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Combinatorics of q-Characters of Finite-Dimensional Representations of Quantum Affine Algebras

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Abstract: We study finite-dimensional representations of quantum affine algebras using q-characters. We prove the conjectures from [FR2] and derive some of their corollaries. In particular, we prove that the tensor product of fundamental representations is reducible if and only if at least one of the pairwise normalized R-matrices has a pole.

Introduction

The intricate structure of the finite-dimensional representations of quantum affine algebras has been extensively studied from different points of view, see, e.g., [CP1, CP2, CP3, CP4, GV, V, KS, AK, FR2]. While a lot of progress has been made, many basic questions remained unanswered. In order to tackle those questions, E. Frenkel and N. Reshetikhin introduced in [FR2] a theory of q-characters for these representations. One of the motivations was the theory of deformed $\mathcal{W}$-algebras developed in [FR1]: the representation ring of a quantum affine algebra should be viewed as a deformed $\mathcal{W}$-algebra, while the q-character homomorphism should be viewed as its free field realization. The study of q-characters in [FR2] was based on two main conjectures. One of the goals of the present paper is to prove these conjectures and to derive some of their corollaries.

Let us describe our results in more detail. Let $\mathfrak{g}$ be a simple Lie algebra, $\widehat{\mathfrak{g}}$ be the corresponding non-twisted affine Kac-Moody algebra, and $U_q\widehat{\mathfrak{g}}$ be its quantized universal enveloping algebra (quantum affine algebra for short). Denote by I the set of vertices of the Dynkin diagram of $\mathfrak{g}$. Let $\operatorname{Rep} U_q\widehat{\mathfrak{g}}$ be the Grothendieck ring of $U_q\widehat{\mathfrak{g}}$. The q-character homomorphism is an injective homomorphism χ_q from $\operatorname{Rep} U_q\widehat{\mathfrak{g}}$ to the ring of Laurent polynomials in infinitely many variables $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times}$. This homomorphism should be viewed as a q-analogue of the ordinary character homomorphism.

Indeed, let G be the connected simply-connected algebraic group corresponding to $\mathfrak{g}$, and let T be its maximal torus. We have a homomorphism $\chi : \operatorname{Rep} G \to \operatorname{Fun} T$ (where $\operatorname{Fun} T$ stands for the ring of regular functions on T), defined by the formula $(\chi(V))(t) = \operatorname{Tr}_V t$, for all $t \in T$. Upon the identification of $\operatorname{Rep} G$ with $\operatorname{Rep} U_q\mathfrak{g}$ and of

Fun T with $\mathbb{Z}[y_i^{\pm 1}]_{i\in I}$, where y_i is the function on T corresponding to the fundamental weight ω_i, we obtain a homomorphism $\chi : \operatorname{Rep} U_q\mathfrak{g} \to \mathbb{Z}[y_i^{\pm 1}]_{i\in I}$. One of the properties of χ_q is that if we replace each $Y_{i,a}^{\pm 1}$ by $y_i^{\pm 1}$ in $\chi_q(V)$, where V is a $U_q\widehat{\mathfrak{g}}$-module, then we obtain $\chi(V|_{U_q\mathfrak{g}})$.

The two conjectures from [FR2] that we prove in this paper may be viewed as q-analogues of the well-known properties of the ordinary characters. The first of them, Theorem 4.1, is the analogue of the statement that the character of any irreducible $U_q\mathfrak{g}$-module W equals the sum of terms which correspond to the weights of the form $\lambda - \sum_{i\in I} n_i\alpha_i, n_i \in \mathbb{Z}_+$, where $\lambda = \sum_{i\in I} l_i\omega_i, l_i \in \mathbb{Z}_+$, is the highest weight of V, and $\alpha_i, i \in I$, are the simple roots. In other words, we have: $\chi(W) = m_+(1 + \sum_p M_p)$, where $m_+ = \prod_{i\in I} y_i^{l_i}$, and each M_p is a product of factors a_j^{-1}, $j \in I$, corresponding to the negative simple roots. Theorem 4.1 says that for any irreducible $U_q\widehat{\mathfrak{g}}$-module V, $\chi_q(V) = m_+(1 + \sum_p M_p)$, where m_+ is a monomial in $Y_{i,a}, i \in I, a \in \mathbb{C}^\times$, with positive powers only (the highest weight monomial), and each M_p is a product of factors $A_{j,c}^{-1}$, $j \in I, c \in \mathbb{C}^\times$, which are the q-analogues of the negative simple roots of $\mathfrak{g}$.

The second statement, Theorem 5.1, gives an explicit description of the image of the q-character homomorphism χ_q. This is a generalization of the well-known fact that the image of the ordinary character homomorphism χ is equal to the subring of invariants of $\mathbb{Z}[y_i^{\pm 1}]_{i\in I}$ under the action of the Weyl group W of $\mathfrak{g}$.

Recall that the Weyl group is generated by the simple reflections s_i, $i \in I$. The subring of invariants of s_i in $\mathbb{Z}[y_i^{\pm 1}]_{i\in I}$ is equal to

$$K_i = \mathbb{Z}[y_j^{\pm 1}]_{j\neq i} \otimes \mathbb{Z}[y_i + y_i a_i^{-1}],$$

and hence we obtain a ring isomorphism $\operatorname{Rep} U_q\widehat{\mathfrak{g}} \simeq \bigcap_{i\in I} K_i$.

In Theorem 5.1 (see also Corollary 5.7) we establish a q-analogue of this isomorphism. Instead of the simple reflections we have the screening operators $S_i, i \in I$, introduced in [FR2]. We show that $\operatorname{Im} \chi_q$ equals $\bigcap_{i\in I} \operatorname{Ker} S_i$. Moreover, $\operatorname{Ker} S_i$ is equal to

$$\mathcal{K}_i = \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j\neq i; a\in\mathbb{C}^\times} \otimes \mathbb{Z}[Y_{i,b} + Y_{i,b}A_{i,bq_i}^{-1}]_{b\in\mathbb{C}^\times}.$$

Thus, we obtain a ring isomorphism $\operatorname{Rep} U_q\widehat{\mathfrak{g}} \simeq \bigcap_{i\in I} \mathcal{K}_i$.

These results allow us to construct in a purely combinatorial way the q-characters of the fundamental representations of $U_q\widehat{\mathfrak{g}}$, see Sect. 5.5.

We derive several corollaries of these results. Here is one of them (see Theorem 6.7 and Proposition 6.15). For each fundamental weight ω_i, there exists a family of $U_q\widehat{\mathfrak{g}}$-modules, $V_{\omega_i}(a), a \in \mathbb{C}^\times$ (see Sect. 1.3 for the precise definition). These are irreducible finite-dimensional representations of $U_q\widehat{\mathfrak{g}}$, which have highest weight ω_i if restricted to $U_q\mathfrak{g}$. They are called the fundamental representations of $U_q\widehat{\mathfrak{g}}$ (of level 0). According to a theorem of Chari-Pressley [CP1, CP3] (see Corollary 1.4 below), any irreducible representation of $U_q\widehat{\mathfrak{g}}$ can be realized as a subquotient of a tensor product of the fundamental representations. The following theorem, which was conjectured, e.g., in [AK], describes under what conditions such a tensor product is reducible.

Denote by $h^\vee$ the dual Coxeter number of $\mathfrak{g}$, and by $r^\vee$ the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$. For the definition of the normalized R-matrix, see Sect. 2.3.

Theorem. *Let $\{V_k\}_{k=1,\dots,n}$, where $V_k = V_{\omega_{s(k)}}(a_k)$, be a set of fundamental representations of $U_q\widehat{\mathfrak{g}}$.*

The tensor product $V_1 \otimes \ldots \otimes V_n$ is reducible if and only if for some $i, j \in \{1, \dots, n\}$, $i \neq j$, the normalized R-matrix $\overline{R}_{V_i,V_j}(z)$ has a pole at $z = a_j/a_i$.

In that case a_j/a_i is necessarily equal to q^k, where k is an integer, such that $2 \leq |k| \leq r^\vee h^\vee$.

The paper is organized as follows. In Sect. 1 we recall the main definitions and results on quantum affine algebras and their finite-dimensional representations. In Sect. 2 we give the definition of the q-character homomorphism and list some of its properties. In Sect. 3 we develop our main technical tool: the restriction homomorphisms τ_J. Sections 4 and 5 contain the proofs of Conjectures 1 and 2 from [FR2], respectively. In Sect. 6 we use these results to describe the structure of the q-characters of the fundamental representations and to prove the above Theorem.

The results of this paper can be generalized to the case of the twisted quantum affine algebras.

In the course of writing this paper we were informed by H. Nakajima that he obtained an independent proof of Conjecture 1 from [FR2] in the ADE case using a geometric approach.

1. Preliminaries on Finite-Dimensional Representations of $U_q\widehat{\mathfrak{g}}$

1.1. Root data. Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ. Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$. Let $\langle\cdot,\cdot\rangle$ be the invariant inner product on $\mathfrak{g}$, normalized as in [K], so that the square of the length of the maximal root equals 2 with respect to the induced inner product on the dual space to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (also denoted by $\langle\cdot,\cdot\rangle$). Denote by I the set $\{1, \dots, \ell\}$. Let $\{\alpha_i\}_{i\in I}$ and $\{\omega_i\}_{i\in I}$ be the sets of simple roots and of fundamental weights of $\mathfrak{g}$, respectively. We have:

$$\langle\alpha_i, \omega_j\rangle = \frac{\langle\alpha_i, \alpha_i\rangle}{2}\delta_{ij}.$$

Let $r^\vee$ be the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$. Thus, $r^\vee = 1$ for simply-laced $\mathfrak{g}$, $r^\vee = 2$ for B_ℓ, C_ℓ, F_4, and $r^\vee = 3$ for G_2.

In this paper we will use the rescaled inner product

$$(\cdot,\cdot) = r^\vee\langle\cdot,\cdot\rangle$$

on $\mathfrak{h}^*$. Set

$$D = \operatorname{diag}(r_1, \dots, r_\ell),$$

where

$$r_i = \frac{(\alpha_i, \alpha_i)}{2} = r^\vee \frac{\langle\alpha_i, \alpha_i\rangle}{2}. \tag{1.1}$$

The r_i's are relatively prime integers. For simply-laced $\mathfrak{g}$, all r_i's are equal to 1 and D is the identity matrix.

Now let $C = (C_{ij})_{1\leq i,j\leq\ell}$ be the *Cartan matrix* of $\mathfrak{g}$,

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

Let $B = (B_{ij})_{1\le i,j\le \ell}$ be the symmetric matrix

$$B = DC,$$

i.e., $B_{ij} = (\alpha_i, \alpha_j) = r^\vee \langle \alpha_i, \alpha_j \rangle$.

Let $q \in \mathbb{C}^\times$ be such that $|q| < 1$. Set $q_i = q^{r_i}$, and

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Following [FR1,FR2], define the $\ell \times \ell$ matrices $B(q)$, $C(q)$, $D(q)$ by the formulas

$$\begin{aligned} B_{ij}(q) &= [B_{ij}]_q, \\ C_{ij}(q) &= (q_i + q_i^{-1})\delta_{ij} + (1 - \delta_{ij})[C_{ij}]_q, \\ D_{ij}(q) &= [D_{ij}]_q = \delta_{ij}[r_i]_q. \end{aligned}$$

We have:

$$B(q) = D(q)C(q).$$

Let $\widetilde{C}(q)$ be the inverse of the Cartan matrix $C(q)$, $C(q)\widetilde{C}(q) = \mathrm{Id}$. We will need the following property of matrix $\widetilde{C}(q)$.

Lemma 1.1. *All coefficients of the matrix $\widetilde{C}(q)$ can be written in the form*

$$\widetilde{C}_{ij}(q) = \frac{\widetilde{C}'_{ij}(q)}{d(q)}, \qquad i, j \in I, \tag{1.2}$$

where $\widetilde{C}'_{ij}(q)$, $d(q)$ are Laurent polynomials in q with non-negative integral coefficients, symmetric with respect to the substitution $q \to q^{-1}$. Moreover,

$$\deg \widetilde{C}'_{ij}(q) < \deg d(q), \qquad i, j \in I.$$

Proof. We write here the minimal choice of $d(q)$, which we use in Sect. 3.2:

$$\begin{aligned} A_\ell &: \ d(q) = q^\ell + q^{\ell-2} + \cdots + q^{-\ell}, \\ B_\ell &: \ d(q) = q^{2\ell-1} + q^{2\ell-3} + \cdots + q^{-2\ell-1}, \\ C_\ell &: \ d(q) = q^{\ell+1} + q^{-\ell-1}, \\ D_\ell &: \ d(q) = (q + q^{-1})(q^{\ell-1} + q^{-\ell+1}), \\ E_6 &: \ d(q) = (q^2 + 1 + q^{-2})(q^6 + q^{-6}), \\ E_7 &: \ d(q) = (q + q^{-1})(q^9 + q^{-9}), \\ E_8 &: \ d(q) = (q + q^{-1})(q^{15} + q^{-15}), \\ F_4 &: \ d(q) = q^9 + q^{-9}, \\ G_2 &: \ d(q) = q^6 + q^{-6}. \end{aligned}$$

For Lie algebras of classical series, the statement of the lemma with the above $d(q)$ follows from the explicit formulas for the entries $\widetilde{C}_{ij}(q)$ of the matrix $\widetilde{C}(q)$ given in Appendix C of [FR1]. For exceptional types, the lemma follows from a case by case inspection of the matrix $\widetilde{C}(q)$. $\square$

1.2. Quantum affine algebras. The quantum affine algebra $U_q\widehat{\mathfrak{g}}$ in the Drinfeld–Jimbo realization [Dr1,J] is an associative algebra over $\mathbb{C}$ with generators $x_i^{\pm}$, $k_i^{\pm 1}$ ($i = 0, \dots, \ell$), and relations:

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i k_j = k_j k_i,$$
$$k_i x_j^{\pm} k_i^{-1} = q^{\pm B_{ij}} x_j^{\pm},$$
$$[x_i^+, x_j^-] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{r=0}^{1-C_{ij}} (-1)^r \begin{bmatrix} 1 - C_{ij} \\ r \end{bmatrix}_{q_i} (x_i^{\pm})^r x_j^{\pm} (x_i^{\pm})^{1-C_{ij}-r} = 0, \quad i \neq j.$$

Here $(C_{ij})_{0 \le i,j \le \ell}$ denotes the Cartan matrix of $\widehat{\mathfrak{g}}$.

The algebra $U_q\widehat{\mathfrak{g}}$ has a structure of a Hopf algebra with the comultiplication Δ and the antipode S given on the generators by the formulas:

$$\Delta(k_i) = k_i \otimes k_i,$$
$$\Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+,$$
$$\Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-,$$

$$S(x_i^+) = -x_i^+ k_i, \qquad S(x_i^-) = -k_i^{-1} x_i^-, \qquad S(k_i^{\pm 1}) = k_i^{\mp 1}.$$

We define a $\mathbb{Z}$-gradation on $U_q\widehat{\mathfrak{g}}$ by setting: $\deg x_0^{\pm} = \pm 1$, $\deg x_i^{\pm} = \deg k_i = 0$, $i \in I = \{1, \dots, \ell\}$.

Denote the subalgebra of $U_q\widehat{\mathfrak{g}}$ generated by $k_i^{\pm 1}$, x_i^+ (resp., $k_i^{\pm 1}$, x_i^-), $i = 0, \dots, \ell$, by $U_q\mathfrak{b}_+$ (resp., $U_q\mathfrak{b}_-$).

The algebra $U_q\mathfrak{g}$ is defined as the subalgebra of $U_q\widehat{\mathfrak{g}}$ with generators $x_i^{\pm}$, $k_i^{\pm 1}$, where $i \in I$.

We will use Drinfeld's "new" realization of $U_q\widehat{\mathfrak{g}}$, see [Dr2], described by the following theorem.

Theorem 1.2 ([Dr2,KT,LSS,B]**).** *The algebra $U_q\widehat{\mathfrak{g}}$ has another realization as the algebra with generators $x_{i,n}^{\pm}$ ($i \in I$, $n \in \mathbb{Z}$), $k_i^{\pm 1}$ ($i \in I$), $h_{i,n}$ ($i \in I$, $n \in \mathbb{Z} \setminus 0$) and central elements $c^{\pm 1/2}$, with the following relations:*

$$k_i k_j = k_j k_i, \quad k_i h_{j,n} = h_{j,n} k_i,$$
$$k_i x_{j,n}^{\pm} k_i^{-1} = q^{\pm B_{ij}} x_{j,n}^{\pm},$$
$$[h_{i,n}, x_{j,m}^{\pm}] = \pm \frac{1}{n} [n B_{ij}]_q c^{\mp |n|/2} x_{j,n+m}^{\pm},$$
$$x_{i,n+1}^{\pm} x_{j,m}^{\pm} - q^{\pm B_{ij}} x_{j,m}^{\pm} x_{i,n+1}^{\pm} = q^{\pm B_{ij}} x_{i,n}^{\pm} x_{j,m+1}^{\pm} - x_{j,m+1}^{\pm} x_{i,n}^{\pm},$$
$$[h_{i,n}, h_{j,m}] = \delta_{n,-m} \frac{1}{n} [n B_{ij}]_q \frac{c^n - c^{-n}}{q - q^{-1}},$$
$$[x_{i,n}^+, x_{j,m}^-] = \delta_{ij} \frac{c^{(n-m)/2} \phi_{i,n+m}^+ - c^{-(n-m)/2} \phi_{i,n+m}^-}{q_i - q_i^{-1}},$$

$$\sum_{\pi\in\Sigma_s}\sum_{k=0}^{s}(-1)^k\begin{bmatrix} s\\ k\end{bmatrix}_{q_i} x^{\pm}_{i,n_{\pi(1)}}\dots x^{\pm}_{i,n_{\pi(k)}}x^{\pm}_{j,m}x^{\pm}_{i,n_{\pi(k+1)}}\dots x^{\pm}_{i,n_{\pi(s)}}=0,$$
$$s=1-C_{ij},$$

for all sequences of integers $n_1,\dots,n_s$, and $i\neq j$, where Σ_s is the symmetric group on s letters, and $\phi^{\pm}_{i,n}$'s are determined by the formula

$$\Phi^{\pm}_i(u):=\sum_{n=0}^{\infty}\phi^{\pm}_{i,\pm n}u^{\pm n}=k_i^{\pm 1}\exp\left(\pm(q-q^{-1})\sum_{m=1}^{\infty}h_{i,\pm m}u^{\pm m}\right). \tag{1.3}$$

For any $a\in\mathbb{C}^\times$, there is a Hopf algebra automorphism τ_a of $U_q\widehat{\mathfrak{g}}$ defined on the generators by the following formulas:

$$\begin{aligned}\tau_a(x^{\pm}_{i,n})&=a^n x^{\pm}_{i,n}, &\quad \tau_a(\phi^{\pm}_{i,n})&=a^n\phi^{\pm}_{i,n},\\ \tau_a(c^{1/2})&=c^{1/2}, &\quad \tau_a(k_i)&=k_i,\end{aligned} \tag{1.4}$$

for all $i\in I, n\in\mathbb{Z}$. Given a $U_q\widehat{\mathfrak{g}}$-module V and $a\in C^\times$, we denote by $V(a)$ the pull-back of V under τ_a.

Define new variables $\widetilde{k}_i^{\pm 1}$, $i\in I$, such that

$$k_j=\prod_{i\in I}\widetilde{k}_i^{C_{ij}},\qquad \widetilde{k}_i\widetilde{k}_j=\widetilde{k}_j\widetilde{k}_i. \tag{1.5}$$

Thus, while k_i corresponds to the simple root α_i, $\widetilde{k}_i$ corresponds to the fundamental weight ω_i. We extend the algebra $U_q\widehat{\mathfrak{g}}$ by replacing the generators $k_i^{\pm 1}, i\in I$ with $\widetilde{k}_i^{\pm 1}, i\in I$. From now on $U_q\widehat{\mathfrak{g}}$ will stand for the extended algebra.

Let $q^{2\rho}=\widetilde{k}_1^2\dots\widetilde{k}_\ell^2$. The square of the antipode acts as follows (see [Dr3]):

$$S^2(x)=\tau_{q^{-2r^\vee h^\vee}}(q^{-2\rho}xq^{2\rho}),\qquad \forall x\in U_q\widehat{\mathfrak{g}}. \tag{1.6}$$

Let w_0 be the longest element of the Weyl group of $\mathfrak{g}$. Let $i\to\bar{i}$ be the bijection $I\to I$, such that $w_0(\alpha_i)=-\alpha_{\bar{i}}$. Define the algebra automorphism $w_0:U_q\widehat{\mathfrak{g}}\to U_q\widehat{\mathfrak{g}}$ by

$$w_0(\widetilde{k}_i)=\widetilde{k}_{\bar{i}},\qquad w_0(h_{i,n})=h_{\bar{i},n},\qquad w_0(x^{\pm}_{i,n})=x^{\pm}_{\bar{i},n}. \tag{1.7}$$

We have: $w_0^2=\mathrm{Id}$. Actually, w_0 is a Hopf algebra automorphism, but we will not use this fact.

1.3. Finite-dimensional representations of $U_q\widehat{\mathfrak{g}}$. In this section we recall some of the results of Chari and Pressley [CP1, CP2, CP3, CP4] on the structure of finite-dimensional representations of $U_q\widehat{\mathfrak{g}}$.

Let P be the weight lattice of $\mathfrak{g}$. It is equipped with the standard *partial order*: the weight λ is higher than the weight μ if $\lambda-\mu$ can be written as a combination of the simple roots with positive integral coefficients.

A vector w in a $U_q\mathfrak{g}$-module W is called a vector of weight $\lambda\in P$, if

$$k_i\cdot w=q^{(\lambda,\alpha_i)}w,\qquad i\in I. \tag{1.8}$$

A representation W of $U_q\mathfrak{g}$ is said to be of type 1 if it is the direct sum of its weight spaces $W = \oplus_{\lambda \in P} W_\lambda$, where $W_\lambda = \{w \in W | k_i \cdot w = q^{(\lambda,\alpha_i)} w\}$. If $W_\lambda \neq 0$, then λ is called a weight of W.

A representation V of $U_q\widehat{\mathfrak{g}}$ is called of type 1 if $c^{1/2}$ acts as the identity on V, and if V is of type 1 as a representation of $U_q\mathfrak{g}$. According to [CP1], every finite-dimensional irreducible representation of $U_q\widehat{\mathfrak{g}}$ can be obtained from a type 1 representation by twisting with an automorphism of $U_q\widehat{\mathfrak{g}}$. Because of that, we will only consider type 1 representations in this paper.

A vector $v \in V$ is called a *highest weight vector* if

$$x^+_{i,n} \cdot v = 0, \qquad \phi^\pm_{i,n} \cdot v = \psi^\pm_{i,n} v, \qquad c^{1/2} v = v, \qquad \forall i \in I, n \in \mathbb{Z}, \tag{1.9}$$

for some complex numbers $\psi^\pm_{i,n}$. A type 1 representation V is a *highest weight representation* if $V = U_q\widehat{\mathfrak{g}} \cdot v$, for some highest weight vector v. In that case the set of generating functions

$$\Psi^\pm_i(u) = \sum_{n=0}^{\infty} \psi^\pm_{i,\pm n} u^{\pm n}, \qquad i \in I,$$

is called the *highest weight* of V.

Warning. The above notions of highest weight vector and highest weight representation are different from standard. Sometimes they are called pseudo-highest weight vector and pseudo-highest weight representation.

Let $\mathcal{P}$ be the set of all I-tuples $(P_i)_{i \in I}$ of polynomials $P_i \in \mathbb{C}[u]$, with constant term 1.

Theorem 1.3 ([CP1, CP3]**).**

(1) *Every finite-dimensional irreducible representation of $U_q\widehat{\mathfrak{g}}$ of type 1 is a highest weight representation.*

(2) *Let V be a finite-dimensional irreducible representation of $U_q\widehat{\mathfrak{g}}$ of type 1 and highest weight $(\Psi^\pm_i(u))_{i \in I}$. Then, there exists $\mathbf{P} = (P_i)_{i \in I} \in \mathcal{P}$ such that*

$$\Psi^\pm_i(u) = q_i^{\deg(P_i)} \frac{P_i(uq_i^{-1})}{P_i(uq_i)}, \tag{1.10}$$

as an element of $\mathbb{C}[[u^{\pm 1}]]$.

Assigning to V the I-tuple $\mathbf{P} \in \mathcal{P}$ defines a bijection between $\mathcal{P}$ and the set of isomorphism classes of finite-dimensional irreducible representations of $U_q\widehat{\mathfrak{g}}$ of type 1. The irreducible representation associated to $\mathbf{P}$ will be denoted by $V(\mathbf{P})$.

(3) *The highest weight of $V(\mathbf{P})$ considered as a $U_q\mathfrak{g}$-module is $\lambda = \sum_{i \in I} \deg P_i \cdot \omega_i$, the lowest weight of $V(\mathbf{P})$ is $\bar{\lambda} = -\sum_{i \in I} \deg P_i \cdot \omega_{\bar{i}}$, and each of them has multiplicity 1.*

(4) *If $\mathbf{P} = (P_i)_{i \in I} \in \mathcal{P}$, $a \in \mathbb{C}^\times$, and if $\tau_a^*(V(\mathbf{P}))$ denotes the pull-back of $V(\mathbf{P})$ by the automorphism τ_a, we have $\tau_a^*(V(\mathbf{P})) \cong V(\mathbf{P}^a)$ as representations of $U_q\widehat{\mathfrak{g}}$, where $\mathbf{P}^a = (P_i^a)_{i \in I}$ and $P_i^a(u) = P_i(ua)$.*

(5) *For $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$ denote by $\mathbf{P} \otimes \mathbf{Q} \in \mathcal{P}$ the I-tuple $(P_i Q_i)_{i \in I}$. Then $V(\mathbf{P} \otimes \mathbf{Q})$ is isomorphic to a quotient of the subrepresentation of $V(\mathbf{P}) \otimes V(\mathbf{Q})$ generated by the tensor product of the highest weight vectors.*

An analogous classification result for Yangians has been obtained earlier by Drinfeld [Dr2]. Because of that, the polynomials $P_i(u)$ are called Drinfeld polynomials.

Note that in our notation the polynomials $P_i(u)$ correspond to the polynomials $P_i(uq_i^{-1})$ in the notation of [CP1,CP3].

For each $i \in I$ and $a \in \mathbb{C}^\times$, define the irreducible representation $V_{\omega_i}(a)$ as $V(\mathbf{P}_a^{(i)})$, where $\mathbf{P}_a^{(i)}$ is the I-tuple of polynomials, such that $P_i(u) = 1 - ua$ and $P_j(u) = 1, \forall j \neq i$. We call $V_{\omega_i}(a)$ the i^{th} *fundamental representation* of $U_q\widehat{\mathfrak{g}}$. Note that in general $V_{\omega_i}(a)$ is reducible as a $U_q\mathfrak{g}$-module.

Theorem 1.3 implies the following

Corollary 1.4 ([CP3]**).** *Any irreducible finite-dimensional representation V of $U_q\widehat{\mathfrak{g}}$ occurs as a quotient of the submodule of the tensor product $V_{\omega_{i_1}}(a_1) \otimes \ldots \otimes V_{\omega_{i_n}}(a_n)$, generated by the tensor product of the highest weight vectors. The parameters (ω_{i_k}, a_k), $k = 1, \ldots, n$, are uniquely determined by V up to permutation.*

2. Definition and First Properties of q-Characters

2.1. Definition of q-characters. Let us recall the definition of the q-characters of finite-dimensional representations of $U_q\widehat{\mathfrak{g}}$ from [FR2].

The completed tensor product $U_q\widehat{\mathfrak{g}} \mathbin{\widehat{\otimes}} U_q\widehat{\mathfrak{g}}$ contains a special element $\mathcal{R}$ called the universal R-matrix (at level 0). It actually lies in $U_q\mathfrak{b}_+ \mathbin{\widehat{\otimes}} U_q\mathfrak{b}_-$ and satisfies the following identities:

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}, \qquad \forall x \in U_q\widehat{\mathfrak{g}},$$
$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{23}, \qquad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}^{13}\mathcal{R}^{12}.$$

For more details, see [Dr3, EFK].

Now let (V, π_V) be a finite-dimensional representation of $U_q\widehat{\mathfrak{g}}$. Define the transfer-matrix corresponding to V by

$$t_V = t_V(z) = \mathrm{Tr}_V\ (\pi_{V(z)} \otimes \mathrm{id})(\mathcal{R}). \tag{2.1}$$

Thus we obtain a map $\nu_q : \mathrm{Rep}\, U_q\widehat{\mathfrak{g}} \to U_q\mathfrak{b}_-[[z]]$, sending V to $t_V(z)$.

Remark 2.1. Note that in [FR2] there was an extra factor $q^{2\rho}$ in formula (2.1). This factor is inessential for the purposes of this paper, and therefore can be dropped.

Denote by $U_q\widetilde{\mathfrak{g}}$ the subalgebra of $U_q\widehat{\mathfrak{g}}$ generated by $x^\pm_{i,n}, \widetilde{k}_i, h_{i,r}, n \leq 0, r < 0, i \in I$. It follows from the proof of Theorem 1.2 that $U_q\mathfrak{b}_- \subset U_q\widetilde{\mathfrak{g}}$. As a vector space, $U_q\widetilde{\mathfrak{g}}$ can be decomposed as follows: $U_q\widetilde{\mathfrak{g}} = U_q\widetilde{\mathfrak{n}}_- \otimes U_q\widetilde{\mathfrak{h}} \otimes U_q\widetilde{\mathfrak{n}}_+$, where $U_q\widetilde{\mathfrak{n}}_\pm$ (resp., $U_q\widetilde{\mathfrak{h}}$) is generated by $x^\pm_{i,n}, i \in I, n \leq 0$ (resp., $\widetilde{k}_i, h_{i,n}, i \in I, n < 0$). Hence

$$U_q\widetilde{\mathfrak{g}} = U_q\widetilde{\mathfrak{h}} \oplus \left(U_q\widetilde{\mathfrak{g}} \cdot (U_q\widetilde{\mathfrak{n}}_+)_0 + (U_q\widetilde{\mathfrak{n}}_-)_0 \cdot U_q\widetilde{\mathfrak{g}}\right),$$

where $(U_q\widetilde{\mathfrak{n}}_\pm)_0$ stands for the augmentation ideal of $U_q\widetilde{\mathfrak{n}}_\pm$. Denote by $\mathbf{h}_q$ the projection $U_q\widetilde{\mathfrak{g}} \to U_q\widetilde{\mathfrak{h}}$ along the last two summands (this is an analogue of the Harish-Chandra homomorphism). We denote by the same letter its restriction to $U_q\mathfrak{b}_-$.

Now we define the map $\chi_q : \mathrm{Rep}\, U_q\widehat{\mathfrak{g}} \to U_q\widetilde{\mathfrak{h}}[[z]]$ as the composition of ν_q : $\mathrm{Rep}\, U_q\widehat{\mathfrak{g}} \to U_q\mathfrak{b}_-[[z]]$ and $\mathbf{h}_q[[z]] : U_q\mathfrak{b}_-[[z]] \to U_q\widetilde{\mathfrak{h}}[[z]]$.

To describe the image of χ_q we need to introduce some more notation.
Let

$$\widetilde{h}_{i,m} = \sum_{j \in I} \widetilde{C}_{ji}(q^m) h_{j,m}, \tag{2.2}$$

where $\widetilde{C}(q)$ is the inverse matrix to $C(q)$ defined in Sect. 1.1. Set

$$Y_{i,a} = \widetilde{k}_i^{-1} \exp\left(-(q - q^{-1}) \sum_{n>0} \widetilde{h}_{i,-n} z^n a^n \right), \qquad a \in \mathbb{C}^\times. \tag{2.3}$$

We assign to $Y_{i,a}^{\pm 1}$ the weight $\pm\omega_i$.

We have the ordinary character homomorphism $\chi : \operatorname{Rep} U_q\mathfrak{g} \to \mathbb{Z}[y_i^{\pm 1}]_{i \in I}$: if $V = \oplus_\mu V_\mu$ is the weight decomposition of V, then $\chi(V) = \sum_\mu \dim V_\mu \cdot y^\mu$, where for $\mu = \sum_{i \in I} m_i \omega_i$ we set $y^\mu = \prod_{i \in I} y_i^{m_i}$. Define the homomorphism

$$\beta : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times} \to \mathbb{Z}[y_i^{\pm 1}]_{i \in I}$$

sending $Y_{i,a}^{\pm 1}$ to $y_i^{\pm 1}$, and denote by

$$\operatorname{res} : \operatorname{Rep} U_q\widehat{\mathfrak{g}} \to \operatorname{Rep} U_q\mathfrak{g}$$

the restriction homomorphism.

Given a polynomial ring $\mathbb{Z}[x_\alpha^{\pm 1}]_{\alpha \in A}$, we denote by $\mathbb{Z}_+[x_\alpha^{\pm 1}]_{\alpha \in A}$ its subset consisting of all linear combinations of monomials in $x_\alpha^{\pm 1}$ with positive integral coefficients.

Theorem 2.2 ([FR2]**).**

(1) χ_q *is an injective homomorphism from* $\operatorname{Rep} U_q\widehat{\mathfrak{g}}$ *to* $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times} \subset U_q\widetilde{\mathfrak{h}}[[z]]$.

(2) *For any finite-dimensional representation* V *of* $U_q\widehat{\mathfrak{g}}$, $\chi_q(V) \in \mathbb{Z}_+[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times}$.

(3) *The diagram*

$$\begin{array}{ccc} \operatorname{Rep} U_q\widehat{\mathfrak{g}} & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times} \\ \downarrow \operatorname{res} & & \downarrow \beta \\ \operatorname{Rep} U_q\mathfrak{g} & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm 1}]_{i \in I} \end{array}$$

is commutative.

(4) $\operatorname{Rep} U_q\widehat{\mathfrak{g}}$ *is a commutative ring that is isomorphic to* $\mathbb{Z}[t_{i,a}]_{i \in I; a \in \mathbb{C}^\times}$, *where* $t_{i,a}$ *is the class of* $V_{\omega_i}(a)$.

The homomorphism

$$\chi_q : \operatorname{Rep} U_q\widehat{\mathfrak{g}} \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times}$$

is called the *q-character homomorphism*. For a finite-dimensional representation V of $U_q\widehat{\mathfrak{g}}$, $\chi_q(V)$ is called the *q-character* of V.

2.2. Spectra of $\Phi^\pm(u)$. According to Theorem 2.2(1), the q-character of any finite-dimensional representation V of $U_q\widehat{\mathfrak{g}}$ is a linear combination of monomials in $Y_{i,a}^{\pm 1}$ with positive integral coefficients. The proof of Theorem 2.2 from [FR2] allows us to relate the monomials appearing in $\chi_q(V)$ to the spectra of the operators $\Phi_i^\pm(u)$ on V as follows.

It follows from the defining relations that the operators $\phi_{i,n}^\pm$ commute with each other. Hence we can decompose any representation V of $U_q\widehat{\mathfrak{g}}$ into a direct sum $V = \oplus V_{(\gamma_{i,n}^\pm)}$ of generalized eigenspaces

$$V_{(\gamma_{i,n}^\pm)} = \{x \in V \,|\, \text{there exists } p, \text{ such that } (\phi_{i,n}^\pm - \gamma_{i,n}^\pm)^p \cdot x = 0, \forall i \in I, n \in \mathbb{Z}\}.$$

Since $\phi_0^\pm = k_i^{\pm 1}$, all vectors in $V_{(\gamma_{i,n}^\pm)}$ have the same weight (see formula (1.8) for the definition of weight). Therefore the decomposition of V into a direct sum of subspaces $V_{(\gamma_{i,n}^\pm)}$ is a refinement of its weight decomposition.

Given a collection $(\gamma_{i,n}^\pm)$ of generalized eigenvalues, we form the generating functions

$$\Gamma_i^\pm(u) = \sum_{n \geq 0} \gamma_{i,\pm n}^\pm u^{\pm n}.$$

We will refer to each collection $\{\Gamma_i^\pm(u)\}_{i\in I}$ occurring on a given representation V as the *common (generalized) eigenvalues* of $\Phi_i^\pm(u), i \in I$, on V, and to dim $V_{(\gamma_{i,n}^\pm)}$ as the *multiplicity* of this eigenvalue.

Let $\mathfrak{B}_V$ be a Jordan basis of $\phi_{i,n}^\pm, i \in I, n \in \mathbb{Z}$. Consider the module $V(z) = \tau_z^*(V)$, see formula (1.4). Then $V(z) = V$ as a vector space. Moreover, the decomposition in the direct sum of generalized eigenspaces of operators $\phi_{i,n}^\pm$ does not depend on z, because the action of $\phi_{i,n}^\pm$ on V and on $V(z)$ differs only by scalar factors z^n. In particular, $\mathfrak{B}_V$ is also a Jordan basis for $\phi_{i,n}^\pm$ acting on $V(z)$ for all $z \in \mathbb{C}^\times$. If $v \in \mathfrak{B}_V$ is a generalized eigenvector with common eigenvalues $\{\Gamma_i^\pm(u)\}_{i\in I}$, then the corresponding common eigenvalues on v in $V(z)$ are $\{\Gamma_i^\pm(zu)\}_{i\in I}$

The following result is a generalization of Theorem 1.3.

Proposition 2.3 ([FR2]**).** *The eigenvalues* $\Gamma_i^\pm(u)$ *of* $\Phi_i^\pm(u)$ *on any finite-dimensional representation of* $U_q\widehat{\mathfrak{g}}$ *have the form:*

$$\Gamma_i^\pm(u) = q_i^{\deg Q_i - \deg R_i} \frac{Q_i(uq_i^{-1})R_i(uq_i)}{Q_i(uq_i)R_i(uq_i^{-1})}, \tag{2.4}$$

as elements of $\mathbb{C}[[u^{\pm 1}]]$*, where* $Q_i(u)$, $R_i(u)$ *are polynomials in* u *with constant term* 1.

Now we can relate the monomials appearing in $\chi_q(V)$ to the common eigenvalues of $\Phi_i^\pm(u)$ on V.

Proposition 2.4. *Let* V *be a finite-dimensional* $U_q\widehat{\mathfrak{g}}$*-module. There is a one-to-one correspondence between the monomials occurring in* $\chi_q(V)$ *and the common eigenvalues of* $\Phi_i^\pm(u), i \in I$*, on* V*. Namely, the monomial*

$$\prod_{i\in I}\left(\prod_{r=1}^{k_i} Y_{i,a_{ir}} \prod_{s=1}^{l_i} Y_{i,b_{is}}^{-1}\right) \tag{2.5}$$

corresponds to the common eigenvalues (2.4)*, where*

$$Q_i(z) = \prod_{r=1}^{k_i}(1 - za_{ir}), \qquad R_i(z) = \prod_{s=1}^{l_i}(1 - zb_{is}), \qquad i \in I. \tag{2.6}$$

The weight of each monomial equals the weight of the corresponding generalized eigenspace. Moreover, the coefficient of each monomial in $\chi_q(V)$ equals the multiplicity of the corresponding common eigenvalue.

Proof. Denote by $U_q\widehat{\mathfrak{n}}_\pm$ the subalgebra of $U_q\widehat{\mathfrak{g}}$ generated by $x^\pm_{i,n}$, $i \in I, n \in \mathbb{Z}$. Let $\widetilde{B}(q)$ be the inverse matrix to $B(q)$ from Sect. 1.1. The following formula for the universal R-matrix has been proved in [KT, LSS, Da]:

$$\mathcal{R} = \mathcal{R}^+\mathcal{R}^0\mathcal{R}^-T, \tag{2.7}$$

where

$$\mathcal{R}^0 = \exp\left(-\sum_{n>0}\sum_{i\in I}\frac{n(q-q^{-1})^2}{q_i^n - q_i^{-n}}h_{i,n}\otimes\widetilde{h}_{i,-n}z^n\right) \tag{2.8}$$

(here we use the notation (2.2)), $\mathcal{R}^\pm \in U_q\widehat{\mathfrak{n}}_\pm \otimes U_q\widetilde{\mathfrak{n}}_\mp$, and T acts as follows: if x, y satisfy $k_i \cdot x = q^{(\lambda,\alpha_i)}x$, $k_i \cdot y = q^{(\mu,\alpha_i)}y$, then

$$T \cdot x \otimes y = q^{-(\lambda,\mu)}x \otimes y. \tag{2.9}$$

By definition, $\chi_q(V)$ is obtained by taking the trace of $(\pi_{V(z)} \otimes \mathrm{id})(\mathcal{R})$ over V and then projecting it on $U_q\widetilde{\mathfrak{h}}[[z]]$ using the projection operator $\mathbf{h}_q$. This projection eliminates the factor $\mathcal{R}^-$, and then taking the trace eliminates $\mathcal{R}^+$ (recall that $U_q\widetilde{\mathfrak{n}}_+$ acts nilpotently on V). Hence we obtain:

$$\chi_q(V) = \mathrm{Tr}_V\left[\exp\left(-\sum_{n>0}\sum_{i\in I}\frac{n(q-q^{-1})^2}{q_i^n - q_i^{-n}}\pi_V(h_{i,n})\otimes\widetilde{h}_{i,-n}z^n\right)(\pi_V\otimes 1)T\right]. \tag{2.10}$$

The trace can be written as the sum of terms m_v corresponding to the (generalized) eigenvalues of $h_{i,n}$ on the vectors v of the Jordan basis $\mathfrak{B}_V$ of V for the operators $\phi^\pm_{i,n}$ (and hence for $h_{i,n}$).

The eigenvalues of $\Phi^\pm_i(u)$ on each vector $v \in \mathfrak{B}_V$ are given by formula (2.4). Suppose that $Q_i(u)$ and $R_i(u)$ are given by formula (2.6). Then the eigenvalue of $h_{i,n}$ on v equals

$$\frac{q_i^n - q_i^{-n}}{n(q-q^{-1})}\left(\sum_{r=1}^{k_i}(a_{ir})^n - \sum_{s=1}^{l_i}(b_{is})^n\right), \qquad n > 0. \tag{2.11}$$

Substituting into formula (2.10) and recalling the definition (2.3) of $Y_{i,a}$ we obtain that the corresponding term m_v in $\chi_q(V)$ is the monomial (2.5). □

Let $V = V(\mathbf{P})$, where

$$P_i(u) = \prod_{k=1}^{n_i} (1 - ua_k^{(i)}), \qquad i \in I. \tag{2.12}$$

Then by Theorem 1.3(3), the module V has highest weight $\lambda = \sum_{i \in I} \deg P_i \cdot \omega_i$, which has multiplicity 1. Proposition 2.4 implies that $\chi_q(V)$ contains a unique monomial of weight λ. This monomial equals

$$\prod_{i \in I} \prod_{k=1}^{n_i} Y_{i,a_k^{(i)}}. \tag{2.13}$$

We call it the *highest weight monomial* of V. All other monomials in $\chi_q(V)$ have lower weight than λ.

A monomial in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ is called *dominant* if it does not contain factors $Y_{i,a}^{-1}$ (i.e., if it is a product of $Y_{i,a}$'s in positive powers only). The highest weight monomial is dominant, but in general the highest weight monomial is not the only dominant monomial occurring in $\chi_q(V)$. Nevertheless, we prove below in Corollary 4.5 that the only dominant monomial contained in the q-character of a fundamental representation $V_{\omega_i}(a)$ is its highest weight monomial $Y_{i,a}$.

Note that a dominant monomial has dominant weight but not all monomials of dominant weight are dominant.

Similarly, a monomial in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ is called *antidominant* if it does not contain factors $Y_{i,a}$ (i.e., if it is a product of $Y_{i,a}^{-1}$'s in negative powers only). The roles of dominant and antidominant monomials are similar, see, e.g., Remark 6.19. By Corollary 6.9, the lowest weight monomial is antidominant.

Remark 2.5. The statement analogous to Proposition 2.3 in the case of the Yangians has been proved by Knight [Kn]. Using this statement, he introduced the notion of character of a representation of Yangian. □

2.3. Connection with the entries of the R-matrix. We already described the q-character of $U_q\widehat{\mathfrak{g}}$ module V in terms of universal R-matrix and in terms of generalized eigenvalues of operators $\phi_{i,n}^{\pm}$. It allows us to describe the q-character of V in terms of diagonal entries of R-matrices acting on the tensor products $V \otimes V_{\omega_i}(a)$ with fundamental representations. We will use this description in Sect. 6.

Define

$$A_{i,a} = k_i^{-1} \exp\left(-(q - q^{-1}) \sum_{n>0} h_{i,-n} z^n a^n \right), \qquad a \in \mathbb{C}^\times. \tag{2.14}$$

Using formula (2.2), we can express $A_{i,a}$ in terms of $Y_{j,b}$'s:

$$A_{i,a} = Y_{i,aq_i} Y_{i,aq_i^{-1}} \prod_{C_{ji}=-1} Y_{j,a}^{-1} \prod_{C_{ji}=-2} Y_{j,aq}^{-1} Y_{j,aq^{-1}}^{-1} \prod_{C_{ji}=-3} Y_{j,aq^2}^{-1} Y_{j,a}^{-1} Y_{j,aq^{-2}}^{-1}. \tag{2.15}$$

Thus, $A_{i,a} \in \mathbb{Z}[Y_{j,b}^{\pm 1}]_{j \in I; b \in \mathbb{C}^\times}$, and the weight of $A_{i,a}$ equals α_i.

Let V and W be irreducible finite-dimensional representations of $U_q\widehat{\mathfrak{g}}$ with highest weight vectors v and w. Let $\overline{R}_{VW}(z) \in \mathrm{End}(V \otimes W)$ be the normalized R-matrix,

$$\overline{R}_{VW}(z) = f_{VW}^{-1}(z)(\pi_{V(z)} \otimes \pi_W)(\mathcal{R}),$$

where $f_{VW}(z)$ is the scalar function, such that

$$\overline{R}_{VW}(z)(v \otimes w) = w \otimes v. \tag{2.16}$$

In what follows we always consider the normalized R-matrix $\overline{R}_{VW}(z)$ written in the basis $\mathfrak{B}_V \otimes \mathfrak{B}_W$.

Recall the definition of the fundamental representation $V_{\omega_i}(a)$ from Sect. 1.3. Denote its highest weight vector by v_{ω_i}.

Lemma 2.6. *Let $v \in \mathfrak{B}_V$ and suppose that the corresponding monomial m_v in $\chi_q(V)$ is given by*

$$m_v = m_+ M \prod_k A_{i,a_k}^{-1}, \tag{2.17}$$

where M is a product of factors $A_{j,b}^{-1}$, $b \in \mathbb{C}^\times$, $j \in I$, $j \neq i$. Then the diagonal entry of the normalized R-matrix $\overline{R}_{V,V_{\omega_i}(b)}(z)$ corresponding to the vector $v \otimes v_{\omega_i}$ is

$$\left(\overline{R}_{V,V_{\omega_i}(b)}(z)\right)_{v\otimes v_{\omega_i}, v\otimes v_{\omega_i}} = \prod_k q_i \frac{1 - a_k z b^{-1} q_i^{-1}}{1 - a_k z b^{-1} q_i}. \tag{2.18}$$

Proof. Recall formula (2.7) for $\mathcal{R}$. We have: $\mathcal{R}^-(v \otimes v_{\omega_i}) = 0$; $v \otimes v_{\omega_i}$ is a generalized eigenvector of $\mathcal{R}^0$; and $\mathcal{R}^+(v \otimes v_{\omega_i})$ is a linear combination of tensor products $x \otimes y \in \mathfrak{B}_V \otimes \mathfrak{B}_{V_{\omega_i}(b)}$, where y has a lower weight than v_{ω_i}. Therefore the diagonal matrix element of $\mathcal{R}$ on $v \otimes v_{\omega_i} \in V(z) \otimes V_{\omega_i}(b)$ equals the generalized eigenvalue of $(\pi_{V(z)} \otimes \pi_{V_{\omega_i}(b)})(\mathcal{R}^0)$ on $v \otimes v_{\omega_i}$.

On the other hand, as explained in the proof of Proposition 2.4, the monomial m_v is equal to the diagonal matrix element of $(\pi_{V(z)} \otimes 1)(\mathcal{R}^0)$ corresponding to v. Therefore the diagonal matrix element of $\mathcal{R}$ corresponding to $v \otimes v_{\omega_i}$ equals the eigenvalue of m_v (considered as an element of $U_q\widetilde{\mathfrak{h}}[[z]]$) on v_{ω_i}.

In particular, if v is the highest weight vector, then the corresponding monomial m_v is the highest weight monomial m_+. Therefore we find that the diagonal matrix element of the non-normalized R-matrix corresponding to $v \otimes v_{\omega_i}$ equals the eigenvalue of m_+ on v_{ω_i}. By formula (2.16) the diagonal matrix element of the normalized R-matrix equals 1. Therefore the eigenvalue of m_+ on v_{ω_i} equals the scalar function $f_{V,V_{\omega_i}(b)}(z)$. Therefore we obtain that the diagonal matrix element of the normalized R-matrix $\overline{R}_{V,V_{\omega_i}(b)}(z)$ corresponding to the vector $v \otimes v_{\omega_i}$ is equal to the eigenvalue of $m_v m_+^{-1}$ on v_{ω_i}. According to formula (2.14), $A_{i,a} = \Phi_i^-(z^{-1}a^{-1})$. Therefore, if m_v is given by formula (2.17), we obtain from formula (1.10) that this matrix element is given by formula (2.18). □

Note that by Theorem 4.1 below every monomial occurring in the q-character of an irreducible representation V can be written in the form (2.17).

3. The Homomorphisms τ_J and Restrictions

3.1. Restriction to $U_q\widehat{\mathfrak{g}}_J$. Given a subset J of I, we denote by $U_q\widehat{\mathfrak{g}}_J$ the subalgebra of $U_q\widehat{\mathfrak{g}}$ generated by $x^{\pm}_{i,n}, \widetilde{k}_i^{\pm 1}, h_{i,r}, i \in J, n \in \mathbb{Z}, r \in \mathbb{Z}\backslash 0$. Let

$$\mathrm{res}_J : \mathrm{Rep}\, U_q\widehat{\mathfrak{g}} \to \mathrm{Rep}\, U_q\widehat{\mathfrak{g}}_J$$

be the restriction map and β_J be the homomorphism

$$\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times} \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in J; a\in\mathbb{C}^\times},$$

sending $Y_{i,a}^{\pm 1}$ to itself for $i \in J$ and to 1 for $i \notin J$.

According to Theorem 3(3) of [FR2], the diagram

$$\begin{array}{ccc} \mathrm{Rep}\, U_q\widehat{\mathfrak{g}} & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times} \\ \big\downarrow \mathrm{res}_J & & \big\downarrow \beta_J \\ \mathrm{Rep}\, U_q\widehat{\mathfrak{g}}_J & \xrightarrow{\chi_{q,J}} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in J; a\in\mathbb{C}^\times} \end{array}$$

is commutative.

We will now refine the homomorphisms β_J and res_J.

3.2. The homomorphism τ_J. Consider the elements $\widetilde{h}_{i,n}$ defined by formula (2.2) and $\widetilde{k}_i^{\pm 1}$ defined by formula (1.5).

Lemma 3.1.

$$\widetilde{k}_i x^{\pm}_{j,n} \widetilde{k}_i^{-1} = q^{\pm r_i \delta_{ij}} x^{\pm}_{j,n},$$

$$[\widetilde{h}_{i,n}, x^{\pm}_{j,m}] = \pm\delta_{ij} \frac{[nr_i]_q}{n} c^{\mp|n|/2} x^{\pm}_{j,n+m},$$

$$[\widetilde{h}_{i,n}, h_{j,m}] = \delta_{i,j}\delta_{n,-m} \frac{[nr_i]_q}{n} \frac{c^n - c^{-n}}{q - q^{-1}}.$$

In particular, $\widetilde{k}_i^{\pm 1}, \widetilde{h}_{i,n}, i \in \overline{J}, n \in \mathbb{Z}\backslash 0$, where $\overline{J} = I - J$, commute with the subalgebra $U_q\widehat{\mathfrak{g}}_J$ of $U_q\widehat{\mathfrak{g}}$.

Proof. These formulas follow from the relations given in Theorem 1.2 and the formula $B(q)\widetilde{C}(q) = D(q)$. □

Denote by $U_q\widehat{\mathfrak{h}}_J^{\perp}$ the subalgebra of $U_q\widehat{\mathfrak{g}}$ generated by $\widetilde{k}_i^{\pm 1}, \widetilde{h}_{i,n}, i \in \overline{J}, n \in \mathbb{Z}\backslash 0$. Then $U_q\widehat{\mathfrak{g}}_J \otimes U_q\widehat{\mathfrak{h}}_J^{\perp}$ is naturally a subalgebra of $U_q\widehat{\mathfrak{g}}$. We can therefore refine the restriction from $U_q\widehat{\mathfrak{g}}$-modules to $U_q\widehat{\mathfrak{g}}_J$-modules by considering the restriction from $U_q\widehat{\mathfrak{g}}$-modules to $U_q\widehat{\mathfrak{g}}_J \otimes U_q\widehat{\mathfrak{h}}_J^{\perp}$-modules.

Thus, we look at the common (generalized) eigenvalues of the operators $k_i^{\pm 1}, h_{i,n}, i \in J$, and $\widetilde{k}_i^{\pm 1}, \widetilde{h}_{i,n}, i \in \overline{J}$. We know that the eigenvalues of $h_{i,n}$ have the form (2.11). The corresponding eigenvalue of $\widetilde{h}_{i,n}$ equals

$$\frac{[n]_q}{n} \sum_{j\in I} \widetilde{C}_{ji}(q^n)[r_j]_{q^n} \left(\sum_{r=1}^{k_j} (a_{jr})^n - \sum_{s=1}^{l_j} (b_{js})^n \right), \qquad n > 0. \tag{3.1}$$

According to Lemma 1.1, $\widetilde{C}_{ji}(x) = \widetilde{C}'_{ji}(x)/d(x)$, where $\widetilde{C}'_{ji}(x)$ and $d(x)$ are certain polynomials with positive integral coefficients (we fix a choice of such $d(x)$ once and for all). Therefore formula (3.1) can be rewritten as

$$\frac{[n]_q}{nd(q^n)} \left(\sum_{m=1}^{u_i} (c_{im})^n - \sum_{p=1}^{t_i} (d_{ip})^n \right), \tag{3.2}$$

where c_{im} and d_{ip} are certain complex numbers (they are obtained by multiplying a_{jr} and b_{js} with all monomials appearing in $\widetilde{C}'_{ji}(q)[r_j]_q$).

According to Proposition 2.4, to each monomial (2.5) in $\chi_q(V)$ corresponds a generalized eigenspace of $h_{i,n}, i \in I, n \in \mathbb{Z} \setminus 0$, with the common eigenvalues given by formula (2.11) (note that the eigenvalues of k_i, $i \in I$, can be read off from the weight of the monomial). Using formula (3.1) we find the corresponding eigenvalues of $\widetilde{h}_{i,n}, i \in \overline{J}$ in the form (3.2). Now we attach to these common eigenvalues the following monomial in the letters $Y^{\pm 1}_{i,a}$, $i \in J$, and $Z^{\pm 1}_{j,c}$, $j \in \overline{J}$:

$$\left(\prod_{i \in J} \prod_{r=1}^{k_i} Y_{i,a_{ir}} \prod_{s=1}^{l_i} Y^{-1}_{i,b_{is}} \right) \cdot \left(\prod_{k \in \overline{J}} \prod_{m=1}^{u_k} Z_{k,c_{km}} \prod_{p=1}^{t_k} Z^{-1}_{k,d_{kp}} \right).$$

The above procedure can be interpreted as follows. Introduce the notation

$$\mathcal{Y} = \mathbb{Z}[Y^{\pm 1}_{i,a}]_{i \in I, a \in \mathbb{C}^\times}, \tag{3.3}$$

$$\mathcal{Y}^{(J)} = \mathbb{Z}[Y^{\pm 1}_{i,a}]_{i \in J, a \in \mathbb{C}^\times} \otimes \mathbb{Z}[Z^{\pm 1}_{k,c}]_{k \in \overline{J}, c \in \mathbb{C}^\times}. \tag{3.4}$$

Write

$$(D(q)\widetilde{C}'(q))_{ij} = \sum_{k \in \mathbb{Z}} p_{ij}(k) q^k.$$

Definition 3.2. *The homomorphism $\tau_J : \mathcal{Y} \to \mathcal{Y}^{(J)}$ is defined by the formulas*

$$\tau_J(Y_{i,a}) = Y_{i,a} \cdot \prod_{j \in \overline{J}} \prod_{k \in \mathbb{Z}} Z^{p_{ij}(k)}_{j,aq^k}, \qquad i \in J, \tag{3.5}$$

$$\tau_J(Y_{i,a}) = \prod_{j \in \overline{J}} \prod_{k \in \mathbb{Z}} Z^{p_{ij}(k)}_{j,aq^k}, \qquad i \in \overline{J}. \tag{3.6}$$

Observe that the homomorphism β_J can be represented as the composition of τ_J and the homomorphism $\mathcal{Y}^{(J)} \to \mathbb{Z}[Y^{\pm 1}_{i,a}]_{i \in J, a \in \mathbb{C}^\times}$ sending all $Z_{k,c}, k \in \overline{J}$, to 1. Therefore τ_J is indeed a refinement of τ_J, and so the restriction of τ_J to the image of $\operatorname{Rep} U_q\widehat{\mathfrak{g}}$ in $\mathcal{Y}$ is a refinement of the restriction homomorphism res_J.

3.3. Properties of τ_J. The main advantage of τ_J over β_J is the following.

Lemma 3.3. *The homomorphism τ_J is injective.*

Proof. The statement of the lemma follows from the fact that the matrix $\widetilde{C}'(q)$ is non-degenerate. $\square$

Lemma 3.4. *Let us write $\chi_q(V)$ as the sum $\sum_k P_k Q_k$, where $P_k \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in J, a \in \mathbb{C}^\times}$, Q_k is a monomial in $\mathbb{Z}[Z_{j,c}^{\pm 1}]_{j \in \overline{J}, c \in \mathbb{C}^\times}$, and all monomials Q_k are distinct. Then the restriction of V to $U_q\widehat{\mathfrak{g}}_J$ is isomorphic to $\oplus_k V_k$, where V_k's are $U_q\widehat{\mathfrak{g}}_J$-modules with $\chi_q^J(V_k) = P_k$. In particular, there are no extensions between different V_k's in V.*

Proof. The monomials in $\chi_q(V) \in \mathfrak{Y}$ encode the common eigenvalues of $h_{i,n}, i \in I$ on V. It follows from Sect. 3.2 that the monomials in $\tau_J(\chi_q(V))$ encode the common eigenvalues of $h_{i,n}, i \in J$, and $\widetilde{h}_{j,n}, j \in \overline{J}$, on V.

Therefore we obtain that the restriction of V to $U_q\widehat{\mathfrak{g}}_J \otimes U_q\widehat{\mathfrak{h}}_J^\perp$ has a filtration with the associated graded factors $V_k \otimes W_k$, where V_k is a $U_q\widehat{\mathfrak{g}}_J$-module with $\chi_q^J(V_k) = P_k$, and W_k is a one-dimensional $U_q\widehat{\mathfrak{h}}_J^\perp$-module, which corresponds to Q_k. By our assumption, the modules W_k over $U_q\widehat{\mathfrak{h}}_J^\perp$ are pairwise distinct. Because $U_q\widehat{\mathfrak{h}}_J^\perp$ commutes with $U_q\widehat{\mathfrak{g}}_J$, there are no extensions between $V_k \otimes W_k$ and $V_l \otimes W_l$ for $k \neq l$, as $U_q\widehat{\mathfrak{g}}_J \otimes U_q\widehat{\mathfrak{h}}_J^\perp$-modules. Hence the restriction of V to $U_q\widehat{\mathfrak{g}}_J$ is isomorphic to $\oplus_k V_k$. $\square$

Write

$$d(q)[r_i]_q = \sum_{k \in \mathbb{Z}} s_i(k) q^k.$$

Set

$$B_{i,a} = \prod_{k \in \mathbb{Z}} Z_{i,aq^k}^{s_i(k)}.$$

Lemma 3.5. *We have:*

$$\tau_J(A_{i,a}) = \beta_J(A_{i,a}), \qquad i \in J, \tag{3.7}$$

$$\tau_J(A_{i,a}) = \beta_J(A_{i,a}) B_{i,a}, \qquad i \in \overline{J}. \tag{3.8}$$

Proof. This follows from the formula $D(q)\widetilde{C}'(q)C(q) = D(q)d(q)$. $\square$

In the case when J consists of a single element $j \in I$, we will write $\mathfrak{Y}^{(J)}$, τ_J and β_J simply as $\mathfrak{Y}^{(j)}$, τ_j and β_j. Consider the diagram (we use the notation (3.3), (3.4)):

$$\begin{array}{ccc} \mathfrak{Y} & \xrightarrow{\tau_j} & \mathfrak{Y}^{(j)} \\ \downarrow & & \downarrow \ \overline{A}_{j,x}^{-1} \\ \mathfrak{Y} & \xrightarrow{\tau_j} & \mathfrak{Y}^{(j)} \end{array} \tag{3.9}$$

where the map corresponding to the right vertical row is the multiplication by $\beta_j(A_{j,x})^{-1} \otimes 1$.

The following result will allow us to reduce various statements to the case of $U_q\widehat{\mathfrak{sl}}_2$.

Lemma 3.6. *There exists a unique map $\mathcal{Y} \to \mathcal{Y}$, which makes the diagram* (3.9) *commutative. This map is the multiplication by $A_{j,x}^{-1}$.*

Proof. The fact that multiplication by $A_{j,x}^{-1}$ makes the diagram commutative follows from formula (3.7). The uniqueness follows from the fact that τ_j and the multiplication by $\beta_j(A_{j,x})^{-1} \otimes 1$ are injective maps. □

4. The Structure of q-Characters

In this section we prove Conjecture 1 from [FR2].

Let V be an irreducible finite-dimensional $U_q\widehat{\mathfrak{g}}$ module V generated by highest weight vector v. Then by Proposition 3 in [FR2],

$$\chi_q(V) = m_+(1 + \sum_p M_p), \tag{4.1}$$

where each M_p is a monomial in $A_{i,c}^{\pm 1}$, $c \in \mathbb{C}^\times$ and m_+ is the highest weight monomial.

In what follows, by a monomial in $\mathbb{Z}[x_\alpha^{\pm 1}]_{\alpha \in A}$ we will always understand a monomial in reduced form, i.e., one that does not contain factors of the form $x_\alpha x_\alpha^{-1}$. Thus, in particular, if we say that a monomial M contains x_α, it means that there is a factor x_α in M which can not be cancelled.

Theorem 4.1. *The q-character of an irreducible finite-dimensional $U_q\widehat{\mathfrak{g}}$ module V has the form* (4.1)*, where each M_p is a monomial in $A_{i,c}^{-1}$, $i \in I$, $c \in \mathbb{C}^\times$ (i.e., it does not contain any factors $A_{i,c}$).*

Proof. The proof follows from a combination of Lemmas 3.3, 3.6 and 1.1.

First, we observe that it suffices to prove the statement of Theorem 4.1 for fundamental representations $V_{\omega_i}(a)$. Indeed, then Theorem 4.1 will be true for any tensor product of the fundamental representations. By Corollary 1.4, any irreducible representation V can be represented as a quotient of a submodule of a tensor product W of fundamental representations, which is generated by the highest weight vector. Therefore each monomial in a q-character of V is also a monomial in the q-character of W. In addition, the highest weight monomials of the q-characters of V and W coincide. This implies that Theorem 4.1 holds for V.

Second, Theorem 4.1 is true for $\mathfrak{g} = U_q\widehat{\mathfrak{sl}}_2$. Indeed, by the argument above, it suffices to check the statement for the fundamental representation $V_1(a)$. But its q-character is known explicitly (see [FR2], formula (4.3)):

$$\chi_q(V_1(a)) = Y_a + Y_{aq^2}^{-1} = Y_a(1 + A_{aq}^{-1}), \tag{4.2}$$

and it satisfies the required property.

For general quantum affine algebra $U_q\widehat{\mathfrak{g}}$, we will prove Theorem 4.1 (for the case of the fundamental representations) by contradiction.

Suppose that the theorem fails for some fundamental representation $V_{\omega_{i_0}}(a_0) = V$ and denote by χ its q-character $\chi_q(V)$. Denote by m_+ the highest weight monomial $Y_{i_0,a}$ of χ.

Recall from Sect. 1.3 that we have a partial order on the weight lattice. It induces a partial order on the monomials occurring in χ. Let m be the highest weight monomial in χ, such that m can not be written as a product of m_+ with a monomial in $A_{i,c}^{-1}$, $i \in I$, $c \in \mathbb{C}^\times$. This means that

$$\text{any monomial } m' \text{ in } \chi, \text{ such that } m' > m, \text{ is a product of } m_+ \text{ and } A_{i,c}^{-1}\text{'s.} \tag{4.3}$$

In Lemmas 4.2 and 4.3 we will establish certain properties of m and in Lemma 4.4 we will prove that these properties can not be satisfied simultaneously.

Recall that a monomial in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I, a \in \mathbb{C}^\times}$ is called dominant if does not contain factors $Y_{i,a}^{-1}$ (i.e., if it is a product of $Y_{i,a}$'s in positive powers only).

Lemma 4.2. *The monomial m is dominant.*

Proof. Suppose m is not dominant. Then it contains a factor of the form $Y_{i,a}^{-1}$, for some $i \in I$. Consider $\tau_i(\chi)$. By Lemma 3.4, we have

$$\tau_i(\chi) = \sum_p \chi_{q_i}(V_p) \cdot N_p,$$

where V_p's are representation of $U_{q_i}\widehat{\mathfrak{sl}}_2 = U_q\widehat{\mathfrak{g}}_{\{i\}}$ and N_p's are monomials in $Z_{j,a}^{\pm 1}$, $j \neq i$. We have already shown that Theorem 4.1 holds for $U_{q_i}\widehat{\mathfrak{sl}}_2$, so

$$\tau_i(\chi) = \sum_p \left(m_p(1 + \sum_r \overline{M}_{r,p}) \right) \cdot N_p, \tag{4.4}$$

where each m_p is a product of $Y_{i,b}$'s (in positive powers only), and each $\overline{M}_{r,p}$ is a product of several factors $\overline{A}_{i,c}^{-1} = Y_{i,cq^{-1}}^{-1} Y_{i,cq}^{-1}$ (note that $\overline{M}_{r,p} = \tau_i(M_{r,p})$.

Since m contains $Y_{i,a}^{-1}$ by our assumption, the monomial $\tau_i(m)$ is not among the monomials $\{m_p \cdot N_p\}$. Hence

$$\tau_i(m) = m_{p_0}\overline{M}_{r_0,p_0} \cdot N_{p_0},$$

for some p_0, r_0 and $\overline{M}_{r_0,p_0} \neq 1$. There exists a monomial m' in χ, such that $\tau_i(m') = m_{p_0} \cdot N_{p_0}$. Therefore using Lemma 3.6 we obtain that

$$m = m' M_{r_0,p_0},$$

where M_{r_0,p_0} is obtained from $\overline{M}_{r_0,p_0}$ by replacing all $\overline{A}_{i,c}^{-1}$ by $A_{i,c}^{-1}$. In particular, $m' > m$ and by our assumption (4.3) it can be written as $m' = m_+ M'$, where M' is a product of $A_{k,c}^{-1}$. But then $m = m' M_{r_0,p_0} = m_+ M' M_{r_0,p_0}$, and so m can be written as a product of m_+ and a product of factors $A_{k,c}^{-1}$. This is a contradiction. Therefore m has to be dominant. □

Lemma 4.3. *The monomial m can be written in the form*

$$m = m_+ M \prod_p A_{j_0,a_p}, \tag{4.5}$$

where M is a product of factors $A_{i,c}^{-1}$, $i \in I$, $c \in \mathbb{C}^\times$. In other words, if m contains factors $A_{j,a}$, then all such $A_{j,a}$ have the same index $j = j_0$.

Proof. Suppose that $m = m_+ M$, where M contains a factor $A_{i,c}$. Let V_m be the generalized eigenspace of the operators $k_j^{\pm 1}, h_{j,n}$, $j \in I$, corresponding to the monomial m. We claim that for all $v \in V_m$ we have:

$$x_{j,n}^+ \cdot v = 0, \qquad j \in I, j \neq i, \qquad n \in \mathbb{Z}. \tag{4.6}$$

Indeed, let $\tau_j(m) = \beta_j(m) \cdot N$ (recall that $\beta_j(m)$ is obtained from m by erasing all $Y_{s,c}$ with $s \neq j$ and N is a monomial in $Z_{s,c}^{\pm 1}$, $s \in I$, $s \neq j$). By Lemma 3.4, $x_{j,n}^+ \cdot v$ belongs to the direct sum of the generalized eigenspaces $V_{m'_p}$, corresponding to the monomials m'_p in χ such that $\tau_j(m'_p) = \beta_j(m'_p) \cdot N$ (with the same N as in $\tau_j(m) = \beta_j(m) \cdot N$). By formula (3.8),

$$\tau_j \left(m_+ \prod A_{i_k,c_k}^{\pm 1} \right) = \tau_j(m_+) \prod \beta_j(A_{i_k,c_k})^{\pm 1} \prod_{i_k \neq j} B_{i_k,c_k}^{\pm 1}.$$

In particular, N contains a factor $B_{i,c}$, and therefore all monomials m'_p with the above property must contain a factor $A_{i,c}$. By our assumption (4.3), the weight of each m'_p can not be higher than the weight of m. But the weight of $x_{j,n}^+ \cdot v$ should be greater than the weight of m. Therefore we obtain formula (4.6).

Now, if M contained factors $A_{i,c}$ and $A_{j,d}$ with $i \neq j$, then any non-zero eigenvector (not generalized) in the generalized eigenspace V_m corresponding to m would be a highest weight vector (see formula (1.9)). Such vectors do not exist in V, because V is irreducible. The statement of the lemma now follows. □

Lemma 4.4. *Let m be any monomial in the q-character of a fundamental representation that can be written in the form* (4.5). *Then m is not dominant.*

Proof. We say a monomial $M \in \mathcal{Y}$ (see (3.3)) has *lattice support with base* $a_0 \in \mathbb{C}^\times$ if $M \in \mathbb{Z}[Y_{i,a_0 q^k}^{\pm 1}]_{i \in I, k \in \mathbb{Z}}$.

Any monomial $m \in \mathcal{Y}$ can be uniquely written as a product $m = m^{(1)} \ldots m^{(s)}$, where each monomial $m^{(i)}$ has lattice support with a base a_i, and $a_i/a_j \notin q^{\mathbb{Z}}$ for $i \neq j$. Note that a non-constant monomial in $A_{i,bq^k}^{\pm 1}$, $i \in I$, $k \in \mathbb{Z}$, can not be equal to a monomial in $A_{i,cq^k}^{\pm 1}$, $i \in I$, $k \in \mathbb{Z}$ if $b/c \notin q^{\mathbb{Z}}$. Therefore if m can be written in the form (4.5), then each $m^{(i)}$ can be written in the form (4.5), where $m_+ = Y_{i_0,a}$ if $a_i = a$, and $m_+ = 1$ if $a/a_i \notin q^{\mathbb{Z}}$ (note that the product over p in (4.5) may be empty for some $m^{(i)}$). We will prove that none of $m^{(i)}$'s is dominant unless $m^{(i)} = m_+$ or $m^{(i)} = 1$.

Consider first the case of $m^{(1)}$, which has lattice support with base a. Then

$$m^{(1)} = \prod_{i \in I} \prod_{n \in \mathbb{Z}} Y_{i,aq^n}^{p_i(n)}.$$

Define Laurent polynomials $P_i(x)$, $i \in I$ by

$$P_i(x) = \sum_{n \in \mathbb{Z}} p_i(n) x^n.$$

If $m^{(1)}$ can be written in the form (4.5), then

$$P_i(x) = -\sum_{j \in I} C_{ij}(x) R_j(x) + \delta_{i,i_0}, \qquad \forall i \in I, \tag{4.7}$$

where $R_j(x)$'s are some polynomials with integral coefficients. All of these coefficients are non-negative if $j \neq j_0$. Now suppose that $m^{(1)}$ is a dominant monomial. Then each $P_i(x)$ is a polynomial with non-negative coefficients. We claim that this is possible only if all $R_i(x) = 0$.

Indeed, according to Lemma 1.1, the coefficients of the inverse matrix to $C(x)$, $\widetilde{C}(x)$, can be written in the form (1.2), where $\widetilde{C}'_{jk}(x)$, $d(x)$ are polynomials with non-negative coefficients. Multiplying (4.7) by $\widetilde{C}'(x)$, we obtain

$$\sum_{j \in I} P_j(x)\widetilde{C}'_{jk}(x) + d(x)R_k(x) = \widetilde{C}'_{i_0,k}(x), \qquad \forall k \in I. \tag{4.8}$$

Given a Laurent polynomial

$$p(x) = \sum_{-r \leq i \leq s} p_i x^i, \qquad p_{-r} \neq 0, \, p_s \neq 0,$$

we will say that the length of $p(x)$ equals $r + s$. Clearly, the length of the sum and of the product of two polynomials with non-negative coefficients is greater than or equal to the length of each of them. Therefore if $k \neq j_0$, and if $R_k(x) \neq 0$, then the length of the LHS is greater than or equal to the length of $d(x)$, which is greater than the length of $\widetilde{C}'_{i_0,k}$ by Lemma 1.1. This implies that $R_k(x) = 0$ for $k \neq j_0$.

Hence $m^{(1)}$ can be written in the form

$$m^{(1)} = Y_{i,a} \prod_{n \in \mathbb{Z}} A^{c_n}_{j_0, aq^n}.$$

But such a monomial can not be dominant because its weight is $\omega_i - n\alpha_{j_0}$, where $n > 0$, and such a weight is not dominant. This proves the required statement for the factor $m^{(1)}$ of m (which has lattice support with base a).

Now consider a factor $m^{(i)}$ with lattice support with base b, such that $b/a \notin q^{\mathbb{Z}}$. In this case we obtain the following equation: the LHS of formula (4.8) $= 0$. The previous discussion immediately implies that there are no solutions of this equation with non-zero polynomials $R_k(x)$ satisfying the above conditions. This completes the proof of the lemma. $\square$

Theorem 4.1 now follows from Lemmas 4.2, 4.3 and 4.4. $\square$

Corollary 4.5. *The only dominant monomial in* $\chi_q(V_{\omega_i}(a))$ *is the highest weight monomial* $Y_{i,a}$.

Proof. This follows from the proof of Lemma 4.4. $\square$

5. A Characterization of *q*-Characters in Terms of the Screening Operators

In this section we prove Conjecture 2 from [FR2].

5.1. Definition of the screening operators. First we recall the definition of the screening operators on $\mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times}$ from [FR2] and state the main result.

Consider the free $\mathcal{Y}$-module with generators $S_{i,x}$, $x \in \mathbb{C}^\times$,

$$\widetilde{\mathcal{Y}}_i = \bigoplus_{x \in \mathbb{C}^\times} \mathcal{Y} \cdot S_{i,x}.$$

Let $\mathcal{Y}_i$ be the quotient of $\widetilde{\mathcal{Y}}_i$ by the relations

$$S_{i,xq_i^2} = A_{i,xq_i} S_{i,x}. \tag{5.1}$$

Clearly,

$$\mathcal{Y}_i \simeq \bigoplus_{x \in (\mathbb{C}^\times / q_i^{2\mathbb{Z}})} \mathcal{Y} \cdot S_{i,x},$$

and so $\mathcal{Y}_i$ is also a free $\mathcal{Y}$-module.

Define a linear operator $\widetilde{S}_i : \mathcal{Y} \to \widetilde{\mathcal{Y}}_i$ by the formula

$$\widetilde{S}_i(Y_{j,a}) = \delta_{ij} Y_{i,a} S_{i,a}$$

and the Leibniz rule: $\widetilde{S}_i(ab) = b\widetilde{S}_i(a) + a\widetilde{S}_i(b)$. In particular,

$$\widetilde{S}_i(Y_{j,a}^{-1}) = -\delta_{ij} Y_{i,a}^{-1} S_{i,a}.$$

Finally, let

$$S_i : \mathcal{Y} \to \mathcal{Y}_i$$

be the composition of $\widetilde{S}_i$ and the projection $\widetilde{\mathcal{Y}}_i \to \mathcal{Y}_i$. We call S_i the i^{th} *screening operator*.

The following statement was conjectured in [FR2] (Conjecture 2).

Theorem 5.1. *The image of the homomorphism χ_q equals the intersection of the kernels of the operators S_i, $i \in I$.*

In [FR2] this theorem was proved in the case of $U_q\widehat{\mathfrak{sl}}_2$. In the rest of this section we prove it for an arbitrary $U_q\widehat{\mathfrak{g}}$.

5.2. Description of Ker S_i. First, we describe the kernel of S_i on $\mathcal{Y}$. The following result was announced in [FR2], Proposition 6.

Proposition 5.2. *The kernel of $S_i : \mathcal{Y} \to \mathcal{Y}_i$ equals*

$$\mathcal{K}_i = \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j \neq i; a \in \mathbb{C}^\times} \otimes \mathbb{Z}[Y_{i,b} + Y_{i,b} A_{i,bq_i}^{-1}]_{b \in \mathbb{C}^\times}. \tag{5.2}$$

Proof. A simple computation shows that $\mathcal{K}_i \subset \mathrm{Ker}_{\mathcal{Y}} S_i$. Let us show that $\mathrm{Ker}_{\mathcal{Y}} S_i \subset \mathcal{K}_i$.

For $x \in \mathbb{C}^\times$, denote by $\mathcal{Y}(x)$ the subring $\mathbb{Z}[Y_{j,xq^n}^{\pm 1}]_{j \in I, n \in \mathbb{Z}}$ of $\mathcal{Y}$. We have:

$$\mathcal{Y} \simeq \bigotimes_{x \in (\mathbb{C}^\times / q^{\mathbb{Z}})} \mathcal{Y}(x).$$

Lemma 5.3.

$$\mathrm{Ker}_{\mathcal{Y}} S_i = \bigotimes_{x \in (\mathbb{C}^\times / q^{\mathbb{Z}})} \mathrm{Ker}_{\mathcal{Y}(x)} S_i.$$

Proof. Let $P \in \mathcal{Y}$, and suppose it contains $Y_{j,a}^{\pm 1}$ for some $a \in \mathbb{C}^\times$ and $j \in I$. Then we can write P as the sum $\sum_k R_k Q_k$, where Q_k's are distinct monomials, which are products of the factors $Y_{s,aq^n}^{\pm 1}$, $s \in I, n \in \mathbb{Z}$ (in particular, one of the Q_k's could be equal to 1), and R_k's are polynomials which do not contain $Y_{s,aq^n}^{\pm 1}$, $s \in I, n \in \mathbb{Z}$. Then

$$S_i(P) = \sum_k (Q_k \cdot S_i(R_k) + R_k \cdot S_i(Q_k)).$$

By definition of S_i, $S_i(Q_k)$ belongs to $\mathcal{Y} \cdot S_{i,a}$, while $S_i(R_k)$ belongs to the direct sum of $\mathcal{Y} \cdot S_{i,b}$, where $b \notin aq^{\mathbb{Z}}$.

Therefore if $P \in \mathrm{Ker}_{\mathcal{Y}} S_i$, then $\sum_k Q_k \cdot S_i(R_k) = 0$. Since Q_k's are distinct, we obtain that $R_k \in \mathrm{Ker}_{\mathcal{Y}} S_i$. But then $S_i(P) = \sum_k R_k \cdot S_k(Q_k)$. Therefore P can be written as $\sum_l R_l \widetilde{Q}_l$, where each $\widetilde{Q}_l$ is a linear combination of the Q_k's, such that $\widetilde{Q}_l \in \mathrm{Ker}_{\mathcal{Y}} S_i$. This proves that

$$P \in \mathrm{Ker}_{\mathcal{Y}(\neq a)} S_i \otimes \mathrm{Ker}_{\mathcal{Y}(a)} S_i,$$

where $\mathcal{Y}(\neq a) = \mathbb{Z}[Y_{j,b}^{\pm 1}]_{j \in I, b \notin aq^{\mathbb{Z}}}$. By repeating this procedure we obtain the lemma (because each polynomial contains a finite number of variables $Y_{j,a}^{\pm 1}$, we need to apply this procedure finitely many times). $\square$

According to Lemma 5.3, it suffices to show that $\mathrm{Ker}_{\mathcal{Y}(x)} S_i \subset \mathcal{K}_i(x)$, where

$$\mathcal{K}_i(x) = \mathbb{Z}[Y_{j,xq^n}^{\pm 1}]_{j \neq i; n \in \mathbb{Z}} \otimes \mathbb{Z}[Y_{i,xq^n} + Y_{i,xq^n} A_{i,xq^n q_i}^{-1}]_{n \in \mathbb{Z}}.$$

Denote Y_{j,xq^n} by $y_{j,n}$, A_{j,xq^n} by $a_{j,n}$, and $A_{j,xq^n} Y_{j,xq^n q_j}^{-1} Y_{j,xq^n q_j^{-1}}^{-1}$ by $\overline{a}_{j,n}$. Note that $\overline{a}_{j,n}$ does not contain factors $y_{j,m}^{\pm 1}$, $m \in \mathbb{Z}$.

Let T be the shift operator on $\mathcal{Y}(x)$ sending $y_{j,n}$ to $y_{j,n+1}$ for all $j \in I$. It follows from the definition of S_i that $P \in \mathrm{Ker}_{\mathcal{Y}(x)} S_i$ if and only if $T(P) \in \mathrm{Ker}_{\mathcal{Y}(x)} S_i$. Therefore (applying T^m with large enough m to P) we can assume without loss of generality that $P \in \mathbb{Z}[y_{i,n}, y_{i,n+2r_i}^{-1}]_{n \geq 0} \otimes \mathbb{Z}[y_{j,n}^{\pm 1}]_{j \neq i, n \geq 0}$.

We find from the definition of S_i:

$$S_i(y_{j,n}) = 0, \qquad j \neq i,$$

$$S_i(y_{i,2r_i n + \epsilon}) = y_{i,\epsilon} \prod_{k=1}^{n} y_{i,2r_i k + \epsilon}^2 \overline{a}_{i,r_i(2k-1)+\epsilon} \cdot S_{i,xq^\epsilon}, \tag{5.3}$$

where $\epsilon \in \{0, 1, \ldots, 2r_i - 1\}$. Therefore each $P \in \mathrm{Ker}_{\mathcal{Y}(x)} S_i$ can be written as a sum $P = \sum P_\epsilon$, where each $P_\epsilon \in \mathrm{Ker}_{\mathcal{Y}(x)} S_i$ and

$$P_\epsilon \in \mathbb{Z}[y_{i,2r_i n + \epsilon}, y_{i,2r_i(n+1)+\epsilon}^{-1}]_{n \geq 0} \otimes \mathbb{Z}[y_{j,n}^{\pm 1}]_{j \neq i, n \geq 0}.$$

It suffices to consider the case $\epsilon = 0$. Thus, we show that if

$$P \in \mathcal{Y}_i^{\geq 0}(x) = \mathbb{Z}[y_{i,2r_i n}, y_{i,2r_i(n+1)}^{-1}]_{n \geq 0} \otimes \mathbb{Z}[y_{j,n}^{\pm 1}]_{j \neq i, n \geq 0},$$

then

$$P \in \mathcal{K}_i^{\geq 0}(x) = \mathbb{Z}[t_n]_{n \geq 0} \otimes \mathbb{Z}[y_{j,n}^{\pm 1}]_{j \neq i, n \geq 0},$$

where

$$t_n = y_{i,2r_in} + y_{i,2r_in}a^{-1}_{i,r_i(2n+1)} = y_{i,2r_in} + y^{-1}_{i,2r_i(n+1)}\overline{a}^{-1}_{i,r_i(2n+1)}.$$

Consider a homomorphism $\mathcal{K}_i^{\geq 0}(x) \otimes \mathbb{Z}[y_{i,2r_in}]_{n\geq 0} \to \mathcal{Y}_i^{\geq 0}(x)$ sending $y^{\pm 1}_{j,n}$, $j \neq i$ to $y^{\pm 1}_{j,n}$, $y_{i,2r_in}$ to $y_{i,2r_in}$, and t_n to $y_{i,2r_in} + y^{-1}_{i,2r_i(n+1)}\overline{a}^{-1}_{i,r_i(2n+1)}$. This homomorphism is surjective, and its kernel is generated by the elements

$$(t_n - y_{i,2r_in})\overline{a}_{i,r_i(2n+1)}y_{i,2r_i(n+1)} - 1. \tag{5.4}$$

Therefore we identify $\mathcal{Y}_i^{\geq 0}(x)$ with the quotient of $\mathcal{K}_i^{\geq 0}(x) \otimes \mathbb{Z}[y_{i,2r_in}]_{n\geq 0}$ by the ideal generated by elements of the form (5.4).

Consider the set of monomials

$$t_{n_1}\dots t_{n_k}y_{i,2r_im_1}\dots y_{i,2r_im_l}\prod_{j\neq i,p_j\geq 0} y^{\pm 1}_{j,p_j},$$

where all $n_1 \geq n_2 \geq \dots n_k \geq 0, m_1 \geq m_2 \geq \dots m_l \geq 0$, and also $m_j \neq n_i + 1$ for all i and j. We call these monomials *reduced*. It is easy to see that the set of reduced monomials is a basis of $\mathcal{Y}_i^{\geq 0}(x)$.

Now let P be an element of the kernel of S_i on $\mathcal{Y}_i^{\geq 0}(x)$. Let us write it as a linear combination of the reduced monomials. We represent P as $y^a_{i,2r_iN}Q + R$. Here N is the largest integer, such that $y_{i,2r_iN}$ is present in at least one of the basis monomials appearing in its decomposition; $a > 0$ is the largest power of $y_{i,2r_iN}$ in P; $Q \neq 0$ does not contain $y_{i,2r_iN}$, and R is not divisible by $y^a_{i,2r_iN}$. Recall that here both $y^a_{i,2r_iN}Q$ and R are linear combinations of reduced monomials.

Recall that $S_i(t_n) = 0$, $S_i(y^{\pm 1}_{j,n}) = 0$, $j \neq i$, and $S_i(y_{i,2r_in})$ is given by formula (5.3).

Suppose that $N > 0$. According to formula (5.3),

$$S_i(P) = ay^{a+1}_{i,2r_iN}\prod_{k=1}^{N-1} y_{i,2r_ik}\prod_{l=1}^{N}\overline{a}_{i,r_i(2l-1)}y_{i,0}\ Q\cdot S_{i,x} + \dots, \tag{5.5}$$

where the dots represent the sum of terms that are not divisible by $y^{a+1}_{i,2r_iN}$. Note that the first term in (5.5) is non-zero because the ring $\mathcal{Y}_i^{\geq 0}(x)$ has no divisors of zero.

The monomials appearing in (5.5) are not necessarily reduced. However, by construction, Q does not contain t_{N-1}, for otherwise $y^a_{i,2r_iN}Q$ would not be a linear combination of reduced monomials. Therefore when we rewrite (5.5) as a linear combination of reduced monomials, each reduced monomial occurring in this linear combination is still divisible by $y^{a+1}_{i,2r_iN}$. On the other hand, no reduced monomials occurring in the other terms of $S_i(P)$ (represented by dots) are divisible by $y^{a+1}_{i,2r_iN}$. Hence for P to be in the kernel, the first term of (5.5) has to vanish, which is impossible. Therefore P does not contain $y_{i,2r_im}$'s with $m > 0$.

But then $P = \sum_k y^{p_k}_{i,0}R_k$, where $R_k \in \mathcal{K}_{i\geq 0}(x)$, and $S_i(P) = \sum_k p_k y^{p_k-1}_{i,0}R_k \cdot S_{i,x}$. Such P is in the kernel of S_i if and only if all $p_k = 0$ and so $P \in \mathcal{K}_{i\geq 0}(x)$. This completes the proof of Proposition 5.2. □

Set

$$\mathcal{K} = \bigcap_{i\in I}\mathcal{K}_i = \bigcap_{i\in I}\left(\mathbb{Z}[Y^{\pm 1}_{j,a}]_{j\neq i;a\in\mathbb{C}^\times}\otimes\mathbb{Z}[Y_{i,b} + Y_{i,b}A^{-1}_{i,bq_i}]_{b\in\mathbb{C}^\times}\right). \tag{5.6}$$

Now we will prove that the image of the q-character homomorphism χ_q equals $\mathcal{K}$.

5.3. The image of χ_q is a subspace of $\mathcal{K}$. First we show that the image of Rep $U_q\widehat{\mathfrak{g}}$ in $\mathcal{Y}$ under the q-character homomorphism belongs to the kernel of S_i.

Recall the ring $\mathcal{Y}^{(i)} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{a\in\mathbb{C}^\times} \otimes \mathbb{Z}[Z_{j,c}^{\pm 1}]_{j\neq i, c\in\mathbb{C}^\times}$ and the homomorphism $\tau_i : \mathcal{Y} \to \mathcal{Y}^{(i)}$ from Sect. 3.2.

Let $\overline{\mathcal{Y}}_i$ be the quotient of $\bigoplus_{x\in\mathbb{C}^\times} \mathbb{Z}[Y_{i,a}^{\pm 1}]_{a\in\mathbb{C}^\times} \cdot S_{i,x}$ by the submodule generated by the elements of the form $S_{i,xq_i^2} - \overline{A}_{i,xq_i} S_{i,x}$, where $\overline{A}_{i,xq_i} = Y_{i,x}Y_{i,xq_i^2}$. Define a derivation $\overline{S}_i : \mathbb{Z}[Y_{i,a}^{\pm 1}]_{a\in\mathbb{C}^\times} \to \overline{\mathcal{Y}}_i$ by the formula $\overline{S}_i(Y_{i,a}) = Y_{i,a}S_{i,a}$. Thus, $\overline{\mathcal{Y}}_i$ coincides with the module $\mathcal{Y}_i$ in the case of $U_{q_i}\widehat{\mathfrak{sl}}_2$ and $\overline{S}_i$ is the corresponding screening operator.

Set

$$\mathcal{Y}_i^{(i)} = \mathbb{Z}[Z_{j,c}^{\pm 1}]_{j\neq i, c\in\mathbb{C}^\times} \otimes \overline{\mathcal{Y}}_i.$$

The map $\overline{S}_i$ can be extended uniquely to a map $\mathcal{Y}^{(i)} \to \mathcal{Y}_i^{(i)}$ by $\overline{S}_i(Z_{j,c}) = 0$ for all $j \neq i, c \in \mathbb{C}^\times$ and the Leibniz rule. We will also denote it by $\overline{S}_i$. The embedding τ_i gives rise to an embedding $\mathcal{Y}_i \to \mathcal{Y}_i^{(i)}$ which we also denote by τ_i.

Lemma 5.4. *The following diagram is commutative*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{\tau_i} & \mathcal{Y}^{(i)} \\ \Big\downarrow{\scriptstyle S_i} & & \Big\downarrow{\scriptstyle \overline{S}_i} \\ \mathcal{Y}_i & \xrightarrow{\tau_i} & \mathcal{Y}_i^{(i)} \end{array}$$

Proof. Since τ_i is a ring homomorphism and both S_i, $\overline{S}_i$ are derivations, it suffices to check commutativity on the generators. Let us choose a representative x in each $q_i^{2\mathbb{Z}}$-coset of $\mathbb{C}^\times$. Then we can write:

$$\mathcal{Y}_i = \bigoplus_{x\in\mathbb{C}^\times/q_i^{2\mathbb{Z}}} \mathcal{Y} \cdot S_{i,x}, \qquad \mathcal{Y}_i^{(i)} = \bigoplus_{x\in\mathbb{C}^\times/q_i^{2\mathbb{Z}}} \mathcal{Y}^{(i)} \cdot S_{i,x}.$$

By definition,

$$S_i(Y_{j,xq_i^{2n}}) = \delta_{ij} Y_{i,x} \prod_m A_{i,xq_i^{2m+1}}^{\pm 1}\; S_{i,x},$$

$$\overline{S}_i(Y_{i,xq_i^{2n}}) = Y_{i,x} \prod_m \overline{A}_{i,xq_i^{2m+1}}^{\pm 1}\; S_{i,x},$$

$$\overline{S}_i(Z_{j,c}) = 0, \qquad \forall j \neq i.$$

Recall from formula (3.5) that $\tau_i(Y_{i,x})$ equals $Y_{i,x}$ times a monomial in $Z_{j,c}^{\pm 1}$, $j \neq i$, and from formula (3.8) that $\tau_i(A_{i,b}^{\pm 1}) = \overline{A}_{i,b}^{\pm 1}$. Using these formulas we obtain:

$$(\tau_i \circ S_i)(Y_{i,xq_i^{2n}}) = (\overline{S}_i \circ \tau_i)(Y_{i,xq^{2n}}) = \tau_i(Y_{i,x}) \prod \overline{A}_{i,xq_i^{2m+1}}^{\pm 1}\; S_{i,x}.$$

On the other hand, when $j \neq i$, $\tau_i(Y_{j,x})$ is a monomial in $Z_{k,c}^{\pm 1}, k \neq i$, according to formula (3.6). Therefore

$$(\tau_i \circ S_i)(Y_{j,x}) = (\overline{S}_i \circ \tau_i)(Y_{j,x}) = 0, \qquad j \neq i.$$

This proves the lemma. □

Corollary 5.5. *The image of the q-character homomorphism $\chi_q : \operatorname{Rep} U_q\widehat{\mathfrak{g}} \to \mathcal{Y}$ is contained in the kernel of S_i on $\mathcal{Y}$.*

Proof. Let V be a finite-dimensional representation of $U_q\widehat{\mathfrak{g}}$. We need to show that $S_i(\chi_q(V)) = 0$. By Lemma 3.4, we can write $\chi_q(V)$ as the sum $\sum_k P_k Q_k$, where each $P_k \in \mathbb{Z}[Y_{i,a}^{\pm 1}]_{a\in\mathbb{C}^\times}$ is in the image of the homomorphism $\chi_q^{(i)} : \operatorname{Rep} U_{q_i}\widehat{\mathfrak{sl}}_2 \to \mathbb{Z}[Y_{i,a}^{\pm 1}]_{a\in\mathbb{C}^\times}$, and Q_k is a monomial in $Z_{j,c}^{\pm 1}$, $j \neq i$.

The image of $\chi_q^{(i)}$ lies in the kernel of the operator $\overline{S}_i$ (in fact, they are equal, but we will not use this now). This immediately follows from the fact that $\operatorname{Rep} U_q\widehat{\mathfrak{sl}}_2 \simeq \mathbb{Z}[\chi_q(V_1(a))]$ and $\overline{S}_i(\chi_q(V_1(a))) = 0$, which is obtained by a straightforward calculation. We also have: $\overline{S}_i(Z_{j,c}) = 0, \forall j \neq i$. Therefore $(\overline{S}_i \circ \tau_i)(\chi_q(V)) = 0$. By Lemma 5.4, $(\tau_i \circ S_i)(\chi_q(V)) = 0$. Since τ_i is injective by Lemma 3.3, we obtain: $S_i(\chi_q(V)) = 0$. $\square$

5.4. $\mathcal{K}$ is a subspace of the image of χ_q. Let $P \in \mathcal{K}$. We want to show that $P \in \operatorname{Im} \chi_q$.

A monomial m contained in $P \in \mathcal{Y}$ is called *highest monomial* (resp., *lowest monomial*), if its weight is not lower (resp., not higher) than the weight of any other monomial contained in P.

Lemma 5.6. *Let $P \in \mathcal{K}$. Then any highest monomial in P is dominant and any lowest weight monomial in P is antidominant.*

Proof. First we prove that the highest monomials are dominant.

By Proposition 5.2,

$$P \in \mathcal{K}_i = \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j\neq i;a\in\mathbb{C}^\times} \otimes \mathbb{Z}[Y_{i,b} + Y_{i,b}A_{i,bq_i}^{-1}]_{b\in\mathbb{C}^\times}.$$

The statement of the lemma will follow if we show that a highest weight monomial contained in any element of $\mathcal{K}_i$ does not contain factors $Y_{i,a}^{-1}$.

Indeed, the weight of $Y_{i,a}$ is ω_i, and the weight of $Y_{i,b}A_{i,bq_i}^{-1}$ is $\omega_i - \alpha_i$. Denote $t_b = \mathbb{Z}[Y_{i,b} + Y_{i,b}A_{i,bq_i}^{-1}]_{b\in\mathbb{C}^\times}$. Given a polynomial $Q \in \mathbb{Z}[t_b]_{b\in\mathbb{C}^\times}$, let $m_1, \ldots , m_k$ be its monomials (in t_b) of highest degree. Clearly, the monomials of highest weight in Q (considered as a polynomial in $Y_{j,a}^{\pm 1}$) are $m_1, \ldots , m_k$, in which we substitute each t_b by $Y_{i,b}$. These monomials do not contain factors $Y_{i,a}^{-1}$.

The statement about the lowest weight monomials is proved similarly, once we observe that

$$\mathcal{K}_i = \mathbb{Z}[Y_{j,a}^{\pm 1}]_{j\neq i;a\in\mathbb{C}^\times} \otimes \mathbb{Z}[Y_{i,b}^{-1} + Y_{i,bq_i^{-2}}A_{i,bq_i^{-1}}]_{b\in\mathbb{C}^\times}. \qquad \square$$

Let m be a highest monomial in P, and suppose that it enters P with the coefficient $\nu_m \in \mathbb{Z} \setminus 0$. Then m is dominant by Lemma 5.2. According to Theorem 1.3(2) and formula (2.13), there exists an irreducible representation V_1 of $U_q\widehat{\mathfrak{g}}$, such that m is the highest weight monomial in $\chi_q(V_1)$. Since $\chi_q(V_1) \in \mathcal{K}$ by Corollary 5.5, we obtain that $P_1 = P - \nu_m \cdot \chi_q(V_1) \in \mathcal{K}$.

For $P \in \mathcal{Y}$, denote by $\Lambda(P)$ the (finite) set of dominant weights λ, such that P contains a monomial of weight greater than or equal to λ. By Proposition 5.2, if $P \in \mathcal{K}$ and $\Lambda(P)$ is empty, then P is necessarily equal to 0.

Note that for any irreducible representation V of $U_q\widehat{\mathfrak{g}}$ of highest weight μ, $\Lambda(\chi_q(V))$ is the set of all dominant weights which are less than or equal to μ. Therefore $\Lambda(P_1)$ is properly contained in $\Lambda(P)$. By applying the above subtraction procedure finitely many times, we obtain an element $P_k = P - \sum_{i=1}^{k} \chi_q(V_i)$, for which $\Lambda(P_k)$ is empty. But then $P_k = 0$.

This shows that $\mathcal{K} \subset \operatorname{Im} \chi_q$. Together with Lemma 5.5, this gives us Theorem 5.1 and the following corollary.

Corollary 5.7. *The q-character homomorphism,*

$$\chi_q : \operatorname{Rep} U_q\widehat{\mathfrak{g}} \to \mathcal{K},$$

where $\mathcal{K}$ is given by (5.6), is a ring isomorphism.

5.5. Application: Algorithm for constructing q-characters. Consider the following problem: Give an algorithm which for any dominant monomial m_+ constructs the q-character of the irreducible $U_q\widehat{\mathfrak{g}}$-module whose highest weight monomial is m_+. In this section we propose such an algorithm. We prove that our algorithm produces the q-characters of the fundamental representations (in this case $m_+ = Y_{i,a}$). We conjecture that the algorithm works for any irreducible module.

Roughly speaking, in our algorithm we start from m_+ and gradually expand it in all possible $U_{q_i}\widehat{\mathfrak{sl}}_2$ directions. (Here we use the explicit formulas for q-characters of $U_q\widehat{\mathfrak{sl}}_2$ and Lemma 3.6.) In the process of expansion some monomials may come from different directions. We identify them in the maximal possible way.

First we introduce some terminology.

Let $\chi \in \mathbb{Z}_{\geq 0}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times}$ be a polynomial and m a monomial in χ occurring with coefficient $s \in \mathbb{Z}_{>0}$. By definition, a *coloring* of m is a set $\{s_i\}_{i\in I}$ of non-negative integers such that $s_i \leq s$. A polynomial χ in which all monomials are colored is called a *colored polynomial*.

We think of s_i as the number of monomials of type m which have come from direction i (or by expanding with respect to the i^{th} subalgebra $U_{q_i}\widehat{\mathfrak{sl}}_2$).

A monomial m is called *i-dominant* if it does not contain variables $Y_{i,a}^{-1}$, $a \in \mathbb{C}^\times$.

A monomial m occurring in a colored polynomial χ with coefficient s is called *admissible* if m is j-dominant for all j such that $s_j < s$. A colored polynomial is called admissible if all of its monomials are admissible.

Given an admissible monomial m occurring with coefficient s in a colored polynomial χ, we define a new colored polynomial $i_m(\chi)$, called the i-expansion of χ with respect to m, as follows.

If $s_i = s$, then $i_m(\chi) = \chi$. Suppose that $s_i < s$ and let $\overline{m}$ be obtained from m by setting $Y_{j,a}^{\pm 1} = 1$, for all $j \neq i$. Since m is admissible, $\overline{m}$ is a dominant monomial. Therefore there exists an irreducible $U_{q_i}\widehat{\mathfrak{sl}}_2$ module V, such that the highest weight monomial of V is $\overline{m}$. We have explicit formulas for the q-characters of all irreducible $U_q\widehat{\mathfrak{sl}}_2$-modules (see, e.g., [FR2, Sect. 4.1]). We write $\chi_{q_i}(V) = \overline{m}(1 + \sum_p \overline{M}_p)$, where $\overline{M}_p$ is a product of $\overline{A}_{i,a}^{-1}$. Let

$$\mu = m(1 + \sum_p M_p), \tag{5.7}$$

where M_p is obtained from $\overline{M}_p$ by replacing all $\overline{A}_{i,a}^{-1}$ by $A_{i,a}^{-1}$.

The colored polynomial $i_m(\chi)$ is obtained from χ by adding monomials occurring in μ by the following rule. Let monomial n occur in μ with coefficient $t \in \mathbb{Z}_{>0}$. If n does not occur in χ then it is added with the coefficient $t(s - s_i)$ and we set the i^{th} coloring of n to be $t(s - s_i)$, and the other colorings to be 0. If n occurs in χ with coefficient r and coloring $\{r_i\}_{i \in I}$, then the new coefficient of n in $i_m(\chi)$ is $\max\{r, r_i + t(s - s_i)\}$. In this case the i^{th} coloring is changed to $r_i + t(s - s_i)$ and other colorings are not changed.

Obviously, the i-expansions of χ with respect to m commute for different i. To expand a monomial m in all directions means to compute $\ell_m(\dots 2_m(1_m(\chi))\dots)$, where $\ell = rk(\mathfrak{g})$.

Now we describe the algorithm. We start with the colored polynomial m_+ with all colorings set equal zero. Let the $U_q\mathfrak{g}$-weight of m_+ be λ. The set of weights of the form $\lambda - \sum_i a_i\alpha_i$, $a_i \in \mathbb{Z}_{\geq 0}$ has a natural partial order. Choose any total order compatible with this partial order, so we have $\lambda = \lambda_1 > \lambda_2 > \lambda_3 > \dots$.

At the first step we expand m_+ in all directions. Then we expand in all directions all monomials of weight λ_1 obtained at the first step. Then we expand in all directions all monomials of weight λ_2 obtained at the previous steps, and so on. Since the monomials obtained in the expansion of a monomial of $U_q\mathfrak{g}$-weight μ have weights less than μ, the result does not depend on the choice of the total order.

Note that for any monomial m except for m_+ occurring with coefficient s at any step, we have $\max_i\{s_i\} = s$. This property means that we identify the monomials coming from different directions in the maximal possible way.

The algorithm stops if all monomials have been expanded. We say that the algorithm fails at a monomial m if m is the first non-admissible monomial to be expanded.

Let m_+ be a dominant monomial and V the corresponding irreducible module.

Conjecture 5.8. *The algorithm never fails and stops after finitely many steps. Moreover, the final result of the algorithm is the q-character of V.*

Theorem 5.9. *Suppose that $\chi_q(V)$ does not contain dominant monomials other than m_+. Then Conjecture 5.8 is true. In particular, Conjecture 5.8 is true in the case of fundamental representations.*

Proof. For $i \in I$, let D_i be a decomposition of the set of monomials in $\chi_q(V)$ with multiplicities into a disjoint union of subsets such that each subset forms the q-character of an irreducible $U_{q_i}\widehat{\mathfrak{sl}}_2$ module. We refer to this decompostion D_i as the i^{th} decomposition of $\chi_q(V)$. Denote D the collection of D_i, $i \in I$.

Consider the following colored oriented graph $\Omega_V(D)$. The vertices are monomials in $\chi_q(V)$ with multiplicities. We draw an arrow of color i from a monomial m_1 to a monomial m_2 if and only if m_1 and m_2 are in the same subset of the i^{th} decomposition and $m_2 = A_{i,a}^{-1}m_1$ for some $a \in \mathbb{C}^\times$.

We call an oriented graph a tree (with one root) if there exists a vertex v (called root), such that there is an oriented path from v to any other vertex. The graph $\Omega_W(D)$, where W is an irreducible $U_q\widehat{\mathfrak{sl}}_2$-module is always a tree and its root corresponds to the highest weight monomial.

Consider the full subgraph of $\Omega_V(D)$ whose vertices correspond to monomials from a given subset of the i^{th} decomposition of $\chi_q(V)$. All arrows of this subgraph are of color

i. By Lemma 3.6, this subgraph is a tree isomorphic to the graph of the corresponding irreducible $U_{q_i}\widehat{\mathfrak{sl}}_2$-module. Moreover, its root corresponds to an i-dominant monomial. Therefore if a vertex of $\Omega_V(D)$ has no incoming arrows of color i, then it corresponds to an i-dominant monomial. In particular, if m has no incoming arrows in $\Omega_V(D)$, then m is dominant. Since by our assumption $\chi_q(V)$ does not contain any dominant monomials except for m_+, the graph $\Omega_V(D)$ is a tree with root m_+.

Choose a sequence of weights $\lambda_1 > \lambda_2 > \dots$ as above. We prove by induction on r the following statement S_r:

The algorithm does not fail during the first r steps. Let χ_r be the resulting polynomial after these steps. Then the coefficient of each monomial m in χ_r is not greater than that in $\chi_q(V)$ and the coefficients of monomials of weights $\lambda_1, \dots, \lambda_r$ in χ_r and $\chi_q(V)$ are equal. Furthermore, there exists a decomposition D of $\chi_q(V)$, such that monomials in χ_r can be identified with vertices in $\Omega_V(D)$ in such a way that all outgoing arrows from vertices with $U_q\mathfrak{g}$-weights $\lambda_1, \dots, \lambda_r$ go to vertices of χ_r. Finally, the j^{th} coloring of a monomial m in χ_r is just the number of vertices of type m in χ_r which have incoming arrows of color j in $\Omega_V(D)$.

The statement S_0 is obviously true. Assume that the statement S_r is true for some $r \geq 0$. Recall that at the $(r+1)^{\text{st}}$ step we expand all monomials of χ_r of weight λ_{r+1}.

Let m be a monomial of weight λ_{r+1} in χ_r, which enters with coefficient s and coloring $\{s_i\}_{i\in I}$.

Then the monomial m enters $\chi_q(V)$ with coefficient s as well. Indeed, $\Omega_V(D)$ is a tree, so all vertices m have incoming arrows from vertices of larger weight. By the statement S_r theses arrows go to vertices corresponding to monomials in χ_r.

Suppose that $s_j < s$ for some $j \in I$. Then m is j-dominant. Indeed, otherwise each vertex of type m in $\Omega_V(D)$ has an incoming arrow of color j coming from a vertex of higher weight. Then by the last part of the statement S_r, $s_j = s$.

Therefore the monomial m is admissible, and the algorithm does not fail at m.

Consider the expansion $j_m(\chi_r)$. Let μ be as in (5.7). In the j^{th} decomposition of $\chi_q(V)$, m corresponds to a root of a tree whose vertices can be identified with monomials in μ. We fix such an identification. Then monomials in μ get identified with vertices in $\Omega_V(D)$.

Let v be the vertex in $\Omega_V(D)$, corresponding to a monomial n in μ. Denote the coefficient of n in χ_r by p and the coloring by $\{p_i\}_{i\in I}$. We have two cases:

a) $p_j = p$. Then the last part of the statement S_r implies that the vertex v does not belong to χ_r. We add the monomial n to χ_r and increase p_j by one (we have already identified it with v).
b) $p_j < p$. Then by S_r there exists a vertex w in χ_r of type n with no incoming arrows of color j. We change the decomposition D_j by switching the vertices v and w and identify n with the new v. We also increase p_j by one. (Thus, in this case we do not add n to χ_r.)

In both cases, the statement S_{r+1} follows.

Since the set of weights of monomials occurring in $\chi_q(V)$ is contained in a finite set $\lambda_1, \lambda_2, \dots, \lambda_N$, the statement S_N proves the first part of the theorem.

Corollary 4.5 then implies the second part of the theorem. $\square$

We plan to use the above algorithm to compute explicitly the q-characters of the fundamental representations of $U_q\widehat{\mathfrak{g}}$ and to obtain their decompositions under $U_q\mathfrak{g}$.

Remark 5.10. There is a similar algorithm for computing the ordinary characters of finite-dimensional $\mathfrak{g}$-modules (equivalently, $U_q\mathfrak{g}$-modules). That algorithm works for those representations (called miniscule) whose characters do not contain dominant weights other than the highest weight (for other representations the algorthim does not work). However, there are very few miniscule representations for a general simple Lie algebra $\mathfrak{g}$. In contrast, in the case of quantum affine algebras there are many representations whose characters do not contain any dominant monomials except for the highest weight monomials (for example, all fundamental representations), and our algorithm may be applied to them.

6. The Fundamental Representations

In this section we prove several theorems about the irreducibility of tensor products of fundamental representations.

6.1. Reducible tensor products of fundamental representations and poles of R-matrices. In this section we prove that the reducibility of a tensor product of the fundamental representations is always caused by a pole in the R-matrix.

We say that a monomial m has *positive lattice support with base a* if m is a product $Y^{\pm 1}_{i,aq^n}$ with $n \geq 0$.

Lemma 6.1. *All monomials in $\chi_q(V_{\omega_i}(a))$ have positive lattice support with base a.*

Proof. For $U_q\widehat{\mathfrak{sl}}_2$, the statement follows from the explicit formula (4.2) for $\chi_q(V_1(a))$. The q-character of any irreducible representation V of $U_q\widehat{\mathfrak{sl}}_2$ is a subsum of a product of the q-characters of $V_1(b)$'s. Moreover, this subsum includes the highest monomial. Hence if the highest weight monomial of $\chi_q(V)$ has positive lattice support with base a, then so do all monomials in $\chi_q(V)$.

Now consider the case of general $U_q\widehat{\mathfrak{g}}$. Suppose there exists a monomial in $\chi = \chi_q(V_{\omega_i}(a))$, which does not have positive lattice support with base a. Let m be a highest among such monomials (with respect to the partial ordering by weights).

By Corollary 4.5, the monomial m is not dominant. In other words, if we rewrite m as a product of $Y^{\pm 1}_{i,b}$, we will have at least one generator in negative power, say $Y^{-1}_{i_0,b_0}$.

Write $\tau_{i_0}(\chi)$ in the form (4.4). The monomial $\tau_{i_0}(m)$ can not be among the monomials $\{m_p N_p\}$, since m contains $Y^{-1}_{i_0,b_0}$. Therefore $\tau_{i_0}(m) = m_{p_0} N_{p_0} \overline{M}_{r_0,p_0}$ for some $\overline{M}_{r_0,p_0} \neq 1$, which is a product of factors $\overline{A}^{-1}_{i,c}$. Let m_1 be a monomial in χ, such that $\tau_{i_0}(m_1) = m_{p_0} N_{p_0}$. Then by Lemma 3.6, $m = m_1 M_{r_0,p_0}$, where M_{r_0,p_0} is obtained from $\overline{M}_{r_0,p_0}$ by replacing all $\overline{A}^{-1}_{i,c}$ with $A^{-1}_{i,c}$.

By construction, the weight of m_1 is higher than the weight of m, so by our assumption, m_1 has positive lattice support with base a. But then m_{p_0} also has positive lattice support with base a. Therefore all monomials in $m_{p_0}(1 + \sum_r \overline{M}_{r,p})$ have positive lattice support with base a. This implies that M_{r_0,p_0}, and hence $m = m_1 M_{r_0,p_0}$, has positive lattice support with base a. This is a contradiction, so the lemma is proved. □

Remark 6.2. From the proof of Lemma 6.1 is clear that the only monomial in $\chi_q(V_{\omega_i}(a))$ which contains $Y^{\pm 1}_{j,aq^n}$ with $n = 0$ is the highest weight monomial $Y_{i,a}$.

Let V be a $U_q\widehat{\mathfrak{g}}$-module with the q-character $\chi_q(V)$. Define the oriented graph Γ_V as follows. The vertices of Γ_V are monomials in $\chi_q(V)$ with multiplicities. Thus, there are $\dim V$ vertices. We denote the monomial corresponding to a vertex α by m_α. We draw an arrow from the vertex α to the vertex β if and only if $m_\beta = m_\alpha A_{i,x}^{-1}$ for some $i \in I$, $x \in \mathbb{C}^\times$.

If V is an irreducible $U_q\widehat{\mathfrak{sl}}_2$-module, then the graph Γ_V is connected. Indeed, every irreducible $U_q\widehat{\mathfrak{sl}}_2$-module is isomorphic to a tensor product of evaluation modules. The graph associated to each evaluation module is connected according to the explicit formulas for the corresponding q-characters (see formula (4.3) in [FR2]). Clearly, a tensor product of two modules with connected graphs also has a connected graph.

Lemma 6.3. *Let $\alpha \in \Gamma_V$ be a vertex with no incoming arrows. Then m_α is a dominant monomial.*

Proof. Let α contain $Y_{i,b}^{-1}$ for some $i \in I$, $b \in \mathbb{C}^\times$. We write the restricted q-character $\tau_i(\chi_q(V))$ in the form (4.4), where each $m_p(1 + \sum_r \overline{M}_{r,p})$ is a q-character of an irreducible $U_{q_i}\widehat{\mathfrak{sl}}_2$ module.

The monomial $\tau_i(m)$ contains $Y_{i,b}^{-1}$ and therefore can not be among the monomials $\{m_p N_p\}$. But the graphs of irreducible $U_q\widehat{\mathfrak{sl}}_2$-modules are connected. So we obtain that $\tau_i(m) = \tau_i(A_{i,c}^{-1})\tau_i(m')$ for some monomial m' in $\chi_q(V)$, and some $c \in \mathbb{C}^\times$. By Lemma 3.6, we have $m = A_{i,c}^{-1}m'$ which is a contradiction. □

Now Corollary 4.5 implies:

Corollary 6.4. *The graphs of all fundamental representations are connected.*

Let a monomial m have lattice support with base a. We call m *right negative* if the factors Y_{i,aq^k} appearing in m, for which k is maximal, have negative powers.

Lemma 6.5. *All monomials in the q-character of the fundamental representation $V_{\omega_i}(a)$, except for the highest weight monomial, are right negative.*

Proof. Let us show first that from the highest weight monomial m_+ there is only one outgoing arrow to the monomial $m_1 = m_+ A_{i,aq_i}^{-1}$. Indeed, the weight of a monomial that is connected to m_+ by an arrow has to be equal to $\omega_i - \alpha_j$ for some $j \in I$. The restriction of $V_{\omega_i}(a)$ to $U_q\widehat{\mathfrak{g}}$ is isomorphic to the direct some of its i^{th} fundamental representation V_{ω_i} and possibly some other irreducible representations with dominant weights less than ω_i. However, the weight $\omega_i - \alpha_j$ is not dominant for any i and j. Therefore this weight has to belong to the set of weights of V_{ω_i}, and the multiplicity of this weight in $V_{\omega_i}(a)$ has to be the same as that in V_{ω_i}. It is clear that the only weight of the form $\omega_i - \alpha_j$ that occurs in V_{ω_i} is $\omega_i - \alpha_i$, and it has multiplicity one. By Theorem 4.1, this monomial must have the form $m_1 = m_+ A_{i,aq_i}^{-1}$.

Now, the graph $\Gamma_{V_{\omega_i}(a)}$ is connected. Therefore each monomial m in $\chi_q(V_{\omega_i}(a))$ is a product of m_1 and factors $A_{j,b}^{-1}$. Note that m_1 is right negative and all $A_{j,b}^{-1}$ are right negative (this follows from the explicit formula (2.15)). The product of two right negative monomials is right negative. This implies the lemma. □

Remark 6.6. It follows from the proof of the lemma that the rightmost factor of each non-highest weight monomial occurring in $\chi_q(V_{\omega_i}(a))$ equals Y_{j,aq^n}^{-1}, where $n \geq 2r_i$. Moreover, the equality holds only for the above monomial m_1 (in that case $j = i$).

Recall the definition of the normalized R-matrix $\overline{R}_{V,W}(z)$ from Sect. 2.3. The following theorem was conjectured, e.g., in [AK].

Theorem 6.7. *Let $\{V_k\}_{k=1,\ldots,n}$, where $V_k = V_{\omega_{s(k)}}(a_k)$, be a set of fundamental representations of $U_q\widehat{\mathfrak{g}}$. The tensor product $V_1 \otimes \ldots \otimes V_n$ is reducible if and only if for some $i, j \in \{1, \ldots, n\}$, $i \neq j$, the normalized R-matrix $\overline{R}_{V_i,V_j}(z)$ has a pole at $z = a_j/a_i$.*

Proof. The "if" part of the theorem is obvious. Let us explain the case when $n = 2$. Let $\sigma : V_1 \otimes V_2 \to V_2 \otimes V_1$ be the transposition. By definition of $\overline{R}_{V_1,V_2}(z)$, the linear map $\sigma \circ \overline{R}_{V_1,V_2}(z)$ is a homomorphism of $U_q\widehat{\mathfrak{g}}$-modules $V_1 \otimes V_2 \to V_2 \otimes V_1$. Therefore if $\overline{R}_{V_1,V_2}(z)$ has a pole at $z = a_2/a_1$, then $V_1 \otimes V_2$ is reducible. It is easy to generalize this argument to general n.

Now we prove the "only if" part.

If the product $V_1 \otimes \cdots \otimes V_n$ is reducible, then the product of the q-characters $\prod_{i=1}^n \chi_q(V_i)$ contains a dominant monomial m that is different from the product of the highest weight monomials. Therefore m is not right negative and m is a product of some monomials m_i' from $\chi_q(V_i)$. Hence at least one of the factors $m_i' = m_i$ must be the highest weight monomial and it has to cancel with the rightmost $Y_{i,b}^{-1}$ appearing in, say, m_j'.

According to Lemma 6.1, $m_j' = m_j M$ where M is a product of $A_{s,a_jq^n}^{-1}$. By our assumption, the maximal n_0 occurring among n is such that $a_jq^{n_0} = a_iq_i^{-1}$. Using Lemma 2.6 we obtain that one of the diagonal entries of $\overline{R}_{V_i,V_j}$ has a factor $1/(1 - a_ia_j^{-1}z)$, which can not be cancelled. Therefore $\overline{R}_{V_i,V_j}$ has a pole at $z = a_j/a_i$. This proves the "only if" part. Moreover, we see that the pole necessarily occurs in a diagonal entry. □

6.2. The lowest weight monomial. Our next goal is to describe (see Proposition 6.15 below) the possible values of the spectral parameters of the fundamental representations for which the tensor product is reducible.

First we develop an analogue of the formalism of Sect. 4 from the point of view of the lowest weight monomials. Recall the involution $I \to I$, $i \to \bar{i}$ from Sect. 1.2. According to Theorem 1.3(3), there is a unique lowest weight monomial m_- in $\chi_q(V_{\omega_i}(a))$, and its weight is $-\omega_{\bar{i}}$.

Lemma 6.8. *The lowest weight monomial of $\chi_q(V_{\omega_i}(a))$ equals $Y_{\bar{i},aq^{r^\vee h^\vee}}^{-1}$.*

Proof. By Lemma 5.6, m_- must be antidominant. Thus, by Lemma 6.1, $m_- = Y_{\bar{i},aq^{n_i}}^{-1}$ for some $n_i > 0$.

Recall the automorphism w_0 defined in (1.7). The module $V_{\omega_{\bar{i}}}(a)$ is obtained from $V_{\omega_i}(a)$ by pull-back with respect to w_0. From the interpretation of the q-character in terms of the eigenvalues of $\Phi_i^{\pm}(u)$, it is clear that the q-character of $V_{\omega_{\bar{i}}}(a)$ is obtained from the q-character of $V_{\omega_i}(a)$ by replacing each $Y_{j,b}^{\pm 1}$ by $Y_{\bar{j},b}^{\pm 1}$. Therefore we obtain: $n_i = n_{\bar{i}}$.

Consider the dual module $V_{\omega_i}(a)^*$. By Theorem 1.3(3), its highest weight equals $\omega_{\bar{i}}$. Hence $V_{\omega_i}(a)^*$ is isomorphic to $V_{\omega_{\bar{i}}}(b)$ for some $b \in \mathbb{C}^\times$. Since $U_q\widehat{\mathfrak{g}}$ is a Hopf algebra, the module $V_{\omega_i}(a)^{\otimes} V_{\omega_i}(a)^*$ contains a one–dimensional trivial submodule. Therefore the product of the corresponding q-characters contains the monomial $m = 1$. According to Lemma 6.5, it can be obtained only as a product of the highest weight monomial in one q-character and the lowest monomial in another. Therefore, $b = aq^{\pm n_i}$.

In the same way we obtain that $V_{\omega_{\bar{i}}}(a)^*$ is isomorphic to $V_{\omega_i}(aq^{\pm n_i})$.

From formula (1.6) for the square of the antipode, we obtain that the double dual, $V_{\omega_i}(a)^{**}$, is isomorphic to $V_{\omega_i}(aq^{-2r^\vee h^\vee})$. Since $n_i > 0$, we obtain that $n_i = r^\vee h^\vee$. □

Having found the lowest weight monomial in the q-characters of the fundamental representations, we obtain using Theorem 1.3 the lowest weight monomial in the q-character of any irreducible module.

Corollary 6.9. *Let V be an irreducible $U_q\widehat{\mathfrak{g}}$-module. Let the highest weight monomial in $\chi_q(V)$ be*

$$m_+ = \prod_{i \in I} \prod_{k=1}^{s_k} Y_{i,a_k^{(i)}}.$$

Then the lowest weight monomial in $\chi_q(V)$ is given by

$$m_- = \prod_{i \in I} \prod_{k=1}^{s_k} Y^{-1}_{\bar{i},a_k^{(i)} q^{r^\vee h^\vee}}.$$

We also obtain a new proof of the following corollary, which has been previously proved in [CP1], Proposition 5.1(b):

Corollary 6.10.

$$V_{\omega_i}(a)^* \simeq V_{\omega_{\bar{i}}}(aq^{-r^\vee h^\vee}).$$

Now we are in position to develop the theory of q-characters based on the lowest weight and antidominant monomials as opposed to the highest weight and dominant ones.

Proposition 6.11. *The q-character of an irreducible finite-dimensional $U_q\widehat{\mathfrak{g}}$ module V has the form*

$$\chi_q(V) = m_-(1 + \sum N_p),$$

where m_- is the lowest weight monomial and each N_p is a monomial in $A_{i,c}$, $i \in I$, $c \in \mathbb{C}^\times$ (i.e., it does not contain any factors $A^{-1}_{i,c}$).

Proof. First we prove the following analogue of formula (4.1):

$$\chi_q(V) = m_-(1 + \sum_p N_p),$$

where each N_p is a monomial in $A^{\pm 1}_{i,c}$, $c \in \mathbb{C}^\times$. The proof of this formula is exactly the same as the proof of Proposition 3 in [FR2]. The rest of the proof is completely parallel to the proof of Theorem 4.1. □

Lemma 6.12. *The only antidominant monomial of q-character of a fundamental representation is the lowest weight monomial.*

Proof. The proof is completely parallel to the proof of Lemma 4.5. $\square$

Lemma 6.13. *All monomials in a q-character of a fundamental representation are products $Y_{i,aq^n}^{\pm 1}$ with $n \leq r^\vee h^\vee$.*

Proof. The proof is completely parallel to the proof of Lemma 6.1. $\square$

The combination of Lemmas 6.1 and 6.13 yields the following result.

Corollary 6.14. *Let the highest weight monomial m_+ of the q-character of an irreducible $U_q\widehat{\mathfrak{g}}$-module V be a product of monomials $m_+^{(i)}$ which have positive lattice support with bases a_i. Let s_i be the maximal integer s, such that Y_{k,a_iq^s} is present in $m_+^{(i)}$ for some $k \in I$. Then any monomial m in $\chi_q(V)$ can be written as a product of monomials $m^{(i)}$, where each $m^{(i)}$ is a product of Y_{j,a_iq^n} with $n \in \mathbb{Z}$, $0 \leq n \leq s_i + r^\vee h^\vee$*

6.3. Restrictions on the values of spectral parameters of reducible tensor products of fundamental representations. It was proved in [KS] that $V_{\omega_i}(a) \otimes V_{\omega_j}(b)$ is irreducible if a/b does not belong to a countable set. As M. Kashiwara explained to us, one can show that this set is then necessarily finite. The following proposition, which was conjectured, e.g., in [AK], gives a more precise description of this set.

Proposition 6.15. *Let $a_i \in \mathbb{C}$, $i = 1, \dots, n$, and suppose that the tensor product of fundamental representations $V_{\omega_{i_1}}(a_1) \otimes \dots \otimes V_{\omega_{i_n}}(a_n)$ is reducible. Then there exist $m \neq j$ such that $a_m/a_j = q^k$, where $k \in \mathbb{Z}$ and $2 \leq k \leq r^\vee h^\vee$.*

Proof. If $V_{\omega_{i_1}}(a_1) \otimes \dots \otimes V_{\omega_{i_n}}(a_n)$ is reducible, then $\chi_q(V_{\omega_{i_1}}(a_1)) \dots \chi_q(V_{\omega_{i_n}}(a_n))$ should contain a dominant term other than the product of the highest weight terms. But for that to happen, for some m and j, there have to be cancellations between some $Y_{p,a_mq^n}^{-1}$ appearing in $\chi_q(V_{\omega_{i_m}}(a_m))$ and some Y_{p,a_jq^l} appearing in $\chi_q(V_{\omega_{i_j}}(a_j))$. These cancellations may only occur if $a_m/a_j = q^{\pm k}$, $k \in \mathbb{Z}$, and $0 \leq k \leq r^\vee h^\vee$, by Lemmas 6.1 and 6.13. Moreover, $k \geq 2$ according to Remark 6.6. $\square$

Note that combining Theorem 6.7, Proposition 6.15 and Remark 6.6 we obtain:

Corollary 6.16. *The set of poles of the normalized R-matrix $\overline{R}_{V_{\omega_i}(a),V_{\omega_j}(a)}(z)$ is a subset of the set $\{q^k | k \in \mathbb{Z}, 2r_i \leq |k| \leq r^\vee h^\vee\}$, if $i = j$; $\{q^k | k \in \mathbb{Z}, 2r_i < k \leq r^\vee h^\vee$ or $2r_j < -k \leq r^\vee h^\vee\}$.*

6.4. The q-characters of the dual representations. In this subsection we show a simple way to obtain the q-character of the dual representation.

Recall that $\mathcal{K}$ is given by (5.6).

Lemma 6.17. *Let $\chi_1, \chi_2 \in \mathcal{K}$. Assume that all dominant monomials in χ_1 are the same as in χ_2 (counted with multiplicities). Then $\chi_1 = \chi_2$.*

Proof. Consider $\chi = \chi_1 - \chi_2$. We have $\chi \in \mathcal{K}$ and χ has no dominant monomials. Then $\chi = 0$ by Lemma 5.6. □

Note that the similar statement is true for antidominant monomials.

Proposition 6.18. *Let $V_{\omega_i}(a)$ be a fundamental representation. Then the q-character of the dual representation $V_{\omega_i}(a)^* \simeq V_{\omega_{\bar{i}}}(aq^{-r^\vee h^\vee})$ is obtained from the q-character of $V_{\omega_i}(a)$ by replacing each $Y^{\pm 1}_{i,aq^n}$ by $Y^{\mp 1}_{i,aq^{-n}}$.*

Proof. Let $\chi_1 = \chi_q(V_{\omega_{\bar{i}}}(aq^{-r^\vee h^\vee}))$ and χ_2 is obtained from $\chi(V_{\omega_i}(a))$ by replacing $Y^{\pm 1}_{i,aq^n}$ by $Y^{\mp 1}_{i,aq^{-n}}$. Then χ_1 and χ_2 are elements in $\mathcal{K}$ with the only dominant monomial $Y_{\bar{i},aq^{-r^\vee h^\vee}}$ by Corollary 4.5 and Lemma 6.12. Therefore $\chi_1 = \chi_2$ by Lemma 6.17. □

Remark 6.19. One can define a similar procedure for obtaining the q-character of the dual to any irreducible $U_q\widehat{\mathfrak{g}}$-module V. Namely, by Theorem 1.3, $\chi_q(V)$ is a subsum in the product of q-characters of fundamental representations. In particular, any monomial m in $\chi_q(V)$ is a product of monomials $m^{(i)}$ from the q-characters of these fundamental representations and Proposition 6.18 tells us what to do with each $m^{(i)}$. This procedure is consistent because $\chi_q((V \otimes W)^*) = \chi_q(V^*) \cdot \chi_q(W^*)$.

Note that under this procedure the dominant monomials go to the antidominant monomials and vice versa.

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