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Tropical Refined Curve Counting with Descendants

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Abstract: We prove a q-refined tropical correspondence theorem for higher genus descendant logarithmic Gromov–Witten invariants with a λ_g class in toric surfaces. Specifically, a generating series of such logarithmic Gromov–Witten invariants agrees with a q-refined count of rational tropical curves satisfying higher valency conditions. As a corollary, we obtain a geometric proof of the deformation invariance of this tropical count. In particular, our results give an algebro-geometric meaning to the tropical count defined by Blechman and Shustin. Our strategy is to use the logarithmic degeneration formula, and the key new technique is to reduce to computing integrals against double ramification cycles and connect these integrals to the non-commutative KdV hierarchy.

Contents

Introduction

We study the logarithmic Gromov–Witten theory of toric varieties, relative their full toric boundary. Our results continue a line of inquiry which connects Hodge integrals [51], the local Gromov–Witten theory of curves [7], and refined tropical curve counting [5]. Our main theorem is a tropical correspondence theorem which relates these

logarithmic Gromov–Witten invariants of toric surfaces to refined tropical enumerative invariants introduced in work of Blechman and Shustin [3]. The Gromov–Witten theory of threefolds is particularly interesting: it connects to sheaf-counting theories, and the dimension of the moduli spaces associated to the enumerative count are independent of genus. Our theorem can be viewed as a result for the "local surface threefolds" given by $X \times \mathbb{A}^1$, where X is a toric surface.

Fix a multiset Δ° of vectors in $\mathbb{Z}^2 \backslash \{(0, 0)\}$ with sum zero, together with non-negative integers n and $k_1, \ldots, k_n$ such that

$$n - 1 + |\Delta^\circ| = 2n + \sum_{i=1}^{n} k_i.$$

Associated to this discrete data are two enumerative invariants.

(1) The datum Δ° determines a toric surface X_Δ and a curve class β_Δ on this toric surface. For every genus $g \geq 0$, there is an associated *logarithmic Gromov–Witten* invariant with λ_g insertion defined as the following intersection product

$$N_{g,\Delta}^{\mathbf{k}} = \int_{[\mathsf{M}_{g,\Delta}]^{\mathrm{vir}}} (-1)^g \lambda_g \prod_{i=1}^{n} \mathsf{ev}_i^\star(\mathsf{pt}) \, \psi_i^{k_i}.$$

Our notation is defined in Sect. 2.1. This logarithmic Gromov–Witten invariant captures information about algebraic curves passing through a generic collection of n points in X_Δ subject to stationary descendant constraints. The multiset Δ° controls the tangency constraints as well as the curve class corresponding to this Gromov–Witten invariant.

(2) Fixing a generic ordered tuple of n points $p = (p_1, \ldots, p_n)$ in $\mathbb{R}^2$, the data $(\Delta^\circ, \mathbf{k})$ defines a finite set of genus zero tropical curves $T_{\Delta, p}^{\mathbf{k}}$. To each tropical curve $h \in T_{\Delta, p}^{\mathbf{k}}$ we assign a rational function $m_h(q)$ of formal variable $q^{1/2}$. We define a count of tropical curves

$$N_{\mathrm{trop}}^{\Delta, \mathbf{k}}(q) = \sum_{h \in T_{\Delta, p}^{\mathbf{k}}} m_h(q).$$

The rational function $N_{\mathrm{trop}}^{\Delta, \mathbf{k}}(q)$ is an invariant defined in terms of polyhedral geometry.

A precise statement of the following theorem appears in Sect. 2.2.

Theorem A. *After the change of variables $q = e^{iu}$ we have the equality*

$$\sum_{g \geq 0} N_{g,\Delta}^{\mathbf{k}} \, u^{2g-2+|\Delta^\circ|-\sum_i k_i} = N_{\mathrm{trop}}^{\Delta, \mathbf{k}}(q).$$

The tropical curve count on the right of Theorem A can be computed in two steps.

(1) A combinatorial process to enumerate all tropical curves.
(2) Compute the multiplicity of each tropical curve as a product over its vertices.

The first step is possible through floor diagrams, lattice path algorithms [44] or with software [16]. Although tropical curves are studied in higher dimensions, we emphasise that, in dimension two, these techniques provide an efficient method to enumerate all tropical curves. See [3, Appendix, p. 26–28] for an explicit computation of $N_{\text{trop}}^{\Delta,\mathbf{k}}$ in a particular example using a lattice path algorithm.

Remarkably, the tropical invariants that arise in Theorem A were discovered through combinatorial considerations by Blechman–Shustin. Our work sheds light on several features. Notably, a priori $N_{\text{trop}}^{\Delta,\mathbf{k}}(q)$ depends on the choice of points p in $\mathbb{R}^2$. The left hand side of the equation in Theorem A does not depend on p. Blechman and Shustin showed that this deformation invariance imposes a severe restriction on $m_h(q)$ and were thus able to write down a multiple of $m_h(q)$ and so a multiple of $N_{\text{trop}}^{\Delta,\mathbf{k}}$, without establishing its relation to Gromov–Witten invariants. Theorem A thus provides a new proof of the following result of Blechman and Shustin [3].

Corollary B. *The count of tropical curves* $N_{\text{trop}}^{\Delta,k}$ *is independent of the choice of points* p.

Our results suggest a generalisation of the Blechman–Shustin multiplicity to tropical curves of genus greater than zero, see Sect. 0.4.4. The count of genus g tropical curves with this generalised multiplicity is independent of p.

0.1. Tropical correspondence theorems. Tropical geometry is a combinatorial shadow of algebro-geometric problems, well suited to capturing enumerative information. Mikhalkin pioneered the connection between tropical and enumerative geometry by establishing an equality between counts of algebraic curves in toric surfaces of fixed degree and genus and a weighted count of tropical curves, of the same degree and genus [44]. A number of subsequent results have exhibited this correspondence principle [11–13, 19, 20, 23, 38, 49]. Notably, Nishinou and Siebert [49] applied degeneration techniques to generalise Mikhalkin's result to counts of rational curves in toric varieties of arbitrary dimension. Contemporary results often connect tropical geometry to logarithmic Gromov–Witten invariants, which are readily accessible through these degeneration techniques.

A logarithmic Gromov–Witten invariant is defined as an intersection product on the moduli space of stable logarithmic maps and thus tautological cohomology classes on this moduli space may be used as *insertions* to define new invariants. Two flavours of tautological cohomology classes play an important role in the sequel: Chern classes of cotangent line bundles denoted ψ_i and the top Chern class of the Hodge bundle denoted λ_g. We recall the definition of these classes in Sect. 2.1.1.

Mikhalkin suggested that under a correspondence theorem ψ-classes should correspond to counts of tropical curves with high valence vertices [45]. Genus zero correspondences with descendants are known for $\mathbb{P}^2$, $\mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^1$ [14,26,41,56], with the most general results coming from [25,38]. More recently, there have been results for λ_g class insertions, but the relationship is more complicated and takes the form of a *refined tropical correspondence*. The significance of the λ_g class is that it allows us to pass from curve counting on a toric surface X, to curve counting on the associated local Calabi–Yau threefold $X \times \mathbb{A}^1$. On any threefold, the virtual dimension of the mapping space is independent of the genus, and in this case the associated invariants are exactly the logarithmic Gromov–Witten invariants of X with a λ_g insertion. By packaging these invariants in an appropriate generating function, one obtains refined curve counts.

Refined tropical correspondence theorems are an example of a ubiquitous phenomena in mathematics: *quantum* analogues to classical results. Such a result depends on a parameter q which recovers the classical result as $q \to 1$. In the setting of tropical correspondence theorems this began with [4]. The authors gave multiplicities of plane tropical curves, depending on a parameter q which refined the ordinary multiplicity appearing in the traditional tropical correspondence theorems. In classical situations, such as Severi degrees, enumerative invariants can be calculated via Euler characteristics of relative Hilbert schemes of points on planar curves. The authors of [22] suggested that refined tropical invariants corresponded to Severi degrees with χ_y genus in place of the Euler characteristic, see also [48].

Correspondence results involving λ_g class insertions and higher genus logarithmic Gromov–Witten theory appeared later in work of Bousseau [5]. This provides an alternative perspective on the algebro-geometric information encoded in q-refined tropical curve counts. Theorem A is parallel to the work of Bousseau, in that we provide the connection between logarithmic Gromov–Witten theory and the tropical curve counts of Blechman and Shustin [3], who extended the refined counts of Block–Göttsche to plane tropical curves with higher valence vertices.

0.2. Relationship to literature. The contribution of the present paper is to handle the simultaneous presence of λ_g and ψ class insertions. The place of our result in the literature may be summarised with the following diagram.

$$
\begin{array}{c}
[5] \\
\Big| {\scriptstyle (1)} \\
[3,21] \xrightarrow{(2)} \text{Present Paper} \xrightarrow{(3)} [25,38,41]
\end{array}
$$
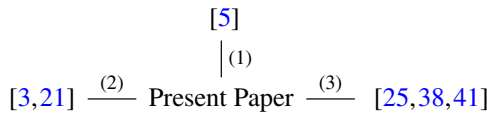
(1) We are not aware of how to generalise Bousseau's argument to the descendant setting, so we must perform certain atomic calculations directly. As a result we provide a new proof of the special case that there are no descendants: a theorem proved in Bousseau's work. Our proof uses three key ingredients. First, the degeneration formula for logarithmic Gromov–Witten theory. This was already used in Bousseau's work, but new subtleties appear in the descendant setting. Secondly, the connection between logarithmic Gromov–Witten invariants of toric surfaces and double ramification cycles [54]. Finally, we use the connection between intersections against double ramification cycles and the KdV hierarchy [8,9].
(2) Our theorem provides an algebro-geometric interpretation of the tropical count defined by Blechman and Shustin. Their invariants, though purely combinatorial, are part of a natural system that contain the geometric refinements of rational curve counts. The generalization of Blechman–Shustin's work to higher genus is a topic of ongoing interest.
(3) Due to the nature of our correspondence theorem, the tropical curves arising in our computations coincide with the ones considered by Markwig–Rau [41]. Our multiplicities coincide with Markwig–Rau's multiplicities when q approaches 1. This is a combinatorial statement. From a geometric viewpoint, the $q \to 1$ specialization simply recovers the correspondence theorems of [25,38], in the special case of plane curves. The multiplicity of the tropical curves in our theorem split as a product over multiplicities assigned to vertices. This contrasts with the general case of [25,38].

0.3. Proof strategy: double-ramification cycles and integrable hierarchies. Degeneration arguments and logarithmic intersection theory allow us to build on [27,53] to prove a simple degeneration formula in our setting. This is parallel to the degeneration arguments of [5,38,49] but additional subtleties arise due to the intersection of the λ_g and ψ conditions. This reduces the proof of Theorem A to computing intersection products of the form

$$\int_{[\mathsf{M}_{g,\Delta}]^{\,\mathrm{vir}}} \lambda_g \, \mathsf{ev}^{\star}(\mathsf{pt}) \, \psi^k$$

which we call *vertex contributions.* Here pt denotes the cohomology class poincare dual to a generic point. These vertex contributions are also descendant logarithmic Gromov–Witten invariants of a toric surface with at most one λ_g class, albeit simpler, with a single point insertion and power of a ψ class.

In joint work with Ranganathan [54] the third author proved a result implying that vertex contributions could be expressed as intersection products on the moduli space $\overline{\mathcal{M}}_{g,n}$ of genus g curves

$$\int_{[\mathsf{M}_{g,\Delta}]^{\,\mathrm{vir}}} \lambda_g \, \mathsf{ev}_1^{\star}(\mathsf{pt}) \, \psi_1^k = \int_{\overline{\mathcal{M}}_{g,n+1}} \lambda_g \, \psi_1^k \mathsf{TC}_g(\Delta).$$

The class $\mathsf{TC}_g(\Delta)$ is the *toric contact cycle,* a higher rank generalisation of the double ramification cycle $\mathsf{DR}_g(\mathbf{a})$. The toric contact sometimes appears in literature, under the moniker *the double double ramification cycle*, see [31,33,46] for background. Both classes are recalled in Sect. 4.

Let Δ be the $2 \times (n + r)$ matrix defined by setting the first n columns zero and the final columns to be the elements of Δ°. When computing vertex contributions we may assume $n = 1$. Writing Δ^x, Δ^y for the rows of the matrix Δ, we establish

$$\lambda_g \cdot \mathsf{TC}_g(\Delta) = \lambda_g \cdot \mathsf{DR}_g(\Delta^x) \cdot \mathsf{DR}_g(\Delta^y)$$

in Proposition 4.5.5. The naive hope that $\mathsf{DR}_g(\Delta^x) \cdot \mathsf{DR}_g(\Delta^y) = \mathsf{TC}_g(\Delta)$ is false, and only true on the compact type locus, so correction terms are required. The content of Proposition 4.5.5 then is that λ_g annihilates the correction terms. See also [32] for a closely related statement. We are left to compute integrals of the form

$$I_{g,d}(a_1, \ldots, a_n; b_1, .., b_n) = \int_{\overline{\mathcal{M}}_{g,n+1}} \lambda_g \psi_1^d \mathsf{DR}_g(0, a_1, \ldots, a_n) \mathsf{DR}_g(0, b_1, \ldots, b_n).$$

$$(1)$$

These integrals are computed with techniques from the theory of integrable hierarchies. Buryak [8] constructed the double ramification hierarchy, whose hamiltonians are generating functions of these integrals. This double ramification hierarchy coincides with the non-commutative KdV hierarchy [9]. This KdV hierarchy is well understood, allowing us to write explicit formulas for generating functions of $I_{g,d}(a_1, \ldots, a_n; b_1, \ldots, b_n)$.

Example 0.3.1. Set

$$\Delta^{\circ} = \{(-1, 0), (-1, 0), (0, -1), (1, 0), (1, 0), (0, 1)\}, \quad \mathbf{k} = (0, 1, 0, 0)$$

and fix points $(p_1, p_2, p_3, p_4) = p$. See Fig. 1 for a tropical curve contributing to $N_{\mathrm{trop}}^{\Delta,\mathbf{k}}$ for a certain choice of p_i. The multiplicity of this tropical curve is a product over
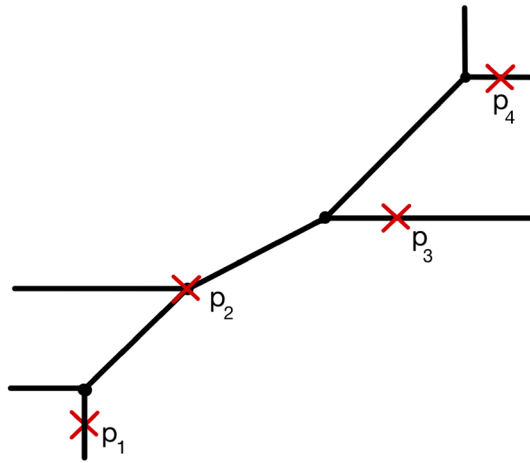
Fig. 1. Above is a tropical curve including edges with primitive directions $(1, 1)$ and $(2, 1)$

vertices. The vertex marked by p_2 contributes $\frac{1}{2}(q^{1/2} + q^{-1/2})$ and all other vertices contribute $(-i)(q^{1/2} - q^{-1/2})$. This tells us the total contribution of this tropical curve is $\frac{i}{2}(q^{1/2} + q^{-1/2})(q^{1/2} - q^{-1/2})^3$.

0.4. Future directions. The long term hope is to compute the descendant partition function of Gromov–Witten invariants of toric threefolds pairs in all genus. To achieve this goal, two generalisations of current results are required: passing to honest threefolds, rather than local surfaces, and higher genus tropical curves. Parker established the primary part of the above computation in the setting of his theory of exploded manifolds [52]. Even assuming the equivalence of logarithmic and exploded invariants, our results are new because we handle descendants.

0.4.1. Logarithmic Gromov–Witten/Donaldson–Thomas invariants The $q = e^{iu}$ change of variables appearing in Theorem A is the same change of variables that controls the logarithmic Gromov–Witten/Donaldson–Thomas correspondence [42,43]. Including descendants in the logarithmic Gromov–Witten/Donaldson–Thomas correspondence is a subtle problem, studied intensely by Moreira, Okounkov, Oblomkov, and Pandharipande [47,50]. Since the Donaldson–Thomas/ Pandharipande–Thomas wall-crossing is expected to be trivial in this setting, our calculations offer a concrete and testable prediction for these new conjectures.

0.4.2. Higher starting genus Fix a multiset Δ° of vectors in $\mathbb{Z}^2 \setminus \{(0, 0)\}$ with sum zero, together with non-negative integers g_0, n and $k_1, \ldots, k_n$ such that

$$n + g_0 - 1 + |\Delta^\circ| = 2n + \sum_{i=1}^{n} k_i.$$

For every genus $g \geq g_0$ we consider the following logarithmic Gromov–Witten invariant

$$N^{\mathbf{k}}_{g,g_0,\Delta} = \int_{[M_{g,\Delta}]^{\mathrm{vir}}} (-1)^{g-g_0} \lambda_{g-g_0} \prod_{i=1}^{n} \mathrm{ev}_i^{\star}(\mathrm{pt}) \, \psi_i^{k_i}.$$

The present paper establishes a tropical correspondence theorem in the the case $g_0 = 0$. The case $g_0 > 0$ without descendant insertions is known [5]. The difficulty with higher genus tropical curves of higher valence is that such curves may be superabundant [38, Remark 2.6]. Consequently, the degeneration and gluing arguments of Sect. 3 fail.

0.4.3. Weak Frobenius Structure Conjecture In Gross–Hacking–Keel's construction of the mirror to a log Calabi–Yau surface X [28], the mirror is constructed as the spectrum of an algebra of *theta functions*: the first example of a theta function is the unit. The *weak frobenius structure conjecture* asserts that the coefficient of the unit in any product of theta functions is a sum of genus zero descendant logarithmic Gromov–Witten invariants of X. The conjecture is known for cluster varieties [37], and Looijenga pairs satisfying [30, Assumptions 1.1] by [34].

A deformation quantization of the Gross–Hacking–Keel mirror, depending on a parameter q, is known [6]. Once again this involves the construction of an algebra, this time non-commutative, generated by *quantum theta functions*. Theorem A provides an avenue to explore the connection between the unit term in any product of quantum theta functions, and descendant logarithmic Gromov–Witten invariants with a λ_g insertion. Products of quantum theta functions are computed with quantum scattering diagrams, which one expects are related to refined counts of tropical curves. Upcoming work of Gräfnitz, Ruddat, Zaslow and Zhou offers progress in this direction [24].

0.4.4. Generalised Blechman–Shustin Our results suggest a generalisation of the multiplicities of Blechman and Shustin to higher genus tropical curves. Indeed, the degeneration arguments of Sect. 3 show that there is a tropical correspondence result for the logarithmic Gromov–Witten invariants

$$N^{\mathbf{k}}_{g,g_0,\Delta} = \int_{[\mathsf{M}_{g,\Delta}]^{\mathrm{vir}}} (-1)^{g-g_0} \lambda_{g-g_0} \prod_{i=1}^{n} \mathsf{ev}_i^{\star}(\mathsf{pt}) \, \psi_i^{k_i}.$$

On the tropical side one sums over tropical curves of genus g_0. One can define the generalised Blechman–Shustin multiplicity as the multiplicity of a tropical curve under such a correspondence theorem. Theorem A shows that the generalised Blechman–Shustin multiplicity coincides with the Blechman–Shustin multiplicity in the case $g_0 = 0$. A version of Corollary B shows counting tropical curves with the generalised Blechman–Shustin multiplicity is independent of the choice of p. This argument does not suggest that the generalised Blechman–Shustin multiplicity of a tropical curve Γ will be a product over vertices of Γ.

1. Tropical Enumerative Geometry

In this section we set up the tropical enumerative problem. Fix $\Delta^{\circ} = \{\delta_1, \ldots, \delta_r\}$ a multiset of non-zero vectors in $\mathbb{Z}^2$ with sum zero. Fix also an ordered tuple $\mathbf{k} = (k_1, \ldots, k_n)$ of n non–negative integers. In the main body, Δ° will record the tangency order of curves with the boundary and $\mathbf{k}$ will record the ψ class power attached to each of the n marked points.

In the sequel we partially record these data in a $2 \times (r + n)$ matrix denoted Δ. The first n columns are zero and the final r columns are the vectors δ_i. In writing down such a matrix we implicitly choose an order on the δ_i; we fix this order without further comment.

1.1. First definitions. Define the *lattice length* ℓ of a vector $\delta_i \in \mathbb{Z}^2$ to be the maximal positive integer ℓ such that one can write $\delta_i = \ell \bar{\delta}_i$ for $\bar{\delta}_i$ a vector in $\mathbb{Z}^2$ called the *direction* of δ_i.

1.1.1. Tropical curves Our definition of tropical curve coincides with the definition presented in [5, Section 2.3]. We refer the reader to [2,38,44,49] for background.

Definition 1.1.1. A graph Γ is a triple consisting of

(1) a finite set of vertices $V(\Gamma)$;
(2) a finite set $E_f(\Gamma)$ of pairs of elements of $V(\Gamma)$ called the *bounded edges*;
(3) and a multiset $E_\infty(\Gamma)$ of elements of $V(\Gamma)$ called the *unbounded edges*.

For us, all graphs are connected. This means for any two elements v, w of $V(\Gamma)$ there exists a sequence of elements $v = u_1, \ldots, u_k = w$ such that $\{u_i, u_{i+1}\}$ lies in $E_f(\Gamma)$ for all i.

An *abstract tropical curve* $|\Gamma|$ is the underlying topological space of a graph Γ.

Definition 1.1.2. A parametrised tropical curve $h : \Gamma \to \mathbb{R}^2$ consists of the following data.

(1) A graph Γ and a non–negative integer g_V assigned to each vertex V of Γ called the *genus*.
(2) A bijective function

$$L : E_\infty(\Gamma) \to \{1, \ldots, r + n\}.$$

(3) A *vector weight* $v_{V,E} \in \mathbb{Z}^2$ for every edge–vertex pair (V, E) with $E \in E_f(\Gamma) \cup E_\infty(\Gamma)$ and $V \in E$ such that for every vertex V, the following balancing condition is satisfied:

$$\sum_{E:V \in E} v_{V,E} = 0.$$

(4) For each bounded edge $E \in E_f(\Gamma)$ a positive real number $\ell(E)$, called the *length* of E.
(5) A map of topological spaces $h : |\Gamma| \to \mathbb{R}^2$ such that restricting h to the edge $\{v_1, v_2\}$ is affine linear to the line segment connecting $h(V_1)$ and $h(V_2)$ and moreover

$$h(V_2) - h(V_1) = \ell(E)v_{V_1,E}.$$

Also, restricting h maps an unbounded edge E associated to vertex V to the ray $h(V) + \mathbb{R}_{\geq 0}v_{V,E}$.

We say h has *degree* Δ if $v_{V,E}$ coincides with the $L(E)^{\text{th}}$ column of Δ whenever $E \in E_\infty(\Gamma)$. The *genus* of a parametrised tropical curve is obtained by adding the sum of g_V to the Betti number of $|\Gamma|$. The *weight* of an edge E, denoted $w(E)$, is the lattice length of $v_{V,E}$.

For a vertex V of Γ write $E_\infty^+(V)$ for the set of unbounded edges E adjacent to V such that $v_E \neq 0$ and $E_f(V)$ for the set of bounded edges adjacent to V. The *valency* val_V of a vertex V is the cardinality of $E_f(V) \cup E_\infty^+(V)$. Write Δ_V° for the multiset of all $v_{V,E}$ for fixed V.

1.2. Multiplicities. In the sequel we count parametrised genus zero tropical curves of degree Δ satisfying certain incidence conditions. Tropical curves are counted with a multiplicity, closely related to the multiplicity of [3]. This multiplicity is a product over multiplicities assigned to each vertex of our tropical curve. Fix for the remainder of the section a parametrised tropical curve $h : \Gamma \to \mathbb{R}^2$.

We will be counting tropical curves passing through a tuple of points $p = (p_1, \dots, p_n)$ in $\mathbb{R}^2$, and thus vertices of Γ come in two flavours. A vertex is *pointed* if its image under h coincides with one of the p_i. Vertices which are not pointed are *unpointed*.

1.2.1. Notation Given two elements $v_1, v_2 \in \mathbb{Z}^2$ we define $v_1 \wedge v_2$ to be the determinant of the matrix with first column v_1 and second column $v - 2$. Following [3], define functions of q

$$[v_1 \wedge v_2]_+ = q^{\frac{1}{2}v_1 \wedge v_2} + q^{-\frac{1}{2}v_1 \wedge v_2} \qquad [v_1 \wedge v_2]_- = q^{\frac{1}{2}|v_1 \wedge v_2|} - q^{-\frac{1}{2}|v_1 \wedge v_2|}.$$

The cyclic group with N elements acts on the set of ordered tuples of N distinct elements from the set $\{1, \dots, N\}$. The action is induced by sending the integer in position i to position $i + 1 \bmod N$. The set of orbits of this action is the set Ω_N of *cyclic permutations*. We sometimes write $\Omega_N = \Omega_N(a_1, \dots, a_N)$ when we wish to think of Ω_N as the set of cyclic permutations of vectors $(a_1, \dots, a_N)$. Where no confusion is likely we omit $(a_1, \dots, a_N)$ from the notation. For ω a cyclic permutation, choose an ordered tuple $\tilde{\omega}$ in the orbit ω. Define

$$k(\omega) = \sum_{2 \le i < j \le N} a_{\tilde{\omega}(i)} \wedge a_{\tilde{\omega}(j)}$$

where $\tilde{\omega}(i)$ sends i to the element in the ith position of the chosen representative $\tilde{\omega}$. As the vectors a_i will always have sum zero, $k(\omega)$ is well defined. Define also a function of formal variable q

$$\mu_N(a_1, \dots, a_N) = \sum_{\omega \in \Omega_N} q^{\frac{k(\omega)}{2}}.$$

1.2.2. Multiplicity of an unpointed vertex Let V be an unpointed trivalent vertex with outgoing vector weights the balanced set of vectors (a_1, a_2, a_3). The multiplicity assigned to V is

$$m_V = (-i) \cdot [a_1 \wedge a_2]_-.$$

1.2.3. Multiplicity of a pointed vertex A vertex V adjacent to edges with vector weights $a_1, \dots, a_N$ defines a balanced set of vectors $\Delta_V = (a_1, \dots, a_N)$ in $\mathbb{R}^2$. From this data we define a function of q

$$m_V = \frac{1}{(N-1)!} \mu_N(a_1, \dots, a_N),$$

which is the multiplicity of V.

1.2.4. Multiplicity of a tropical curve The multiplicity of the parametrised tropical curve $h : \Gamma \to \mathbb{R}$ is the product of m_V over vertices of Γ.

1.2.5. Blechman–Shustin multiplicity The Blechman–Shustin multiplicity of a (pointed) vertex V of valency N with outgoing edges of vector weight $a_1, \ldots, a_N$ denoted $\theta_N(a_1, \ldots, a_N)$ is defined recursively on the valency. When $N = 3$ they defined

$$\theta_3(a_1, a_2, a_3) = [a_1 \wedge a_2]_+$$

and then recursively

$$\theta_N(a_1, \ldots, a_N) = \sum_{i<j} \theta_{N-1}(a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_N, a_i + a_j)\theta_3(a_i, a_j, -(a_i + a_j)).$$

Here a hat denotes omission.

1.2.6. Formula for Blechman–Shustin multiplicity We provide an explicit formula for the Blechman–Shustin multiplicity.

Proposition 1.2.1. *For all choices of a_i, there is an equality*

$$\theta_N(a_1, \ldots, a_N) = \frac{N!}{3!} \mu_N(a_1, \ldots, a_N).$$

The proof of Proposition 1.2.1 was communicated to us by Thomas Blomme.

Proof. The proof proceeds by induction on N. When $N = 3$ there are two cyclic orders which have representatives $\omega_1 = (3, 1, 2)$ and $\omega_2 = (3, 2, 1)$. We thus learn

$$\mu_3(a_1, a_2, a_3) = q^{\frac{1}{2}a_1 \wedge a_2} + q^{-\frac{1}{2}a_1 \wedge a_2} = \theta_3(a_1, a_2, a_3).$$

For the inductive step assume $\theta_{N-1}(a_1, \ldots, a_{N-1}) = \frac{(N-1)!}{3!} \mu_{N-1}(a_1, \ldots, a_{N-1})$. Let W_N be the set of ordered pairs (i, j) where $1 \le i < j \le N$. Define Ω to be the set of cyclic permutations of $(a_1, \ldots, a_N)$ and $\Omega_{i,j}$ the set of cyclic permutations of

$$(a_1, \ldots, \hat{a}_i, \ldots, \hat{a}_j, \ldots, a_N, a_i + a_j)$$

where hat denotes omission. We now proceed with the following chain of equalities.

$$\theta_N(a_1, \ldots, a_N) = \sum_{(i,j) \in W_N} \left(q^{\frac{1}{2}a_i \wedge a_j} + q^{\frac{1}{2}a_j \wedge a_i} \right) \theta_{N-1}(a_1, \ldots \hat{a}_i, \ldots \hat{a}_j, \ldots, a_i + a_j)$$

$$= \frac{(N-1)!}{3!} \sum_{(i,j) \in W_N} \left(q^{\frac{1}{2}a_i \wedge a_j} + q^{\frac{1}{2}a_j \wedge a_i} \right) \mu_{N-1}$$

$$(a_1, \ldots \hat{a}_i, \ldots \hat{a}_j, \ldots, a_i + a_j)$$

$$= \frac{(N-1)!}{3!} \sum_{(i,j) \in W_N} \left(q^{\frac{1}{2}a_i \wedge a_j} + q^{\frac{1}{2}a_j \wedge a_i} \right) \sum_{\omega \in \Omega_{i,j}} q^{\frac{k(\omega)}{2}}$$

$$= \frac{(N-1)!}{3!} \sum_{(i,j) \in W_N} \sum_{\omega \in \Omega_{i,j}} \left(q^{\frac{k(\omega)+a_i \wedge a_j}{2}} + q^{\frac{k(\omega)-a_i \wedge a_j}{2}} \right)$$

In the sequel use $\Omega[i, j]$ for $i \ne j$ to denote the set of cyclic orders of $(a_1, \ldots, a_N)$ in which a_i, a_j are adjacent. There is a two to one map

$$\varphi : \Omega[i, j] \to \Omega_{i,j}$$

obtained by coupling a_i and a_j. The map is two to one to account for the two orders of a_i and a_j and moreover $k(\varphi(x)) = k(x) \pm a_i \wedge a_j$ where the sign depends on the order of a_i, a_j. We deduce,

$$
\begin{aligned}
\theta_n(a_1, \ldots, a_N) &= \frac{(N-1)!}{3!} \sum_{(i,j) \in W_N} \sum_{\omega \in \Omega[i,j]} q^{\frac{k(\omega)}{2}} \\
&= \frac{(N-1)!}{3!} \sum_{\omega \in \Omega} q^{\frac{k(\omega)}{2}} \mathrm{card}\{(i,j) : i < j, \omega \in \Omega[i,j]\} \\
&= \frac{N!}{3!} \sum_{\omega \in \Omega} q^{\frac{k(\omega)}{2}} \\
&= \frac{N!}{3!} \mu_N(a_1, \ldots, a_N).
\end{aligned}
$$

$\square$

1.3. Moduli of tropical curves and maps.

Following [40], note that assuming $n > 0$ a parametrised tropical curve of degree Δ

$$
h : \Gamma \to \mathbb{R}^2
$$

is specified by the following two data:

(1) the image of the first (necessarily contracted) unbounded edge $h(E_1)$;
(2) the data of Γ and the length of its edges.

The slope of each unbounded edge is determined by the degree of h and the slope of bounded edges are determined by the balancing condition.

For m a non–negative integer, there is a cone complex $\mathcal{M}_{0,m}^{\mathrm{trop}}$ whose points biject with abstract tropical curves Γ equipped with m unbounded ends [20, 36]. Datum (1) above is a point in $\mathbb{R}^2$ and the second datum is a point in $\mathcal{M}_{0,n+r}^{\mathrm{trop}}$. In this way $\mathbb{R}^2 \times \mathcal{M}_{0,n+r}^{\mathrm{trop}}$ is a moduli space of parametrised tropical curves [10].

There are n evaluation maps

$$
\mathrm{ev}_i : \mathcal{M}_{0,n+r}^{\mathrm{trop}} \times \mathbb{R}^2 \to \mathrm{Ev}^{\mathrm{trop}} = \mathbb{R}^2
$$

sending a parametrised tropical curve h to the image of the ith unbounded edge under h. This image is necessarily a single point. The cone complex $\mathcal{M}_{0,n+r}^{\mathrm{trop}}$ has a natural embedding into a vector space such that the evaluation maps are pulled back from linear functions [20].

1.3.1. Dimension of a cone

The *combinatorial type* of a tropical curve corresponding to a point p of $\mathbb{R}^2 \times \mathcal{M}_{0,n+r}^{\mathrm{trop}}$ is the data of the corresponding cone σ_p of $\mathcal{M}_{0,n+r}^{\mathrm{trop}}$. Consider now a parametrised tropical curve of degree Δ. Assume our tropical curve has valency 3 at all vertices except the vertices supporting one of the first n marked points $\{q_1, \ldots, q_n\}$ which are $\ell_i + 2$ valent for ℓ_i non–negative integers.

Lemma 1.3.1. *There is an equality*

$$\dim(\sigma_p) = r - 3 - \sum_i (\ell_i - 3)$$

Proof. The dimension of a cone in $\mathcal{M}_{0,n}^{\mathrm{trop}}$ is the number of bounded edges in a tropical curve of the corresponding combinatorial type. We induct on the number of vertices.

If there is a single vertex then there are no interior edges, r coincides with the valency and the equality reads

$$0 = r - 3 - (r - 3),$$

so the result is true.

For the inductive step suppose we add a vertex V_0 of valency k_0 to the graph by replacing an unbounded edge with a bounded one to V_0. This increases $\dim(\sigma_p)$ by one because there is one new unbounded edge. The value of r is increased by $k_0 - 2$ and so the right hand side of our equality increases by one overall. □

1.4. Tropical counting problem. Recall we have fixed a pair $(\Delta^\circ, \mathbf{k})$. In this section we associate a function of q to this data. The function of q is a count of tropical curves with multiplicity passing through a generic tuple of points.

1.4.1. Passing through points Let $p = (p_1, \ldots, p_k)$ be a tuple of points with $p_i \in \mathbb{R}^2$. A parametrised tropical curve h of degree Δ is said to *pass through p with degree* $\mathbf{k}$ if $h(E_i) = p_i$ for $i = 1, \ldots, n$ and E_i is attached to a vertex of valency at least $k_i + 2$. We say a parametrised tropical curve h through p of degree Δ is *rigid* if h is unique among its combinatorial type in having this property.

1.4.2. Weighted count of tropical curves For p a generic tuple of points in $\mathbb{R}^2$, let $T_{\Delta,p}^{\mathbf{k}}$ be the set of rigid parametrised tropical curves of degree Δ passing through p with degree $\mathbf{k}$.

Proposition 1.4.1. *There is an open dense subset U_n^k of $\mathsf{Ev}^{\mathrm{trop}}$ such that whenever $p \in U_n^k$ then $T_{\Delta,p}^{\mathbf{k}}$ is a finite set and the valency of the vertex supporting unbounded edge E_i is $k_i + 2$. Moreover we may choose U_n^k such that all parametrised tropical curves passing through p with degree $\mathbf{k}$ are rigid.*

In the sequel we assume $p_i \neq p_j$ whenever i and j are distinct without further comment.

Remark 1.4.2. Since there are only finitely many combinatorial types of rigid parametrised tropical curves of degree Δ, it is automatic that the set $T_{\Delta,p}^{\mathbf{k}}$ is finite.

Proof. Let T be the set of points in $\mathcal{M}_{0,n} \times \mathbb{R}^2$ corresponding to degree Δ tropical curves passing through p with degree $\mathbf{k}$. Our task is to show that by choosing U_n^k generically we may ensure first $T = T_{\Delta,p}^{\mathbf{k}}$ and second each point of T lies in the interior of cones in which the vertex mapped to marked point p_i has valency precisely $k_i + 2$.

Since $\mathcal{M}_{0,n}^{\mathrm{trop}}$ has finitely many cones, it suffices to identify for each cone σ a dense open U_σ of $\mathsf{Ev}^{\mathrm{trop}}$ such that $T \cap \sigma = T_{\Delta,p}^{\mathbf{k}} \cap \sigma$ and if this set is not empty then the

combinatorial type of tropical curves corresponding to points in the interior of σ have valency $k_i + 2$ at the vertex supporting E_i.

We assume marked point q_i on Γ has valency $k_i' + 2$ for every tropical curve associated to a point in the interior of σ. If a point p of $\mathcal{M}_{0,n} \times \mathbb{R}^2$ corresponds to a tropical curve in $T_{\Delta,p}^{\mathbf{k}}$ then the vertex carrying marked point i must have valency at least k_i. Thus we may assume the vertex of Γ mapped to p_i has valency at least k_i. Assuming the set $\{p_i\}$ of marked points are distinct, Lemma 1.3.1 implies $\mathsf{dim}(\sigma) \leq N - 3 - \sum_i (k_i - 3)$. Thinking of σ as a cone embedded in $\mathbb{R}^\ell$, the restriction of Ev to σ then specifies a linear map $\mathbb{R}^{\mathsf{dim}(\sigma)} \to \mathbb{R}^{2n}$. For a generic choice of $\{p_i\}$ and for fixed σ whenever σ contains a point of T this linear map surjects. Rank-nullity gives a lower bound and the fact $k_i' \geq k_i$ for all i gives the following upper bound

$$N - 3 - \sum_i (k_i - 3) \geq N - 3 - \sum_i (k_i' - 3) \geq N - 3 - \sum_i (k_i - 3).$$

from which we deduce $k_i' = k_i$ for all i.   □

Definition 1.4.3. Recall notation m_V for multiplicities of vertex V defined in Sect. 1.2.3. Fix $p = (p_1, \ldots, p_n)$ in $U_n^{\mathbf{k}}$ and define

$$N_{\mathsf{trop}}^{\Delta,\mathbf{k}}(q) = \sum_{h \in T_{\Delta,p}^{\mathbf{k}}} \prod_{V \in V(\Gamma)} m_V(q).$$

Define also $N_{\mathsf{trop}}^{\Delta,\mathbf{k}}(1) = N_{\mathsf{trop}}^{\Delta,\mathbf{k}}$.

A priori the count $N_{\mathsf{trop}}^{\Delta,\mathbf{k}}(q)$ depends on the choice of p. We suppress this dependence from our notation as it is independent a posteriori.

1.5. Anatomy of tropical curves. We record properties of tropical curves which will be of later use.

Proposition 1.5.1. *The complement of the pointed vertices of a parametrised tropical curve h in $T_{\Delta,p}^{k}$ is a union of trees, each with a single unbounded edge.*

One may specify a component of this complement by the collection κ_i of vertices which lie in its closure. For each κ_i define a parametrised tropical curve

$$h_i : \Gamma_i \to \mathbb{R}^2$$

as follows. First define Γ_i from h by deleting from Γ every vertex not in κ_i, and also deleting every edge which has at least one end not in κ_i. The map h_i is the restriction of h to Γ_i.

Proof. All Γ_i are trees by [44, Proposition 4.19]. We are required to show Γ_i has precisely one unbounded edge which is not contracted by h_i. This is because unbounded edges contracted by h_i are the pointed vertices which are deleted in the proposition statement (see Fig. 2 for an example).

Assume that Γ_i has k contracted unbounded edges and ℓ unbounded edges which are not contracted. The moduli space of parametrised tropical curves with the same combinatorial type as Γ_i is written σ_i and has dimension $2k + \ell - 1$. Since h was rigid,
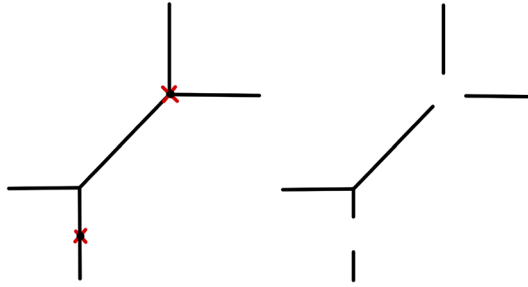
Fig. 2. Left a curve in $T_{\Delta,p}^{(1,0)}$ for p the two red crosses shown and $\Delta^\circ = \{(1,0), (-1,0), (0,1), (0,-1)\}$. Right, the union of trees with a single unbounded edges discussed in Proposition 1.5.1

so is h_i. In order for h_i to be rigid, the dimension of σ_i must equal two times the number of bounded edges. That is,

$$2k = 2k + \ell - 1.$$

Thus there is precisely one unbounded edge which is not contracted.   □

2. Precise Statement of Main Theorem

Fix $(\Delta^\circ, \mathbf{k})$ as in Sect. 1 and recall notation Δ for the associated matrix from the same section. Fixing a lattice direction ρ write n_ρ for the sum of the lattice lengths of the vectors in $\Delta^\circ = \{\delta_1, \ldots, \delta_r\}$ of direction ρ. Associated to $(\Delta^\circ, \mathbf{k})$ is the following data.

(1) Set X_Δ the unique toric surface corresponding under the toric dictionary to the fan with rays in the direction of the vectors of Δ°. See [18] for the toric dictionary.
(2) Set β_Δ the unique curve class on X_Δ whose intersection with the boundary divisor corresponding to the ray ρ is n_ρ.

We consider X_Δ as a logarithmic scheme with the divisorial logarithmic structure from its toric boundary.

2.1. Logarithmic Gromov–Witten Invariants. The moduli space parametrising $(r + n)$-pointed, genus g stable maps to X_Δ of degree β_Δ with the tangency to the toric boundary in the final r markings given by the vectors δ_i, $i = 1, \ldots r$ is not proper. The space of *stable logarithmic maps of type* Δ written $\mathsf{M}_{g,\Delta} = \mathsf{M}_{g,\Delta}^{\log}(X_\Delta|\partial X_\Delta, \beta_\Delta)$ is a compactification [1,15,29]. In the relative situation, such a moduli space exists for any logarithmically smooth proper morphism $X \to S$.

2.1.1. Moduli of curves Writing $\overline{\mathcal{M}}_{g,m}$ for the moduli space of stable genus g curves with m marked points. This space comes equipped with universal curve $p : \mathcal{C} \to \overline{\mathcal{M}}_{g,m}$. Since under the hypotheses of the introduction $m = n + r > 2$, there is a forgetful morphism

$$\pi : \mathsf{M}_{g,\Delta} \to \overline{\mathcal{M}}_{g,n+r}.$$

The moduli space of stable curves carries two flavours of tautological bundle of import to us.

- The *Hodge bundle* $\mathbb{E}_g = p_\star \omega_p$ where ω_p is the relative dualising sheaf of p. We write $\lambda_g = c_g(\mathbb{E}_g)$.
- Note $\overline{\mathcal{M}}_{g,n+r}$ carries $n+r$ tautological sections identifying the marked points. Denote the first n sections as $s_1, \ldots, s_n$ and define

$$\psi_i = c_1(s_i^\star \omega_p).$$

Both ψ_i and λ_g can be pulled back along π to define tautological classes on the moduli space of stable logarithmic maps.

Remark 2.1.1. The ψ_i and λ_g classes on the moduli space of logarithmic stable maps can instead be defined directly as Chern classes of tautological bundles on $\mathsf{M}_{g,\Delta}$. These definitions are equivalent: see [38, Proposition 3.4] for the case of ψ_i classes.

For the λ_g class, we will argue that the Hodge bundle on $\mathsf{M}_{g,\Delta}$ coincides with the pullback of the Hodge bundle on $\overline{\mathcal{M}}_{g,n+r}$ along π. Indeed write $q : C \to \mathsf{M}_{g,\Delta}$ for the domain universal curve and $p' = \pi^\star p : \pi^\star C \to \mathsf{M}_{g,\Delta}$ for the pullback of the universal curve from the moduli of stable curves. There is a stabilisation map $\mathsf{stab} : C \to \pi^\star C$ over $\mathsf{M}_{g,\Delta}$. In this notation, the promised identification of Hodge bundles is an isomorphism between $R^1 p'_\star \mathcal{O}_{\pi^\star C}$ and $R^1 q_\star \mathcal{O}_C$. The identification is immediate from two facts: first the five term exact sequence associated to the relative Leray spectral sequence for the functors $\mathsf{stab}_\star$ and $p'_\star$; second properties of stabilisation for a family of prestable curves [57, TAG 0E8A].

2.1.2. Evaluation maps For each of the $n + r$ marked points $\{q\}$ there is a tautological morphism

$$\mathsf{ev}_q : \mathsf{M}_{g,\Delta} \to X_\Delta$$

called the *evaluation morphism* associated to q. This morphism sends a stable logarithmic map to the image of q in X_Δ. Write $\mathsf{ev}_1, \ldots, \mathsf{ev}_n$ for evaluation maps at the first n sections.

2.1.3. Invariants The moduli space $\mathsf{M}_{g,\Delta}$ carries a virtual fundamental class $\left[\mathsf{M}_{g,\Delta}\right]^{\mathrm{vir}}$ allowing us to define *logarithmic Gromov–Witten invariants*. We will consider the following descendant logarithmic Gromov–Witten invariants with a λ_g insertion.

$$N_{g,\Delta}^{\mathbf{k}} = \int_{[\mathsf{M}_{g,\Delta}]^{\mathrm{vir}}} (-1)^g \lambda_g \prod_i \psi_i^{k_i} \mathsf{ev}_i^\star(\mathsf{pt}).$$

2.2. Main theorem. We are now ready to state our main theorem.

Theorem 2.2.1 (Theorem A). *Set* $q = e^{iu}$. *There is an equality*

$$\sum_{g \geq 0} N_{g,\Delta}^{\mathbf{k}} u^{2g+r-2-\sum_i k_i} = N_{\mathrm{trop}}^{\Delta, \mathbf{k}}(q).$$

Remark 2.2.2. Bousseau [5] proved a tropical correspondence result for integrals of the form

$$\int_{[\mathsf{M}_{g,\Delta}]^{\mathrm{vir}}} (-1)^{g-g_\Delta} \lambda_{g-g_\Delta} \prod_i \mathrm{ev}_i^\star(\mathrm{pt}).$$

We provide a new proof of the special case $g_\Delta = 0$ of Bousseau's result: set $\mathbf{k}$ to be the zero vector in Theorem A. Both our proof and Bousseau's work proceed by reducing to computing *vertex contributions*. Bousseau computes vertex contributions through a consistency argument. We are able to handle descendants because our computation of the vertex contributions in Sects. 4 and 5 involves intersection theory on the moduli space of stable curves. Furthermore, in this case, the vertex contributions are of the same form for $g_\Delta \neq 0$, so the same technique can be used to reprove Bousseau's general result.

3. Decomposition and Gluing

Fix once and for all a generic choice of points p in $\mathbb{R}^2$. By generic we mean $p \in U_n^{\mathbf{k}}$ from Proposition 1.4.1. The arguments of this section follow [5] and proceed in the following steps.

(1) Use the tropical curves in $T_{\Delta,p}^{\mathbf{k}}$ to build a toric degeneration of X_Δ, see Sect. 3.1.
(2) Appeal to the decomposition formula of [2] to turn a computation on the central fibre of this degeneration into a sum over the tropical curves, see Sect. 3.2.
(3) Use the gluing theorem of [53] to decompose the contribution of each tropical curve as a product of contributions from each vertex. See Sect. 3.4.

3.1. Toric degeneration. Following [5, 4.2], [49, Proposition 3.9] and [38, Lemma 3.1], the set of tropical curves $T_{\Delta,p}^{\mathbf{k}}$ determine a polyhedral decomposition $\mathcal{P}$ of $\mathbb{R}^2$ satisfying

- the asymptotic fan of $\mathcal{P}$ is the fan of X_Δ.
- The image of the vertices of any $h : \Gamma \to \mathbb{R}^2$ in $T_{\Delta,p}^{\mathbf{k}}$ are vertices of $\mathcal{P}$ and the image of any edges of h are a union of edges of $\mathcal{P}$.

Moreover by rescaling $\mathbb{R}^2$, we can assume $\mathcal{P}$ to be an integral polyhedral decomposition. This determines a degeneration

$$\nu : X_\mathcal{P} \to \mathbb{A}^1$$

with general fibre X_Δ and special fibre $X_0 = \cup_V X_V$ a union of components indexed by vertices of the decomposition $\mathcal{P}$. Since ν is toric, equipping $X_\mathcal{P}$ and $\mathbb{A}^1$ with the divisorial logarithmic structures from their respective toric boundaries, makes ν into a logarithmically smooth morphism.

Restricting to the central fibre, there is a logarithmically smooth morphism $X_0 \to \mathsf{pt}_\mathbb{N}$. Write $\mathsf{M}_{g,\Delta}(X_0/\mathsf{pt}_\mathbb{N})$ for the moduli space of stable logarithmic maps, where a family over a fine and saturated logarithmic scheme S is a commutative diagram

$$\begin{array}{ccc} C & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathsf{pt}_\mathbb{N} \end{array}$$

with C/S a logarithmic curve, and the contact order data is specified by Δ. The notation highlights that although the discrete data is unchanged, the target has been degenerated.

Each of the p_j determines a section of ν [5, Section 4.2]. The restriction of this section to the special fibre defines a point

$$i_{P^0} : (P_1^0, \ldots, P_n^0) \hookrightarrow X_0^n.$$

Define $\mathsf{M}_{g,\Delta}(X_0/\mathsf{pt}_{\mathbb{N}}, P^0)$ as the fibre product

$$
\begin{array}{ccc}
\mathsf{M}_{g,\Delta}(X_0/\mathsf{pt}_{\mathbb{N}}, P^0) & \longrightarrow & \mathsf{M}_{g,\Delta}(X_0/\mathsf{pt}_{\mathbb{N}}) \\
\downarrow & & \downarrow \\
P^0 = (P_1^0, \ldots, P_n^0) & \xrightarrow{\ i_{P^0}\ } & (X_0)^n.
\end{array}
$$

By deformation invariance of logarithmic Gromov–Witten invariants and [17, Example 6.3.4 (a)],

$$N_{g,\Delta}^{\mathbf{k}} = \int_{[\mathsf{M}_{g,\Delta}(X_0/\mathsf{pt}_{\mathbb{N}}, P^0)]^{\,\mathrm{vir}}} (-1)^g \lambda_g \prod_{i=1}^{n} \psi_i^{k_i}$$

where $[\mathsf{M}_{g,\Delta}(X_0/\mathsf{pt}_{\mathbb{N}}, P^0)]^{\mathrm{vir}} = i_{P^0}^! [\mathsf{M}_{g,\Delta}(X_0/\mathsf{pt}_{\mathbb{N}})]^{\mathrm{vir}}.$

3.2. Decomposition. We now use the decomposition formula of [2] to write $N_{g,\Delta}^{\mathbf{k}}$ of integrals indexed by the tropical curves in $T_{\Delta,p}^{\mathbf{k}}$.

3.2.1. Genus g from genus 0 All tropical curves in $T_{\Delta,p}^{\mathbf{k}}$ have genus zero, however the decomposition formula provides the integral as a sum over genus g tropical curves. Given $h : \Gamma \to \mathbb{R}^2 \in T_{\Delta,p}^{\mathbf{k}}$ we can build genus g tropical curves $\tilde{h} : \tilde{\Gamma} \to \mathbb{R}^2$ in two steps.

(1) Add a genus zero bivalent unpointed vertex to Γ at each point $h^{-1}(V)$ for V a vertex of $\mathcal{P}$.
(2) Distribute an assignment of genus $g_V \in \mathbb{N}$ to the vertices V of $\tilde{\Gamma}$ such that $\sum_{V \in \tilde{\Gamma}} g_V = g$.

We call the resulting set of parametrised tropical curves $T_{\Delta,p}^{g,\mathbf{k}}$.

3.2.2. Maps marked by $\tilde{h}$ Not all rigid parametrised tropical curves lie in $T_{\Delta,p}^{g,\mathbf{k}}$. For *any* rigid, genus g parametrised tropical curve $\tilde{h} : \tilde{\Gamma} \to \mathbb{R}^2$ passing through p, an n-pointed, genus g stable logarithmic map *marked by $\tilde{h}$* is the following data.

(1) An n-pointed genus g stable logarithmic map $f : C/\mathsf{pt}_{\mathcal{M}} \to X_0/\mathsf{pt}_{\mathbb{N}}$ of type Δ passing through P^0.
(2) For each vertex $V \in V(\tilde{\Gamma})$, an ordinary stable map $f_V : C_V \to X_{\Delta_V}$ of class β_{Δ_V} with marked points x_δ for any $\delta \in \Delta_V$ such that $f_V(x_\delta) \in D_\delta$, where D_δ is the toric divisor dual to δ.

These are subject to the following requirements. The underlying curve of C is isomorphic to the curve given by gluing the curves C_V along the points x_δ according to $\tilde{\Gamma}$. Moreover, under the isomorphism above, the scheme-theoretic morphism $C \to X_0$ obtained by gluing the morphisms f_V agrees with the underlying morphism of f.

The moduli space of n-pointed genus g stable logarithmic maps marked by $\tilde{h}$, denoted $\mathsf{M}^{\tilde{h}}_{g,\Delta}$ is a proper Deligne–Mumford stack with a natural perfect obstruction theory and a forgetful morphism

$$i_{\tilde{h}} : \mathsf{M}^{\tilde{h}}_{g,\Delta} \to \mathsf{M}_{g,\Delta}(X_0/\mathsf{pt}_{\mathbb{N}}, P^0).$$

3.2.3. From maps marked by $\tilde{h}$ to Gromov–Witten invariants For each $\tilde{h} \in T^{g,\mathbf{k}}_{\Delta,p}$ define

$$N^{\tilde{h},\mathbf{k}}_{g,\Delta} = \int_{[\mathsf{M}^{\tilde{h}}_{g,\Delta}]^{\mathrm{vir}}} (-1)^g \lambda_g \prod_{i=1}^{n} \psi_i^{k_i}$$

where we abuse notation by writing $\lambda_g = i^\star_{\tilde{h}} \lambda_g$ and $\psi_i = i^\star_{\tilde{h}} \psi_i$. The decomposition formula of [2] tells us that

$$N^{\mathbf{k}}_{g,\Delta} = \sum_{\tilde{h}} \frac{n_{\tilde{h}}}{|\mathsf{Aut}(\tilde{h})|} N^{\tilde{h},\mathbf{k}}_{g,\Delta}.$$

Here $n_{\tilde{h}}$ denotes the smallest positive integer such that $\tilde{h}$ has integral vertices after being rescaled by a factor of $n_{\tilde{h}}$. The number $|\mathsf{Aut}(\tilde{h})|$ is the order of the group of automorphisms of the parametrised tropical curve $\tilde{h}$.

Proposition 3.2.1. *There is an equality of rational numbers*

$$N^{k}_{g,\Delta} = \sum_{\tilde{h} \in T^{g,k}_{\Delta,p}} N^{\tilde{h},\mathbf{k}}_{g,\Delta}.$$

Proof. Any $\tilde{h} \in T^{g,\mathbf{k}}_{\Delta,p}$ is a rigid parametrised tropical curve. Since $\mathcal{P}$ is chosen to be integral all such curves have $n_{\tilde{h}} = 1$. Only tropical curves in $T^{g,\mathbf{k}}_{\Delta,p}$ contribute because all other rigid tropical curves have positive Betti number. The presence of the λ_g insertion ensures that such tropical curves do not contribute, see [5, Lemma 8]. Finally, there are no automorphisms of $\tilde{h}$ because it is rational. $\qquad\square$

3.3. Vertex contribution. Let $\tilde{h} : \tilde{\Gamma} \to \mathbb{R}^2$ be an element of $T^{g,\mathbf{k}}_{\Delta,p}$, there are four types of vertices of $\tilde{\Gamma}$.

(1) Trivalent unpointed vertices, coming from Γ, we call the set of these $V^3(\tilde{\Gamma})$.
(2) Bivalent pointed vertices, coming from Γ, we call the set of these $V^{2p}(\tilde{\Gamma})$.
(3) Bivalent unpointed vertices, not coming from Γ, we call the set of these $V^2(\tilde{\Gamma})$.
(4) m-valent ($m \geq 3$) pointed vertices, coming from Γ, we call the set of these $V^{mp}(\tilde{\Gamma})$.

Recall Δ_V° denotes the balanced multiset of vectors arising from edges adjacent to V. Write Δ_V for the matrix whose columns are the vectors $v_{V,E}$ for E *any* edge adjacent to V. Just as in Sect. 2, this determines a toric surface X_{Δ_V}, a curve class β_{Δ_V} and tangency conditions for β_{Δ_V} with respect to the toric boundary. If the elements of Δ_V° do not span $\mathbb{R}^2$, replace X_{Δ_V} with some toric compactification, the choice is unimportant.

Recall from Proposition 1.5.1 that the complement of the pointed vertices are trees with a single unbounded edge. Consequently we may choose a consistent orientation from pointed vertices to the unbounded edges. We now fix this orientation without further comment.

3.3.1. Trivalent unpointed contribution Let V be an unpointed trivalent vertex of $\tilde{\Gamma}$. For our fixed orientation X_{Δ_V} has two divisors oriented inwards D_1, D_2 with associated edges $E_V^{\mathsf{in},1}$, $E_V^{\mathsf{in},2}$ and one oriented outwards D_{out}. Let $\mathsf{M}_{g_V,\Delta_V}$ be the moduli space of stable logarithmic maps to X_{Δ_V} of genus g_V and of type Δ_V. We have evaluation morphisms with image in the toric boundary of X. We can therefore think of these evaluation maps as morphisms

$$(\mathsf{ev}_{\mathsf{in},1}, \mathsf{ev}_{\mathsf{in},2}, \mathsf{ev}_{\mathsf{out}}) : \mathsf{M}_{g_V,\Delta_V} \to D_1 \times D_2 \times D_{\mathsf{out}}.$$

Define the *trivalent unpointed contribution*

$$N_{g_V,V} = \int_{[\mathsf{M}_{g_V,\Delta_V}]^{\mathsf{vir}}} (-1)^{g_V} \lambda_{g_V}\, \mathsf{ev}_{\mathsf{in},1}^\star(\mathsf{pt}_{D_1})\, \mathsf{ev}_{\mathsf{in},2}^\star(\mathsf{pt}_{D_2}).$$

3.3.2. Bivalent pointed contribution Let V be a pointed bivalent vertex of $\tilde{\Gamma}$. Let $\mathsf{M}_{g_V,\Delta_V}$ be the moduli space of stable log maps to X_{Δ_V} of genus g_V and of type Δ_V. We have an evaluation morphism $\mathsf{ev}_V : \mathsf{M}_{g_V,\Delta_V} \to X_{\Delta_V}$. Define the *bivalent pointed contribution*

$$N_{g_V,V} = \int_{[\mathsf{M}_{g_V,\Delta_V}]^{\mathsf{vir}}} (-1)^{g_V} \lambda_{g_V}\, \mathsf{ev}_V^\star(\mathsf{pt}).$$

3.3.3. Bivalent unpointed contribution Let V be a bivalent unpointed vertex of $\tilde{\Gamma}$. Let $\mathsf{M}_{g_V,\Delta_V}$ be the moduli space of stable log maps to X_{Δ_V} of genus g_V and of type Δ_V. The orientation defines a divisor D_{in} with associated edge E_{in}. Define the *bivalent unpointed contribution*

$$N_{g_V,V} = \int_{[\mathsf{M}_{g_V,\Delta_V}]^{\mathsf{vir}}} (-1)^{g_V} \lambda_{g_V}\, \mathsf{ev}_{\mathsf{in}}^\star(\mathsf{pt}_{D_{\mathsf{in}}}).$$

3.3.4. m-Valent pointed contribution Let V be an m-valent pointed vertex of $\tilde{\Gamma}$ $(m \geq 3)$. Since it is a higher valency pointed vertex there is a corresponding $k_i = m - 2 \geq 1$. Let $\mathsf{M}_{g_V,\Delta_V}$ be the moduli space of stable log maps to X_{Δ_V} of genus g_V and of type Δ_V. We have an evaluation morphism $\mathsf{ev}_V : \mathsf{M}_{g_V,\Delta_V} \to X_V$. Define the *m-valent pointed contribution*

$$N_{g_V,V} = \int_{[\mathsf{M}_{g_V,\Delta_V}]^{\mathsf{vir}}} (-1)^{g_V} \lambda_{g_V}\, \mathsf{ev}_V^\star(\mathsf{pt})\, \psi^{m-2}.$$

3.4. Gluing the vertices. Proposition 3.2.1 reduces computing $N_{g,\Delta}^{\mathbf{k}}$ to integrals over (the virtual class of) $\mathsf{M}_{g,\Delta}^{\tilde{h}}$. We now express these integrals as a product of the vertex contributions defined in Sect. 3.3. The intuitive picture is that curves mapping to X_{Δ_V} glue together to form a map from a curve to X_0. In [5], the author proves a gluing statement at the level of virtual classes on the locus where the curve does not map into the torus fixed points of any X_{Δ_V}. It is then shown that the λ_g insertion will kill any contribution supported away from this locus. This is no longer true by the same arguments in this situation due to the presence of higher valent vertices; the key being that [5, Lemma 17] is false in our situation. Instead, we use the gluing formula of [53]. In general, this gluing formula is somewhat different due to allowing target expansions, but in the special case of degeneration of surfaces with at worst triple points, then the naive gluing formula holds when cutting along a single edge [53, 6.5.3], and we will use this to prove that $N_{g,\Delta}^{\mathbf{k}}$ is equal to a product of the vertex contributions.

3.4.1. Notation For V a pointed vertex of $\tilde{\Gamma}$, there is an $i_V \in \{1, \dots, n\}$ such that $P_{i_V}^0 \in X_{\Delta_V} \subset X_0$. Let $\mathsf{M}_{g_V, \Delta_V}'$ be the fibre product

$$
\begin{array}{ccc}
\mathsf{M}_{g_V,\Delta_V}' & \longrightarrow & \mathsf{M}_{g_V,\Delta_V} \\
\downarrow & & \downarrow {\scriptstyle \mathsf{ev}_V} \\
P_{i_V}^0 & \hookrightarrow & X_{\Delta_V}
\end{array}
$$

As in Sect. 3.1, there is a virtual class $[\mathsf{M}_{g_V,\Delta_V}']^{\mathrm{vir}}$ on $\mathsf{M}_{g_V,\Delta_V}'$ given by Gysin pull-back which pushes forward to $\mathsf{ev}_V^*(\mathsf{pt}) \cap [\mathsf{M}_{g_V,\Delta_V}]^{\mathrm{vir}}$. For V an unpointed vertex, set $\mathsf{M}_{g_V,\Delta_V}' = \mathsf{M}_{g_V,\Delta_V}$

Observe that for each $E \in E_f(\tilde{\Gamma})$, there are two natural maps

$$
\prod_{V \in V(\tilde{\Gamma})} \mathsf{M}_{g_V,\Delta_V}' \to D_E
$$

because each edge is adjacent to two vertices. With this observation we build a map

$$
\mathsf{ev}_{(e)} : \prod_{V \in V(\tilde{\Gamma})} \mathsf{M}_{g_V,\Delta_V}' \to \prod_{E \in E_f(\tilde{\Gamma})} D_E^2.
$$

For $\tilde{h} \in T_{\Delta,p}^{g,\mathbf{k}}$ we define the map:

$$
\mathsf{cut} : \mathsf{M}_{g,\Delta}^{\tilde{h}} \to \prod_{V \in V(\tilde{\Gamma})} \mathsf{M}_{g_V,\Delta_V}',
$$

by partially normalizing the source curve of a stable logarithmic map marked by $\tilde{h}$ (See [5, p.36 − 37] for a detailed construction). Let $\kappa_{\tilde{h}} : \prod_{E \in E_f(\tilde{\Gamma})} D_E \to \prod_{E \in E_f(\tilde{\Gamma})} D_E^2$ be the diagonal embedding and similarly, for an edge $F \in E_f(\tilde{\Gamma})$, let $\kappa_F : D_F \to D_F \times D_F$ the diagonal embedding. We will use the same notation to refer to the cohomology class associated to the diagonal embedding.

Proposition 3.4.1. *There is an equality of Chow cycles*

$$\mathsf{cut}_\star([\mathsf{M}_{g,\Delta}^{\tilde h}]^{vir}) = \left(\prod_{E\in E_f(\tilde\Gamma)} w(E)\right)\left(\mathsf{ev}_{(e)}^\star(\kappa_{\tilde h}) \cap \prod_{V\in V(\tilde\Gamma)} [\mathsf{M}'_{gV,\Delta_V}]^{vir}\right).$$

Proof. We induct on $|V(\tilde\Gamma)|$. The base case $|V(\tilde\Gamma)| = 1$ is vacuous. We now assume the statement is true for $|V(\tilde\Gamma)| < m$ with $m \geq 2$. Let the source graph $\tilde\Gamma$ of $\tilde h$ have exactly m vertices. We cut at a bounded edge $F \in E_f(\tilde\Gamma)$ yielding two rigid parametrised tropical curves

$$\tilde h_1, \tilde h_2 : \tilde\Gamma_1, \tilde\Gamma_2 \to \mathbb{R}^2.$$

Associated to $\tilde h_1, \tilde h_2$ are moduli spaces $M_{\tilde h_i}$ of stable logarithmic maps marked by $\tilde h_i$. Consider the following diagram

By [53, p. 45 penultimate sentence] we have

$$\mathsf{c}_\star([\mathsf{M}_{g,\Delta}^{\tilde h}]^{\mathrm{vir}}) = w(F)\,\mathsf{ev}_F^\star(\kappa_F) \cap \left([\mathsf{M}_{\tilde h_1}]^{\mathrm{vir}} \times [\mathsf{M}_{\tilde h_2}]^{\mathrm{vir}}\right).$$

By the inductive hypothesis

$$(\mathsf{cut}^1)_\star([\mathsf{M}_{\tilde h_1}]^{\mathrm{vir}}) = \left(\prod_{E\in E_f(\tilde\Gamma_1)} w(E)\right)\mathsf{ev}_{\tilde h_1}^\star(\kappa_{\tilde h_1}) \cap \prod_{V\in V(\tilde\Gamma_1)} [\mathsf{M}'_{gV,\Delta_V}]^{\mathrm{vir}},$$

$$(\mathsf{cut}^2)_\star([\mathsf{M}_{\tilde h_2}]^{\mathrm{vir}}) = \left(\prod_{E\in E_f(\tilde\Gamma_2)} w(E)\right)\mathsf{ev}_{\tilde h_2}^\star(\kappa_{\tilde h_2}) \cap \prod_{V\in V(\tilde\Gamma_2)} [\mathsf{M}'_{gV,\Delta_V}]^{\mathrm{vir}}.$$

Combining with a diagram chase, we get

$$\mathsf{cut}_\star([\mathsf{M}_{g,\Delta}^{\tilde h}]^{\mathrm{vir}}) = \left(\prod_{E\in E_f(\tilde\Gamma)} w(E)\right)\mathsf{ev}_{(e)}^\star(\kappa_F \otimes 1 \otimes 1)\mathsf{ev}_{(e)}^\star(1 \otimes \kappa_{\tilde h_1} \otimes 1)\mathsf{ev}_{(e)}^\star$$

$$(1 \otimes 1 \otimes \kappa_{\tilde h_2}) \cap \prod_{V\in V(\tilde\Gamma)} [\mathsf{M}'_{gV,\Delta_V}]^{\mathrm{vir}}$$

$$= \left(\prod_{E\in E_f(\tilde\Gamma)} w(E)\right)\mathsf{ev}_{(e)}^\star(\kappa_{\tilde h}) \cap \prod_{V\in V(\tilde\Gamma)} [\mathsf{M}'_{gV,\Delta_V}]^{\mathrm{vir}}.$$

This completes the proof.   □

Proposition 3.4.2.

$$N_{g,\tilde{h}}^{\Delta,\mathbf{k}} = \prod_{E \in E_f(\tilde{\Gamma})} w(E) \left(\prod_{V \in V^3(\tilde{\Gamma})} N_{g_V,V} \prod_{V \in V^{2u}(\tilde{\Gamma})} N_{g_V,V} \prod_{V \in V^{2p}(\tilde{\Gamma})} N_{g_V,V} \prod_{V \in V^{mp}(\tilde{\Gamma})} N_{g_V,V} \right)$$

Proof. The integrand of $N_{g,\Delta}^{\tilde{h},\mathbf{k}}$ can be written

$$(-1)^g \lambda_g \prod_{i=1}^n \psi_i^{k_i} = \mathsf{cut}^\star \left(\prod_{V \in V(\tilde{\Gamma})} (-1)^{g_V} \lambda_{g_V} \prod_{i=1}^n \psi_i^{k_i} \right)$$

since λ_g classes can be glued [5, Lemma 7] and ψ classes pull back under gluing. Combining with proposition 3.4.1 we learn,

$$N_{g,\Delta}^{\tilde{h},\mathbf{k}} = \left(\prod_{E \in E_f(\tilde{\Gamma})} w(E) \right) \left(\int_{\prod_V [M'_{g_V,\Delta_V}]^{\mathrm{vir}}} \mathsf{ev}_{(e)}^\star(\kappa_{\tilde{h}}) \prod_{V \in V(\tilde{\Gamma})} (-1)^{g_V} \lambda_{g_V} \prod_{i=1}^n \psi_i^{k_i} \right)$$

By definition of the virtual class on M'_{g_V,Δ_V} we have, abusing notation, that

$$N_{g,\Delta}^{\tilde{h},\mathbf{k}} = \left(\prod_{E \in E_f(\tilde{\Gamma})} w(E) \right) \left(\int_{\prod_V [M_{g_V,\Delta_V}]^{\mathrm{vir}}} \prod_{V \in V^{mp}(\tilde{\Gamma}) \cup V^{2p}(\tilde{\Gamma})} \mathsf{ev}_V^\star(\mathsf{pt}) \, \mathsf{ev}_{(e)}^\star(\kappa_{\tilde{h}}) \right.$$
$$\left. \prod_{V \in V(\tilde{\Gamma})} (-1)^{g_V} \lambda_{g_V} \prod_{i=1}^n \psi_i^{k_i} \right) \tag{2}$$

Claim 3.4.3. In Eq. (2) we may replace $\mathsf{ev}_{(e)}^\star(\kappa_{\tilde{h}})$ by

$$\prod_{V \in V^3(\tilde{\Gamma})} \mathsf{ev}_{\mathrm{in},1}^\star(\mathsf{pt}_{D_1}) \cdot \mathsf{ev}_{\mathrm{in},2}^\star(\mathsf{pt}_{D_2}) \prod_{V \in V^{2u}(\tilde{\Gamma})} \mathsf{ev}_{\mathrm{in}}^\star(\mathsf{pt}_{D_{\mathrm{in}}}).$$

After proving Claim 3.4.3 the proposition follows from rearranging equation (2).

Proof. Certainly,

$$\mathsf{ev}_{(e)}^\star(\kappa_{\tilde{h}}) = \prod_{E \in E_f(\tilde{\Gamma})} (\mathsf{ev}_{V_E^s}^E)^\star \mathsf{pt}_E + (\mathsf{ev}_{V_E^t}^E)^\star \mathsf{pt}_E,$$

where V_E^s, V_E^t are the source and target vertices with respect to the orientation on $\tilde{\Gamma}$. By the induction argument of [5, Proposition 22], only the summand $(\mathsf{ev}_{V_E^t}^E)^\star \mathsf{pt}_E$ contributes to our integral. A minor adaptation of op cit is required in our situation: we must check it applies to edges whose source is an m-valent pointed vertex. Indeed, if $V \in V^{mp}(\tilde{\Gamma})$, and E an outgoing edge of $V = V^s$, then $(\mathsf{ev}_{V_E^s}^E)^\star \mathsf{pt}_E$ cannot contribute to the integral because the corresponding term would involve an integral of

$$\lambda_{g_V} \mathsf{ev}_V^\star \mathsf{pt} \cdot (\mathsf{ev}_{V_E^s}^E)^\star \mathsf{pt}_E \cdot \psi^{k_i}$$

but this has degree greater than the virtual dimension of M_{g_V,Δ_V}. □

□

Proposition 3.4.2 expresses $N_{g,\tilde{h}}^{\Delta,\mathbf{k}}$ as a product over all of the vertices of $\tilde{\Gamma}$, the graph appearing in $\tilde{h} : \tilde{\Gamma} \to \mathbb{R}^2 \in T_{\Delta,p}^{g,\mathbf{k}}$. On the other hand, we want to relate this to a product over the vertices of Γ - the graph underlying an element of $T_{\Delta,p}^{\mathbf{k}}$. In [5], the author proved that the contributions from the 'extra vertices' of $\tilde{\Gamma}$ exactly cancel out the contributions of the weights coming from the 'extra edges' of $\tilde{\Gamma}$. More precisely,

Proposition 3.4.4 ([5, Corollary 16]). *Assume* $\tilde{h} : \tilde{\Gamma} \to \mathbb{R}^2$ *is an element of* $T_{\Delta,p}^{g,\mathbf{k}}$.

(1) If there exists a bivalent vertex V *of* $\tilde{\Gamma}$ *with* $g_V \neq 0$ *then*

$$N_{g,\Delta}^{\tilde{h},\mathbf{k}} = 0.$$

(2) If $g_V = 0$ *for all the bivalent vertices* V *of* Γ, *then*

$$N_{g,\Delta}^{\tilde{h},\mathbf{k}} = \left(\prod_{E \in E_f(\Gamma)} w(E) \right) \left(\prod_{V \in V^3(\tilde{\Gamma})} N_{g_V,V} \right) \left(\prod_{V \in V^{\mathsf{mp}}(\tilde{\Gamma})} N_{g_V,V} \right).$$

4. Logarithmic Gromov–Witten Theory and Double Ramification Cycles

Following Proposition 3.4.4, computing $N_{g,\Delta}^{\tilde{h},\mathbf{k}}$ amounts to working out the m-valent pointed vertex contibutions defined in Sect. 3.3.4

$$N_{g_V,V} = \int_{[M_{g_V,\Delta_V}]^{\mathrm{vir}}} (-1)^{g_V} \lambda_{g_V} \, \mathsf{ev}_V^\star(\mathsf{pt})$$

as the trivalent contributions of Sect. 3.3.1 are calculated in [5].

These numbers are logarithmic Gromov–Witten invariants of a toric surface. In this section we show these Gromov–Witten invariants are equal to integrals of tautological classes on $\overline{\mathcal{M}}_{g,n}$ against a product of double ramification cycles. The crucial step for this result is using machinery developed in [54, Section 3.4] to turn integrals against the virtual class of the moduli space of stable logarithmic maps into integrals against the toric contact cycle, a higher rank generalisation of the double ramification cycle. We use the same technique to reprove Bousseau's formula for the trivalent unpointed vertex contribution.

4.1. The double ramification cycle. Fix an ordered tuple of integers with sum zero $\mathbf{a} = (a_1, \ldots, a_n)$. To define the double ramification cycle we study the space of maps to the unique proper toric variety of dimension one: $\mathbb{P}^1$. There is a moduli space $\mathsf{M}_{g,\mathbf{a}}^{\mathsf{rub}}(\mathbb{P}^1)$ of equivalence classes of relative stable map to $\mathbb{P}^1$ for which the ith marked point has contact order a_i [39]. Two maps are identified if there is a torus automorphism of $\mathbb{P}^1$ carrying one map to the other. There is a forgetful map

$$\pi : \mathsf{M}_{g,\mathbf{a}}^{\mathsf{rub}}(\mathbb{P}^1) \to \overline{\mathcal{M}}_{g,n}$$

which forgets the map and stabilises the underlying curve. The moduli space $\mathsf{M}_{g,\mathbf{a}}^{\mathsf{rub}}(\mathbb{P}^1)$ carries a virtual cycle $[\mathsf{M}_{g,\mathbf{a}}^{\mathsf{rub}}]^{\mathrm{vir}}$ [39]. The *double ramification cycle* $\mathsf{DR}(\mathbf{a})$ is the class

$$\pi_\star \left([\mathsf{M}_{g,\mathbf{a}}^{\mathsf{rub}}]^{\mathrm{vir}} \right) \in \mathsf{A}_{2g-3+n}(\overline{\mathcal{M}}_{g,n}).$$

4.2. Toric contact cycle. The toric contact cycle is the analogue of the double ramification cycle replacing $\mathbb{P}^1$ by a dimension r toric variety. The contact order of a marked point is now recorded by an integral vector in $\mathbb{Z}^r$. The tuple **a** is replaced by a $r \times n$ matrix with row sum zero. For us this matrix will always be Δ_V from the previous section.

4.2.1. Rubber stable maps There is a universal compactification $\mathbb{G}_{\mathsf{log}}^r$ of an r-dimensional torus, see [54] for background. This is a stack on the category of logarithmic schemes and admits every choice of two dimensional toric variety as a *subdivision*. See [54, Section 1.1] for the definition of subdivision. There is a moduli space $\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}^r)$ of stable logarithmic maps to rubber $\mathbb{G}_{\mathsf{log}}^r$ with contact data Δ_V tracking stable maps to $\mathbb{G}_{\mathsf{log}}^r$ up to the action of $\mathbb{G}_{\mathsf{log}}^r$, see [54,55].

Remark 4.2.1. The functor $\mathbb{G}_{\mathsf{log}}^r$ is not representable by an algebraic stack with logarithmic structure, however the moduli space $\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}^r)$ is a Deligne–Mumford stack with logarithmic structure.

4.2.2. Virtual fundamental class Consider the cartesian diagram,

$$
\begin{array}{ccc}
\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}^r) & \longrightarrow & \overline{\mathcal{M}}_{g,n} \\
\downarrow & & \downarrow{\scriptstyle 0 \times \cdots \times 0} \\
\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{trop}}^r) & \longrightarrow & \mathsf{Pic}_{g,n} \times \cdots \times \mathsf{Pic}_{g,n}.
\end{array}
$$

The lower horizontal map is the Abel Jacobi section and 0 denotes the section corresponding to the trivial line bundle. The fibre product in the bottom right involves r copies of $\mathsf{Pic}_{g,n}$ and is taken over $\overline{\mathcal{M}}_{g,n}$. The moduli space $\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}^r)$ admits a virtual fundamental class defined as

$$
[\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}^r)]^{\mathsf{vir}} := (0 \times \ldots \times 0)^{!}[\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{trop}}^r)]
$$

where upper shriek denotes refined Gysin pullback. See [39, pp. 22–23] for details in the case $r = 1$ and [54, Section 3.3.3] in general.

4.2.3. Toric contact cycle We define the *toric contact cycle*,

$$
\mathsf{TC}_g(\Delta_V) := \pi_\star([\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}]^{\mathsf{vir}}) \in \mathsf{A}^{rg}(\overline{\mathcal{M}}_{g,n}),
$$

where π denotes the stabilization morphism $\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}} \to \overline{\mathcal{M}}_{g,n}$. In the literature the toric contact cycle is sometimes called the double double ramification cycle, see [31,33,46] for background and development of the theory. In the sequel we will mean $r = 2$ when we say toric contact cycle. Setting $r = 1$ the toric contact cycle coincides with the double ramification cycle defined in Sect. 4.1, see [39].

4.3. Rubber and rigidified geometry. Each column vector δ_i of Δ_V determines a toric stratum Y_i of X_{Δ_V} for $i = 1, \dots, n$. The *rigid evaluation space* is the toric variety

$$\mathsf{Ev}_{\Delta_V} := Y_1 \times \cdots \times Y_n.$$

In the sequel we assume that the dense torus $T = (\mathbb{C}^\star)^2$ of X_{Δ_V} acts effectively on Ev_{Δ_V}. Thus we identify the dense torus of X_{Δ_V} as a subtorus T of the dense torus $\mathsf{Ev}_{\Delta_V}^\circ$ of Ev_{Δ_V}. There is a smooth toric compactification $\mathsf{Ev}_{\Delta_V}^{\mathsf{rub}}$ of $\mathsf{Ev}_{\Delta_V}^\circ / T$ such that the following proposition holds.

Proposition 4.3.1 ([54, Section 3.4, Proof of Theorem B]). *After possibly replacing* $\mathsf{Ev}_{\Delta_V}, \mathsf{M}_{g,\Delta_V}, \mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}$ *by a subdivision, there is a commutative diagram with the right hand square cartesian,*

$$
\begin{array}{ccccc}
\mathsf{M}_{g,\Delta_V} & \xrightarrow{\ \ p\ \ } & P & \xrightarrow{\ \ \mathsf{ev}\ \ } & \mathsf{Ev}_{\Delta_V} \\
& {\scriptstyle \epsilon}\searrow & {\scriptstyle \tilde{\epsilon}}\downarrow & \square & \downarrow{\scriptstyle \delta} \\
& & \mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\log}^2) & \xrightarrow[\mathsf{ev}_{\mathsf{rub}}]{} & \mathsf{Ev}_{\Delta_V}^{\mathsf{rub}}.
\end{array}
$$

Both $\tilde{\epsilon}$ and δ are flat and proper. The morphism δ is toric and on the level of tori restricts to the quotient map

$$\mathsf{Ev}_{\Delta_V}^\circ \to \mathsf{Ev}_{\Delta_V}^\circ / T.$$

Corollary 4.3.2. *For γ an element of $\mathsf{A}^\star(\mathsf{Ev}_{\Delta_V})$ there is an equality*

$$\epsilon_\star(p^\star \mathsf{ev}^\star(\gamma) \cap [\mathsf{M}_{g,\Delta_V}]^{\mathsf{vir}}) = \mathsf{ev}_{\mathsf{rub}}^\star(\delta_\star(\gamma)) \cap [\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\log}^2)]^{\mathsf{vir}}.$$

Proof. From [54, Theorem 3.3.2] we get $[\mathsf{M}_{g,\Delta_V}]^{\mathsf{vir}} = \epsilon^\star[\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\log}^2)]^{\mathsf{vir}}$. Now appeal to the fact that the cartesian square in Proposition 4.3.1 is Cartesian. $\qquad \square$

4.4. Vertex contributions as integrals on $\overline{\mathcal{M}}_{g,n}$. Write pt_{D_i} for the class of a point on D_i in $\mathsf{A}^\star(\mathsf{Ev}_{\Delta_V})$.

4.4.1. m-valent pointed vertices We require a preparatory Lemma. Let n be a natural number and choose a morphism

$$\varphi : \mathbb{G}_{\log}^2 \to \mathbb{G}_{\log}^{n-2} \text{ and write } \varphi' = \mathsf{id} \times \varphi : \mathbb{G}_{\log}^2 \to \mathbb{G}_{\log}^n.$$

Lemma 4.4.1. *There is a Cartesian square*

$$
\begin{array}{ccc}
\mathbb{G}_{\log}^n & \xrightarrow{\ \ p\ \ } & \mathbb{G}_{\log}^2 \\
\downarrow & \square & \downarrow \\
\mathbb{G}_{\log}^n / \varphi'(\mathbb{G}_{\log}^2) & \longrightarrow & \mathsf{Spec}(\mathbb{C}).
\end{array}
\tag{3}
$$

Proof. There is certainly a map

$$\alpha : \mathbb{G}_{\log}^n \to \mathbb{G}_{\log}^n / \varphi'(\mathbb{G}_{\log}^2) \times \mathbb{G}_{\log}^2$$

and our task is to verify this map is an isomorphism. For T an arbitrary test scheme equipped with sheaf of monoids M_T we have

$$\alpha_T : \mathsf{Hom}(T, \mathbb{G}_{\log}^n) = \Gamma(T, M_T)^n \to \mathsf{Hom}(T, \mathbb{G}_{\log}^n / \varphi'(\mathbb{G}_{\log}^2) \times \mathbb{G}_{\log}^2).$$

A point in the set on the right hand side is specified by an element of $(h_1, h_2) \in \Gamma(T, M_T)^n \times \Gamma(T, M_T)^2$, but tuples (h_1, h_2), (h_1', h_2') correspond to the same point of $\mathsf{Hom}(T, \mathbb{G}_{\log}^n / \varphi'(\mathbb{G}_{\log}^2) \times \mathbb{G}_{\log}^2)$ if and only if $h_2 = h_2'$ and $h_1^{-1} h_1'$ lies in the image of φ'. Elementary group theory shows α_T is a bijection and we deduce α is an isomorphism. $\square$

Lemma 4.4.2. *Let V be a pointed m-valent vertex. We then have*

$$\pi_\star(\mathsf{ev}_V^\star(\mathsf{pt}) \cap [\mathsf{M}_{g_V, \Delta_V}(\mathbb{G}_{\log}^2)]^{\mathsf{vir}}) = \mathsf{TC}_g(\Delta_V)$$

Proof. Since Ev_{Δ_V} is toric we may regard it as a subdivision of $\mathbb{G}_{\log}^k$ for some k. The map δ can then be understood as obtained from a quotient map

$$\delta^\circ : \mathbb{G}_{\log}^k \to \mathbb{G}_{\log}^k / \mathbb{G}_{\log}^2$$

by compatible subdivision of source and target.

We now think of δ° as the left vertical arrow of the diagram in Eq. (3). Choose a subdivision of every space in Eq. (3) such that, after replacing Ev_{Δ_V} by a subdivision, we have a cartesian square

$$
\begin{array}{ccc}
\mathsf{Ev}_{\Delta_V} & \xrightarrow{\ p\ } & W \\
\downarrow & \square & \downarrow{\scriptstyle \delta} \\
\mathsf{Ev}_{\Delta_V}^{\mathsf{rub}} & \longrightarrow & \mathsf{Spec}(\mathbb{C})
\end{array}
\qquad (4)
$$

Here W is a toric variety of dimension two. Replacing P and $\mathsf{M}_{g, \Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\log}^2)$ by a subdivision, we may concatenate the Cartesian squares in Eqs. 3 and 4,

$$
\begin{array}{ccccc}
P & \longrightarrow & \mathsf{Ev}_{\Delta_V} & \longrightarrow & W \\
\downarrow & \square & \downarrow & \square & \downarrow \\
\mathsf{M}_{g, \Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\log}^2) & \longrightarrow & \mathsf{Ev}_{\Delta_V}^{\mathsf{rub}} & \longrightarrow & \mathsf{Spec}(\mathbb{C}).
\end{array}
$$

By abstract nonsense we have built a cartesian square of logarithmic schemes which fits into the larger diagram

Let pt be the cohomology class dual to a point in the dense torus of W. There is an equality of Chow cycles

$$\bar{\epsilon}_\star q^\star \mathsf{pt} = s^\star r_\star \mathsf{pt} = 1.$$

Applying both sides to $[\mathsf{M}^{\mathsf{rub}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}}$ we obtain

$$\bar{\epsilon}_\star \left(q^\star \mathsf{pt} \cap \bar{\epsilon}^\star[\mathsf{M}^{\mathsf{rub}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}} \right) = \bar{\epsilon}_\star q^\star \mathsf{pt} \cap [\mathsf{M}^{\mathsf{rub}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}} = [\mathsf{M}^{\mathsf{rub}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}},$$

where the first equality is from the projection formula. To complete the proof, push this formula forward to $\overline{\mathcal{M}}_{g,n}$ along the map

$$\pi' : \mathsf{M}^{\mathsf{rub}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}}) \to \overline{\mathcal{M}}_{g,n}$$

and observe $\pi = \pi' \circ \epsilon$ and by use of [54, Theorem 3.3.2] we also have $\bar{\epsilon}^\star[\mathsf{M}^{\mathsf{rub}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}}$ $= t_\star[\mathsf{M}_{g,\Delta_V}]^{\mathsf{vir}}$.   □

4.4.2. Trivalent unpointed vertices

Lemma 4.4.3. *Let V be a trivalent unpointed vertex. Let $\vec{v}_1$ and $\vec{v}_2$ denote the primitive generators of the rays corresponding to D_1 and D_2.*

$$\pi_\star \left(\mathsf{ev}^\star(\mathsf{pt}_{D_1}) \mathsf{ev}^\star(\mathsf{pt}_{D_2}) \cap [\mathsf{M}_{gV,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}} \right) = |\vec{v}_1 \wedge \vec{v}_2|\mathsf{TC}_g(\Delta_V),$$

where π denotes the forget and stabilise morphism.

Proof. The proof involves explicit computation so we fix coordinates. Consider the morphism

$$\delta : \mathsf{Ev} \to \mathsf{Ev}^{\mathsf{rub}}.$$

Restricting to dense tori δ is the quotient map

$$\delta^\circ : (\mathbb{C}^\star)^3 \to (\mathbb{C}^\star)^3/(\mathbb{C}^\star)^2.$$

Here the action of $(\mathbb{C}^\star)^2$ is the action of the dense torus of X on its toric boundary strata. Therefore, on the level of cocharacters the map δ° is specified by quotienting by the column span W of the matrix with rows v_i. Passing to a subdivision if necessary, we may assume Ev is smooth. The cohomology class $\mathsf{pt}_{D_1} \cup \mathsf{pt}_{D_2}$ is poincare dual to the closure Z in Ev of $V(X-1, Y-1)$. Note Z is just a copy of $\mathbb{P}^1$.

Restricting δ° or δ to the dense torus in Z yields the map $\mathbb{C}^\star \to \mathbb{C}^\star$ which factors as

$$\mathbb{C}^\star \to (\mathbb{C}^\star)^3 \to \mathbb{C}^\star.$$

On the level of cocharacters these maps are

$$1 \mapsto (0, 0, 1) \text{ and } (a, b, c) \mapsto [(a, b, c)] \in Z^3/W.$$

This extends to a map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $|\vec{v}_1 \wedge \vec{v}_2|$ because this is the factor by which cocharacters are scaled. It follows from definitions that on the level of cycles

$$\delta_\star[Z] = |\vec{v}_1 \wedge \vec{v}_2|[\mathbb{P}^1].$$

The rest of the proof follows the steps in the proof of Proposition 4.4.2.   □

4.5. Double ramification and toric contact cycles. We have expressed vertex contributions as integrals over the toric contact cycle, and it remains to compute these integrals. In this section we establish Proposition 4.5.5 which asserts that in the presence of a λ_g class, the toric contact cycle is a product of double ramification cycles.

4.5.1. The compact type locus Define an open subscheme

$$j : \mathcal{M}_{g,n}^{\mathsf{ct}} \hookrightarrow \overline{\mathcal{M}}_{g,n}$$

called the *compact type* locus parametrising curves whose arithmetic and geometric genus coincide. The preimage of $\mathcal{M}_{g,n}^{\mathsf{ct}}$ in $\mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}^2)$ is denoted $k : \mathsf{M}_{g,\Delta_V}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}^2) \hookrightarrow \mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}^2)$. Similarly define

$$k_x : \mathsf{M}_{g,\Delta_V^x}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}) \hookrightarrow \mathsf{M}_{g,\Delta_V^x}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}) \text{ and } k_y : \mathsf{M}_{g,\Delta_V^y}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}) \hookrightarrow \mathsf{M}_{g,\Delta_V^y}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}).$$

Recall Δ_V is a $2 \times n$ matrix of balanced contact order data with rows Δ_V^x, Δ_V^y.

Proposition 4.5.1. *There is an equality in the Chow group of $\mathcal{M}_{g,n}^{\mathsf{ct}}$,*

$$j^\star \left(\mathsf{TC}_g(\Delta_V) - \mathsf{DR}_g(\Delta_V^x)\mathsf{DR}_g(\Delta_V^y) \right) = 0.$$

The proof of Proposition 4.5.1 requires preparatory lemmas. We have maps

$$\pi : \mathsf{M}_{g,\Delta_V}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}^2) \to \overline{\mathcal{M}}_{g,n}, \quad \pi^x : \mathsf{M}_{g,\Delta_V^x}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}) \to \overline{\mathcal{M}}_{g,n}, \quad \pi^y : \mathsf{M}_{g,\Delta_V^y}^{\mathsf{rub}}(\mathbb{G}_{\mathsf{log}}) \to \overline{\mathcal{M}}_{g,n}.$$

These maps restrict to define

$$\overline{\pi} : \mathsf{M}_{g,\Delta_V}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}^2) \to \mathcal{M}_{g,n}^{\mathsf{ct}}, \quad \overline{\pi}_x : \mathsf{M}_{g,\Delta_V^x}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}) \to \mathcal{M}_{g,n}^{\mathsf{ct}}, \quad \overline{\pi}_y : \mathsf{M}_{g,\Delta_V^y}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}) \to \mathcal{M}_{g,n}^{\mathsf{ct}}.$$

Consider the diagram

$$
\begin{array}{ccccc}
\mathsf{M}_{g,\Delta}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}^2) & \longrightarrow & \mathsf{M}_{g,\Delta^x}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}) \times \mathsf{M}_{g,\Delta^y}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}) & \longrightarrow & \mathcal{M}_{g,n}^{\mathsf{ct}} \times \mathcal{M}_{g,n}^{\mathsf{ct}} \\
\downarrow{\scriptstyle \pi} & \square & \downarrow{\scriptstyle \pi_x \times \pi_y} & \square & \downarrow{\scriptstyle 0 \times 0} \\
\mathcal{M}_{g,n}^{\mathsf{ct}} & \xrightarrow{\ \kappa\ } & \mathcal{M}_{g,n}^{\mathsf{ct}} \times \mathcal{M}_{g,n}^{\mathsf{ct}} & \longrightarrow & \mathsf{Pic}_{g,n}^{\mathsf{ct}} \times \mathsf{Pic}_{g,n}^{\mathsf{ct}}
\end{array} \quad . \quad (5)
$$

The first horizontal map is the diagonal embedding; the second is the Abel–Jacobi section on each factor. Define a class

$$[\mathsf{M}_{g,\Delta}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}^2)]^{\mathsf{vir}} = (0 \times 0)^! [\mathcal{M}_{g,n}^{\mathsf{ct}}].$$

Lemma 4.5.2.

$$\kappa^\star (\pi_\star^x [\mathsf{M}_{g,\Delta^x}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}})]^{\mathsf{vir}} \times \pi_\star^y [\mathsf{M}_{g,\Delta^y}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}})]^{\mathsf{vir}}) = \pi_\star [\mathsf{M}_{g,\Delta}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}^2)]^{\mathsf{vir}}$$

Proof. Applying [17, Theorem 6.4] to diagram 5 we learn

$$\kappa^! ([\mathsf{M}_{g,\Delta^x}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}})]^{\mathsf{vir}} \times [\mathsf{M}_{g,\Delta^y}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}})]^{\mathsf{vir}}) = [\mathsf{M}_{g,\Delta}^{\mathsf{rub,ct}}(\mathbb{G}_{\mathsf{log}}^2)]^{\mathsf{vir}}.$$

Push this equality forward along the map π to complete the proof.   □

Lemma 4.5.3. *There is an equality*

$$k^\star[\mathsf{M}^{\mathsf{rub}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}} = [\mathsf{M}^{\mathsf{rub,ct}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}},$$

and similarly for k_x, k_y.

Proof. We prove the statement for k and note k_x, k_y follow similarly. Consider the commutative diagram in which all squares are cartesian

Observe the map $\mathcal{M}^{\mathsf{ct}}_{g,n} \to \mathsf{M}^{\mathsf{rub}}_{g,\Delta}(\mathbb{G}^2_{\mathsf{trop}})$ is an open immersion, and thus the result follows from [17, Theorem 6.2 (b)]. $\qquad\square$

Lemma 4.5.4. *There is an equality in the Chow group of* $\overline{\mathcal{M}}_{g,n}$,

$$j^\star \pi_\star [\mathsf{M}^{\mathsf{rub}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}} = \overline{\pi}_\star [\mathsf{M}^{\mathsf{rub,ct}}_{g,\Delta_V}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}}.$$

Proof. Note $\mathsf{M}^{\mathsf{rub}}_{g,\Delta}(\mathbb{G}^2_{\mathsf{trop}})$ is an open subscheme in a subdivision $\tilde{\mathcal{M}}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$. We thus have a commutative diagram,

The result now follows combining Lemma 4.5.3 and [57, TAG 0EPD]. $\qquad\square$

Proof of Proposition 4.5.1. By Lemma 4.5.2 we know

$$\pi_\star[\mathsf{M}^{\mathsf{rub,ct}}_{g,\Delta}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}} - \pi^x_\star[\mathsf{M}^{\mathsf{rub,ct}}_{g,\Delta^x}(\mathbb{G}_{\mathsf{log}})]^{\mathsf{vir}} \cdot \pi^y_\star[\mathsf{M}^{\mathsf{rub,ct}}_{g,\Delta^y}(\mathbb{G}_{\mathsf{log}})^{\mathsf{vir}}] = 0$$

in the Chow group of $\mathcal{M}^{\mathsf{ct}}_{g,n}$. Combining with Lemma 4.5.4 this equation becomes

$$j^\star\left(\pi_\star[\mathsf{M}^{\mathsf{rub}}_{g,\Delta}(\mathbb{G}^2_{\mathsf{log}})]^{\mathsf{vir}} - \pi^x_\star[\mathsf{M}^{\mathsf{rub}}_{g,\Delta^x}(\mathbb{G}_{\mathsf{log}})]^{\mathsf{vir}} \cdot \pi^y_\star[\mathsf{M}^{\mathsf{rub}}_{g,\Delta^y}(\mathbb{G}_{\mathsf{log}})^{\mathsf{vir}}]\right) = 0.$$

Substituting in the definition of toric contact cycle and double ramification cycle, the proof is complete. $\qquad\square$

4.5.2. Connecting toric contact and double ramification cycles The key result of this subsection follows.

Proposition 4.5.5. *There is an equality in the Chow group of* $\overline{\mathcal{M}}_{g,n}$

$$\lambda_g \mathsf{TC}_g(\Delta_V) = \lambda_g \mathsf{DR}_g(\Delta_V^x)\mathsf{DR}_g(\Delta_V^y).$$

Proof. Our task is to establish

$$R = \lambda_g \left(\mathsf{TC}_g(\Delta_V) - \mathsf{DR}_g(\Delta_V^x)\mathsf{DR}_g(\Delta_V^y) \right) = 0.$$

Combining the excision sequence

$$\mathsf{A}_\star(Z) \to \mathsf{A}_\star(\overline{\mathcal{M}}_{g,n}) \to \mathsf{A}_\star(\mathcal{M}_{g,n}^{\mathsf{ct}})$$

with Proposition 4.5.1 we learn there is some class $R' \in \mathsf{A}_\star(Z)$ which pushes forward to $\mathsf{TC}_g(\Delta_V) - \mathsf{DR}_g(\Delta_V^x)\mathsf{DR}_g(\Delta_V^y)$. Observe $R = \lambda_g \cap \iota_\star R' = \iota_\star(j^\star\lambda_g \cap R')$. Since $j^\star\lambda_g = 0$ [5, Lemma 7,8] we deduce $R = 0$ and the proof is complete.   □

4.6. Vertex contributions and double ramification cycles. To finish this section we apply Proposition 4.5.5 to provide a new formula for the vertex contributions introduced in Sect. 3.3.

Lemma 4.6.1. *Let V be a trivalent unpointed vertex and let $\vec{v}_1$, $\vec{v}_2$ denote the first two columns of Δ_V.*

$$N_{g_V,V} = \mid \vec{v}_1 \wedge \vec{v}_2 \mid \int_{\overline{\mathcal{M}}_{g_V,3}} (-1)^{g_V} \lambda_{g_V} \mathsf{DR}_{g_V}(\Delta_V^x)\mathsf{DR}_{g_V}(\Delta_V^y)$$

Proof. Starting with the definition of the left hand side,

$$N_{g_V,V} = \int_{[M_{g_V,V}]^{\mathsf{vir}}} (-1)^{g_V} \lambda_{g_V} \mathsf{ev}_{\mathsf{in},1}^\star(\mathsf{pt}_{D_1}) \mathsf{ev}_{\mathsf{in},2}^\star(\mathsf{pt}_{D_2})$$

we apply Lemma 4.4.3 and the projection formula to obtain,

$$N_{g_V,V} = |\vec{v}_1 \wedge \vec{v}_2| \int_{\overline{\mathcal{M}}_{g_V,3}} (-1)^{g_V} \lambda_{g_V} \mathsf{TC}_{g_V}(\Delta_V).$$

The result now follows by Proposition 4.5.5.   □

Lemma 4.6.2. *We have an equality*

$$N_{g_V,V} = \int_{\overline{\mathcal{M}}_{g_V,m+1}} (-1)^{g_V} \lambda_{g_V} \psi_1^{m-2} \mathsf{DR}_{g_V}(\Delta_V^x)\mathsf{DR}_{g_V}(\Delta_V^y).$$

Proof. Similar to proof of Lemma 4.6.1, except we apply Lemma 4.4.2 in place of Lemma 4.4.3.   □

5. Results from the Theory of Double Ramification Hierarchies

Set d a positive integer and let $\mathbf{a}, \mathbf{b}$ be tuples of $d + 2$ integers with sum zero. Write $\mathbf{a}_0, \mathbf{b}_0$ the vectors formed from $\mathbf{a}, \mathbf{b}$ by prefixing zero. In this section we evaluate

$$I_{g,d}(\mathbf{a}; \mathbf{b}) = \int_{\overline{\mathcal{M}}_{g,d+3}} (-1)^g \lambda_g \psi_1^d \mathsf{DR}_g(\mathbf{a}_0)\mathsf{DR}_g(\mathbf{b}_0).$$

Combining with Lemma 4.6.2 for $d = m - 2$ we have computed the m-valent pointed vertex contribution $N_{g_V, V}$.

The geometric input in this computation are the WDVV relations on the Losev–Manin space [35]. Buryak and Rossi leveraged these equations to study generating series involving $I_{g,d}(\mathbf{a}; \mathbf{b})$ [8,9]. We use their analysis to understand $I_{g,d}(\mathbf{a}; \mathbf{b})$.

We set up notation. Throughout this section $u_{p,q}, p_b^a, e^{iy}, e^{ix}, \epsilon$ denote formal variables and we often write $u_{0,0} = u$. The heuristic is to study a function on $S^1 \times S^1$ expressed as a fourier series

$$u = u_{0,0} = \sum_{a,b\in\mathbb{Z}} p_b^a e^{i(ay+bx)} \text{ and its derivatives } \partial_x^i \partial_y^j u = u_{i,j}.$$

More formally, define a map

$$T : \mathbb{C}[[u_{\star,\star}, \epsilon]] \to \mathbb{C}[[p_\star^\star, e^{\pm ix}, e^{\pm iy}, \epsilon]] \text{ by sending } u_{k_1,k_2} \mapsto \partial_x^{k_1} \partial_y^{k_2} \left(\sum_{a,b} p_b^a e^{i(ay+bx)} \right).$$

For g an element of $\mathbb{C}[[p_\star^\star, \epsilon, e^{\pm ix}, e^{\pm iy}]]$ we write $\overline{g}$ for its e^{i0} coefficient considered an element of $\mathbb{C}[[p_\star^\star, \epsilon]]$. This is the algebraic incarnation of an integration map. Moreover set $T_0 : \mathbb{C}[[u_{\star,\star}, \epsilon]] \to \mathbb{C}[[p_\star^\star, \epsilon]]$ by composing T with projection to the coefficient of e^{i0}. The next definitions, motivated by the chain rule, complete our setup,

$$\partial_x, \partial_y : \mathbb{C}[[u_{\star,\star}, \epsilon]] \to \mathbb{C}[[u_{\star,\star}, \epsilon]]$$

$$\partial_x : f \mapsto \sum_{k_1,k_2\geq 0} u_{k_1+1,k_2} \frac{\partial f}{\partial u_{k_1,k_2}}, \qquad \partial_y : f \mapsto \sum_{k_1,k_2\geq 0} u_{k_1+1,k_2} \frac{\partial f}{\partial u_{k_1,k_2}}.$$

5.1. The quadratic double ramification integrable hierarchy. Buryak and Rossi study integrals $I_{g,d}(\mathbf{a}; \mathbf{b})$ in the context of integrable hierarchies. We introduce some language from this area so that we may extract Proposition 5.2.1.

5.1.1. Variational derivative Write W for the subspace of $\mathbb{C}[[p_\star^\star, \epsilon]]$ defined by image of T_0. The variational derivative of $\overline{g}$ in $W \subseteq \mathbb{C}[[p_\star^\star, \epsilon]]$ is defined by choosing f in $\mathbb{C}[[u_{\star,\star}, \epsilon]]$ such that $T_0(f) = \overline{g}$. Thinking of $\overline{g}$ as depending on u and its derivatives in this way, we may ask how $\overline{g}$ is affected by changes in $u_{i,j}$. This information is tracked with a *variational derivative* which we define through the formula

$$\frac{\delta \overline{g}}{\delta u} := \sum_{k_1,k_2\geq 0} (-1)^{k_1+k_2} \partial_x^{k_1} \partial_y^{k_2} \frac{\partial f}{\partial u_{k_1,k_2}}.$$

This definition is independent of the choice of f.

Lemma 5.1.1. *For $\bar{g} \in W$ there is an equality*

$$T\left(\frac{\delta \bar{g}}{\delta u}\right) = \sum_{a,b \in \mathbb{Z}} \frac{\partial \bar{g}}{\partial p_b^a} e^{-i(ay+bx)}.$$

Proof. Recall that a bar over a symbol means take coefficient of e^{i0}. We write $\bar{g} = T_0(f)$. Applying the chain rule we learn

$$\sum_{a,b \in \mathbb{Z}} \frac{\partial (T_0(f))}{\partial p_b^a} e^{-i(ay+bx)} = \sum_{k_1,k_2 \geq 0} \sum_{a,b \in \mathbb{Z}} \overline{\frac{\partial T(u_{k_1,k_2})}{\partial p_b^a} T\left(\frac{\partial f}{\partial u_{k_1,k_2}}\right)} e^{-i(ay+bx)}$$

$$= \sum_{k_1,k_2 \geq 0} \sum_{a,b \in \mathbb{Z}} \overline{\partial_x^{k_1} \partial_y^{k_2}(e^{i(ay+bx)}) T\left(\frac{\partial f}{\partial u_{k_1,k_2}}\right)} e^{-i(ay+bx)}$$

$$= \sum_{k_1,k_2 \geq 0} \sum_{a,b \in \mathbb{Z}} (-i)^{k_1+k_2} a^{k_1} b^{k_2} \overline{T\left(\frac{\partial f}{\partial u_{k_1,k_2}}\right) e^{-i(ay+bx)}} e^{i(ay+bx)}.$$

To complete the proof we establish

$$T\left(\partial_x^{k_1} \partial_y^{k_2} h\right) = \sum_{a,b \in \mathbb{Z}} (i)^{k_1+k_2} a^{k_1} b^{k_2} \overline{T(h) e^{-i(ay+bx)}} e^{i(ay+bx)} \qquad (6)$$

for any $h \in \mathbb{C}[[u_{\star,\star}, \epsilon]]$, and apply it to the case $h = \frac{\partial f}{\partial u_{k_1,k_2}}$. We will write $T(h) = \sum_{a,b \in \mathbb{Z}} h_{a,b} e^{i(ay+bx)}$ so that $\partial_x^{k_1} \partial_y^{k_2} h = \sum_{a,b \in \mathbb{Z}} (i)^{k_1+k_2} h_{a,b} a^{k_1} b^{k_2} e^{i(ay+bx)}$. Equation (6) can now be deduced by comparing coefficients of $e^{i(ax+by)}$. □

5.1.2. The noncommutative Moyal product Let f and g be in $\mathbb{C}[[u_{\star,\star}, \epsilon]]$ and define the *non-commutative Moyal product*,

$$f \star g := \sum_{n \geq 0} \sum_{k_1+k_2=n} \frac{(-1)^{k_2}(i\epsilon)^n}{2^n k_1! k_2!} \partial_x^{k_1} \partial_y^{k_2}(f) \partial_x^{k_2} \partial_y^{k_1}(g) \in \mathbb{C}[[u_{\star,\star}, \epsilon]].$$

5.1.3. Generating series The integrals $I_{g,d}$ can be packaged in a generating function

$$\overline{g_d} = \sum_{g \geq 0} \frac{(-\epsilon^2)^g}{(d+2)!} \sum_{a_1,\ldots,a_n,b_1,\ldots,b_n} \int_{\overline{\mathcal{M}}_{g,d+3}} \lambda_g \psi_1^d DR_g(0, a_1, \ldots, a_{d+2})$$

$$DR_g(0, b_1, \ldots, b_{d+2}) \prod_{j=1}^{d+2} p_{b_j}^{a_j},$$

considered as an element of $\mathbb{C}[[p_\star^\star, \epsilon]]$. Buryak and Rossi showed that $\overline{g_d}$ lies in the image of T_0. In particular it will make sense to take a variational derivative.

5.1.4. A result of Buryak and Rossi The variational derivatives of $\overline{g}_d$ are related to the non-commutative Moyal product through the following theorem.

Theorem 5.1.2 ([9, Theorem 4.1]). *There is an equality*

$$\partial_x \frac{\delta \overline{g}_d}{\delta u} = \partial_x \left(\frac{1}{(d+1)!} (u \star \ldots \star u) \right)$$

where there are $d+1$ copies of u on the right hand side.

Remark 5.1.3. Integrable hierarchies are systems of differential equations. Originally such systems arose from studying commuting Hamiltonian flows on a symplectic manifold. The equations in an integrable hierarchy take the form

$$\text{Differential operator} \ = \ \text{a certain variational derivative.}$$

For us, as for Buryak and Rossi, we understand the left hand side as a formal symbol. Buryak has shown that partial cohomological field theories on finite dimensional vector spaces naturally give rise to such an integrable hierarchy [8].

Let V be the free vector space with basis $\{e_a\}_{a \in \mathbb{Z}}$ and define maps $V \mapsto H^\star(\overline{\mathcal{M}}_{g,n})$

$$c_{g,n}(e_{b_1}, \ldots, e_{b_n}) = \mathsf{DR}_g(b_1, \ldots, b_n).$$

Buryak and Rossi execute a version of Buryak's construction for a cohomological field theory on V built from this data. The resulting *quadratic double ramification hierarchy* consists of equations

$$\frac{\partial u}{\partial t_d} = \partial_x \frac{\delta \overline{g}_d}{\delta u}.$$

The left hand side is a formal symbol.

Buryak and Rossi therefore understand Theorem 5.1.2 as follows. The equations of the quadratic double ramification hierarchy coincide with the equations of the hierarchy

$$\frac{\partial u}{\partial t_d} = \partial_x \left(\frac{1}{(d+1)!} (u \star \cdots \star u) \right).$$

This second hierarchy is called the *dispersionless noncommutative KdV hierarchy*.

5.2. Expression for the double ramification integrals. In the remainder of this section we extract an expression for $I_{g,d}(\mathbf{a}; \mathbf{b})$ recorded in the following proposition.

Proposition 5.2.1. *For $a_1, \ldots, a_{d+1}, b_1, \ldots, b_{d+1} \in \mathbb{Z}$ we have*

$$\sum_{g \geq 0} I_{g,d}(\overline{a}, \overline{b}) u^{2g}$$

$$= \frac{1}{(d+1)!} \sum_{\sigma \in S_{d+1}} \prod_{j=1}^{d} \cos \left(u \frac{a_{\sigma(j+1)}(b_{\sigma(1)} + \cdots + b_{\sigma(j)}) - b_{\sigma(j+1)}(a_{\sigma(1)} + \cdots + a_{\sigma(j)})}{2} \right)$$

In the sequel given a tuple $\mathbf{a} = a_1, \ldots, a_n$ define $S_{\mathbf{a}} = -\sum_i a_i$. We write $\bar{\mathbf{a}} = (a_1, \ldots, a_n, S_{\mathbf{a}})$. Applying the map T to the equation of Theorem 5.1.2 and applying Lemma 5.1.1 yields

$$\sum_{g \geq 0} \frac{(-\epsilon^2)^g}{(d+1)!} \sum_{a_1, \ldots, a_{d+1}, b_1, \ldots, b_{d+1}} I_{g,d}(\bar{\mathbf{a}}, \bar{\mathbf{b}}) \prod_{j=1}^{d+1} p_{b_j}^{a_j} e^{i(-S_{\mathbf{a}} y - S_{\mathbf{b}} x)} = \frac{1}{(d+1)!} T(u \star \cdots \star u).$$

(7)

Proof of Proposition 5.2.1. We first make the following claim.

Claim 5.2.2.

$$T(u \star \cdots \star u) = \sum \prod_{j=1}^{d} \cos\left(u\left(\frac{a_{j+1}(b_1 + \cdots + b_j) - b_{j+1}(a_1 + \cdots + a_j)}{2}\right)\right)$$

$$\prod_{k=1}^{d+1} p_{b_k}^{a_k} e^{i(-S_{\mathbf{a}} y - S_{\mathbf{b}} x)}$$

where the first sum on the right hand side is over integers $a_1, \ldots, a_{d+1}, b_1, \ldots, b_{d+1}$. The product on the left is $d+1$ times after substituting $u = i\epsilon$.

We prove Claim 5.2 by induction on d. The base case $d = 1$ is done in [9, Proof of Theorem 4.1], after using the dilaton equation to remove the ψ class. By definition of the Moyal product there is an expression for $\dagger = T((u \star \cdots \star u) \star u)$ as

$$\dagger = \sum_{g' \geq 0} \sum_{k_1 + k_2 = 2g'} \frac{(-1)^{k_2} (i\epsilon)^{2g'}}{2^{2g'} k_1! k_2!} T(\partial_x^{k_1} \partial_y^{k_2}(u \star \cdots \star u))$$

$$\sum_{a_{d+1}, b_{d+1}} (i(b_{d+1}))^{k_2} (i(a_{d+1}))^{k_1} p_{b_{d+1}}^{a_{d+1}} e^{i(a_{d+1} y + b_{d+1} x)}$$

where we have that the summation range only contributes for even indices because $(u \star \cdots \star u) \star u = u \star (u \star \cdots \star u)$. The induction step tells us that $T(\partial_x^{k_1} \partial_y^{k_2}(u \star \cdots \star u))$ introduces the factor of

$$\left(i\left(\sum_{k=1}^{d} b_i\right)\right)^{k_1} \cdot \left(i\left(\sum_{k=1}^{d} a_i\right)\right)^{k_2}$$

which combines with $(-1)^{k_2} (i(b_{d+1}))^{k_2} (i(a_{d+1}))^{k_1}$ to introduce a factor of

$$(a_{d+1}(b_1 + \cdots + b_d))^{k_1} (-b_{d+1}(a_1 + \cdots + a_d))^{k_2}.$$

Now we observe that

$$\sum_{g'\geq0}\sum_{k_1+k_2=2g'}\frac{(i\epsilon)^{2g'}(a_{d+1}(b_1+\cdots+b_d))^{k_1}(-b_{d+1}(a_1+\cdots+a_d))^{k_2}}{2^{2g'}k_1!k_2!}$$

$$=\sum_{g'\geq0}\frac{\left(i\epsilon(\frac{a_{d+1}(b_1+\cdots+b_d)-b_{d+1}(a_1+\cdots+a_d)}{2})\right)^{2g'}}{(2g')!}$$

$$=\cos\left(i\epsilon(\frac{a_{d+1}(b_1+\cdots+b_d)-b_{d+1}(a_1+\cdots+a_d)}{2})\right).$$

Note that this is the cosine factor in the statement of the lemma for $j=d$. It follows now that after substituting $u=i\epsilon$

$$\dagger=\sum_{a_1,\ldots,a_{d+1},b_1,\ldots,b_{d+1}}\prod_{j=1}^{d}\cos\left(u\left(\frac{a_{j+1}(b_1+\cdots+b_j)-b_{j+1}(a_1+\cdots+a_j)}{2}\right)\right)$$

$$\prod_{k=1}^{d+1}p_{b_k}^{a_k}e^{i(-S_{\mathbf{a}}y-S_{\mathbf{b}}x)}.$$

This completes the proof of the claim. Proposition 5.2.1 follows from substituting the formula in the claim into Eq. (7), and taking coefficients. □

6. Completing Proof of Theorem A

We complete the proof of Theorem A, by combining Proposition 3.4.4 with Lemmas 4.6.1, 4.6.2 and Proposition 5.2.1. Recall our goal is to prove the following theorem.

Theorem 6.0.1 (Theorem A). *After the change of variables* $q=e^{iu}$ *we have the equality*

$$\sum_{g\geq0}N_{g,\Delta}^{\mathbf{k}}u^{2g-2+|\Delta^{\circ}|-\sum_i k_i}=\sum_{h\in T_{\Delta,p}^{\mathbf{k}}}\prod_{V\in V(\Gamma)}m_V(q)$$

6.1. Generating series as sums over tropical curves. The following definitions follow [5] and are motivated by Lemma 6.1.1. After fixing an orientation on $\tilde{\Gamma}$ as in Sect. 3.3.1, for a trivalent unpointed vertex $V\in V^3(\tilde{\Gamma})$ define

$$F_V^{3u}(u)=\sum_{g\geq0}N'_{g,V}u^{2g+1}:=\sum_{g\geq0}N_{g,V}w(E_V^{in,1})w(E_V^{in,2})u^{2g+1}$$

and for an m-valent pointed vertex V with $m\geq3$ define

$$F_V^{mp}(u):=\sum_{g\geq0}N_{g,V}u^{2g}.$$

Lemma 6.1.1.

$$\sum_{g \geq 0} N_{g,\Delta}^{\mathbf{k}} u^{2g-2+|\Delta^\circ|-\sum_i k_i} = \sum_{h \in T_{\Delta,p}^{\mathbf{k}}} \left(\prod_{V \in V^3(\Gamma)} F_V^{3\mathsf{u}}(u) \right) \left(\prod_{V \in V^{\mathsf{mp}}(\Gamma)} F^{\mathsf{mp}}(u) \right)$$

Proof. Observe first that by definition

$$\star := \sum_{g \geq 0} N_{g,\Delta}^{\mathbf{k}} u^{2g-2+|\Delta^\circ|-\sum_i k_i} = \sum_{g \geq 0} \sum_{\tilde{h} \in T_{\Delta,p}^{g,\mathbf{k}}} N_{g,\Delta}^{\tilde{h},\mathbf{k}} u^{2g-2+|\Delta^\circ|-\sum_i k_i}$$

By Proposition [3.4.4](#) we know

$$N_{g,\Delta}^{\tilde{h},\mathbf{k}} = \left(\prod_{E \in E_f(\Gamma)} w(E) \right) \left(\prod_{V \in V^3(\tilde{\Gamma})} N_{g_V,V} \right) \left(\prod_{V \in V^{\mathsf{mp}}(\tilde{\Gamma})} N_{g_V,V} \right)$$

whenever $\tilde{h}$ is a parametrised tropical curve with all bivalent vertices having genus 0. Thus we may rewrite

$$\star = \sum_{g \geq 0} \sum_{\substack{\tilde{h} \in T_{\Delta,p}^{g,\mathbf{k}}: \\ g_V = 0 \, \forall \, V \in V^2(\tilde{\Gamma})}} \left(\prod_{E \in E_f(\Gamma)} w(E) \right) \left(\prod_{V \in V^3(\tilde{\Gamma})} N_{g_V,V} \right)$$
$$\left(\prod_{V \in V^{\mathsf{mp}}(\tilde{\Gamma})} N_{g_V,V} \right) u^{2g-2+|\Delta^\circ|-\sum_i k_i}.$$

Since every tropical curve in $T_{\Delta,p}^{g,\mathbf{k}}$ has $|\Delta^\circ| - 2 - \sum_i k_i$ unpointed trivalent vertices we learn,

$$\star = \sum_{g \geq 0} \sum_{\substack{\tilde{h} \in T_{\Delta,p}^{g,\mathbf{k}}: \\ g_V = 0 \, \forall \, V \in V^2(\tilde{\Gamma})}} \left(\prod_{V \in V^3(\tilde{\Gamma})} N_{g_V,V}' u^{2g_V+1} \right) \left(\prod_{V \in V^{\mathsf{mp}}(\tilde{\Gamma})} N_{g_V,V} u^{2g_V} \right).$$

Now we have,

$$\star = \sum_{h \in T_{\Delta,p}^{\mathbf{k}}} \sum_{g \geq 0} \sum_{\substack{g_V: \\ \sum g_V = g}} \left(\prod_{V \in V^3(\Gamma)} N'_{g_V,V} u^{2g_V+1} \right) \left(\prod_{V \in V^{\mathrm{mp}}(\Gamma)} N_{g_V,\Delta_V} u^{2g_V} \right)$$

$$= \sum_{h \in T_{\Delta,p}^{\mathbf{k}}} \left(\sum_{g_1 \geq 0} \sum_{\substack{g_1(V): \\ \sum g_1(V)=g_1}} \prod_{V \in V^3(\Gamma)} N'_{g_1(V),V} u^{2g_1(V)+1} \right)$$

$$\left(\sum_{g_2 \geq 0} \sum_{\substack{g_2(V): \\ \sum g_2(V)=g_2}} \prod_{V \in V^{\mathrm{mp}}(\Gamma)} N_{g_2(V),V} u^{2g_2(V)} \right)$$

$$= \sum_{h \in T_{\Delta,p}^{\mathbf{k}}} \left(\prod_{V \in V^3(\Gamma)} F_V^{3\mathrm{u}}(u) \right) \left(\prod_{V \in V^{\mathrm{mp}}(\Gamma)} F^{\mathrm{mp}}(u) \right)$$

completing the proof.   □

6.2. Formulae for vertex contributions. The right hand side of Lemma 6.1.1 resembles the right hand side of Theorem A. We have formulae for the summands on the right hand side.

Corollary 6.2.1. *Let V be an unpointed trivalent vertex with $\Delta_V^{\circ} = \{v_1, v_2, v_3\}$. After the change of variables $q = e^{iu}$ we have*

$$F_V^{3\mathrm{u}}(u) = (-i) \left(q^{\frac{|v_1 \wedge v_2|}{2}} - q^{-\frac{|v_1 \wedge v_2|}{2}} \right)$$

Proof. By Lemma 4.6.1 we have that

$$F_V^{3\mathrm{u}}(u) = \sum_{g \geq 0} |v_1 \wedge v_2| \int_{\overline{\mathcal{M}}_{g,3}} (-1)^g \lambda_{g_V} \mathsf{DR}_g(\Delta_V^x) \mathsf{DR}_g(\Delta_V^y) u^{2g+1}$$

By combining the dilaton equation with Proposition 5.2.1, or explicitly [9, Theorem 2.1], this is equal to

$$2 \sum_{g \geq 0} \frac{(-1)^g |v_1 \wedge v_2|^{2g+1}}{2^{2g+1}(2g+1)!} u^{2g+1} = 2 \sin \left(u \left(\frac{|v_1 \wedge v_2|}{2} \right) \right).$$

After the substituting $q = e^{iu}$ this gives the result.   □

For an m-valent pointed vertex with $m \geq 3$ write the multiset

$$\Delta_V^{\circ} = \{\delta_1^V, \ldots, \delta_m^V\}.$$

Corollary 6.2.2. *After the change of variables $q = e^{iu}$ we have*

$$F_V^{\mathrm{mp}}(u) = \frac{1}{2^{m-2}(m-1)!} \sum_{\sigma \in S_{m-1}} \prod_{j=1}^{m-2} \left[\delta_{\sigma(j+1)}^V \wedge \sum_{l=1}^{j} \delta_{\sigma(l)}^V \right]_+ .$$

Proof. Follows from Proposition 5.2.1 by comparing coefficients of powers of u, and identifying $d = m - 2$.   □

Lemma 6.2.3 relates the formulae of Corollary 6.2.2 to the tropical curve multiplicity defined in Sect. 1.2.

Lemma 6.2.3. *There is an equality*

$$F_V^{\mathrm{mp}}(u) = \frac{1}{(m-1)!} \mu_m(\delta_1^V, \ldots, \delta_m^V)$$

.

The proof of Lemma 6.2.3 was communicated to us by Thomas Blomme. We record definitions before giving a proof. For ω in Ω_m write $\hat{\omega}$ for the unique representative in the group S_m of permutations of $\{1, \ldots, m\}$ which fixes m. For $\sigma \in S_{m-1}$ let A_σ be the set of cyclic permutations $\omega \in \Omega_m$, such that for any integer $k \leq m - 1$,

$$\max\{\hat{\omega}(\sigma(1)), \ldots, \hat{\omega}(\sigma(k))\} - \min\{\hat{\omega}(\sigma(1)), \ldots, \hat{\omega}(\sigma(k))\} = k - 1.$$

A cyclic permutation ω will be recorded by a unique tuple

$$(\hat{\omega}(m) = m, \hat{\omega}(m - 1), \hat{\omega}(m - 2), \ldots, \hat{\omega}(1)).$$

Write $\{-1, 1\}^{[j]}$ for the set of functions $\epsilon : \{1, \ldots, j\} \to \{-1, 1\}$. There is a bijective function

$$T : \{1, -1\}^{[j]} \to A_{\mathrm{id}}.$$

To define T we recursively define $T_1 = (1)$ and then T_ℓ is $(\ell, T_{\ell-1})$ if $\epsilon(\ell) = -1$ and $(T_{\ell-1}, \ell)$ if $\epsilon(\ell) = 1$. We then set $T(\epsilon) \in A_{\mathrm{id}}$ to be $(m, T_{m-2}(\epsilon))$. It is now possible to define

$$T_\sigma : \{1, -1\}^{[j]} \to A_\sigma$$

by composing T with the action of σ on Ω_n.

Proof. We may rewrite the left hand side as

$$F_V^{\mathrm{mp}}(u) = \frac{1}{2^{m-2}(m-1)!} \sum_{\sigma \in S_{m-1}} \sum_{\epsilon} q^{\frac{1}{2} \sum_{j=1}^{m-2} \epsilon(j) \left(\delta_{\sigma(j+1)}^V \wedge \sum_{l=1}^{j} \delta_{\sigma(l)}^V \right)}.$$

where the second sum is over functions $\epsilon \in \{1, -1\}^{[j]}$. We now claim that for fixed σ

$$\sum_{\epsilon} q^{\frac{1}{2} \sum_{j=1}^{m-2} \epsilon(j) \left(\delta_{\sigma(j+1)}^V \wedge \sum_{l=1}^{j} \delta_{\sigma(l)}^V \right)} = \sum_{\omega \in A_\sigma} q^{\frac{1}{2} k(\omega)}.$$

Indeed,

$$\sum_{j=1}^{m-2} \epsilon(j)\left(\delta^V_{\sigma(j+1)} \wedge \sum_{l=1}^{j} \delta^V_{\sigma(l)}\right) = k(T_\sigma(\epsilon)).$$

Thus we may write

$$F^{\mathsf{mp}}_V(u) = \frac{1}{2^{m-2}(m-1)!} \sum_{\sigma \in S_{m-1}} \sum_{A_\sigma} q^{\frac{1}{2}k(\omega)}$$

$$= \frac{1}{2^{m-2}(m-1)!} \sum_{\omega \in \Omega} q^{\frac{1}{2}k(\omega)} \mathsf{card}\{\sigma \,|\, \omega \in A_\sigma\}.$$

The number $\mathsf{card}\{\sigma \,|\, \omega \in A_\sigma\}$ is independent of ω and so we assume $\omega = \mathrm{id}$. Then note $\mathsf{card}\{\sigma \,|\, \omega \in A_\sigma\} = 2^{m-2}$. We conclude

$$F^{\mathsf{mp}}_V(u) = \frac{1}{(m-1)!} \sum_{\omega \in \Omega} q^{\frac{1}{2}k(\omega)}$$

and the result is proved.   □

6.3. Finishing the proof. In this section we will write $\mu_m(\delta^V_1, \ldots, \delta^V_m) = \mu_V(q)$.

Proof of Theorem A. Substituting Corollary 6.2.2 and Lemma 6.2.3 into Lemma 6.1.1 then rearranging we learn,

$$\sum_{g \geq 0} N^{\mathbf{k}}_{g,\Delta} u^{2g-2+|\Delta^\circ|-\sum_i k_i} = \sum_{h \in T^{\mathbf{k}}_{\Delta,p}} \left(\prod_{V \in V^3(\Gamma)} (-i)(q^{\frac{m(V)}{2}} - q^{-\frac{m(V)}{2}}) \right)$$

$$\cdot \left(\prod_{V \in V^{\mathsf{mp}}(\Gamma)} \frac{1}{(\mathsf{val}_V - 1)!} \mu_V(q) \right)$$

$$= \sum_{h \in T^{\mathbf{k}}_{\Delta,p}} \prod_{V \in V(\Gamma)} m_V(q)$$

$$= N^{\Delta,\mathbf{k}}_{\mathrm{trop}}(q).$$

This completes our proof.   □

Data availability Data sharing not applicable to this article as no datasets were generated or analysed.

Declarations

Conflict of interest There are no Conflict of interest to declare. See Section 0.5 for funding information.

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