Characteristic Polynomials of Random Matrices

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Abstract: Number theorists have studied extensively the connections between the distribution of zeros of the Riemann ζ-function, and of some generalizations, with the statistics of the eigenvalues of large random matrices. It is interesting to compare the average moments of these functions in an interval to their counterpart in random matrices, which are the expectation values of the characteristic polynomials of the matrix. It turns out that these expectation values are quite interesting. For instance, the moments of order $2K$ scale, for unitary invariant ensembles, as the density of eigenvalues raised to the power K^2; the prefactor turns out to be a universal number, i.e. it is independent of the specific probability distribution. An equivalent behaviour and prefactor had been found, as a conjecture, within number theory. The moments of the characteristic determinants of random matrices are computed here as limits, at coinciding points, of multi-point correlators of determinants. These correlators are in fact universal in Dyson's scaling limit in which the difference between the points goes to zero, the size of the matrix goes to infinity, and their product remains finite.

1. Introduction

The correlation function of the eigenvalues of large $N \times N$ matrices are known to exhibit a number of universal features in the large-N limit. For instance in the Dyson limit [1,2], when the distances between these eigenvalues, measured in units of the local spacing, becomes of order $1/N$, the correlation functions, as well as the level spacing distribution, become universal, i.e. independent of the specific probability measure. For finite differences, upon a smoothing of the distribution, the two-point correlation function is again universal [3,4]. The short distance universality was also shown to extend to external source problems [5–8], in which an external matrix is coupled to the random matrix.

In this article, we study the average of the characteristic polynomials, whose zeros are the eigenvalues of the random matrix. The probability distribution of the characteristic

polynomial $\det(\lambda - X)$ of a random matrix X, a polynomial of degree N in λ, may be characterized by its moments $\langle \det^K(\lambda - X) \rangle$, or better by its correlation functions

$$\left\langle \prod_{l=1}^{K} \det(\lambda_l - X) \right\rangle.$$

This study is motivated by various conjectures which appeared recently in number theory for the zeros of the Riemann ζ-function and its generalizations known as L-functions [12]. Indeed the characteristic polynomials, as well as the zeta-fuctions, have their zeros on a straight line, and these zeros obey the same statistical distribution.

For the $2K^{\text{th}}$ moment of the Riemann ζ-function (K is a positive integer), it has been conjectured [9,10] that

$$\frac{1}{T} \int_0^T dt |\zeta\left(\frac{1}{2} + it\right)|^{2K} \simeq \gamma_K a_K (\log T)^{K^2}, \tag{1}$$

where a_K is a number related to the Dirichlet coefficient (the divisor function) $d_K(n)$, and

$$\gamma_K = \prod_{l=0}^{K-1} \frac{l!}{(l+K)!}. \tag{2}$$

The explicit formula for a_K is given in the Appendix, together with summation formulae for the Dirichlet coefficients, which are related to (1). In this work we shall compute the equivalent of (1) for random matrices, show that the density of states $\rho(\lambda)$ replaces $\log T$, and that the same number γ_K is universally present.

For the negative moments, similar conjectures have been proposed, with a cut-off parameter δ for avoiding divergences [11], and we show here how to obtain these negative moments for random matrices.

Several types of L-functions have been introduced [12], which correspond to the three standard classes of random matrices. The conjecture for the average of the moments (1) has been extended to these L-functions [13]. The average is taken as a sum of the discriminant d, for instance, for the Dirichlet $L(\frac{1}{2}, \chi_d)$ function. The relations between the distributions of the eigenvalues of the random matrix theory and the statistical distribution of the zeros of the various L-functions has also been studied [12, 14].

Our aim in this article, is to clarify the universality of the moments of the characteristic polynomials for these three classes. The circular unitary ensemble has been studied earlier by Keating and Snaith [10], who did obtain the γ_K in (2) from their calculation. However this ensemble has a constant density of states, and furthermore it does not allow to study the universality of these properties. In this work we have considered a Gaussian ensemble and non-Gaussian extensions, instead of the circular ensemble, to verify both the explicit dependence in the density of states and the universality of the coefficient γ_K. In the process of the derivation, we have found it necessary to start with the K-point functions $\left\langle \prod_{l=1}^{K} \det(\lambda_l - X) \right\rangle$, which are shown to be themselves universal in the large-N Dyson limit, in which $N(\lambda_i - \lambda_j)$ is held fixed. The moments are then simply the limit of these functions when all the Dyson variables vanish.

2. Correlation Functions of Characteristic Polynomials

We consider random $M \times M$ Hermitian matrices X with a normalized probability distribution

$$P(X) = \frac{1}{Z} \exp -N \mathrm{Tr} V(X), \tag{3}$$

in which V is a given polynomial. It will turn out to be convenient to distinguish here M and N, but we later restrict ourselves to a large N and large M limit, with $\lim M/N = 1$. Let us consider the correlation function of K distinct characteristic polynomials:

$$F_K(\lambda_1, \cdots, \lambda_K) = \left\langle \prod_{\alpha=1}^{K} \det(\lambda_\alpha - X) \right\rangle, \tag{4}$$

in which the bracket denotes an expectation value with the weight (3).

Integrating as usual over the unitary group, we obtain

$$F_K(\lambda_1, \cdots, \lambda_K) = \frac{1}{Z_M} \int \prod_{1}^{M} d\mu(x_i) \, \Delta^2(x_1, \cdots, x_M) \prod_{\alpha=1}^{K} \prod_{i=1}^{M} (\lambda_\alpha - x_i) \tag{5}$$

in which $d\mu(x)$ denotes the measure $d\mu(x) = dx \exp -NV(x)$, Δ the Vandermonde determinant $\Delta(x_1, \cdots, x_M) = \prod_{1 \le i < j \le M} (x_i - x_j)$, and Z_M the normalization constant

$$Z_M = \int \prod_{1}^{M} d\mu(x_i) \, \Delta^2(x_1, \cdots, x_M). \tag{6}$$

We now use the obvious identity

$$\Delta(x_1, \cdots, x_M) \prod_{\alpha=1}^{K} \prod_{i=1}^{M} (\lambda_\alpha - x_i) = \frac{\Delta(x_1, \cdots, x_M; \lambda_1, \cdots, \lambda_K)}{\Delta(\lambda_1, \cdots, \lambda_K)}, \tag{7}$$

and represent the Vandermonde determinants $\Delta(x_1, \cdots, x_M)$ and $\Delta(x_1, \cdots, x_M; \lambda_1, \cdots, \lambda_K)$ as determinants of arbitrary polynomials whose coefficients of highest degree are equal to unity (the so-called monic polynomials)

$$p_n(x) = x^n + \text{lowerdegree.} \tag{8}$$

Then

$$\Delta(x_1, \cdots, x_M) = \det p_n(x_m) \tag{9}$$

(n runs from zero to $M - 1$ and m from one to M), and

$$\Delta(x_1, \cdots, x_M; \lambda_1, \cdots, \lambda_K) = \det p_a(u_b) \tag{10}$$

in which a runs from zero to $M + K - 1$, b from one to $M + K$ and u_b stands for x_b if $b \le M$, or λ_b for $M < b \le M + K$.

Choosing now the polynomials orthogonal with respect to the measure $d\mu$:

$$\int p_n(x)p_m(x)d\mu(x) = h_n\delta_{nm}, \tag{11}$$

we may easily integrate over the M eigenvalues

$$\int \prod_1^M d\mu(x_i)\, \Delta(x_1,\cdots,x_M;\lambda_1,\cdots,\lambda_K)\Delta(x_1,\cdots,x_M)$$

$$= M!\left(\prod_0^{M-1} h_n\right)\det p_\alpha(\lambda_\beta), \tag{12}$$

in which α runs from M to $M+K-1$ and β from 1 to K. Similarly the normalization factor Z_M is given by

$$Z_M = \int \prod_1^M d\mu(x_i)\, \Delta^2(x_1,\cdots,x_M) = M!\left(\prod_0^{M-1} h_n\right). \tag{13}$$

We thus end up with

$$F_K(\lambda_1,\cdots,\lambda_K)$$

$$= \frac{1}{\Delta(\lambda_1,\cdots,\lambda_K)}\det\begin{vmatrix} p_M(\lambda_1) & p_{M+1}(\lambda_1) & \cdots & p_{M+K-1}(\lambda_1) \\ p_M(\lambda_2) & p_{M+1}(\lambda_2) & \cdots & p_{M+K-1}(\lambda_2) \\ \vdots & & & \\ p_M(\lambda_K) & p_{M+1}(\lambda_K) & \cdots & p_{M+K-1}(\lambda_K) \end{vmatrix}. \tag{14}$$

If we are concerned simply with the moments of the distribution of a single characteristic polynomial, we obtain from (14),

$$\mu_K(\lambda) = F_K(\lambda,\cdots,\lambda) = \langle\, [\det(\lambda - X)]^K\, \rangle$$

$$= \frac{(-1)^{K(K-1)/2}}{\prod_{l=0}^{K-1}(l!)}\det\begin{vmatrix} p_M(\lambda) & p_{M+1}(\lambda) & \cdots & p_{M+K-1}(\lambda) \\ p'_M(\lambda) & p'_{M+1}(\lambda) & \cdots & p'_{M+K-1}(\lambda) \\ \vdots & & & \\ p_M^{(K-1)}(\lambda) & p_{M+1}^{(K-1)}(\lambda) & \cdots & p_{M+K-1}^{(K-1)}(\lambda) \end{vmatrix}. \tag{15}$$

These expressions are all exact, but in the next section we shall be concerned with the large N limit. Then (i) the interesting case is that of even K, since for odd K the result is oscillatory (for instance for $K = 1$ $\mu_1(\lambda) = p_M(\lambda)$), (ii) it will turn out that, even if we are interested simply in the moments $\mu_K(\lambda)$, it is more convenient to study first the large N-limit of the F_K with distinct λ_i and afterwards let them approach a single λ.

The results that will be derived later for those F_K's and μ_K's will be shown to be universal in the Dyson limit, in which N goes to infinity, the $\lambda_i - \lambda_j$ goes to zero for any pair i, j, and the products $N(\lambda_i - \lambda_j)$ remain finite. We first derive explicit formulae for the Gaussian case, and show later that they do apply to any random matrix distribution $P(X)$ of the form (3).

3. The Gaussian Case

We now specialize the result (14) of the previous section to the Gaussian distribution of $M \times M$ Hermitian matrices

$$P(X) = \frac{1}{Z_M} \exp -\frac{N}{2} \mathrm{Tr} X^2, \tag{16}$$

with

$$M = N - K, \tag{17}$$

(the reason for this choice of M will be clarified in the next section). Then the polynomials that we have introduced, are Hermite polynomials, and with our normalizations,

$$H_n(x) = \frac{(-1)^n}{N^n} e^{Nx^2/2} \left(\frac{d}{dx}\right)^n e^{-Nx^2/2} = x^n + \text{l.d.}, \tag{18}$$

and

$$h_n = \frac{n!}{N^n} \sqrt{\frac{2\pi}{N}}. \tag{19}$$

The integral representation

$$H_n(x) = \frac{(-1)^n n!}{N^n} \oint \frac{dz}{2i\pi} \frac{e^{-N(z^2/2 + xz)}}{z^{(n+1)}} \tag{20}$$

over a contour which circles around the origin in the z-plane, turns out to be well adapted. Repeated use of this formula in the result (14) yields

$$F_{2K}(\lambda_1, \cdots, \lambda_{2K}) = \frac{(-1)^K}{\Delta(\lambda_1, \cdots, \lambda_{2K})} \frac{\prod_{l=0}^{2K-1}(M+l)!}{N^{K(2M+2K-1)}}$$

$$\times \oint \prod_{l=1}^{2K} \left(\frac{dz_l}{2i\pi z_l^{M+l}}\right) \exp -\left(N \sum_{1}^{2K} \frac{z_l^2}{2}\right) \det(e^{-N\lambda_a z_b}). \tag{21}$$

We can expand the determinant in the r.h.s. and keep only one of the $(2K)!$ terms, antisymmetrizing instead the integration variables z_l. This gives

$$F_{2K}(\lambda_1, \cdots, \lambda_{2K}) = \frac{(-1)^K}{\Delta(\lambda_1, \cdots, \lambda_{2K})} \frac{\prod_{l=0}^{2K-1}(M+l)!}{N^{K(2M+2K-1)}}$$

$$\times \oint \prod_{l=1}^{2K} \left(\frac{dz_l}{2i\pi z_l^{M+2K}}\right) \exp[-N \sum_{1}^{2K} (\frac{z_l^2}{2} + \lambda_l z_l)] \Delta(z_1, \cdots, z_{2K}). \tag{22}$$

This expression for the expectation value of a product of $2K$ characteristic polynomials, as an integral over $2K$ complex variables, is exact for finite N and M.

We are now in position to study the large N-limit through a saddle point integration over each z_l. Since we have chosen $M + K = N$ each z has a weight $\frac{1}{z^K} \exp -N(z^2/2 +$

$\lambda z + \log z$), which presents two saddle points $z_\pm$, solutions of the equation $z^2 + \lambda z + 1 = 0$, i.e. with the parametrization

$$\lambda = 2\sin\phi, \tag{23}$$

when λ lies on the support of the asymptotic Wigner semi-circle of the density of levels,

$$z_+ = ie^{i\phi}, z_- = -ie^{-i\phi}. \tag{24}$$

Therefore there are, a priori 2^{2K} saddle-points at which the moduli of the weight $\exp\left[-N\sum_1^{2K}(\frac{z_l^2}{2} + \lambda_l z_l + \log z_l)\right]$ are the same. However, it is only when $\sum_1^{2K}(\frac{z_l^2}{2} + \lambda_l z_l + \log z_l)$ is real (in the Dyson limit in which the differences between the λ's are small), that the oscillations, which damp the result, are not present. Therefore we keep only the $\binom{2K}{K}$ saddle-points in which we take K solutions of type z_+ and K of type z_-. We are now interested in Dyson's short-distance limit. Defining

$$\lambda = \frac{1}{2K}\sum_{l=1}^{2K}\lambda_l, \tag{25}$$

and the density of eigenvalues at this point

$$\rho(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2} = \frac{1}{\pi}\cos\phi, \tag{26}$$

we introduced the scaling variables

$$x_a = 2\pi N\rho(\lambda)(\lambda_a - \lambda), \text{ with } \sum_{a=1}^{2K} x_a = 0, \tag{27}$$

which are kept finite in this limit. Then the fluctuations around a saddle-point may be taken all at the point λ, and they yield a factor

$$(\frac{2\pi}{N})^K[(1 - z_+^2)(1 - z_-^2)]^{-K/2} = (N\rho(\lambda))^{-K}. \tag{28}$$

We must now take into account the various factors in (22) at these saddle-points. In the Dyson limit the factor $\prod_1^{2K} z_l^K$ which remained in the denominator, may be replaced by one, since at a given λ one has $z_+ z_- = 1$. The only delicate factor is thus $\frac{\Delta(z_1, \cdots, z_{2K})}{\Delta(\lambda_1, \cdots, \lambda_{2K})}\exp\left[-N\sum_1^{2K}(\frac{z_l^2}{2} + \lambda_l z_l + \log z_l)\right]$, which we must first compute at one of the saddle-points, and then take the sum over the $\binom{2K}{K}$ saddle-points. We consider first the saddle-point

$$\begin{aligned} z_l(\lambda_l) &= z_+(\lambda_l) \quad l = 1, \cdots, K, \\ z_l(\lambda_l) &= z_-(\lambda_l) \, l = K+1, \cdots, 2K. \end{aligned} \tag{29}$$

If we expand in the Dyson limit the weight $\exp\left[-N\sum_1^{2K}\left(\frac{z_l^2}{2}+\lambda_l z_l+\log z_l\right)\right]$ one finds

$$\exp\left[-N\sum_1^{2K}\left(\frac{z_l^2}{2}+\lambda_l z_l+\log z_l\right)\right]$$

$$= \exp NK\left(1+\frac{\lambda^2}{2}\right)\times\exp-N\left[\sum_1^K(\lambda_l-\lambda)z_+(\lambda)+\sum_{K+1}^{2K}(\lambda_l-\lambda)z_-(\lambda)\right], \quad (30)$$

(we have used $\frac{d}{d\lambda}\left(\frac{1}{2}z_\pm^2(\lambda)+\lambda z_\pm+\log z_\pm\right)=z_\pm$). Therefore at that saddle-point, in terms of the scaling variables (27),

$$\exp\left[-N\sum_1^{2K}\left(\frac{z_l^2}{2}+\lambda_l z_l+\log z_l\right)\right]=\exp NK\left(1+\frac{\lambda^2}{2}\right)\exp-i\sum_1^K x_l. \quad (31)$$

Let us consider now the ratio of Vandermonde determinants at that same saddle-point:

$$\frac{\Delta(z_1,\cdots,z_{2K})}{\Delta(\lambda_1,\cdots,\lambda_{2K})}=\prod_{1\le l<m\le K}\frac{z_+(\lambda_l)-z_+(\lambda_m)}{\lambda_l-\lambda_m}\prod_{K+1\le l<m\le 2K}\frac{z_-(\lambda_l)-z_-(\lambda_m)}{\lambda_l-\lambda_m}$$

$$\times\prod_{1\le l\le K,K+1\le m\le 2K}\frac{z_+(\lambda_l)-z_-(\lambda_m)}{\lambda_l-\lambda_m}. \quad (32)$$

In the scaling limit, this factor becomes

$$\frac{\Delta(z_1,\cdots,z_{2K})}{\Delta(\lambda_1,\cdots,\lambda_{2K})}=\left(\frac{dz_+}{d\lambda}\frac{dz_-}{d\lambda}\right)^{K(K-1)/2}(2i\cos\phi)^{K^2}\prod_{1\le l\le K,K+1\le m\le 2K}\frac{1}{\lambda_l-\lambda_m}$$

$$= (Ni)^{K^2}(2\pi\rho(\lambda))^{K+K^2}\prod_{1\le l\le K,K+1\le m\le 2K}\frac{1}{x_l-x_m}. \quad (33)$$

Leaving aside for the moment the overall factors which do not change at the various saddle-points, we note the result from this particular one which is

$\exp-i\sum_1^K x_l\prod_{1\le l\le K,K+1\le m\le 2K}\frac{1}{x_l-x_m}$, and consider summing over the $\binom{2K}{K}$ saddle-points. The sum is best done under the form of an integral over K variables. Indeed, if we consider

$$I(x_1,\cdots,x_{2K})=\frac{(-1)^{K(K-1)/2}}{K!}\oint\prod_1^K\frac{du_\alpha}{2i\pi}\exp-i\left(\sum_{l=1}^K u_\alpha\right)\frac{\Delta^2(u_1,\cdots,u_K)}{\prod_{\alpha=1}^K\prod_{l=1}^{2K}(u_\alpha-x_l)} \quad (34)$$

over a contour in which each u_α circles around the x's, we recover exactly the contribution previous saddle-point by choosing $u_1=x_1,\cdots,u_K=x_K$, or any permutation of those K x's. In view of the Vandermonde in the numerator, all the u's have to be different, and

thus there are indeed $\binom{2K}{K}$ poles to be added, which reconstruct exactly the sum on the saddle-points that we needed to perform.

Collecting the various factors that came on the way, we end up with the final formula

$$
\exp -\left(\frac{N}{2}\sum_{l=1}^{2K} V(\lambda_l)\right) F_{2K}(\lambda_1,\cdots,\lambda_{2K}) = (2\pi N\rho(\lambda))^{K^2}\frac{\exp(-NK)}{K!}
$$

$$
\times \oint \prod_1^K \frac{du_\alpha}{2\pi}\exp -i\left(\sum_{\alpha=1}^K u_\alpha\right)\frac{\Delta^2(u_1,\cdots,u_K)}{\prod_{\alpha=1}^K\prod_{l=1}^{2K}(u_\alpha-x_l)}. \tag{35}
$$

If we specialize to $K=1$ one finds

$$
\exp\left\{-\frac{N}{2}(V(\lambda_1)+V(\lambda_2))\right\} F_2(\lambda_1,\lambda_2) = e^{-N}(2\pi N\rho(\lambda))\frac{\sin x}{x} \tag{36}
$$

with $x=\pi N\rho(\lambda)(\lambda_1-\lambda_2)$, in which we recover the well-known Dyson kernel, which characterizes the correlation between eigenvalues, whose universality has been very much discussed over recent years. Note the dependence in $(N\rho(\lambda))^{K^2}$ of this function. This K=1 result (36) is indeed equal to $(2\pi e^{-N})K(\lambda_1,\lambda_2)$, where the kernel $K(\lambda_1,\lambda_2)$ is

$$
K(\lambda_1,\lambda_2) = \frac{\sin[\pi N\rho(\lambda)(\lambda_1-\lambda_2)]}{\pi(\lambda_1-\lambda_2)}. \tag{37}
$$

(In the next section we return to the normalizations. It will be explained how the extra-factor $2\pi e^{-N}$ is cancelled by the normalization constant h_{N-1}.)

We can now specialize this formula to the moments of the distribution of the characteristic polynomial, by letting all the λ's approach each other, i.e. letting the x's vanish. Before we do that, we should point out that the procedure to obtain these moments is in fact subtle. In principle we could have set all the λ's equal at an early stage of the calculation. If we returned for instance to (21) we might have replaced the limit of $\frac{\det(e^{-N\lambda_a z_b})}{\Delta(\lambda_1,\cdots,\lambda_{2K})}$ by $\Delta(z_1,\cdots,z_{2K})$ (up to a factor), but then the saddle-point method to obtain the large N-limit becomes quite problematic. Indeed the Vandermonde of the z's at the saddle-point vanishes and it is necessary to go far beyond the Gaussian integration. However it is now straightforward to obtain this moment from (35). We obtain

$$
\exp -(NKV(\lambda))F_{2K}(\lambda,\cdots,\lambda)
$$

$$
= (2\pi N\rho(\lambda))^{K^2}\frac{\exp(-NK)}{K!}\oint\prod_1^K\frac{du_\alpha}{2\pi}\exp -i\left(\sum_{\alpha=1}^K u_\alpha\right)\frac{\Delta^2(u_1,\cdots,u_K)}{\prod_{\alpha=1}^K u_\alpha^{2K}}. \tag{38}
$$

Expanding the Vandermonde determinant into a sum over permutations, we find

$$
\oint\prod_1^K\frac{du_\alpha}{2\pi}\exp -i\left(\sum_{\alpha=1}^K u_\alpha\right)\frac{\Delta^2(u_1,\cdots,u_K)}{\prod_{\alpha=1}^K u_\alpha^{2K}} = (-1)^{K(K-1)/2}
$$

$$
\times\sum_{P,Q}(-1)^{(P+Q)}\frac{1}{(2K-P_0-Q_0-1)!}\cdots\frac{1}{(2K-P_{K-1}-Q_{K-1}-1)!}, \tag{39}
$$

in which P and Q are permutations of the integers $(0, \cdots, K - 1)$. Therefore

$$\oint \prod_1^K \frac{du_\alpha}{2\pi} \exp -i \left(\sum_{\alpha=1}^K u_\alpha \right) \frac{\Delta^2(u_1, \cdots, u_K)}{\prod_{\alpha=1}^K u_\alpha^{2K}}$$

$$= (-1)^{K(K-1)/2} K! \det_{0 \le i,j \le K-1} \frac{1}{(2K - i - j - 1)!} = K! \prod_0^{K-1} \frac{l!}{(K+l)!}, \quad (40)$$

and thus finally

$$\exp -(NKV(\lambda)) F_{2K}(\lambda, \cdots, \lambda) = (2\pi N\rho(\lambda))^{K^2} e^{-NK} \prod_0^{K-1} \frac{l!}{(K+l)!}. \quad (41)$$

4. Normalizations and Universality

We have studied in the previous section a Gaussian ensemble of random matrices and found that the result (41) for the moment involved $(2\pi N\rho(\lambda))^{K^2}$ times a number and one would like to see how general this result is, as far as the dependence in the density of states is concerned as well as for the normalization. We shall see that this behaviour is quite general, and given a proper normalization, that the prefactor is also universal. Indeed let us recall how the K-point correlation function of the eigenvalues are defined in an ensemble of hermitian $N \times N$ matrices X with a probability weight proportional to $\exp -N \text{Tr} V(X)$. In [1] one finds

$$R_K(\lambda_1, \cdots, \lambda_K) = \frac{N!}{(N-K)!} \frac{1}{Z_N} \int d\lambda_{(K+1)} \cdots d\lambda_N \left\{ \exp -N \sum_1^N V(\lambda_l) \right\}$$

$$\times \Delta^2(\lambda_1, \cdots \lambda_N). \quad (42)$$

Comparing with our initial definitions (5) we see that one has the relation

$$R_K(\lambda_1, \cdots, \lambda_K) = \frac{N!}{(N-K)!} \frac{Z_{N-K}}{Z_N} \left\{ \exp -N \sum_1^K V(\lambda_l) \right\} \Delta^2(\lambda_1, \cdots \lambda_K)$$

$$\times F_{2K}(\lambda_1, \lambda_1, \cdots, \lambda_K, \lambda_K); \quad (43)$$

the r.h.s. reduces, up to a normalization, to our previous product of characteristic functions of matrices $(N - K) \times (N - K)$, each one beeing repeated twice. On the other hand it is well known ([1]) that this K-point function may be expressed in terms of a kernel K_N as

$$R_K(\lambda_1, \cdots, \lambda_K) = \det_{1 \le i,j \le K} K_N(\lambda_i, \lambda_j), \quad (44)$$

and without entering into the precise definition of K_N in terms of orthogonal polynomials, one should simply recall that $\frac{K_N(\lambda, \mu)}{\rho((\lambda + \mu)/2))}$ is universal in the Dyson limit ([4]) ($\lambda - \mu$ goes to zero, N goes to infinity, $N(\lambda - \mu)$ finite), i.e. it is independent of the polynomial V which defines the probability measure.

Therefore we define a modified weight, and modified moments,

$$\Phi_{2K}(\lambda_1, \lambda_2, \cdots, \lambda_{2K})$$

$$= \frac{N!}{(N-K)!} \frac{Z_{N-K}}{Z_N} \left\{ \exp -\frac{N}{2} \sum_1^{2K} V(\lambda_l) \right\} F_{2K}(\lambda_1, \lambda_2, \cdots, \lambda_{2K}) \quad (45)$$

and

$$M_{2K}(\lambda) = \frac{N!}{(N-K)!} \frac{Z_{N-K}}{Z_N} \{\exp -NKV(\lambda)\} F_{2K}(\lambda, \lambda, \cdots, \lambda). \quad (46)$$

The universality of level correlations implies the universality of M_{2K}. Therefore we have to return to the Gaussian case, in order to take into account this new normalization, and then the result will be universal.

From (13) we have

$$\frac{N!}{(N-K)!} \frac{Z_{N-K}}{Z_N} = \frac{1}{\prod_{N-K}^{N-1} h_n}, \quad (47)$$

and, given the explicit expression (19) of h_n for the Gaussian case, we find, in the large N limit,

$$\frac{N!}{(N-K)!} \frac{Z_{N-K}}{Z_N} = (2\pi)^{-K} e^{NK}. \quad (48)$$

With this normalization the universal moment $M_{2K}(\lambda)$ is given by

$$M_{2K}(\lambda) = (2\pi)^{-K} (2\pi N\rho(\lambda))^{K^2} \prod_0^{K-1} \frac{l!}{(K+l)!}. \quad (49)$$

In fact this connection between the usual correlation functions and the expectation values of a product of characteristic functions, (43) and (44), allows one to recover directly the moment $M_{2K}(\lambda)$, by using the universal expression for the kernel $K(\lambda_i, \lambda_j)$ in the Dyson limit,

$$K(\lambda_i, \lambda_j) = \frac{\sin[\pi N\rho(\lambda_i - \lambda_j)]}{\pi(\lambda_i - \lambda_j)}. \quad (50)$$

The integral representation, over $2K$ variables describing contours around the K poles λ_l,

$$\frac{\det_{1 \le i < j \le K} K(\lambda_i, \lambda_j)}{\Delta^2(\lambda_1, \cdots, \lambda_K)}$$

$$= \frac{1}{K!} \oint \prod_1^K \frac{du_l}{2\pi i} \oint \prod_1^K \frac{dv_l}{2\pi i} \frac{\Delta(u_1, \cdots, u_K)\Delta(v_1, \cdots, v_K)}{\prod_{i=1}^K \prod_{j=1}^K (u_i - \lambda_j)(v_i - \lambda_j)} \prod_{i=1}^K K(u_i, v_i) \quad (51)$$

allows one to write easily the limit in which all the λ's are equal:

$$\lim \frac{\det_{1 \le i < j \le K} K(\lambda_i, \lambda_j)}{\Delta^2(\lambda_1, \cdots, \lambda_K)}$$

$$= \frac{1}{K!} \oint \prod_1^K \frac{du_l}{2\pi i} \oint \prod_1^K \frac{dv_l}{2\pi i} \frac{\Delta(u_1, \cdots, u_K)\Delta(v_1, \cdots, v_K)}{\prod_{i=1}^K (u_i - \lambda)^K (v_i - \lambda)^K} \times \prod_{i=1}^K K(u_i, v_i). \quad (52)$$

Since the kernel is a Toeplitz matrix, i.e. $K(\lambda_i, \lambda_j) = K(\lambda_i - \lambda_j)$, one can shift the u's and the v's of λ and the r.h.s. becomes independent of λ. In the case of the sine kernel we obtain, in the limit in which all the λ's are equal,

$$\frac{1}{K!} \oint \prod_1^K \frac{dx_l}{2\pi i} \oint \prod_1^K \frac{dv_l}{2\pi i} \frac{\Delta(v_1, \cdots, v_K)\Delta(x_1, \cdots, x_K)}{\prod_{i=1}^K [(v_i + x_i)^K v_i^K]} \prod_{i=1}^K \frac{\sin(\pi N\rho x_i)}{\pi x_i}$$

$$= \frac{(2\pi\rho N)^{K^2}}{(2\pi)^K} \prod_{l=0}^{K-1} \frac{l!}{(l+K)!}. \quad (53)$$

We have indeed recovered, for any function V defining the probability distribution, the universal moment (49)

5. Large N Asymptotics

Rather than starting, as in the previous sections, with an exact expression for the correlation functions of characteristic functions, and at the end letting N go to infinity, we may use a different method to investigate directly the large N limit for the moments of their distribution. This method applies for a general probability distribution of the form (3) and it may also be used to the more general case of an external matrix source coupled to the matrix X [7] in this distribution. It turns out that here again it is neccessary to consider first F_{2K} for different λ_j's, and let all the λ_j's approach the same λ at the end of the calculation.

From (5), we have

$$\frac{\partial \ln F_{2K}}{\partial \lambda_i} = MG_\lambda(\lambda_i), \quad (54)$$

where $G_\lambda(\lambda_i)$ is the resolvent,

$$G_\lambda(\lambda_i) = \frac{1}{M} \left\langle \mathrm{Tr} \frac{1}{\lambda_i - X} \right\rangle. \quad (55)$$

The bracket here denotes an expectation value with a weight which includes both $P(X)$ and $\prod_1^{2K} \det(\lambda_l - X)$. We assume that the asymptotic spectrum of the eigenvalues x_i of X fill a single interval $[\alpha, \beta]$ in the large M limit. (It is sufficient to consider the single cut case, since we are interested in Dyson short distance universality, which involves only the local statistics). Therefore $G_\lambda(z)$ is also analytic in a plane cut from the interval $[\alpha, \beta]$, and

$$G_\lambda(x \pm i0) = \hat{G}_\lambda(x) \mp i\pi\rho_\lambda(x), \quad (56)$$

where $\hat{G}_\lambda(x) = [G_\lambda(x + i0) + G_\lambda(x - i0)]/2$. The saddle point equation in the large M limit becomes

$$2MG_\lambda(z) - NV'(z) + \sum_{j=1}^{2K} \frac{1}{z - \lambda_j} = 0. \tag{57}$$

The last term of (57) is of relative order $1/N$ and thus we have to solve this Riemann-Hilbert problem to this order. At leading order, we have $2\hat{G}(x) = V'(x)$, and up to order $1/N$,

$$G_\lambda(z) = G(z) + \frac{1}{N}\left(C_G(z) + \sum_{i=1}^{2K} C_{\lambda_i}(z)\right). \tag{58}$$

From the saddle point equation (57), we have $\hat{C}_G(x) = (N - M)\hat{G}(x)$ and $\hat{C}_{\lambda_i}(x) = \frac{1}{2(\lambda_i - x)}$. We now set $M = N - K$. The functions $C_G(z)$ and $C_{\lambda_i}(z)$ are uniquely determined from their analyticity in a plane cut from α to β, and their fall-off as $1/z^2$ for large z (since both $G_\lambda(z)$ and $G(z)$ behave as $1/z$ at infinity). The result is

$$C_G(z) = KG(z) - \frac{K}{\sqrt{(z - \alpha)(z - \beta)}},$$
$$C_{\lambda_i}(z) = \frac{1}{2}\frac{1}{\sqrt{(z - \alpha)(z - \beta)}}\left(1 - \frac{\sqrt{(z - \alpha)(z - \beta)} - \sqrt{(\lambda_i - \alpha)(\lambda_i - \beta)}}{z - \lambda_i}\right). \tag{59}$$

These expressions lead to

$$(N - K)G_\lambda(\lambda_i) = NG(\lambda_i) - \frac{1}{2}\frac{d}{d\lambda_i}\log\sqrt{(\lambda_i - \alpha)(\lambda_i - \beta)}$$
$$-\frac{1}{2}\sum_{j=1,j\neq i}^{2K}\frac{1}{\lambda_i - \lambda_j}\left(1 - \sqrt{\frac{(\lambda_j - \alpha)(\lambda_j - \beta)}{(\lambda_i - \alpha)(\lambda_i - \beta)}}\right). \tag{60}$$

Since there is a branch cut between α and β, one must specify whether λ_i approaches the real axis from above or from below. The sign of the square root on both sides of the cut will be denoted ϵ_i. There are then a priori 2^{2K} saddle points corresponding to the different choices of ϵ_i. For each choice of the ϵ_i's, we have

$$\frac{\partial}{\partial\lambda_i}\log\tilde{F}_\epsilon = \epsilon_i Ni\pi\rho(\lambda_i) + -\frac{1}{2}\frac{d}{d\lambda_i}\log\sqrt{(\lambda_i - \alpha)(\beta - \lambda_i)}$$
$$-\frac{1}{2}\sum_{j=1,j\neq i}^{2K}\frac{1}{\lambda_i - \lambda_j}\left(1 - \frac{\epsilon_j\sqrt{(\lambda_j - \alpha)(\beta - \lambda_j)}}{\epsilon_i\sqrt{(\lambda_i - \alpha)(\beta - \lambda_i)}}\right), \tag{61}$$

where $\tilde{F}_\epsilon$ means the value of F_{2K} for given ϵ_j's multiplied by a factor $\exp(-\frac{N}{2}\sum V(\lambda_i))$. Introducing the parametrization $\phi(x)$, defined by $x = \frac{1}{2}(\alpha + \beta) - \frac{1}{2}(\beta - \alpha)\cos\phi(x)$

and $\frac{1}{2}(\beta - \alpha) \sin \phi(x) = \sqrt{(x - \alpha)(\beta - x)}$, we have

$$
\frac{d}{d\lambda_i} \log \sin \left(\frac{\epsilon_i \phi(\lambda_i) - \epsilon_j \phi(\lambda_j)}{2} \right)
$$
$$
= \frac{\epsilon_i}{2} \frac{1}{\sqrt{(\lambda_i - \alpha)(\beta - \lambda_i)}} \frac{\epsilon_i \sqrt{(\lambda_i - \alpha)(\beta - \lambda_i)} + \epsilon_j \sqrt{(\lambda_j - \alpha)(\beta - \lambda_j)}}{\lambda_i - \lambda_j}. \quad (62)
$$

Thus we obtain $\tilde{F}_\epsilon$ by integration,

$$
\tilde{F}_\epsilon = C_\epsilon \prod_{i<j}^{2K} \frac{\sin \left(\frac{\epsilon_i \phi(\lambda_i) - \epsilon_j \phi(\lambda_j)}{2} \right)}{\lambda_i - \lambda_j} \prod_{i=1}^{2K} \frac{1}{\sqrt{\sin \phi(\lambda_i)}}
$$
$$
\times \prod_{i=1}^{2K} \exp \left(\epsilon_i i N\pi \int_{\lambda_0}^{\lambda_i} \rho(x) dx \right). \quad (63)
$$

We have to sum over all the saddle-point contributions, i.e. sum over all the different choices of ϵ_j's. We focus now on the Dyson limit in which the differences $\lambda_i - \lambda_j$ are all of order 1/N. Among the 2^{2K} possibilities, we retain only the $\binom{2K}{K}$ solutions in which half of K among ϵ_l are positive, and the remaining halves are negative. Otherwise, the exponential factor in the final result gives very rapid oscillations in the large N limit. This situation is thus exactly similar to that of the previous section.

Again the sum over the $\binom{2K}{K}$ saddle-points is conveniently written as a contour integral

$$
\tilde{F} = \frac{1}{K!} \oint \cdots \oint \frac{du_1 du_2 \cdots du_K}{(2\pi i)^K} \frac{\prod_{n<m} (u_n - u_m)^2}{\prod_{n=1}^{K} \prod_{j=1}^{2K} (u_n - \lambda_j)} \cos \left(\sum_{j=1}^{2K} \lambda_j - 2 \sum_{n=1}^{K} u_n \right). \quad (64)
$$

When we set all the $\lambda_j = \lambda$, this becomes

$$
\tilde{F} = \frac{1}{K!} \oint \frac{\prod du_i}{(2\pi i)^K} \frac{\prod_{i<j} (u_i - u_j)^2}{\prod_{i=1}^{K} u_i^{2K}} \cos \left(2 \sum_{n=1}^{K} u_n \pi N\rho \right) \quad (65)
$$

and we recover the result (38). However in this method, since we re-integrated the logarithmic derivative of F_{2K}, the constant of integration remains undetermined. We may fix this constant by the same requirement that we have used in the previous section, and the final result agrees then with the previous calculation.

6. Symplectic Group $Sp(N)$

We have studied up to now unitary invariant measures, characterized for the probability law of the eigenvalues by the factor $|\Delta(x_1, \cdots, x_M)|^\beta$. We could also consider the Gaussian orthogonal ensemble (GOE, with $\beta = 1$) or Gaussian symplectic (GSE, with $\beta = 4$). If we took the GOE for instance, we could immediately relate the correlation

functions of characteristic determinants, to the correlations of the eigenvalues, as in (43) (except that since β is one no doubling of the λ's is needed), and therefore relate the moments universality to the Dyson universal limit. Remaining still with the unitary $\beta = 2$ class, in Cartan's classification of symmetric spaces, we find ensembles which are invariant under $Sp(N)$ or $O(N)$. One of the physical applications of random $Sp(N)$ matrices, is the statistics of the energy levels inside a superconductor vortex [8]. In number theory, it is known that some generalizations of Riemann's ζ-functions, such as the Dirichlet L-function $L(s, \chi_d)$, where χ_d is a quadratic Dirichlet character of mod $|d|$, present a spectrum of low lying zeros on the line Re $s = 1/2$, which agrees with the statistics of the eigenvalues of the Sp(N) random matrix theory [12, 14]. In this Sp(N) invariant symmetric space, the eigenvalues appear always in pairs of positive and negative real numbers. Due to this fact, a new universality class governs the correlations of the eigenvalues near the origin, i.e. near $s = 1/2$ (whereas in the bulk one recovers the previous unitary class).

Therefore we study now the new universality class, which governs the new scaling near the origin. We thus consider random Hermitian matrices X, which are $2M \times 2M$ and satisfy the condition

$$X^T J + JX = 0, \tag{66}$$

where J is

$$J = \begin{pmatrix} 0 & 1_M \\ -1_M & 0 \end{pmatrix}. \tag{67}$$

The unitary symplectic group is the subgroup of $SU(2M)$ consisting of $2M \times 2M$ unitary matrices, satisfying the symplectic constraint

$$U^T = -JU^\dagger J. \tag{68}$$

The integration over this unitary symplectic group for $F_K(\lambda_1, \cdots, \lambda_K)$ gives [8]

$$F_K(\lambda_1, \cdots, \lambda_K) = \left\langle \prod_{\alpha=1}^{K} \det(\lambda_\alpha - X) \right\rangle$$

$$= \frac{1}{Z_M} \int \prod_{1}^{M} d\mu(x_i) \, \Delta^2(x_1^2, \cdots, x_M^2) \prod_{i=1}^{M} x_i^2 \prod_{\alpha=1}^{K} \prod_{i=1}^{M} (\lambda_\alpha^2 - x_i^2). \tag{69}$$

Repeating the analysis of Sect. 2, $F_K(\lambda_1, \cdots, \lambda_K)$ is given again by a determinantal form as (14). Changing x_1 to $x_i^2 = y_i$ and denoting $\mu_i = \lambda_i^2$, we have

$$F_K(\mu_1, \cdots, \mu_K) = \int_0^\infty \prod_{i=1}^{K} dy_i \prod_{i=1}^{K} y_i^{\frac{1}{2}} \prod_{i<j} (y_i - y_j)^2 \prod_{\alpha=1}^{K} \prod_{i=1}^{K} (\mu_\alpha - y_i) e^{-N \sum y_i}. \tag{70}$$

The orthogonal monic polynomials for this measure are the Laguerre polynomials $L_n^{(\frac{1}{2})}(y)$, which is defined by

$$L_n^{(\frac{1}{2})}(y) = \frac{(-1)^n}{\sqrt{y}} \frac{e^{Ny}}{N^n} (\frac{d}{dy})^n (y^{n+\frac{1}{2}} e^{-Ny})$$

$$= \frac{(-1)^n}{N^n} n! \oint \frac{du}{2\pi i} \frac{(1+u)^{n+\frac{1}{2}}}{u^{n+1}} e^{-Nuy}, \tag{71}$$

normalized as required to $L_n^{(\frac{1}{2})}(y) = y^n + \text{lowerdegree}$. The orthogonality condition is

$$\int_0^\infty dy e^{-Ny} \sqrt{y} L_n^{(\frac{1}{2})}(y) L_m^{(\frac{1}{2})}(y) = h_n \delta_{n,m} \tag{72}$$

with $h_n = n! \Gamma(n + \frac{3}{2})/N^{2n+\frac{3}{2}}$, and $h_{N-1} \simeq 2\pi e^{-2N}$ in the large N limit.

From (14), we have

$$F_K(\mu_1, \cdots, \mu_K) = (-1)^{K\left(M + \frac{K-1}{2}\right)} \frac{\prod_{l=0}^{K-1}(M + l)!}{N^{K(M+\frac{K}{2}-\frac{1}{2})}} \frac{1}{\Delta(\mu)}$$

$$\times \oint \prod_{i=1}^K \left(\frac{dz_i}{2\pi i}\right) \prod_{l=1}^K \frac{(1 + z_l)^{M+K-\frac{1}{2}}}{z_l^{M+K}} e^{-N \sum z_\alpha \mu_\alpha} \prod_{i<j}^K \left(\frac{z_i}{1 + z_i} - \frac{z_j}{1 + z_j}\right). \tag{73}$$

We now set $M = N - K$, and the factor $\prod_0^{K-1}(M + l)!/N^{K(M+K/2-1/2)}$ is equal to $(2\pi N)^K e^{-KN}$, up to corrections of relative order $1/N$ in the large N limit. The large N limit is governed by the saddle-point equations $z_l^2 + z_l + \frac{1}{\mu_l} = 0$. In the following we study the scaling vicinity of the origin, in which all the μ_l's scale as $1/N^2$. Then z_l^2 at the saddle-point may be expanded

$$z_l \simeq \frac{i\epsilon_l}{\sqrt{\mu_l}} - \frac{1}{2} + O(\sqrt{\mu_l}), \tag{74}$$

where $\epsilon_l = \pm 1$.

Noting that $\prod_{i<j} \left(\frac{z_i}{1 + z_i} - \frac{z_j}{1 + z_j}\right) = \prod_{i<j}(-z_i^2 \mu_i + z_j^2 \mu_j) \simeq i^{K(K-1)/2} \prod(\epsilon_i \lambda_i - \epsilon_j \lambda_j)$ $(\epsilon_i = \pm 1)$, and combining it with the Vandermonde $\Delta(\lambda^2)$, we are left with a factor $\prod_{i<j} \frac{1}{\epsilon_i \lambda_i + \epsilon_j \lambda_j}$ in this scaling limit. We have also the exponential $e^{-N \sum z_\alpha \mu_\alpha} = e^{-i \sum \epsilon_i \lambda_i}$.

We have again to sum over all the saddle-points, which are characterized by the sign of $\epsilon_i = \pm 1$, and to include the factor due to the fluctuations near the saddle-point. The Gaussian fluctuations yield a factor $(2\pi/(-2i\epsilon_i \lambda_i^3 N))^{1/2}$. Then $(1 + z_l)^{-1/2} \simeq (\lambda_i/(\epsilon_i i))^{1/2}$. There is a $1/(2\pi i)^K$ in addition. We have an extra ϵ_i due to the contour direction, which goes through two saddle points; one is in the positive imaginary plane and the other in the negative half-plane. When $K = 2$, and λ_1 and λ_2 are of order $1/N$, we obtain

$$F_2(\lambda_1, \lambda_2) = \frac{2\pi e^{-2N}}{\lambda_1 \lambda_2} K_{SP}(\lambda_1, \lambda_2), \tag{75}$$

with the kernel $K_{SP}(\lambda_1, \lambda_2)$ given by

$$K_{SP}(\lambda_1, \lambda_2) = \frac{\sin[N(\lambda_1 - \lambda_2)]}{2\pi(\lambda_1 - \lambda_2)} - \frac{\sin[N(\lambda_1 + \lambda_2)]}{2\pi(\lambda_1 + \lambda_2)}. \tag{76}$$

The coefficient $(2\pi)e^{-2N}$ is cancelled by the normalization factor $1/h_{N-1}$. Putting $\lambda_1 = \lambda_2 = 0$, we have, neglecting the factor $2\pi e^{-2N}$, $F_2(0) \simeq \frac{1}{2\pi} \frac{4}{3!} N^3$.

For general K, $F_K(\lambda_1, \cdots, \lambda_k)$ becomes in the scaling limit

$$F_K(\lambda_1, \cdots, \lambda_K) = (-1)^{K(N-K+\frac{K-1}{2})} (2\pi N)^{\frac{K}{2}} e^{-NK} (i)^{\frac{K}{2}(K-1)} \left(\frac{\pi}{N}\right)^{\frac{K}{2}} \frac{1}{(2\pi i)^K}$$

$$\times \sum_\epsilon \frac{e^{-iN \sum_i \epsilon_i \lambda_i}}{\prod_{i=1}^{K} \epsilon_i \lambda_i \prod_{i<j}(\epsilon_i \lambda_i + \epsilon_j \lambda_j)}. \tag{77}$$

The sum over all the saddle-points, characterized by $\epsilon_i \pm 1$, is conveniently written as a contour integral,

$$I = \sum_\epsilon \frac{1}{\prod_{i<j}(\epsilon_i \lambda_i + \epsilon_j \lambda_j) \prod(\epsilon_i \lambda_i)} e^{-iN \sum \epsilon_i \lambda_i}$$

$$= (-1)^{\frac{K}{2}(K-1)} \frac{2^k}{k!} \oint \cdots \oint \prod_{i=1}^{k} \left(\frac{du_i}{2\pi i}\right) \frac{\Delta(u^2)\Delta(u)}{\prod_{i=1}^{k} \prod_{j=1}^{k}(u_i^2 - \lambda_j^2)} e^{-iN\left(\sum_{i=1}^{k} u_i\right)}, \tag{78}$$

where the contour encloses $u_i = \pm\lambda_j$. We may now set $\lambda_j = \lambda$, and keeping track of various coefficients, we obtain the Kth moment $F_K(\lambda, \cdots, \lambda)$. For general λ, the result has a complicated form, but when $\lambda = 0$, it becomes a number

$$F_K(0, \cdots, 0) = \frac{2^{k/2} e^{-NK}}{k!} N^{\frac{K}{2}(K+1)} (i)^{\frac{K}{2}(K-3)} (-1)^{K(N-1)}$$

$$\times \oint \prod_{i=1}^{k} \left(\frac{du_i}{2\pi i}\right) \frac{\Delta(u^2)\Delta(u)}{\prod_{i=1}^{k} u_i^{2K}} e^{-i\sum_{i=1}^{k} u_i}. \tag{79}$$

This representation allows one to compute the Kth moment at the origin. By the expansion of the VanderMonde determinants, similarly to (39), (79) is reduced to a determinant form. We have by the normalization; $\tilde{F}_K(0) = (2\pi)^{-\frac{K}{2}} e^{KN} F_K(0)$,

$$\tilde{F}_K(0) = (-1)^{KN} \prod_{l=1}^{K} \frac{l!}{(2l)!} \frac{(2N)^{\frac{K}{2}(K+1)}}{\pi^{\frac{K}{2}}}. \tag{80}$$

Comparing to the result of the unitary case in (49), we notice that the exponent of N is different and the universal coefficient is given also by the product of the ratio of the factorizations.

For $F_{2K}(\lambda_1, \lambda_1, \cdots, \lambda_K, \lambda_K)$, the even $2K^{\text{th}}$ moment may be obtained again from $\tilde{F}_{2K}(\lambda_1, \lambda_1, \cdots, \lambda_K, \lambda_K) = \det[K(\lambda_i, \lambda_j)]/(\Delta^2(\lambda^2) \prod \lambda_i^2)$; using the expression for the kernel (76), we have for the $2K^{\text{th}}$ moment,

$$\frac{\det[K(\lambda_i, \lambda_j)]}{\Delta^2(\lambda^2) \prod \lambda_i^2}$$

$$= \frac{2^K}{K!} \oint \prod \frac{du_i}{2\pi i} \oint \prod \frac{dv_i}{2\pi i} \frac{\Delta(u^2)\Delta(v^2)}{\prod_{i=1}^{K} \prod_{j=1}^{K}(u_i^2 - \lambda_j^2) \prod_{i=1}^{K} \prod_{j=1}^{K}(v_i^2 - \lambda_j^2)}$$

$$\times \frac{1}{(2\pi)^K} \prod_{i=1}^{K} \frac{\sin[N(u_i - v_i)]}{u_i - v_i}. \tag{81}$$

For general λ, the result has a complicated form, but again here one can compute from there the values at $\lambda = 0$. The result agrees with the previous expression of $\tilde{F}_{2K}(0)$ in (80). One may also use the large N asymptotic analysis as in Sect. 5 and rederive the results as the same sum (78) over the saddle points.

7. Orthogonal Group O(N)

We discuss here the $O(2N)$ case, which is different from $Sp(N)$ (whereas $O(2N + 1)$ has a structure which is similar to $Sp(N)$ [8]). In number theory, for example, the twisted L-function, $L_\tau(s, \chi_d)$, presents a spectrum of low lying zeros, which agrees with the statistics of the eigenvalues of the $O(2N)$ random matrix theory [12,14]. Then in terms of eigenvalues

$$
F_K(\lambda_1, \cdots, \lambda_K) = < \prod_{\alpha=1}^K \det(\lambda_\alpha - X) >
$$

$$
= \frac{1}{Z_M} \int \prod_1^M d\mu(x_i) \, \Delta^2(x_1^2, \cdots, x_M^2) \prod_{\alpha=1}^K \prod_{i=1}^M (\lambda_\alpha^2 - x_i^2). \quad (82)
$$

The difference between the symplectic and orthogonal case is due to the absence of the factor $\prod x_i^2$. Using the analysis of Sect. 2, $F_K(\lambda_1, \cdots, \lambda_K)$ is given by the determinantal form as in (14). Changing x_i^2 to y_i and denoting $\mu_i = \lambda_i^2$, we have

$$
F_K(\mu_1, \cdots, \mu_K) = \int_0^\infty \prod_{i=1}^K dy_i \prod_{i=1}^K y_i^{-\frac{1}{2}} \prod_{i<j} (y_i - y_j)^2 \prod_{\alpha=1}^K \prod_{i=1}^K (\mu_\alpha - y_i) e^{-N \sum y_i}.
$$

$$(83)$$

The orthogonal polynomials for this case are Laguerre polynomials $L_n^{(-\frac{1}{2})}(y)$, which is defined by

$$
L_n^{(-\frac{1}{2})}(y) = (-1)^n \sqrt{y} \frac{e^{Ny}}{N^n} (\frac{d}{dy})^n (y^{n-\frac{1}{2}} e^{-Ny})
$$

$$
= \frac{(-1)^n}{N^n} n! \oint \frac{du}{2\pi i} \frac{(1+u)^{n-\frac{1}{2}}}{u^{n+1}} e^{-Nuy} \quad (84)
$$

normalized as $L_n^{(-\frac{1}{2})}(y) = y^n + $ lowerdegree. The orthogonality condition is

$$
\int_0^\infty dy e^{-Ny} \frac{1}{\sqrt{y}} L_n^{(-\frac{1}{2})}(y) L_m^{(-\frac{1}{2})}(y) = h_n \delta_{n,m} \quad (85)
$$

with $h_n = n! \Gamma(n + \frac{1}{2})/N^{2n+\frac{1}{2}}$, and $h_{N-1} \simeq 2\pi e^{-2N}$ in the large N limit. From (14), we have similar to the $Sp(N)$ case,

$$
F_K(\mu_1, \cdots, \mu_K) = (-1)^{K\left(M + \frac{K-1}{2}\right)} \frac{\prod_{l=0}^{K-1}(M+l)!}{N^{K(M+\frac{K}{2}-\frac{1}{2})}} \frac{1}{\Delta(\mu)}
$$

$$
\times \oint \prod_{i=1}^K \left(\frac{dz_i}{2\pi i}\right) \prod_{l=1}^K \frac{(1+z_l)^{M+K-\frac{3}{2}}}{z_l^{M+K}} e^{-N \sum z_\alpha \mu_\alpha} \prod_{i<j}^K \left(\frac{z_i}{1+z_i} - \frac{z_j}{1+z_j}\right). \quad (86)
$$

We set $M = N - K$, and the factor $\prod_{0}^{K-1} (M + l)!/N^{K(M+K/2-1/2)}$ is equal to $(2\pi N)^K e^{-KN}$, up to corrections of relative order $1/N$ in the large N limit. The saddle point z_l is same as (74). The only difference is the extra factor $(1 + z_l)^{-1} \simeq \frac{\lambda_l}{i\epsilon_l}$. When K=2, we obtain

$$F_2(\lambda_1, \lambda_2) = 2\pi e^{-2N} K_O(\lambda_1, \lambda_2), \tag{87}$$

with the kernel $K_O(\lambda_1, \lambda_2)$ given by

$$K_O(\lambda_1, \lambda_2) = \frac{\sin[N(\lambda_1 - \lambda_2)]}{2\pi(\lambda_1 - \lambda_2)} + \frac{\sin[N(\lambda_1 + \lambda_2)]}{2\pi(\lambda_1 + \lambda_2)}. \tag{88}$$

The factor $(2\pi)e^{-2N}$ is cancelled by the normalization factor $1/h_{N-1} \simeq (2\pi)^{-1}e^{2N}$. Putting $\lambda_1 = \lambda_2 = 0$, we have neglecting the factor $2\pi e^{-2N}$, $F_2(0) \simeq \frac{1}{2\pi}(2N)$.

For general K, $F_K(\lambda_1, \cdots, \lambda_k)$ becomes in the scaling limit

$$F_K(\lambda_1, \cdots, \lambda_K) = (-1)^{K(N-K+\frac{K-1}{2})}(2\pi N)^{\frac{K}{2}} e^{-NK}(i)^{\frac{K}{2}(K-3)} \left(\frac{\pi}{N}\right)^{\frac{K}{2}} \frac{1}{(2\pi i)^K}$$

$$\times \sum_{\epsilon} \frac{e^{-iN\sum_i \epsilon_i \lambda_i}}{\prod_{i<j}(\epsilon_i \lambda_i + \epsilon_j \lambda_j)}. \tag{89}$$

The sum over all the saddle-points, characterized by $\epsilon_i \pm 1$, is conveniently written as a contour integral,

$$I = \sum_{\epsilon} \frac{1}{\prod_{i<j}(\epsilon_i \lambda_i + \epsilon_j \lambda_j)\prod(\epsilon_i \lambda_i)} e^{-iN\sum \epsilon_i \lambda_i}$$

$$= (-1)^{\frac{K}{2}(K-1)} \frac{2^k}{k!} \oint \cdots \oint \prod_{i=1}^{k} \left(\frac{du_i}{2\pi i}\right) \frac{\Delta(u^2)\Delta(u)\prod_{i=1}^{K} u_i}{\prod_{i=1}^{k}\prod_{j=1}^{k}(u_i^2 - \lambda_j^2)} e^{-iN\left(\sum_{i=1}^{k} u_i\right)}, \tag{90}$$

where the contour encloses $u_i = \pm\lambda_j$. We may now set $\lambda_j = \lambda$, and keeping track of various coefficients, we obtain the Kth moment $F_K(\lambda, \cdots, \lambda)$. For general λ, the result has a complicated form, but when $\lambda = 0$, it becomes a number

$$F_K(0, \cdots, 0) = \frac{2^{k/2}e^{-NK}}{k!} N^{\frac{K}{2}(K-1)}(i)^{\frac{K}{2}(K-5)}(-1)^{K(N-1)}$$

$$\times \oint \prod_{i=1}^{k} \left(\frac{du_i}{2\pi i}\right) \frac{\Delta(u^2)\Delta(u)}{\prod_{i=1}^{k} u_i^{2K-1}} e^{-i\sum_{i=1}^{k} u_i}. \tag{91}$$

The normalization factor is $(2\pi)^{-\frac{K}{2}} e^{KN}$ for $F_K(\lambda)$. Denoting the normalized K^{th} moment by $\tilde{F}_K(\lambda)$, we have

$$\tilde{F}_K(0) = (-1)^{KN} \prod_{l=1}^{K-1} \frac{l!}{(2l)!} \frac{(2N)^{\frac{K}{2}(K-1)}}{\pi^{\frac{K}{2}}}. \tag{92}$$

We have $\tilde{F}_{2K}(\lambda_1, \lambda_1, \cdots, \lambda_K, \lambda_K) = \det[K(\lambda_i, \lambda_j)]/\Delta^2(\lambda^2)$. Using the expression for the kernel (88), we obtain for the $2K^{\text{th}}$ moment in the orthogonal $O(2N)$ case,

$$
\frac{\det[K(\lambda_i, \lambda_j)]}{\Delta^2(\lambda^2)}
$$
$$
= \frac{2^K}{K!} \oint \prod \frac{du_i}{2\pi i} \oint \prod \frac{dv_i}{2\pi i} \frac{\Delta(u^2)\Delta(v^2)\prod_{i=1}^K (u_i v_i)}{\prod_{i=1}^K \prod_{j=1}^K (u_i^2 - \lambda_j^2) \prod_{i=1}^K \prod_{j=1}^K (v_i^2 - \lambda_j^2)}
$$
$$
\times \frac{1}{(2\pi)^K} \prod_{i=1}^K \frac{\sin[N(u_i - v_i)]}{u_i - v_i}. \quad (93)
$$

Inserting $\lambda_i = 0$, we find the consistent result with (91).

8. Negative Moments

In number theory literature one finds various moments in which powers of the zeta-functions appear in the denominator [11]. The equivalent for random matrices would be to consider expectation values of the form $\left\langle \prod_1^K \det(\lambda_l - X)^{\epsilon_l} \right\rangle$ in which the ϵ's are ± 1. One cannot use the techniques introduced hereabove any more but, at least in the Gaussian case, it is easy to obtain exact expressions through the use of auxiliary integrations, over both commuting and anti-commuting variables.

We first rederive our previous results for positive moments (i.e. $\epsilon_l = +1$ for all l's) . Let us introduce M Grassmann variables $\bar{c}_a, c_a$ and an integration normalized to

$$
\int \frac{d\bar{c}dc}{\pi} \bar{c}c = 1. \quad (94)
$$

Then, for an hermitian $M \times M$ matrix X, one has

$$
\det(\lambda - X) = N^{-M} \int \prod_1^M \frac{d\bar{c}_a dc_a}{i\pi} \exp i N \sum_{a,b} [\bar{c}_a(\lambda \delta_{a,b} - X_{a,b})c_b]. \quad (95)
$$

A product $\prod_1^K \det(\lambda_l - X)$ is represented by a product of K integrals of the type (95). At the end the random matrix X appears in an expression of the form

$$
\exp -i N \sum_{l=1}^K \sum_{a,b=1}^M X_{ab} \bar{c}_a^{(l)} c_b^{(l)}. \quad (96)
$$

With the Gaussian probability weight (16) we have

$$
\langle \exp i N \mathrm{Tr} A X \rangle = \exp -\frac{N}{2} \mathrm{Tr} A^2, \quad (97)
$$

and thus

$$\left\langle \prod_1^K \det(\lambda_l - X) \right\rangle = N^{-M} \int \prod_1^M \frac{d\bar{c}_a dc_a}{\pi} \exp \left(N \sum_{l=1}^K i\lambda_l \gamma_{ll} + \frac{N}{2} \sum_{l,m=1}^K \gamma_{lm} \gamma_{ml} \right) \tag{98}$$

with

$$\gamma_{lm} = \sum_{a=1}^N \bar{c}_a^{(l)} c_a^{(m)}. \tag{99}$$

We can use an auxiliary $K \times K$ hermitian matrix B to replace the quadratic terms in γ by

$$\exp \frac{N}{2} \mathrm{Tr} \gamma^2 = \left(\frac{N}{2\pi} \right)^{K^2/2} \int d^{K^2} B \exp \left(N \mathrm{Tr} \gamma \mathrm{B} - \frac{N}{2} \mathrm{Tr} \mathrm{B}^2 \right). \tag{100}$$

We are left with an integral over the Grassmannian variables

$$N^{-MK} \int \prod_{a=1}^M \prod_{l=1}^K \frac{d\bar{c}_a^{(l)} dc_a^{(l)}}{i\pi} \exp N \sum_{l,m=1}^K (i\lambda_l \delta_{lm} + B_{lm}) \sum_{a=1}^M \bar{c}_a^{(l)} c_a^{(m)}$$

$$= \left(\det_{1 \leq l,m \leq K} (\lambda_l \delta_{lm} - i B_{lm}) \right)^M. \tag{101}$$

We end up with an integral over a $K \times K$ hermitian matrix B:

$$\left\langle \prod_1^K \det(\lambda_l - X) \right\rangle = \left(\frac{N}{2\pi} \right)^{K^2/2} \int d^{K^2} B \, \{\det(\lambda_l \delta_{lm} - i B_{lm})\}^M \exp -\frac{N}{2} \mathrm{Tr} \mathrm{B}^2. \tag{102}$$

Therefore, from this method as well, we have reduced the correlations of the characteristic functions of the matrix, to an integral over K^2 variables. If one is interested in the moments, i.e. $\lambda_l = \lambda$ for all l's, one may take as variables the eigenvalues b_l of B (which yields a factor $\Delta^2(\lambda_1, \cdots, \lambda_K)$), and recover the previous expressions. For the λ_l's non-equal, one must first shift the matrix B of the diagonal matrix $i\lambda_l \delta_{lm}$, and then integrate out the unitary group $SU(K)$ by the Itzykson–Zuber formula [15–17], to reduce it, as before, to an integral over K variables (a slightly different integral, but which may be handled in the large N-limit in an identical fashion).

In case of negative moments the method is identical, except that we need now ordinary commuting variables, instead of Grassmannian. Indeed starting from

$$\frac{1}{\det(\lambda - X \pm i\epsilon)}$$

$$= N^M \int \prod_1^M \frac{d\phi_a^* d\phi_a}{\pm i\pi} \exp \pm iN \sum_{a,b} [\phi_a^* (\lambda \delta_{a,b} - X_{a,b} \pm i\epsilon \delta_{a,b}) \phi_b], \tag{103}$$

one can introduce, for each factor $(\det(\lambda_l - X))^{\epsilon_l}$ an integration over M complex variables (ϕ_a^*, ϕ_a) if $\epsilon_l = -1$, or over M complex Grassmannian variables $(\bar{c}_a, c_a)$ if $\epsilon_l = +1$. The expectation value with the Gaussian weight $P(X)$ is then immediate. Of course for the negative moments, one must pay attention to the sign of the infinitesimal imaginary part of the λ's since there is a cut on the real axis along the support of Wigner's semi-circle.

Although the method is obvious and elementary, the notations can become cumbersome and, rather than working out the most general case, and arbitrary choices for the signs of the imaginary parts, we restrict ourselves to an example. If we consider only negative powers, we may follow identical steps as hereabove with positive powers, and we find

$$\left\langle \prod_1^K \frac{1}{\det(\lambda_l - X + i\epsilon)} \right\rangle = \left(\frac{N}{2\pi} \right)^{\frac{K^2}{2}} \int d^{K^2} B \, \{\det(\lambda_l \delta_{lm} - B_{lm} + i\epsilon \delta_{lm})\}^{-M}$$

$$\times \exp - \frac{N}{2} \text{TrB}^2. \tag{104}$$

When all the λ's are equal the r.h.s. simplifies to an integral over K variables

$$\int \prod_1^K db_l \frac{\Delta^2(b_1, \cdots, b_K)}{\prod_1^K (\lambda - b_l + i\epsilon)^M} \exp - \left(\frac{N}{2} \sum_1^K b_l^2 \right).$$

For the λ_l's non-equal, after a shift of the matrix B and the integration over $SU(K)$, one obtains

$$\left\langle \prod_1^K \frac{1}{\det(\lambda_l - X + i\epsilon)} \right\rangle = \left(\frac{N}{2\pi} \right)^{K(K+1)/2} \exp - \frac{N}{2} \sum_1^K \lambda_l^2$$

$$\times \int \prod_1^K \frac{db_l}{(b_l - i\epsilon)^M} \frac{\Delta(b_1, \cdots, b_K)}{\Delta(\lambda_1, \cdots, \lambda_K)} \exp - N \sum_1^K \left(\frac{1}{2} b_l^2 + b_l \lambda_l \right), \tag{105}$$

from which one could repeat easily the analysis of Sect. 3.

9. Discussion

We have discussed the universal expressions for the moments of the characteristic polynomials in a random matrix theory, where the ensembles belong to the unitary family ($\beta = 2$).

We have shown that these universalities are related to the universality of the kernel in Dyson's short distance limit. Since the statistics of the zeros of the ζ-function follows the universal behavior of the Gaussian unitary ensemble (GUE) [12, 19], the power moment of the ζ-function also has to follow the universal behavior of GUE. We have studied here the characteristic polynomial, which corresponds to the ζ-function on the critical line, and we have found a universal behavior for the moments of the characteristic polynomial. The universal number (49) appears indeed in the average of the moment of the ζ-function, which was conjectured as (2), $\gamma_K = \prod_0^{K-1} l!/(K+l)!$.

Our method of splitting the singularity by the introduction of the distinct λ_i may be applied directly to the average of the power moment of the Riemann ζ-function. We consider the average of the product of $\zeta(s_i)$, $s_i = \frac{1}{2} \pm i(\lambda_i + t)$,

$$F = \frac{1}{T} \int_0^T \prod_{i=1}^{2K} \zeta(s_i) dt, \tag{106}$$

where we choose K positive λ_i's and K negative ones. If, at the end of the calculation, we set all the positive λ_i's equal to λ and the negative ones to $-\lambda$, one recovers the $2K^{\text{th}}$ moment of the modulus of the ζ-function. When T is large, the leading and the next leading terms of the derivative of $\ln F$ with respect to λ_i, are presumably given by

$$\frac{\partial \ln F}{\partial \lambda_i} \sim \pm i \ln T - \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \tag{107}$$

where the pole in the second term appears when two distinct λ_i coincide, one of the λ_i's with a plus sign and the other one with a minus sign. In the Appendix a discussion of the assumptions leading to (107) is given. Then, after integration, we have, following a line of arguments similar to those of Sect. 5,

$$F = \frac{c}{K!} \oint \prod_{i=1}^K \frac{du_i}{2\pi i} \frac{\Delta^2(u)}{\prod_{i=1}^K \prod_{j=1}^{2K} (u_i - \lambda_j)} e^{-i \sum u_i \ln T}. \tag{108}$$

Therefore, if we let the λ_i's coincide, we recover the integral (39), which provides the universal coefficient γ_K. The coefficient c is not determined by this method, which starts with the logarithmic derivative of F, and an extra normalization condition is needed. In (A.7) it will be argued that a coefficient a_K is present in the result, which is the residue at $s = 1$ of a function $g_K(s)$ defined in the Appendix; it is thus plausible that the coefficient c in (108) is nothing but $c = a_K$.

We have also investigated negative moments as (105). This result may apply to the mean value of negative moments of the ζ-function. Indeed, the exponent K^2 of $\log T$ for the negative integer K, has been conjectured [11].

For the symplectic and orthogonal case, $Sp(N)$ and $O(2N)$ ensembles, there may be also be a correspondence between the random matrix results (80), (92) and the average values of the certain L-functions, with the same γ_K, as far as there is a universality. Existing conjectures [13] for the moment of the L function shows the same exponent $\frac{K}{2}(K+1)$ and $\frac{K}{2}(K-1)$ for the symplectic and the orthogonal cases, and the conjectured values of γ_K agree with our results (80) and (92).

Appendix: Summation Formula for the Riemann Zeta-Function

The Riemann ζ-function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(\frac{1}{1 - \frac{1}{p^s}} \right), \tag{A.1}$$

where p is a prime number. The Kth power of this function is written as

$$[\zeta(s)]^K = \sum_{n=1}^{\infty} \frac{d_K(n)}{n^s}$$

$$= \prod_p \left(1 + \frac{d_K(p)}{p^s} + \frac{d_K(p^2)}{p^{2s}} + \cdots \right), \qquad (A.2)$$

where $d_K(n)$ is the Kth Dirichlet coefficient. When n is a power of the prime number, $d_K(p^j) = \Gamma(K+j)/\Gamma(K)j!$ (this follows easily from the definition of the Dirichlet coefficient $d_K(n) = \sum_{n_1 \cdots n_K = n} 1$).

We consider now the average of (A.2) on the critical line $s = \frac{1}{2} + it$ over a large interval T,

$$\frac{1}{T} \int_0^T dt \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2K} = \frac{1}{T} \int_0^T dt \left| \sum_{n=1}^{\infty} \frac{d_K(n)}{n^{\frac{1}{2}+it}} \right|^2. \qquad (A.3)$$

Expanding the sum $\left| \sum_{n=1}^{\infty} \frac{d_K(n)}{n^s} \right|^2$, which appears in (A.3), we first examine the diagonal terms,

$$\sum_{n=1}^{\infty} \frac{d_K^2(n)}{n^s} = \prod_p \left(1 + \frac{d_K^2(p)}{p^s} + \frac{d_K^2(p^2)}{p^{2s}} + \cdots \right)$$

$$= \prod_p (1 - p^{-s})^{-K^2} \left(1 - \frac{K^2(K-1)^2}{4} p^{-2s} + \cdots \right)$$

$$= [\zeta(s)]^{K^2} g_K(s), \qquad (A.4)$$

where

$$g_K(s) = \prod_p \left[(1 - p^{-s})^{K^2} \sum_{j=0}^{\infty} \frac{d_K^2(p^j)}{p^{js}} \right]. \qquad (A.5)$$

The function $g_K(s)$ is an analytic function of s, including the point $s = 1$.

Let us examine the contribution of these diagonal terms given by (A.4) to (A.3). Their contribution is conveniently found, if we apply the following inversion formula (Perron formula):

$$B(s) = \sum_{n=1}^{\infty} b_n n^{-s},$$

$$f(x) = \sum_{n \le x} b_n. \qquad (A.6)$$

Then, we have

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} B(s) x^s s^{-1} ds, \qquad (A.7)$$

in which c is some arbitrary real positive number. Substituting $b_n = d_K^2(n)$, and $B(s) = \zeta^{K^2}(s)g_K(s)$, we obtain, from the residue of the singularity at $s = 1$,

$$\sum_{n \leq x} d_K^2(n) = \frac{g_K(1)}{\Gamma(K^2)} x \log^{K^2-1} x + O(x \log^{K^2-2} 3x). \tag{A.8}$$

By a partial summation, this approximate calculation yields,

$$\sum_{n \leq T} \frac{d_K^2(n)}{n} \sim \frac{a_K}{\Gamma(K^2+1)} \log^{K^2} T, \tag{A.9}$$

where $a_K = g_K(1)$.

From these formulae, it is seen that the contribution of the diagonal terms to the average of the Kth power moment of the ζ-function does take the asymptotic form of (1). However, neglecting the off-diagonal terms, we failed to reproduce the proper coefficient γ_K, whose understanding clearly requires the off-diagonal products in (A.4) as well.

A lower bound for the $2K^{\text{th}}$ moment is known [18]

$$\int_{T-Y}^{T+Y} |\zeta\left(\frac{1}{2}+it\right)|^{2K} dt \gg Y \log^{K^2} Y, \tag{A.10}$$

where $\log^\epsilon T \leq Y \leq T$. An upper bound seems difficult to obtain, and (1) remains as a conjecture, except for the $K = 1$ and $K = 2$ cases, for which it has been derived.

We note here the results and the conjecture of Montgomery [19] about the density of the zeros of Riemann ζ-function and their correlation. When γ is a zero on the critical line, $\zeta(\frac{1}{2}+i\gamma) = 0$,

$$\sum_{0<\gamma \leq T} 1 \geq \left(\frac{2}{3}+o(1)\right)\frac{T}{2\pi}\log T, \tag{A.11}$$

$$\sum_{0<\gamma,\gamma' \leq T, \alpha/L \leq \gamma-\gamma' \leq \beta/L} 1 = (1+o(1))\left[\int_\alpha^\beta \left(1-(\frac{\sin \pi u}{\pi u})^2\right) du + \delta(\alpha, \beta)\right] TL, \tag{A.12}$$

where $L = \log T/(2\pi)$, and $\delta(\alpha, \beta) = 1$ for $0 \in [\alpha, \beta]$, and otherwise zero. Then (A.11) is equivalent to the average density of state in (1), with for $K = 1$, $\gamma_K = a_K = 1$ and (A.12) is equivalent to the pair correlation function in random matrix theory.

Let us present the arguments which lead to the conjectured formula (107); we first assume that $\lambda_1 - \lambda_2 \sim O((\ln T)^{-1})$ for large T. The diagonal approximation for the product of $\zeta(s_1)$ and $\zeta(s_2)$, which earlier gave the expected behaviour for the moment, but with a wrong coefficient, may thus be applied here again, since we are taking a logarithmic derivative, which is unsensitive to overall normalizations. Within this assumption,

we obtain

$$
\frac{\partial}{\partial \lambda_1} \log F = \frac{\partial}{\partial \lambda_1} \ln \left[\frac{1}{T} \int_1^T dt \left(\sum_{n=1}^\infty \frac{1}{n^{\frac{1}{2}+i\lambda_1+it}} \right) \left(\sum_{n=1}^\infty \frac{1}{n^{\frac{1}{2}-i\lambda_2-it}} \right) \zeta(s_3) \cdots \zeta(s_{2K}) \right]
$$

$$
\sim \frac{\partial}{\partial \lambda_1} \ln \left[\left(\sum_{n<T} \frac{1}{n^{1+i(\lambda_1-\lambda_2)}} \right) \frac{1}{T} \int_1^T dt\, \zeta(s_3) \cdots \zeta(s_{2K}) \right]
$$

$$
\sim \frac{\partial}{\partial \lambda_1} \ln \left[\int_{T_0}^T dx \frac{1}{x^{1+i(\lambda_1-\lambda_2)}} \right]
$$

$$
\sim -i \ln T - \frac{1}{\lambda_1 - \lambda_2}. \tag{A.13}
$$

We have considered up to now what happens when $\lambda_1 - \lambda_2$ is small, but we should repeat the same arguments for the Dyson limit in which all pairs $\lambda_1 - \lambda_j$ are of order $(\log T)^{-1}$. Therefore, when one sums over all possible combinations, one obtains (107).

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