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Limiting Case of the Sobolev Inequality in BMO, with Application to the Euler Equations

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Dedicated to Professor Masayasu Mimura on the occasion of his 60th birthday

Abstract: We shall prove a logarithmic Sobolev inequality by means of the BMO-norm in the critical exponents. As an application, we shall establish a blow-up criterion of solutions to the Euler equations.

1. Introduction

The purpose of this paper is to establish an L^∞-estimate of functions in terms of the BMO norm and the logarithm of a norm of higher derivatives. It is well known that in $\mathbb{R}^n$ the Sobolev space $W^{s,p}$ with $sp > n$ is continuously embedded into L^∞. This is not true in the space $W^{k,r}$ for $kr = n$. Brezis–Gallouet [3] and Brezis-Wainger [4] investigated the relation between L^∞, $W^{k,r}$ and $W^{s,p}$ and proved that there holds the embedding

$$\|f\|_{L^\infty} \le C\left(1 + \log^{\frac{r-1}{r}}(1 + \|f\|_{W^{s,p}})\right), \quad sp > n \tag{1.1}$$

provided $\|f\|_{W^{k,r}} \le 1$ for $kr = n$. Then Ozawa [10,11] gave deep and systematic treatments and clarified the relation between (1.1), the Gagliardo-Nirenberg inequality and the Trudinger-Moser one. The estimate (1.1) was applied to prove existence of global solutions to the nonlinear Schrödinger equation([3,5]). Similar embedding for vector functions u with div $u = 0$ was investigated by Beale–Kato–Majda [1],

$$\|\nabla u\|_{L^\infty} \le C\left(1 + \|\text{rot }u\|_{L^\infty}(1 + \log^+ \|u\|_{W^{s+1,p}}) + \|\text{rot }u\|_{L^2}\right), \quad sp > n, \tag{1.2}$$

where $\log^+ a = \log a$ if $a \ge 1$, $= 0$ if $0 < a < 1$. In [1], they made use of (1.2) to give a blow-up criterion of solutions to the Euler equations.

The difference between these two embeddings stems from the bound of f in $W^{k,r}$ for $kr = n$ and that of rot u in L^∞. However, both of these bounds control f and ∇u in the common space BMO. In this paper, we will show a corresponding embedding estimate in L^∞ by means of the BMO-norm which covers (1.2). As an application of our estimate, we will extend the blow-up criterion of solutions to the Euler equations which was originally given by Beale–Kato–Majda [1]. It is proved in [1] that the L^∞ norm of vorticity controls breakdown of smooth solutions for the 3-D Euler equations. We will generalize such a criterion to the BMO-norm. The advantage to use BMO-space consists of the fact that Riesz transforms are bounded in BMO, but not in L^∞. This fact enables us to prove the same criterion not only by the vorticity but also by the deformation tensor (see Ponce [12]).

Our first result now reads:

Theorem 1. *Let* $1 < p < \infty$ *and let* $s > n/p$. *There is a constant* $C = C(n, p, s)$ *such that the estimate*

$$\|f\|_{L^\infty} \leq C \left(1 + \|f\|_{\mathrm{BMO}}(1 + \log^+ \|f\|_{W^{s,p}})\right) \tag{1.3}$$

holds for all $f \in W^{s,p}$.

Remark. Compared with (1.2), we do not need to add $\|f\|_{L^2}$ to the right-hand side of (1.3). This makes it easier to derive an apriori estimate of solutions to the Euler equations than Beale–Kato–Majda [1].

We next consider the Euler equations for the incompressible fluid motion in $\mathbb{R}^n$ for $n \geq 2$;

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } x \in \mathbb{R}^n, t > 0, \\ u\,|_{t=0} = a. \end{cases} \tag{E}$$

It is proved by Kato–Lai [7] and Kato–Ponce [8] that for every $a \in W^{s,p}$ for $s > n/p+1$ with $\operatorname{div} a = 0$, there are $T > 0$ and a unique solution u of (E) on the interval $[0, T)$ in the class

$$u \in C([0, T); W^{s,p}) \cap C^1([0, T); W^{s-2,p}). \tag{1.4}$$

The time interval T of existence of the solution u depends only on $\|a\|_{W^{s,p}}$. It is an interesting question whether the solution $u(t)$ really blows up as $t \uparrow T$.

Our result on (E) reads as follows.

Theorem 2. *Let* $1 < p < \infty$, $s > n/p + 1$. *Suppose that* u *is the solution of (E) in the class (1.4). If either*

$$\int_0^T \|\mathrm{rot}\, u(t)\|_{\mathrm{BMO}} dt (\equiv M_0) < \infty \tag{1.5}$$

or

$$\int_0^T \|\mathrm{Def}\, u(t)\|_{\mathrm{BMO}} dt (\equiv M_1) < \infty \tag{1.6}$$

holds, then u *can be continued to the solution in the class (1.4) on the interval* $[0, T')$ *for some* $T' > T$, *where* rot u *and* Def u *denote the vorticity and the deformation tensor of* u, *respectively.*

An immediate consequence of the above theorem is

Corollary 1. *Let u be the solution of (E) in the class (1.4) on the interval $[0, T)$ for $1 < p < \infty$, $s > n/p + 1$. Assume that T is maximal, i.e., u cannot be continued to the solution in the class (1.4) on $[0, T')$ for any $T' > T$. Then both*

$$\int_0^T \|\text{rot}\, u(t)\|_{\text{BMO}} dt = \infty \quad \textit{and} \quad \int_0^T \|\text{Def}\, u(t)\|_{\text{BMO}} dt = \infty$$

hold. In particular, we have

$$\limsup_{t\uparrow T} \|\text{rot}\, u(t)\|_{\text{BMO}} = \infty \quad \textit{and} \quad \limsup_{t\uparrow T} \|\text{Def}\, u(t)\|_{\text{BMO}} = \infty.$$

Remarks. 1. Beale–Kato–Majda [1], Ponce [12] and Kato–Ponce [8] obtained the same continuation principle of solutions as in Theorem 2 under the stronger assumption in L^∞.

2. Theorem 2 also holds for the Navier–Stokes equations. However, in the Navier–Stokes equations, on account of the viscosity term, a sharper estimate of solutions holds than for (2.17) below. Moreover, we can formulate the continuation principle for u itself in $L^2(0, T; \text{BMO})$. For details, see [9].

2. Proof of the Theorems

2.1. Proof of Theorem 1. We shall make use of the Littlewood-Paley decomposition; there exists a non-negative function $\varphi \in \mathcal{S}$ ($\mathcal{S}$; the Schwartz class) such that supp $\varphi \subset \{2^{-1} \le |\xi| \le 2\}$ and such that $\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1$ for $\xi \neq 0$. See Bergh-Löfström [2, Lemma 6.1.7]. Let us define ϕ_0 and ϕ_1 as

$$\phi_0(\xi) = \sum_{k=1}^{\infty} \varphi(2^k\xi) \quad \text{and} \quad \phi_1(\xi) = \sum_{k=-\infty}^{-1} \varphi(2^k\xi),$$

respectively. Then we have that $\phi_0(\xi) = 1$ for $|\xi| \le 1/2$, $\phi_0(\xi) = 0$ for $|\xi| \ge 1$ and that $\phi_1(\xi) = 0$ for $|\xi| \le 1$, $\phi_1(\xi) = 1$ for $|\xi| \ge 2$. It is easy to see that for every positive integer N there holds the identity

$$\phi_0(2^N\xi) + \sum_{k=-N}^{N} \varphi(2^{-k}\xi) + \phi_1(2^{-N}\xi) = 1, \quad \xi \neq 0. \tag{2.1}$$

Since C_0^∞ is dense in $W^{s,p}$ and since $W^{s,p}$ is continuously embedded in BMO, it suffices to prove (1.3) for $f \in C_0^\infty$. For such f we have the representation

$$f(x) = \int_{y\in\mathbb{R}^n} K(x-y) \cdot \nabla f(y) dy \quad \text{with} \quad K(y) = \frac{1}{n\omega_n} \frac{y}{|y|^n},$$

for all $x \in \mathbb{R}^n$, where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. By (2.1) we decompose f into three parts:

$$\begin{aligned}
& f(x) \\
&= \int_{y\in\mathbb{R}^n} K(x-y) \times \\
&\quad \times \left(\phi_0(2^N(x-y)) + \sum_{k=-N}^{N} \varphi(2^{-k}(x-y)) + \phi_1(2^{-N}(x-y))\right) \cdot \nabla f(y)dy \\
&\equiv f_0(x) + g(x) + f_1(x)
\end{aligned} \tag{2.2}$$

for all $x \in \mathbb{R}^n$.

Step 1. Estimate of f_0. Let us first consider the case $s \geq 1$. Since $s > n/p$, we can take q and q' so that $1/p - (s-1)/n \leq 1/q < 1/n$, $1/q' = 1 - 1/q$. Then there holds the Sobolev embedding $W^{s,p} \subset W^{2,q}$, so we have by integration by parts that

$$\begin{aligned}
|f_0(x)| &= \left(\int_{y\in\mathbb{R}^n} |K(x-y)\phi_0(2^N(x-y))|^{q'} dy\right)^{\frac{1}{q'}} \left(\int_{y\in\mathbb{R}^n} |\nabla f(y)|^q dy\right)^{\frac{1}{q}} \\
&\leq C \left(\int_{y\in\mathbb{R}^n} \frac{\phi_0(2^N(x-y))^{q'}}{|x-y|^{(n-1)q'}} dy\right)^{\frac{1}{q'}} \|\nabla f\|_{L^q} \\
&\leq C \left(\int_{|x-y|\leq 2^{-N}} \frac{1}{|x-y|^{(n-1)q'}} dy\right)^{\frac{1}{q'}} \|f\|_{W^{s,p}} \\
&\leq C \left(\int_0^{2^{-N}} r^{-(n-1)q'+n-1} dr\right)^{\frac{1}{q'}} \|f\|_{W^{s,p}} \\
&= C 2^{-N(1-n/q)} \|f\|_{W^{s,p}}
\end{aligned} \tag{2.3}$$

for all $x \in \mathbb{R}^n$.

We next consider the case $n/p < s < 1$. Let $H(y) \equiv K(y)\phi_0(y)$. For $\lambda > 0$, we define $H_\lambda(y) = H(\lambda y)$. Then we have

$$\begin{aligned}
f_0(x) &= \int_{y\in\mathbb{R}^n} K(y)\phi_0(2^N y)\nabla f(x-y)dy \\
&= 2^{N(n-1)} \int_{y\in\mathbb{R}^n} K(2^N y)\phi_0(2^N y)\sigma\tau_{-x}\nabla f(y)dy \\
&= -2^{N(n-1)} \left(H_{2^N}, \nabla\sigma\tau_{-x} f\right)_{L^2}
\end{aligned}$$

for all $x \in \mathbb{R}^n$, where $(\tau_h f)(y) = f(y-h)$, $(\sigma f)(y) = f(-y)$ and $(\cdot,\cdot)_{L^2}$ denotes the inner product in $L^2(\mathbb{R}^n)$. By integration by parts, from the above identity we obtain the identity

$$f_0(x) = 2^{N(n-1)} \left(\operatorname{div} (-\Delta)^{-\frac{s}{2}} H_{2^N}, \sigma\tau_{-x}(-\Delta)^{\frac{s}{2}} f\right)_{L^2}, \quad x \in \mathbb{R}^n. \tag{2.4}$$

On the other hand, there holds

$$\nabla(-\Delta)^{-\frac{s}{2}} H \in L^{p'} \quad \text{for } p' = p/(p-1). \tag{2.5}$$

Indeed, since

$$(-\Delta)^{-\frac{s}{2}} H(x) = C \int_{|y|\le 1} \frac{1}{|x-y|^{n-s}} K(y)\phi_0(y)dy, \quad x \in \mathbb{R}^n, \tag{2.6}$$

we have for $|x| \ge 2$,

$$\begin{aligned} |\nabla(-\Delta)^{-\frac{s}{2}} H(x)| &= C \left| \int_{|y|\le 1} \frac{x-y}{|x-y|^{n-s+2}} K(y)\phi_0(y)dy \right| \\ &\le C \int_{y \le 1} \frac{1}{|x-y|^{n-s+1}} \frac{1}{|y|^{n-1}} dy \\ &\le \frac{C}{|x|^{n-s+1}} \end{aligned} \tag{2.7}$$

with $C = C(n, s)$. For $|x| < 1$, we make use of another representation of (2.6) such as

$$(-\Delta)^{-\frac{s}{2}} H(x) = \frac{1}{|x|^{n-1-s}} \cdot h(x) \quad with \tag{2.8}$$

$$h(x) = C \int_{y\in\mathbb{R}^n} \frac{1}{|\frac{x}{|x|} - y|^{n-s}} \frac{y}{|y|^n} \phi_0(|x|y)dy.$$

For each $0 < |x| < 1$, we denote by Π_x the 2-dimensional plane spanned by x and $e_1 \equiv (1, 0, \cdots, 0)$. Let e_x be a unit vector in Π_x with $e_1 \cdot e_x = 0$ so that the pair $\{e_1, e_x\}$ is the orthonormal basis of Π_x. Furthermore, taking another $n-2$ unit vectors $e_x^{(3)}, \cdots, e_x^{(n)}$, we may assume that $\{e_1, e_x, e_x^{(3)}, \cdots, e_x^{(n)}\}$ is an orthonormal basis in $\mathbb{R}^n$. Let us define an orthogonal linear transformation S_x in such a way that

$$\begin{aligned} S_x e_1 &= \cos\theta_x \cdot e_1 - \sin\theta_x \cdot e_x, \\ S_x e_x &= \sin\theta_x \cdot e_1 + \cos\theta_x \cdot e_x, \\ S_x e_x^{(j)} &= e_x^{(j)}, \quad j = 3, \cdots, n, \end{aligned}$$

where θ_x is the angle between x and e_1. Since ϕ_0 is a radial symmetric function, we have by changing the variable $y \to y' = S_x y$ that

$$h(x) = \int_{y\in\mathbb{R}^n} \frac{1}{|e_1 - y|^{n-s}} \frac{S_x^t y}{|y|^n} \phi_0(|x|y)dy,$$

and hence there holds

$$|h(x)| \le C \int_{y\in\mathbb{R}^n} \frac{1}{|e_1 - y|^{n-s}} \frac{1}{|y|^{n-1}} dy \le C \tag{2.9}$$

for all $0 < |x| < 1$ with $C = C(n, s)$. Since $\cos\theta_x = x_1/|x|$, $\sin\theta_x = \pm\sqrt{\sum_{j=2}^n x_j^2}/|x|$, there holds

$$\left| \frac{\partial}{\partial x_j} \left(\frac{S_x^t y}{|y|^n} \right) \right| \le \frac{C}{|x|} \frac{1}{|y|^{n-1}}, \quad j = 1, \cdots, n, \quad \text{for all } 0 < |x| < 1,\ y \in \mathbb{R}^n$$

with $C = C(n)$ independent of x, y, which yields

$$
\begin{aligned}
|\nabla h(x)| &\le \int_{y\in\mathbb{R}^n} \frac{1}{|e_1 - y|^{n-s}} \left(\left| \nabla_x \frac{S_x^t y}{|y|^n} \right| + \frac{1}{|y|^{n-2}} |\nabla \phi_0(|x|y)| \right) dy \\
&\le \frac{C}{|x|} \int_{y\in\mathbb{R}^n} \frac{1}{|e_1 - y|^{n-s}} \frac{1}{|y|^{n-1}} dy + C \int_{\frac{1}{2|x|} \le |y| \le \frac{1}{|x|}} \frac{1}{|e_1 - y|^{n-s}} \frac{1}{|y|^{n-2}} dy \\
&\le \frac{C}{|x|} \qquad (2.10)
\end{aligned}
$$

for all $0 < |x| < 1$ with $C = C(n, s)$. Notice that $\operatorname{supp} \nabla\phi_0 \subset \{1/2 < |\xi| < 1\}$. Then it follows from (2.8), (2.9) and (2.10) that

$$
|\nabla(-\Delta)^{-\frac{s}{2}} H(x)| \le \frac{C}{|x|^{n-s}} \quad \text{for all } |x| < 1. \qquad (2.11)
$$

Since $n/p < s < 1$, from (2.7) and (2.11) we obtain (2.5). Since

$$
\begin{aligned}
\|(-\Delta)^{\frac{1-s}{2}} H_{2^N}\|_{L^{p'}} &= 2^{N(1-s-n/p')} \|(-\Delta)^{\frac{1-s}{2}} H\|_{L^{p'}} \\
&= 2^{N(1+n/p-n-s)} \|(-\Delta)^{\frac{1-s}{2}} H\|_{L^{p'}},
\end{aligned}
$$

it follows from (2.4) and the Hölder inequality that

$$
\begin{aligned}
|f_0(x)| &\le 2^{N(n-1)} \|\mathrm{div}\, (-\Delta)^{-\frac{s}{2}} H_{2^N}\|_{L^{p'}} \|\sigma\tau_{-x}(-\Delta)^{\frac{s}{2}} f\|_{L^p} \\
&\le C 2^{N(n-1)} \|(-\Delta)^{\frac{1-s}{2}} H_{2^N}\|_{L^{p'}} \|(-\Delta)^{\frac{s}{2}} f\|_{L^p} \\
&\le C 2^{-N(s-n/p)} \|f\|_{W^{s,p}} \qquad (2.12)
\end{aligned}
$$

for all $x \in \mathbb{R}^n$. Now, by (2.3) and (2.12) we have in both cases

$$
\|f_0\|_{L^\infty} \le C 2^{-N\beta} \|f\|_{W^{s,p}} \quad \text{with } \beta = \mathrm{Min.}\{1 - n/q, s - n/p\}, \qquad (2.13)
$$

where $C = C(n, p, s)$ is independent of N.

Step 2. Estimate of g. For each $x \in \mathbb{R}^n$, we take $b_k(x)$ so that

$$
b_k(x) = \frac{1}{|B(x, 2^{k+1})|} \int_{B(x,2^{k+1})} f(y) dy, \quad k = 0, \pm 1, \cdots, \pm N,
$$

where $B(x, R)$ denotes the ball centered at x with radius R and $|B|$ is the volume of B. Since $\operatorname{supp} \varphi(2^k(x - \cdot)) \subset \{y \in \mathbb{R}^n; 2^{k-1} \le |y - x| \le 2^{k+1}\}$, we have by integration by parts

$$
\begin{aligned}
|g(x)| &= \left| \sum_{k=-N}^{N} \int_{y\in\mathbb{R}^n} K(x-y)\varphi(2^{-k}(x-y))\nabla_y\,(f(y)-b_k(x))\,dy \right| \\
&= \left| \sum_{k=-N}^{N} \int_{y\in\mathbb{R}^n} \mathrm{div}_{\,y}\left(K(x-y)\varphi(2^{-k}(x-y))\right)(f(y)-b_k(x))\,dy \right| \\
&\le C \sum_{k=-N}^{N} \int_{2^{k-1}\le|y-x|\le 2^{k+1}} \left(\frac{1}{|x-y|^n} + \frac{2^{-k}}{|x-y|^{n-1}}\right)|f(y)-b_k(x)|dy \\
&\le C \sum_{k=-N}^{N} \frac{1}{2^{(k+1)n}} \int_{2^{k-1}\le|y-x|\le 2^{k+1}} |f(y)-b_k(x)|dy \\
&\le C \sum_{k=-N}^{N} \frac{1}{|B(x,2^{k+1})|} \int_{B(x,2^{k+1})} |f(y)-b_k(x)|dy \\
&\le CN\|f\|_{\mathrm{BMO}}
\end{aligned}
$$

for all $x\in\mathbb{R}^n$, which implies that

$$
\|g\|_{L^\infty} \le CN\|f\|_{\mathrm{BMO}} \tag{2.14}
$$

with $C = C(n)$ independent of N.

Step 3. Estimate of f_1. Integrating by parts, we have by a direct calculation

$$
\begin{aligned}
&|f_1(x)| \\
&= \left| \int_{y\in\mathbb{R}^n} \mathrm{div}_{\,y}\left(K(x-y)\phi_1(2^{-N}(x-y))\right) f(y)dy \right| \\
&\le \left| \int_{y\in\mathbb{R}^n} \mathrm{div}\,K(x-y)\phi_1(2^{-N}(x-y))f(y)dy \right| \\
&\quad + 2^{-N} \left| \int_{y\in\mathbb{R}^n} K(x-y)\cdot\nabla\phi_1(2^{-N}(x-y))f(y)dy \right| \\
&\le C\int_{2^N\le|x-y|} |x-y|^{-n}|f(y)|dy \\
&\quad + C2^{-N}\int_{2^N\le|x-y|\le 2^{N+1}} |x-y|^{1-n}|f(y)|dy \\
&\le C\left(\int_{2^N\le|x-y|} |x-y|^{-np'}dy\right)^{1/p'}\|f\|_{L^p} \\
&\quad + C2^{-N}\left(\int_{2^N\le|x-y|\le 2^{N+1}} |x-y|^{-(n-1)p'}dy\right)^{1/p'}\|f\|_{L^p} \\
&\le C\left\{\left(\int_{2^N}^{\infty} r^{-np'+n-1}dr\right)^{1/p'} + 2^{-N}\left(\int_{2^N}^{2^{N+1}} r^{-(n-1)p'+n-1}dr\right)^{1/p'}\right\}\|f\|_{L^p} \\
&\le C2^{-N\cdot\frac{n}{p}}\|f\|_{L^p}
\end{aligned}
$$

for all $x \in \mathbb{R}^n$, which yields

$$\|f_1\|_{L^\infty} \leq 2^{-N\cdot\frac{n}{p}}\|f\|_{L^p} \tag{2.15}$$

with $C = C(n, p)$ independent of N.

Now it follows from (2.2) and (2.13)-(2.15) that

$$\|f\|_{L^\infty} \leq C(2^{-\gamma N}\|f\|_{W^{s,p}} + N\|f\|_{\mathrm{BMO}}) \tag{2.16}$$

with $\gamma = \mathrm{Min.}\{1 - n/q, s - n/p, n/p\}$, where $C = C(n, s, p)$ is independent of N and f. If $\|f\|_{W^{s,p}} \leq 1$, then we may take $N = 1$; otherwise, we take N so large that the first term of the right-hand side of (2.16) is dominated by 1, i.e., $N \equiv \left[\frac{\log\|f\|_{W^{s,p}}}{\gamma\log 2}\right] + 1$ ($[\cdot]$; Gauss symbol) and (2.16) becomes

$$\|f\|_{L^\infty} \leq C\left\{1 + \|f\|_{\mathrm{BMO}}\left(\frac{\log\|f\|_{W^{s,p}}}{\gamma\log 2} + 1\right)\right\}.$$

In both cases, (1.3) holds. This proves Theorem 1.

Remark. There is a simple alternative proof for (2.14). Indeed, we have

$$g(x) = \sum_{k=-N}^{N}(\mathrm{div}\ \Psi)_{\Theta^k} * f(x), \quad x \in \mathbb{R}^n,$$

where $\Psi(x) = K(x)\varphi(x)$ and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Since $\Psi \in \mathcal{S}$ with the property that

$$\int_{\mathbb{R}^n}\mathrm{div}\ \Psi(x)dx = 0,$$

it follows from Stein [13, Chap. IV, 4.3.3] that

$$\begin{aligned}\|g\|_{L^\infty} &\leq \sum_{k=-N}^{N}\|(\mathrm{div}\ \Psi)_{\Theta^k} * f\|_{L^\infty}\\ &\leq \sum_{k=-N}^{N}\sup_{t>0}\|(\mathrm{div}\ \Psi)_t * f\|_{L^\infty}\\ &\leq CN\|f\|_{\mathrm{BMO}},\end{aligned}$$

which yields (2.14).

2.2. Proof of Theorem 2. It is proved by Kato–Lai [7] and Kato–Ponce [8] that for the given initial data $a \in W^{s,p}$ for $s > 1 + n/p$, the time interval T of the existence of the solution u to (E) in the class (1.4) depends only on $\|a\|_{W^{s,p}}$. Hence by the standard argument of continuation of local solutions, it suffices to establish an apriori estimate for u in $W^{s,p}$ in terms of a, T, M_0 or a, T, M_1 according to (1.5) or (1.6). Indeed, we shall show that the solution $u(t)$ in the class (1.4) is subject to the following estimate:

$$\sup_{0<t<T} \|u(t)\|_{W^{s,p}} \le (\|a\|_{W^{s,p}} + e)^{\alpha_j} \exp(CT\alpha_j) \quad \text{with } \alpha_j = e^{CM_j}, \quad j = 0, 1, \tag{2.17}$$

where $C = C(n, p, s)$ is a constant independent of a and T.

We shall first prove (2.17) under (1.5). It follows from the commutator estimate in L^p given by Kato–Ponce [8, Proposition 4.2] that

$$\|u(t)\|_{W^{s,p}} \le \|a\|_{W^{s,p}} \exp\left(C \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau\right), \quad 0 < t < T, \tag{2.18}$$

where $C = C(n, p, s)$. In case $p = 2$, i.e., in the $W^{s,2}$-estimate, this can be done more directly as in Beale–Kato–Majda [1, p. 64, Eq. (14)].

By the Biot-Savard law, we have a representation of ∇u in terms of $\omega \equiv \text{rot}\, u$ as

$$\frac{\partial u}{\partial x_j} = R_j(R \times \omega), \quad j = 1, \cdots, n, \tag{2.19}$$

where $R = (R_1, \cdots, R_n)$, $R_j = \dfrac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}}$ denote the Riesz transforms. Since R is a bounded operator in BMO, this yields

$$\|\nabla u\|_{\text{BMO}} \le C\|\omega\|_{\text{BMO}} \tag{2.20}$$

with $C = C(n)$. Hence it follows from (2.20) and Theorem 1 that

$$\|\nabla u(t)\|_{L^\infty} \le C\left(1 + \|\omega(t)\|_{\text{BMO}}(1 + \log^+ \|u(t)\|_{W^{s,p}})\right) \tag{2.21}$$

for all $0 < t < T$ with $C = C(n, p, s)$. Substituting (2.21) to (2.18), we have

$$\begin{aligned} &\|u(t)\|_{W^{s,p}} + e \\ &\le (\|a\|_{W^{s,p}} + e) \exp\left(C \int_0^t \{1 + \|\omega(\tau)\|_{\text{BMO}} \log(\|u(\tau)\|_{W^{s,p}} + e)\} d\tau\right) \end{aligned}$$

for all $0 < t < T$. Defining $z(t) \equiv \log(\|u(t)\|_{W^{s,p}} + e)$, we obtain from the above estimate

$$z(t) \le z(0) + CT + C \int_0^t \|\omega(\tau)\|_{\text{BMO}} z(\tau) d\tau, \quad 0 < t < T.$$

Now (1.5) and the Gronwall inequality yield

$$\begin{aligned} z(t) &\le (z(0) + CT) \exp\left(C \int_0^t \|\omega(\tau)\|_{\text{BMO}} d\tau\right) \\ &\le (z(0) + CT)\, \alpha_0 \end{aligned}$$

for all $0 < t < T$ with $C = C(n, p, s)$, which implies (2.17) for $j = 0$.

Next, assume (1.6). Instead of (2.19) we make use of another representation

$$\frac{\partial u^l}{\partial x_j} = R_j(\sum_{k=1}^n R_k \text{Def}\, u_{kl}), \quad j, l = 1, \cdots, n, \quad \text{where Def}\, u_{kl} = \frac{\partial u^k}{\partial x_l} + \frac{\partial u^l}{\partial x_k}.$$

Hence again by the boundedness of Riesz transforms in BMO, there holds

$$\|\nabla u\|_{\mathrm{BMO}} \le C\|\mathrm{Def}\, u\|_{\mathrm{BMO}}. \tag{2.22}$$

Then by (2.22) and Theorem 1 we have similarly to (2.21) that

$$\|\nabla u(t)\|_{L^\infty} \le C\left(1 + \|\mathrm{Def}\, u(t)\|_{\mathrm{BMO}}(1 + \log^+ \|u(t)\|_{W^{s,p}})\right)$$

for all $0 < t < T$ with $C = C(n, p, s)$. It is easy to see that the rest of the argument is parallel to that of the case when (1.5) holds, so we get also (2.17) for $j = 1$. This proves Theorem 2.

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