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Sufficient Statistic and Recoverability via Quantum Fisher Information

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Abstract: We prove that for a large class of quantum Fisher information, a quantum channel is sufficient for a family of quantum states, i.e., the input states can be recovered from the output by some quantum operation, if and only if, the quantum Fisher information is preserved under the quantum channel. This class, for instance, includes Winger–Yanase–Dyson skew information. On the other hand, interestingly, the SLD quantum Fisher information, as the most popular example of quantum analogs of Fisher information, does not satisfy this property. Our recoverability result is obtained by studying monotone metrics on the quantum state space, i.e. Riemannian metrics non-increasing under the action of quantum channels, a property often called data processing inequality. For two quantum states, the monotone metric gives the corresponding quantum χ^2 divergence. We obtain an approximate recovery result in the sense that, if the quantum χ^2 divergence is approximately preserved by a quantum channel, then two states can be approximately recovered by the Petz recovery map. We also obtain a universal recovery bound for the $\chi_{\frac{1}{2}}$ divergence. Finally, we discuss applications in the context of quantum thermodynamics and the resource theory of asymmetry.

1. Introduction

Quantum metrology studies high-resolution measurements of physical parameters of quantum systems. In both classical and quantum metrology, the Fisher information metric plays an important role measuring the amount of information a system carries about a parameter θ. The concept of the Fisher information goes back to mathematical statistics [Fis22]: let (Ω, μ) be a probability space and $X(\theta) : \Omega \to \mathbb{R}$ be a family of random variables depending on an unknown parameter θ. The Fisher information of X at θ is defined as

$$I_X(\theta) := \mathbb{E}\Big[\big(\partial_\theta \log p_X(\theta, X)\big)^2 \Big| \theta \Big] = \int_\Omega \frac{|\partial_\theta p_X(\theta, \omega)|^2}{p_X(\theta, \omega)} \, d\mu(\omega), \tag{1.1}$$

where $\omega \mapsto p_X(\theta, \omega)$ is the probability density function of $X(\theta)$ conditioning on θ. By the famous Cramér–Rao bound [Rao45,Cra16], the Fisher information gives a fundamental limit on the precision of parameter estimation: for any unbiased estimator $\hat{\theta}$ of θ, [1] it holds that

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{I_X(\theta)}.$$

The Cramér–Rao bound has been extended to the quantum setting [Hol11,BC94,BCM96]: for a family of quantum states ρ_θ the variance of any unbiased estimator $\hat{\theta}$ of θ satisfies

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{I_{\mathrm{SLD},\rho}(\theta)} . \tag{1.2}$$

Here, $I_{\mathrm{SLD},\rho}(\theta) = \mathrm{tr}(\dot{\rho}_\theta\, \mathbb{J}_{\rho_\theta}(\dot{\rho}_\theta))$ and $\mathbb{J}_\rho$ is the inverse of the symmetric multiplication map $\mathbb{J}_\rho^{-1}(A) = \frac{1}{2}(\rho A + A\rho)$. In the physics literature I_{SLD} is often called the quantum Fisher information (QFI). Following the quantum statistics literature, we call this quantity the SLD (symmetric logarithmic derivative) QFI.

A nice property of the quantum Cramér–Rao bound with SLD QFI is that, similar to its classical version, it is asymptotically achievable; that is, given state $\rho_\theta^{\otimes n}$ there exists a measurement for which the above bound becomes tight in the limit $n \to \infty$ [BNG00,BC94]. This essentially follows from the fact that (i) both quantum and classical Fisher information are additive, and (ii) there exists a measurement on a single copy of $\rho(\theta)$, for which the classical Fisher information of the outcome is equal to $I_{\mathrm{SLD},\rho}(\theta)$. [2] Hence SLD QFI has a distinguished role in statistics that puts it on par with classical Fisher information.

1.1. Sufficient statistic. Another important property of the classical Fisher information is in the context of sufficient statistic. For a family of random variables X with an unknown parameter θ, a statistic $t = T(X)$ is called sufficient for the parameter θ if the conditional distribution of X given t does not depend on θ. It is known (see c.f. [Sch12]) that under certain regularity conditions, e.g., if the density function $p_X(\theta, \omega)$ has full support for all θ, then

$$t = T(X) \text{ sufficient } \theta \iff I_{T(X)}(\theta) = I_X(\theta) . \tag{1.3}$$

Note that $I_T(\theta) \leq I_X(\theta)$ for any statistic $T = t(X)$ by the data processing inequality (DPI). Thus a statistic is sufficient if and only if the DPI of the Fisher information is saturated.

In the quantum setting, the notion of sufficient statistic can be defined in terms of recoverability with quantum channels: given a family of quantum states ρ_θ, a quantum channel Φ is sufficient for the family $\{\rho_\theta\}$ if and only if there exists a quantum channel $\mathcal{R}$ such that $\mathcal{R} \circ \Phi(\rho_\theta) = \rho_\theta$ for all θ. Such $\mathcal{R}$ is called a recovery map, which means the original family ρ_θ can be fully recovered from the channel output $\Phi(\rho_\theta)$. Jenčová and

[1] Here an unbiased estimator satisfies $\mathbb{E}(\hat{\theta}|\theta) = \theta$.

[2] Combining these facts with the asymptotic achievability of the classical Cramér–Rao bound, one can establish the achievability of the Quantum Cramér–Rao bound. As it was noted in [BNG00], the measurement achieving this bound, in general, depends on the unknown parameter θ. [BNG00] shows how this measurement can be determined by consuming a sublinear number of copies.

Petz [JP06] showed that such quantum sufficiency can be characterized via the relative entropy [3]: Φ is sufficient if and only if there exists some state σ such that

$$D(\rho_\theta\|\sigma) = D(\Phi(\rho_\theta)\|\Phi(\sigma)) \ , \ \forall\theta. \tag{1.4}$$

Petz's work [Pet88,Pet86] showed that there is a canonical recovery map,

$$\mathcal{R}_{\sigma,\Phi}(\cdot) = \sigma^{\frac{1}{2}}\Phi^\dagger(\Phi(\sigma)^{-\frac{1}{2}} \cdot \Phi(\sigma)^{-\frac{1}{2}})\sigma^{\frac{1}{2}}, \tag{1.5}$$

called Petz recovery map.

1.2. Failure of SLD QFI in characterizing sufficiency. A natural question is whether, similar to the classical case, the sufficiency of statistics can be determined by preservation of SLD QFI. In other words, does $I_{\mathrm{SLD},\rho}(\theta) = I_{\mathrm{SLD},\Phi(\rho)}(\theta)$ imply that there exists a recovery map $\mathcal{R}$ such that $\mathcal{R}\circ\Phi(\rho_\theta) = \rho_\theta$? Surprisingly, it turns out that the answer is negative.

Proposition 1.1. *There exists a smooth family of full-rank qubit state ρ_θ and a quantum channel Φ,*

$$I_{\mathrm{SLD},\rho}(\theta) = I_{\mathrm{SLD},\Phi(\rho)}(\theta) \ , \ \forall\theta$$

yet there does not exist a recovery channel $\mathcal{R}$, such that $\mathcal{R}\circ\Phi(\rho_\theta) = \rho_\theta$ for all θ.

Without the full-rank assumption, such non-recovery examples has been observed in [Mar22] (See [KS05,Pol13] for a classical example demonstrating non-recoverability for probability distributions with restricted support in an infinite-dimensional space). It was shown in [Mar22] that for any system A with density operator ρ_A and Hamiltonian H_A, there exists a purification $|\psi\rangle_{AB}$ and Hamiltonian H_B on the purifying system B, such that the SLD QFI for the family of pure states

$$|\psi(t)\rangle_{AB} = (e^{-iH_At}\otimes e^{-iH_Bt})|\psi\rangle_{AB} : t\in\mathbb{R}$$

is equal to the SLD QFI for the family of reduced density $\rho_A(t) = e^{-iH_At}\rho_A e^{iH_At}$, which can be obtained from the first family by discarding system B. However, despite preservation of SLD QFI under partial trace, it is impossible to recover the original state $|\psi(t)\rangle_{AB}$ from $\rho_A(t)$. Proposition 1.1 shows that this phenomenon also happens for smooth families of full-rank states, in contrast to the classical Fisher information (1.3).

1.3. Sufficiency via regular QFI metrics. The failure of SLD QFI in characterizing sufficient statistic motivates us to consider other quantum generalizations of classical Fisher information that may satisfy this property. Indeed, quantum extensions of the Fisher information in the context of information geometry have been intensively studied in [Pet96,PH96,LR99,PG11,Hay02,Hol11,Kos05,Pet02]. Viewing the space of all positive quantum states as a manifold, SLD QFI induces a Riemmanian metric, defined for any quantum state ρ as

$$\gamma_\rho^{\mathrm{SLD}}(A) := \mathrm{tr}(A\,\mathbb{J}_\rho(A)) \ , \tag{1.6}$$

[3] For two quantum states with density operators ρ and σ, $D(\rho\|\sigma) = \mathrm{tr}(\rho\log\rho - \rho\log\sigma)$ if $\mathrm{supp}(\rho)\subseteq\mathrm{supp}(\sigma)$, and is infinite otherwise.

for all traceless, Hermitian operators A, interpreted as tangent vectors at ρ. This metric is of special interests, as it is monotone under any quantum channel Φ,

$$\gamma_\rho^{\mathrm{SLD}}(A) \geq \gamma_{\Phi(\rho)}^{\mathrm{SLD}}(\Phi(A)) , \tag{1.7}$$

which is the data processing inequality of SLD QFI at infinitesimal level.

In the classical setting, it was proved by Čencov [Čen78] that the Fisher information metric is the unique Riemannian metric (up to scaling) satisfying the monotonicity as in (1.7). In contrast, there are more than one quantum analog of the classical Fisher information whose corresponding metrics satisfy DPI. This family of metrics, called monotone metrics, were first proposed by Čencov and Morozova [MC89], and later fully classified by Petz [Pet96]. It was observed by Lesniewski and Ruskai [LR99] that any QFI metric corresponds to the Hessian of a given quantum divergence. In the special case where all the density operators in the family are diagonal in a fixed basis, QFI metrics all reduce to the classical Fisher information of the probability distribution defined by the eigenvalues of the density operators. Let $\gamma_\rho^g(A)$ be a QFI metric determined by the operator-monotone function g, see Sect. 3 for the definition. The associated QFI is defined via $I_\rho^g(\theta) := \gamma_{\rho_\theta}^g(\dot\rho_\theta)$.

Interestingly, it turns out that SLD QFI is the smallest QFI, which explains its special role in the Cramér–Rao bound. There is also a largest QFI, namely the Right-Logarithmic Derivative (RLD) Fisher information defined by

$$I_{\mathrm{RLD},\rho}(\theta) = \mathrm{tr}(\rho^{-1}(\theta)|\dot\rho(\theta)|^2) . \tag{1.8}$$

The RLD Fisher information has long been an important figure in single- and multi-parameter quantum estimation as well as other applications [Hay11, Suz16, Mar20, KW21].

While SLD QFI fails to characterize sufficient statistics, our first main result shows that a large class of QFI metrics, which we call them "regular" metrics, characterize sufficiency (see Sect. 3 for the definition of regular QFI metrics).

Theorem 1.2. *Given a smooth family of quantum states $(\rho_\theta)_{\theta\in(a,b)}$ with full support, a quantum channel Φ is sufficient for θ if and only if*

$$I_\rho^g(\theta) = I_{\Phi(\rho)}^g(\theta) , \quad \forall\theta \in (a, b)$$

holds for all/any **regular** *quantum Fisher information I^g. In particular, for any $o \in (a, b)$ in this family, the corresponding Petz recovery map of state ρ_o defined in Eq. (1.5) recovers*[4] *the full family, i.e.,*

$$\mathcal{R}_{\rho_o,\Phi} \circ \Phi(\rho_\theta) = \rho_\theta, \quad \forall\theta \in (a, b).$$

The following well-known metrics are all regular QFI and therefore, according to our theorem, characterize sufficient statistic.

a) Wigner-Yanase-Dyson (WYD) skew information: Given a density operator ρ and Hermitian operator H, the WYD skew information is defined as

$$W_H^{(\alpha)}(\rho) = -\frac{1}{2}\,\mathrm{tr}([\rho^\alpha, H][\rho^{1-\alpha}, H]) , \ 0 < \alpha < 1 . \tag{1.9}$$

which is a QFI of the family $\rho_t = e^{-itH}\rho e^{itH}, t \in \mathbb{R}$.

[4] As we show in Theorem 5.1, this result also holds for rotated Petz maps.

b) x^α-Fisher information: for $0 < \alpha < 1$,

$$I_{\alpha,\rho}(\theta) = \mathrm{tr}(\dot{\rho}_\theta \rho^{-\alpha} \dot{\rho}_\theta \rho^{\alpha-1}) \ .$$

c) Bogoliubov-Kubo-Mori (BKM) Fisher information: BKM QFI is the negative Hessian of the relative entropy, defined as

$$I_{\mathrm{BKM},\rho}(\theta) := -\left.\frac{\partial^2}{\partial\theta_1 \partial\theta_2} D(\rho_{\theta_1} \| \rho_{\theta_2})\right|_{\theta_1=\theta_2=\theta} \tag{1.10}$$

$$= \int_0^\infty \mathrm{tr}(\dot{\rho}_\theta (\rho_\theta + r1)^{-1} \dot{\rho}_\theta (\rho_\theta + r1)^{-1})\, dr \ , \tag{1.11}$$

In Sect. 1.6 we discuss implications of this result in the context of quantum thermodynamics and the resource theory of asymmetry. In conclusion, while to this date most applications of QFI in physics have been based on the special case of SLD QFI, our results clearly demonstrate operational and physical relevance of general QFI metrics, beyond this special case.

1.4. Approximate recoverability. Over the last decade, a series of works established a stronger notion of recoverability, namely approximate recoverability [FR15,JRS18, SFR16,SBT17,CV20,GW21]. This line of research was initiated by the work of Fawzi and Renner [FR15] on approximate quantum Markov chains. The notion of approximate recoverability has found various applications in different areas of physics, including high energy physics and condensed matter theory [CHP19,HPS21,HP19]. A notable result is by Junge et al. [JRS18], which proved that

$$D(\rho\|\sigma) - D(\Phi(\rho)\|\Phi(\sigma)) \geq -2\log F(\rho, \mathcal{R}^{\mathrm{uni}}_{\sigma,\Phi} \circ \Phi(\rho)) \geq \| \rho - \mathcal{R}^{\mathrm{uni}}_{\sigma,\Phi} \circ \Phi(\rho) \|_1^2, \tag{1.12}$$

where $F(\rho, \sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1$ is the fidelity between two quantum states ρ, σ and $\|\cdot\|_1$ denotes the trace norm. Here $\mathcal{R}^{\mathrm{uni}}_{\sigma,\Phi}$ is called the universal recovery map given by

$$\mathcal{R}^{\mathrm{uni}}_{\sigma,\Phi} = \int_{\mathbb{R}} \mathcal{R}^{\frac{t}{2}}_{\sigma,\Phi}\, d\beta(t) \ , \qquad d\beta(t) = \frac{\pi}{2(\cosh(\pi t) + 1)}\, dt \tag{1.13}$$

where $\mathcal{R}^t_{\sigma,\Phi}$ is the rotated version of Petz Recovery map,

$$\mathcal{R}^t_{\sigma,\Phi}(\cdot) = \sigma^{-it} \mathcal{R}_{\sigma,\Phi}\big(\Phi(\sigma)^{it} \cdot \Phi(\sigma)^{-it}\big)\sigma^{it} \ . \tag{1.14}$$

Despite the progress on approximate recoverability via various entropic quantities [GW21,CS22,Ver19], it remains open whether QFI also characterizes approximate recoverability. Our second main result addresses this question for BKM QFI.

Theorem 1.3. *Suppose $\rho_\theta \geq \lambda 1$ for some $\lambda > 0$ and all $\theta \in (a, b)$, then for any $a < s < r < b$*

$$\lambda^{-\frac{1}{2}} \int_s^r \sqrt{I_{\mathrm{BKM},\rho}(\theta) - I_{\mathrm{BKM},\Phi(\rho)}(\theta)}\, d\theta \geq D(\rho_r\|\rho_s) - D(\Phi(\rho_r)\|\Phi(\rho_s)) \ .$$

Combining this with the existing results on approximate recoverability in terms of relative entropy, such as Eq. (1.12), one can obtain recovery bounds in terms of BKM QFI.

1.5. Quantum χ^2-divergence. The proof of the above (approximate) recoverability results follows from a simpler setting, namely, quantum χ^2 divergence. The classical χ^2 divergence of two distributions P and Q is defined by

$$\chi^2(P,Q) = \mathbb{E}_Q \left| \frac{dP}{dQ} - 1 \right|^2 . \tag{1.15}$$

For two quantum states ρ, σ and a given QFI metric γ, the quantum analog of χ^2 divergence is

$$\chi^2_g(\rho,\sigma) = \gamma^g_\sigma(\rho - \sigma) , \tag{1.16}$$

which can be understood as the QFI for the linear interpolation family $\rho_t = t\sigma + (1-t)\rho$. In the special case where ρ and σ commute this quantity reduces to the classical χ^2 divergence in Eq. (1.15) for the distributions defined by the eigenvalues of ρ and σ. The above definition based on monotone metrics guarantees that the χ^2-divergence associated to each metric inherits monotonicity under DPI. These χ^2 divergences have found applications in characterizing the mixing time of a quantum Markov process [TKR10, GR22].

It is also natural to ask whether the recoverability can be characterized with quantum χ^2 divergences or the corresponding QFI metric γ. Again, the answer can be negative or positive depending on the choice of the metric γ. For a large class of quantum χ^2-divergences, Jenčova [Jen12] proved that a quantum channel Φ is sufficient for two states $\{\rho, \sigma\}$ if and only if

$$\chi^2_g(\rho,\sigma) = \chi^2_g(\Phi(\rho), \Phi(\sigma)) ,$$

provided the support condition $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$. On the other hand, the above property does not generally hold for two notable examples—the SLD and RLD χ^2-divergences

$$\chi^2_{\mathrm{SLD}}(\rho,\sigma) = \mathrm{tr}(\mathbb{J}_\sigma(\rho)^2\sigma) - 1 , \ \chi^2_{\mathrm{RLD}}(\rho,\sigma) = \mathrm{tr}(\rho^2\sigma^{-1}) - 1 ,$$

even though both satisfy DPI inequality (see Remark 3.7).

In terms of approximate recoverability, we obtain the following estimate for quantum χ^2 divergence corresponding to BKM and x^α metric.

Theorem 1.4. *Let Φ be a quantum channel. Given a quantum state $\sigma > \lambda I > 0$, we have*

i) for $\varepsilon \in (0, \frac{1}{2})$

$$\left(\chi^2_{\mathrm{BKM}}(\rho,\sigma) - \chi^2_{\mathrm{BKM}}(\Phi(\rho), \Phi(\sigma))\right)^{\frac{1-2\varepsilon}{4}} \geq \frac{\pi}{\cosh(\pi t)} \frac{\|\rho - \mathcal{R}^t_{\sigma,\Phi} \circ \Phi(\rho)\|_1}{(4(\sqrt{\lambda} + \lambda) + 1 + (\epsilon e)^{-\frac{1}{2}})}$$

i) for $\alpha \in (0, \frac{1}{2})$

$$\left(\chi^2_{\alpha}(\rho,\sigma) - \chi^2_{\alpha}(\Phi(\rho), \Phi(\sigma))\right)^{\frac{2}{\alpha}} \geq \frac{\pi}{\cosh(\pi t)} \frac{\|\rho - \mathcal{R}^t_{\sigma,\Phi} \circ \Phi(\rho)\|_1}{4(\sqrt{\lambda} + \lambda) + \sqrt{\frac{\pi}{\alpha \sin(\pi\alpha)}}}$$

where $\mathcal{R}^t_{\sigma,\Phi}$ is the rotated Petz map defined in (1.14).

In the special case of symmetric inverse metric

$$\chi^2_{\frac{1}{2}}(\rho,\sigma) = \mathrm{tr}((\rho-\sigma)\sigma^{-\frac{1}{2}}(\rho-\sigma)\sigma^{-\frac{1}{2}}) = \mathrm{tr}(\rho\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}}) - 1,$$

we achieve a universal approximate recovery bound.

Theorem 1.5. *For two states ρ and σ,*

$$\chi^2_{\frac{1}{2}}(\rho,\sigma) - \chi^2_{\frac{1}{2}}(\Phi(\rho),\Phi(\sigma)) \geq \| \rho - \mathcal{R}_{\sigma,\Phi} \circ \Phi(\rho) \|_1^2 \ ,$$

where $\mathcal{R}_{\sigma,\Phi}$ is the Petz recovery map defined in (1.5).

The $\chi^2_{\frac{1}{2}}$ divergence is known to enjoy some tensorization properties (see [CL19]), and it is closely related to the Sandwiched 2-Rényi relative entropy $D_2(\rho\|\sigma) = \log\mathrm{tr}(\rho\sigma^{-\frac{1}{2}}\rho\sigma^{-\frac{1}{2}})$. Indeed, our bound improves the D_2 approximate recovery bound in [CS22] by removing a factor of $\| \sigma \|_2 \cdot \sqrt{\| \sigma^{-1} \|}$, which is a state dependent constant that makes their bound trivial in infinite dimensions. We also note that here our recovery map $\mathcal{R}_{\sigma,\Phi}$ is the original Petz map, while it remains open whether the recovery bound (1.12) of relative entropy can be achieved with $\mathcal{R}_{\sigma,\Phi}$.

1.6. Applications: quantum thermodynamics and the resource theory of asymmetry. An important application of QFI is in the context of quantum thermodynamics and the closely related resource theory of asymmetry. QFI metrics provide a useful way of quantifying the amount of coherence of a system with respect to its energy eigenbasis and, more generally, the amount of asymmetry (symmetry-breaking) of the system with respect to a given symmetry group [Mar12,MS14,Gir14,YV16,KJJ18,Mar22]. Below we illustrate the application of our results for coherence, and refer to Sect. 6 for the more general setting of asymmetry.

For a system with density operator ρ and Hamiltonian H, consider the family of time evolution of this system, namely states $\rho(t) = e^{-iHt}\rho e^{iHt}$ for $t \in \mathbb{R}$. For any QFI metric γ^g, the QFI of this family with respect to the time parameter t is time-independent, that is, for any $t \in \mathbb{R}$,

$$I^g_\rho(t) = \gamma^g_{\rho(t)}(\dot\rho(t),\dot\rho(t)) = \gamma^g_\rho(i[H,\rho], i[H,\rho]) := I^g_H(\rho) \ . \tag{1.17}$$

This quantity determines the asymmetry of the system with respect to the time translation symmetry, or equivalently, the energetic coherence of the system with respect to the eigenbasis of Hamiltonian H. In particular, $I^g_H(\rho) = 0$ if and only if $[\rho, H] = 0$, namely, the state is diagonal in the energy-eigenbasis.

An important and useful property of this function is its monotonicity under any *covariant* quantum operation $\mathcal{E}$ (CPTP map) that

$$\mathcal{E}\big(e^{-iH_{\mathrm{in}}t}(\cdot)e^{iH_{\mathrm{in}}t}\big) = e^{-iH_{\mathrm{out}}t}\mathcal{E}\big(\,\cdot\,\big)e^{iH_{\mathrm{out}}t} \ \ \forall\, t \in \mathbb{R}, \tag{1.18}$$

where H_{in} and H_{out} are Hamiltonians for the input and output system, respectively. Operations satisfying this property are also called time-translation invariant operations. This property implies that under channel $\mathcal{E}$ the family of states $e^{-iH_{\mathrm{in}}t}\rho e^{iH_{\mathrm{in}}t}$ is mapped to $e^{-iH_{\mathrm{out}}t}\mathcal{E}(\rho)e^{iH_{\mathrm{out}}t}$. Then, the DPI for QFI metrics immediately implies that I^g_H is monotone under any such map $\mathcal{E}$,

$$I^g_{H_{\mathrm{out}}}(\mathcal{E}(\rho)) \leq I^g_{H_{\mathrm{in}}}(\rho) \ .$$

Any function satisfying this monotonicity is called a measure of asymmetry with respect to the symmetry group under consideration, which in this case is the time translation symmetry.[5]

While all QFIs can be used to quantify asymmetry and coherence, recent work [Mar22] has singled out SLD QFI, as the measure of asymmetry with an operational interpretation: namely, it quantifies the *coherence cost* of preparing a general mixed state from pure coherent states. RLD QFI has also shown to be useful for characterizing the distillation of energetic coherence, i.e., time-translation asymmetry [Mar20].

The present work reveals that, in addition to SLD and RLD QFI, regular QFI are particularly useful for quantifying asymmetry and energetic coherence. In particular, they can determine whether the resourcefulness of the system has been degraded by noise or any process that respects the symmetry. We show that

Theorem 1.6. *Consider systems A and B with Hamiltonians H_A and H_B, respectively. Let $\rho \in \mathcal{B}(\mathcal{H}_A)$ be a full-rank density operator and let $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be time-translation invariant quantum channel. Then, there exists a time-translation invariant channel $\mathcal{R}$ such that $\mathcal{R}(\mathcal{E}(\rho)) = \rho$, if and only if*

$$I^g_{H_A}(\rho) = I^g_{H_B}(\mathcal{E}(\rho)) \, , \tag{1.19}$$

where I^g_H is a QFI defined in Eq. (1.17) with respect to some/all regular monotone metric γ^g.

An important example of regular QFI is Wigner-Araki-Yanase skew information

$$W^{(\alpha)}_H(\rho) := \mathrm{Tr}(\rho H^2) - \mathrm{Tr}(\rho^{1-\alpha} H \rho^\alpha H) \, , \tag{1.20}$$

for $0 < \alpha < 1$, which have been previously studied as a measure of coherence and asymmetry [Mar12, MS14, Tak19]. It is worth mentioning that the above property of regular QFIs, that their conservation implies reversibility, was shown [ML16] to hold for the relative entropy of asymmetry (also known as the asymmetry).[6]

Theorem 1.6 follows from applying Theorem 1.2 for the family $\rho(t) = e^{-iH_At}\rho e^{iH_At}$ with $t \in \mathbb{R}$. By the time-translation invariance of $\mathcal{E}$, $\mathcal{E}(\rho(t)) = e^{-iH_Bt}\mathcal{E}(\rho)e^{iH_Bt} : t \in \mathbb{R}$. Furthermore, since QFI $I^g_\rho(t) = I^g_{H_A}(\rho)$ is time-independent, Eq. (1.19) implies that the QFI preserves under $I^g_\rho(t) = I^g_{H_A}(\rho) = I^g_{H_B}(\rho) = I^g_{\mathcal{E}(\rho)}(t)$ for all $t \in \mathbb{R}$. Then, our Theorem 1.2 implies that the Petz recovery of ρ recovers the states for all time t, i.e.,

$$\mathcal{R}_{\rho,\mathcal{E}}(\mathcal{E}(\rho(t))) = \rho(t) \, , \ \forall t \in \mathbb{R} \tag{1.21}$$

Furthermore, for any finite $T > 0$ the time-averaged version of Petz map

$$\mathcal{R}_{\mathrm{avg},T}(\cdot) = \frac{1}{T}\int_{-T/2}^{T/2} dt \; e^{iH_At}\,\mathcal{R}_{\rho,\mathcal{E}}\left(e^{-iH_Bt}(\cdot)e^{iH_Bt}\right)\, e^{-iH_At} \, , \tag{1.22}$$

also satisfies (1.21) (Note that $\rho(t+s) = e^{iH_At}\rho(s)e^{-iH_At}$ and similar for $\mathcal{E}(\rho(t))$). In general, $\mathcal{R}_{\mathrm{avg},T}$ is not covariant for finite T. Nevertheless, since for finite-dimensional

[5] It is often also required that a measure of asymmetry should vanish for all states that are invariant under the action of symmetry, which in this case are states satisfying $e^{-iHt}\rho e^{iHt} = \rho$ for all $t \in \mathbb{R}$. It can be easily seen that function I^g_H satisfies this property.

[6] Unlike metrics is not additive in tensor-product states, the relative entropy of asymmetry grow logarithmically with the number of copies.

Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$, the set of CPTP maps is compact, there exists a limit point $\mathcal{R}_{\text{avg},\infty} = \lim_{n\to\infty} \mathcal{R}_{\text{avg},T_n}$. It is straightforward to show that $\mathcal{R}_{\text{avg},\infty}$ satisfies the time translation invariant condition Eq. (1.18). This proves the Theorem 1.6. We refer to Sect. 6 for the more general case of asymmetry of compact Lie groups.

Outline of the rest of the paper. We first discuss the non-recoverability of SLD and RLD QFI in the Sect. 2. We briefly review in Sect. 3 the definitions of monotone metrics, quantum Fisher information and χ^2-divergences. Section 4 is devoted to the approximate recoverability of regular χ^2-divergences (Theorem 1.4) and the universal recoverability bound for $\chi^2_{\frac{1}{2}}$ (Theorem 1.4). Based on that we prove the recoverability (Theorem 1.2) and approximate recoverability (Theorem 1.3) of regular QFI in Sect. 5. Section 6 discusses the application of our results in quantum coherence and asymmetry.

Notations. We write $\mathbb{M}_n$ for the set of $n \times n$ complex matrices. Given a finite dimensional Hilbert space $\mathcal{H}$, we denote $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{H})_{\text{sa}}$ and $\mathcal{B}(\mathcal{H})_+$ as the set of bounded, Hermitian, and positive (semi-definite) operators respectively. We write $\langle A, B\rangle = \operatorname{tr}(A^*B)$ for the Hilbert–Schmidt inner product, where tr stands for the standard matrix trace. The Schatten norm of order $p \geq 1$ is defined as $\|A\|_p := \operatorname{tr}(|A|^p)^{1/p}$ and $\mathcal{S}_p(\mathcal{H})$ denotes the Schatten-p space. We denote by $\mathcal{D}(\mathcal{H})$ the subset of density operators (positive semi-definite and trace 1) on $\mathcal{H}$, $\mathcal{D}_+(\mathcal{H})$ by the subset of invertible density operators on $\mathcal{H}$. We use I for the identity operator in $\mathcal{B}(\mathcal{H})$ and $\iota^n_{\infty,2}$ for the identity map on $\mathcal{B}(\mathcal{H})$. We write A^* as the adjoint of an operator A and $\Phi^\dagger$ as the adjoint of a map Φ with respect to Hilbert–Schmidt inner product. Given two finite dimensional Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a quantum channel $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a completely positive trace preserving map. In particular, $\Phi(\mathcal{D}(\mathcal{H})) \subset \mathcal{D}(\mathcal{K})$ preserves the density operators.

2. Insuffcient Statistic Preserving SLD Quantum Fisher Information

In this section, we present the counter-example that quantum sufficiency cannot be characterized via SLD QFI. To show this it is useful to compare SLD and RLD QFI. For a family of states ρ_θ, recall that

$$I_{\text{RLD},\rho}(\theta) = \begin{cases} \operatorname{tr}(\rho_\theta^{-1}|\dot{\rho}_\theta|^2), & \text{if } \operatorname{supp}(|\dot{\rho}_\theta|) \subset \operatorname{supp}(\rho_\theta), \\ \infty, & \text{otherwise,} \end{cases} \quad \text{and} \quad I_{\text{SLD},\rho}(\theta) = \operatorname{tr}(L_\theta^2 \rho_\theta) ,$$

where the generalized inverse ρ_θ^{-1} is taken on $\operatorname{supp}(\rho_\theta)$ and L_θ is the symmetric logarithmic derivative, i.e., the Hermitian operator satisfying

$$\dot{\rho}_\theta = \frac{1}{2}(L_\theta \rho_\theta + \rho_\theta L_\theta) . \tag{2.1}$$

Note that I_{RLD} is finite whenever $\operatorname{supp}(|\dot{\rho}_\theta|) \subseteq \operatorname{supp}(\rho_\theta)$, but I_{SLD} can also be finite as long as $\operatorname{supp}(\dot{\rho}_\theta)^\perp \subseteq \operatorname{supp}(\rho_\theta)^\perp$. Moreover, RLD QFI can be rewritten as

$$I_{\text{RLD},\rho}(\theta) = \operatorname{tr}(\rho_\theta^{-1}|\dot{\rho}_\theta|^2) = \frac{1}{4}\operatorname{tr}\left(\rho_\theta^{-1}(L_\theta\rho_\theta + \rho_\theta L_\theta)^2\right) \tag{2.2}$$

$$= \frac{3}{4} I_{\text{SLD},\rho}(\theta) + \frac{1}{4}\operatorname{tr}(\rho_\theta^{-1} L_\theta \rho_\theta^2 L_\theta) . \tag{2.3}$$

We conclude that

$$I_{\mathrm{RLD},\rho}(\theta) - I_{\mathrm{SLD},\rho}(\theta) = \frac{1}{4} P_{L_\theta}(\rho_\theta)\ , \tag{2.4}$$

where the quantity

$$P_L(\rho) = \mathrm{tr}(\rho^{-1} L \rho^2 L) - \mathrm{Tr}(\rho L^2) = -\,\mathrm{tr}(\rho^{-1}[\rho, L]^2)\ , \tag{2.5}$$

is indeed the RLD QFI for the family of states $e^{itL}\rho e^{-itL}$ with respect to parameter t, called the *purity of coherence* of ρ with respect to L [Mar20]. In particular, $P_L(\rho) \geq 0$ always, and it is zero if and only if $[\rho, L] = 0$ commute.

Fix a parameter value θ_o and let $\mathcal{L}_o$ be the pinching map that dephases its input with respect to the spectrum of L_{θ_o}. The pinching map fulfills two properties we need: (1) $\mathcal{L}_o(x)$ commutes with L_{θ_o}; (2) $\mathrm{tr}(\mathcal{L}_o(x) L_{\theta_o}^k) = \mathrm{tr}(x L_{\theta_o}^k)$ for any x and $k \geq 0$. Define $\sigma_\theta = \mathcal{L}_o(\rho_\theta)$. Note that

$$\dot{\sigma}_{\theta_o} = \mathcal{L}_o(\dot{\rho}_{\theta_o}) = \frac{1}{2}\mathcal{L}_o(L_{\theta_o}\rho_{\theta_o} + \rho_{\theta_o} L_{\theta_o}) = \frac{1}{2}(L_{\theta_o}\sigma_{\theta_o} + \sigma_{\theta_o} L_{\theta_o})$$

Thus, L_{θ_o} remains the same for σ_θ at $\theta = \theta_o$ and the SLD QFI at θ_o does not change under this map, i.e.,

$$I_{\mathrm{SLD},\sigma}(\theta_o) = \mathrm{tr}(L_{\theta_o}^2 \sigma_{\theta_o}) = \mathrm{tr}(L_{\theta_o}^2 \rho_{\theta_o}) = I_{\mathrm{SLD},\rho}(\theta_o)\ . \tag{2.6}$$

Furthermore, the state σ_{θ_o} commutes with L_{θ_o} and therefore $P_{L_{\theta_o}}(\sigma_{\theta_o}) = 0$, which means the gap between RLD and SLD QFI vanishes, i.e.,

$$I_{\mathrm{RLD},\sigma}(\theta_o) = I_{\mathrm{SLD},\sigma}(\theta_o) = I_{\mathrm{SLD},\rho}(\theta_o)\ . \tag{2.7}$$

Note that unless $[L_{\theta_o}, \rho_{\theta_o}] = 0$, then $I_{\mathrm{SLD},\rho}(\theta_o) < I_{\mathrm{RLD},\rho}(\theta_o)$ and hence $\mathcal{L}_o$ is not sufficient because $I_{\mathrm{RLD},\sigma}(\theta_o) < I_{\mathrm{RLD},\rho}(\theta_o)$. Thus, we have the following observation similar to [Jen12, Remark 4]

Proposition 2.1. *For a family of states ρ_θ, the difference between RLD and SLD QFI equals to $P_{L_\theta}(\rho_\theta)$, which is zero if and only if $[L_\theta, \rho_\theta] = 0$, where $L_\theta = \mathbb{J}_{\rho_\theta}(\dot{\rho}_\theta)$ is the symmetric logarithm derivative of ρ_θ.*

Fix a parameter value value θ_o, the dephasing map $\mathcal{L}_o$ relative to the eigen-subspaces of L_{θ_o} always preserved SLD QFI at θ_o, but strictly decrease RLD QFI if $[L_{\theta_o}, \rho_{\theta_o}] \neq 0$, hence not recoverable.

In general, because SLD operator L_{θ_o} depends on the parameter o, so does the map $\mathcal{L}_o$. However, as we will see in the following example, it is possible to have a family of states ρ_θ such that the SLD operator L_{θ_o} share the same eigen-subspaces, which means the dephasing map $\mathcal{L}_\theta = \mathcal{L}$ is independent of θ. Furthermore, this family can be chosen to be full-rank. Such a family satisfies the full-rank condition in Theorem 1.2 while SLD QFI remains conserved under the map $\mathcal{L}$, but its conservation does not imply sufficiency of the output statistic. We emphasize that this contrasts with the classical case, where conservation of the Fisher information for probability distributions with full-support implies sufficiency [Sch12, Theorem 2.8].

A Qubit Counterexample for SLD QFI. Consider the family of qubit density operators

$$\rho_\theta = \begin{pmatrix} p(\theta) & \epsilon \times r(\theta) \\ \epsilon \times r(\theta) & 1 - p(\theta) \end{pmatrix} : \quad \theta \in [a, b]$$

defined for $\theta \in [a, b]$, where $p : [a, b] \to (0, 1)$ is an arbitrary function with finite, non-zero derivative $\dot{p}(\theta)$. The function r is determined by p via equation

$$r(\theta) = \exp \int_a^\theta \frac{\dot{p}(s)(1 - 2p(s))}{2p(s)(1 - p(s))} ds , \tag{2.8}$$

and ϵ is chosen such that

$$0 < \epsilon^2 < \min_{\theta \in [a,b]} \frac{p(\theta)(1 - p(\theta))}{r^2(\theta)} . \tag{2.9}$$

The latter condition guarantees that ρ_θ is positive and full-rank for all $\theta \in [a, b]$.

Then, one can easily check that the Hermitian operator

$$L_\theta = \begin{pmatrix} \dot{p}(\theta)/p(\theta) & 0 \\ 0 & -\dot{p}(\theta)/(1 - p(\theta)) \end{pmatrix}$$

satisfies the equation

$$\dot{\rho}_\theta = \frac{1}{2}\Big[\rho_\theta L_\theta + L_\theta \rho_\theta\Big] \tag{2.10}$$

and therefore is the SLD of the family ρ_θ. Note that L_θ is always diagonal. Furthermore, the assumption that $\dot{p}(\theta) \neq 0$ implies that L_θ in non-degenerate. Let $\mathcal{L}$ be the dephasing map in $\{|0\rangle, |1\rangle\}$ basis. Applying this map to state ρ_θ we obtain the family of states

$$\sigma_\theta = \mathcal{L}(\rho_\theta) = \begin{pmatrix} p(\theta) & 0 \\ 0 & 1 - p(\theta) \end{pmatrix} : \quad \theta \in [a, b].$$

Note that L_θ is also the symmetric logarithmic derivative of σ_θ. Under this dephasing, SLD QFI remains conserved. However, because the original family ρ_θ is not diagonal in $\{|0\rangle, |1\rangle\}$ basis, RLD QFI for the family of states σ_θ is strictly less than the RLD QFI for the family of states ρ_θ. Therefore, even though SLD QFI is preserved under $\mathcal{L}$, the RLD QFI decreases strictly. Hence $\sigma_\theta = \mathcal{L}(\rho_\theta)$ is not a sufficient statistic for the original family ρ_θ.

Preservation of RLD QFI under quantum measurement. It remains open whether the preservation of RLD QFI characterize the sufficiency of quantum channel. Below we give a partial answer for quantum to classical channels. Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a quantum channel. We say Φ is quantum to classical if the range of Φ is contained in a classical subsystem $l_\infty(X) \subset \mathcal{B}(\mathcal{K})$, where X is a finite alphabeta and $l_\infty(X)$ is the space of functions on X. The quantum to classical channels are essentially quantum measurement,

$$\Phi(\rho) = \sum_x \operatorname{tr}(\rho A_x)|x\rangle\langle x| .$$

given by some POVM (positive operator valued measurement) $\sum_{x \in X} A_x = 1$, $A_x \geq 0$. Conversely, a classical to quantum channel $\Psi : l_\infty(X) \to \mathcal{B}(\mathcal{H})$ is a state preparation process that $\Psi(|x\rangle\langle x|) = \rho_x$ for a family of quantum states $\{\rho_x\}_{x \in X} \subset \mathcal{B}(\mathcal{H})$.

Proposition 2.2. *(i) Let $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a quantum to classical channel and $\{\rho_\theta\} \subset \mathcal{B}(\mathcal{H})$ be a smooth family of quantum states. Then Φ is sufficient to $\{\rho_\theta\}$ if and only if*

$$I_{RLD,\theta}(\rho) = I_\theta(\Phi(\rho)).$$

(ii) Let $\Psi : l_\infty(X) \to \mathcal{B}(\mathcal{H})$ be a classical to quantum channel and $\{p_\theta\} \subset l_\infty(X)$ be a smooth family of probabilities densities. Then Ψ is sufficient to $\{p_\theta\}$ if and only if

$$I_\theta(p) = I_{SLD,\theta}(\Psi(p)).$$

Proof. For a general quantum channel Φ and a family of states ρ_θ, we have the following two chains of inequalities

$$I_{RLD,\theta}(\rho) \geq I_{BKM,\theta}(\rho) \geq I_{SLD,\theta}(\rho) \geq I_{SLD,\theta}(\Phi(\rho)),$$
$$I_{RLD,\theta}(\rho) \geq I_{RLD,\theta}(\Phi(\rho)) \geq I_{BKM,\theta}(\Phi(\rho)) \geq I_{SLD,\theta}(\Phi(\rho)).$$

For (i), suppose Φ is classical to quantum and the RLD Fisher information is preserved, then

$$I_{\mathrm{RLD},\theta}(\rho) = I_{RLD,\theta}(\Phi(\rho)) = I_{BKM,\theta}(\Phi(\rho)) = I_{SLD,\theta}(\Phi(\rho))$$

where we used the fact all QFI coincide for classical states $\{\Phi(\rho)_\theta\}$. This implies the first chain is also all equalities

$$I_{RLD,\theta}(\rho) = I_{BKM,\theta}(\rho) = I_{SLD,\theta}(\rho) = I_{SLD,\theta}(\Phi(\rho)) \ .$$

In particular, we have $I_{\mathrm{BKM},\theta}(\rho) = I_{\mathrm{BKM},\theta}(\Phi(\rho))$ for every θ, which implies the sufficiency of Φ by Theorem 5.2. The proof for (ii) is similar. If Ψ is classical to quantum and the SLD Fisher information is preserved,

$$I_{RLD,\theta}(p) = I_{BKM,\theta}(p) = I_{SLD,\theta}(p) = I_{SLD,\theta}(\Psi(p)) \ .$$

This implies the second chain are all equalities and hence $I_{\mathrm{BKM},\theta}(p) = I_{\mathrm{BKM},\theta}(\Psi(p))$, which implies the the sufficiency of Ψ. □

3. Monotone Metrics

The faithful state space $\mathcal{D} := \mathcal{D}_+(\mathcal{H})$ can be viewed as a submanifold of $\mathcal{B}(\mathcal{H})$. At each point $\rho \in \mathcal{D}$, the tangent space

$$T_\rho\mathcal{D} = \{A \in \mathcal{B}(\mathcal{H}) \mid A = A^* \ , \ \mathrm{tr}(A) = 0\}$$

is the subspace of traceless Hermitian operators. A Riemannian metric on $\mathcal{D}$ is a smooth assignment $\rho \mapsto \gamma_\rho$ to a positive bilinear form $\gamma_\rho : T_\rho\mathcal{D} \times T_\rho\mathcal{D} \to \mathbb{R}$.

Definition 3.1. We say a Riemannian metric γ is a monotone metric, if for any quantum channel $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$,

$$\gamma_\rho(A, A) \geq \gamma_{\Phi(\rho)}(\Phi(A), \Phi(A)) \ , \quad \forall A \in T_\rho\mathcal{D} \ .$$

We will use the short notation $\gamma^g_\rho(A) := \gamma^g_\rho(A, A)$.

The monotone metrics are classified by operator anti-monotone (i.e. decreasing) functions $g : (0, \infty) \to (0, \infty)$. Given $\rho \in \mathcal{D}$, we define

$$\mathbb{J}_\rho^g = g(L_\rho R_\rho^{-1}) R_\rho^{-1} \ ,$$

where $L_\rho(A) = \rho A$, $R_\rho(A) = A\rho$ are left and right multiplications respectively. Based on the work of Chentsov and Morozova [MC89], Petz in [Pet96] proved that every monotone metric admits the following form

$$\gamma_\rho^g(A, B) = \langle A, \mathbb{J}_\rho^g(B)\rangle = \langle A\rho^{-\frac{1}{2}}, g(L_\rho R_\rho^{-1})(B\rho^{-\frac{1}{2}})\rangle \ . \tag{3.1}$$

If $\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$, the metric is explicit with matrix coefficients

$$\gamma_\rho(A, A) = \sum_{i,j} c(\lambda_i, \lambda_j) |\langle\phi_i|A|\phi_j\rangle|^2$$

where $c(x, y) = y^{-1} g(xy^{-1})$ is called Morozova-Chentsov function. If we assume that the definition of γ_ρ^g for $A, B \in \mathcal{B}(\mathcal{H})$ has the symmetric property

$$\gamma_\rho(A, B) = \gamma_\rho(B^*, A^*),$$

this corresponds to operator anti-monotone functions satisfying

$$g(x^{-1}) = x g(x) \ , \ \ g(1) = 1 \ . \tag{3.2}$$

These operator anti-monotone functions admit the following integral form (see [LR99])

$$g(x) = \int_0^\infty \frac{1}{x+s} \nu_g(s) ds,$$

where $\nu_g(s^{-1}) = s\nu_g(s)$. Then $\mathbb{J}_\rho^g$ can be written as

$$\mathbb{J}_\rho^g = R_\rho^{-1} \int_0^\infty \frac{1}{s + \Delta_\rho} \nu_g(s) \, ds \tag{3.3}$$

where $\Delta_\rho = L_\rho R_\rho^{-1}$ is the relative modular operator. The monotone metric γ^g is uniquely determined by the associated measure $\nu_g(s)ds$:

$$\gamma_\rho^g(A, B) = \int_0^\infty \langle A, \frac{R_\rho^{-1}}{s + \Delta_\rho}(B)\rangle \nu_g(s) ds \ .$$

We emphasis that the measure $\nu_g(s)ds$ is a key object in our discussion, which plays an important role in obtaining the recoverability of monotone metrics and sufficiency of quantum Fisher information.

We will encounter the following special cases in our discussion.

Example 3.2. (1) SLD metric: for $g(x) = \frac{2}{x+1}$ and $\nu_g = 2\delta_1$ being the point mass at 1,

$$\gamma_\rho^{\mathrm{SLD}}(A, B) = 2\,\mathrm{tr}(A^*(L_\rho + R_\rho)^{-1}B)\ . \tag{3.4}$$

γ^{SLD} is also called Bures metric in the literature. (2)RLD metric: for $g(x) = \frac{1}{2x}$ and $\nu_g = \frac{1}{2}\delta_0$ being the point mass at $s = 0$

$$\gamma_\rho^{\mathrm{RLD}}(A, B) = \frac{1}{2}\,\mathrm{tr}(A^*\rho^{-1}B) \tag{3.5}$$

This corresponds to the RLD Fisher information introduced in (1.8). (3) BKM metric: for $g(x) = \frac{\log x}{x-1}$ and $\nu_g(s) = \frac{1}{s+1}$,

$$\gamma_\rho^{\mathrm{BKM}}(A, B) = \int_0^\infty \mathrm{tr}(A^*(\rho + s1)^{-1}B(\rho + s1)^{-1})\,ds\ . \tag{3.6}$$

(4) x^α-metrics: for $g(x) = \frac{1}{2}(x^{-\alpha} + x^{\alpha-1})$, $\alpha \in (0, 1)$ and $\nu_g(s) = \frac{\sin(\pi\alpha)}{2\pi}\big(s^{-\alpha} + s^{\alpha-1}\big)$,

$$\gamma_\rho^\alpha(A, B) = \frac{1}{2}\,\mathrm{tr}(A^*\rho^{-\alpha}B\rho^{\alpha-1}) + \frac{1}{2}\,\mathrm{tr}(A^*\rho^{\alpha-1}B\rho^{-\alpha})\ . \tag{3.7}$$

A special case is $\alpha = \frac{1}{2}$:

$$\gamma_\rho^{\frac{1}{2}}(A, B) = \mathrm{tr}(A^*\rho^{-\frac{1}{2}}B\rho^{-\frac{1}{2}})\ . \tag{3.8}$$

(5) WYD metric: for $\alpha \in (0, 1)$, $g(x) = \frac{(1-x^\alpha)(1-x^{1-\alpha})}{\alpha(1-\alpha)(1-x^2)}$ and $\nu_g(s) = \frac{\sin(\pi s)}{\pi}(1+s)^{(\alpha-2)} + \frac{\sin(\pi s^{-1})}{s\pi}(1 + s^{-1})^{(\alpha-2)}$,

$$\gamma_\rho^{\mathrm{WYD}}(A, B) = \frac{\partial^2}{\partial s\partial t}\,\mathrm{tr}\left((\rho + sA)^\alpha(\rho + tB)^{1-\alpha}\right)|_{s=t=0} \tag{3.9}$$

This gives the WYD skew information as in Eq. (1.9).

It is clear from functional calculus that if $g_1 \le g_2$, then

$$\gamma_\rho^{g_1}(A) \le \gamma_\rho^{g_2}(A)\ ,\ \forall A \in \mathcal{B}(\mathcal{H})\ .$$

From this perspective, for any monotone metric γ^g,

$$\gamma_\rho^{\mathrm{SLD}}(A) \le \gamma_\rho^g(A) \le \gamma_\rho^{\mathrm{RLD}}(A)$$

and the $\frac{1}{2}$-metric is the smallest among all α-metrics.

The definition of monotone metric is one to one correspondent to quantum χ^2 divergences.

Definition 3.3. Let γ^g be a monotone metric associated to an operator anti-monotone function g satisfying (3.2). For two quantum states $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, the associated quantum χ^2 divergence is

$$\chi_g^2(\rho, \sigma) := \gamma_\sigma^g(\rho - \sigma)\ . \tag{3.10}$$

It is clear from the definition that all quantum χ^2 divergences satisfies the data processing inequality: for any quantum channel Φ,

$$\chi_g^2(\rho, \sigma) \geq \chi_g^2(\Phi(\rho), \Phi(\sigma)).$$

Jenčova [Jen12] proves the following recoverability of monotone metric. Here we state a slightly weaker form.

Theorem 3.4 (Proposition 4 of [Jen12]). *Let $\rho \in \mathcal{D}_+(\mathcal{H})$ and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a quantum channel. Then for any $A \in T_\rho\mathcal{D}$, the following are equivalent*

(i) $\gamma_\rho^g(A) = \gamma_{\Phi(\rho)}^g(\Phi(A))$ for all monotone metric γ_g.

(ii) $\gamma_\rho^g(A) = \gamma_{\Phi(\rho)}^g(\Phi(A))$ for some monotone metric γ_g such that $\nu_g(s)ds$ has full support on $\mathbb{R}$.

(iii) $A = \mathcal{R}_{\rho,\Phi} \circ \Phi(A)$ where $\mathcal{R}_{\rho,\Phi}$ is the Petz map.

(iv) There exists a quantum channel $\mathcal{R}$ such that $\rho = \mathcal{R} \circ \Phi(\rho)$ and $A = \mathcal{R} \circ \Phi(A)$.

Remark 3.5. We note that in the condition (iii), the the Petz map can be replaced by rotated petz map

$$\mathcal{R}_{\rho,\Phi}^t(A) = \rho^{\frac{1}{2}-it}\Phi^\dagger(\Phi(\rho)^{-\frac{1}{2}+it}A\Phi(\rho)^{-\frac{1}{2}-it})\rho^{\frac{1}{2}+it}. \tag{3.11}$$

Indeed, consider the state $\rho_\varepsilon = \rho + \varepsilon A$ for some small $\varepsilon > 0$. The condition (iii) implies Φ is sufficient to $\{\rho_\varepsilon, \rho\}$. It follows from (1.12) that for any t

$$\mathcal{R}_{\rho,\Phi}^t \circ \Phi(\rho_\varepsilon) = \rho_\varepsilon \ , \mathcal{R}_{\rho,\Phi}^t \circ \Phi(\rho) = \rho.$$

By linearity, this implies $\mathcal{R}_{\rho,\Phi}^t \circ \Phi(A) = A$.

Motivated by the above result, we have the following definition.

Definition 3.6. We say a monotone metric γ^g, the quantum χ_g^2 divergence or its associated operator anti-monotone function g (1) is **regular** if $\nu_g(s)\, ds$ has full support on $\mathbb{R}$; (2) is **strongly regular** if the Lebesgue measure ds is absolutely continuous with respect to $\nu_g(s)\, ds$.

We see that the metric γ^{BKM}, $\gamma^{(\alpha)}$ and γ^{WYD} are (strongly) regular but γ_ρ^{SLD} and γ_ρ^{RLD} are not. This is the reason for the non-recoverability for the latter two.

Remark 3.7. As observed in [Jen12, Remark 4] and also Sect. 2 that if $[A, L] \neq 0$,

$$\gamma_{\mathcal{L}(\rho)}^{\text{RLD}}(\mathcal{L}(A)) = \gamma_{\mathcal{L}(\rho)}^{\text{SLD}}(\mathcal{L}(A)) = \gamma_\rho^{\text{SLD}}(A) < \gamma_\rho^{\text{RLD}}(A) \ ,$$

where $\mathcal{L}$ is the pinching map for the spectrum of L. Thus γ^{SLD} is preserved under $\mathcal{L}$ but $\mathcal{L}$ is not sufficient for ρ and A. For RLD metric, we note that

$$\chi_{\text{RLD}}^2(\rho, \sigma) = \gamma_\sigma^{\text{RLD}}(\rho - \sigma) = \text{tr}((\rho - \sigma)^2\sigma^{-1}) = \text{tr}(\rho^2\sigma^{-1}) - 1$$

which is essentially the Petz-Rényi 2-divergence $Q_2(\rho\|\sigma) = \text{tr}(\rho^2\sigma^{-1})$. It has been observed in [HM17, Example 4.8] there exists states ρ, σ and quantum channel Φ such that

$$Q_2(\rho\|\sigma) = Q_2(\Phi(\rho)\|\Phi(\sigma)) \ ,$$

but Φ is not sufficient for $\{\rho, \sigma\}$.

We note that for a general density operator ρ, the monotone metric γ^g are well-defined and finite for A with $s(A) \leq s(\rho)$, where $s(\rho)$ is the support of ρ. For example, the RLD metric $\gamma_\rho^{\text{RLD}}(A) = +\infty$ as long as $s(A) \nleq s(\rho)$. If in addition, $\lim_{x\to 0^+} g(x)$ exists and finite, γ^g is also finite for self-adjoint A if $(1 - s(\rho))A(1 - s(\rho)) = 0$. This is the case for SLD metric.

4. Approximate Recovery via Monotone Metrics

4.1. Approximate recovery for strongly regular monotone metrics. As we see in the above discussion, the preservation of regular monotone metric characterizes the recoverability that for a quantum channel,

$$\gamma_\rho^g(A) = \gamma_{\Phi(\rho)}^g(\Phi(A)) \Longleftrightarrow A = \mathcal{R}_{\rho,\Phi}^t \circ \Phi(A)$$

for any/all $t \in \mathbb{R}$, where $\mathcal{R}_{\rho,\Phi}^t$ is the rotated Petz map of ρ and channel Φ. In this section, we prove an approximate version of the above result for strongly regular monotone. Our argument is inspired from the previous works [CV20], [CV18] and [GW21] on the approxiamte recovery of quantum divergences.

We start with a lemma on the Stinespring dilation of the rotated Petz map.

Lemma 4.1. *Let $\rho \in \mathcal{D}_+(\mathcal{H})$ and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a quantum channel. Denote $e = supp(\Phi(\rho)) \in B(K)$ as the support projection of $\Phi(\rho)$ in $\mathcal{B}(e\mathcal{K})$. For any $t \in \mathbb{R}$, we define the linear map*

$$V_{\rho,t} : \mathcal{B}(e\mathcal{K}) \to \mathcal{B}(\mathcal{H}), \qquad V_{\rho,t}(A) = \Phi^\dagger(A\Phi(\rho)^{-\frac{1}{2}-it})\rho^{\frac{1}{2}+it}.$$

Then

i) $V_{\rho,t}$ is a contraction on $\mathcal{S}_2(e\mathcal{K})$, i.e. $V_{\rho,t}^* V_{\rho,t} \le I$.
ii) $V_{\rho,t}^* \Delta_\rho V_{\rho,t} \le \Delta_{\Phi(\rho)}$ as positive operators on $\mathcal{S}_2(e\mathcal{K})$.

Proof. For any $A \in \mathcal{B}(e\mathcal{K})$, we have

$$\begin{aligned}
\langle A, V_{\rho,t}^* V_{\rho,t}(A)\rangle &= \langle V_{\rho,t}(A), V_{\rho,t}(A)\rangle \\
&= \langle \Phi^\dagger(A\Phi(\rho)^{-\frac{1}{2}-it})\rho^{\frac{1}{2}+it}, \Phi^\dagger(A\Phi(\rho)^{-\frac{1}{2}-it})\rho^{\frac{1}{2}+it}\rangle \\
&= \mathrm{tr}\left(\rho^{\frac{1}{2}-it}\Phi^\dagger(\Phi(\rho)^{-\frac{1}{2}+it}A^*)\Phi^\dagger(A\Phi(\rho^{-\frac{1}{2}-it}))\rho^{\frac{1}{2}+it}\right) \\
&\le \mathrm{tr}\left(\rho\Phi^\dagger\left(\Phi(\rho)^{-\frac{1}{2}+it}A^*A\Phi(\rho)^{-\frac{1}{2}-it}\right)\right) \\
&= \langle \Phi(\rho), \Phi(\rho)^{-\frac{1}{2}+it}A^*A\Phi(\rho)^{-\frac{1}{2}-it}\rangle = \langle A, A\rangle,
\end{aligned}$$

where the above inequality follows from the operator Schwarz inequality

$$\Phi^\dagger(X^*)\Phi^\dagger(X) \le \Phi^\dagger(X^*X), \quad \forall X \in \mathcal{B}(\mathcal{H}).$$

Similarly,

$$\begin{aligned}
\langle A, V_{\rho,t}^* \Delta_\rho V_{\rho,t}(A)\rangle &= \langle V_{\rho,t}(A), \Delta_\rho V_{\rho,t}(A)\rangle \\
&= \langle \Phi^\dagger(A\Phi(\rho)^{-\frac{1}{2}-it})\rho^{\frac{1}{2}+it}, \rho\Phi^\dagger(A\Phi(\rho)^{-\frac{1}{2}-it})\rho^{\frac{1}{2}+it}\rho^{-1})\rangle \\
&= \mathrm{tr}\left(\Phi^\dagger(\Phi(\rho)^{-\frac{1}{2}+it}A^*)\rho\Phi^\dagger(A\Phi(\rho)^{-\frac{1}{2}-it})\right) \\
&= \mathrm{tr}\left(\rho\Phi^\dagger(A\Phi(\rho)^{-\frac{1}{2}-it})\Phi^\dagger(\Phi(\rho)^{-\frac{1}{2}+it}A^*)\right) \\
&\le \mathrm{tr}\left(\rho\Phi^\dagger(A\Phi(\rho)^{-1}A^*)\right) = \langle A, \Delta_{\Phi(\rho)}(A)\rangle,
\end{aligned}$$

where again the inequality follows from the operator Schwarz inequality. □

Our next lemma is a modification of [CV20, Lemma 2.1].

Lemma 4.2. *Let* $\Delta : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, $\tilde{\Delta} : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$, $V : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ *be positive linear maps and* V *be a contraction. If* $V^*\Delta V \leq \tilde{\Delta}$ *as positive operators on* $\mathcal{S}_2(\mathcal{H})$*, then for any* $s \geq 0$ *and* $h \in \mathcal{B}(\mathcal{H})$*,*

$$\langle h,\ (s+\Delta)^{-1}(h)\rangle - \langle h,\ V(s+\tilde{\Delta})^{-1}V^*(h)\rangle \geq \langle h_s,\ (s+\Delta)(h_s)\rangle,$$

where

$$h_s = (s+\Delta)^{-1}h - V(s+\tilde{\Delta})^{-1}V^*(h)\ . \tag{4.1}$$

Proof. Let us first calculate the right hand side using the definition of h_s,

$$\begin{aligned}\langle h_s,\ (s+\Delta)(h_s)\rangle &= \langle h,\ (s+\Delta)^{-1}(h)\rangle - 2\langle h,\ V(s+\tilde{\Delta})^{-1}V^*(h)\rangle \\ &\quad + \langle V(s+\tilde{\Delta})^{-1}V^*(h),\ (s+\Delta)\,V(s+\tilde{\Delta})^{-1}V^*(h)\rangle.\end{aligned}$$

Then since $V^*V \leq I$ and $V^*(\Delta+s)V \leq (\tilde{\Delta}+s)$ by the assumption, we obtain an upper bound of the last term

$$\begin{aligned}&\langle V(s+\tilde{\Delta})^{-1}V^*(h), (s+\Delta)V(s+\tilde{\Delta})^{-1}V^*(h)\rangle \\ &\quad = \langle (s+\tilde{\Delta})^{-1}V^*(h),\ V^*(s+\Delta)V(s+\tilde{\Delta})^{-1}V^*(h)\rangle \\ &\quad \leq \langle (s+\tilde{\Delta})^{-1}V^*(h),\ (s+\tilde{\Delta})(s+\tilde{\Delta})^{-1}V^*(h)\rangle \\ &\quad = \langle h,\ V(s+\tilde{\Delta})^{-1}V^*(h)\rangle\ ,\end{aligned}$$

which yields the desired inequality in the lemma. □

Applying the above lemmas to $V_{\rho,t}$ and Δ_ρ, we have

$$\langle A,\ (s+\Delta_\rho)^{-1}(A)\rangle - \langle A,\ V_{\rho,t}(s+\Delta_{\Phi(\rho)})^{-1}V_{\rho,t}^*(A)\rangle \geq \langle A_{s,t},\ (s+\Delta_\rho)(A_{s,t})\rangle,$$

where

$$A_{s,t} = (s+\Delta_\rho)^{-1}A - V_{\rho,t}(s+\Delta_{\Phi(\rho)})^{-1}V_{\rho,t}^*(A). \tag{4.2}$$

The next lemma shows the above expression upper bounds the approximate recoverability.

Lemma 4.3. *Let* $\rho \in \mathcal{D}_+(\mathcal{H})$ *and* $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *be a quantum channel. Let* $\mathcal{R}^t_{\rho,\Phi}$ *be the rotated Petz recovery map as defined in* (3.11).

$$\begin{aligned}\|A - \mathcal{R}^t_{\rho,\Phi}(\Phi(A))\|_1 &\leq \left\| \Big(A - \mathcal{R}^t_{\rho,\Phi}(\Phi(A))\Big)\rho^{-\frac{1}{2}} \right\|_2 \\ &\leq \frac{\cosh(\pi t)}{\pi} \left\| \int_0^\infty s^{-\frac{1}{2}+it}\Delta_\rho^{\frac{1}{2}}\tilde{A}_{s,t}\,ds \right\|_2\ ,\end{aligned}$$

where

$$\tilde{A}_{s,t} = (s+\Delta_\rho)^{-1}(A\rho^{-\frac{1}{2}+it}) - V_{\rho,t}(s+\Delta_{\Phi(\rho)})^{-1}V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}). \tag{4.3}$$

Proof. By the integral representation $r^{-\frac{1}{2}+it} = \dfrac{\cosh(\pi t)}{\pi}\displaystyle\int_0^\infty \frac{s^{-\frac{1}{2}+it}}{r+s}ds$ from [Kom66], we have

$$\begin{aligned}
&\frac{\cosh(\pi t)}{\pi}\int_0^\infty s^{-\frac{1}{2}+it}\Delta_\rho^{\frac{1}{2}}\tilde{A}_{s,t}ds\\
&= \Delta_\rho^{\frac{1}{2}}\frac{\cosh(\pi t)}{\pi}\int_0^\infty s^{-\frac{1}{2}+it}(s+\Delta_\rho)^{-1}(A\rho^{-\frac{1}{2}+it})ds\\
&\quad - \Delta_\rho^{\frac{1}{2}}V_{\rho,t}\frac{\cosh(\pi t)}{\pi}\int_0^\infty s^{-\frac{1}{2}+it}(s+\Delta_{\Phi(\rho)})^{-1}V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it})ds\\
&= \Delta_\rho^{it}(A\rho^{-\frac{1}{2}+it}) - \Delta_\rho^{\frac{1}{2}}V_{\rho,t}\Delta_{\Phi(\rho)}^{-\frac{1}{2}+it}V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it})\\
&= \rho^{it}A\rho^{-\frac{1}{2}} - \rho^{\frac{1}{2}}\Phi^\dagger(\Phi(\rho)^{-\frac{1}{2}+it}\Phi(A)\Phi(\rho)^{-\frac{1}{2}-it})\rho^{it}\\
&= \rho^{it}(A-\mathcal{R}_{\rho,\Phi}^t(\Phi(A)))\rho^{-\frac{1}{2}}.
\end{aligned}$$

Then the inequality follows from the Hölder inequality,

$$\|\rho^{it}(A-\mathcal{R}_{\rho,\Phi}^t(\Phi(A)))\rho^{-\frac{1}{2}}\|_2\|\rho^{\frac{1}{2}}\|_2 \ge \|A-\mathcal{R}_{\rho,\Phi}^t(\Phi(A))\|_1\ .$$

□

We now estimate the above recoverability bounds via the monotone metrics. This is our main technical lemma.

Lemma 4.4. *Let γ^g be a monotone metric given by the integral representation (3.3) with measure $\nu_g(s)ds$. For any $0 < a \le b < \infty$, suppose there exists a function $w : [0,\infty) \to \mathbb{R}_+$ such that*

$$W_{a,b} := \int_a^b \frac{w(s)}{s}ds \ < \infty$$

and a constant $C_g(a,b) > 0$ such that on the interval $[a,b]$

$$\frac{1}{w(s)}ds \le C_g(a,b)\nu_g(s)ds.$$

Then we have for any $\rho \in \mathcal{D}_+(\mathcal{H})$, $A \in \mathcal{B}(\mathcal{H})$ and $t \in \mathbb{R}$

$$\|A-\mathcal{R}_{\rho,\Phi}^t(\Phi(A))\|_1 \le \frac{\cosh(\pi t)}{\pi}\left(4\sqrt{a}h_1 + \frac{4}{\sqrt{b}}h_2 + C_g(a,b)^{\frac{1}{2}}W_{a,b}^{\frac{1}{2}}h_3\right),$$

where

$$h_1 := \mathrm{tr}(A^*\rho^{-1}A)^{\frac{1}{2}},\ h_2 := \frac{1}{2}\Big(\mathrm{tr}(\rho^{-2}A^*\rho A)^{\frac{1}{2}} + \mathrm{tr}(\Phi(\rho)^{-2}\Phi(A)^*\Phi(\rho)\Phi(A))^{\frac{1}{2}}\Big)$$
$$h_3 := \Big(\gamma_\rho^g(A) - \gamma_{\Phi(\rho)}^g(\Phi(A))\Big)^{\frac{1}{2}}\ .$$

Moreover, such constants $W_{a,b}$ and $C_g(a,b)$ always exist if the monotone metric γ^g is strongly regular.

Proof. By Lemma 4.3, it is enough to bound $\| \int_0^\infty s^{-\frac{1}{2}+it} \Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t} ds \|_2$ from above. We split the integral into three terms

$$\int_0^\infty s^{-\frac{1}{2}+it} \Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t} ds = \int_0^a s^{-\frac{1}{2}+it} \Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t} ds + \int_a^b s^{-\frac{1}{2}+it} \Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t} ds + \int_b^\infty s^{-\frac{1}{2}+it} \Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t} ds = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Let $h_{t,a}(x) = \int_0^a \frac{s^{-\frac{1}{2}+it} x^{\frac{1}{2}}}{(x+s)} ds$. Then

$$\|\mathrm{I}\|_2 \le \|h_{t,a}(\Delta_\rho)(A\rho^{-\frac{1}{2}+it})\|_2 + \left\| \int_0^a s^{-\frac{1}{2}+it} \Delta_\rho^{\frac{1}{2}} V_{\rho,t}(s+\Delta_{\Phi(\rho)})^{-1} V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}) ds \right\|_2 .$$

Next, we notice that

$$\begin{aligned}
&\left\| \int_0^a s^{-\frac{1}{2}+it} \Delta_\rho^{\frac{1}{2}} V_{\rho,t}(s+\Delta_{\Phi(\rho)})^{-1} V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}) ds \right\|_2^2 \\
&= \langle V_{\rho,t}^* \Delta_\rho V_{\rho,t} \int_0^a \frac{s^{-\frac{1}{2}+it}}{s+\Delta_{\Phi(\rho)}} V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}) ds, \int_0^a \frac{s^{-\frac{1}{2}+it}}{s+\Delta_{\Phi(\rho)}} V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}) ds \rangle \\
&\overset{(1)}{\le} \langle \Delta_{\Phi(\rho)} \int_0^a \frac{s^{-\frac{1}{2}+it}}{s+\Delta_{\Phi(\rho)}} V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}) ds, \int_0^a \frac{s^{-\frac{1}{2}+it}}{s+\Delta_{\Phi(\rho)}} V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}) ds \rangle \\
&\le \langle \int_0^a \frac{s^{-\frac{1}{2}+it} \Delta_{\Phi(\rho)}^{\frac{1}{2}}}{s+\Delta_{\Phi(\rho)}} V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}) ds, \int_0^a \frac{s^{-\frac{1}{2}+it} \Delta_{\Phi(\rho)}^{\frac{1}{2}}}{s+\Delta_{\Phi(\rho)}} V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}) ds \rangle \\
&= \|h_{t,a}(\Delta_{\Phi(\rho)}) V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it})\|_2^2,
\end{aligned}$$

where the inequality in (1) above follows from Lemma 4.1. Moreover, for $x > 0$, we have

$$|h_{t,a}(x)| \le \int_0^a \left| \frac{s^{-\frac{1}{2}+it} x^{\frac{1}{2}}}{(x+s)} \right| ds = 2 \arctan \sqrt{\frac{a}{x}} \le 2\sqrt{\frac{a}{x}}.$$

Together with the inequality above and another use of Lemma 4.1, we can hence bound $\|I\|_2$ from above:

$$\begin{aligned}
\|\mathrm{I}\|_2 &\le 2\sqrt{a} \langle A\rho^{-\frac{1}{2}+it},\ \Delta_\rho^{-1}(A\rho^{-\frac{1}{2}+it}) \rangle^{\frac{1}{2}} + 2\sqrt{a} \langle V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}),\ \Delta_{\Phi(\rho)}^{-1} V_{\rho,t}^*(A\rho^{-\frac{1}{2}+it}) \rangle^{\frac{1}{2}} \\
&= 2\sqrt{a}\, \mathrm{tr}(A^*\rho^{-1}A)^{\frac{1}{2}} + 2\sqrt{a}\, \mathrm{tr}(\Phi(A^*)\Phi(\rho)^{-1}\Phi(A))^{\frac{1}{2}} \\
&\le 4\sqrt{a}\, \mathrm{tr}(A^*\rho^{-1}A)^{\frac{1}{2}},
\end{aligned}$$

where the last inequality follows from the Schwarz-type operator inequality (see [LR74], [Wol12, Theorem 5.3]):

$$\Phi(A^*)\Phi(\rho)^{-1}\Phi(A) \le \Phi(A^*\rho^{-1}A).$$

(or simply the monotonicity of RLD metric). Let us now consider $\|\mathrm{III}\|_2$. Define $\tilde{h}_{t,b}(x) = \int_b^\infty \frac{x^{\frac{1}{2}} s^{-\frac{1}{2}+it}}{(x+s)} ds$. Then for $x > 0$,

$$|\tilde{h}_{t,b}(x)| \le \int_b^\infty \left| \frac{x^{\frac{1}{2}} s^{-\frac{1}{2}+it}}{(x+s)} \right| ds = 2\arctan(\sqrt{\frac{x}{b}}) \le 2\sqrt{\frac{x}{b}}$$

Similar to I, we obtain that

$$\|\mathrm{III}\|_2 \le \frac{2}{\sqrt{b}} \operatorname{tr}(\rho^{-2} A^* \rho A)^{\frac{1}{2}} + \frac{2}{\sqrt{b}} \operatorname{tr}(\Phi(\rho)^{-2} \Phi(A^*) \Phi(\rho) \Phi(A))^{\frac{1}{2}} .$$

Finally, we consider the second integral:

$$\begin{aligned} \|\mathrm{II}\|_2 &\le \int_a^b \sqrt{\frac{1}{s}} \, \|\Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t}\|_2 \, ds \\ &\le \left(\int_a^b \frac{w(s)}{s} ds \right)^{\frac{1}{2}} \left(\int_a^b \frac{1}{w(s)} \|\Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t}\|_2^2 \, ds \right)^{\frac{1}{2}} \\ &= W_{a,b}^{\frac{1}{2}} \left(\int_a^b \frac{1}{w(s)} \|\Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t}\|_2^2 \, ds \right)^{\frac{1}{2}} . \end{aligned}$$

Now, since by assumption $\frac{1}{w(s)} ds \le C_g(a,b) \nu_g(s) ds$ on the interval $[a, b]$, we have

$$\begin{aligned} \|\mathrm{II}\|_2 &\le W_{a,b}^{\frac{1}{2}} C_g(a,b)^{\frac{1}{2}} \left(\int_0^\infty \|\Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t}\|_2^2 \nu_g(s) \, ds \right)^{\frac{1}{2}} \\ &\overset{(2)}{\le} W_{a,b}^{\frac{1}{2}} C_g(a,b)^{\frac{1}{2}} \left(\gamma_\rho^g(A) - \gamma_{\Phi(\rho)}^g(\Phi(A)) \right)^{\frac{1}{2}} . \end{aligned}$$

The inequality (2) above is justified by rewriting $\gamma_\rho^g(A) - \gamma_{\Phi(\rho)}^g(\Phi(A))$ as follows:

$$\begin{aligned} &\gamma_\rho^g(A, A) - \gamma_{\Phi(\rho)}^g(\Phi(A), \Phi(A)) \\ &= \int_0^\infty \left(\langle A, \frac{R_\rho^{-1}}{s + \Delta_\rho}(A) \rangle - \langle \Phi(A), \frac{R_{\Phi(\rho)}^{-1}}{s + \Delta_{\Phi(\rho)}}(\Phi(A)) \rangle \right) \nu_g(s) ds \\ &= \int_0^\infty \langle A\rho^{-\frac{1}{2}+it}, \, (s + \Delta_\rho)^{-1} (A\rho^{-\frac{1}{2}+it}) \rangle \nu_g(s) \, ds \\ &\quad - \int_0^\infty \langle A\rho^{-\frac{1}{2}+it}, \, V_{\rho,t} (s + \Delta_{\Phi(\rho)})^{-1} V_{\rho,t}^* (A\rho^{-\frac{1}{2}+it}) \rangle \nu_g(s) \, ds. \end{aligned}$$

Then by Lemma 4.2,

$$\begin{aligned} \gamma_\rho^g(A, A) - \gamma_{\Phi(\rho)}^g(\Phi(A), \Phi(A)) &\ge \int_0^\infty \langle \tilde{A}_{s,t}, \, (s + \Delta_\rho)(\tilde{A}_{s,t}) \rangle \nu_g(s) \, ds \\ &\ge \int_0^\infty \langle \tilde{A}_{s,t}, \, \Delta_\rho(\tilde{A}_{s,t}) \rangle \nu_g(s) \, ds \\ &\ge \int_a^b \|\Delta_\rho^{\frac{1}{2}} \tilde{A}_{s,t}\|_2^2 \nu_g(s) \, ds \end{aligned}$$

from which (2) follows. This finishes the proof by adding up the bounds found for I, II and III.

When γ^g is strongly regular. Then $\frac{1}{\nu_g} \in L_1((0,\infty))$. A natural choice of w is $w(s) = \frac{1}{\nu_g(s)}$. Then $W_{a,b} = \int_a^b \frac{w(s)}{s} ds \leq \frac{1}{a} \int_0^\infty \frac{1}{\nu_g(s)} ds < \infty$, we have the global constant $C_g(a,b) = 1$. □

Remark 4.5. The quantity $h_1 = \mathrm{tr}(A^*\rho^{-1}A)^{\frac{1}{2}} = \gamma_\rho^{\mathrm{RLD}}(A)$ is the RLD metric, which satisfies the monotonicity. On the other hand,

$$\mathrm{tr}(\rho^{-2}A^*\rho A) = \langle A, \Delta_\rho R_\rho^{-1} A\rangle$$

corresponds to γ^g with $g(x) = x$, which is not a monotone metric. So we have to also include the term $\mathrm{tr}(\Phi(\rho)^{-2}\Phi(A)^*\Phi(\rho)\Phi(A))$ in the estimate. When $\rho \geq \lambda 1$, both quantities are bounded by the 2-norm of A,

$$\begin{aligned} \mathrm{tr}(A^*\rho^{-1}A) &\leq \lambda^{-1} \| A \|_2^2, \\ \mathrm{tr}(\Phi(\rho)^{-2}\Phi(A)^*\Phi(\rho)\Phi(A)) &\leq \mathrm{tr}(\Phi(\rho)^{-2}\Phi(A)^*\Phi(A)) \\ &\leq \mathrm{tr}(\rho^{-2}A^*A) \leq \lambda^{-2} \| A \|_2^2 \end{aligned}$$

We now apply the above lemma to a concrete example of our approximate recovery results. Recall the Bogolyubov-Kubo-Mori metric is

$$\gamma_\rho^{\mathrm{BKM}}(A) = \int_0^\infty \mathrm{tr}(A^*(\rho+s)^{-1}A(\rho+s)^{-1})ds.$$

The associated measure is $\nu(s)ds = \frac{1}{1+s}ds$.

Corollary 4.6. *For all $\rho \in \mathcal{D}_+(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$, we have*

$$\gamma_\rho^{\mathrm{BKM}}(A) - \gamma_{\Phi(\rho)}^{\mathrm{BKM}}(\Phi(A)) \geq \sup_{0<\epsilon<\frac{1}{2}} \left(\frac{\pi}{\cosh(\pi t)} \frac{\|A - \mathcal{R}_{\rho,\Phi}^t(\Phi(A))\|_1}{K(\rho, A, \epsilon)} \right)^{\frac{4}{1-2\epsilon}},$$

where

$$\begin{aligned} K(\rho, A, \epsilon) :=\, & 4\,\mathrm{tr}(A^*\rho^{-1}A)^{\frac{1}{2}} + 2\,\mathrm{tr}(\rho^{-2}A^*\rho A)^{\frac{1}{2}} + 2\,\mathrm{tr}(\Phi(\rho)^{-2}\Phi(A)^*\Phi(\rho)\Phi(A))^{\frac{1}{2}} \\ & + 1 + (\epsilon e)^{-\frac{1}{2}}. \end{aligned}$$

If $\sigma \geq \lambda I$, then

$$\chi^2_{\mathrm{BKM}}(\rho,\sigma) - \chi^2_{\mathrm{BKM}}(\Phi(\rho), \Phi(\sigma)) \geq \left(\frac{\pi}{\cosh(\pi t)} \frac{\|\rho - \mathcal{R}_{\sigma,\Phi}^t \circ \Phi(\rho)\|_1}{(4(\sqrt{\lambda}+\lambda) + 1 + (\epsilon e)^{-\frac{1}{2}})} \right)^{\frac{4}{1-2\epsilon}},$$

Proof. By choosing $w(s) = 1 + s$, Lemma 4.4 applies with the constant $C_g(a, b) = 1$ and $W_{a,b} = b - a + \ln b - \ln a$. With the notations of Lemma 4.4, we hence have that, by choosing $\delta := \min\{h_3,\ 1\}$, $a = b^{-1} = \delta$,

$$\begin{aligned}\|A - \mathcal{R}^t_{\rho,\Phi}(\Phi(A))\|_1 &\le \frac{\cosh(\pi t)}{\pi}\delta^{\frac12}\left(4h_1 + 4h_2 + \Big(-2\ln\delta - \delta + \frac{1}{\delta}\Big)^{\frac12}\delta^{\frac12}\right)\\ &\le \frac{\cosh(\pi t)}{\pi}\delta^{\frac12}\left(4h_1 + 4h_2 + \Big(-2\ln\delta + \frac{1}{\delta}\Big)^{\frac12}\delta^{\frac12}\right)\\ &\overset{(1)}{\le} \frac{\cosh(\pi t)}{\pi}\delta^{\frac12}\left(4h_1 + 4h_2 + (-2\ln\delta)^{\frac12}\delta^{\frac12} + 1\right)\end{aligned}$$

where in (1) we have used the two-point inequality $(a+b)^{\frac12} \le a^{\frac12} + b^{\frac12}$ for any $a, b \ge 0$. Finally, since $2\ln(\delta) + (\epsilon e)^{-1}\delta^{-2\epsilon} \ge 0$ for any $\epsilon > 0$ and $\delta \in (0, 1]$. Together with the fact that $\delta^{-\epsilon} \ge 1$ for $\delta \in (0, 1]$ and $\epsilon > 0$, we obtain that

$$\|A - \mathcal{R}^t_{\rho,\Phi}(\Phi(A))\|_1 \le \frac{\cosh(\pi t)}{\pi}\delta^{\frac12-\epsilon}\left(4h_1 + 4h_2 + (\epsilon e)^{-\frac12} + 1\right),$$

which completes the proof. □

Another example is the α-metric. Recall that

$$\gamma^\alpha_\rho(A) = \frac12\,\mathrm{tr}(A^*\rho^{\alpha-1}A\rho^{-\alpha}) + \frac12\,\mathrm{tr}(A^*\rho^{-\alpha}B\rho^{\alpha-1})\ .$$

Let χ^2_α denote the associated quantum χ^2 divergence.

Corollary 4.7. *Let $\alpha \in (0, 1)$. Then, for all $\rho \in \mathcal{D}_+(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$, we have*

$$\gamma^\alpha_\rho(A) - \gamma^\alpha_\rho(\Phi(A)) \ge \left(\frac{\pi}{\cosh(\pi t)}\frac{\|A - \mathcal{R}^t_{\rho,\Phi}(\Phi(A))\|_1}{K(\rho, A, \alpha)}\right)^{\frac{2}{\alpha}},$$

where

$$\begin{aligned}K(\rho, A, \alpha) &= 4\,\mathrm{tr}(A^*\rho^{-1}A)^{\frac12} + 2\,\mathrm{tr}(\rho^{-2}A^*\rho A)^{\frac12} + 2\,\mathrm{tr}(\Phi(\rho)^{-2}\Phi(A)^*\Phi(\rho)\Phi(A))^{\frac12}\\ &\quad + \sqrt{\frac{\pi}{\alpha\sin(\pi\alpha)}}.\end{aligned}$$

If $\sigma \ge \lambda I$, then

$$\chi^2_\alpha(\rho, \sigma) - \chi^2_\alpha(\Phi(\rho), \Phi(\sigma)) \ge \left(\frac{\pi}{\cosh(\pi t)}\frac{\|\rho - \mathcal{R}^t_{\sigma,\Phi}\circ\Phi(\rho)\|_1}{(4(\sqrt{\lambda} + \lambda) + \sqrt{\frac{2\pi}{\alpha\sin(\pi\alpha)}})}\right)^{\frac{4}{1-\alpha}},$$

Proof. We take $w(s) = s^\alpha$ and $C_\alpha = \frac{2\pi}{\sin(\pi\alpha)}$ so that $W_{a,b} = \int_a^b x^{\alpha-1}dx = \frac{1}{\alpha}(b^\alpha - a^\alpha)$. Then by Lemma 4.4, we have that, given $\delta := \min\{h_3,\ 1\}$ and $a = b^{-1} = \delta$,

$$\begin{aligned}&\|A - \mathcal{R}^t_{\rho,\Phi}(\Phi(A))\|_1\\ &\quad\le \frac{\cosh(\pi t)}{\pi}\Big(4h_1\delta^{1/2} + 4h_2\delta^{1/2} + \sqrt{\alpha^{-1}C_\alpha(\delta^{-\alpha} - \delta^\alpha)\delta}\Big)\\ &\quad\le \frac{\cosh(\pi t)}{\pi}\Big(4h_1 + 4h_2 + \sqrt{\frac{2\pi}{\alpha\sin(\pi\alpha)}}\Big)\delta^{\frac{1-\alpha}{2}}\ .\end{aligned}$$

□

4.2. Approximate recovery for $x^{\frac{1}{2}}$ metrics. In this section, we discuss approximate recovery bounds for $x^{\frac{1}{2}}$ metrics. Our starting point is a simple argument for general monotone metrics. Given a quantum channel Φ, the data processing inequality

$$\langle \Phi(A), \mathbb{J}^g_{\Phi(\rho)}(\Phi(A))\rangle = \gamma^g_{\Phi(\rho)}(\Phi(A), \Phi(A)) \le \gamma^g_\rho(A, A) = \langle A, \mathbb{J}^g_\rho(A)\rangle \ . \tag{4.4}$$

can be interpreted as

$$\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi \le \mathbb{J}^g_\rho \ ,$$

as positive operators on the Hilbert–Schmidt space $\mathcal{S}_2(\mathcal{H})$. By the operator anti-monotonicity of $x \mapsto x^{-1}$, we have

$$(\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi)^{-1} \ge (\mathbb{J}^g_\rho)^{-1} \ .$$

This leads to the following lemma.

Lemma 4.8. *For any density operator $\rho \in \mathcal{D}_+(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$,*

$$\gamma^g_\rho(A) - \gamma^g_{\Phi(\rho)}(\Phi(A)) \ge \gamma^g_\rho(A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A)) \ge \| A - \mathbb{J}^g_\rho \Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A) \|_1^2 \tag{4.5}$$

Proof. We have

$$\begin{aligned}
\gamma^g_\rho(A) - \gamma^g_{\Phi(\rho)}(\Phi(A)) &= \langle A, \mathbb{J}^g_\rho A\rangle - \langle \Phi(A), \mathbb{J}^g_{\Phi(\rho)}\Phi(A)\rangle \\
&= \langle A, \mathbb{J}^g_\rho(A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A))\rangle \\
&\overset{(1)}{\ge} \langle A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A), \mathbb{J}^g_\rho(A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A))\rangle \\
&= \gamma^g_\rho\Big(A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A)\Big) \\
&\overset{(2)}{\ge} \| A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A) \|_1^2
\end{aligned}$$

Here the first inequality follows from

$$\begin{aligned}
&\langle A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A), \mathbb{J}^g_\rho(A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A))\rangle \\
&\quad = \langle A, \mathbb{J}^g_\rho(A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi)\rangle - \langle (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A), \\
&\quad \mathbb{J}^g_\rho(A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A))\rangle \\
&\quad = \langle A, \mathbb{J}^g_\rho(A - (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi)\rangle - \langle \Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A), A\rangle + \langle \Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A), \\
&\quad (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A))\rangle
\end{aligned}$$

and for the last two terms we have

$$\begin{aligned}
\langle \Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A), (\mathbb{J}^g_\rho)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A)\rangle &\le \langle \Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A), \Big(\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi\Big)^{-1}\Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A)\rangle \\
&= \langle A, \Phi^\dagger \mathbb{J}^g_{\Phi(\rho)}\Phi(A)\rangle \, .
\end{aligned}$$

The second inequality was found in [TKR10], the proof of which we include for completeness: it is sufficient to consider the Bures (SLD) metric

$$\gamma_\rho^{\mathrm{Bures}}(X) = \langle X, \mathbb{J}_\rho^b(X)\rangle \ , \ \mathbb{J}_\rho^b = 2(L_\rho + R_\rho)^{-1} \ ,$$

since $\gamma_\rho^{\mathrm{Bures}}(X) \le \gamma_\rho^g(X)$ for any g. Then

$$\begin{aligned}\| X \|_1^2 &= \sup_U |\operatorname{tr}(XU)|^2 = \sup_U |\langle U, X\rangle|^2 \\ &= \sup_U \langle (\mathbb{J}_\rho^b)^{-\frac{1}{2}} U, (\mathbb{J}_\rho^b)^{\frac{1}{2}} X\rangle \\ &\le |\sup_U \langle U, (\mathbb{J}_\rho^b)^{-1}(U)\rangle| \cdot \langle X, \mathbb{J}_\rho^b(X)\rangle \\ &= |\sup_U \frac{1}{2}\operatorname{tr}(U^*U\rho + U^*\rho U))| \cdot \gamma_\rho^{\mathrm{Bures}}(X) = \gamma_\rho^{\mathrm{Bures}}(X).\end{aligned}$$

That completes the proof. □

The above estimate is nice but not necessarily gives a recovery bound. The reason is that the map

$$(\mathbb{J}_\rho^g)^{-1}\Phi^\dagger \mathbb{J}_{\Phi(\rho)}^g$$

is not necessarily a channel. Indeed, such a map is always trace preserving because the adjoint is unital

$$\mathbb{J}_{\Phi(\rho)}^g \Phi (\mathbb{J}_\rho^g)^{-1}(1) = \mathbb{J}_{\Phi(\rho)}^g(\Phi(\rho)) = 1 \ .$$

But the map $(\mathbb{J}_\rho^g)^{-1}$ may not be positive, as it is the case for the Bures metric

$$(\mathbb{J}_\rho^b)^{-1}(A) = \frac{1}{2}(\rho A + A\rho)$$

Nevertheless, the situation simplifies in the case of the $x^{\frac{1}{2}}$-metric:

$$\gamma_\rho^{\frac{1}{2}}(A, B) = \operatorname{tr}(A^*\rho^{-\frac{1}{2}} B\rho^{-\frac{1}{2}}) = \langle A, \mathbb{J}_\rho^{\frac{1}{2}}(B)\rangle \ , \quad \chi_{\frac{1}{2}}^2(\rho, \sigma) = \gamma_\sigma^{\frac{1}{2}}(\rho - \sigma, \rho - \sigma) \ ,$$

where the multiplication operator is

$$\mathbb{J}_\rho^{\frac{1}{2}}(A) = \rho^{-\frac{1}{2}} A \rho^{-\frac{1}{2}} \ .$$

The inverse operator is

$$(\mathbb{J}_\rho^{\frac{1}{2}})^{-1}(A) = \rho^{\frac{1}{2}} A \rho^{\frac{1}{2}} \ .$$

Then it is clear that for $x^{\frac{1}{2}}$-metric,

$$(\mathbb{J}_\rho^{\frac{1}{2}})^{-1}\Phi^\dagger \mathbb{J}_{\Phi(\rho)}^{\frac{1}{2}} = \mathcal{R}_{\rho,\Phi}$$

gives the Petz recovery map. Thus we have the following simple recovery bound for $x^{\frac{1}{2}}$-metric

Theorem 4.9. *For any density operator $\rho \in \mathcal{D}_+(\mathcal{H})$ and $A \in \mathcal{B}(\mathcal{H})$,*

$$\gamma_\rho^{\frac{1}{2}}(A) - \gamma_{\Phi(\rho)}^{\frac{1}{2}}(\Phi(A)) \geq \gamma_\rho^{\frac{1}{2}}(A - \mathcal{R}_{\rho,\Phi} \circ \Phi(A)) \geq \| A - \mathcal{R}_{\rho,\Phi} \circ \Phi(A) \|_1^2 \qquad (4.6)$$

In terms of quantum χ^2 divergence, for two quantum states ρ and σ with $\text{supp}(\rho) \leq \text{supp}(\sigma)$*,*

$$\chi^2_{\frac{1}{2}}(\rho, \sigma) - \chi^2_{\frac{1}{2}}(\Phi(\rho), \Phi(\sigma)) \geq \| \rho - \mathcal{R}_{\sigma,\Phi} \circ \Phi(\rho) \|_1^2,$$

where $\mathcal{R}_{\sigma,\Phi}$ is the Petz recovery map.

Remark 4.10. A weaker inequality

$$\| \rho - \mathcal{R}_{\sigma,\Phi} \circ \Phi(\rho) \|_1^2 \leq \|\rho\|_2^2 \, \| \rho^{-1} \|_\infty \left(\chi^2_{\frac{1}{2}}(\rho, \sigma) - \chi^2_{\frac{1}{2}}(\Phi(\rho), \Phi(\sigma)) \right)$$

was obtained in [CS22]. Here we removed the singularity term $\|\rho\|_2^2 \, \| \rho^{-1} \|$. Note that for any density operators ρ, $\|\rho\|_2^2 \, \| \rho^{-1} \| \geq \|\rho\|_1 = 1$ but in infinite dimensional $\mathcal{B}(\mathcal{H})$, $\|\rho\|_2^2 \, \| \rho^{-1} \|_= +\infty$ always.

By Corollary 4.7 and choosing $t = 0$, we have

$$\| \rho - \mathcal{R}_{\sigma,\Phi} \circ \Phi(\rho) \|_1^2 \leq \frac{K(\sigma, \rho - \sigma, \frac{1}{2})^2}{\pi^2} \left(\chi^2_{\frac{1}{2}}(\rho, \sigma) - \chi^2_{\frac{1}{2}}(\Phi(\rho), \Phi(\sigma)) \right)^{\frac{1}{2}} ,$$

where $K(\sigma, \rho - \sigma, \frac{1}{2}) \geq \sqrt{2\pi}$. Therefore, the recovery bound in Theorem 4.9 is much tighter than Corollary 4.7 when $\chi^2_{\frac{1}{2}}(\rho, \sigma) - \chi^2_{\frac{1}{2}}(\Phi(\rho), \Phi(\sigma))$ is small.

5. Sufficiency via Quantum Fisher Information

In this section, we discuss sufficiency of quantum Fisher information. Let (a, b) be an interval and $\{\rho_\theta\}_{\theta \in (a,b)}$ be a smooth one-parameter family of quantum states. Given a monotone metric γ^g in Definition 3.1, the associated quantum Fisher information of the family $\{\rho_\theta\}$ is defined as

$$I_\rho^g(\theta) = \gamma_{\rho_\theta}^g(\dot{\rho}_\theta) \ ,$$

where $\dot{\rho}_\theta = \frac{d}{d\theta}\rho_\theta$ is the derivative of ρ_θ with respect to the parameter θ. It is inherited from γ^g that I^g satisfies the data processing inequality: for any quantum channel Φ

$$I_\rho^g(\theta) \geq I_{\Phi(\rho)}^g(\theta)$$

where the right-hand side is the Fisher information of the family $\theta \mapsto \Phi(\rho_\theta)$. Recall that we say a channel Φ is sufficient for $\{\rho_\theta\}_{\theta \in (a,b)}$ if there exists a recovery channel $\mathcal{R}$ such that $\mathcal{R} \circ \Phi(\rho_\theta) = \rho_\theta$ for any $\theta \in (a, b)$.

Theorem 5.1. *Let $(\rho_\theta)_{\theta \in (a,b)}$ be a smooth family of full-rank quantum states in $\mathcal{D}_+(\mathcal{H})$. A quantum channel $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is sufficient for $\{\rho_\theta\}_\theta$ if and only if*

$$I_\rho^g(\theta) = I_{\Phi(\rho)}^g(\theta) \ , \ \forall\theta$$

for all/some regular quantum Fisher information I_ρ^g. Moreover, the recovery map $\mathcal{R}$ can be chosen as the (rotated) Petz recovery map $\mathcal{R}^t_{\rho_o,\Phi}$ for any $o \in (a, b)$.

Proof. As Theorem 3.4 is valid on the support of ρ_θ, $I^g_\rho(\theta) = I^g_{\Phi(\rho)}(\theta)$ implies that for any $t \in \mathbb{R}$ and $\theta \in (a, b)$,

$$\dot{\rho}_\theta = \mathcal{R}^t_{\rho_\theta,\Phi}(\Phi(\dot{\rho}_\theta)).$$

Thus for each $t \in \mathbb{R}$,

$$\rho_\theta^{-\frac{1}{2}+it}\dot{\rho}_\theta\rho_\theta^{-\frac{1}{2}-it} = \Phi^*\Big(\Phi(\rho_\theta)^{-\frac{1}{2}+it}\Phi(\dot{\rho}_\theta)\Phi(\rho)_\theta^{-\frac{1}{2}-it}\Big)$$

Recall that

$$\frac{d}{d\theta}\log\rho_\theta = \int_{\mathbb{R}} \rho_\theta^{-\frac{1+it}{2}}\dot{\rho}_\theta\rho_\theta^{-\frac{1-it}{2}}d\beta(t)$$

where t is integrating over the probablity measure $d\beta(t) = \frac{\pi}{2(\cosh(\pi t)+1)}dt$, for example see [SBT17]. Thus we have

$$\frac{d}{d\theta}\log\rho_\theta = \Phi^*\Big(\frac{d}{d\theta}\log\Phi(\rho_\theta)\Big).$$

Integrating the above equality over any finite interval $[o, \theta] \subset (a, b)$, we have

$$\log\rho_\theta - \log\rho_o = \Phi^*(\log\Phi(\rho_\theta) - \log\Phi(\rho_o)).$$

We therefore obtain

$$D(\rho_\theta\|\rho_o) = D(\Phi(\rho_\theta)\|\Phi(\rho_o))$$

which implies $\rho_\theta = \mathcal{R}^t_{\rho_o,\Phi}\circ\Phi(\rho_\theta)$ for any θ and $t \in \mathbb{R}$. □

The above result can be easily extended to the case of multivariate parameters. Let $\Theta \subset \mathbb{R}^n$ be a simple connected region and $\{\rho_\theta\}_{\theta\in\Theta}$ be a smooth family of quantum states. The Fisher information matrix is defined as

$$I^g_\rho(\theta) := [I^g_\rho(\theta)]_{ij} = \left[\gamma^g_{\rho_\theta}(\partial_i\rho_\theta, \partial_j\rho_\theta)\right]_{1\le i,j\le n} .$$

$I^g_\rho(\theta)$ is a real $n\times n$ positive semi-definite matrix. It is known that for any quantum channel Φ,

$$I^g_\rho(\theta) \ge I^g_{\Phi(\rho)}(\theta)$$

as positive semi-definite matrix for every θ.

Theorem 5.2. *Let $(\rho_\theta)_{\theta\in\Theta}$ be a smooth family of full-rank quantum states in $\mathcal{D}(\mathcal{H})$. A quantum channel Φ is sufficient for $\{\rho_\theta\}$ if and only if*

$$I^g_\rho(\theta) = I^g_{\Phi(\rho)}(\theta)\ ,\ \forall\theta\ ,$$

for all/some regular quantum Fisher information I^g.

Proof. Take a smooth path $\eta : [0,1] \to \Theta$ such that $\eta(0) = o$ and $\eta(1) = \theta$. Then $\rho_s := \rho_{\eta(s)}$ is a one parameter family of states and

$$\dot{\eta}(s) = D_{\dot{\eta}(s)}\rho_\theta = \sum_j \dot{\eta}_j(s)\partial_j \rho_\theta$$

where $\dot{\eta}(s)$ is the derivative of η at s. Then

$$I^g_\rho(s) = \gamma_{\rho_s}(D_{\dot{\eta}(s)}\rho_\theta, D_{\dot{\eta}(s)}\rho_\theta) \ .$$

If

$$I_\rho(\theta) = I_{\Phi(\rho)}(\theta)$$

as positive semi-definite matrix, we have

$$I_\rho(s) = \sum_{ij}[I_\rho(\theta)]_{ij}\dot{\eta}_i(s)\dot{\eta}_j(s) = \sum_{ij}[I_{\Phi(\rho)}(\theta)]_{ij}\dot{\eta}_i(s)\dot{\eta}_j(s) = I_{\Phi(\rho)}(s)$$

where the right hand side is the QFI for the family $\Phi(\rho)_s := \Phi(\rho)_{\eta(s)}$. Then the assertion follows from the one dimensional case. □

We now discuss an approximate recovery bound using BKM Fisher information. Recall the BKM metric is

$$\gamma^{\mathrm{BKM}}_\rho(X) = \langle X, \mathbb{J}^{\mathrm{BKM}}_\rho(X)\rangle = \int_0^\infty \mathrm{tr}(X^*(\rho+s)^{-1}X(\rho+s)^{-1})ds \ .$$

The inverse map is

$$(\mathbb{J}^{\mathrm{BKM}}_\rho)^{-1}(X) = \int_0^1 \rho^t X \rho^{1-t} dt$$

It is easy to see that if $\rho \le C\sigma$ for $C > 0$, then

$$\langle X, (\mathbb{J}^{\mathrm{BKM}}_\rho)^{-1}X\rangle \le C\langle X, (\mathbb{J}^{\mathrm{BKM}}_\sigma)^{-1}X\rangle$$

Note that here the optimal constant C is the $D_{\max}$ relative entropy up to a logarithm

$$D_{\max}(\rho\|\sigma) = \log\inf\{C > 0 \mid \rho \le C\sigma\}$$

Theorem 5.3. *Let* $\{\rho_\theta\}_{\theta\in[0,1]}$ *be a smooth family of quantum states. Then for any quantum channel* Φ*, denoting* $\Phi(\rho)_\theta := \Phi(\rho_\theta)$*,*

$$D(\rho_t\|\rho_0) - D(\Phi(\rho)_t\|\Phi(\rho)_0) \le \int_0^t e^{\frac{1}{2}D_{\max}(\rho_0\|\rho_\theta)}\sqrt{I^{\mathrm{BKM}}_\rho(\theta) - I^{\mathrm{BKM}}_{\Phi(\rho)}(\theta)}\,d\theta \ . \quad (5.1)$$

In particular, if $\rho_\theta \ge \lambda 1$ *for any* θ*,*

$$D(\rho_1\|\rho_0) - D(\Phi(\rho)_1\|\Phi(\rho)_0) \le \lambda^{-\frac{1}{2}}\int_0^1 \sqrt{I^{\mathrm{BKM}}_\rho(\theta) - I^{\mathrm{BKM}}_{\Phi(\rho)}(\theta)}\,d\theta \ .$$

Proof. In the following, we use the short notations $\gamma = \gamma^{\mathrm{BKM}}$, $I = I^{\mathrm{BKM}}$ and $\mathbb{J} = \mathbb{J}^{\mathrm{BKM}}$. By Lemma 4.8, we have

$$\begin{aligned} I_\rho(\theta) - I_{\Phi(\rho)}(\theta) &\geq \| \mathbb{J}_{\rho_\theta}^{-\frac{1}{2}} (\mathbb{J}_{\rho_\theta}(\dot{\rho}_\theta) - \Phi^\dagger \mathbb{J}_{\Phi(\rho_\theta)} \Phi(\dot{\rho}_\theta)) \|_2^2 \\ &\geq e^{-D_{\max}(\rho_0 \| \rho_\theta)} \| \mathbb{J}_{\rho_0}^{-\frac{1}{2}} (\mathbb{J}_{\rho_\theta}(\dot{\rho}_\theta) - \Phi^\dagger \mathbb{J}_{\Phi(\rho_\theta)} (\dot{\Phi(\rho)}_\theta)) \|_2^2 \\ &= e^{-D_{\max}(\rho_0 \| \rho_\theta)} \| \mathbb{J}_{\rho_0}^{-\frac{1}{2}} ((\dot{\log \rho_\theta}) - \Phi^\dagger (\log \dot{\Phi}(\rho)_\theta) \|_2^2 \end{aligned}$$

Then

$$\begin{aligned} &\left\| \mathbb{J}_{\rho_0}^{-\frac{1}{2}} \Big(\log \rho_1 - \log \rho_0 - \Phi^\dagger (\log \Phi(\rho_1) - \log \Phi(\rho_0)) \Big) \right\|_2 \\ &\leq \int_0^1 \| \mathbb{J}_{\rho_0}^{-\frac{1}{2}} ((\dot{\log \rho_\theta}) - \Phi^\dagger (\log \dot{\Phi}(\rho)_\theta)) \|_2 \, d\theta \\ &\leq \int_0^1 e^{\frac{1}{2} D_{\max}(\rho_0 \| \rho_\theta)} \sqrt{I_\rho(\theta) - I_{\Phi(\rho)}(\theta)} \, d\theta \, . \end{aligned}$$

Therefore,

$$\begin{aligned} D(\rho_t \| \rho_0) - D(\Phi(\rho)_t \| \Phi(\rho)_0) &= \mathrm{tr} \Big(\rho_t \big(\log \rho_t - \log \rho_0 - \Phi^\dagger (\log \Phi(\rho_t) - \log \Phi(\rho_0)) \big) \Big) \\ &\leq \left\| \mathbb{J}_{\rho_t}^{-\frac{1}{2}} \Big(\log \rho_t - \log \rho_0 - \Phi^\dagger \big(\log \Phi(\rho_t) - \log \Phi(\rho_0) \big) \Big) \right\|_2 \\ &\leq \int_0^t e^{\frac{1}{2} D_{\max}(\rho_t \| \rho_\theta)} \sqrt{I_\rho(\theta) - I_{\Phi(\rho)}(\theta)} \, d\theta \end{aligned}$$

□

Remark 5.4. The above estimate gives a recovery bound via

$$D(\rho \| \sigma) - D(\Phi(\rho) \| \Phi(\sigma)) \geq \| \rho - \mathcal{R}_{\sigma,\Phi}^{\mathrm{uni}} \circ \Phi(\rho) \|_1^2, \tag{5.2}$$

where $\mathcal{R}_{\sigma,\Phi}^{\mathrm{uni}}$ is the universal recovery map. In the case when $\rho_1 = \rho_t$ is a small perturbation of ρ_0 and $t \to 0$, the estimate (5.1) is not of optimal asymptotic order, because when t is small,

$$D(\rho_t \| \rho_0) - D(\Phi(\rho)_t \| \Phi(\rho)_0) = (I_\rho(0) - I_{\Phi(\rho)}(0)) t^2 + O(t^3),$$

but (5.1) gives $\sim \sqrt{I_\rho(0) - I_{\Phi(\rho)}(0)} t$.

6. Recoverability of Asymmetry

In general, the resource theory of asymmetry can be defined for all symmetry groups. Let G be a compact Lie group and $\mathfrak{g}$ be its Lie algebra. We assume each system under consideration has a given unitary representation of G. To quantify the asymmetry of a state ρ with respect to symmetry G, we consider the QFI metric for the family of states $\{\rho_g = U(g)\rho U^*, g \in G\}$, where U is the unitary representation on the system. To express the metric, it suffices to consider a neighborhood of the identity element e of the Lie group, which can be smoothly parameterized by n real parameters, denoted by $\Theta = (\theta_1, \cdots, \theta_n) \in \Omega \subset \mathbb{R}^n$, where $n = \dim(\mathfrak{g})$. Then for the parametrized family $\rho_\Theta = U(g(\Theta))\rho U^*(g(\Theta))$, the QFI matrix at ρ is the $n \times n$ matrix

$$I_G(\rho) := \Big[\gamma_\rho(\partial_i \rho_g, \partial_j \rho_g)\Big|_{g=e}\Big]_{1\le i,j\le n} = \Big[\gamma_\rho([L_i, \rho], [L_j, \rho])\Big]_{1\le i,j\le n} \tag{6.1}$$

where $\partial_i \rho_g = \frac{\partial \rho_{g(\Theta)}}{\partial \theta_i}$ is the partial derivatives and the skew-Hermitian operators

$$L_i = \frac{\partial U(g(\Theta))}{\partial \theta_i}\Big|_{g=e} \quad i = 1, \cdots, n \tag{6.2}$$

form a basis for the representation of the Lie algebra $\mathfrak{g}$ induced by the unitary representation of G.

As an example, one can choose the local parametrization by the exponential map such that for $\Theta = (\theta_1, \cdots, \theta_n)$, the corresponding unitary is $U(g(\Theta)) = \exp(\sum_i \theta_i L_i)$.[7] For instance, in the case of $SO(3)$ symmetry corresponding to rotations in $3D$ space, the parametrization can be chosen with the angular momentum operators L_1, L_2, L_3 in x, y, z directions. Then, the unitary $\exp(\phi \sum_i n_i L_i)$ for a real vector $\hat{n} = (n_1, n_2, n_3)$ with the normalization $\|\hat{n}\|_2 = 1$, corresponds to the rotation by angle ϕ around the axis $\hat{n}$. In this case the QFI metric will be a 3×3 real matrix that determines the sensitivity of state ρ under rotations around x, y, z axes.

As we saw in Eq. (1.17) in the case of time translation symmetry, the value of QFI with respect to the time parameter t is constant for the entire family of time-evolved versions of state. However, this is not the case for QFI matrix on a general group G, which will depend on the group element $g \in G$. Nevertheless, it turns out that for compact connected Lie groups, the QFI matrix $I_G(\rho)$ determines the QFI matrix for the entire family $\rho_g = U(g)\rho U^*(g) : g \in G$. Let $\eta = (\eta_1, \cdots, \eta_n) \in \mathbb{R}^n$ be a local coordinate at point $g \in G$. Then, the QFI matrix defined relative to this coordinate system is related to the QFI matrix at the identity via the congruence transformation

$$\Big[\gamma_\rho(\partial_i \rho_g, \partial_j \rho_g)\Big]_{i,j} = V^T(g)\Big[\gamma_\rho(\partial_i \rho_g, \partial_j \rho_g)\Big|_{g=e}\Big]_{i,j} V(g) = V^T(g) I_G(\rho) V(g) \tag{6.3}$$

where $\partial_i \rho_g = \partial \rho_{g(\Phi)}/\partial \eta_i$ and $V(g)$ is an invertible $n \times n$ real matrix defined by equation

$$U^*(g)\frac{\partial}{\partial \eta_i} U(g(\Phi)) = \sum_{r=1}^{n} V_{ri}(g) L_r. \tag{6.4}$$

Note that here the matrix V is only determined by the local coordinate (or equivalently the basis in Lie algebra) but independent of the representation U.[8]

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[7] Note that in the case of compact connected Lie groups, the exponential map from the Lie algebra to the Lie group is surjective.

[8] The exact form of matrix $V(g)$ can be obtained, e.g., via equation

$$\frac{d}{ds}\exp(X(s)) = \exp(X(s))\frac{1 - \exp(-\mathrm{ad}_{X(s)})}{\mathrm{ad}_{X(s)}}\frac{d}{ds}X(s)$$

where $\mathrm{ad}_X(Y) = [X, Y]$.

A CPTP map $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ respects this symmetry, or is called covariant, if it satisfies the covariance condition

$$\mathcal{E}\big(U_A(g)(\cdot)U_A^*(g)\big) = U_B(g)\mathcal{E}(\cdot)U_B^*(g)\,\forall g \in G \tag{6.5}$$

where U_A and U_B are the given representations of G on the input system $\mathcal{H}_A$ and output systems $\mathcal{H}_B$, respectively. Then, for the family $\rho_g = U_A(g)\rho U_A^*(g)$, $\rho \in \mathcal{B}(\mathcal{H}_A)$,

$$\mathcal{E}(\rho_g) = \mathcal{E}\big(U_A(g)\rho U_A^*(g)\big) = U(g)\mathcal{E}(\rho)U^*(g) = \mathcal{E}(\rho)_g$$

The monotonicity of the QFI matrix under data processing then implies

$$I_G(\rho) \geq I_G(\mathcal{E}(\rho)). \tag{6.6}$$

as matrices, for any covariant map $\mathcal{E}$ and any QFI metric I_g. In the case of regular QFI metrics, our result in Theorem 5.1 implies that conservation of QFI metric guarantees reversibility.

Theorem 6.1. *Let G be a compact connected Lie group and let $\mathcal{H}_A$ and $\mathcal{H}_B$ be two systems with representations U_A and U_B of the group G. Let $\rho \in \mathcal{B}(\mathcal{H}_A)$ be a full rank density operator and $\mathcal{E} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$ be a covariant quantum channel. Then, this process is reversible with a covariant operation $\mathcal{R}$ from B to A, such that $\mathcal{R}(\mathcal{E}(\rho)) = \rho$, if and only if,*

$$I_G(\rho) = I_G(\mathcal{E}(\rho))$$

where I_G can be the QFI matrix in Eq. (6.1) defined by some/all regular QFI metric

The proof of this theorem is similar to the case of time-translation symmetry for the special case of a periodic representation, in which it suffices to consider the average Petz map Eq. 1.22 with the time T being the period.

Proof. By the congruence transformation (6.3), we see $I_G(\rho) = I_G(\mathcal{E}(\rho))$ implies the QFI matrix

$$I_\rho(g) = I_{\mathcal{E}(\rho)}(g)\ ,$$

are preserved at $\rho_g = U_A(g)\rho U_A(g)^*$ for every $g \in G$. By Theorem 5.1, this implies the Petz recovery map $\mathcal{R}_{\rho,\mathcal{E}}$ satisfies

$$\mathcal{R}_{\rho,\mathcal{E}}\big(U_B(g)\mathcal{E}(\rho)U_B(g)^*\big) = U_A(g)\rho\, U_A(g)^*\ , \tag{6.7}$$

for all $g \in G$. In general, this Petz map $\mathcal{R}_{\rho,\mathcal{E}}$ is not covariant. By twirling this map with the uniform (Haar) measure over the group G, we obtain the map

$$\mathcal{R}_{\text{avg}}(\cdot) := \int_G U_A(g)^*\mathcal{R}_{\rho,\mathcal{E}}\left(U_B(g)(\cdot)U_B(g)^*\right)U_A(g)d\mu(g)\ , \tag{6.8}$$

which is covariant and satisfies $\mathcal{R}_{\text{avg}}(\mathcal{E}(\rho)) = \rho$. This completes the proof. □

7. Final Discussion

In this work, we discover a dichotomy that the preservation of quantum analogs of Fisher information can or cannot characterize the sufficiency of quantum channels, depending on the associated measures in integral representations. This divides the quantum Fisher information into two categories: the regular ones, such as BMK metric and WYD metric, are proven to be able to characterize channel sufficiency; the most-known SLD QFI, surprisingly, does not satisfy this property. Such phenomenon is completely new in the quantum regime, as there is only one classical Fisher information whose preservation guarantees channel sufficiency. For SLD QFI, we construct an explicit counterexample of a parameterized family of qubit states, whose SLD QFI does not decrease under a pinching map but the family is not recoverable from the pinched states. It remains open that whether a similar counter-example can be constructed for RLD quantum Fisher information.

Another important contribution of our work is the first approximate recovery bound of strongly regular monotone metric and χ_g^2 divergence in Lemma 4.4 and Corollary 4.6, as well as BKM Fisher information in Theorem 5.3. For the special case of $\chi_{\frac{1}{2}}^2$ divergence, we obtain a universal recovery bound in Theorem 4.9 with a surprisingly simple method. It significantly improves the main result of [CS22], which is not tight in large dimensions and fails for infinite dimensions.

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