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Vortex Condensates for the SU(3) Chern–Simons Theory*

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Abstract: We investigate $SU(3)$-periodic vortices in the self-dual Chern–Simons theory proposed by Dunne in [13,15]. At the first admissible non-zero energy level $E = 2\pi$, and for *each* (broken and unbroken) *vacuum* state $\phi_{(0)}$ of the system, we find a family of periodic vortices asymptotically gauge equivalent to $\phi_{(0)}$, as the Chern–Simons coupling parameter $k \to 0$. At higher energy levels, we show the existence of *multiple* gauge distinct periodic vortices with at least one of them asymptotically gauge equivalent to the (broken) *principal embedding vacuum*, when $k \to 0$.

1. Introduction

In recent years several Chern–Simons field theories [14] have been proposed largely motivated by their possible applications to the physics of high critical temperature superconductivity. In fact, the corresponding Chern–Simons vortex theory has revealed a much richer structure compared with that described by the "classical" Yang–Mills framework [21,26].

The study of Chern–Simons vortices has been particularly succesful in the *abelian* situation, see [31,30,4,34,29,37,8,9,6] and [7], where the existence and multiplicity of vortices with different nature (e.g. topological, nontopological, periodically constrained, etc.) have been established for the models proposed in [18,20] and [23].

Only recently some progress has been made towards the existence of *nonabelian* CS-vortices, concerning the self-dual Chern–Simons theory proposed by Dunne in [13, 15], see also [22,25] and [24].

Dunne's model is defined in the $(2+1)$-Minkowski space $\mathbb{R}^{1,2}$, with metric tensor $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1)$. The gauge group is given by a compact Lie group G equipped with a semi-simple Lie algebra $(\mathcal{G}, [\ ,\ ])$. The relative Chern–Simons Lagrangian density

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is given by

$$\mathcal{L} = -Tr\{(D_\mu\phi)^\dagger D^\mu\phi\} - k\epsilon^{\mu\nu\alpha}Tr\{\partial_\mu A_\nu A_\alpha + \frac{2}{3}A_\mu A_\nu A_\alpha\} - V(\phi,\phi^\dagger), \quad (1.1)$$

where $A = (A_0, A_1, A_2)$ is the gauge connection of the principal bundle over $\mathbb{R}^{2,1}$ with structure group G that, together with the Higgs field ϕ, takes values in $\mathcal{G}$ through the adjoint representation of G. The gauge-covariant derivative $D_\mu = \partial_\mu + [A_\mu, \cdot\ ]$ is used to weakly couple the Higgs field ϕ with the gauge potential $A = (A_0, A_1, A_2)$. Furthermore, the Levi–Civita antisymmetric tensor $\epsilon^{\mu\nu\alpha}$ is chosen with $\epsilon^{012} = 1$, $k > 0$ is the Chern–Simons coupling parameter and "$Tr\{\dots\}$" refers to the trace in the matrix representation of $\mathcal{G}$. The gauge-invariant scalar potential $V(\phi,\phi^\dagger)$ is defined by,

$$V(\phi,\phi^\dagger) = \frac{1}{4k^2}Tr\{([[\phi,\phi^\dagger],\phi] - v^2\phi)^\dagger([[\phi,\phi^\dagger],\phi] - v^2\phi)\},$$

where the constant v^2 plays the role of a mass parameter. In the following we set $v^2 = 1$.

Vortices for $\mathcal{L}$ correspond to *static* solutions (with "finite" energy) for the Euler–Lagrange equations corresponding to (1.1).

Over $\mathbb{R}^2$, topological and nontopological vortices (see below) have been established in [38] and [36] respectively.

Here we look for *periodic* vortices or *condensates*, namely for those static solutions which satisfy appropriate periodic boundary conditions to be specified according to the gauge invariance of $\mathcal{L}$.

We note immediately that the Euler-Lagrange equations corresponding to $\mathcal{L}$ (see (5.31a-b) in [14]) are very difficult to handle directly, even when we restrict to consider "Bogomol'nyi" type vortices, which are obtained by solving the reduced (first order) "*relativistic self-dual Chern–Simons equations*":

$$\begin{cases} D_-\phi = 0 \\ F_{+-} = \frac{1}{k^2}[\phi - [[\phi,\phi^\dagger],\phi],\phi^\dagger], \end{cases} \quad (1.2)$$

where $D_- = D_1 - iD_2$ and $F_{+-} = \partial_+A_- - \partial_-A_+ + [A_+, A_-]$, with $A_\pm = A_1 \pm iA_2$ and $\partial_\pm = \partial_1 \pm i\partial_2$.

Solutions to (1.2) define a sort of "energy minimizers" for the system as they saturate the lower bound:

$$\mathcal{E} = \frac{1}{2k^2}\mathrm{tr}(\phi^\dagger(\phi - [[\phi,\phi^\dagger],\phi])) \quad (1.3)$$

(modulo some negligible surface terms) for the energy density $\mathcal{E}$ corresponding to $\mathcal{L}$ (see [14,15]).

From (1.2) it is easy to determine the *zero*-energy vortices (*vacua* states), where the gauge field vanishes (modulo gauge transformations) while the Higgs field $\phi_{(0)}$ corresponds to a *zero* of the potential V, namely, it is gauge equivalent to a solution of the algebraic equation (see [14,15]):

$$\Big[[\phi,\phi^\dagger],\phi\Big] = \phi. \quad (1.4)$$

To be able to determine other *non-zero* energy solutions of (1.2), Dunne in [13] has proposed a simplified form of the self-dual system (1.2) in which the fields are algebrically restricted as follows.

Let r be the rank of the Lie algebra $\mathcal{G}$, $\{H_a\}$ the generators of the Cartan subalgebra and $\{E_{\pm a}\}$ the family of the simple root step operators (with $E_{-a} = E_a^\dagger$), normalized according to a Chevalley basis [5, 19]. Hence, they satisfy the following commutation relations:

$$
\begin{aligned}
[H_a, H_b] &= 0,\\
[H_a, E_{\pm b}] &= \pm K_{ab} E_{\pm b},\\
[E_a, E_{-b}] &= \delta_{ab} H_a,
\end{aligned}
$$

$a, b = 1, \ldots, r$, and are subject to the normalization conditions:

$$
\begin{aligned}
Tr\{H_a H_b\} &= K_{ab},\\
Tr\{E_a E_{-b}\} &= \delta_{ab},\\
Tr\{H_a E_{\pm b}\} &= 0,
\end{aligned}
$$

$a, b = 1, \ldots, r$, where $K = (K_{ab})$ is the Cartan matrix.

We assume that the fields take the form:

$$A_\mu = -i \sum_{a=1}^{r} A_\mu^a H_a, \tag{1.5}$$

$$\phi = \sum_{a=1}^{r} \phi^a E_a \tag{1.6}$$

with A_μ^a $(a = 1, \ldots, r; \mu = 0, 1, 2)$ real-valued functions and ϕ^a $(a = 1, \ldots, r)$ complex-valued functions.

In view of (1.5) and (1.6) the gauge invariance of $\mathcal{L}$ can be expressed in terms of the gauge group $\mathcal{H} = \text{span}\, \{\mathrm{e}^{i \sum_{a=1}^{r} w_a(x) H_a}\}$ (where w_a $a = 1, \ldots, r$ are real-valued smooth functions) generated by the Cartan subalgebra generators $\{H_a\}$. In other words, the gauge transformation laws for the components of the gauge potential and the Higgs field take the following simplified form:

$$
\begin{aligned}
A_\mu^a &\longrightarrow A_\mu^a + \partial_\mu w_a,\\
\phi^a &\longrightarrow \mathrm{e}^{i \sum_{b=1}^{r} K_{ba} w_b} \phi^a,
\end{aligned}
\tag{1.7}
$$

$a = 1, \ldots, r$.

With the algebraic restriction on the fields (1.5) and (1.6), the Lagrangian density $\mathcal{L}$ and the potential V are simplified as well. The Chern–Simons term decomposes into r copies of an *abelian* Chern–Simons term, and we have

$$
\begin{aligned}
\mathcal{L}_{\text{restricted}} = &- \sum_{a=1}^{r} \left| \partial_\mu \phi^a - i (\sum_{b=1}^{r} K_{ba} A_\mu^b) \phi^a \right|^2\\
&- k \sum_{a=1}^{r} \epsilon^{\mu\nu\rho} \partial_\mu A_\nu^a A_\rho^a - V_{\text{restricted}},
\end{aligned}
\tag{1.8}
$$

where the *restricted* potential becomes

$$V_{\text{restricted}} = \frac{1}{4k^2}\sum_{a=1}^{r}|\phi^a|^2 - \frac{1}{2k^2}\sum_{a,b=1}^{r}|\phi^a|^2 K_{ab}|\phi^b|^2 + \frac{1}{4k^2}\sum_{a,b,c=1}^{r}|\phi^a|^2 K_{ab}|\phi^b|^2 K_{bc}|\phi^c|^2. \tag{1.9}$$

Furthermore, the "relativistic self-dual Chern–Simons equations" (1.2) (away from the zeroes of ϕ^a) combine into the single set of coupled equations:

$$\partial_+\partial_- \ln|\phi^a|^2 = -\frac{1}{k^2}\sum_{b=1}^{r}K_{ab}|\phi^b|^2 + \frac{1}{k^2}\sum_{b,c=1}^{r}|\phi^b|^2 K_{bc}|\phi^c|^2 K_{ac}, \quad a = 1,\dots r. \tag{1.10}$$

To recover the $A_{\mu=0}$ component of the gauge potential, we must supplement (1.2) with the Gauss Law constraint of the system, which componentwise reads as follows:

$$kF_{12}^a = J_0^a \quad a = 1,\dots,r, \tag{1.11}$$

where $F_{12}^a = \partial_1 A_2^a - \partial_2 A_1^a$ and J_0^a define (after multiplication by $-i$) respectively, the components (in the Cartan subalgebra) of the gauge curvature and the current density $J_0 = -i([\phi^\dagger, D^0\phi] - [(D^0\phi)^\dagger, \phi])$.

The selfdual equations (1.2), imply that $D_0\phi = \frac{i}{2k}([[\phi,\phi^\dagger],\phi] - \phi)$ and, by direct calculation, for the energy density (1.3) we find the expression

$$\mathcal{E} = \sum_{a=1}^{r} F_{12}^a \tag{1.12}$$

(modulo negligible surface terms).

Therefore under the decomposition (1.5) and (1.6), the system may be described in terms of r-*(abelian)* Chern–Simons fields A_μ^a coupled to r complex scalar Higgs fields ϕ^a with the couplings determined by the Cartan matrix $K = (K_{ab})$.

Note that, by means of (1.5) and (1.6), the algebraic equation (1.4) may be solved explicitly in terms of the components $\phi_{(0)}^a$ of $\phi_{(0)}$ in the Cartan subalgebra. When $\phi_{(0)}^a \neq 0$, we find that,

$$|\phi_{(0)}^a|^2 = \sum_{b=1}^{r}(K^{-1})_{ab} \quad a = 1,\dots r, \tag{1.13}$$

where K^{-1} is the inverse of the Cartan matrix K.

Notice that, on the base of (1.3), vortex solutions of (1.2) in $\mathbb{R}^2$ (with sufficiently fast decay as $|x| \to +\infty$) satisfies the energy relation:

$$E = \int_{\mathbb{R}^2}\mathcal{E} = \frac{1}{2k^2}\int_{\mathbb{R}^2}\operatorname{tr}\left(\phi^\dagger\left(\phi - [[\phi,\phi^\dagger],\phi]\right)\right). \tag{1.14}$$

Therefore, it is to be expected that *non*-zero energy vortex solutions in $\mathbb{R}^2$ become gauge equivalent to the vacua states as $|x| \to +\infty$.

Results in this direction have been obtained by Yang [38] and Wang-Zhang [36] when the gauge group $G = SU(N)$. Yang in [38] shows that there exist solutions for (1.2) satisfying the ansatz (1.5) and (1.6), for every *prescribed* configuration of zeros for ϕ^a, and such that $|\phi^a|$ uniformly converges to $|\phi^a_{(0)}|$ in (1.13) as $|x| \to +\infty$, $a = 1, \ldots, r$.

Thus, Yang's vortices are asymptotically equivalent (as $|x| \to +\infty$) to the so called *principal embedding vacuum*. Usually, one refers to those as the *topological* solutions for (1.2) in $\mathbb{R}^2$. More difficult to derive are instead the *nontopological* solutions, namely those asymptotically gauge-equivalent to the other vacua states, as $|x| \to +\infty$. A class of nontopological solutions has been derived recently in [36]. The solutions in [36] are shown to be asymptotically gauge-equivalent to the *unbroken* vacuum $\phi_{(0)} = 0$, as $|x| \to +\infty$. We observe that, when $N = 2$, the $SU(N)$-gauge theory corresponding to $\mathcal{L}$ in (1.1) reduces to the *abelian* Chern–Simons–Higgs theory introduced by Hong–Kim–Pac [18] and Jackiw–Weinberg [20]. Thus, the above mentioned results [38] and [36] extend the work of [31, 30] and [6] on topological and nontopological *abelian* Chern–Simons vortices. In the *abelian* contest we also mention the work of [7] on nontopological Maxwell–Chern–Simons vortices for the Lee–Lee–Min model [23].

However, to establish condensate-type vortices whose feature more closely resemble those of the mixed states predicted by Abrikosov in superconductivity [1], it is necessary to derive solutions for (1.2) subject to gauge invariant periodic boundary condition.

For this purpose note that, in the stationary case, the functions w_a ($a = 1, \ldots, r$) in (1.7) expressing the gauge invariance of (1.1), depend only on the space-variables $x = (x_1, x_2)$ and the gauge transformation laws reduce to:

$$\begin{aligned} &A_0^a \to A_0^a \quad A_j^a \to A_j^a + \partial_j w_a, \quad j = 1, 2, \\ &\phi^a \longrightarrow \mathrm{e}^{i \sum_{b=1}^r K_{ba} w_b} \phi^a, \quad a = 1, \ldots, r. \end{aligned} \tag{1.15}$$

Therefore, following 't Hooft [33], for each of the r-components of the fields we require appropriate periodic boundary conditions to hold in the periodic cell domain:

$$\Omega = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \mid -\frac{a}{2} \le x_1 \le \frac{a}{2}, -\frac{b}{2} \le x_2 \le \frac{b}{2} \right\},$$

as follows.

Let $e_1 = (a, 0)$ and $e_2 = (0, b)$ and decompose the boundary of Ω by setting

$$\partial\Omega = \Gamma^1 \cup \Gamma^2 \cup \{e_1 + \Gamma^2\} \cup \{e_2 + \Gamma^1\} \cup \{0, e_1, e_2, e_1 + e_2\},$$

with

$$\begin{aligned} \Gamma^1 &= \left\{ x \in \mathbb{R}^2 \mid x = \frac{1}{2}(s e_1 - e_2) \quad |s| < 1 \right\}, \\ \Gamma^2 &= \left\{ x \in \mathbb{R}^2 \mid x = \frac{1}{2}(s e_2 - e_1) \quad |s| < 1 \right\}. \end{aligned}$$

We require that each component A^a_μ and ϕ^a ($a = 1, \ldots, r$) of the vortex condensates (A, ϕ) satisfies:

$$\begin{cases} \mathrm{e}^{i \sum_{b=1}^r K_{ba} \xi_k^b (x + e_k)} \phi^a (x + e_k) = \mathrm{e}^{i \sum_{b=1}^r K_{ba} \xi_k^b (x)} \phi^a (x), & \text{(a)} \\ A_0^a (x + e_k) = A_0^a (x), & \text{(b)} \\ (A_j^a + \partial_j \xi_k^a)(x + e_k) = (A_j^a + \partial_j \xi_k^a)(x), \quad j = 1, 2, & \text{(c)} \\ x \in \Gamma^1 \cup \Gamma^2 \setminus \Gamma^k, \quad k = 1, 2, \quad a = 1, \ldots, r, & \end{cases} \tag{1.16}$$

where ξ_1^a, ξ_2^a $(a = 1, \dots, r)$ are smooth functions defined in a neighborhood of $\Gamma^2 \cup \{e_1 + \Gamma^2\}$ and $\Gamma^1 \cup \{e_2 + \Gamma^1\}$, respectively.

Notice that, in analogy to the abelian case, the set of boundary conditions (1.16) produce a "quantization" effect on the "charges" (see [4] and [35]).

In fact, in view of (1.16)-(a), to any vortex condensate we can associate r-integers $N_a \in \mathbb{Z}$ $(a = 1, \dots, r)$ (vortex numbers), corresponding to the phase shift of ϕ^a around $\partial\Omega$. More precisely, for $a = 1, \dots, r$ and $k = 1, 2$, set

$$\hat{\xi}_k^a \left(s^1, s^2\right) = \sum_{b=1}^{r} K_{ba} \xi_k^b \left(s^1 e_1 + s^2 e_2\right) \quad s^j \in (0, 1)\,, \; j = 1, 2,$$

we have:

$$\begin{aligned} &\hat{\xi}_1^a \left(1, 0^+\right) - \hat{\xi}_1^a \left(0, 0^+\right) + \hat{\xi}_2^a \left(0^+, 0\right) - \hat{\xi}_2^a \left(0^+, 1\right) \\ &+ \hat{\xi}_1^a \left(0, 1^-\right) - \hat{\xi}_1^a \left(1, 1^-\right) + \hat{\xi}_2^a \left(1^-, 1\right) - \hat{\xi}_2^a \left(1^-, 0\right) = 2\pi N_a. \end{aligned}$$

Consequently, by means of (1.16)-(c) and (1.11), for the "magnetic flux"-component $\Phi_a = \int_\Omega F_{12}^a$ and the "electric charge"-component $Q_a = \int_\Omega J_0^a$ we obtain the relations,

$$\sum_{b=1}^{r} K_{ba} \Phi_b = 2\pi N_a, \quad Q_a = k\Phi_a.$$

Hence, they obey to the following "quantization" rules:

$$\Phi_a = 2\pi \sum_{b=1}^{r} \left(K^{-1}\right)_{ba} N_b, \quad Q_a = 2\pi k \sum_{b=1}^{r} \left(K^{-1}\right)_{ba} N_b. \tag{1.17}$$

Accordingly, for the *energy*

$$E = \int_\Omega \mathcal{E} = \frac{1}{2k^2} \int_\Omega \mathrm{tr} \left(\phi^\dagger (\phi - [[\phi, \phi^\dagger], \phi]\right), \tag{1.18}$$

we may use (1.14) to derive

$$E = \sum_{a=1}^{r} \int_\Omega F_{12}^a = 2\pi \sum_{a,b=1}^{r} \left(K^{-1}\right)_{ba} N_b = 2\pi \sum_{b=1}^{r} |\phi_{(0)}^b|^2 N_b, \tag{1.19}$$

where $\phi_{(0)}^b$ expresses the components (in the Cartan subalgebra) of the principal embedding vacuum as given in (1.13).

Since $D_-\phi = 0$, or equivalently $\partial_- \ln \phi^a = i \sum_{b=1}^{r} A_-^b K_{ab}$ $(a = 1, \dots, r)$, as in [4] and [34], each vortex number N_a has the (topological) interpretation of counting the number of zeroes (according to their multiplicity) of the Higgs scalar component ϕ^a in Ω.

By virtue of (1.18), we now expect doubly periodic vortex solutions to become asymptotically gauge-equivalent to the vacua-states, when $k \to 0^+$.

Thus, in the same spirit of the results [38] and [36] mentioned above, we are going to establish periodic vortex condensates for $\mathcal{L}$, where the asymptotic behavior of the Higgs field, as $k \to 0^+$, is prescribed according to *any* fixed zero of the gauge potential V.

For this purpose, we shall focus our attention on the simplest non-abelian case of physical relevance and take the gauge group $G = SU(3)$. Our results should be compared with those obtained in [4,34] for the abelian case (i.e. $G = SU(2)$) and in [29] for the Maxwell-Chern–Simons-Higgs model of Lee–Lee–Min [23].

Note that for the Lie group $SU(3)$ the Cartan matrix is $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Thus, the *restricted* potential takes the form:

$$
\begin{aligned}
V_{\text{restricted}}\left(|\phi^1|, |\phi^2|\right) &= \frac{1}{4k^2}\left(4|\phi^1|^6 + 4|\phi^2|^6 - 3|\phi^1|^4|\phi^2|^2 - 3|\phi^1|^2|\phi^2|^4\right.\\
&\quad \left. + 4|\phi^1|^4 + 4|\phi^2|^4 - 4|\phi^1|^2|\phi^2|^2 + |\phi^1|^2 + |\phi^2|^2\right)\\
&= \frac{1}{4k^2}\left(|\phi^1|^2\left(2|\phi^1|^2 - |\phi^2|^2 - 1\right)^2 + |\phi^2|^2\left(2|\phi^2|^2 - |\phi^1|^2 - 1\right)^2\right)
\end{aligned} \tag{1.20}
$$

whose zeroes coincide with the pairs $(|\phi^1|^2, |\phi^2|^2) = (0, 0)$ unbroken vacuum,

$(|\phi^1|^2, |\phi^2|^2) = (1, 1)$ principal embedding vacuum, $(|\phi^1|^2, |\phi^2|^2) = (0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$.

Thus, for $k > 0$ small, we will be interested in deriving $SU(3)$- vortex solutions for (1.2) - (1.16), under the ansatz (1.5) and (1.6), such that the components ϕ^a $(a = 1, 2)$ of the Higgs field satisfy one of the following:

- $|\phi^1| \to 1$ and $|\phi^2| \to 1$ (type I)
- $|\phi^1| \to 0$ and $|\phi^2| \to 0$ (type II)
- $\begin{cases} |\phi^1| \to \frac{1}{\sqrt{2}} \quad \text{and} \quad |\phi^2| \to 0 & \text{(a)} \\ |\phi^1| \to 0 \quad \text{and} \quad |\phi^2| \to \frac{1}{\sqrt{2}} & \text{(b)} \end{cases}$ (type III)

in some suitable norm as $k \to 0^+$.

In this direction, we prove that there exist *two* gauge distinct family of solutions for any prescribed pair of "vortex numbers" N_a and relative set of *vortex points* $\mathcal{Z}_a = \{p_1^a, \dots, p_{N_a}^a\} \subset \Omega$ corresponding to the *zeroes* of the component ϕ^a of the Higgs field $(a = 1, 2)$, provided we take $k > 0$ sufficiently small.

Only one of these two families of solutions we can characterize as having the prescribed asymptotic behavior of type I, as $k \to 0$.

The existence of solutions of type II and III is proved only when $N_1 + N_2 = 1$, that is, a single vortex point is prescribed. In particular, we may conclude that at the energy level $E = 2\pi$ (see (1.19)) there exist $SU(3)$- periodic vortices for each of the prescribed type I, II and III. More precisely, we obtain the following results.

Theorem 1.1. *Let N_a be a nonnegative integer and $\mathcal{Z}_a = \{p_1^a, \dots, p_{N_a}^a\} \subset \Omega$ be an assigned set of N_a-points (not necessarily distinct) in Ω, $a = 1, 2$. For $0 < k < \sqrt{\frac{3|\Omega|}{8\pi \max\{2N_1+N_2, 2N_2+N_1\}}}$ sufficiently small, there exist two gauge distinct $SU(3)$-periodic vortex solutions of* (1.2)–(1.16) *satisfying the ansatz* (1.5)–(1.6) *and such that:*

(i) *the component ϕ^a of the Higgs field satisfies: $|\phi^a| < 1$ in Ω, and ϕ^a vanishes exactly at each $p_j^a \in \mathcal{Z}_a$ with the multiplicity given by the repetition of p_j^a in $\mathcal{Z}_a$, $a = 1, 2$.*

(ii) *The induced "magnetic flux"-component Φ_a and "electric charge"-component Q_a, satisfy:*

$$\Phi_a = \frac{1}{k} Q_a = \frac{2\pi}{3}(2N_a + N_b) \quad a \neq b = 1, 2.$$

(iii) *The energy E satisfies:*

$$E = 2\pi(N_1 + N_2).$$

Furthermore, one of these two solutions is always of type I, in the sense that the components $\phi^a_{(1)}$ $(a = 1, 2)$ of the Higgs field satisfy:

$$|\phi^a_{(1)}| \to 1, \quad \text{as } k \to 0^+, \tag{1.21}$$

pointwise a.e. in Ω and strongly in $L^p(\Omega)$, $\forall\ p \geq 1$.

While, when $N_1 + N_2 = 1$, then the other solution is of type II, and the corresponding components $\phi^a_{(2)}$ $(a = 1, 2)$ of the Higgs field satisfy:

$$|\phi^a_{(2)}| \to 0, \quad \text{as } k \to 0^+, \text{ uniformly in } \Omega. \tag{1.22}$$

For $k > \sqrt{\frac{3|\Omega|}{8\pi \max\{2N_1+N_2, 2N_2+N_1\}}}$ it is not possible to have $SU(3)$-periodic vortex solutions for (1.2)–(1.16) *satisfying the ansatz* (1.5)–(1.6) *together with the properties (ii) and (iii).*

Remark 1.2. In the statement above, it is understood that in case $N_a = 0$ for some $a = 1, 2$ then the corresponding set of vortex points $\mathcal{Z}_a$ is taken to be the empty set. Furthermore, a bootstrap argument shows that, in fact, the convergence in (1.22) holds $C^m(\Omega)$-uniformly, for every $m \in \mathbb{N}$, and in any other relevant norm.

Theorem 1.1 may be considered as the complete analogue of the results on *abelian* Chern–Simons periodic vortices (corresponding to $G = SU(2)$) obtained by Caffarelli–Yang [4] and Tarantello [34].

In fact, if $G = SU(2)$ then $r = 1$, and the restricted gauge-potential takes the form $V(|\phi|) = \frac{1}{4k^2}|\phi|^2(1 - 2|\phi|^2)^2$. So, in this case, only type I and II-vortices are allowed. To insure the existence of type II-vortices, the given restriction on the vortex numbers not to exceed the value 1 appears as a technical condition, which is required also in the abelian case [34].

Indeed, in the spirit of [34], we derive our results by considering different "constrained" variational principles for each type I, II and III-vortex.

This point of view was inspired by a "constrained" variational approach introduced by Caffarelli–Yang [4] to treat 1-periodic vortices in the abelian situation. The restriction on the vortex number $N = 1$ was needed in [4] in order to derive the existence of a minimum for the relative variational problem, as a direct consequence of the Moser–Trudinger inequality (see [3,16]). On the other hand, our variational problems take a system-form for which Moser–Trudinger's inequality no longer suffices to yield directly the existence of a minimizer even under the given restriction on the vortex numbers. Instead, we show (see also [27]) that, on the constrained set, an "improved" form of the Moser–Trudinger inequality holds, which enables us to obtain a minimum for all the variational problems under examination regardless of the values of the vortex numbers.

However, for type II (and even more so for type III) vortices we need to restrict the sum of the vortex numbers, as above, in order to insure that these minima actually lie on the “interior” of the constrained set, and thus yield to the desired vortex-solution. At the moment, it is not clear how to remove such a restriction even for the simpler abelian situation where only recently some progress has been made in this direction, see [28, 12] and [10].

Concerning the type III-vortices, specific to $SU(3)$-theory, we have the following:

Theorem 1.3. *For each fixed point* $p \in \Omega$ *and* $0 < k < \frac{1}{4}\sqrt{\frac{3|\Omega|}{\pi}}$ *sufficiently small, there exists an* $SU(3)$*-periodic-vortex solution for* (1.2)–(1.16) *satisfying the ansatz* (1.5)–(1.6) *such that for the component* ϕ^a $(a = 1, 2)$ *of the Higgs field the following holds:*

(i) *(first component):* $|\phi^1| < 1$*;* ϕ^1 *never vanishes in* Ω *and,*

$$|\phi^1| \to \frac{1}{\sqrt{2}}, \quad \text{as } k \to 0^+, \tag{1.23}$$

pointwise a.e. in Ω *and strongly in* $\mathcal{H}^1(\Omega)$.
The corresponding induced first-component of the “magnetic field” Φ_1 *and “electric charge”* Q_1 *satisfy:* $\Phi_1 = \frac{1}{k} Q_1 = \frac{2\pi}{3}$*;*

(ii) *(second component):* $|\phi^2| < 1$*;* ϕ^2 *admits a simple zero at* $p \in \Omega$ *and,*

$$|\phi^2| \to 0, \quad \text{as } k \to 0^+, \tag{1.24}$$

pointwise a.e. in Ω *and strongly in* $\mathcal{H}^1(\Omega)$.
The corresponding induced second-component of the “magnetic field” Φ_2 *and “electric charge”* Q_2 *satisfy:* $\Phi_2 = \frac{1}{k} Q_2 = \frac{4\pi}{3}$.

(iii) *The energy* $E = 2\pi$.

Remark 1.4. In words, Theorem 1.3 states the existence of a type III (a) vortex with *first* vortex number $N_1 = 0$ and *second* vortex $N_2 = 1$.

Due to the complete symmetry of (1.10) with respect to the indices $a = 1, 2$, we can also claim the existence of a $SU(3)$-periodic vortex of type III (b) simply by exchanging the role between the indices. Thus, Theorem 1.3 may be completed with the existence of another $SU(3)$-periodic vortex whose component ϕ^a of the Higgs field satisfy:

(i) (*first* component): $|\phi^1| < 1$; ϕ^1 admits a simple zero at $p \in \Omega$ and,

$$|\phi^1| \to 0 \quad \text{as } k \to 0^+, \tag{1.25}$$

pointwise a.e. in Ω and strongly in $\mathcal{H}^1(\Omega)$.
The corresponding induced first component of the “magnetic field” $\Phi_1 = \frac{4\pi}{3}$ and “electric charge” $Q_1 = \frac{4\pi}{3} k$;

(ii) (*second* component): $|\phi^2| < 1$; ϕ^2 *never* vanishes in Ω and,

$$|\phi^2| \to \frac{1}{\sqrt{2}}, \quad \text{as } k \to 0^+, \tag{1.26}$$

pointwise a.e. in Ω and strongly in $\mathcal{H}^1(\Omega)$.

The corresponding induced second component of the "magnetic field" Φ_2 and "electric charge" Q_2 satisfy $\Phi_2 = \frac{1}{k} Q_2 = \frac{2\pi}{3}$.

(iii) The energy $E = 2\pi$.

For such a solution the corresponding vortex numbers are given by $N_1 = 1$ and $N_2 = 0$.

Thus, in case $N_1 + N_2 = 1$, we can combine the results above and conclude:

Corollary 1.5. *For given $p^1, p^2 \in \Omega$ and $0 < k < \frac{1}{4}\sqrt{\frac{3|\Omega|}{\pi}}$ sufficiently small, at the energy level $E = 2\pi$ there exists a $SU(3)$-periodic vortex, satisfying* (1.2)–(1.16) *and the ansatz* (1.5)–(1.6)*, for each of the asymptotic behaviors prescribed by the type I, II and III (a),(b), as $k \to 0^+$. Furthermore, either the first component of the Higgs field ϕ^1 admits a simple zero at p^1 and the second component ϕ^2 never vanishes; or ϕ^1 never vanishes and ϕ^2 admits a simple zero at p^2.*

Note that, at the moment, no existence result is available concerning vortices in $\mathbb{R}^2$ with the asymptotic behavior of the type III, as $|x| \to +\infty$.

To establish the results above, we take advantage of the equation $D_-\phi = 0$, which we may write componentwise as follows:

$$\partial_- \ln \phi^a = i \sum_{b=1}^{r} A_-^b K_{ba}, \quad a = 1, \ldots, r. \tag{1.27}$$

In fact, by virtue of (1.27), we can follow an approach introduced by Taubes ([35]) for the study of self-dual Ginzburg–Landau vortices, and derive from (1.10) a system of nonlinear elliptic equations for the real variable functions $u_a = \ln |\phi^a|^2$ $(a = 1, 2)$ of the following form (see also [38]):

$$\begin{cases} \Delta u_a = -\frac{1}{k^2}\left(\sum_{b=1}^r K_{ab} e^{u_b} - \sum_{b,c=1}^r e^{u_b} K_{bc} e^{u_c} K_{ac}\right) + 4\pi \sum_{j=1}^{N_a} \delta_{p_j^a} \quad \text{on } \Omega, \\ u_a \quad \text{doubly periodic on } \partial\Omega, \quad a = 1, \ldots, r \end{cases} \tag{1.28}$$

where the points $p_1^a, \ldots, p_{N_a}^a \in \Omega$ are the prescribed zeroes of the scalar fields ϕ^a $(a = 1, \ldots, r)$ repeated according to their multiplicity.

In fact, from each solution u_a $(a = 1, \ldots, r)$ of (1.28) we may recover, under the ansatz (1.5) and (1.6), the whole vortex-solution for (1.2), by setting:

$$\begin{aligned} \phi^a(x) &= e^{\frac{1}{2}u_a(x) + i\sum_{j=1}^{N_a} \operatorname{Arg}\left(x - p_j^a\right)}, \\ A_1^a - iA_2^a &= -i \sum_{b=1}^r \partial_- \ln \phi^b \left(K^{-1}\right)_{ba}, \\ A_0^a &= -\frac{1}{2k}\left(|\phi^a|^2 - \sum_{b=1}^r \left(K^{-1}\right)_{ba}\right), \end{aligned} \tag{1.29}$$

where K^{-1} is the inverse of the Cartan matrix K. Clearly, from (1.29) we have that ϕ^a vanishes exactly at each p_j^a with the multiplicity corresponding to the repetition of p_j^a in $\mathcal{Z}_a$.

We shall devote the following sections to the analysis of the elliptic system (1.28) in case $G = SU(3)$.

2. Variational Formulation and Preliminary Results

We study the system (1.28) when the gauge group considered is $SU(3)$. Recalling that the Cartan matrix for $SU(3)$ is given by $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, the system (1.28) takes the following form:

$$\begin{cases} \Delta u_1 = \lambda(4e^{2u_1} - 2e^{2u_2} - 2e^{u_1} + e^{u_2} - e^{u_1+u_2}) + 4\pi \sum_{j=1}^{N_1} \delta_{p_j^1} & \text{on } \Omega \\ \Delta u_2 = \lambda(4e^{2u_2} - 2e^{2u_1} - 2e^{u_2} + e^{u_1} - e^{u_1+u_2}) + 4\pi \sum_{j=1}^{N_2} \delta_{p_j^2} & \text{on } \Omega \\ u_1, u_2 \quad \text{doubly periodic on } \partial\Omega. \end{cases} \tag{2.1}$$

where we have set

$$\lambda = \frac{1}{k^2} > 0. \tag{2.2}$$

Concerning problem (2.1) we shall prove the following results:

Theorem 2.1. (a) *For* $0 < \lambda < \frac{8\pi}{3|\Omega|} \max\{2N_1 + N_2, 2N_2 + N_1\}$ *problem* (2.1) *admits no solutions .*

(b) *Every solution* (u_1, u_2) *for* (2.1) *satisfies :*

$$e^{u_a} \le 1, \quad \text{in } \Omega \quad (a = 1, 2). \tag{2.3}$$

(c) *There exists* $\lambda_0 > 0$ *sufficiently large such that,* $\forall\lambda > \lambda_0$ *problem* (2.1) *admits, at least, two distinct solutions, one of which always satisfies:*

$$e^{u_a} \to 1, \quad \text{as } \lambda \to +\infty \tag{2.4}$$

pointwise a.e. in Ω *and strongly in* $L^p(\Omega)$, $\forall\, p \ge 1$.

We point out that, contrary to part (a) of Theorem 2.1 where the estimate on the *non-existence* range of λ's is given independently of the position of the vortex points, the range $(\lambda_0, +\infty)$ of existence as established in part (c), depends on the position of such points as can be seen already by the rough estimate given in (2.21).

Clearly, (2.4) insures the existence of a periodic vortex solution of type I. Concerning the existence of type II and III vortices, we have to limit our attention to consider the case where the vortex-numbers (N_1, N_2) satisfy: $N_1 + N_2 = 1$.

Thus, we consider problem (2.1) in the simpler form:

$$\begin{cases} \Delta u_1 = \lambda(4e^{2u_1} - 2e^{2u_2} - 2e^{u_1} + e^{u_2} - e^{u_1+u_2}) + 4\pi N_1 \delta_{p^1} & \text{on } \Omega \\ \Delta u_2 = \lambda(4e^{2u_2} - 2e^{2u_1} - 2e^{u_2} + e^{u_1} - e^{u_1+u_2}) + 4\pi N_2 \delta_{p^2} & \text{on } \Omega \\ u_1, u_2 \quad \text{doubly periodic on } \partial\Omega \end{cases} \tag{2.5}$$

with assigned points $p^a \in \Omega$, $a = 1, 2$.

We prove:

Theorem 2.2. *For* $N_1 + N_2 = 1$ *and* $\lambda > \frac{16\pi}{3|\Omega|}$ *sufficiently large, problem* (2.10) *admits a (weak) solution* (u_1^-, u_2^-) *satisfying:*

$$e^{u_a^-} \to 0 \quad \text{as } \lambda \to +\infty, \ \text{uniformly in } \Omega \quad (a = 1, 2), \tag{2.6}$$

(and in any other relevant norm).

Furthermore, there always exists a second solution (u_1^*, u_2^*) *such that,*

(i) *if* $N_1 = 0$ *and* $N_2 = 1$, *then*

$$e^{u_1^*} \to \frac{1}{2}, \quad e^{u_2^*} \to 0 \quad as\ \lambda \to +\infty \tag{2.7}$$

pointwise a.e. in Ω *and strongly in* $\mathcal{H}^1(\Omega)$*;*

(ii) *if* $N_1 = 1$ *and* $N_2 = 0$, *then*

$$e^{u_1^*} \to 0, \quad e^{u_2^*} \to \frac{1}{2} \quad as\ \lambda \to +\infty \tag{2.8}$$

pointwise a.e. in Ω *and strongly in* $\mathcal{H}^1(\Omega)$.

It is clear that, by means of the transformations in (1.29) and (2.2), we obtain Theorem 1.1, 1.3 as well as Corollary 1.5 as an immediate consequence of Theorem 2.1 and 2.2.

To establish Theorem 2.1 and 2.2, it will be convenient to distinguish between the singular and regular part of the solutions in (2.1). For this purpose denote by u_0^a $(a = 1, 2)$ the unique solution for the problem (see [3])

$$\begin{cases} \Delta u_0^a = -\frac{4\pi N_a}{|\Omega|} + 4\pi \sum_{j=1}^{N_a} \delta_{p_j^a} & \text{on } \Omega \\ \int_\Omega u_0^a = 0 \quad u_0^a \text{ doubly periodic on } \partial\Omega \end{cases} \tag{2.9}$$

$a = 1, 2$.

As is well known, $u_0^a \in C^\infty(\Omega \setminus \{p_1^a \dots p_{N_i}^a\})$ and if n_j^a is the multiplicity of p_j^a then u_0^a behaves like: $\ln |x - p_j^a|^{2n_j^a}$, as $x \to p_j^a$.

Setting $u_a = u_0^a + v_a$ and $h_a = e^{u_0^a}$ we have that (u_1, u_2) is a solution for (2.1) if and only if (v_1, v_2) is a *smooth* solution for the following system:

$$\begin{cases} \Delta v_1 = \lambda \left(4h_1^2 e^{2v_1} - 2h_2^2 e^{2v_2} - 2h_1 e^{v_1} + h_2 e^{v_2} - h_1 h_2 e^{v_1+v_2}\right) + \frac{4\pi N_1}{|\Omega|} & \text{on } \Omega \\ \Delta v_2 = \lambda \left(4h_2^2 e^{2v_2} - 2h_1^2 e^{2v_1} - 2h_2 e^{v_2} + h_1 e^{v_1} - h_1 h_2 e^{v_1+v_2}\right) + \frac{4\pi N_2}{|\Omega|} & \text{on } \Omega \\ v_1, v_2 \quad \text{doubly periodic on } \partial\Omega. \end{cases} \tag{2.10}$$

As a preliminary result, we start to derive part (b) of Theorem 2.1.

Proof of (2.3). First of all notice that, in view of (2.9) and (2.10), $\lim_{x \to p_j^a} u_a = -\infty$ for $a = 1, 2$ and $j = 1, \dots, N_a$. Therefore u_a attains its maximum value at some point $\bar{x}_a \in \Omega \setminus \{p_1^a, \dots, p_{N_a}^a\}$. Set $\bar{u}_a = \max_\Omega u_a = u_a(\bar{x}_a)$ $(a = 1, 2)$. By symmetry, we can assume without loss of generality, that $\bar{u}_1 \geq \bar{u}_2$. Since (u_1, u_2) is a solution of (2.1), we derive

$$\begin{aligned} 0 &\geq 4e^{2\bar{u}_1} - 2e^{2u_2(\bar{x}_1)} - 2e^{\bar{u}_1} + e^{u_2(\bar{x}_1)} - e^{\bar{u}_1 + u_2(\bar{x}_1)} \geq \\ &\geq 2e^{2\bar{u}_1} - 2e^{\bar{u}_1} + e^{u_2(\bar{x}_1)} - e^{\bar{u}_1 + u_2(\bar{x}_1)} \\ &= 2e^{\bar{u}_1}\left(e^{\bar{u}_1} - 1\right) - e^{u_2(\bar{x}_1)}\left(e^{\bar{u}_1} - 1\right) = \left(2e^{\bar{u}_1} - e^{u_2(\bar{x}_1)}\right)\left(e^{\bar{u}_1} - 1\right) \end{aligned}. \tag{2.11}$$

Thus, $e^{\bar{u}_1} - 1 \leq 0$ and consequently

$$e^{u_a(x)} \leq 1 \quad \text{for any } x \in \Omega, \quad \text{and } a = 1, 2. \qquad \square \tag{2.12}$$

We notice that (2.10) admits a variational formulation on $\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$. Here $\mathcal{H}^1(\Omega)$ denotes the space of doubly periodic functions $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ with periodic cell domain Ω. It defines an Hilbert space equipped with the standard $H^1(\Omega)$-scalar product.

We shall denote by $\|\cdot\|$ the usual norm on $\mathcal{H}^1(\Omega)$ as given by $\|v\|^2 = \|\nabla v\|_2^2 + \|v\|_2^2 = \int_\Omega |\nabla v|^2 + \int_\Omega |v|^2$.

It is easy to check that (weak) solutions for (2.10) correspond to critical points in $\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$ for the (unbounded) functional

$$\begin{aligned} I_\lambda(v_1, v_2) = &\frac{1}{3}\left(\|\nabla v_1\|^2 + \|\nabla v_2\|^2 + \int_\Omega \nabla v_1 \cdot \nabla v_2\right) + \lambda \int_\Omega W(v_1, v_2) \\ &+ \frac{4\pi}{3}(2N_1 + N_2)\int_\Omega v_1 + \frac{4\pi}{3}(2N_2 + N_1)\int_\Omega v_2, \quad v_1, v_2 \in \mathcal{H}^1(\Omega) \end{aligned} \tag{2.13}$$

with $W(v_1, v_2)$ given by

$$\begin{aligned} W(v_1, v_2) &= h_1^2 \mathrm{e}^{2v_1} + h_2^2 \mathrm{e}^{2v_2} - h_1 \mathrm{e}^{v_1} - h_2 \mathrm{e}^{v_2} - h_1 h_2 \mathrm{e}^{v_1+v_2} + 1 \\ &= \frac{1}{4}(2h_1\mathrm{e}^{v_1} - h_2\mathrm{e}^{v_2} - 1)^2 + \frac{3}{4}(h_2\mathrm{e}^{v_2} - 1)^2 \\ &= \frac{1}{4}(2h_2\mathrm{e}^{v_2} - h_1\mathrm{e}^{v_1} - 1)^2 + \frac{3}{4}(h_1\mathrm{e}^{v_1} - 1)^2 \geq 0. \end{aligned} \tag{2.14}$$

To simplify notations, from now on we shall assume, without loss of generality, that $|\Omega| = 1$.

Integrating (2.10) over Ω, we find that any solution (v_1, v_2) for (2.10) satisfies the following constraint conditions:

$$\begin{cases} 4\int_\Omega h_1^2 \mathrm{e}^{2v_1} - 2\int_\Omega h_2^2 \mathrm{e}^{2v_2} - 2\int_\Omega h_1 \mathrm{e}^{v_1} + \int_\Omega h_2 \mathrm{e}^{v_2} - \int_\Omega h_1 h_2 \mathrm{e}^{v_1+v_2} + \frac{4\pi N_1}{\lambda} = 0 \\ 4\int_\Omega h_2^2 \mathrm{e}^{2v_2} - 2\int_\Omega h_1^2 \mathrm{e}^{2v_1} - 2\int_\Omega h_2 \mathrm{e}^{v_2} + \int_\Omega h_1 \mathrm{e}^{v_1} - \int_\Omega h_1 h_2 \mathrm{e}^{v_1+v_2} + \frac{4\pi N_2}{\lambda} = 0. \end{cases} \tag{2.15}$$

Conditions (2.15) may be more clearly interpreted if we set $v_i = c_i + w_i$ with $\int_\Omega w_i = 0$ and $c_i = \int_\Omega v_i$ $(i = 1, 2)$. Indeed, after some simple algebraic manipulation, from (2.15) we get a quadratic system for the variables e^{c_i} $(i = 1, 2)$ as follows:

$$\begin{cases} 2\mathrm{e}^{2c_1}\int_\Omega h_1^2 \mathrm{e}^{2w_1} - \mathrm{e}^{c_1}(\int_\Omega h_1 \mathrm{e}^{w_1} + \mathrm{e}^{c_2}\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}) + \frac{4\pi}{3\lambda}(2N_1 + N_2) = 0 \\ 2\mathrm{e}^{2c_2}\int_\Omega h_2^2 \mathrm{e}^{2w_2} - \mathrm{e}^{c_2}(\int_\Omega h_2 \mathrm{e}^{w_2} + \mathrm{e}^{c_1}\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}) + \frac{4\pi}{3\lambda}(2N_2 + N_1) = 0. \end{cases} \tag{2.16}$$

Consequently, a solution $u_i = u_0^i + c_i + w_i$ $(i = 1, 2,)$ for (2.1) must satisfy:

$$\frac{\left(\int_\Omega h_i \mathrm{e}^{w_i} + \mathrm{e}^{c_j}\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}\right)^2}{\int_\Omega h_i^2 \mathrm{e}^{2w_i}} \geq \frac{32\pi\left(2N_i + N_j\right)}{3\lambda} \quad i \neq j = 1, 2. \tag{2.17}$$

Thus, taking into account (2.12), by (2.17) and Hölder inequality we obtain,

$$\frac{32\pi\left(2N_i + N_j\right)}{3\lambda} \leq \frac{4\left(\int_\Omega h_i \mathrm{e}^{w_i}\right)^2}{\int_\Omega h_i^2 \mathrm{e}^{2w_i}} \leq 4|\Omega|, \quad i \neq j = 1, 2. \tag{2.18}$$

Hence, the condition

$$\lambda \geq \frac{8\pi}{3|\Omega|} \max\{2N_1 + N_2; 2N_2 + N_1\} \tag{2.19}$$

is necessary for the solvability of (2.1), and part (a) of Theorem 2.1 follows.

Set

$$E = \{w \in \mathcal{H}^1(\Omega) \ : \ \int_\Omega w = 0\}.$$

We shall see that, for each fixed pair $(w_1, w_2) \in E \times E$ satisfying:

$$\left(\int_\Omega h_i \mathrm{e}^{w_i}\right)^2 \geq \frac{32\pi}{3\lambda}\left(2N_i + N_j\right)\int_\Omega h_i^2 \mathrm{e}^{2w_i}, \quad i \neq j, \quad i, j = 1, 2, \tag{2.20}$$

the system (2.16) admits four distinct solutions (c_1, c_2).

For this purpose, from now on, we take

$$\lambda > \frac{32\pi}{3} \max\left\{(2N_1 + N_2)\frac{\int_\Omega h_1^2}{\left(\int_\Omega h_1\right)^2}; (2N_2 + N_1)\frac{\int_\Omega h_2^2}{\left(\int_\Omega h_2\right)^2}\right\} \tag{2.21}$$

and define the set

$$\mathcal{A}_\lambda = \{(w_1, w_2) \in E \times E \ : \ w_i \text{ satisfies } \quad (2.20), \quad i = 1, 2\}. \tag{2.22}$$

Note that $(0, 0) \in \mathcal{A}_\lambda$.

For given $(w_1, w_2) \in \mathcal{A}_\lambda$, we introduce the smooth functions $g_i^\pm : [0, +\infty) \to \mathbb{R}$ $(i = 1, 2)$ defined as follows:

$$\begin{aligned} g_1^\pm(X) \equiv & \frac{\int_\Omega h_1 \mathrm{e}^{w_1} + X\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}}{4\int_\Omega h_1^2 \mathrm{e}^{2w_1}} \\ & \pm \frac{\sqrt{\left(\int_\Omega h_1 \mathrm{e}^{w_1} + X\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}\right)^2 - \frac{32\pi}{3\lambda}(2N_1 + N_2)\int_\Omega h_1^2 \mathrm{e}^{2w_1}}}{4\int_\Omega h_1^2 \mathrm{e}^{2w_1}} \\ g_2^\pm(X) \equiv & \frac{\int_\Omega h_2 \mathrm{e}^{w_2} + X\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}}{4\int_\Omega h_2^2 \mathrm{e}^{2w_2}} \\ & \pm \frac{\sqrt{\left(\int_\Omega h_2 \mathrm{e}^{w_2} + X\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}\right)^2 - \frac{32\pi}{3\lambda}(2N_2 + N_1)\int_\Omega h_2^2 \mathrm{e}^{2w_2}}}{4\int_\Omega h_2^2 \mathrm{e}^{2w_2}}, \end{aligned} \tag{2.23}$$

and set

$$\begin{cases} F^+(X) \equiv X - g_1^+(g_2^+(X)), & F^-(X) \equiv X - g_1^-(g_2^-(X)) \\ F^\pm(X) \equiv X - g_1^+(g_2^-(X)), & F^\mp(X) \equiv X - g_1^-(g_2^+(X)) \end{cases}.$$

It is easy to check that solutions of (2.16) correspond to the zeroes of the smooth functions $F^* : [0, +\infty) \to \mathbb{R}$, with $* = +, -, \pm, \mp$.

Notice that $g_i^{\pm}(X) > 0$ for any $X \geq 0$ $(i = 1, 2)$ and so $F^*(0) < 0$ $(* = +, -, \pm, \mp)$. Moreover,

$$\lim_{X \to +\infty} g_i^-(X) = 0 \quad (i = 1, 2), \tag{2.24}$$

while

$$\frac{g_i^+(X)}{X} = \frac{\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}}{2\int_\Omega h_i^2 \mathrm{e}^{2w_i}} + o(1), \quad \text{as } X \to +\infty \quad (i = 1, 2). \tag{2.25}$$

Therefore,

$$\frac{F^+(X)}{X} = \left(1 - \frac{\left(\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}\right)^2}{4\int_\Omega h_1^2 \mathrm{e}^{2w_1} \int_\Omega h_2^2 \mathrm{e}^{2w_2}}\right) + o(1), \quad \text{as } X \to +\infty.$$

For the remaining F^*, $(* = -, \pm, \mp)$ we have

$$\frac{F^*(X)}{X} = 1 + o(1), \quad \text{as } X \to +\infty.$$

Therefore, for any choice $* = +, -, \pm, \mp$, it follows:

$$\lim_{X \to +\infty} F^*(X) = +\infty,$$

and hence, by continuity, $F^*(X^*) = 0$, for some $X^* > 0$ and $* = +, -, \pm, \mp$.

Furthermore, for $i, j = 1, 2$, $i \neq j$,

$$\frac{dg_i^{\pm}}{dX}(X) = \pm g_i^{\pm}(X) \frac{\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}}{\sqrt{\left(\int_\Omega h_i \mathrm{e}^{w_i} + X\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2}\right)^2 - \frac{32\pi}{3\lambda}\left(2N_i + N_j\right)\int_\Omega h_i^2 \mathrm{e}^{2w_i}}}, \tag{2.26}$$

and so, $F^{\pm}$ and $F^{\mp}$ are strictly increasing. Moreover, for $(w_1, w_2) \in \mathcal{A}_\lambda$ $(i = 1, 2)$ and $X > 0$, we have

$$\begin{aligned}\frac{dF^+}{dX}(X) = &- \frac{\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2} g_1^+(g_2^+(X))}{\sqrt{(\int_\Omega h_1 \mathrm{e}^{w_1} + g_2^+(X)\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2})^2 - \frac{32\pi}{3\lambda}(2N_1 + N_2)\int_\Omega h_1^2 \mathrm{e}^{2w_1}}} \\ &\cdot \frac{\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2} g_2^+(X)}{\sqrt{(\int_\Omega h_2 \mathrm{e}^{w_2} + X\int_\Omega h_1 h_2 \mathrm{e}^{w_1+w_2})^2 - \frac{32\pi}{\lambda}(2N_2 + N_1)\int_\Omega h_2^2 \mathrm{e}^{2w_2}}} + 1 \\ &> 1 - \frac{g_1^+(g_2^+(X))}{X} = \frac{F^+(X)}{X},\end{aligned}$$

and, analogously,

$$\frac{dF^-}{dX}(X) > 1 - \frac{g_1^-(g_2^-(X))}{X} = \frac{F^-(X)}{X}.$$

So, $\frac{F^*}{X}$ is strictly increasing, for $* = +, -$, and $X > 0$.

In conclusion, for any $(w_1, w_2) \in \mathcal{A}_\lambda$ and $* = +, -, \pm, \mp$ there exists a unique $X^* > 0$ such that $F^*(X^*) = 0$. Set $e^{c_1^*(w_1,w_2)} = X^*$, and observe that, by the strict monotonicity of g_i^+ and g_i^-, $i = 1, 2$ (see (2.26)), there exists a unique $c_i^* = c_i^*(w_1, w_2)$ $(i = 1, 2; * = +, -, \pm, \mp)$ satisfying

$$\begin{aligned} e^{c_1^+} &= g_1^+(e^{c_2^+}), \quad & e^{c_2^+} &= g_2^+(e^{c_1^+}); \\ e^{c_1^\pm} &= g_1^+(e^{c_2^\pm}), \quad & e^{c_2^\pm} &= g_2^-(e^{c_1^\pm}); \\ e^{c_1^\mp} &= g_1^-(e^{c_2^\mp}), \quad & e^{c_2^\mp} &= g_2^+(e^{c_1^\mp}); \\ e^{c_1^-} &= g_1^-(e^{c_2^-}), \quad & e^{c_2^-} &= g_2^-(e^{c_1^-}). \end{aligned} \tag{2.27}$$

Consequently, for given $(w_1, w_2) \in \mathcal{A}_\lambda$, setting $v_i^* = w_i + c_i^*(w_1, w_2)$, $i = 1, 2$, $* = +, -, \pm, \mp$ we have that the pair (v_1^*, v_2^*) satisfy (2.15).

Our goal will be to derive solutions (v_1, v_2) for (2.10) which decompose as $v_i = w_i + c_i$ with $\int_\Omega w_i = 0$, $c_i = \int_\Omega v_i$ and $c_i = c_i^*(w_1, w_2)$ $(i = 1, 2)$ with prescribed $*$ to coincide with either $+, -, \pm$ or $\mp$.

This will yield to solutions of (2.10) with specific asymptotic behavior as $\lambda \to +\infty$.

Note that, by the complete symmetry of the problem, the case $* = \pm$ and $* = \mp$ are similar in nature. So, we shall limit our attention to the case $* = \pm$ with the understanding that, by changing the role between the indices, analogous considerations hold also when $* = \mp$.

We start with the following:

Lemma 2.3. *For every* $(w_1, w_2) \in \mathcal{A}_\lambda$ *we have*

(i) $e^{c_i^+} \int_\Omega h_i e^{w_i} \le 1$ *for* $i = 1, 2$*;*

(ii) $e^{c_i^-} \int_\Omega h_i e^{w_i} \le \frac{8\pi}{\lambda}(2N_i + N_j)$ *for* $i = 1, 2$, $i \neq j$*;*

(iii) $e^{c_1^\pm} \int_\Omega h_1 e^{w_1} \le \frac{1}{2} + \frac{16\pi}{3\lambda}(2N_2 + N_1)$ *and* $e^{c_2^\pm} \int_\Omega h_2 e^{w_2} \le \frac{8\pi}{\lambda}(2N_2 + N_1)$.

Proof. (i) In view of (2.23), from (2.27) for $i, j = 1, 2$ $i \neq j$ we have

$$e^{c_i^+} \le \frac{\int_\Omega h_i e^{w_i} + e^{c_j^+} \int_\Omega h_1 h_2 e^{w_1 + w_2}}{2 \int_\Omega h_i^2 e^{2w_i}}.$$

Iterating such an inequality, by means of Hölder inequality, we get,

$$\begin{aligned} e^{c_1^+} &\le \frac{\int_\Omega h_1 e^{w_1}}{2 \int_\Omega h_1^2 e^{2w_1}} + \frac{\int_\Omega h_1 h_2 e^{w_1+w_2}}{4 \int_\Omega h_1^2 e^{2w_1} \int_\Omega h_2^2 e^{2w_2}} \left(\int_\Omega h_2 e^{w_2} + e^{c_1^+} \int_\Omega h_1 h_2 e^{w_1+w_2} \right) \\ &\le \frac{\int_\Omega h_1 e^{w_1}}{2 \int_\Omega h_1^2 e^{2w_1}} + \frac{\int_\Omega h_1 h_2 e^{w_1+w_2} \int_\Omega h_2 e^{w_2}}{4 \int_\Omega h_1^2 e^{2w_1} \int_\Omega h_2^2 e^{2w_2}} + \frac{1}{4} e^{c_1^+} . \end{aligned}$$

By symmetry, an analogous estimate holds for $e^{c_2^+}$. Hence,

$$e^{c_i^+} \int_\Omega h_i e^{w_i} \le \frac{4}{3} \left(\frac{(\int_\Omega h_i e^{w_i})^2}{2 \int_\Omega h_i^2 e^{2w_i}} + \frac{\int_\Omega h_1 h_2 e^{w_1+w_2} \int_\Omega h_1 e^{w_1} \int_\Omega h_2 e^{w_2}}{4 \int_\Omega h_1^2 e^{2w_1} \int_\Omega h_2^2 e^{2w_2}} \right), \quad i = 1, 2, \tag{2.28}$$

and, using Hölder inequality, we derive the desired estimate.

To obtain (ii), we use again (2.27) together with (2.23). Thus, for $i = 1, 2, i \neq j$, we have

$$e^{c_i^-} \leq \frac{8\pi(2N_i + N_j)}{3\lambda} \frac{1}{\int_\Omega h_i e^{w_i} + e^{c_j^-} \int_\Omega h_1 h_2 e^{w_1+w_2}} \tag{2.29}$$

and (ii) easily follows.

As above, to obtain (iii), note that from (2.27) and (2.23), we have

$$e^{c_2^\pm} \leq \frac{8\pi(2N_2 + N_1)}{3\lambda} \frac{1}{\int_\Omega h_2 e^{w_2} + e^{c_1^\pm} \int_\Omega h_1 h_2 e^{w_1+w_2}}, \tag{2.30}$$

and so,

$$e^{c_2^\pm} \leq \frac{8\pi(2N_i + N_j)}{\lambda \int_\Omega h_2 e^{w_2}}. \tag{2.31}$$

On the other hand,

$$e^{c_1^\pm} \leq \frac{\int_\Omega h_1 e^{w_1} + e^{c_2^\pm} \int_\Omega h_1 h_2 e^{w_1+w_2}}{2\int_\Omega h_1^2 e^{2w_1}}, \tag{2.32}$$

while from (2.30) we also have,

$$e^{c_2^\pm} \leq \frac{8\pi(2N_2 + N_1)}{3\lambda} \frac{1}{e^{c_1^\pm} \int_\Omega h_1 h_2 e^{w_1+w_2}}. \tag{2.33}$$

Furthermore, from (2.27) and (2.23) it is easy to check that

$$e^{c_1^\pm} \geq \frac{\int_\Omega h_1 e^{w_1}}{4\int_\Omega h_1^2 e^{2w_1}}. \tag{2.34}$$

Combining (2.33) with (2.34), from (2.32) we get

$$\begin{aligned} e^{c_1^\pm} \int_\Omega h_1 e^{w_1} &\leq \frac{\left(\int_\Omega h_1 e^{w_1}\right)^2}{2\int_\Omega h_1^2 e^{2w_1}} + \frac{e^{c_2^\pm} \int_\Omega h_1 h_2 e^{w_1+w_2} \int_\Omega h_1 e^{w_1}}{2\int_\Omega h_1^2 e^{2w_1}} \\ &\leq \frac{\left(\int_\Omega h_1 e^{w_1}\right)^2}{2\int_\Omega h_1^2 e^{2w_1}} + \frac{4\pi\,(2N_2 + N_1) \int_\Omega h_1 e^{w_1}}{3\lambda e^{c_1^\pm} \int_\Omega h_1^2 e^{2w_1}} \\ &\leq \frac{\left(\int_\Omega h_1 e^{w_1}\right)^2}{2\int_\Omega h_1^2 e^{2w_1}} + \frac{16\pi\,(2N_2 + N_1)}{3\lambda}, \end{aligned} \tag{2.35}$$

and the desired estimate follows by Hölder inequality. □

Remark 2.4. Note that, by Jensen's inequality, $\int_\Omega h_i e^{w_i} \geq 1$ $(i = 1, 2)$, which combined with the estimates in Lemma 2.3, gives in particular that,

$$\begin{aligned} &e^{c_i^+} \leq 1, \quad e^{c_i^-} \leq O\left(\frac{1}{\lambda}\right), \quad i = 1, 2; \\ &e^{c_1^\pm} \leq \frac{1}{2} + O\left(\frac{1}{\lambda}\right) \quad \text{and} \quad e^{c_2^\pm} \leq O\left(\frac{1}{\lambda}\right). \end{aligned} \tag{2.36}$$

This suggests that solutions of (2.10) with the prescribed asymptotic behavior of type I, II and III (see the Introduction) should correspond to those with mean-values as given by c_i^+, c_i^- and $c_i^\pm$, respectively.

With this aim, we consider the functionals J_λ^+, J_λ^-, $J_\lambda^\pm$ and $J_\lambda^\mp$ defined on $\mathcal{A}_\lambda$ and obtained by inserting the constraint (2.27) into I_λ. More precisely, for $(w_1, w_2) \in \mathcal{A}_\lambda$, define

$$J_\lambda^*(w_1, w_2) = I_\lambda(w_1 + c_1^*(w_1, w_2), w_2 + c_2^*(w_1, w_2)), \quad \text{with } * = +, -, \pm, \mp.$$

Using (2.16) we find:

$$\begin{aligned} J_\lambda^*(w_1, w_2) = &\frac{1}{3}\left(\|\nabla w_1\|^2 + \|\nabla w_2\|^2 + \int_\Omega \nabla w_1 \cdot \nabla w_2\right) \\ &+ \frac{\lambda}{2}\int_\Omega \left(1 - e^{c_1^*} h_1 e^{w_1}\right) + \frac{\lambda}{2}\int_\Omega \left(1 - e^{c_2^*} h_2 e^{w_2}\right) - 4\pi\,(N_1 + N_2) \\ &+ \frac{4\pi}{3}\,(2N_1 + N_2)\, c_1^* + \frac{4\pi}{3}\,(2N_2 + N_1)\, c_2^*, \end{aligned} \tag{2.37}$$

with $* = +, -, \pm$ and $\mp$.

Remark 2.5. It is easy to check that J_λ^* is Frechét differentiable in the interior of $\mathcal{A}_\lambda$. Moreover if (w_1, w_2) (in the interior of $\mathcal{A}_\lambda$) is a critical point for J_λ^*, then $(w_1 + c_1^*, w_2 + c_2^*)$ defines a critical point for I_λ.

Concerning the existence of critical points for J_λ^*, we will show that such a functional is bounded below and attains its infimum on $\mathcal{A}_\lambda$. However, only when $* = +$, we can prove that the corresponding minimum point belongs to the interior of $\mathcal{A}_\lambda$, for $\lambda > 0$ large (see Proposition 3.4 below). In the other cases $* = -, \pm$ and $\mp$, we can prove that this occurs only when $N_1 + N_2 = 1$ (see Sect. 5). It is an interesting open question to know what happens for the remaining cases.

3. Multivortex Solutions of the I Type

In this section we analyze the functional J_λ^+, and the corresponding minimization problem in $\mathcal{A}_\lambda$.

We start with a preliminary lemma already derived in [27]:

Lemma 3.1. *If* $(w_1, w_2) \in \mathcal{A}_\lambda$, *then* $\forall \tau \in (0, 1]$ *we have*

$$\int_\Omega h_i e^{w_i} \le \left(\frac{3\lambda}{32\pi\,(2N_i + N_j)}\right)^{\frac{1-\tau}{\tau}} \left(\int_\Omega h_i^\tau e^{\tau w_i}\right)^{\frac{1}{\tau}}, \quad i, j = 1, 2 \quad i \ne j. \tag{3.1}$$

Proof. Let $\tau \in (0, 1)$ and let $a = \frac{1}{2-\tau}$ so that $\tau a + 2(1-a) = 1$. By the interpolation inequality we have

$$\begin{aligned} \int_\Omega h_i e^{w_i} &\le \left(\int_\Omega h_i^\tau e^{\tau w_i}\right)^a \left(\int_\Omega h_i^2 e^{2w_i}\right)^{1-a} \\ &\le \left(\int_\Omega h_i^\tau e^{\tau w_i}\right)^a \left(\frac{3\lambda}{32\pi\,(2N_i + N_j)}\right)^{1-a} \left(\int_\Omega h_i e^{w_i}\right)^{2(1-a)}, \quad i \ne j, \end{aligned}$$

Consequently, for $i \neq j$, $i, j = 1, 2$,

$$\left(\int_\Omega h_i e^{w_i}\right)^{2a-1} \leq \left(\frac{3\lambda}{32\pi\left(2N_i + N_j\right)}\right)^{1-a} \left(\int_\Omega h_i^\tau e^{\tau w_i}\right)^a,$$

that is,

$$\int_\Omega h_i e^{w_i} \leq \left(\frac{3\lambda}{32\pi\left(2N_i + N_j\right)}\right)^{\frac{1-\tau}{\tau}} \left(\int_\Omega h_i^\tau e^{\tau w_i}\right)^{\frac{1}{\tau}}. \qquad \square$$

In view of Lemma 3.1 we derive the coerciveness of J_λ^+ as follows:

Proposition 3.2. *For $\lambda > 0$ sufficiemtly large, there exist constants $\alpha, C > 0$ (independent of λ) such that*

$$J_\lambda^+(w_1, w_2) \geq \alpha\left(\|\nabla w_1\|^2 + \|\nabla w_2\|^2\right) - C\,(ln\,\lambda + 1) \tag{3.2}$$

for all $(w_1, w_2) \in \mathcal{A}_\lambda$.

Moreover, J_λ^+ attains its infimum on $\mathcal{A}_\lambda$.

Proof. From (2.23) and (2.27) it follows immediately that

$$e^{c_i^+} \geq \frac{\int_\Omega h_i e^{w_i}}{4\int_\Omega h_i^2 e^{2w_i}}, \quad i = 1, 2.$$

Hence, from (2.20) we find a suitable constant $C > 0$, independent of λ, such that for every $(w_1, w_2) \in \mathcal{A}_\lambda$, we have

$$c_i^+(w_1, w_2) \geq -\ln\lambda - \ln\int_\Omega h_i e^{w_i} - C, \quad i = 1, 2. \tag{3.3}$$

Whence, using Lemma 3.1, for fixed $\tau \in (0, 1)$ and any $(w_1, w_2) \in \mathcal{A}_\lambda$, we obtain:

$$\begin{aligned}
J_\lambda^+(w_1, w_2) \geq{}& \frac{1}{6}\sum_{i=1}^{2}\|\nabla w_i\|_2^2 - \sum_{i\neq j=1,2}\frac{4\pi}{3}\left(2N_i + N_j\right)\,\ln\int_\Omega h_i e^{w_i} \\
&- \sum_{i\neq j=1,2}\frac{4\pi}{3}\left(2N_i + N_j\right)\,\ln\lambda - C \geq \frac{1}{6}\sum_{i=1}^{2}\|\nabla w_i\|_2^2 \\
&- \sum_{i\neq j=1,2}\frac{4\pi}{3}\left(2N_i + N_j\right)\,\ln[\left(\frac{3\lambda}{16\pi\left(2N_i + N_j\right)}\right)^{\frac{1-\tau}{\tau}}\left(\int_\Omega e^{\tau\left(w_i + u_0^i\right)}\right)^{\frac{1}{\tau}}] \\
&- \sum_{i\neq j=1,2}\frac{4\pi}{3}\left(2N_i + N_j\right)\,\ln\lambda - C \\
\geq{}& \frac{1}{6}\sum_{i=1}^{2}\|\nabla w_i\|_2^2 - \sum_{i\neq j=1,2}\frac{4\pi}{3\tau}\left(2N_i + N_j\right)\,\left(\ln\int_\Omega e^{\tau w_i} + \max_\Omega u_0^i\right) \\
&- \sum_{i\neq j=1,2}\frac{4\pi}{3}\left(\frac{1}{\tau}\right)\left(2N_i + N_j\right)\ln\lambda - C,
\end{aligned}$$

for some constant $C > 0$ independent of λ.

Recall that, by Moser–Trudinger's inequality [3](see [16,11] and [27], for alternative proofs) we have that

$$\int_\Omega e^w \leq C \exp\left(\frac{1}{16\pi}\|\nabla w\|_2^2\right), \quad \forall w \in E, \tag{3.4}$$

with C a positive constant depending only on Ω. Thus, for any $(w_1, w_2) \in \mathcal{A}_\lambda$, we obtain

$$\begin{aligned} J_\lambda^+(w_1, w_2) \geq \frac{1}{6} \sum_{i \neq j = 1,2} \left(1 - \frac{\tau\left(2N_i + N_j\right)}{2}\right) \|\nabla w_i\|_2^2 \\ - \frac{4\pi}{3} \sum_{i \neq j = 1,2} \frac{\left(2N_i + N_j\right)}{\tau} \ln \lambda - C_\tau, \end{aligned} \tag{3.5}$$

with $C_\tau > 0$ a suitable constant independent of λ.

Hence, it suffices to take $0 < \tau < \frac{2}{\max_{i \neq j = 1,2}(2N_i + N_j)}$ in (3.5), to derive (3.2) and conclude that J_λ^+ is coercive on $\mathcal{A}_\lambda$. Since J_λ^+ is weakly lower semicontinuous on the weakly closed set $\mathcal{A}_\lambda$, we immediately conclude that the infimum of J_λ^+ is attained on $\mathcal{A}_\lambda$. □

Our next goal is to prove that, for λ sufficiently large, such a minimum point lies in the interior of $\mathcal{A}_\lambda$.

To this purpose we will estimate the functional J_λ^+ on the boundary $\partial\mathcal{A}_\lambda$ of $\mathcal{A}_\lambda$.

Lemma 3.3. *For $\lambda > 0$ sufficiently large,*

$$\inf_{(w_1, w_2) \in \partial\mathcal{A}_\lambda} J_\lambda^+(w_1, w_2) \geq \frac{\lambda}{2} - C(\sqrt{\lambda} + \ln \lambda + 1), \tag{3.6}$$

with $C > 0$ a suitable constant independent of λ.

Proof. For $(w_1, w_2) \in \partial\mathcal{A}_\lambda$, we have that the identity

$$\left(\int_\Omega h_i e^{w_i}\right)^2 = \frac{32\pi}{3\lambda}(2N_i + N_j) \int_\Omega h_i^2 e^{2w_i}, \quad i \neq j \tag{3.7}$$

necessarily holds for $i = 1$ or 2. Without loss of generality assume that (3.7) holds for $i = 2$.

Using (3.7) into (2.28), by means of Hölder's inequality, we find

$$\begin{aligned} e^{c_2^+} \int_\Omega h_2 e^{w_2} &\leq \frac{4}{3}\left(\frac{(\int_\Omega h_2 e^{w_2})^2}{2\int_\Omega h_2^2 e^{2w_2}} + \frac{\int_\Omega h_1 e^{w_1} \int_\Omega h_2 e^{w_2}}{4(\int_\Omega h_1^2 e^{2w_1})^{\frac{1}{2}} (\int_\Omega h_2^2 e^{2w_2})^{\frac{1}{2}}}\right) \\ &\leq C(\frac{1}{\lambda} + \frac{1}{\sqrt{\lambda}}), \end{aligned} \tag{3.8}$$

for some constant $C > 0$ independent of λ.

Hence, for $\lambda > 0$ sufficiently large, we have

$$e^{c_2^+} \int_\Omega h_2 e^{w_2} \le \frac{2C}{\sqrt{\lambda}}. \tag{3.9}$$

Using (2.37) with $* = +$, the arguments of Proposition 3.2 with $\tau = \frac{2}{\max_{i \neq j = 1,2}(2N_i + N_j)}$, Lemma 2.3-(i) and (3.9), we get

$$\begin{aligned} J_\lambda^+(w_1, w_2) &\ge \frac{\lambda}{2} \int_\Omega \left(1 - e^{c_1^+} h_1 e^{w_1}\right) + \frac{\lambda}{2} \int_\Omega \left(1 - e^{c_2^+} h_2 e^{w_2}\right) \\ &\quad - \frac{2\pi}{3} \max_{i,j=1,2; i \neq j} \left(2N_i + N_j\right)^2 \ln \lambda - C \\ &\ge \frac{\lambda}{2} - C\left(\sqrt{\lambda} + \ln \lambda + 1\right), \end{aligned}$$

for any $(w_1, w_2) \in \partial \mathcal{A}_\lambda$, with $C > 0$ a suitable constant independent of λ. □

In order to find suitable test-functions in the interior of $\mathcal{A}_\lambda$, where the reverse estimate in (3.6) holds, we recall here some results obtained in [34] concerning the Abelian Chern–Simons–Higgs equation.

In [34] (*Proposition 3.1*) it is proved that for $\mu > 0$ sufficiently large there exist $\bar{v}_\mu^i = \bar{c}_\mu^i + \bar{w}_\mu^i$, with $\bar{c}_\mu^i = \int_\Omega \bar{v}_\mu^i$ and the $\int_\Omega \bar{w}_\mu^i = 0$ $(i = 1, 2)$ solution of

$$\begin{cases} \Delta v = \mu e^{v + u_0^i}(e^{v + u_0^i} - 1) + 4\pi N_i \\ v \in H^1(\Omega), \end{cases} \tag{3.10}$$

such that $u_0^i + \bar{v}_\mu^i < 1$ in Ω, $\bar{c}_\mu^i \to 0$ and $\bar{w}_\mu^i \to -u_0^i$ pointwise a.e., as $\mu \to +\infty$ $(i = 1, 2)$. Since $h_i = e^{u_0^i} \in L^\infty(\Omega)$, by dominated convergence, we have that $h_i e^{\bar{w}_\mu^i} \to 1$ strongly in $L^p(\Omega)$ for any $p \ge 1$. In particular,

$$\int_\Omega h_1 h_2 e^{\bar{w}_\mu^1 + \bar{w}_\mu^2} \to 1, \quad \text{as } \mu \to +\infty. \tag{3.11}$$

Hence, for fixed $\lambda_0 > 0$ large and $\epsilon \in (0, 1)$, we can find $\mu_\epsilon > 0$ sufficiently large to insure that, setting $\bar{w}_{i,\epsilon} = \bar{w}_{\mu_\epsilon}^i$ $(i = 1, 2)$, we have $(\bar{w}_{1,\epsilon}, \bar{w}_{2,\epsilon}) \in \mathcal{A}_\lambda$ for every $\lambda \ge \lambda_0$ and

$$\max_{j=1,2} \frac{2\int_\Omega h_j^2 e^{2\bar{w}_{j,\epsilon}} + 1}{4\int_\Omega h_1^2 e^{2\bar{w}_{1,\epsilon}} \int_\Omega h_2^2 e^{2\bar{w}_{2,\epsilon}} - 1} > 1 - \epsilon. \tag{3.12}$$

Recalling that, by Jensen's inequality, $\int_\Omega h_1 h_2 e^{\bar{w}_{1,\epsilon}+\bar{w}_{2,\epsilon}} \geq 1$ and $\int_\Omega h_j^2 e^{2\bar{w}_{j,\epsilon}} \geq (\int_\Omega h_j e^{2\bar{w}_{j,\epsilon}})^2 \geq 1$, $j = 1, 2$, by means of (2.27) and (2.23) we obtain,

$$\begin{aligned}
e^{c_i^+(\bar{w}_{1,\epsilon},\bar{w}_{2,\epsilon})} &\geq \frac{\left(\int_\Omega h_i e^{\bar{w}_{1,\epsilon}} + e^{c_j^+(\bar{w}_{1,\epsilon},\bar{w}_{2,\epsilon})} \int_\Omega h_1 h_2 e^{\bar{w}_{1,\epsilon}+\bar{w}_{1,\epsilon}}\right)}{4\int_\Omega h_i^2 e^{2\bar{w}_{i,\epsilon}}} \\
&\quad \times \left(1 + \sqrt{1 - \frac{32\pi}{3\lambda}(2N_i + N_j)\frac{\int_\Omega h_i^2 e^{2\bar{w}_{i,\epsilon}}}{(\int_\Omega h_i e^{\bar{w}_{i,\epsilon}})^2}}\right) \\
&\geq \frac{1 + e^{c_j^+(\bar{w}_{1,\epsilon},\bar{w}_{2,\epsilon})}}{2\int_\Omega h_i^2 e^{2\bar{w}_{i,\epsilon}}} + \frac{\int_\Omega h_i e^{\bar{w}_{i,\epsilon}}}{4\int_\Omega h_i^2 e^{2\bar{w}_{i,\epsilon}}} \\
&\quad \times \left(\sqrt{1 - \frac{32\pi}{3\lambda}(2N_i + N_j)\frac{\int_\Omega h_i^2 e^{2\bar{w}_{i,\epsilon}}}{(\int_\Omega h_i e^{\bar{w}_{i,\epsilon}})^2}} - 1\right) \\
&\geq \frac{1 + e^{c_j^+(\bar{w}_{1,\epsilon},\bar{w}_{2,\epsilon})}}{2\int_\Omega h_i^2 e^{2\bar{w}_{i,\epsilon}}} - \frac{8\pi(2N_i + N_j)}{3\lambda}, \quad \text{for } i, j = 1, 2 \quad \text{and } i \neq j.
\end{aligned} \tag{3.13}$$

Thus, setting $c_{i,\epsilon}^+ = c_i^+(\bar{w}_{1,\epsilon}, \bar{w}_{2,\epsilon})$ $i = 1, 2$, an iteration of the estimate above yields to

$$\begin{aligned}
e^{c_{i,\epsilon}^+} &\geq \frac{1}{2\int_\Omega h_i^2 e^{2\bar{w}_{i,\epsilon}}} + \frac{1}{4\int_\Omega h_1^2 e^{2\bar{w}_{1,\epsilon}} \int_\Omega h_2^2 e^{2\bar{w}_{2,\epsilon}}}(1 + e^{c_{i,\epsilon}^+}) \\
&\quad - \frac{1}{2\int_\Omega h_i^2 e^{2\bar{w}_{i,\epsilon}}}\frac{8\pi(2N_j + N_i)}{3\lambda} - \frac{8\pi(2N_i + N_j)}{3\lambda} \\
&\geq \frac{2\int_\Omega h_j^2 e^{2\bar{w}_{j,\epsilon}} + 1}{4\int_\Omega h_1^2 e^{2\bar{w}_{1,\epsilon}} \int_\Omega h_2^2 e^{2\bar{w}_{2,\epsilon}}} + \frac{e^{c_{i,\epsilon}^+}}{4\int_\Omega h_1^2 e^{2\bar{w}_{1,\epsilon}} \int_\Omega h_2^2 e^{2\bar{w}_{2,\epsilon}}} - \frac{4\pi(4N_j + 5N_i)}{3\lambda},
\end{aligned}$$

for $i \neq j = 1, 2$.

Consequently, for $i \neq j = 1, 2$,

$$e^{c_{i,\epsilon}^+} \geq \frac{2\int_\Omega h_j^2 e^{2\bar{w}_{j,\epsilon}} + 1}{4\int_\Omega h_1^2 e^{2\bar{w}_{1,\epsilon}} \int_\Omega h_2^2 e^{2\bar{w}_{2,\epsilon}} - 1} - \frac{4\pi(4N_j + 5N_i)}{9\lambda}.$$

In view of (3.12), we conclude that

$$e^{c_{i,\epsilon}^+} \geq 1 - \epsilon - \frac{4\pi}{\lambda}\max_{j=1,2} N_j, \quad i = 1, 2,$$

that gives

$$(1 - e^{c_{i,\epsilon}^+} \int_\Omega h_i e^{\bar{w}_{i,\epsilon}}) \leq \epsilon + \frac{4\pi}{\lambda}\max_{j=1,2} N_j, \quad i = 1, 2 \tag{3.14}$$

for all $\lambda \geq \lambda_0$.

Now, we are ready to prove:

Proposition 3.4. *For* $\lambda > 0$ *sufficiently large,*

$$\inf_{(w_1,w_2)\in\partial\mathcal{A}_\lambda} J_\lambda^+(w_1,w_2) > \inf_{(w_1,w_2)\in\mathcal{A}_\lambda} J_\lambda^+(w_1,w_2). \tag{3.15}$$

Proof. Fix $\epsilon \in (0,\frac{1}{2})$ and consider $(\bar{w}_{1,\epsilon},\bar{w}_{2,\epsilon})$ satisfying (3.14) for $\lambda \geq \lambda_0$.

Since $c_i^+ \leq 0$, $i = 1,2$ (see Remark 2.4), by (2.37) (with $* = +$) and (3.14), we have:

$$J_\lambda^+(\bar{w}_{1,\epsilon},\bar{w}_{2,\epsilon}) \leq \|\nabla\bar{w}_{1,\epsilon}\|^2 + \|\nabla\bar{w}_{2,\epsilon}\|^2 + \lambda\epsilon \leq C_\epsilon + \lambda\epsilon, \tag{3.16}$$

with $C_\epsilon > 0$ a suitable constant depending on ϵ only.

Comparing with Lemma 3.3, it follows

$$\begin{aligned}\inf_{(w_1,w_2)\in\partial\mathcal{A}_\lambda} J_\lambda^+(w_1,w_2) - \inf_{(w_1,w_2)\in\mathcal{A}_\lambda} J_\lambda^+(w_1,w_2) \geq \inf_{(w_1,w_2)\in\partial\mathcal{A}_\lambda} J_\lambda^+(w_1,w_2)\\ - J_\lambda^+(\bar{w}_{1,\epsilon},\bar{w}_{2,\epsilon}) \geq \lambda(\frac{1}{2}-\epsilon) - C(\sqrt{\lambda}+\ln\lambda) - C_\epsilon \to +\infty,\end{aligned}$$

as $\lambda \to +\infty$, and the proposition is proved. □

From Proposition 3.2 and 3.4 we may conclude that, for $\lambda > 0$ sufficiently large and $(N_1,N_2) \in \mathbb{N}\times\mathbb{N}$, there exists $(w_{1,\lambda},w_{2,\lambda})$ in the interior of $\mathcal{A}_\lambda$, where J_λ^+ attains its infimum. Consequently, $(J_\lambda^+)'(w_{1,\lambda},w_{2,\lambda}) = 0$, and

$$\begin{cases} v_{1,\lambda}^+ = w_{1,\lambda} + c_1^+(w_{1,\lambda},w_{2,\lambda})\\ v_{2,\lambda}^+ = w_{2,\lambda} + c_2^+(w_{1,\lambda},w_{2,\lambda})\end{cases} \tag{3.17}$$

defines a critical point for I_λ, namely a (weak) solution of (2.10).

Next we prove that, as in the Abelian Chern–Simons-Higgs theory (see [34]), the solution characterized by the choice of the "plus" sign in (2.23), namely $(v_{1,\lambda}^+, v_{2,\lambda}^+)$, give rise to a periodic vortex of type I.

More precisely, we prove

Proposition 3.5. *For* λ *sufficiently large, let* $(v_{1,\lambda}^+, v_{2,\lambda}^+)$ *be the solution of* (2.10) *as given by* (3.17)*, then,*

- $e^{u_0^i + v_{i,\lambda}^+} \to 1$ *as* $\lambda \to +\infty$*, pointwise a.e. in* Ω *and in* $L^p(\Omega)$*,* $\forall\, p \geq 1$ $(i = 1,2)$.

In order to prove Proposition 3.5 we start with the following lemma:

Lemma 3.6. *Let* $(v_{1,\lambda}^+, v_{2,\lambda}^+)$ *be given by* (3.17)*, then,*

$$\int_\Omega W(v_{1,\lambda}^+, v_{2,\lambda}^+) \to 0, \quad \text{as } \lambda \to +\infty. \tag{3.18}$$

Proof. In view of (3.16) we may conclude that $\forall\epsilon > 0\ \exists\lambda_\epsilon > 0$ and $C_\epsilon > 0$ such that $\forall\lambda \geq \lambda_\epsilon$ we have

$$\inf_{\mathcal{A}_\lambda} J_\lambda^+ \leq \epsilon\lambda + C_\epsilon. \tag{3.19}$$

On the other hand, following the argument of Proposition 3.2 (with $\tau = 2\ (\max_{i\neq j=1,2} (2N_j + N_i))^{-1}$) we obtain

$$\inf_{\mathcal{A}_\lambda} J_\lambda^+ = J_\lambda^+(w_{1,\lambda}^+, w_{2,\lambda}^+) \geq \lambda \int_\Omega W(v_{1,\lambda}^+, v_{2,\lambda}^+) - \frac{4\pi}{3} \max_{i\neq j}(2N_i + N_j)^2 \ln\lambda - C, \tag{3.20}$$

for $C > 0$ a suitable constant independent of λ.

So putting together (3.19) and (3.20), we obtain that

$$\limsup_{\lambda\to+\infty} \int_\Omega W(v_{1,\lambda}^+, v_{2,\lambda}^+) \leq \epsilon, \quad \forall\epsilon > 0$$

and the conclusion follows. □

Proof of Proposition 3.5. Recalling (2.14), by Lemma 3.6 we have $h_i e^{v_{i,\lambda}^+} \to 1$ in $L^2(\Omega)$ as $\lambda \to +\infty$ and $i = 1, 2$. Since $(v_{1,\lambda}^+ + u_0^1, v_{2,\lambda}^+ + u_0^2)$ is a solution of (2.1), by (2.3), we have that

$$e^{v_{i,\lambda}^+ + u_0^i} \leq 1, \quad \text{in } \Omega, \quad i = 1, 2.$$

Hence, $e^{v_i^+ + u_0^i} \to 1$ pointwise a.e., and, by dominated convergence, strongly in $L^p(\Omega)$, $\forall\, p \geq 1$, as $\lambda \to +\infty$. □

We conclude this section by observing that the type I multivortex solution $(v_{1,\lambda}^+, v_{2,\lambda}^+)$ can be characterized variationally as follows:

Lemma 3.7. *Let* $(v_{1,\lambda}^+, v_{2,\lambda}^+)$ *be given by* (3.17). *Then* $(v_{1,\lambda}^+, v_{2,\lambda}^+)$ *is a local minimum for the functional* I_λ.

Proof. For any $(w_1, w_2) \in \mathcal{A}_\lambda$ observe that

$$\partial_{c_i} I_\lambda(w_1 + c_1^+(w_1, w_2), w_2 + c_2^+(w_1, w_2)) = 0, \quad i = 1, 2. \tag{3.21}$$

Moreover, for $i = 1, 2$, $i \neq j$ we have

$$\partial_{c_i^2}^2 I_\lambda\,(w_1 + c_1, w_2 + c_2) = \lambda \int_\Omega \left(4h_i^2 e^{2(w_i+c_i)} - h_i e^{w_i+c_i} - h_1h_2 e^{c_1+c_2} e^{w_1+w_2}\right),$$
$$\partial_{c_1c_2}^2 I_\lambda\,(w_1 + c_1, w_2 + c_2) = -\lambda \int_\Omega h_1h_2 e^{c_1+c_2} e^{w_1+w_2}.$$

In view of (2.23) we have

$$\partial_{c_i^2}^2 I_\lambda(v_{1,\lambda}^+, v_{2,\lambda}^+) = \lambda((\int_\Omega h_i e^{v_{i,\lambda}^+} + h_1h_2 e^{v_{1,\lambda}^+ + v_{2,\lambda}^+})^2 - \frac{32\pi}{3\lambda}(2N_i + N_j) \int_\Omega h_i^2 e^{2v_{i,\lambda}^+})^{\frac{1}{2}}, \tag{3.22}$$

and since $(w_{1,\lambda}, w_{2,\lambda})$ lies in the interior of $\mathcal{A}_\lambda$,

$$\partial_{c_i^2}^2 I_\lambda(v_{1,\lambda}^+, v_{2,\lambda}^+) > \lambda \int_\Omega h_1h_2 e^{v_{1,\lambda}^+ + v_{2,\lambda}^+}, \quad i = 1, 2.$$

Therefore, at the point $(v^+_{1,\lambda}, v^+_{2,\lambda})$, the Hessian matrix of $I_\lambda(w_1 + c_1, w_2 + c_2)$ w.r.t. the variables (c_1, c_2) is strictly positive defined. Let $v_i = w_i + c_i$ $(i = 1, 2)$; by continuity, there exists $\delta > 0$ such that for any $\sum_{i=1,2}\|v_i - v^+_{i,\lambda}\| \le \delta$, we have $(w_1, w_2) \in \mathcal{A}_\lambda$ and

$$I_\lambda(v_1, v_2) \ge I_\lambda(w_1 + c^+_1(w_1, w_2), w_2 + c^+_2(w_1, w_2)) = J^+_\lambda(w_1, w_2). \tag{3.23}$$

Therefore,

$$I_\lambda\,(v_1, v_2) \ge J^+_\lambda\,(w_1, w_2) \ge \inf_{(w_1,w_2)\in\mathcal{A}_\lambda} J^+_\lambda\,(w_1, w_2) = I_\lambda \left(v^+_{1,\lambda}, v^+_{2,\lambda}\right), \tag{3.24}$$

and so $(v^+_{1,\lambda}, v^+_{2,\lambda})$ defines a local minimum for I_λ. □

4. The Mountain Pass Solution

In this section we prove the existence of a second solution of "mountain pass" type and obtain the proof of Theorem 2.1.

We start to show that the functional I_λ satisfies the compactness condition of Palais–Smale.

Lemma 4.1. *Let* $\{(v_{1,n}, v_{2,n})\}$ *be a sequence in* $\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$ *satisfying:*

(1) $I_\lambda(v_{1,n}, v_{2,n}) \to \alpha$ *as* $n \to +\infty$,
(2) $\|I'_\lambda(v_{1,n}, v_{2,n})\| \to 0$ *as* $n \to +\infty$, *then* $\{(v_{1,n}, v_{2,n})\}$ *admits a convergent subsequence in* $\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$.

Proof. Set $v_{i,n} = w_{i,n} + c_{i,n}$ $(i = 1, 2)$, where $\int_\Omega w_{i,n} = 0$ and $c_{i,n} = \int_\Omega v_{i,n}$.

For any $(\psi_1, \psi_2) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$,

$$\begin{aligned} I'_\lambda\left(v_{1,n}, v_{2,n}\right)[\psi_1, \psi_2] &= \frac{2}{3}\left(\int_\Omega \nabla w_{1,n}\nabla\psi_1 + \int_\Omega \nabla w_{2,n}\nabla\psi_2\right) \\ &+ \frac{1}{3}\left(\int_\Omega \nabla w_{1,n}\nabla\psi_2 + \int_\Omega \nabla w_{2,n}\nabla\psi_1\right) \\ &+ \lambda\int_\Omega\left(2h_1^2 e^{2v_{1,n}} - h_1 e^{v_{1,n}} - h_1h_2 e^{v_{1,n}+v_{2,n}} + \frac{4\pi}{3}\,(2N_1 + N_2)\right)\psi_1 \\ &+ \lambda\int_\Omega\left(2h_2^2 e^{2v_{2,n}} - h_2 e^{v_{2,n}} - h_1h_2 e^{v_{1,n}+v_{2,n}} + \frac{4\pi}{3}\,(2N_2 + N_1)\right)\psi_2. \end{aligned} \tag{4.1}$$

Choosing $\psi_i = 1$ and $\psi_j = 0$ $(i, j = 1, 2; i \ne j)$ in (4.1) we get, as $n \to +\infty$,

$$\left|\lambda\int_\Omega(2h_i^2 e^{2v_{i,n}} - h_i e^{v_{i,n}} - h_1h_2 e^{v_{1,n}+v_{2,n}}) + \frac{4\pi}{3}(2N_i + N_j)\right| \le o(1). \tag{4.2}$$

By (2.14), it implies:

$$2\lambda\int_\Omega W(v_{1,n}, v_{2,n}) + \lambda\int_\Omega h_1 e^{v_{1,n}} + \lambda\int_\Omega h_2 e^{v_{2,n}} \le 2\lambda + o(1).$$

Hence, as $n \to +\infty$, we have

$$\begin{aligned}&\int_\Omega W(v_{1,n}, v_{2,n}) \le 1 + o(1);\\&\int_\Omega h_i \mathrm{e}^{v_{i,n}} \le 2 + o(1) \quad i = 1, 2.\end{aligned} \tag{4.3}$$

From (4.3), and Jensen's inequality, we also get

$$\mathrm{e}^{c_{i,n}} \le 2 + o(1), \quad \text{as } n \to +\infty. \tag{4.4}$$

Furthermore, using (2.14) together with (4.3) we derive, as $n \to +\infty$, that

$$\int_\Omega (h_i \mathrm{e}^{v_{i,n}} - 1)^2 \le 2 + o(1), \quad i = 1, 2, \tag{4.5}$$

$$\int_\Omega h_i^2 \mathrm{e}^{2v_{i,n}} \le C, \quad i = 1, 2, \tag{4.6}$$

with $C > 0$ a suitable constant.

Set $(w_{1,n} + w_{2,n})^+ = \max\{w_{1,n} + w_{2,n}, 0\}$ and take (4.1) with $\psi_1 = \psi_2 = (w_{1,n} + w_{2,n})^+$, then

$$\begin{aligned}&I'_\lambda\left(v_{1,n}, v_{2,n}\right)[\psi_1, \psi_2] = \|\nabla\left(w_{1,n} + w_{2,n}\right)^+\|^2\\&+ \lambda\int_\Omega\left(2h_1^2\mathrm{e}^{2v_{1,n}} + 2h_2^2\mathrm{e}^{2v_{2,n}} - 4h_1h_2\mathrm{e}^{v_{1,n}+v_{2,n}}\right)\left(w_{1,n} + w_{2,n}\right)^+\\&+ 2\lambda\int_\Omega h_1h_2\mathrm{e}^{v_{1,n}+v_{2,n}}\left(w_{1,n} + w_{2,n}\right)^+ - \lambda\sum_{i=1,2}\int_\Omega h_i\mathrm{e}^{v_{i,n}}\left(w_{1,n} + w_{2,n}\right)^+\\&+ 4\pi\left(N_1 + N_2\right)\int_\Omega\left(w_{1,n} + w_{2,n}\right)^+ \ge 2\lambda\int_\Omega h_1h_2\mathrm{e}^{v_{1,n}+v_{2,n}}\left(w_{1,n} + w_{2,n}\right)^+\\&- \lambda\left(\int_\Omega\left(\left(w_{1,n} + w_{2,n}\right)^+\right)^2\right)^{\frac{1}{2}}\sum_{i=1,2}\left(\int_\Omega h_i^2\mathrm{e}^{2v_{i,n}}\right)^{\frac{1}{2}}.\end{aligned} \tag{4.7}$$

Therefore, by (4.6) and assumption (2) we get,

$$\begin{aligned}\int_\Omega h_1h_2\mathrm{e}^{v_{1,n}+v_{2,n}}(w_{1,n} + w_{2,n})^+ &\le C\|(w_{1,n} + w_{2,n})^+\|_2 + \epsilon_n\|(w_{1,n} + w_{2,n})^+\|\\&\le C(\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2),\end{aligned} \tag{4.8}$$

with $\epsilon_n \to 0$ as $n \to +\infty$ and $C > 0$ a suitable constant independent of $n \in \mathbb{N}$. Note that in (4.8) we have used the well known estimate:

$$\begin{aligned}\|(w_{1,n} + w_{2,n})^+\|_2 \le \|(w_{1,n} + w_{2,n})^+\| &\le \|w_{1,n} + w_{2,n}\|_2\\&\le C_0(\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2),\end{aligned}$$

with a suitable constant $C_0 > 0$ (independent of $n \in \mathbb{N}$).

Now, take $\psi_1 = w_{1,n}$, $\psi_2 = w_{2,n}$ in (4.1) and use (4.6), (4.8), and the Poincaré inequality to derive

$$\begin{aligned} I'_\lambda(v_{1,n}, v_{2,n})[w_{1,n}, w_{2,n}] &\geq \frac{1}{3}(\|\nabla w_{1,n}\|^2 + \|\nabla w_{2,n}\|^2) \\ &\quad + \lambda \int_\Omega \sum_{i=1}^{2} (2h_i^2 \mathrm{e}^{2v_{i,n}} - h_i \mathrm{e}^{v_{i,n}}) w_{i,n} \\ &\quad - \lambda \int_\Omega h_1 h_2 \mathrm{e}^{v_{1,n}+v_{2,n}} (w_{1,n} + w_{2,n})^+ \\ \geq \frac{1}{3}(\|\nabla w_{1,n}\|^2 + \|\nabla w_{2,n}\|^2) &+ 2\lambda \sum_{i=1}^{2} \int_\Omega h_i^2 \mathrm{e}^{2c_{i,n}} (\mathrm{e}^{2w_{i,n}} - 1) w_{i,n} \\ +2\lambda \sum_{i=1}^{2} \int_\Omega h_i^2 \mathrm{e}^{2c_{i,n}} w_{i,n} &- \lambda \sum_{i=1}^{2} \left(\int_\Omega h_i^2 \mathrm{e}^{2v_{i,n}} \right)^{\frac{1}{2}} \|w_{i,n}\|_2 \\ -C(\|\nabla w_{1,n}\|_2 + \|\nabla w_{2,n}\|_2) &\geq \frac{1}{3} \sum_{i=1,2} \|\nabla w_{i,n}\|_2^2 - C \sum_{i=1,2} \|\nabla w_{i,n}\|_2, \end{aligned}$$

where we used that $(\mathrm{e}^{2w_{i,n}} - 1) w_{i,n} \geq 0$, $i = 1, 2$ a.e. in Ω.

Hence, by assumption (2), we conclude that $\|\nabla w_{1,n}\|$ and $\|\nabla w_{2,n}\|$ are bounded sequences. Moreover, from assumption (1), $I_\lambda(v_{1,n}, v_{2,n})$ is bounded below uniformly on $n \in \mathbb{N}$ and we get a constant $C > 0$, independent of $n \in \mathbb{N}$, such that

$$\begin{aligned} 4\pi(N_1 + N_2) \min\{c_{1,n}, c_{2,n}\} &\geq -\frac{1}{6}(\|\nabla w_{1,n}\|^2 + \|\nabla w_{2,n}\|^2) \\ &\quad - \lambda \int_\Omega W(v_{1,n}, v_{2,n}) + I_\lambda(v_{1,n}, v_{2,n}) \geq -C, \end{aligned} \tag{4.9}$$

that is, the sequence $c_{i,n}$ $(i = 1, 2)$ is also bounded from below.

Therefore, after passing to a subsequence, we get

$$v_{i,n} \rightharpoonup \bar{v}_i, \quad (i = 1, 2) \quad \text{as } n \to +\infty, \tag{4.10}$$

weakly in $\mathcal{H}^1(\Omega)$, strongly in $L^p(\Omega)$, $p \geq 1$ and pointwise a.e. in Ω. Moreover, $\mathrm{e}^{v_{i,n}} \to \mathrm{e}^{\bar{v}_i}$ strongly in $L^p(\Omega)$ $p \geq 1$, and $c_{i,n} \to \int_\Omega \bar{v}_i = \bar{c}_i$.

Consequently, for any $(\psi_1, \psi_2) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$ we derive

$$I'_\lambda \left(v_{1,n}, v_{2,n}\right)[\psi_1, \psi_2] \to I'_\lambda \left(\bar{v}_1, \bar{v}_2\right)[\psi_1, \psi_2] = 0, \tag{4.11}$$

namely $(\bar{v}_1, \bar{v}_2)$ defines a critical point for I_λ.

In order to obtain strong convergence in $\mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$ we choose $\psi_1 = v_{1,n} - \bar{v}_1$ and $\psi_2 = v_{2,n} - \bar{v}_2$ into (4.1). By assumption (2) and (4.11), we obtain

$$\begin{aligned} &|(I'_\lambda(v_{1,n}, v_{2,n}) - I'_\lambda(\bar{v}_1, \bar{v}_2))[v_{1,n} - \bar{v}_1, v_{2,n} - \bar{v}_2]| \\ &\qquad \leq \epsilon_n(\|v_{1,n} - \bar{v}_1\| + \|v_{2,n} - \bar{v}_2\|) = o(1), \end{aligned} \tag{4.12}$$

as $n \to +\infty$.

Consequently,

$$\frac{1}{3}\sum_{i=1}^{2}\|\nabla\left(w_{i,n}-\bar{w}_i\right)\|^2 \le -2\lambda\sum_{i=1}^{2}\int_\Omega h_i^2\left(\mathrm{e}^{2v_{i,n}}-\mathrm{e}^{2\bar{v}_i}\right)\left(v_{i,n}-\bar{v}_i\right)$$
$$+\lambda\sum_{i=1}^{2}\int_\Omega h_i\left(\mathrm{e}^{v_{i,n}}-\mathrm{e}^{\bar{v}_i}\right)\left(v_{i,n}-\bar{v}_i\right)+$$
$$+\lambda\int_\Omega h_1h_2\left(\mathrm{e}^{v_{1,n}+v_{2,n}}-\mathrm{e}^{\bar{v}_1+\bar{v}_2}\right)\left(v_{1,n}+v_{2,n}-(\bar{v}_1+\bar{v}_2)\right)+o\,(1)=o\,(1)\,,$$

as $n\to+\infty$, and the desired conclusion follows. □

Proof of Theorem 2.1(c). In Sect. 3 we proved, for $\lambda>0$ sufficiently large, the existence of a solution of problem (2.10) with the desired asymptotic behavior (2.4) as $\lambda\to+\infty$ (see Proposition 3.5). Moreover, in Lemma 3.7, we have shown that such a solution $(v_{1,\lambda}^+, v_{2,\lambda}^+)$ defines a local minimum for I_λ, namely

$$\exists\,\delta_0>0\ :\ I_\lambda(v_1,v_2)\ge I_\lambda(v_{1,\lambda}^+,v_{2,\lambda}^+),\quad \text{provided}\quad \sum_{i=1,2}\|v_i-v_{i,\lambda}^+\|\le\delta_0. \tag{4.13}$$

In order to find a second solution for (2.10), we observe that I_λ admits a “mountain pass” structure. In fact, there exists a constant $C_\lambda>0$ (depending only on λ) such that for $c\in\mathbb{R}$ we have:

$$I_\lambda(v_{1,\lambda}^+,w_{2,\lambda}+c)-I_\lambda(v_{1,\lambda}^+,v_{2,\lambda}^+)\le C_\lambda+\frac{4\pi}{3}(2N_2+N_1)c. \tag{4.14}$$

We distinguish two cases.

(i). If $(v_{1,\lambda}^+,v_{2,\lambda}^+)$ is not a strict local minimum for I_λ, namely

$$\forall\,0<\delta<\delta_0\qquad \inf_{\sum_{i=1,2}\|v_i-v_{i,\lambda}^+\|=\delta} I_\lambda=I_\lambda(v_{1,\lambda}^+,v_{2,\lambda}^+), \tag{4.15}$$

then, by an application of Ekeland’s lemma (see [17], Corollary 1.6), we obtain a local minimum $(v_{1,\lambda}^\delta,v_{2,\lambda}^\delta)$ for I_λ, such that $\sum_{i=1,2}\|v_{i,\lambda}^\delta-v_{i,\lambda}^+\|=\delta$ for every $\delta\in(0,\delta_0)$. Therefore, in this case, we find a one-parameter family of (weak) solutions for (2.10). Otherwise,

(ii). if $(v_{1,\lambda}^+,v_{2,\lambda}^+)$ is a strict local minimum for I_λ, then

$$\exists\,\delta_1\in(0,\delta_0)\ :\ \inf_{\sum_{i=1,2}\|v_i-v_{i,\lambda}^+\|=\delta_1} I_\lambda(v_1,v_2)>I_\lambda(v_{1,\lambda}^+,v_{2,\lambda}^+). \tag{4.16}$$

Moreover, in view of (4.14), there exists $\bar{c}<0$ such that $|\bar{c}-c_2^+(w_{1,\lambda},w_{2,\lambda})|>\delta_1$ and

$$I_\lambda(v_{1,\lambda}^+,w_{2,\lambda}+\bar{c})<I_\lambda(v_{1,\lambda}^+,v_{2,\lambda}^+). \tag{4.17}$$

We introduce the class of paths

$$\Gamma_\lambda=\{\gamma\in C([0,1],\mathcal{H}^1(\Omega)\times\mathcal{H}^1(\Omega)):\gamma(0)=(v_{1,\lambda}^+,v_{2,\lambda}^+);\ \gamma(1)=(v_{1,\lambda}^+,w_{2,\lambda}+\bar{c})\}$$

and define

$$\alpha_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > I_\lambda(v^+_{1,\lambda}, v^+_{2,\lambda}).$$

In view of Lemma 4.1, (4.16) and (4.17), we can apply the Mountain Pass Theorem of Ambrosetti–Rabinowitz [2] to conclude that α_λ defines a critical level for I_λ. Namely, there exists $(\bar{v}_{1,\lambda}, \bar{v}_{2,\lambda}) \in \mathcal{H}^1(\Omega) \times \mathcal{H}^1(\Omega)$ such that

$$I'_\lambda(\bar{v}_{1,\lambda}, \bar{v}_{2,\lambda}) = 0 \quad \text{and} \quad I_\lambda(\bar{v}_{1,\lambda}, \bar{v}_{2,\lambda}) = \alpha_\lambda > I_\lambda(v^+_{1,\lambda}, v^+_{2,\lambda}).$$

Hence $(\bar{v}_{1,\lambda}, \bar{v}_{2,\lambda})$ defines a (weak) solution of (2.10) distinct from the local minimum $(v^+_{1,\lambda}, v^+_{2,\lambda})$. □

5. Vortex Solutions of the II and III-Type

In this section we are going to establish Theorem 2.2, by proving the existence of one-vortex solutions from minima on $\mathcal{A}_\lambda$ of the functionals J^-_λ, $J^\pm_\lambda$ and $J^\mp_\lambda$ (see remark 2.5).

We start to discuss the minimization problem for J^-_λ.

To this purpose we prove a preliminary lemma:

Lemma 5.1. *There exists a constant $C > 0$, independent of λ, such that for any $(w_1, w_2) \in \mathcal{A}_\lambda$,*

$$e^{c_i^-} \geq \frac{C}{\lambda \int_\Omega h_i e^{w_i}}, \quad i = 1, 2. \tag{5.1}$$

Proof. By symmetry, it suffices to show (5.1) for $i = 1$.

From (2.27) and (2.23), we have

$$e^{c_1^-} \geq \frac{4\pi(2N_1 + N_2)}{3\lambda(\int_\Omega h_1 e^{w_1} + e^{c_2^-} \int_\Omega h_1 h_2 e^{w_1 + w_2})}. \tag{5.2}$$

On the other hand, by the Hölder inequality, (2.20) and Lemma 2.3 (ii) we have

$$\begin{aligned} e^{c_2^-} \int_\Omega h_1 h_2 e^{w_1+w_2} &\leq e^{c_2^-} \left(\int_\Omega h_1^2 e^{2w_1}\right)^{\frac{1}{2}} \left(\int_\Omega h_2^2 e^{2w_2}\right)^{\frac{1}{2}} \\ &\leq \frac{3\lambda}{32\pi\sqrt{(2N_1+N_2)\,(2N_2+N_1)}} e^{c_2^-} \int_\Omega h_2 e^{w_2} \int_\Omega h_1 e^{w_1} \\ &\leq \frac{3}{4}\sqrt{\frac{2N_2+N_1}{2N_1+N_2}} \int_\Omega h_1 e^{w_1}. \end{aligned} \tag{5.3}$$

Combining (5.2) and (5.3), we obtain

$$e^{c_1^-} \geq \frac{4\pi(2N_1+N_2)}{3(1 + \frac{3}{4}\sqrt{\frac{2N_2+N_1}{2N_1+N_2}})} \frac{1}{\lambda \int_\Omega h_1 e^{w_1}},$$

and the desired estimate is established. □

Lemma 5.1 permits to obtain the following:

Proposition 5.2. *If $N_1 + N_2 = 1$, then there exists a constant $C > 0$, independent of λ, such that for all $(w_1, w_2) \in \mathcal{A}_\lambda$ we have:*

$$J_\lambda^-(w_1, w_2) \geq \frac{1}{30}(\|\nabla w_1\|^2 + \|\nabla w_2\|^2) + \lambda - 4\pi \ln \lambda - C. \tag{5.4}$$

Moreover, J_λ^- attains its infimum on $\mathcal{A}_\lambda$.

Proof. Recalling (2.37), from Lemma 5.1 we get

$$\begin{aligned} J_\lambda^-(w_1, w_2) \geq{}& \frac{1}{3}\left(\|\nabla w_1\|^2 + \|\nabla w_2\|^2 + \int_\Omega \nabla w_1 \nabla w_2\right) + \lambda \\ &+ \frac{4\pi}{3}(2N_1 + N_2)\ln\frac{1}{\lambda\int_\Omega h_1 e^{w_1}} + \frac{4\pi}{3}(2N_2 + N_1)\ln\frac{1}{\lambda\int_\Omega h_2 e^{w_2}} - C, \end{aligned} \tag{5.5}$$

for any $(w_1, w_2) \in \mathcal{A}_\lambda$ and for some constant $C > 0$ independent of λ.

Using the estimate $\int_\Omega |\nabla w_1||\nabla w_2| \leq \frac{1}{2\epsilon}\|\nabla w_1\|^2 + \frac{\epsilon}{2}\|\nabla w_2\|^2$, valid for any $\epsilon > 0$, and the Moser–Trudinger inequality (3.4), for some constant $C > 0$ (independent of λ), we obtain

$$\begin{aligned} J_\lambda^-(w_1, w_2) \geq{}& \frac{1}{3}\left(1 - \frac{1}{2\epsilon} - \frac{2N_1 + N_2}{4}\right)\|\nabla w_1\|^2 \\ &+ \frac{1}{3}\left(1 - \frac{\epsilon}{2} - \frac{2N_2 + N_1}{4}\right)\|\nabla w_2\|^2 \\ &+ \lambda - 4\pi(N_1 + N_2)\ln\lambda - C \\ \geq{}& \frac{1}{12}\left(3 - \frac{2}{\epsilon} - N_1\right)\|\nabla w_1\|^2 \\ &+ \frac{1}{12}(3 - 2\epsilon - N_2)\|\nabla w_2\|^2 + \lambda - 4\pi\ln\lambda - C, \end{aligned}$$

provided $N_1 + N_2 = 1$.

Thus (5.4) follows by choosing $\epsilon = \frac{5}{4}(1 - N_2) + \frac{4}{5}(1 - N_1)$.

Therefore J_λ^- is coercive on $\mathcal{A}_\lambda$. By its weak lower semicontinuity on the weakly closed set $\mathcal{A}_\lambda$, we immediately conclude that J_λ^- attains its infimum on $\mathcal{A}_\lambda$. □

Let $(w_{1,\lambda}, w_{2,\lambda}) \in \mathcal{A}_\lambda$ satisfy $J_\lambda^-(w_{1,\lambda}, w_{2,\lambda}) = \inf_{\mathcal{A}_\lambda} J_\lambda^-$, in order to prove that

$$\begin{cases} v_{1,\lambda}^- = w_{1,\lambda} + c_1^-(w_{1,\lambda}, w_{2,\lambda}) \\ v_{2,\lambda}^- = w_{2,\lambda} + c_2^-(w_{1,\lambda}, w_{2,\lambda}) \end{cases} \tag{5.6}$$

defines a (weak) solution of (2.10), it suffices to show that $(w_{1,\lambda}, w_{2,\lambda})$ lies in the interior of $\mathcal{A}_\lambda$. Indeed, we have

Proposition 5.3. *For $N_1 + N_2 = 1$ and $\lambda > 0$ sufficiently large, we have*

$$\inf_{(w_1,w_2)\in\partial\mathcal{A}_\lambda} J_\lambda^-(w_1, w_2) > \inf_{(w_1,w_2)\in\mathcal{A}_\lambda} J_\lambda^-(w_1, w_2). \tag{5.7}$$

Proof. For $(w_1, w_2) \in \partial\mathcal{A}_\lambda$ the identity

$$\left(\int_\Omega h_i e^{w_i}\right)^2 = \frac{32\pi\left(2N_i + N_j\right)}{\lambda}\int_\Omega h_i^2 e^{2w_i}, \quad i \neq j \tag{5.8}$$

holds for $i = 1$ or 2.

Now let $(w_{1,\lambda}, w_{2,\lambda})$ satisfies $J_\lambda^-(w_{1,\lambda}, w_{2,\lambda}) = \inf_{\mathcal{A}_\lambda} J_\lambda^-$ and by contradiction assume that $(w_{1,\lambda}, w_{2,\lambda}) \in \partial\mathcal{A}_\lambda$. W.l.o.g. we may suppose that (5.8) holds for $i = 2$. By Jensen's inequality it follows that,

$$\int_\Omega h_2^2 e^{2w_{2,\lambda}} \to +\infty, \quad \text{as } \lambda \to +\infty,$$

and, by the Moser–Trudinger inequality (3.4), necessarily

$$\|\nabla w_{2,\lambda}\| \to +\infty, \quad \text{as } \lambda \to +\infty. \tag{5.9}$$

Furthermore, by Proposition 5.2 we get

$$J_\lambda^-(w_{1,\lambda}, w_{2,\lambda}) \geq \frac{1}{30}(\|\nabla w_{1,\lambda}\|^2 + \|\nabla w_{2,\lambda}\|^2) + \lambda - 4\pi\ln\lambda - C, \tag{5.10}$$

with $C > 0$ a suitable constant independent of λ.

On the other hand, by Lemma 2.3 (ii) and Lemma 5.1 there exist constants $C_1, C_2 > 0$, independent of λ, such that

$$\begin{aligned} J_\lambda^-(0,0) \leq{} & \frac{\lambda}{2}\left(1 - \frac{C_1}{\lambda}\right) + \frac{\lambda}{2}\left(1 - \frac{C_1}{\lambda}\right) + \frac{4\pi}{3}(2N_1 + N_2)\ln\frac{C_2}{\lambda} \\ & + \frac{4\pi}{3}(2N_2 + N_1)\ln\frac{C_2}{\lambda} \leq \lambda - 4\pi(N_1 + N_2)\ln\lambda + C, \end{aligned} \tag{5.11}$$

with $C > 0$ a suitable constant independent of λ.

Thus, if $N_1 + N_2 = 1$, from (5.10) and (5.11) we obtain a contradiction since,

$$0 \geq J_\lambda^-(w_{1,\lambda}, w_{2,\lambda}) - J_\lambda^-(0,0) \geq \frac{1}{30}\|\nabla w_{2,\lambda}\|^2 - C \to +\infty, \quad \text{as } \lambda \to +\infty.$$

Hence, $(w_{1,\lambda}, w_{2,\lambda})$ must belong to the interior of $\mathcal{A}_\lambda$. □

Remark 5.4. Note that, if we use estimate (3.1) into (5.5) we can derive, as in Proposition 3.2, that the functional J_λ^- is bounded below and attains its infimum on $\mathcal{A}_\lambda$ for *all* integers N_1 and N_2. However, in the general situation, the estimate (5.4) gets worse with respect to λ and becomes:

$$J_\lambda^-(w_1, w_2) \geq \alpha\left(\|\nabla w_1\|_2^2 + \|\nabla w_2\|_2^2\right) + \lambda - 4\pi c_\alpha \ln\lambda - C, \tag{5.12}$$

for some $\alpha > 0$ and some constant $c_\alpha = c_\alpha(N_1, N_2) \geq 0$, depending on N_1 and N_2 in such a way that $\frac{c_\alpha}{N_1+N_2} \to +\infty$, as $N_1 + N_2 \to +\infty$.

Thus, the estimate (5.12) is no longer sufficient for the arguments in the proof of Proposition 5.3 to yield (5.7).

So we have established that (5.6) defines a solution for (2.10) provided $\lambda > 0$ is sufficiently large. Next, we show that $(v^-_{1,\lambda}, v^-_{2,\lambda})$ exhibits a different asymptotic behavior as $\lambda \to +\infty$ w.r.t. the family $(v^+_{1,\lambda}, v^+_{2,\lambda})$ obtained in Proposition 3.5 above. Indeed, we have

Proposition 5.5. *Let $N_1 + N_2 = 1$ and $(v^-_{1,\lambda}, v^-_{2,\lambda})$ be given by* (5.6). *Then,*

(i) $c^-_i(w_{1,\lambda}, w_{2,\lambda}) \to -\infty$, *as* $\lambda \to +\infty$.
(ii) *There exists a constant $C > 0$ (independent of λ) such that $\|\nabla w_{i,\lambda}\| \le C$ ($i = 1, 2$).*

Furthermore, any sequence $\lambda_n \to +\infty$ admits a subsequence (still denoted by λ_n) such that for $w_{i,n} = w_{i,\lambda_n}$ $(i = 1, 2)$ we have

$$w_{i,n} \to \bar{w}_i \quad \textit{strongly in } \mathcal{H}^1(\Omega), \quad (i = 1, 2)$$

and $(\bar{w}_1, \bar{w}_2)$ satisfies:

$$\begin{cases} -\Delta w_1 = 4\pi N_1 \left(\frac{4}{3} \frac{h_1 e^{w_1}}{\int_\Omega h_1 e^{w_1}} - \frac{1}{3} \frac{h_2 e^{w_2}}{\int_\Omega h_2 e^{w_2}} - 1 \right) + \frac{8\pi}{3} N_2 \left(\frac{h_1 e^{w_1}}{\int_\Omega h_1 e^{w_1}} - \frac{h_2 e^{w_2}}{\int_\Omega h_2 e^{w_2}} \right) \\ -\Delta w_2 = 4\pi N_2 \left(\frac{4}{3} \frac{h_2 e^{w_2}}{\int_\Omega h_2 e^{w_2}} - \frac{1}{3} \frac{h_1 e^{w_1}}{\int_\Omega h_1 e^{w_1}} - 1 \right) + \frac{8\pi}{3} N_1 \left(\frac{h_2 e^{w_2}}{\int_\Omega h_2 e^{w_2}} - \frac{h_1 e^{w_1}}{\int_\Omega h_1 e^{w_1}} \right) \\ \int_\Omega w_i = 0, \quad w_i \in \mathcal{H}^1(\Omega) \quad i = 1, 2. \end{cases} \tag{5.13}$$

Proof. By Remark 2.4 and Lemma 5.1 we have

$$e^{c^-_i(w_{1,\lambda}, w_{2,\lambda})} = O\left(\frac{1}{\lambda}\right), \quad i = 1, 2,$$

and (i) immediately follows.

(ii). Recalling (5.10) and (5.11) we have

$$0 \ge J^-_\lambda(w_{1,\lambda}, w_{2,\lambda}) - J^-_\lambda(0, 0) \ge \frac{1}{30}\left(\|\nabla w_{1,\lambda}\|^2 + \|\nabla w_{2,\lambda}\|^2 \right) - C,$$

hence, we immediately derive $\|\nabla w_{i,\lambda}\| \le C$ $(i = 1, 2)$ for some suitable constant $C > 0$ independent of λ.

Therefore, passing to subsequences if necessary, we derive $w_{i,n} = w_{i,\lambda_n} \to \bar{w}_i$ weakly in $\mathcal{H}^1(\Omega)$, strongly in $L^p(\Omega)$ $\forall p \ge 1$ and pointwise a.e. in Ω. Furthermore, by dominated convergence, we also have

$$\int_\Omega e^{\alpha(u^i_0 + w_{i,n})} \to \int_\Omega e^{\alpha(u^i_0 + \bar{w}_i)}, \quad i = 1, 2, \quad \text{and } \alpha > 0. \tag{5.14}$$

Consequently, taking into account (i), and the definition of $c^-_{i,n} = c^-_{i,n}(w_{1,n}, w_{2,n})$ in (2.27), as $n \to +\infty$, we get

$$\lambda_n e^{c^-_{i,n}} \to \frac{4\pi}{3} \frac{2N_i + N_j}{\int_\Omega h_i e^{\bar{w}_i}}, \quad i, j = 1, 2, \quad i \ne j. \tag{5.15}$$

The weak convergence in $\mathcal{H}^1(\Omega)$, together with (5.14) and (5.15), yield the conclusion that $(\bar{w}_1, \bar{w}_2)$ is a weak solution for (5.13).

Finally, to prove that $w_{i,n} \to \bar{w}_i$ $(i = 1, 2)$ strongly in $\mathcal{H}^1(\Omega)$, notice that

$$\int_\Omega |\nabla w_{i,n} - \nabla \bar{w}_i|^2 = -\int_\Omega \left(\Delta w_{i,n} - \Delta \bar{w}_i\right)\left(w_{i,n} - \bar{w}_i\right) = \int_\Omega h_{i,n}\left(w_{i,n} - \bar{w}_i\right),$$

where

$$\begin{aligned} h_{1,n} =& \lambda_n e^{c^-_{1,n}}\left(2h_1 e^{w_{1,n}} - 2h_1 e^{\bar{w}_1}\right) + 2h_1 e^{\bar{w}_1}\left(\lambda_n e^{c^-_{1,n}} - \frac{4\pi}{3}\frac{2N_1+N_2}{\int_\Omega h_1 e^{\bar{w}_1}}\right) \\ &- \lambda_n e^{c^-_{2,n}}\left(h_2 e^{w_{2,n}} - h_2 e^{\bar{w}_2}\right) - h_2 e^{\bar{w}_2}\left(\lambda_n e^{c^-_{2,n}} - \frac{4\pi}{3}\frac{2N_2+N_1}{\int_\Omega h_2 e^{\bar{w}_2}}\right) \\ &- \lambda_n\left(4e^{2c^-_{1,n}} h_1^2 e^{2w_{1,n}} - 2e^{2c^-_{2,n}} h_2^2 e^{2w_{2,n}} - e^{c^-_{1,n}} e^{c^-_{2,n}} h_1 e^{w_{1,n}} h_2 e^{w_{2,n}}\right), \end{aligned} \tag{5.16}$$

and $h_{2,n}$ is given by the symmetric expression. Hence, $\|h_n\|_p$ is uniformly bounded for any $p \geq 1$ and $\int_\Omega |\nabla w_{i,n} - \nabla \bar{w}_i|^2 \leq \|h_n\|_2 \|w_{i,n} - \bar{w}_i\|_2 \to 0$. □

Corollary 5.6. *Let* $(v^-_{1,\lambda}, v^-_{2,\lambda})$ *be given by* (5.6). *Then as* $\lambda \to +\infty$,

- $e^{v^-_{i,\lambda}} \to 0$ *uniformly in* $C^k(\Omega)$, $\forall k > 0$, *i=1,2.*

Proof. In view of (i) in Proposition 5.5 it is enough to prove that any sequence $\lambda_n \to +\infty$ admits a subsequence (which we still denote by λ_n) such that $w_{i,n} = w_{i,\lambda_n} \to \bar{w}_i$ in $C^k(\Omega)$ for any $k \geq 0$.

This is readily established since, $-\Delta(w_{i,n} - \bar{w}_i) = h_{i,n}$, with $h_{i,n}$ given in (5.16), and $\|h_{i,n}\|_p \to 0$ as $n \to +\infty$ for any $p \geq 1$. Consequently, $\|w_{i,n} - \bar{w}_i\|_{C^{1,\alpha}} \to 0$ as $n \to +\infty$ and $\alpha \in (0, 1)$. A bootstrap argument then gives $\|w_{i,n} - \bar{w}_i\|_{C^k} \to 0$, for any $k \in \mathbb{N}$. □

To conclude the proof of Theorem 2.2, we consider the analogous minimization problem for $J^{\pm}_\lambda$ and $J^{\mp}_\lambda$ on $\mathcal{A}_\lambda$. We start with the following:

Proposition 5.7. *Let* $N_1 + N_2 = 1$ *and* $* = \pm$ *or* $\mp$, *there exists a constant* $C > 0$, *independent of* λ, *such that*

$$J^*_\lambda(w_1, w_2) \geq \frac{1}{30}\left(\|\nabla w_1\|^2 + \|\nabla w_2\|^2\right) + \frac{3}{4}\lambda - 4\pi \ln \lambda - C, \tag{5.17}$$

for all $(w_1, w_2) \in \mathcal{A}_\lambda$.

Moreover, J^*_λ *attains its infimum on* $\mathcal{A}_\lambda$.

Proof. It suffices to prove (5.17) with $* = \pm$, the other case $* = \mp$ follows analogously by exchanging the role between the indices.

In view of (2.23), from (2.27) it follows immediately that

$$e^{c^{\pm}_1} \geq \frac{\int_\Omega h_1 e^{w_1}}{4\int_\Omega h_1^2 e^{2w_1}}, \geq \frac{8\pi\ (2N_1 + N_2)}{3\lambda}\frac{1}{\int_\Omega h_1 e^{w_1}}; \tag{5.18}$$

while,

$$e^{c_2^{\pm}} \geq \frac{8\pi}{3\lambda\left(\int_\Omega h_2 e^{w_2} + e^{c_1^{\pm}}\int_\Omega h_1 h_2 e^{w_1+w_2}\right)} \geq \frac{4\pi}{3\lambda \max\left\{\int_\Omega h_2 e^{w_2}, \int_\Omega h_1 h_2 e^{w_1+w_2}\right\}}, \tag{5.19}$$

where we have used that $e^{c_1^{\pm}} \leq 1$ (see Remark 2.4).

In case $\int_\Omega h_2 e^{w_2} \geq \int_\Omega h_1 h_2 e^{w_1+w_2}$, by (2.37) we can use Lemma 2.3 (iii), (5.18), (5.19) and the Moser–Trudinger inequality (3.4), to derive

$$\begin{aligned}
J_\lambda^{\pm}(w_1, w_2) \geq{}& \frac{1}{3}\left(\|\nabla w_1\|^2 + \|\nabla w_2\|^2 + \int_\Omega \nabla w_1 \nabla w_2\right) + \frac{3}{4}\lambda \\
&- \frac{4\pi}{3}(2N_1 + N_2)\ln\int_\Omega h_1 e^{w_1} - \frac{4\pi}{3}(2N_2 + N_1)\ln\int_\Omega h_2 e^{w_2} \\
&- 4\pi(N_1 + N_2)\ln\lambda - C \\
\geq{}& \frac{1}{3}\left(1 - \frac{1}{2\epsilon} - \frac{2N_1 + N_2}{4}\right)\|\nabla w_1\|^2 \\
&+ \frac{1}{3}\left(1 - \frac{\epsilon}{2} - \frac{2N_2 + N_1}{4}\right)\|\nabla w_2\|^2 \\
&+ \frac{3}{4}\lambda - 4\pi(N_1 + N_2)\ln\lambda - C
\end{aligned} \tag{5.20}$$

for every $\epsilon > 0$ and $C > 0$ independent of λ. Since $N_1 + N_2 = 1$, as in Proposition 5.2, we can certainly make a choice of $\epsilon > 0$ in (5.20) in order to insure (5.17).

Now suppose that $\int_\Omega h_2 e^{w_2} < \int_\Omega h_1 h_2 e^{w_1+w_2}$, proceeding as above, in this case we get

$$\begin{aligned}
J_\lambda^{\pm}(w_1, w_2) \geq{}& \frac{1}{3}\left(\|\nabla w_1\|^2 + \|\nabla w_2\|^2 + \int_\Omega \nabla w_1 \nabla w_2\right) + \frac{3}{4}\lambda - 4\pi(N_1 + N_2)\ln\lambda \\
&- \frac{4\pi}{3}(2N_1 + N_2)\ln\int_\Omega h_1 e^{w_1} - \frac{4\pi}{3}(2N_2 + N_1)\ln\int_\Omega h_1 h_2 e^{w_1+w_2} - C \\
\geq{}& \frac{1}{6}\left(\|\nabla w_1\|^2 + \|\nabla w_2\|^2 + \|\nabla w_1 + \nabla w_2\|^2\right) - \frac{(2N_1 + N_2)}{12}\|\nabla w_1\|^2 \\
&- \frac{(2N_2 + N_1)}{12}\|\nabla w_1 + \nabla w_2\|^2 + \frac{3}{4}\lambda - 4\pi(N_1 + N_2)\ln\lambda - C \\
\geq{}& \frac{1}{24}\left(\|\nabla w_1\|^2 + \|\nabla w_2\|^2\right) + \frac{3}{4}\lambda - 4\pi\ln\lambda - C,
\end{aligned} \tag{5.21}$$

provided $N_1 + N_2 = 1$.

In any case we get the desired estimate (5.17). Thus $J_\lambda^{\pm}$ is coercive on $\mathcal{A}_\lambda$. Since it is weakly lower semicontinuous on the weakly closed set $\mathcal{A}_\lambda$, we immediately conclude that $J_\lambda^{\pm}$ attains its infimum on $\mathcal{A}_\lambda$. □

Remark 5.8. By similar considerations to those of Remark 5.4, we can assert that, in fact, the functional J_λ^*, $* = \pm$ or $\mp$, is bounded below and attains its infimum on $\mathcal{A}_\lambda$ for *all* integers N_1 and N_2. However, we need to restrict to the case $N_1 + N_2 = 1$ in order to insure a sharp form of estimate (5.17) (see (5.36) below), which is crucial for the existence of a minimizer in the *interior* of $\mathcal{A}_\lambda$.

Let $* = \pm$ or $\mp$ and denote by $(w^*_{1,\lambda}, w^*_{2,\lambda}) \in \mathcal{A}_\lambda$ a minimum of J^*_λ in $\mathcal{A}_\lambda$, namely $J^*_\lambda(w_{1,\lambda}, w_{2,\lambda}) = \inf_{\mathcal{A}_\lambda} J^*_\lambda$. Define

$$\begin{cases} v^*_{1,\lambda} = w^*_{1,\lambda} + c^*_1(w^*_{1,\lambda}, w^*_{2,\lambda}) \\ v^*_{2,\lambda} = w^*_{2,\lambda} + c^*_2(w^*_{1,\lambda}, w^*_{2,\lambda}), \end{cases} \tag{5.22}$$

$* = \pm$ or $\mp$.

To show that $(w^*_{1,\lambda}, w^*_{2,\lambda})$ lies in the interior of $\mathcal{A}_\lambda$, and hence that $(v^*_{1,\lambda}, v^*_{1,\lambda})$ defines a (weak) solution of (2.10), we prove the following preliminary result which holds for any choice of $N_1, N_2 \in \mathbb{N}$.

Lemma 5.9. (i) *Let* $(v^\pm_{1,\lambda}, v^\pm_{2,\lambda})$ *be given by* (5.22), *then*

$$\int_\Omega h_1 \mathrm{e}^{v^\pm_{1,\lambda}} \to \frac{1}{2}.$$

(ii) *Let* $(v^\mp_{1,\lambda}, v^\mp_{2,\lambda})$ *be given by* (5.22), *then*

$$\int_\Omega h_2 \mathrm{e}^{v^\mp_{2,\lambda}} \to \frac{1}{2}.$$

Proof. By symmetry, we only need to establish (i).

As in the proof of Lemma 3.3, let $\bar{v}^1_\mu = \bar{c}^1_\mu + \bar{w}^1_\mu$, with $\bar{c}^1_\mu = \int_\Omega \bar{v}^1_\mu$ and $\int_\Omega \bar{w}^1_\mu = 0$ be the solution of

$$\begin{cases} \Delta v = \mu \mathrm{e}^{v+u^1_0}(\mathrm{e}^{v+u^1_0} - 1) + 4\pi N_1 \\ v \in \mathcal{H}^1(\Omega) \end{cases}$$

satisfying $\bar{w}^1_\mu \to -u^1_0$ pointwise a.e. in Ω and

$$h_1 \mathrm{e}^{\bar{w}^1_\mu} \to 1, \quad \text{in } L^p(\Omega),\ \forall p \geq 1, \tag{5.23}$$

as $\mu \to +\infty$ (see [34, Proposition 3.1]).

Observe that, from (2.27) and (2.23), we have

$$\begin{aligned} \mathrm{e}^{c^\pm_1} \int_\Omega h_1 \mathrm{e}^{w_1} &\geq \frac{\left(\int_\Omega h_1 \mathrm{e}^{w_1}\right)^2}{4 \int_\Omega h_1^2 \mathrm{e}^{2w_1}} \left(1 + \sqrt{1 - \frac{32\pi\ (2N_1 + N_2)}{3\lambda} \frac{\int_\Omega h_1^2 \mathrm{e}^{2w_1}}{\left(\int_\Omega h_1 \mathrm{e}^{w_1}\right)^2}}\right) \\ &\geq \frac{1}{2} \frac{\left(\int_\Omega h_1 \mathrm{e}^{w_1}\right)^2}{\int_\Omega h_1^2 \mathrm{e}^{2w_1}} - \frac{8\pi}{3\lambda}\,(2N_1 + N_2)\,. \end{aligned} \tag{5.24}$$

Recalling (5.23) and Lemma 2.3 (iii), we find $\lambda_0 > 0$ sufficiently large and $c_0 > 0$ such that for any $\epsilon > 0$ there exist $\mu_\epsilon > 0$ with the property that $(\bar{w}^1_{\mu_\epsilon}, 0) \in \mathcal{A}_\lambda$ and

$$\begin{aligned} &\mathrm{e}^{c^\pm_1\left(\bar{w}^1_{\mu_\epsilon}, 0\right)} \int_\Omega h_1 \mathrm{e}^{\bar{w}^1_{\mu_\epsilon}} \geq \frac{1}{2} - \epsilon - \frac{8\pi}{3\lambda}\,(2N_1 + N_2)\,, \\ &\mathrm{e}^{c^\pm_2\left(\bar{w}^1_{\mu_\epsilon}, 0\right)} \int_\Omega h_2 \geq \frac{c_0}{\lambda}, \end{aligned} \tag{5.25}$$

for every $\lambda \geq \lambda_0$.

Consequently,

$$J_\lambda^\pm\left(\bar{w}^1_{\mu_\epsilon},0\right)\le\frac{1}{3}\|\nabla\bar{w}^1_{\mu_\epsilon}\|^2+\lambda\left(1-\frac{1}{4}+\frac{\epsilon}{2}\right)+O\left(\ln\lambda\right),\quad\text{as }\lambda\to+\infty.\tag{5.26}$$

On the other hand, we have

$$J_\lambda^\pm\left(w_{1,\lambda},w_{2,\lambda}\right)\ge\frac{\lambda}{2}\left(1-\mathrm{e}^{c_1^\pm}\int_\Omega h_1\mathrm{e}^{w_{1,\lambda}}\right)+\frac{\lambda}{2}+O\left(\ln\lambda\right),\quad\text{as }\lambda\to+\infty.\tag{5.27}$$

Therefore,

$$0\ge J_\lambda^\pm\left(w_{1,\lambda},w_{2,\lambda}\right)-J_\lambda^\pm\left(\bar{w}^1_{\mu_\epsilon},0\right)\ge\frac{\lambda}{2}\left(1-\mathrm{e}^{c_1^\pm}\int_\Omega h_1\mathrm{e}^{w_{1,\lambda}}\right)+\frac{\lambda}{2}$$
$$-\left(\frac{1}{3}\|\nabla\bar{w}^1_{\mu_\epsilon}\|^2+\lambda\left(1-\frac{1}{4}+\frac{\epsilon}{2}\right)\right)+O\left(\ln\lambda\right),$$

from which we derive

$$\limsup_{\lambda\to+\infty}\left(\frac{1}{2}-\mathrm{e}^{c_1^\pm}\int_\Omega h_1\mathrm{e}^{w_{1,\lambda}}\right)\le\epsilon,\quad\forall\,\epsilon>0.$$

At this point, taking into account Lemma 2.3 (iii), we conclude

$$\int_\Omega h_1\mathrm{e}^{v_{1,\lambda}^\pm}\to\frac{1}{2},\quad\text{as }\lambda\to+\infty.\qquad\square$$

Remark 5.10. Putting together (2.35) and (5.24), we have that necessarily

$$\frac{\left(\int_\Omega h_1\mathrm{e}^{w_{1,\lambda}}\right)^2}{\int_\Omega h_1^2\mathrm{e}^{2w_{1,\lambda}}}\to1,\quad\text{as }\lambda\to+\infty.\tag{5.28}$$

Using Lemma 5.9, we derive:

Proposition 5.11. (i) *If* $N_1=0$ *and* $N_2=1$ *then, for* $\lambda>0$ *sufficiently large,*

$$\inf_{(w_1,w_2)\in\partial\mathcal{A}_\lambda}J_\lambda^\pm(w_1,w_2)>\inf_{(w_1,w_2)\in\mathcal{A}_\lambda}J_\lambda^\pm(w_1,w_2).\tag{5.29}$$

(ii) *If* $N_1=1$ *and* $N_2=0$ *then, for* $\lambda>0$ *sufficiently large,*

$$\inf_{(w_1,w_2)\in\partial\mathcal{A}_\lambda}J_\lambda^\mp(w_1,w_2)>\inf_{(w_1,w_2)\in\mathcal{A}_\lambda}J_\lambda^\mp(w_1,w_2).\tag{5.30}$$

Proof. Again by symmetry we only need to establish (i). Let us suppose that $(w_{1,\lambda},w_{2,\lambda})$ satisfies $J^\pm(w_{1,\lambda},w_{2,\lambda})=\inf_{\mathcal{A}_\lambda}J^\pm$ and $(w_{1,\lambda},w_{2,\lambda})\in\partial\mathcal{A}_\lambda$. In view of Remark 5.10 necessarily

$$\left(\int_\Omega h_2\mathrm{e}^{w_{2,\lambda}}\right)^2=\frac{32\pi\,(2N_2+N_1)}{\lambda}\int_\Omega h_2^2\mathrm{e}^{2w_{2,\lambda}}.\tag{5.31}$$

As a consequence of (5.31) we get

$$\int_\Omega h_2^2\mathrm{e}^{2w_{2,\lambda}}\to+\infty\quad\text{as }\lambda\to+\infty$$

and, by the Moser–Trudinger inequality (3.4), necessarily

$$\|\nabla w_{2,\lambda}\| \to +\infty \quad \text{as } \lambda \to +\infty. \tag{5.32}$$

Now, note that if $N_1 = 0$ then $u_0^1 = 0$, and in particular $h_1 = e^{u_0^1} = 1$. By explicit calculation we see that,

$$\begin{aligned} e^{c_1^\pm(0,0)} &= \frac{1}{2} + O\left(\frac{1}{\lambda}\right), \\ e^{c_2^\pm(0,0)} &= O\left(\frac{1}{\lambda}\right), \quad \text{as } \lambda \to +\infty. \end{aligned} \tag{5.33}$$

Therefore, in view of (2.14) and (5.33) we get

$$\begin{aligned} J_\lambda^\pm(0,0) &\le \frac{\lambda}{2}\left(1 - e^{c_1^\pm(0,0)}\right) + \frac{\lambda}{2}\left(1 - e^{c_2^\pm(0,0)} \int_\Omega h_2\right) - \frac{8\pi}{3}\ln\lambda + C \\ &\le \frac{3}{4}\lambda - \frac{8\pi}{3}\ln\lambda + C \quad \text{as } \lambda \to +\infty. \end{aligned} \tag{5.34}$$

for a suitable constant $C > 0$, independent of λ.

On the other hand, by Lemma 5.9, for λ sufficiently large, we can insure that,

$$c_1^\pm(w_{1,\lambda}, w_{2,\lambda}) \ge -\ln \int_\Omega e^{w_{1,\lambda}} - \ln 4. \tag{5.35}$$

Thus, using the same arguments of Proposition 5.7, by Lemma 2.3 (iii) we find constants $\alpha, C > 0$ (independent of λ) such that

$$\begin{aligned} J_\lambda^\pm\left(w_{1,\lambda}, w_{2,\lambda}\right) \ge{}& \alpha\left(\|\nabla w_{1,\lambda}\|^2 + \|\nabla w_{2,\lambda}\|^2\right) + \frac{\lambda}{2}\int_\Omega \left(1 - e^{c_1^\pm} e^{w_{1,\lambda}}\right) \\ &+ \frac{\lambda}{2}\int_\Omega \left(1 - e^{c_2^\pm} h_2 e^{w_{2,\lambda}}\right) - \frac{8\pi}{3}\ln\lambda - C \\ \ge{}& \alpha\left(\|\nabla w_{1,\lambda}\|^2 + \|\nabla w_{2,\lambda}\|^2\right) + \frac{3}{4}\lambda - \frac{8\pi}{3}\ln\lambda - C. \end{aligned} \tag{5.36}$$

Combining (5.34) and (5.36) we conclude that,

$$0 \ge J_\lambda^\pm(w_{1,\lambda}, w_{2,\lambda}) - J_\lambda^\pm(0,0) \ge \alpha(\|\nabla w_{1,\lambda}\|_2^2 + \|\nabla w_{2,\lambda}\|_2^2) - C, \tag{5.37}$$

and, in view of (5.32), we reach a contradiction. □

To conclude we determine the asymptotic behavior, as $\lambda \to +\infty$, of the family of solutions given by (5.22).

Proposition 5.12. *Let* $(v_{1,\lambda}^*, v_{2,\lambda}^*)$ *be given by* (5.22) *with* $* = \pm$ *or* $\mp$. *We have*

- *(case* $* = \pm$*):* $c_1^\pm(w_{1,\lambda}^\pm, w_{2,\lambda}^\pm) \to \ln\frac{1}{2}$, $c_2^\pm(w_{1,\lambda}^\pm, w_{2,\lambda}^\pm) \to -\infty$ $w_{1,\lambda}^\pm \to 0$ *strongly in* $\mathcal{H}^1(\Omega)$ *as* $\lambda \to +\infty$*; along any sequences* $\lambda_n \to +\infty$*, there exists a subsequence (still denoted by* λ_n*) such that* $w_{2,\lambda_n}^\pm \to w_\pm$ *strongly in* $\mathcal{H}^1(\Omega)$ *with* $w_\pm$ *satisfying:*

$$\begin{cases} -\Delta w = 4\pi\left(\dfrac{h_2 e^w}{\int_\Omega h_2 e^w} - \dfrac{1}{|\Omega|}\right) & \text{in } \Omega \\ \int_\Omega w = 0. \end{cases} \tag{5.38}$$

In particular,

$$e^{v^{\pm}_{1,\lambda}} \to \frac{1}{2} \quad \textit{and} \quad e^{v^{\pm}_{2,\lambda}} \to 0, \quad \textit{strongly in } \mathcal{H}^1(\Omega), \quad \textit{as } \lambda \to +\infty. \tag{5.39}$$

- *(case $* = \mp$): $c_1^{\mp}(w^{\mp}_{1,\lambda}, w^{\mp}_{2,\lambda}) \to -\infty$, $c_2^{\mp}(w^{\mp}_{1,\lambda}, w^{\mp}_{2,\lambda}) \to \ln \frac{1}{2}$, $w^{\mp}_{2,\lambda} \to 0$ strongly in $\mathcal{H}^1(\Omega)$ as $\lambda \to +\infty$; along any sequences $\lambda_n \to +\infty$, there exists a subsequence (still denoted by λ_n) such that $w^{\mp}_{1,\lambda_n} \to w_{\mp}$ strongly in $\mathcal{H}^1(\Omega)$ with $w_{\mp}$ satisfying:*

$$\begin{cases} -\Delta w = 4\pi \left(\dfrac{h_1 e^w}{\int_\Omega h_1 e^w} - \dfrac{1}{|\Omega|} \right) & \textit{in } \Omega \\ \int_\Omega w = 0. \end{cases} \tag{5.40}$$

In particular,

$$e^{v^{\mp}_{1,\lambda}} \to 0 \quad \textit{and} \quad e^{v^{\mp}_{2,\lambda}} \to \frac{1}{2}, \quad \textit{strongly in } \mathcal{H}^1(\Omega), \quad \textit{as } \lambda \to +\infty. \tag{5.41}$$

Proof. As usual we only need to prove the result in case $* = \pm$, the other case follows by exchanging the role between the indices. Recall that in this case $N_1 = 0$, and hence $h_1 = 1$, $N_2 = 1$. By (5.37) we have

$$\|\nabla w^{\pm}_{i,\lambda}\|_2^2 \le C, \tag{5.42}$$

for suitable $C > 0$ (independent of λ) and $i = 1, 2$. Consequently,

$$1 \le \int_\Omega h_i e^{w^{\pm}_{i,\lambda}} \le C \quad (i = 1, 2) \tag{5.43}$$

with $C > 0$ independent of λ.

Thus, by setting $c^{\pm}_{i,\lambda} = c_i^{\pm}(w^{\pm}_{1,\lambda}, w^{\pm}_{2,\lambda})$, we have:

$$c^{\pm}_{2,\lambda} = \frac{8\pi}{9\lambda} \frac{(2N_2 + N_1)}{\int_\Omega h_2 e^{w^{\pm}_{2,\lambda}}} + o\left(\frac{1}{\lambda}\right) \tag{5.44}$$

and

$$c^{\pm}_{1,\lambda} = \frac{1}{2} \frac{\int_\Omega e^{w^{\pm}_{1,\lambda}}}{\int_\Omega e^{2w^{\pm}_{1,\lambda}}} + \frac{4\pi}{9\lambda}(N_1 + N_2)(N_2 - 9N_1) + o\left(\frac{1}{\lambda}\right) \tag{5.45}$$

as $\lambda \to +\infty$.

From (5.43) we derive immediately that $c^{\pm}_{2,\lambda} \to -\infty$, as $\lambda \to +\infty$.

Furthermore, in view of (5.28) and (5.42), along any sequence $\lambda_n \to +\infty$, we find a subsequence (still denoted by λ_n) such that,

$$w_{1,n} := w^{\pm}_{1,\lambda_n} \to 0 \text{ weakly in } \mathcal{H}^1(\Omega),$$
$$\text{strongly in } L^p(\Omega),\ p \ge 1 \text{ and pointwise a.e. in } \Omega.$$

Analogously, for suitable $w_{\pm} \in \mathcal{H}^1(\Omega)$, we may claim that,

$$w_{2,n} := w^{\pm}_{2,\lambda_n} \to w_{\pm} \text{ weakly in } \mathcal{H}^1(\Omega),$$
$$\text{strongly in } L^p(\Omega),\ p \ge 1 \text{ and pointwise a.e. in } \Omega,$$

Note in particular that $e^{w_{1,n}} \to 1$, $e^{w_{2,n}} \to e^{w_\pm}$ in $L^p(\Omega)$, $\forall\ p \geq 1$.

Thus, using (2.10) (with $|\Omega| = 1$) together with (5.44)–(5.45), we find:

$$\begin{aligned} -\Delta(w_{1,n} + 2w_{2,n}) &= 3\lambda_n h_2 e^{c^{\pm}_{2,n}} e^{w_{2,n}} (1 + e^{c^{\pm}_{1,n}} e^{w_{1,n}} - 2e^{c^{\pm}_{2,n}} e^{w_{2,n}}) - 4\pi(2N_2 + N_1) \\ &= 4\pi(2N_2 + N_1) \left(\frac{h_2 e^{w_{2,n}}}{\int_\Omega h_2 e^{w_{2,n}}} - \frac{1}{|\Omega|} \right) + \phi_n \quad \text{in } \Omega \end{aligned}$$

with $c^{\pm}_{i,n} := c^{\pm}_{i,\lambda_n}$ $(i = 1, 2)$ and $\phi_n \to 0$ strongly in $L^p(\Omega)$, $\forall\ p \geq 1$.

Consequently, by elliptic regularity theory, we obtain (after taking a subsequence if necessary)

$$\frac{1}{2} w_{1,n} + w_{2,n} \to w_\pm \quad \textit{strongly} \text{ in } C^{1,\alpha}(\Omega),\ \alpha \in (0, 1) \tag{5.46}$$

and $w_\pm$ satisfies:

$$\begin{cases} -\Delta w = 2\pi(2N_2 + N_1) \left(\dfrac{h_2 e^{w}}{\int_\Omega h_2 e^{w}} - \dfrac{1}{|\Omega|} \right) & \text{in } \Omega \\ \int_\Omega w = 0. \end{cases} \tag{5.47}$$

with $N_2 = 1$ and $N_1 = 0$, namely (5.40).

On the other hand, if we insert (5.44)-(5.45) into the first equation in (2.10) (with $N_1 = 0$ and $N_2 = 1$) we get:

$$\begin{aligned} \Delta w_{1,n} = \lambda_n \frac{(\int_\Omega e^{w_{1,n}})^2}{\int_\Omega e^{2w_{1,n}}} \left(\frac{e^{2w_{1,n}}}{\int_\Omega e^{2w_{1,n}}} - \frac{e^{w_{1,n}}}{\int_\Omega e^{w_{1,n}}} \right) + \frac{8\pi}{9} \frac{h_2 e^{w_{2,n}}}{\int_\Omega h_2 e^{w_{2,n}}} (2 - e^{w_{1,n}}) \\ + \frac{8\pi}{9} e^{w_{1,n}} (2e^{w_{1,n}} - 1) + \psi_n \quad \text{in } \Omega \end{aligned}$$

with $\psi_n \to 0$ strongly in $L^p(\Omega)$, $\forall p \geq 1$.

Therefore,

$$\|\nabla w_{1,n}\|_2^2 = -\lambda_n \frac{(\int_\Omega e^{w_{1,n}})^2}{\int_\Omega e^{2w_{1,n}}} \int_\Omega \left(\frac{e^{2w_{1,n}}}{\int_\Omega e^{2w_{1,n}}} - \frac{e^{w_{1,n}}}{\int_\Omega e^{w_{1,n}}} \right) w_{1,n} - \int_\Omega f_n w_{1,n}, \tag{5.48}$$

with $f_n \to \frac{8\pi}{9}(1 + \frac{h_2 e^{w_\pm}}{\int_\Omega h_2 e^{w_\pm}})$ in $L^p(\Omega)$, $\forall\ p \geq 1$.

Note that the function $h(t) := \int_\Omega \frac{e^{t w_{1,n}}}{\int_\Omega e^{t w_{1,n}}} w_{1,n}$ is increasing in $t \in \mathbb{R}$, since

$$\begin{aligned} h'(t) &= \int_\Omega \frac{e^{t w_{1,n}}}{\int_\Omega e^{t w_{1,n}}} w_{1,n}^2 - (\int_\Omega \frac{e^{t w_{1,n}}}{\int_\Omega e^{t w_{1,n}}} w_{1,n})^2 \\ &= \int_\Omega \frac{e^{t w_{1,n}}}{\int_\Omega e^{t w_{1,n}}} (w_{1,n} - \int_\Omega \frac{e^{t w_{1,n}}}{\int_\Omega e^{t w_{1,n}}} w_{1,n})^2 \geq 0 \quad \forall\ t \in \mathbb{R}. \end{aligned}$$

Thus,

$$\int_\Omega (\frac{e^{2w_{1,n}}}{\int_\Omega e^{2w_{1,n}}} - \frac{e^{w_{1,n}}}{\int_\Omega e^{w_{1,n}}}) w_{1,n} = h(2) - h(1) \geq 0,$$

and from (5.48) we derive

$$\|\nabla w_{1,n}\|_2 \to 0, \quad \text{as } n \to +\infty; \tag{5.49}$$

$$\lambda_n \int_\Omega \left(\frac{e^{2w_{1,n}}}{\int_\Omega e^{2w_{1,n}}} - \frac{e^{w_{1,n}}}{\int_\Omega e^{w_{1,n}}} \right) w_{1,n} \to 0 \quad \text{as } n \to +\infty. \tag{5.50}$$

Taking into account (5.46), we can also assert that,

$$w_{2,n} \to w_\pm \quad \text{strongly in } \mathcal{H}^1(\Omega).$$

Since (5.49) holds along any sequence $\lambda_n \to +\infty$, we may conclude that, $w^{\pm}_{1,\lambda} \to 0$ strongly in $\mathcal{H}^1(\Omega)$ as $\lambda \to +\infty$.

Finally, from (5.45) we get $c^{\pm}_{1,\lambda} \to \ln \frac{1}{2}$, as $\lambda \to +\infty$.

This concludes the proof. □

Clearly, Theorem 2.2 it is now an immediate consequence of Corollary 5.6 and Proposition 5.12.

Final remarks. It is an interesting open problem to know if Theorem 2.2 remains valid without the restriction $N_1 + N_2 = 1$.

To test whether or not our approach could be generalized for a more general choice of $N_1, N_2 \in \mathbb{N}$, we could start by investigating the existence question for problems (5.13) and (5.47). While problem (5.47) has appeared already in abelian theory, see [34], and it has been studied in [32] and [11], the elliptic system (5.13) is a novelty of the $SU(3)$-theory and it is certainly worthwhile investigating.

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