $k = \mathbb{C}$. Nevertheless, some results remain valid for $\operatorname{char}(k) = 0$. (We refer the reader to Appendix B for notations and the main results on Lie theory for formal group schemes).

Theorem 4.2. *The functor $\mathfrak{Lie}$ induces an injective group homomorphism:*

$$\operatorname{Ext}^1(G, \mathbb{G}_m) \hookrightarrow \operatorname{Ext}^1(\mathfrak{Lie}(G), \widehat{\mathbb{G}}_a).$$

Proof. Here $\operatorname{Ext}^1(G, \mathbb{G}_m)$ denotes the group of equivalence classes of central extensions of G by $\mathbb{G}_m$ as formal groups, and $\operatorname{Ext}^1(\mathfrak{Lie}(G), \widehat{\mathbb{G}}_a)$ denotes the group of equivalence classes of central extensions of Lie algebras.

Given an extension of G, $\widetilde{G}$, the restriction of the group functors $\mathbb{G}_m$, $\widetilde{G}$ and G to the category $\mathcal{C}_a$ ($\widehat{\mathbb{G}}_m$, $\widehat{\widetilde{G}}$ and $\widehat{G}$ respectively) gives rise to a class in $\operatorname{Ext}^1(\widehat{G}, \widehat{\mathbb{G}}_m)$. Observe that this map is injective. Recalling that $\mathfrak{Lie}(G) = \mathfrak{Lie}(\widehat{G})$, $\widehat{\mathbb{G}}_a \simeq \widehat{\mathbb{G}}_m$ and Theorem B.4, one concludes. $\square$

4.3. Some canonical isomorphisms.

Theorem 4.3.
$$L_\beta \simeq \Lambda_\beta.$$

Proof. Observe that Eq. (4.1) implies that:

$$p_1^* L_\beta \simeq \text{Isom}\left(\mu_{\beta_\bullet}^* p_2^* \text{Det}_V, \, p_2^* \text{Det}_V\right),$$

and hence L_β is the line bundle associated with the central extension $\tilde{G}_\beta$. □

Theorem 4.4. *There are canonical isomorphisms:*

$$m^* \Lambda_\beta \xrightarrow{\sim} p_1^* \Lambda_\beta \otimes p_2^* \Lambda_\beta \quad \forall \beta \in \mathbb{Z}.$$

Proof. This is a consequence of Subsect. 3.6.

Theorem 4.5 (Local Mumford formula)**.** *There exist canonical isomorphisms of invertible sheaves:*

$$\Lambda_\beta \xrightarrow{\sim} \Lambda_1^{\otimes(1-6\beta+6\beta^2)} \quad \forall \beta \in \mathbb{Z}.$$

Proof. This is a consequence of Theorem 4.2 and formula (3.2). □

Remark 4. This theorem is a local version of Mumford's formula. The next subsection will throw some light on the relation between this formula and the original global one. It is worth pointing out that the calculations performed in Subsect. 4.1 throw light on the explicit expression of the above isomorphism. This can be done with procedures similar to those of [BM].

Corollary 4.6. *Let H be the subgroup of G consisting of series $\sum_{i\geq 0} a_i z^i$, where a_0 is nilpotent and $a_1 = 1$.*
    There is a canonical isomorphism:

$$(L_2|_H)^{\otimes 12} \simeq \mathcal{O}_H$$

(see [Se] §6 for explicit formulae).

4.4. Orbits of G: Relation with the moduli space of curves. Recall from [MP] the definition (which follows the ideas of [KNTY, Ue]):

Definition 4.7. *Set a k-scheme S. Define the functor $\widetilde{\mathcal{M}}_g^\infty$ over the category of k-schemes by:*

$$S \rightsquigarrow \widetilde{\mathcal{M}}_g^\infty(S) = \{\text{families } (C, D, z) \text{ over } S\},$$

where these families satisfy:

1. *$\pi : C \to S$ is a proper flat morphism, whose geometric fibres are integral curves of arithmetic genus g,*
2. *$\sigma : S \to C$ is a section of π, such that when considered as a Cartier divisor D over C it is smooth, of relative degree 1, and flat over S. (We understand that $D \subset C$ is smooth over S, iff for every closed point $x \in D$ there exists an open neighborhood U of x in C such that the morphism $U \to S$ is smooth.)*

3. ϕ is an isomorphism of $\mathcal{O}_S$-algebras:

$$\widehat{\Sigma}_{C,D} \xrightarrow{\sim} \mathcal{O}_S((z)).$$

On the set $\widetilde{\mathcal{M}}_g^\infty(S)$ one can define an equivalence relation, $\sim$: (C, D, z) and (C', D', z') are said to be equivalent, if there exists an isomorphism $C \to C'$ (over S) such that the first family goes to the second under the induced morphisms. Let us define the moduli functor of pointed curves of genus g, $\mathcal{M}_g^\infty$, as the sheafication of $\widetilde{\mathcal{M}}_g^\infty(S)/\sim$. We know from Theorem 6.5 of [MP] that it is representable by a k-scheme $\mathcal{M}_g^\infty$. The following theorems are now standard results:

Theorem 4.8. *Let g, β be two non-negative integer numbers. The "Krichever morphism":*

$$K_\beta : \mathcal{M}_g^\infty \longrightarrow \mathrm{Gr}(k((t))(dt)^{\otimes\beta}),$$
$$(C, p, z) \longmapsto H^0(C - p, \omega_C^{\otimes\beta})$$

is injective in a (formal) neighborhood of every geometric point. The image will be denoted by $\mathcal{M}_{g,\beta}^\infty$.

Theorem 4.9. *The action μ_β of G on $\mathrm{Gr}(V)$ induces an action $\mathcal{M}_{g,\beta}^\infty$.*

Proof. Recall that $G(R) = \mathrm{Aut}_{R-alg} R((t))$ and that the points of $\mathcal{M}_g^\infty(R)$ are certain sub-R-algebras of $R((t))$ (R being a commutative ring with identity). We thus have that the Krichever morphism is equivariant with respect to the canonical action of G on $\mathcal{M}_g^\infty$ and μ_0 on $\mathrm{Gr}(V)$. This implies the $\beta = 0$ case. The claim is now a straightforward generalization. $\square$

In order to study the deformations of a given datum, more definitions are needed. First, let $\mathcal{M}_g'$ be the subscheme of $\mathcal{M}_g^\infty$ defined by the same conditions as in Definition 4.7 except that the third one is replaced by:

- z is a formal trivialization of C along D; that is, a family of epimorphisms of rings:

$$\mathcal{O}_C \longrightarrow \sigma_* \left(\mathcal{O}_S[t]/t^m \mathcal{O}_S[t] \right) \quad m \in \mathbb{N}$$

compatible with respect to the canonical projections

$$\mathcal{O}_S[t]/t^m \mathcal{O}_S[t] \to \mathcal{O}_S[t]/t^{m'} \mathcal{O}_S[t]$$

(for $m \geq m'$), and such that that corresponding to $m = 1$ equals σ.

Analogously, we introduce the moduli space of pointed curves with an n-order trivialization, $\mathcal{M}_g^n$ ($n \geq 1$), as the k-scheme representing the sheafication of the following functor over the category of k-schemes:

$$S \rightsquigarrow \{ \text{ families } (C, D, z) \text{ over } S \}/ \sim,$$

where these families satisfy the same conditions except for the third which is replaced by:

- z is a n-order trivialization of C along D; that is, an isomorphism:

$$\mathcal{O}_C/\mathcal{O}_C(-nD) \longrightarrow \sigma_* \left(\mathcal{O}_S[t]/t^n \mathcal{O}_S[t] \right).$$

The canonical projections $\mathcal{M}_g^\infty \to \mathcal{M}_g^n$ will be denoted by p_n. Observe that the natural projections $\mathcal{M}_g^m \to \mathcal{M}_g^n$ $(m > n)$ render $\{\mathcal{M}_g^n | n \geq 0\}$ an inverse system and that $\mathcal{M}_g'$ is its inverse limit. In particular, we have:

$$\mathcal{M}_g' = \varprojlim_n \mathcal{M}_g^n$$

The deformation functor of a rational point X of $\mathcal{M}_g^\infty$, D_X, is the following functor over $\mathcal{C}_a$ (local rational and artinian k-algebras):

$$A \rightsquigarrow \mathcal{M}_g^\infty(A) \underset{\mathcal{M}_g^\infty(k)}{\times} \{X\}.$$

Similarly, define D_X' (resp. $D_{X_n}^n$), the deformation functor of X (resp. $X_n := p_n(X)$) in $\mathcal{M}_g'$ (resp. $\mathcal{M}_g^n$). Since all the $\mathcal{M}$'s are schemes, the corresponding deformation functors are representable by the completion of the local rings.

Lemma 4.10. *Let $X \in \mathcal{M}_g'(k)$ be a triplet (C, p, z) with C smooth. Then, the following sequence:*

$$0 \to H^0(C - p, \mathbb{T}_C) \to k((t))\partial_t \to \varprojlim_n H^1(C, \mathbb{T}_C(-np)) \to 0$$

(where $\mathbb{T}_C$ is the tangent sheaf on C) is exact.

Proof. Let m, n be two positive integers. Let us consider the exact sequence:

$$0 \to \mathcal{O}_C(-np) \to \mathcal{O}_C(mp) \to \mathcal{O}_C(mp)/\mathcal{O}_C(-np) \to 0$$

Since z is a formal trivialization and p is smooth, it induces an isomorphism

$$\mathcal{O}_C(mp)/\mathcal{O}_C(-np) \xrightarrow{\sim} t^{-m}k[[t]]/t^n k[[t]].$$

Twisting the sequence with $\mathbb{T}_C$ and taking cohomology one obtains:

$$0 \to H^0(\mathbb{T}_C(-np)) \to H^0(\mathbb{T}_C(mp)) \to t^{-m}k[[t]]\partial_t/t^n k[[t]]\partial_t \to$$
$$\to H^1(\mathbb{T}_C(-np)) \to H^1(\mathbb{T}_C(mp)) \to 0$$

since $\mathcal{O}_p \otimes_{\mathcal{O}_C} \mathbb{T}_C \simeq < \partial_t >$. Taking direct limit on m and inverse limit on n, the result follows.   □

Theorem 4.11. *Let k be a field of characteristic 0. Fix a rational point $X \in \mathcal{M}_g^\infty(k)$ corresponding to a smooth curve.*

The morphism of functors:

$$\widehat{G} \longrightarrow D_X$$

induced by Theorem 4.9 is surjective.

Proof. Let $\mathcal{O}_X$ be the local ring of $\mathcal{M}_g^\infty$ at X. The statement is equivalent to showing the surjectivity of the induced maps:

$$\widehat{G}(A) \to D_X(A) = \text{Spf}(\widehat{\mathcal{O}}_X)(A)$$

for all $A \in \mathcal{C}_a$. Now, Lemma A.2 reduces the problem to the case $A = k[\epsilon]/\epsilon^2$:

$$\pi : \widehat{G}(k[\epsilon]/\epsilon^2) \to T_X \mathcal{M}_g^\infty$$

(where T denotes the tangent space).

Observe that given X there exists an element $g \in \widehat{G}$ such that the transform of X under g, X^g, belongs to $\mathcal{M}_g'$. Then, the proof is equivalent to showing that:

$$T_{X^g} \mathcal{M}_g' \subseteq \text{Im}\,\pi.$$

From Lemma 4.10, it follows that the action of $\widehat{G}(k[\epsilon]/\epsilon^2) = k((t))\partial_t = \mathfrak{Lie}(G)$ on $k((t))$ and that of $\text{Der}(H^0(C - p, \mathcal{O}_C)) = H^0(C - p, \mathbb{T}_C)$ on $H^0(C - p, \mathcal{O}_C)$ are compatible; further, the isotropy of X under $k((t))\partial_t$ is precisely $H^0(C - p, \mathbb{T}_C)$. One can now check that the above sequence induces a map:

$$\varprojlim_n H^1(C, \mathbb{T}_C(-np)) \hookrightarrow T_X \mathcal{M}_g^\infty$$

whose image is naturally identified with $T_{X^g} \mathcal{M}_g' = \varprojlim_n T_{X_n} \mathcal{M}_g^n$ via the Kodaira–Spencer isomorphism. And the theorem follows. $\square$

Remark 5. Let us now compare Theorem 4.5 and the standard Mumford formula. Let $\mathcal{M}_g$ denote the moduli space of genus g curves, $\pi_g : \mathcal{C}_g \to \mathcal{M}_g$ the universal curve, and ω the relative dualizing sheaf. Let us consider the family of invertible sheaves:

$$\lambda_\beta := \text{Det}(R^\bullet \pi_{g,*} \omega^{\otimes \beta}) \quad \beta \in \mathbb{Z}.$$

Let $p : \mathcal{M}_g^\infty \to \mathcal{M}_g$ be the canonical projection. Then, it holds that:

$$K_\beta^* \text{Det}_V \xrightarrow{\sim} p^* \lambda_\beta.$$

Furthermore, choose a rational point $X \in \mathcal{M}_g^\infty$ and let p_β be the composite:

$$\widehat{G} \to D_X^\beta \to \mathcal{M}_g.$$

Then, it holds that there exist isomorphisms:

$$\Lambda_\beta \xrightarrow{\sim} p_\beta^* \lambda_\beta.$$

The compatibility of these isomorphisms with those of the Mumford formula, $\lambda_\beta \xrightarrow{\sim} \lambda_1^{\otimes(1-6\beta+6\beta^2)}$, should follow from the proof Theorem 4.5 and the computations of [BM, BS].

5. Application to a Non-Perturbative Approach to Bosonic Strings

Two standard approaches to Conformal Field Theories are based on moduli spaces of Riemann Surfaces (with additional structure) and on the representation theory of the Virasoro algebra, respectively. It is thus natural to attempt to "unify" both interpretations (e.g. [KNTY]).

In our setting, Subsects. 3.4 and 4.4 unveil the important role of the group G in both approaches. Motivated by this fact and by the suggestions of [BR] and [Mo], we propose G as a "universal moduli space" which will allow a formulation of a non-perturbative string theory. Let us remark that in the formal geometric setting developed in [MP], the group G is the moduli space of formal curves.

Let us sketch how this construction should be carried out, although details and proofs will be given in a forthcoming paper.

Let us consider the vector space $V_d = \mathbb{C}^d \otimes_{\mathbb{C}} \mathbb{C}((t))$. The natural representation, μ_1, of G on V_1 induces a representation of G on V_d, given by $\mu_1 \oplus .^d. \oplus \mu_1$. Following the procedure given in Sect. 3, it is easily proved that this representation yields an action, ρ_d, of G on the Grassmannian $\text{Gr}(V_d)$ preserving the determinant bundle. The corresponding central extension determines a line bundle $\mathcal{L}_{\rho_d}$ on G with a bitorsor structure.

In order to clarify the physical meaning of this higher dimensional picture, it is worth pointing out that the Fock space corresponding to string theory in the space-time $\mathbb{R}^{2d-1,1}$ is naturally interpreted as a subspace of $H^0(\text{Gr}(V_d), \text{Det}^*)$, the space of global sections of the dual of the determinant bundle. Moreover, the actions of the Virasoro algebra on the Fock space and that of $\mathfrak{Lie}(G)$ on $H^0(\text{Gr}(V_d), \text{Det}^*)$ are compatible.

The calculations in Sect. 5 of [BR] can now be restated in the following form: there exists a canonical isomorphism of invertible sheaves:

$$\mathcal{L}_{\rho_d} \overset{\sim}{\to} \Lambda_1^{\otimes d}.$$

This isomorphism, together with the Local Mumford Formula (Theorem 4.5), implies that $\mathcal{L}_{\rho_d}$ and Λ_2 are isomorphic if and only if $d = 13$ (complex dimension).

Observe that the group scheme G carries a filtration $\{G_n | n \geq 0\}$, where

$$G_n(R) := \{\phi \in G(R) | \phi(t) = \sum_{i \geq -m} a_i t^i \text{ with } m \leq n\}.$$

The restriction homomorphisms:

$$j_n^* : H^0(G, \Lambda_\beta) \to H^0(G_n, \Lambda_\beta|_{G_n})$$

associated with the inclusions $j_n : G_n \hookrightarrow G$ give

$$j^* : H^0(G, \Lambda_\beta) \to \varprojlim_n H^0(G_n, \Lambda_\beta|_{G_n}).$$

Let X be a rational point of $\mathcal{M}_g^\infty$. The action of G on X induces:

$$\phi_g^n : G_n \longrightarrow \mathcal{M}_g^\infty$$

which takes values in the deformation functor of X, D_X. Moreover, $G_n \to D_X$ happens to be surjective for all $n \geq 3g - 3$ (see Theorem 4.11). Denote by $F_g \in H^0(G_{3g-3}, \Lambda_2|_{G_{3g-3}})$ the inverse image by ϕ_g^{3g-3} of the section of λ_2 corresponding to the partition

function of genus g. Then, there exists a global section $F \in H^0(G, \Lambda_2)$ such that $j^*(F)$ is precisely $\{F_g\}$.

The relationship between hermitian forms on the canonical sheaf of a complex manifold and holomorphic measures on them is well known. The generalization of this relation to infinite-dimensional manifolds would allow us to give a genus-independent Polyakov measure on G constructed in terms of the above introduced F.

Appendix

A. Deformation Theory

Let us recall some notations and give some results on deformation theory as exposed in [Sc].

Let $\mathcal{C}_a$ be the category of local rational Artin k-algebras. An admissible linearly topologized k-algebra $\mathcal{O}$ (see [EGA] §7) canonically defines a functor from $\mathcal{C}_a$ to the category of sets:

$$A \rightsquigarrow h_\mathcal{O}(A) := \mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}, A)$$

(where A is endowed with the discrete topology). Observe that $h_\mathcal{O}(A) = \mathrm{Hom}_{k\text{-alg}}(\mathcal{O}, A)$ for a discrete k-algebra $\mathcal{O}$.

The condition that $h_\mathcal{O}$ consists of only one point is equivalent to saying that $\mathcal{O}$ is local and rational.

The definition below is that given in [Sc] 2.2, which generalizes the concept of "formal smoothness" of [Ma].

Definition A.1. *A functor homomorphism $F \to G$ is smooth iff the morphism:*

$$F(B) \ \to \ F(A) \times_{G(A)} G(B)$$

is surjective for every surjection $B \to A$ in $\mathcal{C}_a$.

Remark 6. The following remarks merit attention:

- if $F \to G$ is smooth, then $F(A) \to G(A)$ is surjective for all A in $\mathcal{C}_a$ ([Sc, (2.4)],
- $h_\mathcal{O} \to h_{\mathcal{O}'}$ is smooth iff $\mathcal{O}$ is a series power ring over $\mathcal{O}'$ ([Sc] 2.5),
- $h_\mathcal{O}$ is said to be smooth iff the canonical morphism $h_\mathcal{O} \to h_k$ is smooth.

The tangent space to a functor over $\mathcal{C}_a$, F, is defined by

$$\mathsf{t}_F \ := \ F(k[\epsilon]/\epsilon^2).$$

Recall Lemma 2.10 of [Sc]: if it holds that

$$F(k[V \oplus W]) \simeq F(k[V]) \times F(k[W])$$

for arbitrary vector spaces V, W (where $k[V]$ is the ring $k \oplus V$ with $V^2 = 0$), then $F(k[V])$ (and in particular t_F) has a canonical vector space structure such that $F(k[V]) \simeq \mathsf{t}_F \otimes V$. Observe that the functor $h_\mathcal{O}$ satisfies the above condition for all $\mathcal{O}$.

Lemma A.2. *Let $\phi : F := h_{\mathcal{O}_F} \to G := h_{\mathcal{O}_G}$ and $F \to h_k$ be two morphisms of functors over $\mathcal{C}_a$ such that:*

- *$F \to h_k$ is smooth,*
- *the sets $F(k)$ and $G(k)$ consist of one element,*

• $t_F := F(k[\epsilon]/\epsilon^2) \to t_G := G(k[\epsilon]/\epsilon^2)$ *is surjective,*

then $F \to G$ *is smooth (and hence surjective).*

Proof. First, we claim that $F(k[V]) \to G(k[V])$ is surjective for every k-vector space V ($k[V]$ denotes the ring $k \oplus V$ in which $V^2 = 0$). Since $F(k[V \oplus W]) \simeq F(k[V]) \times F(k[W])$ and $G(k[V \oplus W]) \simeq G(k[V]) \times G(k[W])$ for vector spaces V, W, Lemma 2.10 of [Sc] holds, and hence there are canonical vector space structures on $F(k[V])$ and $G(k[V])$ such that they are isomorphic to $t_F \otimes V$ and $t_G \otimes V$ (in a functorial way) respectively. Since $t_F \to t_G$ is surjective by hypothesis, the claim follows.

Let A be an object of $\mathcal{C}_a$ and $I \subset A$ an ideal such that $I^2 = 0$. Then, one has a commutative diagram:

$$
\begin{array}{ccc}
F(A) & \xrightarrow{\phi_A} & G(A) \\
{\scriptstyle \rho_F}\downarrow & & {\scriptstyle \rho_G}\downarrow \\
F(A/I) & \xrightarrow{\phi_I} & G(A/I)
\end{array} ,
$$

where we can assume by induction over $\dim_k A$ that ϕ_I is surjective (since $t_F \to t_G$ is surjective).

Let (f, g) be an element of $F(A/I) \times G(A)$ such that $\phi_I(f) = \rho_G(g)$. Since $F \to h_k$ is smooth and $\mathcal{O}$ is local it follows that ρ_F is a surjection. Let $\bar{f} \in F(A)$ be a preimage of f. Then the images of $\phi_A(\bar{f})$ and g under ρ_G coincide; both of them are $\phi_I(f)$. Note that $\rho_G^{-1}(\phi_I(f))$ is an affine space modeled over $\mathrm{Der}_k(\mathcal{O}_G, I)$; or what amounts to the same:

$$ g - \phi_n(\bar{f}) \in \mathrm{Der}_k(\mathcal{O}_G, I). $$

Observe that the bottom arrow of the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Der}_k(\mathcal{O}_F, I) & \longrightarrow & \mathrm{Der}_k(\mathcal{O}_G, I) \\
{\scriptstyle \simeq}\downarrow & & {\scriptstyle \simeq}\downarrow \\
F(k[I]) & \longrightarrow & G(k[I])
\end{array}
$$

is surjective. Let $D \in F(k[I])$ be a preimage of $g - \phi_n(\bar{f})$.

It is now easy to verify that $\bar{f} + D$ is a preimage of (f, g) under the induced morphism:

$$ F(A) \longrightarrow F(A/I) \times G(A), $$

and the statement follows.   □

B. Lie Theory

This appendix aims at generalizing some results of Lie Theory for the case of (infinite) formal groups. To this end, we recall some more results of [Sc] and proceed with ideas quite close to those of [Ha, §14].

Definition B.1. *A functor F from $\mathcal{C}_a$ to the category of groups will be called a group functor. If, moreover, there exists a k-algebra $\mathcal{O}$ and an isomorphism $F \simeq h_{\mathcal{O}}$, then F will be called a formal group functor.*

Let $\mathcal{C}_{\mathrm{gr}}$ and $\mathcal{C}_{\mathrm{for\ gr}}$ denote the categories of group functors and formal group functors over $\mathcal{C}_a$, respectively. Let $\mathcal{C}^0_{\mathrm{for\ gr}}$ denote the full subcategory of $\mathcal{C}_{\mathrm{for\ gr}}$ consisting of those F such that $F(k)$ has only one element and F is smooth.

Remark 7.

- Let F be a formal group functor over $\mathcal{C}_a$. Then, the "tangent space at the neutrum":

$$\mathfrak{Lie}(F) := F(k[\epsilon]/\epsilon^2) \times_{F(k)} \{1\}$$

  (which coincides with $\mathfrak{t}_F$) is a Lie algebra where the Lie bracket is induced by the product of F.
- Finally, for a formal group functor and a morphism $A \to A/I$ with $I^2 = 0$ one has the following exact sequence of groups:

$$0 \to F(k[I]) \to F(A) \to F(A/I) \to 0.$$

Lemma B.2. *Let* $\mathrm{char}(k) = 0$. *Let* F *and* G *be two formal group functors. Assume that* F *is smooth and that* $F(k) = \{e\}$ *(one point). Then, the canonical map:*

$$\mathrm{Hom}_{\mathrm{gr}}(F, G) \to \mathrm{Hom}_{\mathrm{vect.\ sp.}}(\mathfrak{t}_F, \mathfrak{t}_G)$$

is injective.

Proof. Let A be an object of $\mathcal{C}_a$ and $\mathfrak{m} \subset A$ its maximal ideal and n such that $\mathfrak{m}^{n+1} = 0$. Let ϕ, ψ be in $\mathrm{Hom}_{\mathrm{gr}}(F, G)$ such that the induced vector space homomorphisms ϕ_*, ψ_* from $\mathfrak{t}_F$ to $\mathfrak{t}_G$ coincide. One has to prove that $\phi = \psi$.

Let us first deal with the case $n = 1$. By Lemma 2.10 of [Sc], there exist functorial isomorphisms $F(A) \simeq \mathfrak{t}_F \otimes \mathfrak{m}$ and $G(A) \simeq \mathfrak{t}_G \otimes \mathfrak{m}$ ($\mathfrak{m}$ as a k-vector space). It is now clear that both, ϕ and ψ, give the same morphism $F(A) \to G(A)$.

Now assume $n \geq 2$. Using the Nakayama Lemma one obtains a surjection:

$$A_{r,n} := k[x_1, \dots, x_r]/(x_1^{n+1}, \dots, x_r^{n+1}) \to A,$$

and hence a commutative diagram:

$$
\begin{array}{ccc}
F(A_{r,n}) & \longrightarrow & F(A) \\
\phi \downarrow & & \phi \downarrow \\
G(A_{r,n}) & \longrightarrow & G(A)
\end{array}
$$

and similarly for ψ. Observe that the top row is surjective since F is smooth and $A_{r,n} \to A$ is surjective. Therefore, it suffices to prove the statement for $A_{r,n}$.

Note that the injection $A_{r,n} \hookrightarrow A_{r \cdot n, 1}$ ($\mathrm{char}(k) = 0$):

$$k[\{x_i \mid 1 \leq i \leq r\}]/(x_i^{n+1}) \to k[\{x_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n\}]/(x_{ij}^2)$$

$$x_i \longmapsto x_{i1} + \dots + x_{in}$$

induces two commutative diagrams (for ϕ and ψ):

$$
\begin{array}{ccccc}
0 & \longrightarrow & F(A_{r,n}) & \longrightarrow & F(A_{r \cdot n, 1}) \\
& & \downarrow & & \downarrow \\
0 & \longrightarrow & G(A_{r,n}) & \longrightarrow & G(A_{r \cdot n, 1})
\end{array}
$$

It is then enough to check the case of $A_{r,1}$. Let us proceed by induction on r. The case $r = 1$ follows directly from the hypotheses.

We claim that the the following diagram is commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F(k[\ker(p)]) & \longrightarrow & F(A_{r,1}) & \xrightarrow{\ p_F\ } & F(A_{r-1,1}) & \longrightarrow & 0 \\
& & \phi_p \downarrow & & \phi_r \downarrow & & \phi_{r-1} \downarrow & & \\
0 & \longrightarrow & G(k[\ker(p)]) & \longrightarrow & G(A_{r,1}) & \xrightarrow{\ p_G\ } & G(A_{r-1,1}) & \longrightarrow & 0
\end{array}
$$

(and analogously for ψ). The morphisms p_F and p_G are surjective since they have sections, because the natural inclusion $A_{r-1,1} \hookrightarrow A_{r,1}$ is a section of the projection:

$$
\begin{aligned}
p : A_{r,1} &\to A_{r-1,1}, \\
x_r &\mapsto 0.
\end{aligned}
$$

Bearing in mind that $(\ker p)^2 = 0$, the claim follows.

The first case, which we have already proved (the square of the maximal ideal is (0)), implies that the $\phi_p = \psi_p$. The induction's hypothesis implies that $\phi_{r-1} = \psi_{r-1}$.

Now, recalling that both sequences split, one concludes that $\phi_r = \psi_r$ as desired. □

Let us now relate the study of group functors with that of Lie algebras. Let $\mathcal{C}_{Lie}$ denotes the category of Lie k-algebras. Then, there is a functor:

$$
\begin{aligned}
\mathfrak{Lie} : \mathcal{C}_{\text{for gr}} &\longrightarrow \mathcal{C}_{\text{Lie}}, \\
F &\longmapsto \mathfrak{Lie}(F) = \mathfrak{t}_F.
\end{aligned}
$$

For a Lie k-algebra $\mathfrak{L}$ define a functor on $\mathcal{C}_a$:

$$
A \rightsquigarrow \underline{\mathfrak{L}}(A) := \mathfrak{L} \otimes_k \mathfrak{m}_A
$$

(the Lie bracket of $\underline{\mathfrak{L}}(A)$ is that of $\mathfrak{L}$ extended by A-linearity).

Let $CH(x, y)$ denote the Campbell-Hausdorff series (see, for instance, [Ha] 14.4.15):

$$
CH(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]] + \dots \tag{B.1}
$$

then the map:

$$
\begin{aligned}
\underline{\mathfrak{L}}(A) \times \underline{\mathfrak{L}}(A) &\to \underline{\mathfrak{L}}(A) \\
(x, y) &\longmapsto CH(x, y)
\end{aligned}
$$

(note that $CH(x, y)$ is a finite sum since A is artinian) endows $\underline{\mathfrak{L}}(A)$ with a group structure ([Ha] 14.4.13-16). Let us denote this group functor by $\mathfrak{L}^g$. Moreover $\mathfrak{L}^g \to h_k$ is smooth and $\mathfrak{L}^g(k)$ consists of one point. Finally, since $CH(x, y)$ only depends on additions of iterated Lie brackets one has that every morphism of Lie algebras $\mathfrak{L}_1 \to \mathfrak{L}_2$ induces a morphism of group functors $\mathfrak{L}_1^g \to \mathfrak{L}_2^g$. In other words, there is a functor:

$$
\begin{aligned}
\mathfrak{G} : \mathcal{C}_{\text{Lie}} &\longrightarrow \mathcal{C}_{\text{gr}} \\
\mathfrak{L} &\longmapsto \mathfrak{L}^g
\end{aligned}
$$

such that $\mathfrak{Lie} \circ \mathfrak{G} = \text{Id}$.

Example 1. It is now easy to prove that finite dimensional Lie algebras are the Lie algebras of formal groups. Indeed, let $\mathfrak{L}^*$ be the dual vector space of a given Lie algebra $\mathfrak{L}$. Then, it holds that:

$$\mathrm{Hom}_{\mathrm{cont}}(\mathcal{O}, A) = \mathfrak{L} \otimes_k \mathfrak{m}_A,$$

where $\mathcal{O} := \widehat{S}^\bullet \mathfrak{L}^*$ is the completion of the symmetric algebra, $S^\bullet \mathfrak{L}^*$, with respect to the maximal ideal generated by $\mathfrak{L}^*$.

It is now straightforward to see that $\mathfrak{L}^g = h_\mathcal{O}$ and that

$$\mathfrak{Lie}(h_\mathcal{O}) = (\mathfrak{m}_\mathcal{O}/\mathfrak{m}_\mathcal{O}^2)^* = \mathfrak{L}.$$

Lemma B.3. *Let F be an object of $\mathcal{C}_{\mathrm{for\ gr}}^0$. The functor homomorphism (which will be called exponential) defined by:*

$$\underline{t}_F \to F$$

$$D \mapsto \exp(D) := \sum_{i \geq 0} \frac{1}{i!} D^i$$

yields an isomorphism $\mathfrak{t}_F^g \simeq F$.

Proof. Note that the sum is finite since $D \in \underline{t}_F(A) = \mathfrak{t}_F \otimes \mathfrak{m}_A$ (for $A \in \mathcal{C}_a$) is of the type $D = \sum_j m_j D_j$ (where $m_j \in \mathfrak{m}_A$ and $D_j \in \mathfrak{t}_F$) and hence D^i has coefficients in $\mathfrak{m}_A^j$. By the above construction, the exponential is a group homomorphism since it holds that ([Ha] 14.14):

$$\exp(D) \cdot \exp(D') = \exp(CH(D, D')).$$

In the same way that the exponential map has been defined a logarithm can also be introduced. Now the conclusion follows trivially. $\square$

From all these results one has the main theorem of this appendix which is a version for (certain) non-commutative group functors of the standard Lie Third Theorem.

Theorem B.4. *The functor $\mathfrak{Lie}$ renders $\mathcal{C}_{\mathrm{for\ gr}}^0$ a full subcategory of $\mathcal{C}_{\mathrm{Lie}}$.*

Proof. This follows from the following two facts:

- if $F, G \in \mathcal{C}_{\mathrm{gr}}$ have isomorphic Lie algebras $\mathfrak{t}_F \simeq \mathfrak{t}_G$, then they are isomorphic. (Recall that there are group isomorphisms $\mathfrak{t}_F^g \simeq F$ and $\mathfrak{t}_G^g \simeq G$).
- $\mathrm{Hom}_{\mathcal{C}_{\mathrm{gr}}}(F, G) \simeq \mathrm{Hom}_{\mathcal{C}_{\mathrm{Lie}}}(\mathfrak{t}_F, \mathfrak{t}_G)$ (Lemma B.2 proves the injectivity and the equality $\mathfrak{Lie} \circ \mathfrak{E} = \mathrm{Id}$ the surjectivity).

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