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Ground State Energy of the One-Component Charged Bose Gas*

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Dedicated to Leslie L. Foldy on the occasion of his 80th birthday

Abstract: The model considered here is the "jellium" model in which there is a uniform, fixed background with charge density $-e\rho$ in a large volume V and in which $N = \rho V$ particles of electric charge $+e$ and mass m move – the whole system being neutral. In 1961 Foldy used Bogolubov's 1947 method to investigate the ground state energy of this system for bosonic particles in the large ρ limit. He found that the energy per particle is $-0.402\, r_s^{-3/4} me^4/\hbar^2$ in this limit, where $r_s = (3/4\pi\rho)^{1/3} e^2 m/\hbar^2$. Here we prove that this formula is correct, thereby validating, for the first time, at least one aspect of Bogolubov's pairing theory of the Bose gas.

1. Introduction

Bogolubov's 1947 pairing theory [B] for a Bose fluid was used by Foldy [F] in 1961 to calculate the ground state energy of the one-component plasma (also known as "jellium") in the high density regime – which is the regime where the Bogolubov method was thought to be exact for this problem. Foldy's result will be verified rigorously in this paper; to our knowledge, this is the first example of such a verification of Bogolubov's theory in a three-dimensional system of bosonic particles.

Bogolubov proposed his approximate theory of the Bose fluid [B] in an attempt to explain the properties of liquid Helium. His main contribution was the concept of pairing of particles with momenta k and $-k$; these pairs are supposed to be the basic constituents of the ground state (apart from the macroscopic fraction of particles in the "condensate", or $k = 0$ state) and they are the basic unit of the elementary excitations of the system. The pairing concept was later generalized to fermions, in which case the pairing was between

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particles having opposite momenta and, at the same time, opposite spin. Unfortunately, this appealing concept about the boson ground state has neither been verified rigorously in a 3-dimensional example, nor has it been conclusively verified experimentally (but pairing has been verified experimentally for superconducing electrons).

The simplest question that can be asked is the correctness of the prediction for the ground state energy (GSE). This, of course, can only be exact in a certain limit – the "weak coupling" limit. In the case of the charged Bose gas, interacting via Coulomb forces, this corresponds to the *high density* limit. In gases with short range forces the weak coupling limit corresponds to low density instead.

Our system has N bosonic particles with unit positive charge and coordinates x_j, and a uniformly negatively charged "background" in a large domain Ω of volume V. We are interested in the thermodynamic limit. A physical realization of this model is supposed to be a uniform electron sea in a solid, which forms the background, while the moveable "particles" are bosonic atomic nuclei. The particle number density is then $\rho = N/V$ and this number is also the charge density of the background, thus ensuring charge neutrality.

The Hamiltonian of the one-component plasma is

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + U_{pp} + U_{pb} + U_{bb}, \tag{1}$$

where $p = -i\nabla$ is the momentum operator, $p^2 = -\Delta$, and the three potential energies, particle-particle, particle-background and background-background, are given by

$$U_{pp} = \sum_{1\le i<j\le N} |x_i - x_j|^{-1}, \tag{2}$$

$$U_{pb} = -\rho \sum_{j=1}^{N} \int_\Omega |x_j - y|^{-1}\, d^3y, \tag{3}$$

$$U_{bb} = \tfrac{1}{2}\rho^2 \int_\Omega \int_\Omega |x-y|^{-1}\, d^3x d^3y. \tag{4}$$

In our units $\hbar^2/m = 1$ and the charge is $e = 1$. The "natural" energy unit we use is two Rydbergs, $2Ry = me^4/\hbar^2$. It is customary to introduce the dimensionless quantity $r_s = (3/4\pi\rho)^{1/3} e^2 m/\hbar^2$. High density is small r_s.

The Coulomb potential is infinitely long-ranged and great care has to be taken because the finiteness of the energy per particle in the thermodynamic limit depends, ultimately, on delicate cancellations. The existence of the thermodynamic limit for a system of positive and negative particles, with the negative ones being fermions, was shown only in 1972 [LLe] (for the free energy, but the same proof works for the ground state energy). Oddly, the jellium case is technically a bit harder, and this was done in 1976 [LN] (for both bosons and fermions). One conclusion from this work is that neutrality (in the thermodynamic limit) will come about automatically – even if one does not assume it – provided one allows any excess charge to escape to infinity. In other words, given the background charge, the choice of a neutral number of particles has the lowest energy in the thermodynamic limit. A second point, as shown in [LN], is that e_0 is independent of the shape of the domain Ω provided the boundary is not too wild. For Coulomb systems this is not trivial and for real magnetic systems it is not even generally true. We take advantage of this liberty and assume that our domain is a cube $[0, L] \times [0, L] \times [0, L]$ with $L^3 = V$.

We note the well-known fact that the lowest energy of H in (1) without any restriction about "statistics" (i.e., on the whole of $\otimes^N L^2(\mathbb{R}^3)$) is the same as for bosons, i.e., on the symmetric subspace of $\otimes^N L^2(\mathbb{R}^3)$. The fact that bosons have the lowest energy comes from the Perron–Frobenius Theorem applied to $-\Delta$.

Foldy's calculation leads to the following theorem about the asymptotics of the energy for small r_s, which we call Foldy's law.

Theorem 1.1 (Foldy's Law). *Let E_0 denote the ground state energy, i.e., the bottom of the spectrum, of the Hamiltonian H acting in the Hilbert space $\otimes^N L^2(\mathbb{R}^3)$. We assume that $\Omega = [0, L] \times [0, L] \times [0, L]$. The ground state energy per particle, $e_0 = E_0/N$, in the thermodynamic limit $N, L \to \infty$ with $N/V = \rho$ fixed, in units of $me^4/\hbar^2$, is*

$$\begin{aligned} \lim_{V\to\infty} E_0/N = e_0 &= -0.40154\, r_s^{-3/4} + o(\rho^{1/4}) \\ &= -0.40154 \left(\frac{4\pi}{3}\right)^{1/4} \rho^{1/4} + o(\rho^{1/4}), \end{aligned} \tag{5}$$

where the number -0.40154 *is, in fact, the integral*

$$A = \frac{1}{\pi} 6^{1/4} \int_0^\infty \left\{ p^2(p^4+2)^{1/2} - p^4 - 1 \right\} dp = -\frac{3^{1/4} 4\, \Gamma(3/4)}{5\sqrt{\pi}\, \Gamma(5/4)} \approx -0.40154. \tag{6}$$

Actually, our proof gives a result that is more general than Theorem 1.1. We allow the particle number N to be totally arbitrary, i.e., we do not require $N = \rho V$. Our lower bound is still given by (5), where now ρ refers to the background charge density.

In [F] 0.40154 is replaced by 0.80307 since the energy unit there is 1 Ry. The main result of our paper is to prove (5) by obtaining a lower bound on E_0 that agrees with the right side of (5) An upper bound to E_0 that agrees with (5) (to leading order) was given in 1962 by Girardeau [GM], using the variational method of himself and Arnowitt [GA]. Therefore, to verify (5) to leading order it is only necessary to construct a rigorous lower bound of this form and this will be done here. It has to be admitted, as explained below, that the problem that Foldy and Girardeau treat is slightly different from ours because of different boundary conditions and a concommitant different treatment of the background. We regard this difference as a technicality that should be cleared up one day, and do not hesitate to refer to the statement of 1.1 as a theorem.

Before giving our proof, let us remark on a few historical and conceptual points. Some of the early history about the Bose gas, can be found in the lecture notes [L].

Bogolubov's analysis starts by assuming periodic boundary condition on the big box Ω and writing everything in momentum (i.e., Fourier) space. The values of the momentum, k are then discrete: $k = (2\pi/L)(m_1, m_2, m_3)$ with m_i an integer. A convenient tool for taking care of various $n!$ factors is to introduce second quantized operators $a_k^\#$ (where $a^\#$ denotes a or a^*), but it has to be understood that this is only a bookkeeping device. Almost all authors worked in momentum space, but this is neither necessary nor necessarily the most convenient representation (given that the calculations are not rigorous). Indeed, Foldy's result was reproduced by a calculation entirely in x-space [LS]. Periodic boundary conditions are not physical, but that was always chosen for convenience in momentum space.

We shall instead let the particle move in the whole space, i.e., the operator H acts in the Hilbert space $L^2(\mathbb{R}^{3N})$, or rather, since we consider bosons, in the the subspace consisting of the N-fold fully symmetric tensor product of $L^2(\mathbb{R}^3)$. The background potential defined in (2) is however still localized in the cube Ω. We could also have confined the particles to Ω with Dirichlet boundary conditions. This would only raise the ground state energy and thus, for the lower bound, our setup is more general.

There is, however, a technical point that has to be considered when dealing with Coulomb forces. The background never appears in Foldy's calculation; he simply removes the $k = 0$ mode from the Fourier transform, ν of the Coulomb potential (which is $\nu(k) = 4\pi|k|^{-2}$, but with k taking the discrete values mentioned above, so that we are thus dealing with a "periodized" Coulomb potential). The $k = 0$ elimination means that we set $\nu(0) = 0$, and this amounts to a subtraction of the average value of the potential – which is supposed to be a substitute for the effect of a neutralizing background. It does not seem to be a trivial matter to prove that this is equivalent to having a background, but it surely can be done. Since we do not wish to overload this paper, we leave this demonstration to another day. In any case the answers agree (in the sense that our rigorous lower bound agrees with Foldy's answer), as we prove here. If one accepts the idea that setting $\nu(0) = 0$ is equivalent to having a neutralizing background, then the ground state energy problem is finished because Girardeau shows [GM] that Foldy's result is a true upper bound within the context of the $\nu(0) = 0$ problem.

The potential energy is quartic in the operators $a_k^\#$. In Bogolubov's analysis only terms in which there are four or two $a_0^\#$ operators are retained. The operator a_0^* creates, and a_0 destroys particles with momentum 0 and such particles are the constituents of the "condensate". In general there are no terms with three $a_0^\#$ operators (by momentum conservation) and in Foldy's case there is also no four $a_0^\#$ term (because of the subtraction just mentioned).

For the usual short range potential there is a four $a_0^\#$ term and this is supposed to give the leading term in the energy, namely $e_0 = 4\pi\rho a$, where a is the "scattering length" of the two-body potential. Contrary to what would seem reasonable, this number, $4\pi\rho a$ is *not* the coefficient of the four $a_0^\#$ term, and to to prove that $4\pi\rho a$ is, indeed, correct took some time. It was done in 1998 [LY] and the method employed in [LY] will play an essential role here. But it is important to be clear about the fact that the four $a_0^\#$, or "mean field" term is absent in the jellium case by virtue of charge neutrality. The leading term in this case presumably comes from the two $a_0^\#$ terms, and this is what we have to prove. For the short range case, on the other hand, it is already difficult enough to obtain the $4\pi\rho a$ energy that going beyond this to the two $a_0^\#$ terms is beyond the reach of rigorous analysis at the moment.

The Bogolubov ansatz presupposes the existence of Bose–Einstein condensation (BEC). That is, most of the particles are in the $k = 0$ mode and the few that are not come in pairs with momenta k and $-k$. Two things must be said about this. One is that the only case (known to us) in which one can verify the correctness of the Bogolubov picture at weak coupling is the *one-dimensional* delta-function gas [LLi] – in which case there is presumably *no* BEC (because of the low dimensionality). Nevertheless the Bogolubov picture remains correct at low density and the explanation of this seeming contradiction lies in the fact that BEC is not needed; what is really needed is a kind of condensation on a length scale that is long compared to relevant parameters, but which is fixed and need not be as large as the box length L. This was realized in [LY] and the main idea there was to decompose Ω into fixed-size boxes of appropriate length and use Neumann boundary conditions on these boxes (which can only lower the energy, and which is fine since we want a lower bound). We shall make a similar decomposition here, but, unlike the case in [LY] where the potential is purely repulsive, we must deal here with the Coulomb potential and work hard to achieve the necessary cancellation.

The only case in which BEC has been proved to exist is in the hard core lattice gas at half-filling (equivalent to the spin-1/2 XY model) [KLS].

Weak coupling is sometimes said to be a "perturbation theory" regime, but this is not really so. In the one-dimensional case [LLi] the asymptotics near $\rho = 0$ is extremely difficult to deduce from the exact solution because the "perturbation" is singular. Nevertheless, the Bogolubov calculation gives it effortlessly, and this remains a mystery.

One way to get an excessively negative lower bound to e_0 for jellium is to ignore the kinetic energy. One can then show easily (by an argument due to Onsager) that the potential energy alone is bounded below by $e_0 \sim -\rho^{1/3}$. See [LN]. Thus, our goal is to show that the kinetic energy raises the energy to $-\rho^{1/4}$. This was done, in fact, in [CLY], but without achieving the correct coefficient $-0.803(4\pi/3)^{1/4}$. Oddly, the $-\rho^{1/4}$ law was proved in [CLY] by first showing that the *non-thermodynamic* $N^{7/5}$ law for a *two-component* bosonic plasma, as conjectured by Dyson [D], is correct.

The [CLY] paper contains an important innovation that will play a key role here. There, too, it was necessary to decompose $\mathbf{R}^3$ into boxes, but a way had to be found to eliminate the Coulomb interaction *between* different boxes. This was accomplished by *not* fixing the location of the boxes but rather averaging over all possible locations of the boxes. This "sliding localization" will play a key role here, too. This idea was expanded upon in [GG]. Thus, we shall have to consider only one finite box with the particles and the background charge in it independent of the rest of the system. However, a price will have to be paid for this luxury, namely it will not be entirely obvious that the number of particles we want to place in each box is the same for all boxes, i.e., $\rho\ell^3$, where ℓ is the length of box. Local neutrality, in other words, cannot be taken for granted. The analogous problem in [LY] is easier because no attractive potentials are present there. We solve this problem by choosing the number, n, in each box to be the number that gives the lowest energy in the box. This turns out to be close to $n = \rho\ell^3$, as we show and as we know from [LN] must be the case as $\ell \to \infty$.

Finally, let us remark on one bit of dimensional analysis that the reader should keep in mind. One should not conclude from (5) that a typical particle has energy $\rho^{1/4}$ and hence momentum $\rho^{1/8}$ or de Broglie wavelength $\rho^{-1/8}$. This is *not* the correct picture. Rather, a glance at the Bogolubov–Foldy calculation shows that the momenta of importance are of order $\rho^{-1/4}$, and the seeming paradox is resolved by noting that the number of excited particles (i.e., those not in the $k = 0$ condensate) is of order $N\rho^{-1/4}$. This means that we can, hopefully, localize particles to lengths as small as $\rho^{-1/4+\epsilon}$, and cut off the Coulomb potential at similar lengths, without damage, provided we do not disturb the condensate particles. It is this clear separation of scales that enables our asymptotic analysis to succeed.

2. Outline of the Proof

The proof of our Main Theorem 1.1 is rather complicated and somewhat hard to penetrate, so we present the following outline to guide the reader.

2.1. Section 3. Here we localize the system whose size is L into small boxes of size ℓ independent of L, but dependent on the intensive quantity ρ. Neumann boundary conditions for the Laplacian are used in order to ensure a lower bound to the energy. We always think of operators in terms of quadratic forms and the Neumann Laplacian in a box Q is defined for all functions in $\psi \in L^2(Q)$ by the quadratic form

$$(\psi, -\Delta_{\text{Neumann}}\psi) = \int_Q |\nabla\psi(x)|^2 \, dx.$$

The lowest eigenfunction of the Neumann Laplacian is the constant function and this plays the role of the condensate state. This state not only minimizes the, kinetic energy, but it is also consistent with neutralizing the background and thereby minimizing the Coulomb energy. The particles not in the condensate will be called “excited” particles.

To avoid localization errors we take $\ell \gg \rho^{-1/4}$, which is the relevant scale as we mentioned in the Introduction. The interaction among the boxes is controlled by using the sliding method of [CLY]. The result is that we have to consider only interactions among the particles and the background in each little box separately.

The N particles have to be distributed among the boxes in a way that minimizes the total energy. We can therefore not assume that each box is neutral. Instead of dealing with this distribution problem we do a simpler thing which is to choose the particle number in each little box so as to achieve the absolute minimum of the energy in that box. Since all boxes are equivalent this means that we take a common value n as the particle number in each box. The total particle number which is n times the number of boxes will not necessarily equal N, but this is of no consequence for a lower bound. We shall show later, however, that it equality is nearly achieved, i.e., the the energy minimizing number n in each box is close to the value needed for neutrality.

2.2. Section 4. It will be important for us to replace the Coulomb potential by a cutoff Coulomb potential. There will be a short distance cutoff of the singularity at a distance r and a large distance cutoff of the tail at a distance R, with $r \leq R \ll \ell$. One of the unusual features of our proof is that r are R are not fixed once and for all, but are readjusted each time new information is gained about the error bounds.

In fact, already in Sect. 4 we give a simple preliminary bound on n by choosing $R \sim \rho^{-1/3}$, which is much smaller than the relevant scale $\rho^{-1/4}$, although the choice of R that we shall use at the end of the proof is of course much larger than $\rho^{-1/4}$, but less than ℓ.

2.3. Section 5. There are several terms in the Hamiltonian. There is the kinetic energy, which is non-zero only for the excited particles. The potential energy, which is a quartic term in the language of second quantization, has various terms according to the number of times the constant function appears. Since we do not have periodic boundary conditions we will not have the usual simplification caused by conservation of momentum, and the potential energy will be correspondingly more complicated than the usual expression found in textbooks.

In this section we give bounds on the different terms in the Hamiltonian and use these to get a first control on the condensation, i.e., a control on the number of particles $\widehat{n}_+$ in each little box that are not in the condensate state.

The difficult point is that $\widehat{n}_+$ is an operator that does not commute with the Hamiltonian and so it does not have a sharp value in the ground state. We give a simple preliminary bound on its average $\langle\widehat{n}_+\rangle$ in the ground state by again choosing $R \sim \rho^{-1/3}$. In order to control the condensation to an appropriate accuracy we shall eventually need not only a bound on the average, $\langle \widehat{n}_+ \rangle$, but also on the fluctuation, i.e, on $\langle \widehat{n}_+^2 \rangle$. This will be done in Sect. 8 using a novel method developed in Appendix A for localizing off-diagonal matrices.

2.4. Section 6. The part of the potential energy that is most important is the part that is quadratic in the condensate operators $a_0^\#$ and quadratic in the excited variables $a_p^\#$ with $p \neq 0$. This, together with the kinetic energy, which is also quadratic in the $a_p^\#$, is the part of the Hamiltonian that leads to Foldy's law. Although we have not yet managed to eliminate the non-quadratic part up to this point we study the main "quadratic" part of the Hamiltonian. It is in this section that we essentially do Foldy's calculation.

It is not trivial to diagonalize the quadratic form and thereby reproduce Foldy's answer because there is no momentum conservation. In particular there is no simple relation between the resolvent of the Neumann Laplacian and the Coulomb kernel. The former is defined relative to the box and the latter is defined relative to the whole of $\mathbb{R}^3$. It is therefore necessary for us to localize the wavefunction in the little box away from the boundary. On such functions the boundary condition is of no importance and we can identify the kinetic energy with the Laplacian in all of $\mathbb{R}^3$. This allows us to have a simple relation between the Coulomb term and the kinetic energy term since the Coulomb kernel is in fact the resolvent of the Laplacian in all of $\mathbb{R}^3$.

When we cut off the wavefunction near the boundary we have to be very careful because we must not cut off the part corresponding to the particles in the condensate. To do so would give too large a localization energy. Rather, we cut off only functions with sufficiently large kinetic energy so that the localization energy is relatively small compared to the kinetic energy. The technical lemma needed for this is a double commutator inequality given in Appendix B.

2.5. Section 7. At this point we have bounds available for the quadratic part (from Sect. 6) and the annoying non-quadratic part (from Sect. 5) of the Hamiltonian. These depend on r, R, n, $\langle \widehat{n}_+ \rangle$, and $\langle \widehat{n}_+^2 \rangle$. We avail ourselves of the bounds previously obtained for n and $\langle \widehat{n}_+ \rangle$ and now use our freedom to choose different values for r and R to bootstrap to the desired bounds on n and $\langle \widehat{n}_+ \rangle$, i.e., we prove that there is almost neutrality and almost condensation in each little box.

2.6. Section 8. In order to control $\langle \widehat{n}_+^2 \rangle$ we utilize, for the first time, the new method for localizing large matrices given in Appendix A. This method allows us to restrict to states with small fluctuations in $\widehat{n}_+$, and thereby bound $\langle \widehat{n}_+^2 \rangle$, provided we know that the terms that do not commute with $\widehat{n}_+$ have suffciently small expectation values. We then give bounds on these $\widehat{n}_+$ "off-diagonal" terms. Unfortunately, these bounds are in terms of positive quantities coming from the Coulomb repulsion, but for which we actually do not have independent a-priori bounds. Normally, when proving a lower bound to a Hamiltonian, we can sometimes control error terms by absorbing them into positive terms in the Hamiltonian, which are then ignored. This may be done even when we do not have an a-priori bound on these positive terms. If we want to use Theorem A.1 in Appendix A, we will need an absolute bound on the "off-diagonal" terms and we can therefore not use the technique of absorbing them into the positive terms. The decision when to use the theorem in Appendix A or use the technique of absorption into positive terms is resolved in Sect. 9.

2.7. Section 9. Since we do not have an a-priori bound on the positive Coulomb terms as described above we are faced with a dichotomy. If the positive terms are, indeed, so large that enough terms can be controlled by them we do not need to use the localization technique of Appendix A to finish the proof of Foldy's law. The second possibility is that the positive terms are bounded in which case we can use this fact to control the terms that do commute with $\widehat{n}_+$ and this allows us to use the localization technique in Appendix A to finish the proof of Foldy's law. Thus, the actual magnitude of the positive repulsion terms is unimportant for the derivation of Foldy's law.

3. Reduction to a Small Box

As described in the previous sections we shall localize the problem into smaller cubes of size $\ell \ll L$. We shall in fact choose ℓ as a function of ρ in such a way that $\rho^{1/4}\ell \to \infty$ as $\rho \to \infty$.

We shall localize the kinetic energy by using Neumann boundary conditions on the smaller boxes.

We shall first, however, describe how we may control the electostatic interaction between the smaller boxes using the sliding technique of [CLY].

Let t, with $0 < t < 1/2$, be a parameter which we shall choose later to depend on ρ in such a way that $t \to 0$ as $\rho \to \infty$.

The choice of ℓ and t as functions of ρ will be made at the end of Sect. 9 when we complete the proof of Foldy's law.

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ satisfy supp $\chi \subset [(-1+t)/2, (1-t)/2]^3$, $0 \le \chi \le 1$, $\chi(x) = 1$ for x in the smaller box $[(-1+2t)/2, (1-2t)/2]^3$, and $\chi(x) = \chi(-x)$. Assume that all m-th order derivatives of χ are bounded by $C_m t^{-m}$, where the constants C_m depend only on m and are, in particular, independent of t. Let $\chi_\ell(x) = \chi(x/\ell)$. Let $\eta = \sqrt{1-\chi}$. We shall assume that χ is defined such that η is also C^1. Let $\eta_\ell(x) = \eta(x/\ell)$. Using χ we define the constant γ by $\gamma^{-1} = \int \chi(y)^2\, dy$, and note that $1 \le \gamma \le (1-2t)^{-3}$. We also introduce the Yukawa potential $Y_\nu(x) = |x|^{-1}e^{-\nu|x|}$ for $\nu > 0$.

As a preliminary to the following Lemma 3.1 we quote Lemma 2.1 in [CLY].

Lemma. *Let $K : \mathbb{R}^3 \to \mathbb{R}$ be given by*

$$K(z) = r^{-1}\left\{e^{-\nu r} - e^{-\omega r}h(z)\right\}$$

with $r = |z|$ and $\omega > \nu \ge 0$. Let h satisfy (i) *h is a C^4 function of compact support;* (ii) *$h(z) = 1 + ar^2 + \mathrm{O}(r^3)$ near $z = 0$. Let $h(z) = h(-z)$, so that K has a real Fourier transform. Then there is a constant, C_3 (depending on h) such that if $\omega - \nu \ge C_3$ then K has a positive Fourier transform and, moreover,*

$$\sum_{1 \le i < j \le N} e_i e_j K(x_i - x_j) \ge \frac{1}{2}(\nu - \omega)N$$

for all $x_1, \dots x_N \in \mathbb{R}^3$ and all $e_i = \pm 1$.

Lemma 3.1 (Electrostatic decoupling of boxes using sliding). *There exists a function of the form $\omega(t) = Ct^{-4}$ (we assume that $\omega(t) \ge 1$ for $t < 1/2$) and a constant γ with $1 \le \gamma \le (1-2t)^{-3}$ such that if we set*

$$w(x, y) = \chi_\ell(x) Y_{\omega(t)/\ell}(x - y) \chi_\ell(y) \tag{7}$$

then the potential energy satisfies

$$
\begin{aligned}
&U_{pp} + U_{pb} + U_{bb} \\
&\quad \geq \gamma \sum_{\lambda \in \mathbb{Z}^3} \int_{\mu \in [-\frac{1}{2}, \frac{1}{2}]^3} d\mu \Big\{ \sum_{1 \leq i < j \leq N} w\big(x_i + (\mu+\lambda)\ell, x_j + (\mu+\lambda)\ell\big) \\
&\qquad\qquad - \rho \sum_{j=1}^{N} \int_{\Omega} w\big(x_j + (\mu+\lambda)\ell, y + (\mu+\lambda)\ell\big)\, dy \\
&\qquad\qquad + \tfrac{1}{2}\rho^2 \iint_{\Omega \times \Omega} w\left(x + (\mu+\lambda)\ell, y + (\mu+\lambda)\ell\right)\, dx\, dy \Big\} - \frac{\omega(t) N}{2\ell}.
\end{aligned}
$$

Proof. We calculate

$$
\begin{aligned}
&\sum_{\lambda \in \mathbb{Z}^3} \int_{\mu \in [-1/2, 1/2]^3} d\mu\, \gamma \chi(x + (\mu+\lambda)) Y_{\omega}(x - y) \chi(y + (\mu+\lambda)) \\
&\qquad = \int \gamma \chi(x + z) Y_{\omega}(x - y) \chi(y + z)\, dz = h(x - y) Y_{\omega}(x - y),
\end{aligned}
$$

where we have set $h = \gamma \chi * \chi$. Note that $h(0) = 1$ and that h satisfies all the assumptions in Lemma 2.1 in [CLY]. We then conclude from Lemma 2.1 in [CLY] that the Fourier transform of the function $F(x) = |x|^{-1} - h(x) Y_{\omega(t)}(x)$ is non-negative, where ω is a function such that $\omega(t) \to \infty$ as $t \to 0$. [The detailed bounds from [CLY] show that we may in fact choose $\omega(t) = Ct^{-4}$, since $\omega(t)$ has to control the 4th derivative of h.] Note, moreover, that $\lim_{x \to 0} F(x) = \omega(t)$. Hence

$$
\begin{aligned}
&\sum_{1 \leq i < j \leq N} F(y_i - y_j) - \rho \sum_{j=1}^{N} \int_{\ell^{-1}\Omega} F(y_j - y)\, dy \\
&\qquad\qquad + \tfrac{1}{2}\rho^2 \iint_{\ell^{-1}\Omega \times \ell^{-1}\Omega} F(x - y)\, dx\, dy \geq -\frac{N\omega(t)}{2}.
\end{aligned}
$$

The lemma follows by writing $|x|^{-1} = F(x) + h(y) Y_{\omega(t)}(x)$ and by rescaling from boxes of size 1 to boxes of size ℓ. □

As explained above we shall choose the parameters t and ℓ as functions of ρ at the very end of the proof. We shall choose them in such a way that $t \to 0$ and $\rho^{1/4}\ell \to \infty$ as $\rho \to \infty$. Moreover, we will have conditions of the form

$$
\rho^{-\tau}(\rho^{1/4}\ell) \to 0, \quad \text{and } t^{\nu}(\rho^{1/4}\ell) \to \infty
$$

as $\rho \to \infty$, where τ, ν are universal constants.

Consider now the n-particle Hamiltonian

$$
H^{n}_{\mu,\lambda} = -\tfrac{1}{2} \sum_{j=1}^{n} \Delta^{(j)}_{Q_{\mu,\lambda}} + \gamma W_{\mu,\lambda}, \tag{8}
$$

where we have introduced the Neumann Laplacian $\Delta^{(j)}_{Q_{\mu,\lambda}}$ of the cube $Q_{\mu,\lambda} = (\mu + \lambda)\ell + \left[-\frac{1}{2}\ell, \frac{1}{2}\ell\right]^3$ and the potential

$$W_{\mu,\lambda}(x_1, \ldots, x_n) = \sum_{1\le i<j\le n} w\left(x_i + (\mu+\lambda)\ell, x_j + (\mu+\lambda)\ell\right) - \rho \sum_{j=1}^{n} \int_\Omega w\left(x_j + (\mu+\lambda)\ell, y + (\mu+\lambda)\ell\right) dy + \tfrac{1}{2}\rho^2 \iint_{\Omega\times\Omega} w\left(x + (\mu+\lambda)\ell, y + (\mu+\lambda)\ell\right) dx\, dy.$$

Lemma 3.2 (Decoupling of boxes). *Let $E^n_{\mu,\lambda}$ be the ground state energy of the Hamiltonian $H^n_{\mu,\lambda}$ given in (8) considered as a bosonic Hamiltonian. The ground state energy E_0 of the Hamiltonian H in (1) is then bounded below as*

$$E_0 \ge \sum_{\lambda\in\mathbb{Z}^3} \int_{\mu\in[-\frac{1}{2},\frac{1}{2}]^3} \inf_{1\le n\le N} E^n_{\mu,\lambda}\, d\mu - \frac{\omega(t)N}{2\ell}.$$

Proof. If $\Psi(x_1, \ldots, x_N) \in L^2(\mathbb{R}^{3N})$ is a symmetric function. Then

$$(\Psi, H\Psi) \ge \sum_{\lambda\in\mathbb{Z}^3} \int_{\mu\in[-\frac{1}{2},\frac{1}{2}]^3} (\Psi, \widetilde{H}_{\mu,\lambda}\Psi)\, d\mu - \frac{\omega(t)N}{2\ell},$$

where

$$(\Psi, \widetilde{H}_{\mu,\lambda}\Psi) = \sum_{j=1}^{N} \int_{x_j\in Q_{\mu,\lambda}} |\nabla_j \Psi(x_1, \ldots, x_N)|^2\, dx_1 \ldots dx_N + \gamma \int W_{\mu,\lambda}(x_1, \ldots, x_N) |\Psi(x_1, \ldots, x_N)|^2\, dx_1 \ldots dx_N.$$

The lemma follows since it is clear that $(\Psi, \widetilde{H}_{\mu,\lambda}\Psi) \ge \inf_{1\le n\le N} E^n_{\mu,\lambda}$. □

For given μ the Hamiltonians $H^n_{\mu,\lambda}$ fall in three groups depending on λ. The first kind for which $Q_{\lambda,\mu} \cap \Omega = \emptyset$. They describe boxes with no background. The optimal energy for these boxes are clearly achieved for $n = 0$. The second kind for which $Q_{\lambda,\mu} \subset \Omega$. These Hamiltonians are all unitarily equivalent to γH^n_ℓ, where

$$H^n_\ell = \sum_{j=1}^{n} \left(-\tfrac{1}{2}\gamma^{-1}\Delta_{\ell,j} - \rho \int w(x_j, y)\, dy\right) + \sum_{1\le i<j\le n} w(x_i, x_j) + \tfrac{1}{2}\rho^2 \iint w(x, y)\, dx\, dy, \tag{9}$$

where $-\Delta_\ell$ is the Neumann Laplacian for the cube $[-\ell/2, \ell/2]^3$. Finally, there are operators of the third kind for which $Q_{\mu,\lambda}$ intersects both Ω and its complement. In this case the particles only see part of the background. If we artificially add the missing background only the last term in the potential $W_{\mu,\lambda}$ increases. (The first term does not change and the second can only decrease.) In fact it will increase by no more than

$$\tfrac{1}{2}\rho^2 \iint w(x,y)\,dx\,dy \le \tfrac{1}{2}\rho^2 \iint_{\substack{x\in[-\ell/2,\ell/2]^3 \\ y\in[-\ell/2,\ell/2]^3}} |x-y|^{-1}\,dx\,dy \le C\rho^2\ell^5.$$

Thus the operator $H^n_{\mu,\lambda}$ of the third kind are bounded below by an operator which is unitarily equivalent to $\gamma H^n_\ell - C\rho^2\ell^5$.

We now note that the number of boxes of the third kind is bounded above by $C(L/\ell)^2$. The total number of boxes of the second or third kind is bounded above by $(L+\ell)^3/\ell^3 = (1+L/\ell)^3$.

We have therefore proved the following result.

Lemma 3.3 (Reduction to one small box). *The ground state energy E_0 of the Hamiltonian H in (1) is bounded below as*

$$E_0 \ge (1+L/\ell)^3\gamma \inf_{1\le n\le N} \inf\ \mathrm{Spec}\ H^n_\ell - C(L/\ell)^2\rho^2\ell^5 - \frac{\omega(t)N}{2\ell},$$

where H^n_ℓ is the Hamiltonian defined in (9).

In the rest of the paper we shall study the Hamiltonian (9).

4. Long and Short Distance Cutoffs in the Potential

The potential in the Hamiltonian (9) is w given in (7). Our aim in this section to replace w by a function that has long and short distance cutoffs.

We shall replace the function w by

$$w_{r,R}(x,y) = \chi_\ell(x)V_{r,R}(x-y)\chi_\ell(y), \tag{10}$$

where

$$V_{r,R}(x) = Y_{R^{-1}}(x) - Y_{r^{-1}}(x) = \frac{e^{-|x|/R} - e^{-|x|/r}}{|x|}. \tag{11}$$

Here $0 < r \le R \le \omega(t)^{-1}\ell$. Note that for $x \ll r$ then $V_{r,R}(x) \approx r^{-1} - R^{-1}$ and for $|x| \gg R$ then $V_{r,R}(x) \approx |x|^{-1}e^{-|x|/R}$.

In this section we shall bound the effect of replacing w by $w_{r,R}$. We shall not fix the cutoffs r and R, but rather choose them differently at different stages in the later arguments.

We first introduce the cutoff R alone, i.e., we bound the effect of replacing w by $w_R(x,y) = \chi_\ell(x)V_R(x-y)\chi_\ell(y)$, where $V_R(x) = |x|^{-1}e^{-|x|/R} = Y_{R^{-1}}(x)$. Thus, since $R \le \omega(t)^{-1}\ell$, the Fourier transforms satisfy

$$\widehat{Y}_{\omega/\ell}(k) - \widehat{V}_R(k) = 4\pi\left(\frac{1}{k^2+(\omega(t)/\ell)^2} - \frac{1}{k^2+R^{-2}}\right) \ge 0.$$

(We use the convention that $\hat{f}(k) = \int f(x)e^{-ikx}\,dx$.) Hence $w(x,y) - w_R(x,y) = \chi_\ell(x)\left(Y_{\omega/\ell} - V_R\right)(x-y)\chi_\ell(y)$ defines a positive semi-definite kernel. Note, moreover, that $\left(Y_{\omega/\ell} - V_R\right)(0) = R^{-1} - \omega/\ell \leq R^{-1}$ Thus,

$$\begin{aligned}
&\sum_{1\leq i<j\leq n} w(x_i,x_j) - \rho\sum_{j=1}^n \int w(x_j,y)\,dy + \tfrac{1}{2}\rho^2 \iint w(x,y)\,dx\,dy \\
&\quad - \left(\sum_{1\leq i<j\leq n} w_R(x_i,x_j) - \rho\sum_{j=1}^n \int w_R(x_j,y)\,dy + \tfrac{1}{2}\rho^2 \iint w_R(x,y)\,dx\,dy\right) \\
&= \frac{1}{2}\iint \left[\sum_i^n \delta(x-x_i) - \rho\right](w-w_r)(x,y)\left[\sum_i^n \delta(y-x_i) - \rho\right]dx\,dy \\
&\quad - \frac{1}{2}\sum_i^n \chi_\ell(x_i)^2\left(Y_{\omega/\ell} - V_R\right)(0) \geq -\tfrac{1}{2}n\left(Y_{\omega/\ell} - V_R\right)(0) = -\tfrac{1}{2}nR^{-1}. \qquad (12)
\end{aligned}$$

We now bound the effect of replacing w_R by $w_{r,R}$. I.e., we are replacing $V_R(x) = |x|^{-1}e^{-|x|/R}$ by $|x|^{-1}\left(e^{-|x|/R} - e^{-|x|/r}\right)$. This will lower the repulsive terms and for the attractive term we get

$$\begin{aligned}
-\rho\sum_{j=1}^n \int w_R(x_j,y)\,dy \geq\ & -\rho\sum_{j=1}^n \int w_{r,R}(x_j,y)\,dy \\
& - n\rho\sup_x \int \chi_\ell(x)\frac{e^{-|x-y|/r}}{|x-y|}\chi_\ell(y)\,dy \qquad (13)\\
\geq\ & -\rho\sum_{j=1}^n \int w_{r,R}(x_j,y)\,dy - Cn\rho r^2.
\end{aligned}$$

If we combine the bounds (12) and (13) we have the following result.

Lemma 4.1 (Long and short distance potential cutoffs). *Consider the Hamiltonian*

$$\begin{aligned}
H^n_{\ell,r,R} = \sum_{j=1}^n \left(-\tfrac{1}{2}\gamma^{-1}\Delta_{\ell,j} - \rho\int w_{r,R}(x_j,y)\,dy\right) + \sum_{1\leq i<j\leq n} w_{r,R}(x_i,x_j) \qquad (14)\\
+ \tfrac{1}{2}\rho^2\iint w_{r,R}(x,y)\,dx\,dy,
\end{aligned}$$

where $w_{r,R}$ is given in (10) and (11) with $0 < r \leq R \leq \omega(t)^{-1}\ell$ and $-\Delta_\ell$ as before is the Neumann Laplacian for the cube $[-\ell/2,\ell/2]^3$. Then the Hamiltonian H^n_ℓ defined in (9) obeys the lower bound

$$H^n_\ell \geq H^n_{\ell,r,R} - \tfrac{1}{2}nR^{-1} - C_1 n\rho r^2.$$

A similar argument gives the following result.

Lemma 4.2. *With the same notation as above we have for $0 < r' \leq r \leq R \leq R' \leq \omega(t)^{-1}\ell$ that*

$$H^n_{\ell,r',R'} \geq H^n_{\ell,r,R} - \tfrac{1}{2}nR^{-1} - C_1 n\rho r^2.$$

Proof. Simply note that $V_{r',R'}(x) - V_{r,R}(x) = Y_{R'^{-1}}(x) - Y_{R^{-1}}(x) + Y_{r^{-1}}(x) - Y_{r'^{-1}}(x)$ and now use the same arguments as before. □

Corollary 4.3 (The particle number n cannot be too small). *There exists a constant $C > 0$ such that if $\omega(t)^{-1}\rho^{1/3}\ell > C$ then $H_\ell^n \geq 0$ if $n \leq C\rho\ell^3$.*

Proof. Choose $R = \rho^{-1/3}$ and $r = \frac{1}{2}R$. Then we may assume that $R \leq \omega(t)^{-1}\ell$ since $\omega(t)^{-1}\rho^{1/3}\ell$ is large. From Lemma 4.1 we see immediately that

$$H_\ell^n \geq -\sum_{j=1}^{n} \rho \int w_{r,R}(x_j, y)\, dy + \tfrac{1}{2}\rho^2 \iint w_{r,R}(x,y)\, dx\, dy - Cn\rho R^2$$
$$\geq -n \sup_x \rho \int w_{r,R}(x,y)\, dy + \tfrac{1}{2}\rho^2 \iint w_{r,R}(x,y)\, dx\, dy - Cn\rho R^2.$$

The corollary follows since $\sup_x \int w_{r,R}(x,y)\, dy \leq 4\pi R^2$ and with the given choice of R and r it is easy to see that $\frac{1}{2}\iint w_{r,R}(x,y)\, dx\, dy \geq cR^2\ell^3$. □

5. Bound on the Unimportant Part of the Hamiltonian

In this section we shall bound the Hamiltonian $H_{\ell,r,R}^n$ given in (14). We emphasize that we do not necessarily have neutrality in the cube, i.e., n and $\rho\ell^3$ may be different. We are simply looking for a lower bound to $H_{\ell,r,R}^n$, that holds for all n. The goal is to find a lower bound that will allow us to conclude that the optimal n, i.e., the value for which the energy of the Hamiltonian is smallest, is indeed close to the neutral value.

We shall express the Hamiltonian in second quantized language. This is purely for convenience. We stress that we are not in any way changing the model by doing this and the treatment is entirely rigorous and could have been done without the use of second quantization.

Let $u_p, \ell p/\pi \in (\mathbb{N} \cup \{0\})^3$ be an orthonormal basis of eigenfunctions of the Neumann Laplacian $-\Delta_\ell$ such that $-\Delta_\ell u_p = |p|^2 u_p$. I.e.,

$$u_p(x_1, x_2, x_3) = c_p \ell^{-3/2} \prod_{j=1}^{3} \cos\left(\frac{p_j\pi(x_j + \ell/2)}{\ell}\right),$$

where the normalization satisfies $c_0 = 1$ and in general $1 \leq c_p \leq \sqrt{8}$. The function $u_0 = \ell^{-3/2}$ is the constant eigenfunction with eigenvalue 0. We note that for $p \neq 0$ we have

$$(u_p, -\Delta_\ell u_p) \geq \pi^2 \ell^{-2}. \tag{15}$$

We now express the Hamiltonian $H_{\ell,r,R}^n$ in terms of the creation and annihilation operators $a_p = a(u_p)$ and $a_p^* = a(u_p)^*$.

Define

$$\widehat{w}_{pq,\mu\nu} = \iint w_{r,R}(x,y) u_p(x) u_q(y) u_\mu(x) u_\nu(y)\, dx\, dy.$$

We may then express the two-body repulsive potential as

$$\sum_{1\leq i<j\leq n} w_{r,R}(x_i, x_j) = \tfrac{1}{2} \sum_{pq,\mu\nu} \widehat{w}_{pq,\mu\nu} a_p^* a_q^* a_\nu a_\mu,$$

where the right-hand side is considered restricted to the n-particle subspace. Likewise the background potential can be written

$$-\rho \sum_{j=1}^{n} w_{r,R}(x_j, y)\, dy = -\rho \ell^3 \sum_{pq} \widehat{w}_{0p,0q} a_p^* a_q$$

and the background-background energy

$$\tfrac{1}{2}\rho^2 \iint w_{r,R}(x, y)\, dx\, dy = \tfrac{1}{2}\rho^2 \ell^6 \widehat{w}_{00,00}.$$

We may therefore write the Hamiltonian as

$$\begin{aligned} H^n_{\ell,r,R} = {} & \tfrac{1}{2}\gamma^{-1} \sum_p |p|^2 a_p^* a_p + \tfrac{1}{2} \sum_{pq,\mu\nu} \widehat{w}_{pq,\mu\nu} a_p^* a_q^* a_\nu a_\mu \\ & - \rho\ell^3 \sum_{pq} \widehat{w}_{0p,0q} a_p^* a_q + \tfrac{1}{2}\rho^2 \ell^6 \widehat{w}_{00,00}. \end{aligned} \tag{16}$$

We also introduce the operators $\widehat{n}_0 = a_0^* a_0$ and $\widehat{n}_+ = \sum_{p\neq 0}$. These operators represent the number of particles in the condensate state created by a_0^* and the number of particle *not* in the condensate. Note that on the subspace where the total particle number is n, both of these operators are non-negative and $\widehat{n}_+ = n - \widehat{n}_0$.

Using the bounds on the long and short distance cutoffs in Lemma 4.1 we may immediately prove a simple bound on the expectation value of $\widehat{n}_+$.

Lemma 5.1 (Simple bound on the number of excited particles). *There is a constant $C > 0$ such that if $\omega(t)^{-1}\rho^{1/3}\ell > C$ then for any state such that the expectation $\langle H^n_\ell \rangle \leq 0$, the expectation of the number of excited particles satisfies $\langle \widehat{n}_+ \rangle \leq Cn\rho^{-1/6}\left(\rho^{1/4}\ell\right)^2$.*

Proof. We simply choose $r = R = \rho^{-1/3}$ in Lemma 4.1. This is allowed since $R \leq \omega(t)^{-1}\ell$ is ensured from the assumption that $\omega(t)^{-1}\rho^{1/3}\ell$ is large. We then obtain

$$H^n_\ell \geq \sum_{j=1}^{n} -\tfrac{1}{2}\gamma^{-1}\Delta_{\ell,j} - \tfrac{1}{2}nR^{-1} - Cn\rho r^2 \geq \sum_{j=1}^{n} -\tfrac{1}{2}\gamma^{-1}\Delta_{\ell,j} - Cn\rho^{1/3}.$$

The bound on $\langle \widehat{n}_+ \rangle$ follows since the bound on the gap (15) implies that $\langle \sum_{j=1}^n -\Delta_{\ell,j} \rangle \geq \langle \widehat{n}_+ \rangle \pi^2 \ell^{-2}$. □

Motivated by Foldy's use of the Bogolubov approximation it is our goal to reduce the Hamiltonian $H^n_{\ell,r,R}$ so that it has only what we call quadratic terms, i.e., terms which contain precisely two $a_p^\#$ with $p \neq 0$. More precisely, we want to be able to ignore all terms containing the coefficients

- $\widehat{w}_{00,00}$.
- $\widehat{w}_{p0,q0} = \widehat{w}_{0p,0q}$, where $p, q \neq 0$. These terms are in fact quadratic, but do not appear in the Foldy Hamiltonian. We shall prove that they can also be ignored.
- $\widehat{w}_{p0,00} = \widehat{w}_{0p,00} = \widehat{w}_{00,p0} = \widehat{w}_{00,0p}$, where $p \neq 0$.
- $\widehat{w}_{pq,\mu 0} = \widehat{w}_{\mu 0,pq} = \widehat{w}_{qp,0\mu} = \widehat{w}_{0\mu,qp}$, where $p, q, \mu \neq 0$.

- $\widehat{w}_{pq,\mu\nu}$, where $p, q, \mu, \nu \neq 0$. The sum of all these terms form a non-negative contribution to the Hamiltonian and can, when proving a lower bound, either be ignored or used to control error terms.

We shall consider these cases one at a time.

Lemma 5.2 (Control of terms with $\widehat{w}_{00,00}$). *The sum of the terms in $H^n_{\ell,r,R}$ containing $\widehat{w}_{00,00}$ is equal to*

$$\frac{1}{2}\widehat{w}_{00,00}\left[\left(\widehat{n}_0 - \rho\ell^3\right)^2 - \widehat{n}_0\right] = \frac{1}{2}\widehat{w}_{00,00}\left[\left(n - \rho\ell^3\right)^2 + (\widehat{n}_+)^2 - 2\left(n - \rho\ell^3\right)\widehat{n}_+ - \widehat{n}_0\right].$$

Proof. The terms containing $\widehat{w}_{00,00}$ are

$$\frac{1}{2}\widehat{w}_{00,00}\left(a_0^* a_0^* a_0 a_0 - 2\rho\ell^3 a_0^* a_0 + \rho^2\ell^6\right) = \frac{1}{2}\widehat{w}_{00,00}\left(a_0^* a_0 - \rho\ell^3\right)^2 - \frac{1}{2}\widehat{w}_{00,00} a_0^* a_0$$

using the 0commutation relation $[a_p, a_q^*] = \delta_{p,q}$. □

Lemma 5.3 (Control of terms with $\widehat{w}_{p0,q0}$). *The sum of the terms in $H^n_{\ell,r,R}$ containing $\widehat{w}_{p0,q0}$ or $\widehat{w}_{0p,0q}$ with $p, q \neq 0$ is bounded below by*

$$-4\pi[\rho - n\ell^{-3}]_+\widehat{n}_+ R^2 - 4\pi\widehat{n}_+^2\ell^{-3}R^2,$$

where $[t]_+ = \max\{t, 0\}$.

Proof. The terms containing $\widehat{w}_{p0,q0}$ or $\widehat{w}_{0p,0q}$ are

$$\sum_{\substack{p\neq 0\\ q\neq 0}}\left(\frac{1}{2}\widehat{w}_{p0,q0} a_p^* a_0^* a_0 a_q + \frac{1}{2}\widehat{w}_{0p,0q} a_0^* a_p^* a_q a_0 - \rho\ell^3\widehat{w}_{0p,0q} a_p^* a_q\right) = (\widehat{n}_0 - \rho\ell^3)\sum_{\substack{p\neq 0\\ q\neq 0}}\widehat{w}_{p0,q0} a_p^* a_q.$$

Note that $\widehat{n}_0$ commutes with $\sum_{\substack{p\neq 0\\ q\neq 0}}\widehat{w}_{p0,q0} a_p^* a_q$.

We have that

$$\widehat{w}_{p0,q0} = \ell^{-3}\int\int w_{r,R}(x,y)\,dy\,u_p(x)u_q(x)\,dx.$$

Hence

$$\sum_{\substack{p\neq 0\\ q\neq 0}}\widehat{w}_{p0,q0} a_p^* a_q = \ell^{-3}\int\int w_{r,R}(x,y)\,dy\left(\sum_{p\neq 0} u_p(x) a_p^*\right)\left(\sum_{p\neq 0} u_p(x) a_p^*\right)^* dx.$$

$$\leq \ell^{-3}\sup_{x'}\int w_{r,R}(x',y)\,dy\int\left(\sum_{p\neq 0} u_p(x) a_p^*\right)\left(\sum_{p\neq 0} u_p(x) a_p^*\right)^* dx.$$

$$= \ell^{-3}\sup_{x'}\int w_{r,R}(x',y)\,dy\sum_{p\neq 0} a_p^* a_p = \ell^{-3}\sup_{x'}\int w_{r,R}(x',y)\,dy\,\widehat{n}_+.$$

Since

$$\sup_x \int w_{r,R}(x,y)\,dy \le \int V_{r,R}(y)\,dy \le 4\pi R^2$$

we obtain the operator inequality

$$0 \le \sum_{\substack{p\neq 0\\ q\neq 0}} \widehat{w}_{p0,q0} a_p^* a_q \le 4\pi\ell^{-3}R^2\widehat{n}_+,$$

and the lemma follows.

Before treating the last two types of terms we shall need the following result on the structure of the coefficients $\widehat{w}_{pq,\mu\nu}$.

Lemma 5.4. *For all* $p', q' \in (\pi/\ell)\,(\mathbb{N}\cup\{0\})^3$ *and* $\alpha \in \mathbb{N}$ *there exists* $J^\alpha_{p'q'} \in \mathbb{R}$ *with* $J^\alpha_{p'q'} = J^\alpha_{q'p'}$ *such that for all* $p, q, \mu, \nu \in (\pi/\ell)\,(\mathbb{N}\cup\{0\})^3$ *we have*

$$\widehat{w}_{pq,\mu\nu} = \sum_\alpha J^\alpha_{p\mu} J^\alpha_{q\nu}. \tag{17}$$

Moreover we have the operator inequalities

$$0 \le \sum_{p,p'\neq 0} \widehat{w}_{pp',00} a_p^* a_{p'} = \sum_{p,p'\neq 0} \widehat{w}_{p0,0p'} a_p^* a_{p'} \le 4\pi\ell^{-3}R^2\widehat{n}_+ \tag{18}$$

and

$$0 \le \sum_{p,p',m\neq 0} \widehat{w}_{pm,mp'} a_p^* a_{p'} \le r^{-1}\widehat{n}_+.$$

Proof. The operator $\mathcal{A}$ with integral kernel $w_{r,R}(x,y)$ is a non-negative Hilbert–Schmidt operator on $L^2(\mathbb{R}^3)$ with norm less than $\sup_k \hat{V}_{r,R}(k) \le 4\pi R^2$. Denote the eigenvalues of $\mathcal{A}$ by λ_α, $\alpha = 1,2,\ldots$ and corresponding orthonormal eigenfunctions by φ_α. We may assume that these functions are real. The eigenvalues satisfy $0 \le \lambda_\alpha \le 4\pi R^2$. We then have

$$\widehat{w}_{pq,\mu\nu} = \sum_\alpha \lambda_\alpha \int u_p(x)u_\mu(x)\varphi_\alpha(x)\,dx \int u_q(y)u_\nu(y)\varphi_\alpha(y)\,dy.$$

The identity (17) thus follows with $J^\alpha_{p\mu} = \lambda_\alpha^{1/2}\int u_p(x)u_\mu(x)\varphi_\alpha(x)\,dx$.

If P denotes the projection onto the constant functions we may also consider the operator $(I-P)\mathcal{A}(I-P)$. Denote its eigenvalues and eigenfunctions by λ'_α and φ'_α. Then again $0 \le \lambda'_\alpha \le 4\pi R^2$. Hence we may write

$$\widehat{w}_{p0,0p'} = \ell^{-3}\sum_\alpha \lambda'_\alpha \int u_p(x)\varphi'_\alpha(x)\,dx \int u_{p'}(y)\varphi'_\alpha(y)\,dy.$$

Thus, since all φ'_α are orthogonal to constants we have

$$\sum_{p,p'\neq 0}\widehat{w}_{p0,0p'}a_p^*a_{p'}$$
$$=\ell^{-3}\sum_\alpha \lambda'_\alpha\left(\sum_{p\neq 0}\int u_p(x)\varphi'_\alpha(x)\,dx\,a_p^*\right)\left(\sum_{p\neq 0}\int u_p(x)\varphi'_\alpha(x)\,dx\,a_p^*\right)^*$$
$$=\ell^{-3}\sum_\alpha \lambda'_\alpha a^*\left(\varphi'_\alpha\right)a\left(\varphi'_\alpha\right).$$

The inequalities (18) follow immediately from this.

The fact that $\sum_{p,p',m\neq 0}\widehat{w}_{pm,mp'}a_p^*a_{p'}\geq 0$ follows from the representation (17). Moreover, since the kernel $w_{R,r}(x,y)$ is a continuous function we have that $w_{r,R}(x,x)=\sum_\alpha\lambda_\alpha\varphi_\alpha(x)^2$ for almost all x and hence

$$\sum_{m\neq 0}\widehat{w}_{pm,mp'}=\int u_p(x)u_{p'}(x)w_{r,R}(x,x)\,dx-\widehat{w}_{p0,0p'}.$$

We therefore have

$$\sum_{p,p',m\neq 0}\widehat{w}_{pm,mp'}a_p^*a_{p'}\leq\sum_{p,p'\neq 0}\int u_p(x)u_{p'}(x)W_{r,R}(x,x)\,dx\,a_p^*a_{p'}$$
$$=\int w_{r,R}(x,x)\left(\sum_{p\neq 0}u_p(x)a_p^*\right)\left(\sum_{p\neq 0}u_p(x)a_p^*\right)^*dx$$
$$\leq\sup_{x'}w_{r,R}(x',x')\int\left(\sum_{p\neq 0}u_p(x)a_p^*\right)\left(\sum_{p\neq 0}u_p(x)a_p^*\right)^*dx$$
$$=\sup_{x'}w_{r,R}(x',x')\widehat{n}_+$$

and the lemma follows since $\sup_{x'}w_{r,R}(x',x')\leq r^{-1}$. □

Lemma 5.5 (Control of terms with $\widehat{w}_{p0,00}$). *The sum of the terms in $H^n_{\ell,r,R}$ containing $\widehat{w}_{p0,00}$, $\widehat{w}_{0p,00}$,$\widehat{w}_{00,p0}$, or $\widehat{w}_{00,0p}$, with $p\neq 0$ is, for all $\varepsilon>0$, bounded below by*

$$-\varepsilon^{-1}4\pi\ell^{-3}R^2\widehat{n}_0\widehat{n}_+-\varepsilon\widehat{w}_{00,00}(\widehat{n}_0+1-\rho\ell^3)^2, \tag{19}$$

and by

$$\sum_{p\neq 0}\widehat{w}_{p0,00}\left((n-\rho\ell^3)a_p^*a_0+a_0^*a_p(n-\rho\ell^3)\right)$$
$$-\varepsilon^{-1}4\pi\ell^{-3}R^2\widehat{n}_0\widehat{n}_+-\varepsilon\widehat{w}_{00,00}(\widehat{n}_+-1)^2. \tag{20}$$

Proof. The terms containing $\widehat{w}_{p0,00}$, $\widehat{w}_{0p,00}$,$\widehat{w}_{00,p0}$, or $\widehat{w}_{00,0p}$ are

$$\sum_{p\neq 0}\tfrac{1}{2}\widehat{w}_{p0,00}\left(2a_p^*a_0^*a_0a_0+2a_0^*a_0^*a_0a_p-2\rho\ell^3a_0^*a_p-2\rho\ell^3a_p^*a_0\right)$$
$$=\sum_{p\neq 0}\widehat{w}_{p0,00}\left((\widehat{n}_0-\rho\ell^3)a_p^*a_0+a_0^*a_p(\widehat{n}_0-\rho\ell^3)\right)$$
$$=\sum_\alpha\sum_{p\neq 0}J_{p0}^\alpha J_{00}^\alpha\left(a_p^*a_0(\widehat{n}_0+1-\rho\ell^3)+(\widehat{n}_0+1-\rho\ell^3)a_0^*a_p\right).$$

In the last term we have used the representation (17) and the commutation relation $[\widehat{n}_0,a_0]=a_0$. For all $\varepsilon>0$ we get that the above expression is bounded below by

$$\varepsilon^{-1}\sum_\alpha\sum_{p,p'\neq 0}J_{p0}^\alpha J_{p'0}^\alpha\widehat{n}_0a_p^*a_{p'}-\varepsilon\sum_\alpha\left(J_{00}^\alpha\right)^2(\widehat{n}_0+1-\rho\ell^3)^2$$
$$=-\varepsilon^{-1}\sum_{p,p'\neq 0}\widehat{w}_{p0,0p'}\widehat{n}_0a_p^*a_{p'}-\varepsilon\widehat{w}_{00,00}(\widehat{n}_0+1-\rho\ell^3)^2.$$

The bound (19) follows from (18).

The second bound (20) follows in the same way if we notice that the terms containing $\widehat{w}_{p0,00}$, $\widehat{w}_{0p,00}$,$\widehat{w}_{00,p0}$, or $\widehat{w}_{00,0p}$ may be written as

$$\sum_{p\neq 0}\widehat{w}_{p0,00}\left((n-\rho\ell^3)a_p^*a_0+a_0^*a_p(n-\rho\ell^3)\right)$$
$$+\sum_\alpha\sum_{p\neq 0}J_{p0}^\alpha J_{00}^\alpha\left(a_p^*a_0(1-\widehat{n}_+)+(1-\widehat{n}_+)a_0^*a_p\right).\qquad\square$$

Lemma 5.6 (Control of terms with $\widehat{w}_{pq,m0}$). *The sum of the terms in $H_{\ell,r,R}^n$ containing $\widehat{w}_{pq,m0}$, $\widehat{w}_{pq,0m}$,$\widehat{w}_{p0,qm}$, or $\widehat{w}_{0p,qm}$, with $p,q,m\neq 0$ is bounded below by*

$$-\varepsilon^{-1}4\pi\ell^{-3}R^2\widehat{n}_0\widehat{n}_+-\varepsilon\widehat{n}_+r^{-1}-\varepsilon\sum_{p,m,p',m'\neq 0}\widehat{w}_{mp',pm'}a_m^*a_{p'}^*a_{m'}a_p,$$

for all $\varepsilon>0$.

Proof. The terms containing $\widehat{w}_{pq,m0}$, $\widehat{w}_{pq,0m}$,$\widehat{w}_{p0,qm}$, or $\widehat{w}_{0p,qm}$ are

$$\begin{aligned}
\sum_{pqm\neq 0} \widehat{w}_{pqm0}\left(a_p^* a_q^* a_m a_0 + a_0^* a_m^* a_q a_p\right)
&= \sum_\alpha \left(\left(\sum_{q\neq 0} J_{q0}^\alpha a_q^* a_0\right)\left(\sum_{pm\neq 0} J_{pm}^\alpha a_p^* a_m\right)\right.\\
&\quad \left. + \left(\sum_{pm\neq 0} J_{pm}^\alpha a_p^* a_m\right)^* \left(\sum_{q\neq 0} J_{q0}^\alpha a_q^* a_0\right)^*\right)\\
&\geq -\sum_\alpha \left(\varepsilon^{-1}\left(\sum_{q\neq 0} J_{q0}^\alpha a_q^* a_0\right)\left(\sum_{q\neq 0} J_{q0}^\alpha a_0^* a_q\right)\right.\\
&\quad \left. +\varepsilon\left(\sum_{pm\neq 0} J_{pm}^\alpha a_m^* a_p\right)\left(\sum_{pm\neq 0} J_{pm}^\alpha a_p^* a_m\right)\right).
\end{aligned}$$

Using that $J_{pm}^\alpha = J_{mp}^\alpha$ we may write this as

$$\begin{aligned}
&-\varepsilon^{-1}\sum_{qq'\neq 0} \widehat{w}_{q0,0q'} a_q^* a_{q'} a_0 a_0^* - \varepsilon \sum_{p,m,p',m'\neq 0} \widehat{w}_{mp',pm'} a_m^* a_p a_{p'}^* a_{m'}\\
&= -\varepsilon^{-1}\sum_{qq'\neq 0} \widehat{w}_{q0,0q'} a_q^* a_{q'} a_0 a_0^* - \varepsilon \sum_{p,m,p',m'\neq 0} \widehat{w}_{mp',pm'} a_m^* a_{p'}^* a_{m'} a_p\\
&\quad - \varepsilon \sum_{p,m,m'\neq 0} \widehat{w}_{mp,pm'} a_m^* a_{m'}.
\end{aligned}$$

The lemma now follows from Lemma 5.4. □

6. Analyzing the Quadratic Hamiltonian

In this section we consider the main part of the Hamiltonian. This is the "quadratic" Hamiltonian considered by Foldy. It consists of the kinetic energy and all the terms with the coefficients $\widehat{w}_{pq,00}$, $\widehat{w}_{00,pq}$ $\widehat{w}_{p0,0q}$, and $\widehat{w}_{0p,q0}$with $p, q \neq 0$, i.e.,

$$\begin{aligned}
H_{\text{Foldy}} &= \tfrac{1}{2}\gamma^{-1}\sum_p |p|^2 a_p^* a_p\\
&\quad + \tfrac{1}{2}\sum_{pq\neq 0} \widehat{w}_{pq,00}\left(a_p^* a_0^* a_0 a_q + a_0^* a_p^* a_q a_0 + a_p^* a_q^* a_0 a_0 + a_0^* a_0^* a_p a_q\right) \qquad (21)\\
&= \tfrac{1}{2}\gamma^{-1}\sum_p |p|^2 a_p^* a_p + \sum_{pq\neq 0} \widehat{w}_{pq,00}\left(a_p^* a_q a_0^* a_0 + \tfrac{1}{2} a_p^* a_q^* a_0 a_0 + \tfrac{1}{2} a_0^* a_0^* a_p a_q\right).
\end{aligned}$$

In order to compute all the bounds we found it necessary to include the first term in (20) into the "quadratic" Hamiltonian. We therefore define

$$\begin{aligned} H_Q = \tfrac{1}{2}\gamma^{-1}\sum_p |p|^2 a_p^* a_p + \sum_{p\neq 0} \widehat{w}_{p0,00}\left((n-\rho\ell^3)a_p^* a_0 + a_0^* a_p(n-\rho\ell^3)\right) \\ + \sum_{pq\neq 0} \widehat{w}_{pq,00}\left(a_p^* a_q a_0^* a_0 + \tfrac{1}{2}a_p^* a_q^* a_0 a_0 + \tfrac{1}{2}a_0^* a_0^* a_p a_q\right). \end{aligned} \tag{22}$$

Note that $H_{\text{Foldy}} = H_Q$ in the neutral case $n = \rho\ell^3$. Our goal is to give a lower bound on the ground state energy of the Hamiltonian H_Q.

For the sake of convenience we first enlarge the one-particle Hilbert space $L^2\left([-\ell/2, \ell/2]^3\right)$. In fact, instead of considering the symmetric Fock space over $L^2\left([-\ell/2, \ell/2]^3\right)$ we now consider the symmetric Fock space over the one-particle Hilbert space $L^2\left([-\ell/2, \ell/2]^3\right) \oplus \mathbb{C}$. Note that the larger Fock space of course contains the original Fock space as a subspace. On the larger space we have a new pair of creation and annihilation operators that we denote $\widetilde{a}_0^*$ and $\widetilde{a}_0$. These operators merely create vectors in the $\mathbb{C}$ component of $L^2\left([-\ell/2, \ell/2]^3\right) \oplus \mathbb{C}$, and so commute with all other operators.

We shall now write

$$\widetilde{a}_p = \begin{cases} a_p, & \text{if } p \neq 0 \\ \widetilde{a}_0, & \text{if } p = 0 \end{cases} \quad \text{and} \quad \widetilde{a}_p^* = \begin{cases} a_p^*, & \text{if } p \neq 0 \\ \widetilde{a}_0^*, & \text{if } p = 0 \end{cases}. \tag{23}$$

We now define the Hamiltonian

$$\begin{aligned} \widetilde{H}_Q = \tfrac{1}{2}\gamma^{-1}\sum_p |p|^2 \widetilde{a}_p^* \widetilde{a}_p + \sum_p \widehat{w}_{p0,00}\left((n-\rho\ell^3)\widetilde{a}_p^* a_0 + a_0^* \widetilde{a}_p(n-\rho\ell^3)\right) \\ + \sum_{pq} \widehat{w}_{pq,00}\left(\widetilde{a}_p^* \widetilde{a}_q a_0^* a_0 + \tfrac{1}{2}\widetilde{a}_p^* \widetilde{a}_q^* a_0 a_0 + \tfrac{1}{2}a_0^* a_0^* \widetilde{a}_p \widetilde{a}_q\right), \end{aligned} \tag{24}$$

where we no longer restrict p, q to be different from 0. Note that for all states on the larger Fock space for which $\langle \widetilde{a}_0^* \widetilde{a}_0 \rangle = 0$ we have $\langle \widetilde{H}_Q \rangle = \langle H_Q \rangle$.

For any function $\varphi \in L^2\left([-\ell/2, \ell/2]^3\right)$ we introduce the creation operator

$$\widetilde{a}^*(\varphi) = \sum_p (u_p, \varphi)\widetilde{a}_p^*.$$

Note that the sum includes $p = 0$. the difference from $a^*(\varphi)$ is given by $\widetilde{a}^*(\varphi) - a^*(\varphi) = (u_0, \varphi)\left(\widetilde{a}_0^* - a_0^*\right)$.

Then $[\widetilde{a}(\varphi), \widetilde{a}^*(\psi)] = (\varphi, \psi)$. We have introduced the "dummy" operator $\widetilde{a}_0^*$ in order for this relation to hold. One could just as well have stayed in the old space, but then the relation above would hold only for functions orthogonal to constants.

For any $k \in \mathbb{R}^3$ denote $\chi_{\ell,k}(x) = e^{ikx}\chi_\ell(x)$ and define the operators

$$b_k^* = \widetilde{a}^*(\chi_{\ell,k})a_0 \quad \text{and} \quad b_k = \widetilde{a}(\chi_{\ell,k})a_0^*$$

They satisfy the commutation relations

$$\begin{aligned} [b_k, b_{k'}^*] &= a_0^* a_0 \left(\chi_{\ell,k}, \chi_{\ell,k'}\right) - \widetilde{a}(\chi_{\ell,k})\widetilde{a}^*(\chi_{\ell,k'}) \\ &= a_0^* a_0 \widehat{\chi_\ell^2}(k'-k) - \widetilde{a}(\chi_{\ell,k})\widetilde{a}^*(\chi_{\ell,k'}) \end{aligned} \tag{25}$$

We first consider the kinetic energy part of the Hamiltonian. We shall bound it using the double commutator bound in Appendix B. First we need a well known comparisson between the Neumann Laplacian and the Laplacian in the whole space.

Lemma 6.1 (Neumann resolvent is bigger than free resolvent). *Let P_ℓ denote the projection in $L^2(\mathbb{R}^3)$ that projects onto $L^2([-\ell/2,\ell/2]^3)$ (identified as a subspace). Then if $-\Delta$ denotes the Laplacian on all of $\mathbb{R}^3$ and $-\Delta_\ell$ is the Neumann Laplacian on $[-\ell/2,\ell/2]^3$ we have the operator inequality*

$$(-\Delta_\ell + a)^{-1} \geq P_\ell(-\Delta + a)^{-1}P_\ell,$$

for all $a > 0$.

Proof. It is clear that for all $f \in L^2(\mathbb{R}^3)$

$$\|P_\ell(-\Delta_\ell + a)^{1/2}P_\ell(-\Delta + a)^{-1/2}f\|^2 \leq \|f\|^2,$$

and hence

$$\|(-\Delta + a)^{-1/2}P_\ell(-\Delta_\ell + a)^{1/2}P_\ell f\|^2 \leq \|f\|^2.$$

Now simply use this with $f = (-\Delta_\ell + a)^{-1/2}u$. □

Lemma 6.2 (The kinetic energy bound). *There exists a constant $C' > 0$ such that if $C't < 1$, where t is the parameter used in the definition of χ_ℓ in Sect. 3, we have*

$$\left\langle \sum_p |p|^2 \widetilde{a}_p^* \widetilde{a}_p \right\rangle \geq (2\pi)^{-3}(1 - C't)^2 n^{-1} \int_{\mathbb{R}^3} \frac{|k|^4}{|k|^2 + (\ell t^3)^{-2}} \langle b_k^* b_k \rangle \, dk$$

for all states with $\langle \widetilde{a}_0^* \widetilde{a}_0 \rangle = 0$ and particle number equal to n, i.e., $\left\langle \sum_p a_p^* a_p \right\rangle^2 = \left\langle \left(\sum_p a_p^* a_p \right)^2 \right\rangle = n^2$.

Proof. Let s, with $0 < s \leq t$, be a parameter to be chosen below. Recall that t is the parameter used in the definition of χ_ℓ in Section 3. Then since $\chi_\ell^2 + \eta_\ell^2 = 1$ we have

$$\begin{aligned}
-\Delta_\ell &\geq \frac{(-\Delta_\ell)^2}{-\Delta_\ell + ((\ell s)^{-2}} = \tfrac{1}{2}(\chi_\ell^2 + \eta_\ell^2)\frac{(-\Delta_\ell)^2}{-\Delta_\ell + (\ell s)^{-2}} + \tfrac{1}{2}\frac{(-\Delta_\ell)^2}{-\Delta_\ell + (\ell s)^{-2}}(\chi_\ell^2 + \eta_\ell^2) \\
&= \chi_\ell \frac{(-\Delta_\ell)^2}{-\Delta_\ell + (\ell s)^{-2}} \chi_\ell + \eta_\ell \frac{(-\Delta_\ell)^2}{-\Delta_\ell + (\ell s)^{-2}} \eta_\ell \\
&\quad + \left[\left[\frac{(-\Delta_\ell)^2}{-\Delta_\ell + (\ell s)^{-2}}, \chi_\ell\right], \chi_\ell\right] + \left[\left[\frac{(-\Delta_\ell)^2}{-\Delta_\ell + (\ell s)^{-2}}, \eta_\ell\right], \eta_\ell\right] \\
&\geq \chi_\ell \frac{(-\Delta_\ell)^2}{-\Delta_\ell + (\ell s)^{-2}} \chi_\ell + \eta_\ell \frac{(-\Delta_\ell)^2}{-\Delta_\ell + (\ell s)^{-2}} \eta_\ell \\
&\quad - C(\ell t)^{-2} \frac{-\Delta_\ell}{-\Delta_\ell + (\ell s)^{-2}} - C\ell^{-2}s^2 t^{-4},
\end{aligned}$$

where the last inequality follows from Lemma B.1 in Appendix B. We can now repeat this calculation to get

$$
\begin{aligned}
-\Delta_\ell \geq{} & \chi_\ell\left(\frac{(-\Delta_\ell)^2}{-\Delta_\ell+(\ell s)^{-2}}-C(\ell t)^{-2}\frac{-\Delta_\ell}{-\Delta_\ell+(\ell s)^{-2}}\right)\chi_\ell\\
&+\eta_\ell\left(\frac{(-\Delta_\ell)^2}{-\Delta_\ell+(\ell s)^{-2}}-C(\ell t)^{-2}\frac{-\Delta_\ell}{-\Delta_\ell+(\ell s)^{-2}}\right)\eta_\ell-C\ell^{-2}s^2t^{-4}\\
&-C(\ell t)^{-2}\left(\left[\left[\frac{-\Delta_\ell}{-\Delta_\ell+(\ell s)^{-2}},\chi_\ell\right],\chi_\ell\right]+\left[\left[\frac{-\Delta_\ell}{-\Delta_\ell+(\ell s)^{-2}},\eta_\ell\right],\eta_\ell\right]\right).
\end{aligned}
$$

If we therefore use (53) in Lemma B.1 and recall that $s \leq t$ we arrive at

$$
\begin{aligned}
-\Delta_\ell \geq{} & \chi_\ell\left(\frac{(-\Delta_\ell)^2}{-\Delta_\ell+(\ell s)^{-2}}-C(\ell t)^{-2}\frac{-\Delta_\ell}{-\Delta_\ell+(\ell s)^{-2}}\right)\chi_\ell\\
&+\eta_\ell\left(\frac{(-\Delta_\ell)^2}{-\Delta_\ell+(\ell s)^{-2}}-C(\ell t)^{-2}\frac{-\Delta_\ell}{-\Delta_\ell+(\ell s)^{-2}}\right)\eta_\ell-C\ell^{-2}s^2t^{-4}.
\end{aligned}
$$

Note that for $\alpha > 0$ we have

$$
\alpha\frac{(-\Delta_\ell)^2}{-\Delta_\ell+(\ell s)^{-2}}-C(\ell t)^{-2}\frac{-\Delta_\ell}{-\Delta_\ell+(\ell s)^{-2}}\geq -C\alpha^{-1}s^2t^{-4}\ell^{-2}.
$$

Thus if we also assume that $\alpha < 1$ we have

$$
-\Delta_\ell \geq (1-\alpha)\chi_\ell\frac{(-\Delta_\ell)^2}{-\Delta_\ell+(\ell s)^{-2}}\chi_\ell - C\alpha^{-1}s^2t^{-4}\ell^{-2}.
$$

Thus if u is a normalized function on $L^2(\mathbb{R}^3)$ which is orthogonal to constants we have according to the bound on the gap (15) that for all $0 < \delta < 1$

$$
\begin{aligned}
(u,-\Delta_\ell u) \geq{} & (1-\delta)(1-\alpha)\left(u,\chi_\ell\frac{(-\Delta_\ell)^2}{-\Delta_\ell+(\ell s)^{-2}}\chi_\ell u\right)\\
&-C(1-\delta)\alpha^{-1}s^2t^{-4}\ell^{-2}+\delta\pi^2\ell^{-2}.
\end{aligned}
$$

We choose $\alpha = \delta = C' s t^{-2}$ for an appropriately large constant $C' > 0$ and assume that s and t are such that δ is less than 1. Then

$$
(u,-\Delta_\ell u) \geq (1-C'st^{-2})^2\left(u,\chi_\ell\frac{(-\Delta_\ell)^2}{-\Delta_\ell+(\ell s)^{-2}}\chi_\ell u\right).
$$

If we now use Lemma 6.1 we may write this as

$$
\begin{aligned}
(u,-\Delta_\ell u) &\geq (1-C'st^{-2})^2\left(u,\chi_\ell\Delta_\ell\frac{1}{-\Delta+(\ell s)^{-2}}\Delta_\ell\chi_\ell u\right)\\
&=(1-C'st^{-2})^2\left(u,\chi_\ell\frac{(-\Delta)^2}{-\Delta+(\ell s)^{-2}}\chi_\ell u\right),
\end{aligned}
$$

where in the last inequality we have used that $\Delta\chi = \Delta_\ell\chi$ and $\chi\Delta = \chi\Delta_\ell$.

We now choose $s = t^3$ and we may then write this inequality in second quantized form as

$$\left\langle \sum_p |p|^2 \widetilde{a}_p^* \widetilde{a}_p \right\rangle \geq (2\pi)^{-3}(1 - C't)^2 \int_{\mathbb{R}^3} \frac{|k|^4}{|k|^2 + (\ell t^3)^{-2}} \left\langle \widetilde{a}^*(\chi_{\ell,k}) \widetilde{a}(\chi_{\ell,k}) \right\rangle dk$$

using that $\langle \widetilde{a}_0^* \widetilde{a}_0 \rangle = 0$. Since we consider only states with particle number n the inequality still holds if we insert $n^{-1} a_0 a_0^*$ as in the statement of the lemma. □

With the same notation as in the above lemma we may write

$$w_{r,R}(x, y) = (2\pi)^{-3} \int \hat{V}_{r,R}(k) \chi_{\ell,k}(x) \overline{\chi_{\ell,k}(y)}\, dk.$$

The last two sums in the Hamiltonian (24) can therefore be written as

$$(2\pi\ell)^{-3} \int \hat{V}_{r,R}(k) \Big[(n - \rho\ell^3)\ell^{-3/2} \left(\widehat{\chi}_\ell(k) b_k^* + \overline{\widehat{\chi}_\ell(k)} b_k \right) + \tfrac{1}{2} \left(b_k^* b_k + b_{-k}^* b_{-k} + b_k^* b_{-k}^* + b_k b_{-k} \right) \Big] dk - \sum_{pq} \widehat{w}_{pq,00} \widetilde{a}_p^* \widetilde{a}_q .$$

Note that it is important here that the potential $w_{r,R}$ contains the localization function χ_ℓ.

Thus, since $\hat{V}_{r,R}(k) = \hat{V}_{r,R}(-k)$ and $\overline{\widehat{\chi}_\ell(k)} = \widehat{\chi}_\ell(-k)$ we have for states with $\langle \widetilde{a}_0^* \widetilde{a}_0 \rangle = 0$ that

$$\langle \widetilde{H}_Q \rangle \geq \int_{\mathbb{R}^3} \langle h_Q(k) \rangle\, dk - \sum_{pq} \widehat{w}_{pq,00} \left\langle \widetilde{a}_p^* \widetilde{a}_q \right\rangle, \tag{26}$$

where

$$\begin{aligned} h_Q(k) = {} & \frac{(1 - C't)^2}{4(2\pi)^3 \gamma n} \frac{|k|^4}{|k|^2 + (\ell t^3)^{-2}} \left(b_k^* b_k + b_{-k}^* b_{-k} \right) \\ & + \frac{\hat{V}_{r,R}(k)}{2(2\pi\ell)^3} \Big[(n - \rho\ell^3)\ell^{-3/2} \left(\widehat{\chi}_\ell(k)(b_k^* + b_{-k}) + \overline{\widehat{\chi}_\ell(k)}(b_k + b_{-k}^*) \right) \\ & + \left(b_k^* b_k + b_{-k}^* b_{-k} + b_k^* b_{-k}^* + b_k b_{-k} \right) \Big]. \end{aligned} \tag{27}$$

Theorem 6.3 (Simple case of Bogolubov's method). *For arbitrary constants $\mathcal{A} \geq \mathcal{B} > 0$ and $\kappa \in \mathbb{C}$ we have the inequality*

$$\begin{aligned} & \mathcal{A}(b_k^* b_k + b_{-k}^* b_{-k}) + \mathcal{B}(b_k^* b_{-k}^* + b_k b_{-k}) + \kappa(b_k^* + b_{-k}) + \overline{\kappa}(b_k + b_{-k}^*) \\ & \quad \geq -\tfrac{1}{2}(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2})([b_k, b_k^*] + [b_{-k}, b_{-k}^*]) - \frac{2|\kappa|^2}{\mathcal{A} + \mathcal{B}}. \end{aligned}$$

Proof. We may complete the square

$$\begin{aligned}\mathcal{A}(b_k^*b_k + b_{-k}^*b_{-k}) + \mathcal{B}(b_k^*b_{-k}^* + b_kb_{-k}) + \kappa(b_k^* + b_{-k}) + \overline{\kappa}(b_k + b_{-k}^*)\\ = D(b_k^* + \alpha b_{-k} + a)(b_k + \alpha b_{-k}^* + \overline{a}) + D(b_{-k}^* + \alpha b_k + \overline{a})(b_{-k} + \alpha b_{-k}^* + a)\\ - D\alpha^2([b_k, b_k^*] + [b_{-k}, b_{-k}^*]) - 2D|a|^2,\end{aligned}$$

if

$$D(1+\alpha^2) = \mathcal{A}, \quad 2D\alpha = \mathcal{B}, \quad aD(1+\alpha) = \kappa.$$

We choose the solution $\alpha = \mathcal{A}/\mathcal{B} - \sqrt{\mathcal{A}^2/\mathcal{B}^2 - 1}$. Hence

$$D\alpha^2 = \mathcal{B}\alpha/2 = \tfrac{1}{2}(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2}), \quad D|a|^2 = \frac{|\kappa|^2}{D(1+\alpha^2+2\alpha)} = \frac{|\kappa|^2}{\mathcal{A}+\mathcal{B}}. \qquad \square$$

Usually when applying Bogolubov's method the commutator $[b_k, b_k^*]$ is a positive constant. In this case the lower bound in the theorem is actually the bottom of the spectrum of the operator. If moreover, $\mathcal{A} > \mathcal{B}$ the bottom is actually an eigenvalue. In our case the commutator $[b_k, b_k^*]$ is not a constant, but according to (25) we have

$$[b_k, b_k^*] \le \int \chi_\ell(x)^2\, dx a_0^* a_0 \le \ell^3 a_0^* a_0. \tag{28}$$

From this and the above theorem we easily conclude the following bound.

Lemma 6.4 (Lower bound on quadratic Hamiltonian). *On the subspace with n particles we have*

$$H_Q \ge -In^{5/4}\ell^{-3/4} - \tfrac{1}{2}\left(n - \rho\ell^3\right)^2 \widehat{w}_{00,00} - 4\pi n^{5/4}\ell^{-3/4}(n\ell)^{-1/4},$$

where $I = \frac{1}{2}(2\pi)^{-3}\int_{\mathbb{R}^3} f(k) - (f(k)^2 - g(k)^2)^{1/2}\, dk$ *with*

$$g(k) = 4\pi\frac{1}{k^2 + (n^{1/4}\ell^{-3/4}R)^{-2}} - 4\pi\frac{1}{k^2 + (n^{1/4}\ell^{-3/4}r)^{-2}}$$

and

$$f(k) = g(k) + \tfrac{1}{2}\gamma^{-1}(1 - C't)^2\frac{|k|^4}{|k|^2 + (n^{1/4}\ell^{1/4}t^3)^{-2}}.$$

Proof. We consider a state with $\langle\widetilde{a}_0^*\widetilde{a}_0\rangle = 0$. Then $\langle H_Q\rangle = \langle\widetilde{H}_Q\rangle$. We shall use (26). Note first that

$$\left\langle\sum_{pq}\widehat{w}_{pq,00}\widetilde{a}_p^*\widetilde{a}_q\right\rangle = \left\langle\sum_{p,q\neq 0}\widehat{w}_{p0,0q}a_p^*a_q\right\rangle \le 4\pi\ell^{-3}R^2\widehat{n}_+ \le 4\pi\ell^{-1}n$$

by (18) and the fact that $R \le \ell$. We may of course rewrite $\ell^{-1}n = n^{5/4}\ell^{-3/4}(n\ell)^{-1/4}$.

By Theorem 6.3, (27) and (28) we have

$$h_Q(k) \ge -(\mathcal{A}_k - \sqrt{\mathcal{A}_k^2 - \mathcal{B}_k^2})n\ell^3 - \frac{\hat{V}_{r,R}(k)^2(n-\rho\ell^3)^2}{2(2\pi)^6\ell^9(\mathcal{A}_k + \mathcal{B}_k)}\left|\widehat{\chi}_\ell(k)\right|^2,$$

where

$$\mathcal{B}_k = \frac{\hat{V}_{r,R}(k)}{2(2\pi\ell)^3}, \quad \mathcal{A}_k = \mathcal{B}_k + \frac{(1-C't)^2}{4(2\pi)^3\gamma n}\frac{|k|^4}{|k|^2+(\ell t^3)^{-2}}.$$

Since $\mathcal{A}_k > \mathcal{B}_k$ we have that

$$h_Q(k) \geq -(\mathcal{A}_k - \sqrt{\mathcal{A}_k^2 - \mathcal{B}_k^2})n\ell^3 - \frac{\hat{V}_{r,R}(k)(n-\rho\ell^3)^2}{2(2\pi)^3\ell^6}\left|\widehat{\chi}_\ell(k)\right|^2.$$

Note that

$$\int \frac{\hat{V}_{r,R}(k)(n-\rho\ell^3)^2}{2(2\pi)^3\ell^6}\left|\widehat{\chi}_\ell(k)\right|^2 dk$$
$$= \frac{1}{2}\left(\frac{n}{\ell^3}-\rho\right)^2 \iint \chi_\ell(x)V_{r,R}(x-y)\chi_\ell(y)\,dx\,dy = \frac{1}{2}\left(n-\rho\ell^3\right)^2 \widehat{w}_{00,00}.$$

The lemma now follows from (26) by a simple change of variables in the k integral. □

As a consequence we get the following bound for the Foldy Hamiltonian.

Corollary 6.5 (Lower bound on the Foldy Hamiltonian). *The Foldy Hamiltonian in (21) satisfies*

$$H_{\text{Foldy}} \geq -In^{5/4}\ell^{-3/4} - 4\pi n^{5/4}\ell^{-3/4}(n\ell)^{-1/4}. \tag{29}$$

There is constant $C > 0$ such that if $\rho^{1/4}R > C$, $\rho^{1/4}\ell t^3 > C$, and $t < C^{-1}$ then the Foldy Hamiltonian satisfies the bound

$$H_{\text{Foldy}} \geq \frac{1}{4}\sum_p |p|^2 a_p^* a_p - Cn^{5/4}\ell^{-3/4}. \tag{30}$$

Proof. Lemma 6.4 holds for all ρ hence also if we had replaced ρ by n/ℓ^3 in this case we get (29).

The integral I satisfies the bound

$$I \leq \frac{1}{2}(2\pi)^{-3}\int_{\mathbb{R}^3} \max\left\{g(k), \frac{1}{2}g(k)^2(f(k)-g(k))^{-1}\right\} dk.$$

By Corollary 4.3 we may assume that $n \geq c\rho\ell^3$. Hence I is bounded by a constant as long as $\rho^{1/4}R$ and $\rho^{1/4}\ell t^3$ are sufficiently large and t is sufficiently small (which also ensures that γ is close to 1). Note that we do not have to make any assumptions on r. Moreover, if this is true we also have that $n\ell \geq c\rho\ell^4$ is large and hence $(n\ell)^{-1}$ is small. This would give the bound in the corollary except for the first positive term. The above argument, however, also holds (with different constants) if we replace the kinetic energy in the Foldy Hamiltonian by $\frac{1}{2}\left(\gamma^{-1}-\frac{1}{2}\right)\sum_p |p|^2 a_p^* a_p$ (assuming that $\gamma < 2$). This proves the corollary. □

Note that if

$$n^{1/4}\ell^{-3/4}R \to \infty,\ n^{1/4}\ell^{-3/4}r \to 0,\ n^{1/4}\ell^{1/4}t^3 \to \infty,\ \text{and } t \to 0 \tag{31}$$

it follows by dominated convergence that I converges to

$$\begin{aligned}\tfrac{1}{2}(2\pi)^{-3}\int_{\mathbb{R}^3} 4\pi|k|^{-2} + \tfrac{1}{2}|k|^2 - \left((4\pi|k|^{-2} + \tfrac{1}{2}|k|^2)^2 - (4\pi|k|^{-2})^2\right)^{1/2} dk \\ = (2/\pi)^{3/4}\int_0^\infty 1 + x^4 - x^2\left(x^4+2\right)^{1/2} dx = -\left(\frac{4\pi}{3}\right)^{1/4} A,\end{aligned}$$

where A was given in (6). Thus if we can show that $n \sim \rho\ell^3$ we see that the term $-In^{5/4}\ell^{-3/4} \sim -I\rho^{1/4}n$ agrees with Foldy's calculation (5) for the little box of size ℓ.

Our task is now to show that indeed $n \sim \rho\ell^3$, i.e., that we have approximate neutrality in each little box and that the term above containing the integral I is indeed the leading term.

7. Simple Bounds on n and $\widehat{n}_+$

The Lemmas 4.1, 5.2, 5.3, 5.5, and 5.6 together with Lemma 6.4 or Corollary 6.5 control all terms in the Hamiltonian H_ℓ^n except the positive term

$$\tfrac{1}{2}\sum_{p,m,p',m'\neq 0} \widehat{w}_{mp',pm'} a_m^* a_{p'}^* a_{m'} a_p .$$

If we use (30) in Corollary 6.5 together with the other bounds we obtain the following bound if $\rho^{1/4}R$ and $\rho^{1/4}\ell t^3$ are sufficiently large and t is sufficiently small

$$\begin{aligned}H_\ell^n \geq\ & \tfrac{1}{4}\sum_p |p|^2 a_p^* a_p - Cn^{5/4}\ell^{-3/4} - \tfrac{1}{2}nR^{-1} - Cn\rho r^2 \\ & + \tfrac{1}{2}\widehat{w}_{00,00}\left[\left(\widehat{n}_0 - \rho\ell^3\right)^2 - \widehat{n}_0\right] \\ & - 4\pi[\rho - n\ell^{-3}]_+\widehat{n}_+R^2 - 4\pi\widehat{n}_+^2\ell^{-3}R^2 \\ & - \varepsilon^{-1}8\pi\ell^{-3}R^2\widehat{n}_0\widehat{n}_+ - \varepsilon\widehat{w}_{00,00}(\widehat{n}_0 + 1 - \rho\ell^3)^2 \\ & - \varepsilon\widehat{n}_+r^{-1} + (\tfrac{1}{2} - \varepsilon)\sum_{p,m,p',m'\neq 0}\widehat{w}_{mp',pm'} a_m^* a_{p'}^* a_{m'} a_p .\end{aligned}$$

The assumptions on $\rho^{1/4}R$, $\rho^{1/4}\ell t^3$, and t are needed in order to bound the integral I above by a constant. If we choose $\varepsilon = 1/4$, use $\widehat{w}_{00,00} \leq 4\pi R^2\ell^{-3}$ and ignore the last positive term in the bound above we arrive at

$$\begin{aligned} H_\ell^n \geq \tfrac{1}{4}\sum_p |p|^2 a_p^* a_p - Cn^{5/4}\ell^{-3/4} - \tfrac{1}{2}nR^{-1} - Cn\rho r^2 + \tfrac{1}{4}\widehat{w}_{00,00}\left(\widehat{n}_0 - \rho\ell^3\right)^2 \\ - 4\pi[\rho - n\ell^{-3}]_+\widehat{n}_+ R^2 - 4\pi\widehat{n}_+^2\ell^{-3}R^2 \\ - 32\pi\ell^{-3}R^2\widehat{n}_0\widehat{n}_+ - 4\pi R^2\ell^{-3}\left(\widehat{n}_0 - \tfrac{1}{2}\rho\ell^3 + \tfrac{1}{4}\right) - \tfrac{1}{4}\widehat{n}_+ r^{-1} \\ \geq \tfrac{1}{4}\sum_p |p|^2 a_p^* a_p - Cn^{5/4}\ell^{-3/4} - \tfrac{1}{2}nR^{-1} - Cn\rho r^2 + \tfrac{1}{4}\widehat{w}_{00,00}\left(\widehat{n}_0 - \rho\ell^3\right)^2 \\ - 48\pi\ell^{-3}R^2 n\widehat{n}_+ - 4\pi R^2\ell^{-3}\left(\widehat{n}_0 + \tfrac{1}{4}\right) - \tfrac{1}{4}\widehat{n}_+ r^{-1}, \end{aligned} \tag{32}$$

where in the last inequality we have used that $\rho\ell^3 \leq 2n$, $\widehat{n}_0 \leq n$ and $\widehat{n}_+ \leq n$.

Lemma 7.1 (Simple bound on n). *Let $\omega(t)$ be the function described in Lemma 3.1. There is a constant $C > 0$ such that if $(\rho^{1/4}\ell)t^3 > C$ and $(\rho^{1/4}\ell)\rho^{-1/12}$, t, and $\omega(t)(\rho^{1/4}\ell)^{-1}$ are smaller than C^{-1} then for any state with $\langle H_\ell^n\rangle \leq 0$ we have $C^{-1}\rho\ell^3 \leq n \leq C\rho\ell^3$.*

Proof. The lower bound follows from Corollary 4.3. To prove the upper bound on n we choose $R = \omega(t)^{-1}\ell$ (the maximally allowed value) and $r = b\omega(t)^{-1}\ell$, where we shall choose b sufficiently small, in particular $b < 1/2$. We then have that $\rho^{1/4}R = \omega(t)^{-1}\rho^{1/4}\ell$ is large. Moreover $\widehat{w}_{00,00} \geq CR^2\ell^{-3} = C\omega(t)^{-2}\ell^{-1}$ for some constant $C > 0$ and we get from (32) and Lemma 5.1 that

$$\begin{aligned} \langle H_\ell^n\rangle \geq \ell^{-1}[-Cn^{5/4}\ell^{1/4} - \tfrac{1}{2}n\omega(t) - Cb^2\omega(t)^{-2}n^2 + C\omega(t)^{-2}\left(\langle\widehat{n}_0\rangle - \rho\ell^3\right)^2 \\ - 48\pi\omega(t)^{-2}\rho^{-1/6}(\ell\rho^{1/4})^2 n^2 - 4\pi\omega(t)^{-2}\left(n + \tfrac{1}{4}\right) - \tfrac{1}{4}nb^{-1}\omega(t)], \end{aligned}$$

where we have again used that $c\rho\ell^3 \leq n$, $\widehat{n}_0 \leq n$ and $\widehat{n}_+ \leq n$. Note that

$$n^{5/4}\ell^{1/4} \leq C\omega(t)^{-2}n^2(\rho^{1/4}\ell)^{-2}\rho^{-1/4}\omega(t)^2$$

and $n\omega(t) \leq C\omega(t)^{-2}n^2\rho^{-1}\omega(t)^3$. From Lemma 5.1 we know that $\langle\widehat{n}_0\rangle \geq n(1 - C\rho^{-1/6}(\ell\rho^{1/4})^2)$. By choosing b small enough we see immediately that $n \leq C\rho\ell^3$. □

Using this result as an input in (32) we can get a better bound on n than above and a better bound on $\langle\widehat{n}_+\rangle$ than given in Lemma 5.1. In particular, the next lemma in fact implies that we have near neutrality, i.e., that n is nearly $\rho\ell^3$.

Lemma 7.2 (Improved bounds on n and $\langle\widehat{n}_+\rangle$). *There exists a constant $C > 0$ such that if $(\rho^{1/4}\ell)t^3 > C$ and $(\rho^{1/4}\ell)\rho^{-1/12}$, t, and $\omega(t)(\rho^{1/4}\ell)^{-1}$ are smaller than C^{-1} then for any state with $\langle H_\ell^n\rangle \leq 0$ we have $\langle\sum_p |p|^2 a_p^* a_p\rangle \leq C\rho^{5/4}\ell^3(\rho^{1/4}\ell)$ and*

$$\langle\widehat{n}_+\rangle \leq Cn\rho^{-1/4}(\rho^{1/4}\ell)^3 \quad \textit{and} \quad \left(\frac{n - \rho\ell^3}{\rho\ell^3}\right)^2 \leq C\rho^{-1/4}(\rho^{1/4}\ell)^3.$$

For any other state with $\langle H_{\ell,r',R'}^n\rangle' \leq 0$ we have the same bound on $\langle\widehat{n}_+\rangle'$ if $r' \leq \rho^{-3/8}(\rho^{1/4}\ell)^{1/2}$ and $R' \geq a(\rho^{1/4}\ell)^{-2}\ell$ where $a > 0$ is an appropriate constant.

Proof. Inserting the bound $n \leq C\rho\ell^3$ into (32) gives

$$H_\ell^n \geq \tfrac{1}{4}\sum_p |p|^2 a_p^* a_p - C\rho^{5/4}\ell^3 - \tfrac{1}{2}\rho\ell^3 R^{-1} - C\rho^2\ell^3 r^2 + \tfrac{1}{4}\widehat{w}_{00,00}\left(\widehat{n}_0 - \rho\ell^3\right)^2$$
$$- CR^2\rho\widehat{n}_+ - CR^2\left(\rho + \tfrac{1}{4}\ell^{-3}\right) - \tfrac{1}{4}\widehat{n}_+ r^{-1}.$$

We now choose $r = \rho^{-3/8}(\rho^{1/4}\ell)^{1/2}$ and $R = a(\rho^{1/4}\ell)^{-2}\ell$, where we shall choose a below, independently of ρ, $\rho^{1/4}\ell$, and t. Note that since $\omega(t)(\rho^{1/4}\ell)^{-2}$ is small we may assume that $R \leq \omega(t)^{-1}\ell$ as required and since $(\rho^{1/4}\ell)\rho^{-1/12}$ is small we may assume that $r \leq R$. Moreover $r^{-1} = \rho^{-1/8}(\rho^{1/4}\ell)^{3/2}\ell^{-2}$ and $R^2\rho = a^2(\rho^{1/4}\ell)^{-4}\ell^2\rho = a^2\ell^{-2}$. Hence, since $\sum_p |p|^2 a_p^* a_p \geq \pi^2\ell^{-2}\widehat{n}_+$ (see 15), we have

$$H_\ell^n \geq \tfrac{1}{8}\sum_p |p|^2 a_p^* a_p + \left(\tfrac{\pi^2}{8} - a^2 - \tfrac{1}{4}\rho^{-1/8}(\rho^{1/4}\ell)^{3/2}\right)\ell^{-2}\widehat{n}_+$$
$$+ \tfrac{1}{4}\widehat{w}_{00,00}\left(\widehat{n}_0 - \rho\ell^3\right)^2$$
$$- (\tfrac{1}{2a} + C)\rho^{5/4}\ell^3(\rho^{1/4}\ell) - Ca^2\rho^{5/4}\ell^3(\rho^{1/4}\ell)^{-5}(1 + (\rho^{1/4}\ell)^{-3}\rho^{-1/4}).$$

By choosing a appropriately (independently of ρ, $\rho^{1/4}\ell$, and t) we immediately get the bound on $\langle\sum_p |p|^2 a_p^* a_p\rangle$ and the bound $\ell^{-2}\langle\widehat{n}_+\rangle \leq C\rho^{5/4}\ell^3(\rho^{1/4}\ell)$, which implies the stated bound on $\langle\widehat{n}_+\rangle$. The bound on $(n - \rho\ell^3)^2(\rho\ell^3)^{-2}$ follows since we also have $\widehat{w}_{00,00}\langle(\widehat{n}_0 - \rho\ell^3)^2\rangle \leq C\rho^{5/4}\ell^3(\rho^{1/4}\ell)$ and

$$\widehat{w}_{00,00}\langle\left(\widehat{n}_0 - \rho\ell^3\right)^2\rangle \geq CR^2\ell^{-3}\left(\langle\widehat{n}_0\rangle - \rho\ell^3\right)^2$$
$$\geq Ca^2(\rho^{1/4}\ell)^{-4}\ell^2\left(n - \rho\ell^3 - nC\rho^{-1/4}(\ell\rho^{1/4})^3\right)^2,$$

where we have used the bound on $\langle\widehat{n}_+\rangle$ which we have just proved.

The case when $\langle H_{\ell,r',R'}^n\rangle' \leq 0$ follows in the same way because we may everywhere replace H_ℓ^n by $H_{\ell,r',R'}^n$ and use Lemma 4.2 instead of Lemma 4.1. Note that in this case we already know the bound on n since we still assume the existence of the state such that $\langle H_\ell^n\rangle \leq 0$. □

8. Localization of $\widehat{n}_+$

Note that Lemma 7.2 may be interpreted as saying that we have neutrality and condensation, in the sense that $\langle\widehat{n}_+\rangle$ is a small fraction of n, in each little box. Although this bound on $\langle\widehat{n}_+\rangle$ is sufficient for our purposes we still need to know that $\langle\widehat{n}_+^2\rangle \sim \langle\widehat{n}_+\rangle^2$. We shall however not prove this for a general state with negative energy. Instead we shall show that we may change the ground state, without changing its energy expectation significantly, in such a way that the possible $\widehat{n}_+$ values are bounded by $Cn\rho^{-1/4}(\rho^{1/4}\ell)^3$. To do this we shall use the method of localizing large matrices in Lemma A.1 of Appendix A.

We begin with any normalized n-particle wavefunction Ψ of the operator H_ℓ^n. Since Ψ is an n-particle wave function we may write $\Psi = \sum_{m=0}^n c_m\Psi_m$, where for all $m = 1, 2, \ldots, n$, Ψ_m, is a normalized eigenfunctions of $\widehat{n}_+$ with eigenvalue m. We may now consider the $(n+1)\times(n+1)$ Hermitean matrix $\mathcal{A}$ with matrix elements $\mathcal{A}_{mm'} = \left(\Psi_m, H^n_{\ell,r,R}\psi'_m\right)$.

We shall use Lemma A.1 for this matrix and the vector $\psi = (c_0,\ldots,c_n)$. We shall choose M in Lemma A.1 to be of the order of the upper bound on $\langle\widehat{n}_+\rangle$ derived in Lemma 7.2, e.g., M is the integer part of $n\rho^{-1/4}(\rho^{1/4}\ell)^3$. Recall that with the assumption in Lemma 7.2 we have $M \gg 1$. With the notation in Lemma A.1 we have $\lambda = (\psi, \mathcal{A}\psi) = (\Psi, H^n_{\ell,r,R}\Psi)$. Note also that because of the structure of $H^n_{\ell,r,R}$ we have, again with the notation in Lemma A.1, that $d_k = 0$ if $k > 3$. We conclude from Lemma A.1 that there exists a normalized wavefunction $\widetilde{\Psi}$ with the property that the corresponding $\widehat{n}_+$ values belong to an interval of length M and such that

$$\left(\Psi, H^n_{\ell,r,R}\Psi\right) \geq \left(\widetilde{\Psi}, H^n_{\ell,r,R}\widetilde{\Psi}\right) - CM^{-2}(|d_1| + |d_2|).$$

We shall discuss d_1, d_2, which depend on Ψ, in detail below, but first we give the result on the localization of $\widehat{n}_+$ that we shall use.

Lemma 8.1 (Localization of $\widehat{n}_+$). *There is a constant $C > 0$ with the following property. If $(\rho^{1/4}\ell)t^3 > C$ and $(\rho^{1/4}\ell)\rho^{-1/12}$, t, and $\omega(t)(\rho^{1/4}\ell)^{-1}$ are less than C^{-1} and $r \leq \rho^{3/8}(\rho^{1/4}\ell)^{1/2}$, $R \geq C(\rho^{1/4}\ell)^{-2}\ell$, and Ψ is a normalized wavefunction such that*

$$\left(\Psi, H^n_{\ell,r,R}\Psi\right) \leq 0 \quad \textit{and} \quad \left(\Psi, H^n_{\ell,r,R}\Psi\right) \leq -C(n\rho^{-1/4}(\rho^{1/4}\ell)^3)^{-2}(|d_1| + |d_2|) \tag{33}$$

then there exists a normalized wave function $\widetilde{\Psi}$, which is a linear combination of eigenfunctions of $\widehat{n}_+$ with eigenvalues less than $Cn\rho^{-1/4}(\rho^{1/4}\ell)^3$ only, such that

$$\left(\Psi, H^n_{\ell,r,R}\Psi\right) \geq \left(\widetilde{\Psi}, H^n_{\ell,r,R}\widetilde{\Psi}\right) - C(n\rho^{-1/4}(\rho^{1/4}\ell)^3)^{-2}(|d_1| + |d_2|). \tag{34}$$

Here d_1 and d_2, depending on Ψ, are given as explained in Lemma A.1.

Proof. As explained above we choose M to be of order $n\rho^{-1/4}(\rho^{1/4}\ell)^3$. We then choose $\widetilde{\Psi}$ as explained above. Then (34) holds. We also know that the possible $\widehat{n}_+$ values of $\widetilde{\Psi}$ range in an interval of length M. We do not know however, where this interval is located. The assumption (33) will allow us to say more about the location of the interval.

In fact, it follows from (33), (34) that $\left(\widetilde{\Psi}, H^n_{\ell,r,R}\widetilde{\Psi}\right) \leq 0$. It is then a consequence of Lemma 7.2 that $\left(\widetilde{\Psi}, \widehat{n}_+\Psi\right) \leq Cn\rho^{-1/4}(\rho^{1/4}\ell)^3$. This of course establishes that the allowed $\widehat{n}_+$ values are less than $C'n\rho^{-1/4}(\rho^{1/4}\ell)^3$ for some constant $C' > 0$. □

Our final task in this section is to bound d_1 and d_2. We have that $d_1 = (\Psi, H^n_{\ell,r,R}(1)\psi)$, where $H^n_{\ell,r,R}(1)$ is the part of the Hamiltonian $H^n_{\ell,r,R}$ containing all the terms with the coefficents $\widehat{w}_{pq,\mu\nu}$ for which precisely one or three indices are 0. These are the terms bounded in Lemmas 5.5 and 5.6. These lemmas are stated as one-sided bounds. It is clear from the proof that they could have been stated as two sided bounds. Alternatively we may observe that $H^n_{\ell,r,R}(1)$ is unitarily equivalent to $-H^n_{\ell,r,R}(1)$. This follows by applying the unitary transform which maps all operators a^*_p and a_p with $p \neq 0$ to $-a^*_p$ and $-a_p$. From Lemmas 5.5 and 5.6 we therefore immediately get the following bound on d_1.

Lemma 8.2 (Control of d_1). *With the notation above we have for all $\varepsilon > 0$*

$$|d_1| \le \varepsilon^{-1} 8\pi\ell^{-3} R^2 \,(\Psi, \widehat{n}_0\widehat{n}_+\Psi) + \varepsilon\left(\Psi, \left(\widehat{n}_+ r^{-1} + \widehat{w}_{00,00}(\widehat{n}_0 + 1 - \rho\ell^3)^2\right)\Psi\right)$$
$$+ \varepsilon\left(\Psi, \sum_{p,m,p',m'\neq 0} \widehat{w}_{mp',pm'} a_m^* a_{p'}^* a_{m'} a_p \Psi\right).$$

Likewise, we have that $d_2 = (\Psi, H^n_{\ell,r,R}(2)\psi)$, where $H^n_{\ell,r,R}(2)$ is the part of the Hamiltonian $H^n_{\ell,r,R}$ containing all the terms with precisely two a_0 or two a_0^*. i.e., these are the terms in the Foldy Hamiltonian, which do not commute with $\widehat{n}_+$.

Lemma 8.3 (Control of d_2). *There exists a constant $C > 0$ such that if $(\rho^{1/4}\ell)t^3 > C$ and $(\rho^{1/4}\ell)\rho^{-1/12}$, t, and $\omega(t)(\rho^{1/4}\ell)^{-1}$ are less than C^{-1} and Ψ is a wave function with $(\Psi, H^n_\ell\Psi) \le 0$ then with the notation above we have*

$$|d_2| \le C\rho^{5/4}\ell^3(\rho^{1/4}\ell) + 4\pi\ell^{-3}R^2\,(\Psi, \widehat{n}_+\widehat{n}_0\Psi)\,.$$

Proof. If we replace all the operators a_p^* and a_p with $p \neq 0$ in the Foldy Hamiltonian by $-ia_p^*$ and ia_p we get a unitarily equivalent operator. This operator however differs from the Hamiltonian H_{Foldy} only by a change of sign on the part that we denoted $H^n_{\ell,r,R}(2)$. Since both operators satisfy the bound in Corollary 6.5 we conclude that

$$|d_2| \le \left(\Psi, \left[\tfrac{1}{2}\gamma^{-1}\sum_p |p|^2 a_p^* a_p + \tfrac{1}{2}\sum_{pq\neq 0} \widehat{w}_{pq,00}\left(a_p^* a_0^* a_0 a_q + a_0^* a_p^* a_q a_0\right)\right]\Psi\right)$$
$$+ Cn^{5/4}\ell^{-3/4}.$$

Note that both sums above define positive operators. This is trivial for the first sum. For the second it follows from (18) in Lemma 5.4 since $a_0^* a_0$ commutes with all a_p^* and a_p with $p \neq 0$. The lemma now follows from (18) and from Lemma 7.2. □

9. Proof of Foldy's Law

We first prove Foldy's law in a small cube. Let Ψ be a normalized n-particle wave function. We shall prove that with an appropriate choice of ℓ

$$\left(\Psi, H^n_\ell\Psi\right) \ge \left(\tfrac{4\pi}{3}\right)^{1/3} A\rho\ell^3\left(\rho^{1/4} + o\left(\rho^{1/4}\right)\right), \tag{35}$$

where A is given in (6). Note that $A < 0$. It then follows from Lemma 3.3 that

$$E_0 \ge (1 + L/\ell)^3\gamma\left(\tfrac{4\pi}{3}\right)^{1/3} A\rho\ell^3\left(\rho^{1/4} + o\left(\rho^{1/4}\right)\right) - C(L/\ell)^2\rho^2\ell^5 - \frac{\omega(t)N}{2\ell}.$$

Thus, since $N = \rho L^3$ we have

$$\lim_{L\to\infty}\frac{E_0}{N} \ge \gamma\left(\tfrac{4\pi}{3}\right)^{1/3} A\left(\rho^{1/4} + o\left(\rho^{1/4}\right)\right) - C\rho^{1/4}\omega(t)\left(\rho^{1/4}\ell\right)^{-1}.$$

Foldy's law (5) follows since we shall choose (see below) t and ℓ in such a way that as $\rho \to \infty$ we have $t \to 0$ and hence $\gamma \to 1$ and $\omega(t)(\rho^{1/4}\ell)^{-1} \to 0$ (see condition (41) below).

It remains to prove (35). First we fix the long and short distance potential cutoffs

$$R = \omega(t)^{-1}\ell, \quad \text{and} \quad r = \rho^{-3/8}(\rho^{1/4}\ell)^{-1/2}. \tag{36}$$

We may of course assume that $\left(\Psi, H_\ell^n \Psi\right) \leq 0$. Thus n satisfies the bound in Lemma 7.2. We proceed in two steps. In Lemma 9.1 Foldy's law in the small boxes is proved under the restrictive assumption given in (37) below. Finally, in Theorem 9.2 Foldy's law in the small boxes is proved by considering the alternative case that (37) fails. Let us note that, logically speaking, this could have been done in the reverse order. I.e., we could, instead, have begun with the case that (37) fails. At the end of the section we combine Theorem 9.2 with Lemma 3.3 to show that Foldy's law in the small box implies Foldy's law Theorem 1.1.

At the end of this section we show how to choose ℓ and t so that Theorem 9.2 implies (35) and hence Theorem 1.1, as explained above.

Lemma 9.1 (Foldy's law for H_ℓ^n: restricted version). *Let R and r be given by (36). There exists a constant $C > 0$ such that if $(\rho^{1/4}\ell)t^3 > C$ and $(\rho^{1/4}\ell)\rho^{-1/12}$, t, and $\omega(t)(\rho^{1/4}\ell)^{-1}$ are less than C^{-1} then, whenever*

$$\begin{aligned} &n\ell^{-3}R^2\left(\Psi, \widehat{n}_+\Psi\right) \\ &\leq C^{-1}\left(\Psi, \left(\widehat{w}_{00,00}(\widehat{n}_0 - \rho\ell^3)^2 + \sum_{p,m,p',m'\neq 0} \widehat{w}_{mp',pm'} a_m^* a_{p'}^* a_{m'} a_p\right)\Psi\right), \end{aligned} \tag{37}$$

we have that

$$\begin{aligned} \left(\Psi, H_\ell^n\Psi\right) \geq -In^{5/4}\ell^{-3/4} &- C\rho^{5/4}\ell^3 \left(\omega(t)(\rho^{1/4}\ell)^{-1} + \omega(t)^{-2}\rho^{-1/8}(\rho^{1/4}\ell)^{13/2}\right. \\ &\left. + +\rho^{-1/8}(\rho^{1/4}\ell)^{7/2}\right), \end{aligned}$$

with I as in Lemma 6.4.

Proof. We assume $\left(\Psi, H_\ell^n\Psi\right) \leq 0$. We proceed as in the beginning of Sect. 7, but we now use (29) of Corollary 6.5 instead of (30). We then get

$$\begin{aligned} H_\ell^n \geq &- In^{5/4}\ell^{-3/4} - 4\pi n^{5/4}\ell^{-3/4}(n\ell)^{-1/4} - \tfrac{1}{2}nR^{-1} - Cn\rho r^2 \\ &+ \tfrac{1}{2}\widehat{w}_{00,00}\left[\left(\widehat{n}_0 - \rho\ell^3\right)^2 - \widehat{n}_0\right] \\ &- 4\pi[\rho - n\ell^{-3}]_+\widehat{n}_+R^2 - 4\pi\widehat{n}_+^2\ell^{-3}R^2 \\ &- \varepsilon^{-1}8\pi\ell^{-3}R^2\widehat{n}_0\widehat{n}_+ - \varepsilon\widehat{w}_{00,00}(\widehat{n}_0 + 1 - \rho\ell^3)^2 \\ &- \varepsilon\widehat{n}_+r^{-1} + (\tfrac{1}{2} - \varepsilon)\sum_{p,m,p',m'\neq 0}\widehat{w}_{mp',pm'}a_m^*a_{p'}^*a_{m'}a_p. \end{aligned}$$

If we now use the assumption (37) and the facts that $\widehat{n}_+ \leq n$, $\widehat{n}_0 \leq n$, and $\widehat{w}_{00,00} \leq 4\pi R^2\ell^{-3}$ we see with appropriate choices of ε and C that

$$\begin{aligned} H_\ell^n \geq &- In^{5/4}\ell^{-3/4} - 4\pi n^{5/4}\ell^{-3/4}(n\ell)^{-1/4} - \tfrac{1}{2}nR^{-1} - Cn\rho r^2 - CR^2\ell^{-3}(n+1) \\ &- CR^2\ell^{-3}|n - \rho\ell^3|(\widehat{n}_+ + 1) - C\widehat{n}_+r^{-1}. \end{aligned}$$

If we finally insert the choices of R and r and use Lemma 7.2 we arrive at the bound in the lemma. □

Theorem 9.2 (Foldy's law for H_ℓ^n). *There exists a $C > 0$ such that if $(\rho^{1/4}\ell)t^3 > C$ and $(\rho^{1/4}\ell)\rho^{-1/12}$, t, and $\omega(t)(\rho^{1/4}\ell)^{-1}$ are less than C^{-1} then for any normalized n-particle wave function Ψ we have*

$$\left(\Psi, H_\ell^n\Psi\right) \geq -In^{5/4}\ell^{-3/4} - C\rho^{5/4}\ell^3\Big(\omega(t)(\rho^{1/4}\ell)^{-1} + \omega(t)^{-1}\rho^{-1/16}(\rho^{1/4}\ell)^{29/4} + \rho^{-1/8}(\rho^{1/4}\ell)^{7/2}\Big), \quad (38)$$

where I is defined in Lemma 6.4 with r and R as in (36).

Proof. According to Lemma 9.1 we may assume that

$$n\ell^{-3}R^2\,(\Psi, \widehat{n}_+\Psi) \geq C^{-1}\Bigg(\Psi, \Bigg(\widehat{w}_{00,00}(\widehat{n}_0 - \rho\ell^3)^2 + \sum_{p,m,p',m'\neq 0} \widehat{w}_{mp',pm'}a_m^* a_{p'}^* a_{m'} a_p\Bigg)\Psi\Bigg), \quad (39)$$

where C is at least as big as the constant in Lemma 9.1. We still assume that $\left(\Psi, H_\ell^n\Psi\right) \leq 0$.

We begin by bounding d_1 and d_2 using Lemmas 8.2 and 8.3. We have from Lemmas 7.2 and 8.3 that

$$\begin{aligned}|d_2| &\leq C\rho^{5/4}\ell^3(\rho^{1/4}\ell) + C\ell^{-1}\omega(t)^{-2}n^2\rho^{-1/4}(\rho^{1/4}\ell)^3\\ &\leq C[n\rho^{-1/4}(\rho^{1/4}\ell)^3]^2\rho^{5/4}\ell^3\left((\rho^{1/4}\ell)^{-11} + \omega(t)^{-2}(\rho^{1/4}\ell)^{-7}\right)\\ &\leq C[n\rho^{-1/4}(\rho^{1/4}\ell)^3]^2\rho^{5/4}\ell^3\omega(t)^{-2}(\rho^{1/4}\ell)^{-7}.\end{aligned}$$

In order to bound d_1 we shall use (39). Together with Lemma 8.2 this gives (choosing $\varepsilon = 1/2$ say)

$$|d_1| \leq C\ell^{-3}R^2 n\,(\Psi, \widehat{n}_+\Psi) + \tfrac{1}{2}\left(\Psi, \left(\widehat{n}_+ r^{-1} + \widehat{w}_{00,00}(n - \rho\ell^3 + 1)\right)\Psi\right).$$

Inserting the choices for r and R and using Lemma 7.2 gives

$$|d_1| \leq C[n\rho^{-1/4}(\rho^{1/4}\ell)^3]^2\rho^{5/4}\ell^3\left(\omega(t)^{-2}(\rho^{1/4}\ell)^{-7} + \rho^{-1/8}(\rho^{1/4}\ell)^{-17/2}\right),$$

where we have also used that we may assume that $\rho^{-1/8}(\rho^{1/4}\ell)^{-9/2}$ is small. The assumption (33) now reads

$$\left(\Psi, H_{\ell,r,R}^n\Psi\right) \leq -C\rho^{5/4}\ell^3\left(\omega(t)^{-2}(\rho^{1/4}\ell)^{-7} + \rho^{-1/8}(\rho^{1/4}\ell)^{-17/2}\right).$$

If this is not satisfied we see immediately that the bound (38) holds.

Thus from Lemma 8.1 it follows that we can find a normalized n-particle wavefunction $\widetilde{\Psi}$ with

$$\left(\widetilde{\Psi}, \widehat{n}_+\widetilde{\Psi}\right) \leq Cn\rho^{-1/4}(\rho^{1/4}\ell)^3 \quad \text{and} \quad \left(\widetilde{\Psi}, \widehat{n}_+^2\widetilde{\Psi}\right) \leq Cn^2\rho^{-1/2}(\rho^{1/4}\ell)^6 \quad (40)$$

such that

$$\left(\Psi, H_{\ell,r,R}^n\Psi\right) \geq \left(\widetilde{\Psi}, H_{\ell,r,R}^n\widetilde{\Psi}\right) - C\rho^{5/4}\ell^3\left(\omega(t)^{-2}(\rho^{1/4}\ell)^{-7} + \rho^{-1/8}(\rho^{1/4}\ell)^{-17/2}\right).$$

In order to analyze $\left(\widetilde{\Psi}, H^n_{\ell,r,R}\widetilde{\Psi}\right)$ we proceed as in the beginning of Sect. 7. This time we use Lemmas 4.1, 5.2, 5.3, 5.5, and 5.6 together with Lemma 6.4 instead of Corollary 6.5. We obtain

$$\begin{aligned} H^n_{\ell,r,R} \geq \tfrac{1}{2}\widehat{w}_{00,00}\left[\left(n-\rho\ell^3\right)^2 + (\widehat{n}_+)^2 - 2\left(n-\rho\ell^3\right)\widehat{n}_+ - \widehat{n}_0\right] \\ - 4\pi[\rho - n\ell^{-3}]_+\widehat{n}_+R^2 - 4\pi\widehat{n}_+^2\ell^{-3}R^2 - \varepsilon\widehat{n}_+r^{-1} - \varepsilon^{-1}8\pi\ell^{-3}R^2\widehat{n}_0\widehat{n}_+ \\ - \varepsilon\widehat{w}_{00,00}(\widehat{n}_+ - 1)^2 + (\tfrac{1}{2}-\varepsilon)\sum_{p,m,p',m'\neq 0}\widehat{w}_{mp',pm'}a^*_m a^*_{p'}a_{m'}a_p \\ - \tfrac{1}{2}\left(n-\rho\ell^3\right)^2\widehat{w}_{00,00} - 4\pi n^{5/4}\ell^{-3/4}(n\ell)^{-1/4} - In^{5/4}\ell^{-3/4}. \end{aligned}$$

This time we shall however not choose ε small, but rather big. Note that since $w_{r,R}(x,y) \leq r^{-1}$ we have $\sum_{p,m,p',m'\neq 0}\widehat{w}_{mp',pm'}a^*_m a^*_{p'}a_{m'}a_p \leq r^{-1}\widehat{n}_+(\widehat{n}_+ - 1)$, which follows immediately from

$$\begin{aligned} &\sum_{p,m,p',m'\neq 0}\widehat{w}_{mp',pm'}a^*_m a^*_{p'}a_{m'}a_p \\ &= \iint w_{r,R}(x,y)\left(\sum_{p,m\neq 0}u_m(x)u_p(y)a_m a_p\right)^*\sum_{p,m\neq 0}u_m(x)u_p(y)a_m a_p\,dx\,dy. \end{aligned}$$

We therefore have

$$\begin{aligned} H^n_{\ell,r,R} \geq\ & -In^{5/4}\ell^{-3/4} - 4\pi n^{5/4}\ell^{-3/4}(n\ell)^{-1/4} - CR^2\ell^{-3}\widehat{n}_0 \\ & - C\ell^{-3}R^2|\rho\ell^3 - n|\widehat{n}_+ - 4\pi\widehat{n}_+^2\ell^{-3}R^2 - \varepsilon\widehat{n}_+r^{-1} - \varepsilon^{-1}8\pi\ell^{-3}R^2\widehat{n}_0\widehat{n}_+ \\ & - \varepsilon CR^2\ell^{-3}\widehat{n}_+^2 - \varepsilon\widehat{n}_+^2 r^{-1}. \end{aligned}$$

If we now insert the choices of r and R, take the expectation in the state given by $\widetilde{\Psi}$, and use (40) and the bound on n from Lemma 7.2 we arrive at

$$\begin{aligned} \left(\widetilde{\Psi},\ H^n_{\ell,r,R}\widetilde{\Psi}\right) \geq\ & -In^{5/4}\ell^{-3/4} - C\rho^{5/4}\ell^3\Big[(\rho^{1/4}\ell)^{-1} + \omega(t)^{-2}(\rho^{1/4}\ell)^{-1} \\ & + \omega(t)^{-2}\rho^{-1/8}(\rho^{1/4}\ell)^{11/2} + \omega(t)^{-2}\rho^{-1/4}(\rho^{1/4}\ell)^8 + \varepsilon\rho^{-1/8}(\rho^{1/4}\ell)^{7/2} \\ & + \varepsilon^{-1}\omega(t)^{-2}(\rho^{1/4}\ell)^5 + \varepsilon\omega(t)^{-2}\rho^{-1/4}(\rho^{1/4}\ell)^8 + \varepsilon\rho^{-1/8}(\rho^{1/4}\ell)^{19/2}\Big]. \end{aligned}$$

If we now choose $\varepsilon = \omega(t)^{-1}\rho^{1/16}(\rho^{1/4}\ell)^{-9/4}$ we arrive at (38). □

Completion of the proof of Foldy's law, Theorem 1.1. We have accumulated various errors and we want to show that they can all be made small. There are basically two parameters that can be adjusted, ℓ and t. Instead of ℓ it is convenient to use $X = \rho^{1/4}\ell$. We shall choose X as a function of ρ such that $X \to \infty$ as $\rho \to \infty$. From Lemma 7.1 we know that for some fixed $C > 0$ $C^{-1}\rho\ell^3 \leq n \leq C\rho\ell^3$. Hence according to (31) with r and R

given in (36) we have that $I \to -\left(\frac{4\pi}{3}\right)^{1/3} A$ as $\rho \to \infty$ if

$$\omega(t)^{-1} X \to \infty, \tag{41}$$

$$\rho^{1/4} X \to \infty, \tag{42}$$

$$t^3 X \to \infty, \tag{43}$$

$$t \to 0. \tag{44}$$

The hypotheses of Theorem 9.2 are valid if (41), (43), (44), and

$$\rho^{-1/12} X \to 0 \tag{45}$$

hold. From Lemma 7.2, for which the hypotheses are now automatically satisfied, we have that $n = \rho\ell^3(1 + O(\rho^{-1/8} X^{3/2})$ and from (45) we see that n is $\rho\ell^3$ to leading order.

With these conditions we find that the first term on the right side of (38) is, in the limit $\rho \to \infty$, exactly Foldy's law. The conditions that the other terms in (38) are of lower order are

$$(X/\omega(t))^{4/25} \rho^{-1/100} X \to 0, \tag{46}$$

$$\rho^{-1/28} X \to 0 \tag{47}$$

together with (41).

It remains to show that we can satisfy the conditions (41–47). Condition (42) is trivially satisfied since both ρ and X tend to infinity. Since $\omega(t) \sim t^{-4}$ for small t we see that (43) is implied by (41). Condition (45) is implied by (47), which is in turn implied by (41) and (46). The remaining two conditions (41) and (46) are easily satisfied by an approriate choice of X and t as functions for ρ with $X \to \infty$ and $t \to 0$ as $\rho \to \infty$. In fact, we simply need $\rho^{1/116} t^{-16/29} \gg X \gg t^{-4}$.

The bound (35) has now been established. Hence Foldy's law Theorem 1.1 follows as discussed in the beginning of the section.

Appendix

A. Localization of Large Matrices

The following theorem allows us to reduce a big Hermitean matrix, $\mathcal{A}$, to a smaller principal submatrix without changing the lowest eigenvalue very much. (The k^{th} supra- (resp. infra-) diagonal of a matrix $\mathcal{A}$ is the submatrix consisting of all elements $a_{i,i+k}$ (resp. $a_{i+k,i}$).)

Theorem A.1 (Localization of large matrices). *Suppose that $\mathcal{A}$ is an $N \times N$ Hermitean matrix and let $\mathcal{A}^k$, with $k = 0, 1, ..., N-1$, denote the matrix consisting of the k^{th} supra- and infra-diagonal of $\mathcal{A}$. Let $\psi \in \mathbf{C}^N$ be a normalized vector and set $d_k = (\psi, \mathcal{A}^k \psi)$ and $\lambda = (\psi, \mathcal{A}\psi) = \sum_{k=0}^{N-1} d_k$. ($\psi$ need not be an eigenvector of $\mathcal{A}$.)*

Choose some positive integer $M \leq N$. Then, with M fixed, there is some $n \in [0, N - M]$ and some normalized vector $\phi \in \mathbf{C}^N$ with the property that $\phi_j = 0$ unless $n + 1 \leq j \leq n + M$ (i.e., ϕ has length M) and such that

$$(\phi, \mathcal{A}\phi) \leq \lambda + \frac{C}{M^2} \sum_{k=1}^{M-1} k^2 |d_k| + C \sum_{k=M}^{N-1} |d_k|, \tag{48}$$

where $C > 0$ is a universal constant. (Note that the first sum starts with $k = 1$.)

Proof. It is convenient to extend the matrix $\mathcal{A}_{i,j}$ to all $-\infty < i, j < +\infty$ by defining $\mathcal{A}_{i,j} = 0$ unless $1 \le i, j \le N$. Similarly, we extend the vector ψ and we define the numbers d_k and the matrix $\mathcal{A}^k$ to be zero when $k \notin [0, N-1]$. We shall give the construction for M odd, the M even case being similar.

For $s \in \mathbf{Z}$ set $f(s) = A_M[M + 1 - 2|s|]$ if $2|s| < M$ and $f(s) = 0$ otherwise. Thus, $f(s) \neq 0$ for precisely M values of s. Also, $f(s) = f(-s)$. A_M is chosen so that $\sum_s f(s)^2 = 1$.

For each $m \in \mathbf{Z}$ define the vector $\phi^{(m)}$ by $\phi_j^{(m)} = f(j-m)\psi_j$. We then define $K^{(m)} = (\phi^{(m)}, \mathcal{A}\phi^{(m)}) - (\lambda + \sigma)(\phi^{(m)}, \phi^{(m)})$. (The number σ will be chosen later.) After this, we define $K = \sum_m K^{(m)}$. Using the fact that $\sum_s f(s)^2 = 1$, we have that

$$\begin{aligned}\sum_m (\phi^{(m)}, \mathcal{A}\phi^{(m)}) &= \sum_m \sum_{k=0} (\phi^{(m)}, \mathcal{A}^k \phi^{(m)}) = \sum_s \sum_k f(s) f(k+s)(\psi, \mathcal{A}^k \psi)\\ &= \sum_s \sum_{k=0} f(s) f(k+s) d_k\end{aligned}$$

and

$$\lambda = \lambda \sum_m (\phi^{(m)}, \phi^{(m)}) = \sum_s \sum_{k=0} f(s)^2 (\psi, \mathcal{A}^k \psi) = \sum_s \sum_k f(s)^2 d_k \tag{49}$$

Hence

$$K = \sum_m K^{(m)} = -\sigma - \sum_{k=1}^{N-1} d_k \gamma_k \tag{50}$$

with

$$\gamma_k = \frac{1}{2} \sum_s [f(s) - f(s+k)]^2 . \tag{51}$$

Let us choose $\sigma = -\sum_{k=1}^{N-1} d_k \gamma_k$. Then, $\sum_m K^{(m)} = 0$. Recalling that not all of the $\phi^{(m)}$ equal zero, we conclude that there is at least one value of m such that (i) $\phi^{(m)} \neq 0$ and (ii) $(\phi^{(m)}, \mathcal{A}\phi^{(m)}) \le (\lambda + \sigma)(\phi^{(m)}, \phi^{(m)})$.

This concludes the proof of (48) except for showing that $\gamma_k \le C \frac{k^2}{k^2 + M^2}$ for all M and k. This is evident from the easily computable large M asymptotics in (51). $\square$

B. A Double Commutator Bound

Lemma B.1. *Let* $-\Delta_N$ *be the Neumann Laplacian of some bounded open set* $\mathcal{O}$. *Given* $\theta \in C^\infty(\overline{\mathcal{O}})$ *with* supp $|\nabla\theta| \subset \mathcal{O}$ *satisfying* $\|\partial_i \theta\| \le C t^{-1}$, $\|\partial_i \partial_j \theta\| \le C t^{-2}$, $\|\partial_i \partial_j \partial_k \theta\| \le C t^{-3}$, *for some* $0 < t$ *and all* $i, j, k = 1, 2, 3$. *Then for all* $s > 0$ *we have the operator inequality*

$$\left[\left[\frac{(-\Delta_N)^2}{-\Delta_N + s^{-2}}, \theta\right], \theta\right] \ge -C t^{-2} \frac{-\Delta_N}{-\Delta_N + s^{-2}} - C s^2 t^{-4}. \tag{52}$$

We also have the norm bound

$$\left\| \left[\left[\frac{-\Delta_N}{-\Delta_N + s^{-2}}, \theta\right], \theta\right] \right\| \le C(s^2 t^{-2} + s^4 t^{-4}). \tag{53}$$

Proof. We calculate the commutator

$$\left[\frac{(-\Delta_N)^2}{-\Delta_N+s^{-2}},\theta\right]=s^{-2}\frac{1}{-\Delta_N+s^{-2}}[-\Delta_N,\theta]\frac{1}{-\Delta_N+s^{-2}}(-\Delta_N)$$
$$+\frac{-\Delta_N}{-\Delta_N+s^{-2}}[-\Delta_N,\theta].$$

Likewise we calculate the double commutator

$$\begin{aligned}\left[\left[\frac{(-\Delta_N)^2}{-\Delta_N+s^{-2}},\theta\right],\theta\right]= &-\frac{-\Delta_N}{-\Delta_N+s^{-2}}[[-\Delta_N,\theta]\theta]\frac{-\Delta_N}{-\Delta_N+s^{-2}}\\ &+[[-\Delta_N,\theta]\theta]\frac{-\Delta_N}{-\Delta_N+s^{-2}}+\frac{-\Delta_N}{-\Delta_N+s^{-2}}[[-\Delta_N,\theta]\theta]\\ &-2s^{-4}\frac{1}{-\Delta_N+s^{-2}}[-\Delta_N,\theta]\frac{1}{-\Delta_N+s^{-2}}[\theta,-\Delta_N]\frac{1}{-\Delta_N+s^{-2}}.\end{aligned}\tag{54}$$

Note that $[[-\Delta_N,\theta]\theta]=-2(\nabla\theta)^2$ and thus the first term above is positive.

We claim that

$$[-\Delta_N,\theta][\theta,-\Delta_N]\leq -Ct^{-2}\Delta_N+Ct^{-4}.\tag{55}$$

To see this we simply calculate

$$[-\Delta_N,\theta][\theta,-\Delta_N]=-\sum_{i,j}^{3}\Big(4\partial_i(\partial_i\theta)(\partial_j\theta)\partial_j+(\partial_i^2\theta)(\partial_j^2\theta)+2(\partial_i\theta)(\partial_i\partial_j^2\theta)\Big)$$

The last two terms are bounded by Ct^{-4}. For the first term we have by the Cauchy-Schwarz inequality for operators, $BA^*+AB^*\leq\varepsilon^{-1}AA^*+\varepsilon BB^*$, for all $\varepsilon>0$, that

$$-\sum_{i,j}^{3}\partial_i(\partial_i\theta)(\partial_j\theta)\partial_j=\sum_{i,j}^{3}(\partial_i(\partial_i\theta))\left(\partial_j(\partial_j\theta)\right)^*\leq -3\sum_{i}^{3}\partial_i(\partial_i\theta)(\partial_i\theta)\partial_i$$

and this is bounded above by $-3t^{-2}\Delta_N$ and we get (55). Inserting (55) into (54), recalling that the first term is positive, we obtain

$$\left[\left[\frac{(-\Delta_N)^2}{-\Delta_N+s^{-2}},\theta\right],\theta\right]\geq -2(\nabla\theta)^2\frac{-\Delta_N}{-\Delta_N+s^{-2}}-2\frac{-\Delta_N}{-\Delta_N+s^{-2}}(\nabla\theta)^2$$
$$-Ct^{-2}\frac{-\Delta_N}{-\Delta_N+s^{-2}}-Cs^2t^{-4}.$$

Again using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&2(\nabla\theta)^2\frac{-\Delta_N}{-\Delta_N+s^{-2}}+2\frac{-\Delta_N}{-\Delta_N+s^{-2}}(\nabla\theta)^2\\ &\leq 2t^{-2}\left(\frac{-\Delta_N}{-\Delta_N+s^{-2}}\right)^{1/2}(\nabla\theta)^4\left(\frac{-\Delta_N}{-\Delta_N+s^{-2}}\right)^{1/2}+2t^{-2}\left(\frac{-\Delta_N}{-\Delta_N+s^{-2}}\right)\\ &\leq Ct^{-2}\frac{-\Delta_N}{-\Delta_N+s^{-2}},\end{aligned}$$

and (52) follows.

The bound (53) is proved in the same way. Indeed,

$$\left[\left[\frac{-\Delta_N}{-\Delta_N+s^{-2}},\theta\right],\theta\right] = -s^{-2}\frac{1}{-\Delta_N+s^{-2}}[[-\Delta_N,\theta],\theta]\frac{1}{-\Delta_N+s^{-2}} + 2s^{-2}\frac{1}{-\Delta_N+s^{-2}}\,[-\Delta_N,\theta]\,\frac{1}{-\Delta_N+s^{-2}}\,[\theta-\Delta_N]\,\frac{1}{-\Delta_N+s^{-2}},$$

and (53) follows from $[[-\Delta_N,\theta]\,\theta] = -2\,(\nabla\theta)^2$ and (55). □

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