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# Uniform Spectral Properties of One-Dimensional Quasicrystals, III. $\alpha$-Continuity

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**Abstract:** We study the spectral properties of one-dimensional whole-line Schrödinger operators, especially those with Sturmian potentials. Building upon the Jitomirskaya–Last extension of the Gilbert–Pearson theory of subordinacy, we demonstrate how to establish $\alpha$- continuity of a whole-line operator from power-law bounds on the solutions on a half-line. However, we require that these bounds hold uniformly in the boundary condition.

We are able to prove these bounds for Sturmian potentials with rotation numbers of bounded density and arbitrary coupling constant. From this we establish purely $\alpha$-continuous spectrum uniformly for all phases.

Our analysis also permits us to prove that the point spectrum is empty for *all* Sturmian potentials.

## 1. Introduction

In this article we are interested in spectral properties of discrete one-dimensional Schrödinger operators on the whole line, that is, operators $H$ in $\ell^2(\mathbb{Z})$ of the form

$$[Hu](n) = u(n+1) + u(n-1) + V(n)u(n) \tag{1}$$

with arbitrary potential $V : \mathbb{Z} \to \mathbb{R}$.

Among the most powerful tools that have been developed for the investigation of the spectral type of such operators are those which establish a correspondence to the behavior of the solutions of the associated difference equation

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n). \tag{2}$$

In what follows we shall always assume a solution of (2) to be normalized in the sense that

$$|u(0)|^2 + |u(1)|^2 = 1. \tag{3}$$

An elementary observation is that a support of the pure point part of a spectral measure associated to $H$ is given by

$$M_{\mathrm{pp}} = \{E \in \mathbb{R} : \exists \text{ solution } u \text{ of (2) which is } \ell^2 \text{ at both } \pm\infty\}.$$

The following notion, introduced by Gilbert and Pearson [7], allows an analogous description of a support of the singular part. Namely, a solution $u$ of (2) is called subordinate at $+\infty$ if

$$\lim_{L\to\infty} \frac{\|u\|_L}{\|v\|_L} = 0 \tag{4}$$

for any solution $v$ of (2) which is linearly independent of $u$. Here $\|\cdot\|_L$ denotes the norm of the solution over a lattice interval of length $L$, that is,

$$\|u\|_L^2 = \sum_{n=0}^{\lfloor L\rfloor} \big|u(n)\big|^2 \ + \ (L - \lfloor L\rfloor)\big|u(\lfloor L\rfloor + 1)\big|^2. \tag{5}$$

Subordinacy of a solution $u$ at $-\infty$ is defined analogously. Gilbert [6] then proves that

$$M_{\mathrm{sing}} = \{E \in \mathbb{R} : \exists \text{ solution } u \text{ of (2) which is subordinate at both } \pm\infty\}$$

is a support of the singular part of a spectral measure associated to $H$. Hence the standard decomposition of a spectral measure into its pure point, singular continuous, and absolutely continuous part can be investigated by studying solutions of (2). Recall that by the RAGE theorem, each of these standard spectral parts is related to certain quantum dynamical behavior.

*Remark 1.* Note that these support descriptions require a certain condition to hold at both "endpoints" $+\infty$ and $-\infty$. That is, if one can show that for some energy $E$, there is an endpoint such that no solution of (2) satisfies this condition (square-summability and subordinacy, respectively) at this endpoint, then this energy does not belong to the respective support. In this sense the "more continuous half-line dominates" and this picture is consistent with heuristic quantum evolution in one dimension.

Recently, further decompositions of spectral measures have been proposed by Last [18]. These decompositions are motivated by the goal of answering more delicate questions arising in the study of quantum dynamics in the presence of purely singular continuous spectral measures. A finite positive measure $d\Lambda$ is said to be uniformly $\alpha$-Hölder continuous (or U$\alpha$H) if the distribution function

$$\Lambda(E) = \int_{-\infty}^{E} d\Lambda$$

is uniformly $\alpha$-Hölder continuous. A measure is said to be $\alpha$-continuous if it is absolutely continuous with respect to a U$\alpha$H measure. This definition of $\alpha$-continuity is equivalent to the more common "$\mu(S) = 0$ for all sets $S$ of zero $\alpha$-Hausdorff measure" [22]. On the other hand, a measure is called $\alpha$-singular if it is supported on a set of zero $\alpha$-Hausdorff measure. Last discusses the decomposition of a measure into its $\alpha$-continuous and its $\alpha$-singular part and he obtains explicit quantum dynamical bounds in the case where the $\alpha$-continuous part is non-trivial. Moreover, this decomposition is further motivated as there is apparently a very nice interpolation of the Gilbert–Pearson results. Namely, Jitomirskaya and Last introduce in [11] the following notion: A solution $u$ of (2) is called $\alpha$-subordinate at $+\infty$ if, setting $\beta = \frac{\alpha}{2-\alpha}$,

$$\liminf_{L\to\infty} \frac{\|u\|_L}{\|v\|_L^\beta} = 0 \tag{6}$$

for any solution $v$ of (2) which is linearly independent of $u$. Again, $\alpha$-subordinacy at $-\infty$ is defined analogously. In [12] these authors establish this interpolation for half-line operators. The natural whole-line correspondence accompanying the half-line result would be the following interpolation of the Gilbert result.

*Conjecture.* A support of the $\alpha$-singular part of a spectral measure associated to $H$ is given by

$$M_{\alpha\text{-sing}} = \{E \in \mathbb{R} : \exists \text{ sol. } u \text{ of (2) which is } \alpha\text{-subordinate at both } \pm\infty\}.$$

We shall obtain, in Theorem 1 below, a restricted version of this statement. In view of Remark 1 the goal is to establish the following implication: Pick some endpoint. If for all energies in some set $\Sigma$, all solutions of (2) are not $\alpha$-subordinate at the chosen endpoint, then the $\alpha$-singular part of a spectral measure associated to $H$ gives zero weight to $\Sigma$. There is a well-known way to prove non-existence of $\alpha$-subordinate solutions for some fixed energy $E$ and a fixed endpoint which has been exploited in [2,13]. Namely, power-law bounds of the form

$$C_1 L^{\gamma_1} \le \|u\|_L \le C_2 L^{\gamma_2}$$

for all normalized solutions $u$ of (2) imply non-existence of $\alpha$-subordinate solutions at $+\infty$, where $\alpha = \frac{2\gamma_1}{\gamma_1+\gamma_2}$; similarly at $-\infty$. The restriction we have to impose on the conjecture in order to establish the desired connection is twofold. Firstly, we require that non-existence of $\alpha$-subordinate solutions is established by this power-law criterion. Secondly, we need that the bounds are uniform in the solutions corresponding to a fixed energy. Under these assumptions one may conclude purely $\alpha$-continuous spectrum on $\Sigma$.

**Theorem 1.** *Let $\Sigma$ be a bounded set. Suppose there are constants $\gamma_1, \gamma_2$ such that for each $E \in \Sigma$, every normalized solution of* (2) *obeys the estimate*

$$C_1(E) L^{\gamma_1} \le \|u\|_L \le C_2(E) L^{\gamma_2} \tag{7}$$

*for $L > 0$ sufficiently large and suitable constants $C_1(E), C_2(E)$. Let $\alpha = 2\gamma_1/(\gamma_1+\gamma_2)$. Then $H$ has purely $\alpha$-continuous spectrum on $\Sigma$, that is, for any $\phi \in \ell^2$, the spectral measure for the pair $(H,\phi)$ is purely $\alpha$-continuous on $\Sigma$. Moreover, if the constants $C_1(E), C_2(E)$ can be chosen independently of $E \in \Sigma$, then for any $\phi \in \ell^2$ of compact support, the spectral measure for the pair $(H,\phi)$ is uniformly $\alpha$-Hölder continuous on $\Sigma$.*

*Remark 2.* a) We have stated the theorem in “right half-line” form. Of course, there is an analogous “left half-line” version.

b) In particular, the intuition embodied in Remark 1 interpolates. For example, if one is able to establish uniform power-law bounds on the right half-line, then the resulting $\alpha$-continuity is independent of the potential on the left half-line. In this sense the more continuous half-line dominates and bounds the dimensionality of the whole-line problem from below. Note, however, that the naive rule “dim(whole-line) = max(dim(left half-line), dim right half-line))" is wrong. Indeed, using the analysis of sparse potentials by Jitomirskaya and Last in [12], one may construct examples where the two half-line problems have zero-dimensional spectrum (in a certain energy region) and the whole-line problem has one-dimensional spectrum.

c) By combining the results of [12] and the ideas we present to prove Theorem 1, one can prove analogs of this theorem for Jacobi matrices and Schrödinger operators in $L^2(\mathbb{R})$.

Our application of Theorem 1 is to Schrödinger operators with Sturmian potentials. That is, we shall consider the operators

$$[H_{\lambda,\theta,\beta}u](n) = u(n+1) + u(n-1) + \lambda v_{\theta,\beta}(n)u(n), \tag{8}$$

acting in $\ell^2(\mathbb{Z})$, along with the corresponding difference equation

$$(H_{\lambda,\theta,\beta} - E)u = 0. \tag{9}$$

Here

$$v_{\theta,\beta}(n) = \chi_{[1-\theta,1)}\big(n\theta + \beta \bmod 1\big),$$

with coupling constant $\lambda \in \mathbb{R} \setminus \{0\}$, irrational rotation number $\theta \in (0,1)$, and phase $\beta \in [0,1)$.

The family of operators $(H_{\lambda,\theta,\beta})$ is commonly agreed to model a one-dimensional quasicrystal. It provides a natural generalization of the Fibonacci family of operators which corresponds to rotation number $\theta = \theta_F = \frac{\sqrt{5}-1}{2}$, the golden mean. This model was introduced independently by two groups in the early 1980's [16,21] and has been studied extensively since. The review articles [3,25] recount the history of generalizations of the basic Fibonacci model and the results obtained for each of them.

Before stating the result, let us recall some basic notions from continued fraction expansion theory; we mention [15,17] as general references.

Given $\theta \in (0,1)$ irrational, we have an expansion

$$\theta = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

with uniquely determined $a_n \in \mathbb{N}$. The associated rational approximants $\frac{p_n}{q_n}$ are defined by

$$\begin{aligned} p_0 &= 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2}, \\ q_0 &= 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}. \end{aligned}$$

The number $\theta$ is said to have bounded density if

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i < \infty.$$

The set of bounded density numbers is uncountable but has Lebesgue measure zero.

**Theorem 2.** *Let $\theta$ be a bounded density number. Then for every $\lambda$, there exists $\alpha = \alpha(\lambda, \theta) > 0$ such that for every $\beta$ and every $\phi \in \ell^2(\mathbb{Z})$ of compact support, the spectral measure for the pair $(H_{\lambda,\theta,\beta}, \phi)$ is uniformly $\alpha$-Hölder continuous. In particular, $H_{\lambda,\theta,\beta}$ has purely $\alpha$-continuous spectrum.*

In the course of the proof of Theorem 2 we will establish solution estimates which allow us to exclude eigenvalues for *all* parameter values.

**Theorem 3.** *For every $\lambda, \theta, \beta$, the operator $H_{\lambda,\theta,\beta}$ has empty point spectrum.*

*Remark 3.* This is the final result on a question with a long history. Building upon Sütő [23,24], the paper [1] by Bellissard et al. proves zero-measure spectrum and hence absence of absolutely continuous spectrum for all parameter values. Moreover, the authors of [1] implicitly exclude eigenvalues for $\beta = 0$ and arbitrary $\lambda, \alpha$. Absence of eigenvalues for $\beta \neq 0$ is listed as an open problem. Various partial results have been obtained since; see [4] for detailed remarks on the history of the problem and the first result that holds uniformly in the phase. The main improvement in the present article will be discussed in Sect. 4.

Combining Theorem 3 with the results from [1] we obtain a complete identification of the spectral type.

**Corollary 11.** *For every $\lambda, \theta, \beta$, the operator $H_{\lambda,\theta,\beta}$ has purely singular continuous zero- measure spectrum.*

The organization of this article is as follows. Section 2 discusses the transition from half-line eigenfunction estimates to spectral properties of the whole-line operator and so proves Theorem 1. In Sect. 3 we present some crucial properties of Sturmian potentials. We recall in particular the unique decomposition property and the uniform bounds on the traces of certain transfer matrices. Section 4 provides a study of the scaling properties of solutions to (9) with respect to the decomposition of the potentials on various levels and shows how Theorem 3 follows from these scaling properties. Uniform upper and lower power-law bounds on $\|u\|_L$ for certain rotation numbers are established in Sect. 4. In Sect. 5 this information is then combined with Theorem 1 to prove Theorem 2.

## 2. Subordinacy Theory

In this section we demonstrate how the solution estimates discussed in the introduction may be used to prove $\alpha$-continuity of spectral measures for some $\alpha > 0$. Although we shall only be applying Theorem 1 to Sturmian potentials, we believe the result holds a broader interest. Moreover, it will cost us nothing in clarity to treat the operator

$$[Hu](n) = u(n+1) + u(n-1) + V(n)u(n)$$

with arbitrary potential $V\colon \mathbb{Z} \to \mathbb{R}$. To each such whole-line operator we associate two half-line operators, $H_+ = P_+^* H P_+$ and $H_- = P_-^* H P_-$, where $P_\pm$ denote the inclusions $P_+ : \ell^2(\{1, 2, \dots\}) \hookrightarrow \ell^2(\mathbb{Z})$ and $P_- : \ell^2(\{0, -1, -2, \dots\}) \hookrightarrow \ell^2(\mathbb{Z})$.

The spectral properties of $H, H_\pm$ are typically studied via the Weyl $m$-functions. For each $z \in \mathbb{C} \setminus \mathbb{R}$ we define $\psi^\pm(n; z)$ to be the unique solutions to

$$H\psi^\pm = z\psi^\pm, \quad \psi^\pm(0; z) = 1 \quad \text{and} \quad \sum_{n=0}^{\infty} |\psi^\pm(\pm n; z)|^2 < \infty.$$

With this notation we can define the Weyl functions by

$$m^+(z) = \langle \delta_1 | (H_+ - z)^{-1} \delta_1 \rangle = -\psi^+(1; z)/\psi^+(0; z),$$
$$m^-(z) = \langle \delta_0 | (H_- - z)^{-1} \delta_0 \rangle = -\psi^-(0; z)/\psi^-(1; z)$$

for each $z \in \mathbb{C} \setminus \mathbb{R}$. Here and elsewhere, $\delta_n$ denotes the vector in $\ell^2$ supported at $n$ with $\delta_n(n) = 1$. For the whole-line problem, the $m$-function role is played by the $2 \times 2$ matrix $M(z)$:

$$\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]^\dagger M(z) \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right] = \big\langle (a\delta_0 + b\delta_1) \big| (H - z)^{-1} (a\delta_0 + b\delta_1) \big\rangle.$$

Or, more explicitly,

$$\begin{aligned} M &= \frac{1}{\psi^+(1)\psi^-(0) - \psi^+(0)\psi^-(1)} \begin{bmatrix} \psi^+(0)\psi^-(0) & \psi^+(1)\psi^-(0) \\ \psi^+(1)\psi^-(0) & \psi^+(1)\psi^-(1) \end{bmatrix} \\ &= \frac{1}{1 - m^+ m^-} \begin{bmatrix} m^- & -m^+ m^- \\ -m^+ m^- & m^+ \end{bmatrix} \end{aligned}$$

with $z$ dependence suppressed. We define $m(z) = \operatorname{tr}\big(M(z)\big)$, that is, the trace of $M$. These definitions relate the $m$-functions to resolvents and hence to spectral measures. By pursuing these relations, one finds that:

$$\begin{aligned} m^\pm(z) &= \int \frac{1}{t - z} d\rho^\pm(t), \\ m(z) &= \int \frac{1}{t - z} d\Lambda(t), \end{aligned} \tag{10}$$

where $d\rho^+, d\rho^-$ are the spectral measures for the pairs $(H_+, \delta_1)$, $(H_-, \delta_0)$, respectively, and $d\Lambda$ is the sum of the spectral measures for the pairs $(H, \delta_0)$ and $(H, \delta_1)$. An immediate consequence of these representations is that each of the $m$-functions maps $\mathbb{C}^+ = \{x + iy : y > 0\}$ to itself.

The pair of vectors $\{\delta_0, \delta_1\}$ is cyclic for $H$; indeed, if $\phi$ is supported in $\{-N, \ldots, N, N+1\}$, then there exist polynomials $P_0, P_1$ of degree not exceeding $N$ such that $\phi = P_0(H)\delta_0 + P_1(H)\delta_1$. This may be proved readily, by induction, once it is observed that $\phi(-N), \phi(N+1)$ uniquely determine the leading coefficients of $P_0, P_1$, respectively.

Our immediate goal is to prove that $d\Lambda$ is uniformly $\alpha$- Hölder continuous. This will follow quickly from

**Theorem 4.** *Fix $E \in \mathbb{R}$. Suppose every solution of $(H - E)u = 0$ with $|u(0)|^2 + |u(1)|^2 = 1$ obeys the estimate*

$$C_1 L^{\gamma_1} \le \|u\|_L \le C_2 L^{\gamma_2} \tag{11}$$

*for $L > 0$ sufficiently large. Then*

$$\sup_\varphi \left| \frac{\sin(\varphi) + \cos(\varphi) m^+(E + i\epsilon)}{\cos(\varphi) - \sin(\varphi) m^+(E + i\epsilon)} \right| \le C_3 \epsilon^{\alpha - 1}, \tag{12}$$

*where $\alpha = 2\gamma_1/(\gamma_1 + \gamma_2)$.*

*Proof.* This result lies within the Gilbert–Pearson theory of subordinacy [6,7,14]. A concise proof is available in [11,12]. In this context, the $\varphi$ above corresponds to the choice of boundary conditions. □

**Corollary 21.** *Given a Borel set $\Sigma$, suppose that the estimate* (11) *holds for every $E \in \sigma(H)$ with $C_1, C_2$ independent of $E$. Then, given any function $m^- : \mathbb{C}^+ \to \mathbb{C}^+$, and any $E \in \Sigma$,*

$$|m(E+i\epsilon)| = \left| \frac{m^+(E+i\epsilon) + m^-(E+i\epsilon)}{1 - m^+(E+i\epsilon)m^-(E+i\epsilon)} \right| \le C_3\, \epsilon^{\alpha-1} \tag{13}$$

*for all $\epsilon > 0$. Consequently, $\Lambda(E)$ is uniformly $\alpha$-Hölder continuous at all points $E \in \Sigma$. In particular, $d\Lambda$ is $\alpha$-continuous on $\Sigma$.*

*Proof.* Fix $E \in \Sigma$ and $\epsilon > 0$. Then, by introducing new variables $z = e^{2i\varphi}$ and $\mu = (m^+ - i)/(m^+ + i)$, we may rewrite (12) as

$$\sup_{|z|=1} \left| \frac{1+\mu z}{1-\mu z} \right| \le C_3 \epsilon^{\alpha-1}.$$

Note that $\mathrm{Im}(m^+) > 0$ implies $|\mu| < 1$ and so $(1+\mu z)/(1-\mu z)$ defines an analytic function on $\{z : |z| \le 1\}$. The point $z = (i - m^-)/(i + m^-)$ lies inside the unit disk since $\mathrm{Im}(m^-) > 0$. The estimate (13) now follows from the maximum modulus principle and a few simple manipulations. This estimate and the representation (10) provide

$$\Lambda\big([E-\epsilon, E+\epsilon]\big) \le 2\epsilon \mathrm{Im}\big(m(E+i\epsilon)\big) \le 2C_3\, \epsilon^{\alpha} \quad \text{for all } E \in \Sigma, \epsilon > 0,$$

from which $\Lambda(E)$ is uniformly $\alpha$-Hölder continuous on $\Sigma$. □

*Remark 4.* If we permit $C_1, C_2$ to depend on $E$, the only consequence is that now $C_3$ depends on $E$ and so $\Lambda$ need not be *uniformly* Hölder continuous. However, $\alpha$- continuity is still guaranteed.

*Proof of Theorem 1.* Given $\phi \in \ell^2(\mathbb{Z})$ with compact support, the remarks preceding Theorem 4 show that the spectral measure for $\phi$ is bounded by $f(E)d\Lambda(E)$ for some polynomially bounded function $f(E)$. If $C_1, C_2$ are independent of $E$, then, by the above corollary, $d\Lambda$ is uniformly $\alpha$-Hölder continuous, and as $\Sigma$ is bounded, this implies that $f d\Lambda$ is also U$\alpha$H.

In the case that $C_1, C_2$ are permitted to depend on $E$, the remark above shows that $d\Lambda$ is $\alpha$-continuous. Given any $\phi \in \ell^2$, its spectral measure may be written as $f d\Lambda$ and so must be $\alpha$-continuous. □

## 3. Basic Properties of Sturmian Potentials

In this section we recall some basic properties of Sturmian potentials. For further information we refer the reader to [1,3,4,19,20]. We focus, in particular, on the decomposition of Sturmian potentials into canonical words, which obey recursive relations, and on known results on the traces of the transfer matrices associated to these words.

Fix some rotation number, $\theta$, and let $a_n$ denote the coefficients in its continued fraction expansion. Define the words $s_n$ over the alphabet $\mathcal{A} = \{0, 1\}$ by

$$s_{-1} = 1, \quad s_0 = 0, \quad s_1 = s_0^{a_1 - 1} s_{-1}, \quad s_n = s_{n-1}^{a_n} s_{n-2}, \; n \ge 2. \tag{14}$$

In particular, the word $s_n$ has length $q_n$ for each $n \ge 0$. By definition, $s_{n-1}$ is a prefix of $s_n$ for each $n \ge 2$. For later use, we recall the following elementary formula [4].

**Proposition 31.** *For each* $n \geq 2$, $s_n s_{n+1} = s_{n+1} s_{n-1}^{a_n - 1} s_{n-2} s_{n-1}$.

Thus, the word $s_n s_{n+1}$ has $s_{n+1}$ as a prefix. Note that the dependence of $a_n$, $p_n$, $q_n$, $s_n$ on $\theta$ is left implicit. Fix coupling constant $\lambda$ and energy $E$; then, for each $w = w_1 \dots w_n \in \mathcal{A}^n$, we define the transfer matrix $M(\lambda, E, w)$ by

$$M(\lambda, E, w) = \begin{bmatrix} E - \lambda w_n & -1 \\ 1 & 0 \end{bmatrix} \times \cdots \times \begin{bmatrix} E - \lambda w_1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{15}$$

If $u$ is a solution to (9), we have

$$U(n+1) = M\big(\lambda, E, v_{\theta,\beta}(1) \dots v_{\theta,\beta}(n)\big) U(1),$$

where

$$U(n) = \begin{bmatrix} u(n) \\ u(n-1) \end{bmatrix}.$$

When studying the power-law behavior of $\|u\|_L$, one can investigate as well the behavior of

$$\|U\|_L = \left( \sum_{n=1}^{\lfloor L \rfloor} \big\|U(n)\big\|^2 \; + \; (L - \lfloor L \rfloor) \big\|U(\lfloor L \rfloor + 1)\big\|^2 \right)^{\frac{1}{2}}, \tag{16}$$

where

$$\|U(n)\|^2 = |u(n)|^2 + |u(n-1)|^2,$$

since

$$\tfrac{1}{2} \|U\|_L^2 \leq \|u\|_L^2 \leq \|U\|_L^2. \tag{17}$$

Now, the spectrum of $H_{\lambda,\theta,\beta}$ is independent of $\beta$ [1] and can thus be denoted by $\Sigma_{\lambda,\theta}$. Let us define

$$\begin{aligned} x_n &= \operatorname{tr}\big(M(\lambda, E, s_{n-1})\big), \\ y_n &= \operatorname{tr}\big(M(\lambda, E, s_n)\big), \\ z_n &= \operatorname{tr}\big(M(\lambda, E, s_n s_{n-1})\big), \end{aligned}$$

with dependence on $\lambda$ and $E$ suppressed.

**Proposition 32.** *For every* $\lambda$, *there exists* $C_\lambda \in (1, \infty)$ *such that for every irrational* $\theta$, *every* $E \in \Sigma_{\lambda,\theta}$, *and every* $n \in \mathbb{N}$, *we have*

$$\max\{|x_n|, |y_n|, |z_n|\} \leq C_\lambda.$$

*Proof.* This result follows implicitly from [1]. It can be derived from the analysis in [1] by combining their bound on $|x_n|$ and $|y_n|$ with the fact that the traces obey the Fricke-Vogt invariant

$$x_n^2 + y_n^2 + z_n^2 - x_n y_n z_n = \lambda^2 + 4,$$

which was also shown in [1]. □

The words $s_n$ are now related to the sequences $v_{\theta,\beta}$ in the following way. For each pair $(\theta, n)$, every sequence $v_{\theta,\beta}$ may be partitioned into words such that each word is either $s_n$ and $s_{n-1}$. This uniform combinatorial property, together with the uniform trace bounds given in Proposition 3.2, lies at the heart of the results contained in this paper and its precursors [4, 5]. Let us make this property explicit.

**Definition 33.** Let $n \in \mathbb{N}_0$ be given. An $(n, \theta)$-partition of a function $f : \mathbb{Z} \longrightarrow \{0, 1\}$ is a sequence of pairs $(I_j, z_j)$, $j \in \mathbb{Z}$ such that:

i) the sets $I_j \subset \mathbb{Z}$ partition $\mathbb{Z}$;
ii) $1 \in I_0$;
iii) each block $z_j$ belongs to $\{s_n, s_{n-1}\}$; and
iv) the restriction of $f$ to $I_j$ is $z_j$. That is, $f_{d_j} f_{d_j+1} \dots f_{d_{j+1}-1} = z_j$.

Notice that $d_j$ is defined implicitly to be the left-hand endpoint of the interval $I_j$.

We will suppress the dependence on $\theta$ if it is understood to which $\theta$ we refer. In particular, we will write $n$-partition instead of $(n, \theta)$-partition. The unique decomposition property is now given in the following lemma which was proved in [4].

**Lemma 34.** *For every $n \in \mathbb{N}_0$ and every $\beta \in [0, 1)$, there exists a unique $n$-partition $(I_j, z_j)$ of $v_{\theta,\beta}$. Moreover, if $z_j = s_{n-1}$, then $z_{j-1} = z_{j+1} = s_n$. If $z_j = s_n$, then there is an interval $I = \{d, d + 1, \dots, d + l - 1\} \subset \mathbb{Z}$ containing $j$ and of length $l \in \{a_{n+1}, a_{n+1} + 1\}$ such that $z_i = s_n$ for all $i \in I$ and $z_{d-1} = z_{d+l} = s_{n-1}$.*

We finish this section with a short discussion of symmetry properties of the words $v_{\theta,\beta}$. This will show that the considerations below, based on a study of the operators $H_{\lambda,\theta,\beta}$ on the right half-line, could equally well be based on a study of the operators on the left half-line. This particularly implies that for all parameter values, given an energy in the spectrum, both at $+\infty$ and $-\infty$ every solution of (9) does not tend to zero.

For a finite word $w = w_1 \dots w_n$ over $\{0, 1\}$, define the reverse word $w^R$ by $w^R = w_n \dots w_1$ and for a word $w \in \{0, 1\}^{\mathbb{Z}}$, define the reverse word $w^R$ by $w^R = v$ with $v_n = w_{-n}$ for $n \in \mathbb{Z}$. It is not hard to show that every $v_{\theta,\beta}$ allows a unique $n$-$R$-partition [19]. Here, an $n$-$R$-partition is defined by replacing $s_{n-1}$ and $s_n$ by $s_{n-1}^R$ and $s_n^R$, respectively, in the definition of $n$-partition. Mimicking the proof of Lemma 5.1 in [5] with the norm replaced by the trace, immediately gives $x_n^R = x_n$, $y_n^R = y_n$ and $z_n^R = z_n$. Here, $x_n^R$, $y_n^R$ and $z_n^R$ are defined by replacing $s_{n-1}$, $s_n$ and $s_n s_{n-1}$ with their reverse words in the definition of $x_n$, $y_n$ and $z_n$, respectively. Thus, the analog of Proposition 3.2 holds for $x_n^R$, $y_n^R$, $z_n^R$ (in fact, this can also be established by remarking that the underlying trace map system is essentially unchanged by passing from $s_n$ to $s_n^R$). The $n$-$R$-partitions and the bound on the traces allow one to study the operators on the left half-line in exactly the same way as the operators on the right half-line are studied in the following two sections. Alternatively, it is possible to show that the map $R$ leaves the set $\overline{\{v_{\theta,\beta} : \beta \in [0, 1)\}} \subset \{0, 1\}^{\mathbb{Z}}$ invariant, where the bar denotes closure with respect to product topology [19]. This could also be used to show that the two half-lines are equally well accessible.

## 4. Scaling Behavior of Solutions

In this section, we use the trace bounds and the partition lemma to study the growth of $\|U\|_L$ for energies in the spectrum and normalized solutions to (9). For our purposes it will be sufficient to consider this quantity only for $L = q_{8n}$, $n \in \mathbb{N}$. In Lemma 4.1 below it is shown that this growth has a lower bound which is exponential in $n$. In particular, this will imply absence of eigenvalues as claimed in Theorem 3 and it will also be used in our proof of power-law (in $L$) lower bounds for certain rotation numbers which will be given in the next section.

**Lemma 41.** *Let $\lambda, \theta, \beta$ be arbitrary, $E \in \Sigma_{\lambda,\theta}$, and let $u$ be a normalized solution to* (9). *Then, for every $n \geq 8$, the inequality*

$$\|U\|_{q_n} \geq D_\lambda \|U\|_{q_{n-8}}$$

*holds, where*

$$D_\lambda^2 = 1 + \left[\tfrac{1}{2C_\lambda}\right]^2.$$

*Proof of Theorem 3.* It follows immediately from Lemma 4.1 that for all parameter values $\lambda, \theta, \beta$, the operator $H_{\lambda,\theta,\beta}$ has no eigenvalues. □

Before giving the proof of Lemma 4.1, let us recall a basic definition: A word $w = w_1 \dots w_n$ is conjugate to a word $v = v_1 \dots v_n$ if for some $i \in \{1, \dots, n\}$, we have $w_1 \dots w_n = v_i \dots v_n v_1 \dots v_{i-1}$, that is, if $w$ is obtained from $v$ by a cyclic permutation of its symbols.

To prove Lemma 4.1 we shall employ the mass-reproduction technique that was used in [2]. This technique is based on the two-block version of the Gordon argument from [8]. More explicitly we have

**Lemma 42.** *Fix $\lambda, \theta, \beta$. Suppose that $v_{\theta,\beta}(j) \dots v_{\theta,\beta}(j+2k-1)$ is conjugate to $(s_{n-1})^2$, $(s_n)^2$, or $(s_{n-1}s_n)^2$ for some $n \in \mathbb{N}$, $l \leq k$, and every $j \in \{1, \dots, l\}$. Let $E \in \Sigma_{\lambda,\theta}$. Then every normalized solution $u$ to* (9) *satisfies*

$$\|U\|_{l+2k} \geq D_\lambda \|U\|_l.$$

*Proof.* Consider some $j \in \{1, \dots, l\}$. By definition, we have

$$U(j+k) = M\big(\lambda, E, v_{\theta,\beta}(j) \dots v_{\theta,\beta}(j+k-1)\big)U(j),$$
$$\text{and} \quad U(j+2k) = M\big(\lambda, E, v_{\theta,\beta}(j) \dots v_{\theta,\beta}(j+2k-1)\big)U(j).$$

Since $v_{\theta,\beta}(j) \dots v_{\theta,\beta}(j+2k-1)$ is conjugate to a square, it is itself a square, and

$$U(j+2k) = \Big[M\big(\lambda, E, v_{\theta,\beta}(j) \dots v_{\theta,\beta}(j+k-1)\big)\Big]^2 U(j).$$

Hence, applying the Cayley–Hamilton theorem,

$$U(j+2k) - \mathrm{tr}\Big[M\big(\lambda, E, v_{\theta,\beta}(j) \dots v_{\theta,\beta}(j+k-1)\big)\Big]U(j+k) + U(j) = 0. \quad (18)$$

Moreover,

$$\Big|\mathrm{tr}\Big[M\big(\lambda, E, v_{\theta,\beta}(j) \dots v_{\theta,\beta}(j+k-1)\big)\Big]\Big| \leq C_\lambda. \quad (19)$$

Combining (18) and (19), we obtain

$$\max\,\big\{\|U(j+k)\|, \|U(j+2k)\|\big\} \geq \frac{1}{2C_\lambda}\|U(j)\| \quad (20)$$

for all $1 \le j \le l$. We can therefore proceed as follows,

$$\begin{aligned} \|U\|_{l+2k}^2 &= \sum_{m=1}^{l+2k} \|U(m)\|^2 \\ &= \sum_{m=1}^{l} \|U(m)\|^2 + \sum_{m=l+1}^{l+2k} \|U(m)\|^2 \\ &\ge \sum_{m=1}^{l} \|U(m)\|^2 + \left[\tfrac{1}{2C_\lambda}\right]^2 \sum_{m=1}^{l} \|U(m)\|^2 \\ &= \left(1 + \left[\tfrac{1}{2C_\lambda}\right]^2\right) \|U\|_l^2 . \end{aligned}$$

This proves the assertion. □

*Proof of Lemma 4.1.* We make use of the information provided by Lemma 3.4 and exhibit squares in the potentials which are suitable in the sense that they satisfy the assumption of Lemma 4.2. In fact, we shall show

$$\|U\|_{2(q_{n+1}+q_n)+q_{n-1}} \ge D_\lambda \|U\|_{q_{n-4}} \tag{21}$$

for all $\lambda, \theta, \beta$, all $E \in \Sigma_{\lambda,\theta}$, all solutions $u$, and all $n \ge 4$. Since $q_{n+4} \ge 2(q_{n+1} + q_n) + q_{n-1}$, this proves the assertion.

Fix $\lambda, \theta, \beta$ and some $n \ge 4$ and consider the $n$-partition of $v_{\theta,\beta}$. Since we want to exhibit squares close to the origin, we consider the following cases.

*Case 1. $z_0 = s_{n-1}$.* Applying (14) and Proposition 3.1, we see that this block is followed by $s_{n-1}^2 s_{n-4}$. We can therefore apply Lemma 4.2 with $l = q_{n-4}$ and $k = q_{n-1}$. This yields (21) and we are done in this case.

*Case 2. $z_0 = s_n$ and $z_1 = s_n$.* Proposition 3.1 yields that these two blocks are followed by $s_n s_{n-3}$. Lemma 4.2 now applies with $l = q_{n-3}$ and $k = q_n$.

*Case 3. $z_0 = s_n$ and $z_1 = s_{n-1}$.* Let $z'_j$ label the blocks in the $(n+1)$-partition of $v_{\theta,\beta}$. By uniqueness of the $n$-partition we therefore have $z'_0 = s_{n+1}$. Let us consider the following subcases.

*Case 3.1. $z'_1 = s_{n+1}$.* Similarly to Case 2, this implies that $z'_0 z'_1$ is followed by $s_{n+1}s_{n-2}$ and hence Lemma 4.2 applies with $l = q_{n-2}$ and $k = q_{n+1}$.

*Case 3.2. $z'_1 = s_n$.* It follows that $z'_2 = s_{n+1}$. Again we consider two subcases.
*Case 3.2.1. $z'_3 = s_n$.* Of course, this case can only occur if $a_{n+2} = 1$. We infer that $z'_4 = s_{n+1}$. But this implies that we have squares conjugate to $s_n s_{n+1}$ and Lemma 4.2 is applicable with $l = q_{n-1}$ and $k = q_n + q_{n+1}$. Hence, (21) also holds in this case.
*Case 3.2.2. $z'_3 = s_{n+1}$.* Let us consider the consequences of this particular case for the blocks in the $n$- partition. We have

$$z_0 z_1 \ldots z_{2a_{n+1}+4} = s_n s_{n-1} s_n s_n^{a_{n+1}} s_{n-1} s_n^{a_{n+1}} s_{n-1}. \tag{22}$$

Since $s_n$ is a prefix of $s_{n+1}$, this must be followed by $s_n$. We therefore have the sequence of blocks

$$s_n s_{n-1} s_n s_n^{a_{n+1}} s_{n-1} s_n^{a_{n+1}} s_{n-1} s_n,$$

where the site $1 \in \mathbb{Z}$ is contained in the leftmost block. Using Proposition 3.1 this can be rewritten as

$$s_n s_{n-1} s_n s_n^{a_{n+1}} s_{n-1} s_n^{a_{n+1}} s_n s_{n-2}^{a_{n-1}-1} s_{n-3} s_{n-2},$$

which can as well be interpreted as

$$s_n s_{n-1} s_n s_n^{a_{n+1}} s_{n-1} s_n s_n^{a_{n+1}} s_{n-2}^{a_{n-1}-1} s_{n-3} s_{n-2}.$$

Thus, Lemma 4.2 is applicable with $l = q_{n-3}$ and $k = q_n + q_{n+1}$ which closes Case 3.2.2. Between Cases 1, 2, and 3 we have covered all possible choices of $z_0, z_1$. □

*Remark 5.* While our analysis is similar in spirit to the analysis performed in [4], we want to note here that we were able to improve upon essential aspects. Not only are we now able to treat an arbitrary rotation number $\theta$ ([4] had to exclude the case $\limsup a_n = 2$), we are also able to restrict our attention to one half-line which is of course crucial since we are aiming at an application of Theorem 1. The improvement stems from our considering the triple $\{s_{n-1}, s_n, s_{n-1}s_n\}$ as being the set of "good" words. This allows us to conclude as in Case 3.2.2 which is not possible when one is only working with the pair $\{s_{n-1}, s_n\}$ of "good" words as was done in [4].

## 5. Power-Law Upper and Lower Bounds on Solutions

In this section we provide power-law bounds for $\|u\|_L$ in the case where the rotation number $\theta$ has suitable number theoretic properties. Recall that $a_n$ denote the coefficients in the continued fraction expansion of $\theta$ and $q_n$ denote the denominators of the canonical continued fraction approximants to $\theta$.

**Proposition 51.** *Let $\theta$ be such that for some $B < \infty$, $q_n \le B^n$ for every $n \in \mathbb{N}$. Then for every $\lambda$, there exist $0 < \gamma_1, C_1 < \infty$ such that for every $E \in \Sigma_{\lambda,\theta}$ and every $\beta$, every normalized solution $u$ of* (9) *obeys*

$$\|u\|_L \ge C_1 L^{\gamma_1} \tag{23}$$

*for $L$ sufficiently large.*

*Remark 6.* The set of $\theta$'s obeying the assumption of Proposition 5.1 has full Lebesgue measure [15].

*Proof.* The bound (23) can be derived from the exponential lower bound on $\|U\|_{q_{8n}}$, $n \in \mathbb{N}$, given the exponential upper bound on $q_n, n \in \mathbb{N}$. Lemma 4.1 established the power-law bound for $L = q_{8n}$. It can then be interpolated to other values of $L$ (see [2] for details). □

**Proposition 52.** *Let $\theta$ be a bounded density number. Then for every $\lambda$, there exist $0 < \gamma_2, C_2 < \infty$ such that for every $E \in \Sigma_{\lambda,\theta}$ and every $\beta$, every normalized solution $u$ of* (9) *obeys*

$$\|u\|_L \le C_2 L^{\gamma_2} \tag{24}$$

*for all $L$.*

*Proof.* The proof is based upon local partitions and results by Iochum et al. [9,10]. Up to interpolation to non-integer $L$'s, it was given in [5]. □

*Remark 7.* It is easy to see that bounded density numbers obey the assumption of Proposition 5.1. Thus, if $\theta$ is a bounded density number, we have

$$C_1 L^{\gamma_1} \leq \|u\|_L \leq C_2 L^{\gamma_2}$$

with $\lambda$-dependent constants $\gamma_i, C_i$, uniformly for all energies from the spectrum, all phases $\beta$, and all normalized solutions of (9).

We are now fully prepared for the

*Proof of Theorem 2.* We employ Theorem 1. Propositions 5.1 and 5.2 provide the estimate (11) for each $E$ in the spectrum $\Sigma_{\lambda,\theta}$ of $H_{\lambda,\theta,\beta}$. This set is bounded because the potential is bounded and hence, so is the operator $H_{\lambda,\theta,\beta}$. Of course, the spectral measure for the pair $(H_{\lambda,\theta,\beta}, \phi)$ is supported by $\Sigma_{\lambda,\theta}$ and so must be uniformly $\alpha$-Hölder continuous. □

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