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Eulerian and Lagrangian Stability in Zeitlin's Model of Hydrodynamics

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To the memory of Vladimir Zeitlin, whose beautiful model guides us still.

Abstract: The two-dimensional (2-D) Euler equations of a perfect fluid possess a beautiful geometric description: they are reduced geodesic equations on the infinite-dimensional Lie group of symplectomorphims with respect to a right-invariant Riemannian metric. This structure enables insights to Eulerian and Lagrangian stability via sectional curvature and Jacobi equations. The Zeitlin model is a finite-dimensional analogue of the 2-D Euler equations; the only known discretization that preserves the rich geometric structure. Theoretical and numerical studies indicate that Zeitlin's model provides consistent long-time behaviour on large scales, but to which extent it truly reflects the Euler equations is mainly open. Towards progress, we give here two results. First, convergence of the sectional curvature in the Euler–Zeitlin equations on the Lie algebra $\mathfrak{su}(N)$ to that of the Euler equations on the sphere. Second, L^2-convergence of the corresponding Jacobi equations for Lagrangian and Eulerian stability. The results allow geometric conclusions about Zeitlin's model to be transferred to Euler's equations and vice versa, which could expedite the ultimate aim: to characterize the generic long-time behaviour of perfect 2-D fluids.

1. Introduction

The motion of an incompressible fluid can be described in two equivalent ways: either by following the trajectory of a fixed fluid particle—the Lagrangian point of view—or by considering the velocity of fluid particles passing by a fixed point in space—the Eulerian point of view. The two viewpoints give rise to separate notions of stability with respect to perturbation of the initial conditions.

In his pioneering work, Arnold [3] showed that ideal fluid motions describe geodesics on the group of volume-preserving diffeomorphisms endowed with a right-invariant L^2 Riemannian metric. This beautiful description enables tools from Riemannian geometry in the study of ideal fluids. Now, Riemannian geodesic motion is stable in the Lagrangian sense if a perturbation of the initial conditions yields a geodesic that stays close to the unperturbed one. Infinitesimal perturbations of geodesics are given by the Jacobi fields, whose evolution is governed by the sectional curvature. Roughly speaking, negative curvature suggests instability, whereas positive curvature suggests stability. This leads to an interest in sectional curvature of volume-preserving diffeomorphism groups on various domains. Using Fourier series, Arnold first derived a formula for the sectional curvature when the fluid domain is the two-torus. Lukatsky [17], Arakelyan-Savvidy [2], Dowker-Wei [10] and Yoshida [32] then derived corresponding formulae for the sphere, Nakamura [23] for the three-torus case, whereas Lukatsky [18] and Preston [28] exposed general computations for two-dimensional (2-D) compact surfaces. Regarding the link with stability, Misiołek [19] adapted Rauch comparison theorem to prove that negative sectional curvature of a plane spanned by the velocity and a Jacobi field implies that the L^2 norm of the Jacobi field grows at least linearly in time.

Misiołek's result established "slow" (less than exponential) Lagrangian instability in time. Arnold, however, early advocated that mostly negative sectional curvature should imply "fast" (exponential) instability and based thereon he concluded that long-term weather forecasts are intrinsically unreliable (see [4, Preface and Ch. 4B]). The question was further clarified by Preston [26], who demonstrated that, although the sectional curvature is non-positive and mostly strictly negative, the Jacobi fields do not necessarily grow "fast". More precisely, he studied a splitting of the Jacobi equations and proved that (i) the sign of sectional curvature alone could not provide information about exponential Lagrangian instability, and (ii) exponential Eulerian instability always imply Lagrangian exponential instability. In summary, the notion of Lagrangian and Eulerian stability and their connection to sectional curvature is an important tool in the analysis of ideal hydrodynamics.

Another important tool for understanding 2-D ideal hydrodynamics is Zeitlin's model [33,34], which in turn is based on quantization results of Hoppe [12,13]. Zeitlin's model is the only known (spatial) finite-dimensional approximation that fully adopts Arnold's geometric description in that it also describes geodesics on a Lie group equipped with a right-invariant Riemannian metric. This structure gives rise to conservation of Casimirs, such as *enstrophy*, critical for the understanding of 2-D-specific turbulence phenomenology as described by Kraichnan [16]. In this context, Zeitlin's model provide a coherent approach to simulating the qualitative long-time behaviour of 2-D Euler equations [9,20]. Furthermore, since Zeitlin's model establish a link between hydrodynamics and matrix theory, results from the latter enable new techniques [22]. An example is canonical decomposition of the vorticity field along the stabilizer of the stream function, which captures the dynamics of large and small vortex scales formations [21]. Numerical evidences show that Zeitlin's model, contrary to traditional discretization, retain the correct qualitative behaviour, for example the spectral power laws in the inverse energy cascade [8]. Local convergence of solutions to Zeitlin's model to solutions of the Euler equations was established by Gallagher [11]. But a rigorous understanding of the observed superior long-time behaviour remains largely open.

Our objective here is to answer the following question: how well does Zeitlin's model capture the Lagrangian and Eulerian stability properties of the 2-D Euler equations? In particular, does the sectional curvature in Zeitlin's discretization converge to the sectional curvature of the Euler equations? We answer this and related questions for the case when the fluid domain is the sphere; the most relevant domain for applications in geophysical fluid dynamics (*cf.* [25,35]).

In Sect. 2 we recall the geometric formulation of Euler equations on the sphere. In Sect. 3 we then present Zeitlin's finite-dimensional analogue of the continuous Euler equations. In Sect. 4 we recall the notions of Eulerian and Lagrangian stability and the link to sectional curvature. As a first main result, we show in Sect. 6 that the sectional curvature of the Euler–Zeitlin equations converges to the sectional curvature of the Euler equations when the degrees of freedom in the model tend to infinity. In particular, this result implies that for Zeitlin's model with enough degrees of freedom, the sign of the finite- and infinite-dimensional sectional curvatures are the same. Thus, Zeitlin's model preserve the Lagrangian stability behaviour implied by the sectional curvature. The second main result, proved in Sect. 7, concerns stationary solutions of both the Euler–Zeitlin and the Euler equations. We show that Lagrangian perturbations (i.e. Jacobi fields) and Eulerian perturbations of the Zeitlin system converge in a certain sense toward corresponding perturbations of the continuous Euler equations. Thus, Zeitlin's model also preserves the stable or unstable nature of stationary solutions to the Euler equations. We are not aware of any other discretization of the 2-D Euler equations with this property.

2. Euler's Equations on the Sphere

In his seminal work, Arnold [3] described ideal incompressible fluid flows on a Riemannian manifold (M, g) with volume form μ, as an equation for geodesics on the infinite-dimensional group of volume preserving diffeomorphisms $\mathrm{Diff}_\mu(M)$, with respect to the standard L^2 metric. More precisely, Arnold showed that the geodesic equation on $\mathrm{Diff}_\mu(M)$, right translated to the Lie algebra $\mathfrak{X}_\mu(M)$ of divergence-free vector fields, yields the incompressible Euler equations

$$\begin{aligned} \dot{u} + \nabla_u u &= -\nabla p \\ \operatorname{div} u &= 0, \end{aligned} \tag{1}$$

where $\dot{u}:=\partial_t u$ denotes differentiation with respect to time, and $\nabla_u u$ denotes the covariant derivative of u along u. The equivalent Hamiltonian formulation on the cotangent bundle is described by the *kinetic energy* Hamiltonian, which is also right-invariant. Hamilton's equations can then be reduced by translation to the corresponding Lie-Poisson system on the smooth dual of the Lie algebra $\mathfrak{X}_\mu(M)^* \simeq \Omega^1(M)/\Omega^0(M)$ (*cf.* [4]).

When M is the unit sphere $\mathbb{S}^2$, there is a Lie–Poisson isomorphism between $\mathfrak{X}_\mu(M)^*$ and the space $C_0^\infty(\mathbb{S}^2)$ of smooth function with vanishing mean. The isomorphism is given by $\star d$, where $\star$ is the Hodge star and d is the exterior derivative. When identifying a one-form to a divergence-free velocity field via the metric, the aforementioned isomorphism consists in taking the curl of the velocity field which gives the *vorticity function* $\omega \in C_0^\infty(\mathbb{S}^2)$. The significance of the vorticity function is that it is transported by the vector field u. In turn, the divergence free vector field u is the Hamiltonian vector field X_ψ for some Hamiltonian $\psi \in C_0^\infty(\mathbb{S}^2)$ called the *stream function*. The Euler Eq. (1) can then be written entirely in terms of the vorticity and stream function

$$\dot{\omega} + \{\psi, \omega\} = 0, \quad \Delta\psi = \omega. \tag{2}$$

The Poisson bracket on $\mathbb{S}^2$ is given by

$$\{\psi, \omega\}(x) = x \cdot (\nabla\psi \times \nabla\omega)$$

where the gradients are taken in $\mathbb{R}^3$, extending constantly on rays functions on the sphere. The vorticity Eq. (2) is an infinite-dimensional Lie–Poisson system on $C_0^\infty(\mathbb{S}^2)$ for the Lie-Poisson bracket given by

$$\{\mathcal{F}, \mathcal{G}\}_{LP}(\omega) = \int_{\mathbb{S}^2} \omega \left\{ \frac{\delta \mathcal{F}}{\delta \omega}, \frac{\delta \mathcal{G}}{\delta \omega} \right\}$$

where $\mathcal{F}, \mathcal{G}$ are functionals : $C_0^\infty(\mathbb{S}^2) \to \mathbb{R}$, and for the specific Hamiltonian

$$\mathcal{H}(\omega) = \frac{1}{2} \int_{\mathbb{S}^2} \omega (-\Delta)^{-1} \omega. \tag{3}$$

In addition to total energy, and contrary to the 3-D Euler equations, the 2-D system possesses infinitely many constants of motion, called *Casimirs*: for any function $f \in C^\infty(\mathbb{R}, \mathbb{R})$,

$$\mathcal{C}_f(\omega) = \int_{\mathbb{S}^2} f(\omega)$$

is conserved. A distinguished class of Casimirs are those with $f(x) = x^k$ ($k \in \mathbb{N}$). In particular, the quadratic Casimir is called *enstrophy*, and, as mentioned, is critical in 2-D turbulence phenomenology [16]. Higher-order Casimirs also play a role in the formation of large-scale coherent vortex structures [1].

Notice that the Hamiltonian $\mathcal{H}$ in Eq. (3) is equal to the kinetic energy (L^2 norm) of the velocity field which in turn is the H^1 norm of the stream function

$$\mathcal{H}(\omega) = \frac{1}{2} \int_{\mathbb{S}^2} \omega (-\Delta)^{-1} \omega = \frac{1}{2} \int_{\mathbb{S}^2} u \cdot u = \frac{1}{2} \int_{\mathbb{S}^2} \psi (-\Delta) \psi,$$

where $u = X_\psi$. The connection between solutions to the vorticity Eq. (2) and geodesics on $\mathrm{Diff}_\mu(\mathbb{S}^2)$ is established as follows: if $\omega(t)$ is a solution and $\psi(t)$ the corresponding path of stream functions, then a geodesic curve $\gamma(t) \in \mathrm{Diff}_\mu(\mathbb{S}^2)$ is obtained by integrating the corresponding non-autonomous ordinary differential equation

$$\dot{\gamma}(t) = X_{\psi(t)} \circ \gamma(t). \tag{4}$$

3. Zeitlin's Model on the Sphere

Zeitlin's insight was to use quantization theory to spatially discretize the vorticity Eq. (2) by replacing the Poisson algebra of smooth functions with the matrix Lie algebra $\mathfrak{u}(N)$ of skew-Hermitian $N \times N$ complex matrices [33,34]. To achieve this, Zeitlin used an explicit quantization scheme developed by Hoppe [12,13] initially within the context of relativistic membranes. Hoppe's quantization is an example of *Toeplitz quantization* [5,6].

3.1. L_α approximation. Bordemann, Hoppe, Schaller, and Schlichenmaier [5] proposed a set of axioms to characterize a family of matrix algebras $(\mathfrak{g}_N, [\cdot,\cdot]_N)$ as an approximation of an arbitrary Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ (typically an infinite-dimensional Poisson algebra). They called it L_α *approximation*. Given, for each Lie algebra $\mathfrak{g}_N$, a distance d_N and a projection $p_N : \mathfrak{g} \to \mathfrak{g}_N$, the family $(\mathfrak{g}_N, [\cdot,\cdot]_N, d_N, p_N)$ is an L_α approximation of $\mathfrak{g}$ if for each pair $x, y \in \mathfrak{g}$

(1) $d_N(p_N(x), p_N(y)) \to 0$ as $N \to \infty$ implies $x = y$, and
(2) $d_N([p_N(x), p_N(y)]_N - p_N([x, y])) \to 0$ as $N \to \infty$.

3.2. Quantization of the sphere. Recall the L^2 orthonormal basis for $C^\infty(\mathbb{S}^2, \mathbb{C})$ provided by the *spherical harmonics*. Expressed in inclination-azimuthal coordinates $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$ they are

$$\mathcal{Y}_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_{lm}\big(\cos(\theta)\big)e^{im\phi}$$

where P_{lm} are the associated Legendre polynomials for $l \geq 0$ and $m \in \{-l, \cdots, l\}$. The spherical harmonics are eigenfunctions of the Laplace–Beltrami operator on $\mathbb{S}^2$

$$\Delta \mathcal{Y}_{lm} = -l(l+1)\mathcal{Y}_{lm}.$$

Using the spherical harmonics basis, Hoppe [12] gave in his thesis an explicit quantization of the Poisson algebra $\mathfrak{g} = (C^\infty(\mathbb{S}^2, \mathbb{C}), \{\cdot, \cdot\})$ of complex valued smooth functions (see [5, Example 3] for an exposition in terms of L_α approximations). The approximating Lie algebras are $\mathfrak{g}_N = \mathfrak{gl}(N, \mathbb{C})$ with $[\cdot, \cdot]_N = \frac{1}{\hbar_N}[\cdot, \cdot]$, where

$$\hbar_N = \frac{2}{\sqrt{N^2 - 1}}$$

and $[\cdot, \cdot]$ is the matrix commutator.

Let us introduce the following rescaled inner products on $\mathfrak{gl}(N, \mathbb{C})$

$$\langle A, B\rangle_{L^2_N} := \frac{4\pi}{N}\operatorname{tr}(A^\dagger B).$$

The distances d_N are given by the induced norms of the inner products, and the projections are defined via $p_N \mathcal{Y}_{lm} := iT^N_{lm} \in \mathfrak{gl}(N, \mathbb{C})$ for

$$(T^N_{lm})_{m1,m2} = \sqrt{\frac{N}{4\pi}}(-1)^{(N-1)/2-m}\sqrt{2l+1}\begin{pmatrix} (N-1)/2 & l & (N-1)/2 \\ -m_1 & m & m_2 \end{pmatrix}$$

where (:::) is the Wigner 3j-symbol. Note that the quantized harmonics (T^N_{lm}) form an orthonormal basis of $\mathfrak{gl}(N, \mathbb{C})$ with respect to the *rescaled* inner product. Thus, a function expanded in spherical harmonics $f = \sum_{l=0}^\infty \sum_m f^{lm}\mathcal{Y}_{lm}$ is projected onto

$$p_N(f) = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} f^{lm}\, iT^N_{lm}.$$

Recall from Sect. 2 that the vorticity and stream functions for the Euler equations on the sphere are real-valued. That is, we consider the Poisson subalgebra $C^\infty(\mathbb{S}^2)$ of real valued functions. The corresponding Lie subalgebras of $\mathfrak{gl}(N)$ are given by skew-Hermitian complex matrices $\mathfrak{u}(N)$. Furthermore, the smaller, trace free subalgebras $\mathfrak{su}(N) \subset \mathfrak{u}(N)$ correspond to $C^\infty_0(\mathbb{S}^2)$.

The last ingredient we need in order to approximate the vorticity formulation (2) of the 2-D Euler equations is an approximation $\Delta_N : \mathfrak{su}(N) \to \mathfrak{su}(N)$ of the Laplace–Beltrami operator. Given the construction above, it is natural to define it directly in terms of the basis T^N_{lm} as

$$\Delta_N T^N_{lm} = -l(l+1)T^N_{lm}$$

so that, up to truncation $l < N$, it corresponds to the Laplacian on $C^\infty(\mathbb{S}^2)$. It turns out that the quantized Laplacian Δ_N defined this way admits a beautiful, canonical description in the theory of unitary representation theory of $\mathfrak{so}(3)$; see the work of Hoppe and Yau [14] for details. In short, if $x^1, x^2, x^3 \in C^\infty(\mathbb{S}^2)$ denotes the Euclidean coordinate functions (for $\mathbb{S}^2$ embedded as the unit sphere in $\mathbb{R}^3$), and if $X_N^i = p_N x^i$, then X_N^1, X_N^2, X_N^3 are (scaled) generators of a representation of $\mathfrak{so}(3)$ in $\mathfrak{u}(N)$ and the quantized Laplacian for $F \in \mathfrak{u}(N)$ is given by

$$\Delta_N F = \frac{1}{\hbar_N^2} \sum_{i=1}^{3} [X_N^i, [X_N^i, F]]. \tag{5}$$

We now have all the components we need to define Zeitlin's model on the sphere. For more details on the projection operator p_N and the Hoppe–Yau Laplacian Δ_N, including efficient computer implementation, we refer to the work of Modin and Viviani [20] and Cifani, Modin, and Viviani [9].

3.3. The Euler–Zeitlin equations. Using the aforementioned matrix algebra approximation of $\big(C^\infty(\mathbb{S}^2), \{\cdot,\cdot\}\big)$, the *Euler–Zeitlin equations* are given by the following matrix flow for $W = W(t) \in \mathfrak{su}(N)$

$$\dot{W} + \frac{1}{\hbar_N}[P, W] = 0, \quad \Delta_N P = W, \tag{6}$$

where $W = W(t) \in \mathfrak{su}(N)$ is the *vorticity matrix* and $P = P(t) \in \mathfrak{su}(N)$ is the *stream matrix*. The remarkable feature of these equations is that they completely capture Arnold's description, but in a finite-dimensional setting: the Euler–Zeitlin equations constitute a Lie–Poisson system on the dual $\mathfrak{su}(N)^*$, which via the Frobenius inner product is identified with $\mathfrak{su}(N)$, for the Hamiltonian given by

$$\mathcal{H}_N(W) = \frac{1}{2}\left\langle W, -\Delta_N^{-1} W\right\rangle_{L_N^2} = -\frac{2\pi}{N}\operatorname{tr}(W^\dagger \Delta_N^{-1} W).$$

The associated right invariant Riemannian metric on the matrix group is determined from the inner product on the Lie algebra $\mathfrak{su}(N)$ given by

$$\langle A, B\rangle_{H_N^{-1}} := \left\langle A, -\Delta_N^{-1} B\right\rangle_{L_N^2} = -\frac{4\pi}{N}\operatorname{tr}(A^\dagger \Delta_N B).$$

The reconstruction equation for geodesics on the matrix group $\mathrm{SU}(N)$ is

$$\dot{G}(t) = P(t)G(t).$$

This equation is the direct analogue to Eq. (4) for geodesics on the infinite-dimensional group $\mathrm{Diff}_\mu(\mathbb{S}^2)$.

The Euler–Zeitlin system (6) is isospectral, which reflects the transport nature of the vorticity Eq. (2). Indeed, in addition to the Hamiltonian, the system has the following constants of motion (Casimirs)

$$\mathcal{C}_{N,k}(W) = \frac{4\pi}{N}\operatorname{tr}(W^k).$$

As $N \to \infty$ these converge to the corresponding Casimirs $\mathcal{C}_k(\omega)$ of the continuous system (see [29, Corollary 8.1.2]).

4. Stability Results

4.1. Lagrangian stability and sectional curvature. For a fluid domain given by a compact Riemannian manifold M, let us first review the notion of Lagrangian stability on $\mathrm{Diff}_\mu(M)$ and how it is related to sectional curvature. Consider a geodesic $\gamma = \gamma(t)$ on $\mathrm{Diff}_\mu(M)$, and a smooth family γ_s of geodesics such that $\gamma_0 = \gamma$. The corresponding *Jacobi field* is the vector field along γ defined by

$$J = \frac{d}{ds}\Big|_{s=0} \gamma_s \in T_\gamma \,\mathrm{Diff}_\mu(M). \tag{7}$$

It satisfies the *Jacobi equation*, which in abstract notation can be written

$$\bar{\nabla}_{\dot\gamma} \bar{\nabla}_{\dot\gamma} J + R_\gamma(J, \dot\gamma)\dot\gamma = 0, \tag{8}$$

where $\bar{\nabla}_{\dot\gamma}$ is the "big" co-variant derivative on $\mathrm{Diff}_\mu(\mathbb{S}^2)$ (as an infinite-dimensional Riemannian manifold) and R is the corresponding "big" Riemann curvature tensor (see [19] for details on the functional analytic setting). The fluid motion γ is *Lagrangian stable* if every Jacobi field along γ remains bounded in the Riemannian metric along the trajectory $t \mapsto \gamma(t)$.

From the curvature tensor one can extract *sectional curvature* for the plane spanned by two tangent vectors $U, V \in T_\varphi \,\mathrm{Diff}_\mu(M)$ as

$$C_\varphi(U, V) := \frac{\left\langle R_\varphi(U, V)V, U\right\rangle_\varphi}{\|U\|_\varphi^2 \|V\|_\varphi^2 - \langle U, V\rangle_\varphi^2},$$

where $\langle\cdot,\cdot\rangle_\varphi$ denotes the Riemannian metric and $\|\cdot\|_\varphi$ the corresponding norm. Due to right invariance, it follows that $C_\varphi(U, V) = C_{\mathrm{id}}(u, v) =: C(u, v)$ for the divergence free vector fields on M given by $u = U \circ \varphi^{-1}$ and $v = V \circ \varphi^{-1}$. Furthermore, Arnold [3] gave the explicit formula

$$\begin{aligned} C(u, v) =& \frac{1}{4}\|B(u, v) + B(v, u)\|_{L^2}^2 + \frac{1}{2}\langle [u, v], B(u, v) - B(v, u)\rangle_{L^2} \\ &- \frac{3}{4}\|[u, v]\|_{L^2}^2 - \langle B(u, u), B(v, v)\rangle_{L^2}, \end{aligned}$$

where $B\colon \mathfrak{X}_\mu(M) \times \mathfrak{X}_\mu(M) \to \mathfrak{X}_\mu(M)$ is defined by

$$\langle B(u, v), w\rangle_{L^2} = \langle u, [v, w]\rangle_{L^2}, \quad \forall u, v, w \in \mathfrak{X}_\mu(M).$$

For the case $M = \mathbb{S}^2$, with $u = X_f$ and $v = X_g$ for $f, g \in C_0^\infty(\mathbb{S}^2)$, the sectional curvature is

$$\begin{aligned} C(X_f, X_g) = & \frac{1}{4}\|\Delta^{-1}\{\Delta f, g\} + \Delta^{-1}\{\Delta g, f\}\|_{H^1}^2 \\ & + \frac{1}{2}\Big\langle \{f, g\}, \Delta^{-1}\{\Delta f, g\} - \Delta^{-1}\{\Delta g, f\}\Big\rangle_{H^1} \\ & - \frac{3}{4}\|\{f, g\}\|_{H^1}^2 - \Big\langle \Delta^{-1}\{\Delta f, f\}, \Delta^{-1}\{\Delta g, g\}\Big\rangle_{H^1}, \end{aligned} \tag{9}$$

where $\langle\cdot,\cdot\rangle_{H^1} = \langle -\Delta(\cdot), \cdot\rangle_{L^2}$ and correspondingly for $\|\cdot\|_{H^1}$. A direct analogue of this formula in the finite-dimensional case yields the sectional curvature $C_N : \mathfrak{su}(N) \times \mathfrak{su}(N) \to \mathbb{R}$ of $\mathrm{SU}(N)$ in Zeitlin's model (see Sect. 6 below).

Sectional curvature provide important information about the Lagrangian stability of the fluid. For example, Misiołek [19, Lemma 4.2] proved that if the sectional curvature of the plane spanned by $\dot\gamma$ and a Jacobi field J remains *non-positive* along γ, then J grows at least linearly in time, and, consequently, γ is at least weakly Lagrangian unstable. Similar results are valid also in finite dimensions, as pointed out by Preston [26].

We are now ready to state the first main theorem of the paper.

Theorem 1. *The sectional curvature of* $\mathrm{SU}(N)$ *(with Zeitlin's metric) converges to the sectional curvature of* $\mathrm{Diff}_\mu(\mathbb{S}^2)$ *(with Arnold's metric) as follows: for any* $f, g \in H^7(\mathbb{S}^2)$

$$\left|C_N(p_N f, p_N g) - C(X_f, X_g)\right| \leq \hbar_N \alpha \, \|f\|_{H^7}^2 \, \|g\|_{H^7}^2 \,,$$

where the constant $\alpha > 0$ *is independent of* f, g, N.

4.2. Splitting of Jacobi equation and Eulerian stability. Just as the Euler equations are expressed in the Eulerian (right reduced) variable $u = \dot\varphi \circ \varphi^{-1}$, it is natural to express the corresponding Jacobi equation (8) also in Eulerian variables. As explored by Preston [26,27], such a rewriting splits the second order Jacobi equation into two first order equations that exposes the links between Eulerian and Lagrangian stability. For the family of geodesics γ_s as before, let $u_s = \dot\gamma_s \circ \gamma_s^{-1}$ be the associated Eulerian velocity fields, each one a solution to the Euler equations. The right reduced version of the Jacobi field J in Eq. (7) is $j = J \circ \gamma^{-1}$. Since the Jacobi equation is of second order, we also need a variable corresponding to the derivative of J, namely the variation of the velocity field

$$z = \frac{d}{ds}\Big|_{s=0} u_s.$$

Under this change of variables $(J, \dot J) \leftrightarrow (j, z)$, the Jacobi equation (8) turns into the first order system

$$\dot j + [u, j] = z \tag{10a}$$
$$\dot z + P\nabla_u z + P\nabla_z u = 0 \tag{10b}$$

As expected, Eq. (10b) is the linearized Euler equation about the solution u, whereas Eq. (10a) is a reconstruction equation for the reduced Jacobi field j.

On the sphere we obtain a "vorticity formulation" expressed in the functions $\upsilon, \zeta \in C_0^\infty(\mathbb{S}^2)$ defined by $j = X_\upsilon$ and $\zeta = \operatorname{curl} z$. Geometrically, υ is an element of the Lie algebra, whereas ζ is an element of the dual. The Eq. (10) expressed in these variables become

$$\begin{aligned} \partial_t \upsilon - \left\{\Delta^{-1}\omega, \upsilon\right\} &= \Delta^{-1}\zeta \\ \partial_t \zeta - \left\{\Delta^{-1}\omega, \zeta\right\} - \left\{\Delta^{-1}\zeta, \omega\right\} &= 0, \end{aligned} \tag{11}$$

where ω fulfil the vorticity Eq. (2). Now, the solution $\omega = \omega(t)$ is called *Eulerian stable* (with respect to some norm) if every solution ζ of the system (11) is bounded uniformly in time.

Remark 4.1. On an arbitrary Lie group G endowed with a right-invariant metric, the corresponding splitting of the Jacobi equations become the following system of equations on the direct product of $\mathfrak{g} = T_e G$ with itself:

$$\dot{Y} + [\mathcal{A}^{-1}W, Y] = \mathcal{A}^{-1}Z$$
$$\dot{Z} - \mathrm{ad}^*_{\mathcal{A}^{-1}W} Z - \mathrm{ad}^*_{\mathcal{A}^{-1}Z} W = 0,$$

where $W = W(t) \in \mathfrak{g}$ is a solution to the Euler–Arnold equation for geodesic curves on G, ad^*_X is the adjoint of the linear operator $\mathrm{ad}_X = [X, \cdot]$, and $\mathcal{A}\colon \mathfrak{g} \to \mathfrak{g}^*$ is the *inertia operator* that defines the inner product on $\mathfrak{g}$.

For details, see for example [15, sec. 4].

For the Euler–Zeitlin equations (6) on $\mathfrak{su}(N)$ we get the analogue of (11)

$$\dot{Y} - \frac{1}{\hbar_N}\left[\Delta_N^{-1}W, Y\right] = \Delta_N^{-1}Z$$
$$\dot{Z} - \frac{1}{\hbar_N}\left[\Delta_N^{-1}W, Z\right] - \frac{1}{\hbar_N}\left[\Delta_N^{-1}Z, W\right] = 0. \tag{12}$$

The interpretation of these equations is two-fold: they describe at the same time a discretization of Preston's reduced Jacobi equations (11) and, independently of the connection to the Euler equations, the reduced form of the Jacobi equations for the Zeitlin model describing geodesics on $\mathrm{SU}(N)$.

To study the correspondence of Eulerian and Lagrangian stability between the Euler and Euler–Zeitlin systems we now restrict to *stationary solutions* of both systems. It means that

$$\left\{\Delta^{-1}\omega, \omega\right\} = 0 \tag{13}$$
$$\frac{1}{\hbar_N}[\Delta_N^{-1} p_N\omega, p_N\omega] = 0. \tag{14}$$

It is an open problem to characterize in which situations stationary Euler (13) implies stationary Euler–Zeitlin (14) (although we always have that if ω is stationary then Eq. (14) is fulfilled in the limit $N \to \infty$). However, the following situations are straightforward to check (*cf.* Viviani [31]):

- If, for some fixed l, the vorticity is $\omega = \sum_{m=-l}^{l} \omega^{lm}\mathcal{Y}_{lm}$ then ω and $p_N\omega$ are stationary solutions.
- If ω is *zonal* (there exists a choice of north pole on the sphere for which ω is constant on fixed latitudes) then ω and $p_N\omega$ are stationary solutions.

Define the embedding $\iota_N : \mathfrak{u}(N) \to C^\infty(\mathbb{S}^2)$ by

$$\iota_N(T^N_{lm}) = \mathcal{Y}_{lm}, \quad l = 0, \ldots, N-1. \tag{15}$$

Our second main theorem states that corresponding stationary solutions to the Euler and the Euler–Zeitlin equations share the same Eulerian and Lagrangian L^2-stability as $N \to \infty$.

Theorem 2. *Let ω be a stationary solution of the vorticity Eq. (2) such that $W = p_N\omega$ is a stationary solution of the Euler–Zeitlin equation (6). Let $\upsilon(t)$ and $\zeta(t)$ be solutions of the reduced Jacobi equations (11). Furthermore, let $Y(t)$ and $Z(t)$ be corresponding solutions of the finite-dimensional reduced Jacobi equations (12) with $Y(0) = p_N\upsilon(0)$ and $Z(0) = p_N\zeta(0)$. Then, for any fixed t,*

$$\begin{aligned} \|\iota_N Y(t) - \upsilon(t)\|_{L^2} &\longrightarrow 0 \\ \|\iota_N Z(t) - \zeta(t)\|_{L^2} &\longrightarrow 0 \end{aligned} \quad as \quad N \to \infty.$$

Moreover, the convergence is uniform on bounded intervals of t.

One interpretation is that the Zeitlin discretization (12) of the reduced Jacobi equations (11) is convergent in the sense of numerical analysis. Indeed, the proof is based on concepts of stability and consistency and is given in Sect. 7. The theorem implies that Euler–Zeitlin model preserves the stable or unstable nature of stationary solutions of Euler equations, so that results on either the continuous or the discretized system can be transferred to the other. For example, the result by Taylor [30, Thm. 4.1.1], that zonal stationary solutions that are strictly monotonous in the meridional direction are Eulerian stable.

5. Bracket Convergence and Preliminary Estimates

Before proving the two main theorems, we expose a central result used in the proofs, and then give some needed estimates.

The projection p_N can be understood as Toeplitz quantization on the sphere. The central result of interest to us is that the Poisson bracket is approximated by the Lie algebra bracket in a stronger sense than the one shown by Hoppe. To state it we first introduce the matrix operator norm, given for $A \in \mathfrak{u}(N)$ by

$$\|A\|_{L^\infty_N} := \sup_{\|x\|=1} \|Ax\|$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{C}^N$. The notation of the norm is motivated by the following consistency result.

Theorem 3. (Bordemann, Meinrenken, Schlichenmaier [6]) *For every $f \in C^\infty(\mathbb{S}^2)$ there exists $c > 0$ such that*

$$\|f\|_{L^\infty} - c\hbar_N \le \|p_N f\|_{L^\infty_N} \le \|f\|_{L^\infty}.$$

In the same paper the authors also prove convergence of the brackets as $\hbar_N \to 0$. Indeed, they prove the following result:

Theorem 4. (Bordemann, Meinrenken, Schlichenmaier [6]) *For every $f, g \in C^\infty(\mathbb{S}^2)$*

$$\left\| \frac{1}{\hbar_N}[p_N f, p_N g] - p_N\{f, g\} \right\|_{L^\infty_N} = O(\hbar_N).$$

Charles and Polterovich [7, Prop. 3.6, 3.9] exposed a sharper estimate of this quantum-classical correspondence, namely

$$\left\| \frac{1}{\hbar_N}[p_N f, p_N g] - p_N \{f, g\} \right\|_{L^\infty_N} \leq \hbar_N c_0 (\|f\|_{C^1} \|g\|_{C^3} + \|f\|_{C^2} \|g\|_{C^2} + \|f\|_{C^3} \|g\|_{C^1})$$

where $c_0 > 0$ is a constant independent of f and g, and $\|f\|_{C^k} = \max_{i \leq k} \sup \left|\nabla^i f\right|$. Via Sobolev embeddings we then obtain the following bound

$$\left\| \frac{1}{\hbar_N}[p_N f, p_N g] - p_N \{f, g\} \right\|_{L^\infty_N} \leq \hbar_N c_0 \|f\|_{H^5} \|g\|_{H^5}, \tag{16}$$

where $\|\cdot\|_{H^5}$ is a Sobolev H^5 norm (with a suitable, fixed scaling).

In addition to the spectral norm $\|\cdot\|_{L^\infty_N}$, we shall use the following inner products and norms on $\mathfrak{su}(N)$

$$\langle A, B\rangle_{L^2_N} := \frac{4\pi}{N} \operatorname{tr}(A^\dagger B), \qquad \|A\|_{L^2_N} := \sqrt{\langle A, A\rangle_{L^2_N}}$$

$$\|A\|_{L^1_N} := \frac{4\pi}{N} \sum_{k=1}^N |\lambda_k|$$

$$\langle A, B\rangle_{H^q_N} := \left\langle A, (-\Delta)^q B\right\rangle_{L^2_N} \qquad \|A\|_{H^q_N} := \sqrt{\langle A, A\rangle_{H^q_N}}$$

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of $A \in \mathfrak{su}(N)$. The first result is that the inner products converge spectrally to their continuous analogues.

Lemma 5. *Let $q \in \{-1, 0, 1\}$ and $s \geq q$. For every $f, g \in H^s(\mathbb{S}^2)$*

$$\left|\langle p_N f, p_N g\rangle_{H^q_N} - \langle f, g\rangle_{H^q}\right| \leq \hbar_N^{2(s-q)} \|f\|_{H^s} \|g\|_{H^s}$$

Proof. The embedding ι_N, defined in Eq. (15), is isometric for any pair H^q_N, H^q of inner products on $\mathfrak{u}(N)$ and $C^\infty(\mathbb{S}^2)$.

Let Π_N denote the L^2–projection on spherical harmonics with $l \geq N$. Then

$$\begin{aligned}
\left|\langle \Pi_N f, g\rangle_{H^q}\right| &= \left|\sum_{l=N}^\infty \sum_{m=-l}^l (l(l+1))^q f^{lm} g^{lm}\right| \\
&= \left|\sum_{l=N}^\infty \frac{1}{(l(l+1))^{s-q}} \sum_{m=-l}^l (l(l+1))^s f^{lm} g^{lm}\right| \\
&\leq \frac{1}{(N(N+1))^{s-q}} \sum_{l=N}^\infty \left|\sum_{m=-l}^l (l(l+1))^s f^{lm} g^{lm}\right| \\
&\leq \left(\frac{4}{(N-1)(N+1)}\right)^{s-q} \|f\|_{H^s} \|g\|_{H^s} \\
&= \hbar^{2(s-q)} \|f\|_{H^s} \|g\|_{H^s}.
\end{aligned}$$

These (non-sharp) estimates conclude the proof. □

The second lemma concerns the comparison of the different inner products and norms.

Lemma 6. *For every $A, B \in \mathfrak{su}(N)$*

$$\begin{aligned} (i) &\quad \|A\|_{H_N^{-1}} \leq \frac{1}{\sqrt{2}} \|A\|_{L_N^2} \\ (ii) &\quad \left\|\Delta_N^{-1} A\right\|_{L_N^2} \leq \frac{1}{2} \|A\|_{L_N^2} \\ (iii) &\quad \|\Delta_N A\|_{L_N^2} \leq N(N+1) \|A\|_{L_N^2} \\ (iv) &\quad \|A\|_{L_N^1} \leq \sqrt{4\pi} \|A\|_{L_N^2} \leq 4\pi \|A\|_{L_N^\infty} \\ (v) &\quad \left|\langle A, B\rangle_{L_N^2}\right| \leq \|A\|_{L_N^\infty} \|B\|_{L_N^1} . \end{aligned}$$

Proof of Lemma 6. (i) The quantized harmonics $iT_{lm} = p_N \mathcal{Y}_{lm}$ form an $\langle\cdot,\cdot\rangle_{L_N^2}$ -orthonormal basis for $\mathfrak{gl}(N,\mathbb{C})$. If we expand $A \in \mathfrak{su}(N) \subset \mathfrak{gl}(N,\mathbb{C})$ in that basis

$$A = \sum_{l=1}^{N-1} \sum_{m=-l}^{l} a^{lm}\, iT_{lm}$$

we have

$$\begin{aligned} \|A\|^2_{H_N^{-1}} &= \left\langle -\Delta_N^{-1} A, A\right\rangle_{L_N^2} = \sum_{l=1}^{N-1} \sum_{m=-l}^{l} \frac{(a^{lm})^2}{l(l+1)} \\ &\leq \frac{1}{2} \sum_{l=1}^{N-1} \sum_{m=-l}^{l} (a^{lm})^2 = \frac{1}{2} \|A\|^2_{L_N^2} . \end{aligned}$$

The estimates (ii)–(iii) follow since $2 \leq l(l+1) \leq N(N+1)$.

Let $i\lambda_1, \ldots, i\lambda_N$ denote the eigenvalues of A. The estimate (iv) then follows since

$$\begin{aligned} \|A\|_{L_N^1} &= \frac{4\pi}{N} \sum_{k=1}^{N} |\lambda_k| \leq 4\pi \left(\frac{1}{N} \sum_{k=1}^{N} |\lambda_k|^2\right)^{1/2} \\ &= \sqrt{4\pi}\, \|A\|_{L_N^2} \leq 4\pi\, \|A\|_{L_N^\infty} . \end{aligned}$$

The last estimate is direct from the definition of $\|A\|_{L_N^1}$. □

6. Proof of the Convergence of Sectional Curvature

In this section we prove Theorem 1, which states that sectional curvature of the quantized matrix algebras $(\mathfrak{su}(N), [\cdot,\cdot]_N, \langle\cdot,\cdot\rangle_{H_N^1})$ converges as $N \to \infty$ to the sectional curvature of infinite-dimensional system

$$(\mathfrak{X}_\mu(\mathbb{S}^2), [\cdot,\cdot], \langle\cdot,\cdot\rangle_{L^2}) \simeq (C_0^\infty(\mathbb{S}^2), \{\cdot,\cdot\}, \langle\cdot,\cdot\rangle_{H^1}).$$

Arnold [3] gave the general formula for sectional curvature on a Lie group G equipped with a right invariant Riemannian metric determined by an arbitrary inner product $\langle\cdot,\cdot\rangle$ and its corresponding norm $\|\cdot\|$ on the Lie algebra $\mathfrak{g}$:

$$C(\xi,\eta) = \frac{1}{4}\|B(\xi,\eta)+B(\eta,\xi)\|^2 + \frac{1}{2}\langle[\xi,\eta], B(\xi,\eta)-B(\eta,\xi)\rangle \\ -\frac{3}{4}\|[\xi,\eta]\|^2 - \langle B(\xi,\xi), B(\eta,\eta)\rangle, \tag{17}$$

where $B\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is defined by

$$\langle B(\xi,\eta),\nu\rangle = \langle \xi,[\eta,\nu]\rangle, \quad \forall\, \xi,\eta,\nu\in\mathfrak{g}.$$

In our specific case, when $\mathfrak{g}$ is the Poisson algebra of smooth functions on the sphere and the metric is Sobolev H^1, we have

$$\begin{aligned}\langle B(f,g),h\rangle_{H^1} &= \langle f,\{g,h\}\rangle_{H^1} = \langle -\Delta f,\{g,h\}\rangle_{L^2} \\ &= \langle\{-\Delta f,g\},h\rangle_{L^2} = \Big\langle \underbrace{\Delta^{-1}\{\Delta f,g\}}_{B(f,g)}, h\Big\rangle_{H^1}.\end{aligned}$$

Substitution into Arnold's formula (17) yields Eq. (9) above for the sectional curvature $C(X_f,X_g)$. It is convenient to express it in the L^2 inner product (because L^2 is bi-invariant with respect to the Poisson algebra)

$$\begin{aligned}C(X_f,X_g) = \\ &-\frac{1}{4}\Big\langle\{\Delta f,g\}+\{\Delta g,f\},\Delta^{-1}\Big(\{\Delta f,g\}+\{\Delta g,f\}\Big)\Big\rangle_{L^2} &\text{(18a)}\\ &-\frac{1}{2}\langle\{f,g\},\{\Delta f,g\}-\{\Delta g,f\}\rangle_{L^2} &\text{(18b)}\\ &+\frac{3}{4}\langle\{f,g\},\Delta\{f,g\}\rangle_{L^2} &\text{(18c)}\\ &+\Big\langle\{\Delta f,f\},\Delta^{-1}\{\Delta g,g\}\Big\rangle_{L^2}. &\text{(18d)}\end{aligned}$$

The same calculation for $\mathfrak{su}(N)$ equipped with the right invariant metric determined by the inner product $\langle\cdot,\cdot\rangle_{H^1_N}$ yields the analogous formula

$$\begin{aligned}C_N(F,G) = \\ &-\frac{1}{4\hbar_N^2}\Big\langle[\Delta_N F,G]+[\Delta_N G,F],\Delta_N^{-1}\Big([\Delta_N F,G]+[\Delta_N G,F]\Big)\Big\rangle_{L^2_N} &\text{(19a)}\\ &-\frac{1}{2\hbar_N^2}\langle[F,G],[\Delta_N F,G]-[\Delta_N G,F]\rangle_{L^2_N} &\text{(19b)}\\ &+\frac{3}{4\hbar_N^2}\langle[F,G],\Delta_N[F,G]\rangle_{L^2_N} &\text{(19c)}\\ &+\frac{1}{\hbar_N^2}\Big\langle[\Delta_N F,F],\Delta_N^{-1}[\Delta_N G,G]\Big\rangle_{L^2_N}. &\text{(19d)}\end{aligned}$$

Now, the aim is to prove that

$$\left|C(X_f, X_g) - C_N(p_N f, p_N g)\right| \to 0 \qquad \text{as} \qquad N \to \infty.$$

To this end, we carry out estimates for each corresponding term in the formulae (18) and (19). The "problematic" term is the third, since the H^1_N norm is not bounded by L^∞_N.

6.1. The first, second, and fourth terms. By construction of the quantized Laplacian we have that $p_N \circ \Delta = \Delta_N \circ p_N$. Thus,

$$[\Delta_N p_N f, p_N g] = [p_N \Delta f, p_N g] \quad \text{and} \quad [\Delta_N p_N g, p_N f] = [p_N \Delta g, p_N f].$$

If $\Pi_N : C^\infty(\mathbb{S}^2) \to C^\infty(\mathbb{S}^2)$ denotes the projection onto spherical harmonics with $l \geq N$, then for $a, b, c, d \in C^\infty_0(\mathbb{S}^2)$ and $q \in \{0, 1\}$

$$\begin{aligned}
&\left|\Big\langle \underbrace{\frac{1}{\hbar_N}[p_N a, p_N b]}_{\mathrm{I}}, \underbrace{\frac{1}{\hbar_N}[p_N c, p_N d]}_{\mathrm{II}} \Big\rangle_{H_N^{-q}} - \Big\langle \underbrace{\{a, b\}}_{\mathrm{I}'}, \underbrace{\{c, d\}}_{\mathrm{II}'} \Big\rangle_{H^{-q}} \right| = \\
&\left| \langle \mathrm{I}, \mathrm{II} \rangle_{H_N^{-q}} - \langle p_N \mathrm{I}', p_N \mathrm{II}' \rangle_{H_N^{-q}} - \underbrace{\langle \Pi_N \{a, b\}, \{c, d\} \rangle_{H^{-q}}}_{r_N} \right| \leq \\
&\frac{1}{2}\left| \langle \mathrm{I} - p_N \mathrm{I}', \mathrm{II} + p_N \mathrm{II}' \rangle_{H_N^{-q}} \right| + \frac{1}{2}\left| \langle \mathrm{I} + p_N \mathrm{I}', \mathrm{II} - p_N \mathrm{II}' \rangle_{H_N^{-q}} \right| + |r_N| \underset{\text{lem. 6}}{\leq} \\
&\frac{1}{2}\left\| \mathrm{I} - p_N \mathrm{I}' \right\|_{L_N^\infty} \left\| \Delta_N^{-q} (\mathrm{II} + p_N \mathrm{II}') \right\|_{L_N^1} + \\
&\frac{1}{2}\left\| \mathrm{II} - p_N \mathrm{II}' \right\|_{L_N^\infty} \left\| \Delta_N^{-q} (\mathrm{I} + p_N \mathrm{I}') \right\|_{L_N^1} + |r_N| \underset{\text{eq. (16)}}{\leq} \\
&\frac{\hbar_N c_0}{2} \|a\|_{H^5} \|b\|_{H^5} \left\| \Delta_N^{-q} (\mathrm{II} + p_N \mathrm{II}') \right\|_{L_N^1} + \\
&\frac{\hbar_N c_0}{2} \|c\|_{H^5} \|d\|_{H^5} \left\| \Delta_N^{-q} (\mathrm{I} + p_N \mathrm{I}') \right\|_{L_N^1} + |r_N| \leq \\
&\frac{\hbar_N c_0}{2} \|a\|_{H^5} \|b\|_{H^5} \left(\left\| \Delta_N^{-q} (\mathrm{II} - p_N \mathrm{II}') \right\|_{L_N^1} + 2 \left\| \Delta_N^{-q} p_N \mathrm{II}' \right\|_{L_N^1} \right) + \\
&\frac{\hbar_N c_0}{2} \|c\|_{H^5} \|d\|_{H^5} \left(\left\| \Delta_N^{-q} (\mathrm{I} - p_N \mathrm{I}') \right\|_{L_N^1} + 2 \left\| \Delta_N^{-q} p_N \mathrm{I}' \right\|_{L_N^1} \right) + |r_N| \underset{\text{lem. 6}}{\leq} \\
&\frac{2\hbar_N c_0 \sqrt{\pi}}{2^q} \Bigg(\|a\|_{H^5} \|b\|_{H^5} \left(\hbar_N c_0 \sqrt{\pi} \|c\|_{H^5} \|d\|_{H^5} + \left\| p_N \mathrm{II}' \right\|_{L_N^2} \right) + \\
&\|c\|_{H^5} \|d\|_{H^5} \left(\hbar_N c_0 \sqrt{\pi} \|a\|_{H^5} \|b\|_{H^5} + \left\| p_N \mathrm{I}' \right\|_{L_N^2} \right) \Bigg) + |r_N| \underset{\text{lem. 5}}{\leq} \\
&\frac{2\hbar_N c_0 \sqrt{\pi}}{2^q} \Bigg(\|a\|_{H^5} \|b\|_{H^5} \left(\hbar_N c_0 \sqrt{\pi} \|c\|_{H^5} \|d\|_{H^5} + \|\{c, d\}\|_{L^2} \right) + \\
&\|c\|_{H^5} \|d\|_{H^5} \left(\hbar_N c_0 \sqrt{\pi} \|a\|_{H^5} \|b\|_{H^5} + \|\{a, b\}\|_{L^2} \right) \Bigg) + |r_N| \underset{\text{lem. 7}}{\leq}
\end{aligned}$$

$$\frac{2\hbar_N c_0\sqrt{\pi}}{2^q}\Bigg(\|a\|_{H^5}\,\|b\|_{H^5}\left(\hbar_N c_0\sqrt{\pi}\,\|c\|_{H^5}\,\|d\|_{H^5} + \|c\|_{H^1}\,\|d\|_{H^1}\right) +$$

$$\|c\|_{H^5}\,\|d\|_{H^5}\left(\hbar_N c_0\sqrt{\pi}\,\|a\|_{H^5}\,\|b\|_{H^5} + \|a\|_{H^1}\,\|b\|_{H^1}\right)\Bigg) + |r_N|\,.$$

The last inequality follows from the well-known result:

Lemma 7. *Let $f, g \in C^1(\mathbb{S}^2)$. Then*

$$\|\{f, g\}\|_{L^2} \le \|f\|_{H^1}\,\|g\|_{H^1}\,.$$

Proof. Let $J : T\mathbb{S}^2 \to T\mathbb{S}^2$ denote the complex structure on $\mathbb{S}^2$. By Cauchy–Schwartz

$$\|\{f, g\}\|_{L^2}^2 = \int_{\mathbb{S}^2} |\nabla f \cdot J\nabla g|^2 \le \|\nabla f\|_{L^2}^2\,\|J\nabla g\|_{L^2}^2\,.$$

The estimate now follows since $\mathbb{S}^2$ is Kähler, so J is an isometry. □

By choosing a, b, c, d from the set $\{f, g, \Delta f, \Delta g\}$, to match the corresponding terms, using Sobolev embeddings and estimating $|r_N|$ with Lemma 5, we obtain

$$|(19a) - (18a) + (19b) - (18b) + (19d) - (18d)| \le \alpha\hbar_N\|f\|_{H^7}^2\|g\|_{H^7}^2$$

for a constant $\alpha > 0$ independent of f, g, N.

6.2. The third term. The term

$$|(19c) - (18c)| = \frac{3}{4}\left|\frac{1}{\hbar_N^2}\,\|[p_N f, p_N g]\|_{H_N^1}^2 - \|\{f, g\}\|_{H^1}^2\right|$$

cannot be treated directly by the estimate above, since the H_N^1 norm is not controlled by the L_N^∞ norm as $N \to \infty$. To overcome this problem we shall need the following results for the quantized Laplacian Δ_N.

Lemma 8. *For the coordinate functions $x^i \in C^\infty(\mathbb{S}^2)$ and the corresponding matrices $X_N^i = p_N x^i$, let $\nabla^{\perp,i} = \{x^i, \cdot\}$ and $\nabla_N^{\perp,i} = [X_N^i, \cdot]/\hbar_N$. Then $\nabla^{\perp,i}$ and $\nabla_N^{\perp,i}$ are p_N-related:*

$$p_N \circ \nabla^{\perp,i} = \nabla_N^{\perp,i} \circ p_N.$$

Proof. For any $l \ge 1$, the Lie group SO(3) acts on $\{f \in C^\infty(\mathbb{S}^2) \mid \Delta f = l(l+1)f\}$ via its action on $\mathbb{S}^2$. The corresponding infinitesimal representation of $\mathfrak{so}(3)$ is Hamiltonian with respect to the Poisson bracket with generators $x^1, x^2, x^3 \in C^\infty(\mathbb{S}^2)$, i.e., given exactly by $\nabla^{\perp,i}$. From the work of Hoppe and Yau [14] it follows that X_N^1, X_N^2, X_N^3 are corresponding generators for the infinitesimal representation of $\mathfrak{so}(3)$ on $\{F \in \mathfrak{su}(N) \mid \Delta_N F = l(l+1)F\}$ and that p_N is an algebra morphism. This proves the result. □

Lemma 9. *Let* $F, G \in \mathfrak{su}(N)$ *and* $f, g \in C^3(\mathbb{S}^2)$. *Then*

$$\Delta_N[F, G] = [\Delta_N F, G] + [F, \Delta_N G] + 2\sum_{i=1}^{3}[\nabla_N^{\perp,i} F, \nabla_N^{\perp,i} G]$$

and

$$\Delta\{f, g\} = \{\Delta f, g\} + \{f, \Delta g\} + 2\sum_{i=1}^{3}\{\nabla^{\perp,i} f, \nabla^{\perp,i} g\}.$$

Proof. The first result follows from the Hoppe–Yau formula (5) and application of the Jacobi identity twice.

The second result is a direct continuous analogue. □

From these results we obtain

$$|(19c) - (18c)| = \frac{3}{4}\left|\frac{1}{\hbar_N^2}\langle[p_N f, p_N g], \Delta_N[p_N f, p_N g]\rangle_{L_N^2}^2 - \langle\{f, g\}, \Delta\{f, g\}\rangle_{L^2}^2\right|$$

$$\underset{\text{lem. 9}}{=} \frac{3}{4}\left|\frac{1}{\hbar_N^2}\Big\langle[p_N f, p_N g], [\Delta_N p_N f, p_N g] + [p_N f, \Delta_N p_N g]\right.$$

$$+ 2\sum_{i=1}^{3}[\nabla_N^{\perp,i} p_N f, \nabla_N^{\perp,i} p_N g]\Big\rangle_{L_N^2}^2$$

$$\left. - \Big\langle\{f, g\}, \{\Delta f, g\} + \{f, \Delta g\} + 2\sum_{i=1}^{3}\{\nabla^{\perp,i} f, \nabla^{\perp,i} g\}\Big\rangle_{L^2}^2\right|$$

$$\underset{\text{lem. 8}}{=} \frac{3}{4}\left|\frac{1}{\hbar_N^2}\Big\langle[p_N f, p_N g], [p_N \Delta f, p_N g] + [p_N f, p_N \Delta g]\right.$$

$$+ 2\sum_{i=1}^{3}[p_N \nabla^{\perp,i} f, p_N \nabla^{\perp,i} g]\Big\rangle_{L_N^2}^2$$

$$\left. - \Big\langle\{f, g\}, \{\Delta f, g\} + \{f, \Delta g\} + 2\sum_{i=1}^{3}\{\nabla^{\perp,i} f, \nabla^{\perp,i} g\}\Big\rangle_{L^2}^2\right|.$$

Each pair of corresponding quantized and continuous brackets can now be treated by the same estimate as the second term above, which gives

$$|(19c) - (18c)| \le$$
$$\alpha\hbar_N^2 \|f\|_{H^5}(\|f\|_{H^7} + \|f\|_{H^6})\|g\|_{H^5}(\|g\|_{H^7} + \|g\|_{H^6})$$
$$+\beta\hbar_N\left(\sum_{a,b\in\{f,g\}}\|a\|_{H^5}(\|b\|_{H^7} + \|b\|_{H^6})\right)\left(\sum_{a,b\in\{f,g\}}\|a\|_{H^1}(\|b\|_{H^3} + \|b\|_{H^2})\right)$$
$$+\gamma\|\Pi_N\{f, g\}\|_{L^2}\sum_{a,b\in\{f,g\}}\left(\|\Pi_N\{b, \Delta a\}\|_{L^2} + 2\sum_{i=1}^{3}\left\|\Pi_N\{\nabla^{\perp,i} a, \nabla^{\perp,i} b\}\right\|_{L^2}\right)$$

for constants $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ independent of f, g, N. From the Sobolev norm relations $\|f\|_{H^q} \le \|f\|_{H^p}$ for $q \le p$ and from Lemma 5 we then obtain the result in Theorem 1. This concludes the proof.

7. Proof of the Convergence of the Reduced Jacobi Equation

In this section we prove the second main result stated in Theorem 2 above.

Let us first rewrite the continuous and quantized reduced Jacobi equations (11) and (12) for corresponding stationary solutions $\omega_0 \in C_0^\infty(\mathbb{S}^2)$ and $W_0 = p_N\omega_0 \in \mathfrak{su}(N)$ as linear evolution equations on $C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$ and $(\mathfrak{su}(N))^2$ respectively. The continuous reduced Jacobi equation is then

$$\begin{cases} \dot{\boldsymbol{\xi}} = \mathcal{L}\boldsymbol{\xi} \\ \boldsymbol{\xi}(t=0) = \boldsymbol{\xi}_0 \end{cases}$$

where

$$\boldsymbol{\xi} = \begin{bmatrix} \upsilon \\ \zeta \end{bmatrix} \quad \text{and} \quad \mathcal{L} = \begin{bmatrix} \{\Delta^{-1}\omega_0, \ \cdot\ \} & \Delta^{-1} \\ 0 & \{\Delta^{-1}\omega_0, \ \cdot\ \} + \{\omega_0, \Delta^{-1}\ \cdot\ \} \end{bmatrix}.$$

The corresponding quantized reduced Jacobi equation is

$$\begin{cases} \dot{\boldsymbol{X}} = \Lambda_N \boldsymbol{X} \\ \boldsymbol{X}(t=0) = \boldsymbol{X}_0 := p_N\boldsymbol{\xi}_0 \end{cases} \tag{20}$$

where

$$\boldsymbol{X} = \begin{bmatrix} Y \\ Z \end{bmatrix} \quad \text{and}$$

$$\Lambda_N = \begin{bmatrix} \frac{1}{\hbar_N}\left[\Delta_N^{-1}W_0, \ \cdot\ \right] & \Delta_N^{-1} \\ 0 & \frac{1}{\hbar_N}\left[\Delta_N^{-1}W_0, \ \cdot\ \right] + \frac{1}{\hbar_N}\left[\Delta_N^{-1}\ \cdot\ , W_0\right] \end{bmatrix}.$$

Recall that the embedding $\iota_N : \mathfrak{su}(N) \to C^\infty(\mathbb{S}^2)$ maps quantized harmonics to continuous ones. Let $\pi_N : C^\infty(\mathbb{S}^2) \to C^\infty(\mathbb{S}^2)$ be the truncation of the (continuous) spherical harmonics expansion up to $l \le N-1$, i.e., $\pi_N = \iota_N p_N$. In order to directly compare the quantized and continuous Jacobi equations, we introduce the operator $\mathcal{L}_N := \iota_N \circ \Lambda_N \circ p_N$ such that if $\boldsymbol{X}(t)$ is a solution of the matrix dynamical system (20), then $\boldsymbol{\xi}_N(t) = \iota_N\boldsymbol{X}(t) + \Pi_N\boldsymbol{\xi}_0$ is the solution of the continuous system

$$\begin{cases} \dot{\boldsymbol{\xi}}_N = \mathcal{L}_N\boldsymbol{\xi}_N \\ \boldsymbol{\xi}_N(t=0) = \boldsymbol{\xi}_0 \end{cases} \tag{21}$$

In order to prove the convergence $\boldsymbol{\xi}_N(t)$ to $\boldsymbol{\xi}(t)$, we will use the following result of Trotter and Kato (see, e.g., Pazy [24, Thm. 3.4.2]).

Theorem 10. (Trotter, Kato) *For a Banach space $(X, \|\cdot\|)$, let $(L_N)_{N\ge 0}$ and L be linear operators $D(L) \subset X \to X$. Let $\|\cdot\|$ denote also the operator norm and make the following assumptions.*

Well-posedness there exist scalars M, ω such that L is the infinitesimal generator of a C^0 semigroup $T(t)$ satisfying $\|T(t)\| \le Me^{\omega t}$.

Stability for any $N \ge 0$, L_N is the infinitesimal generator of a C^0 semigroup $T_N(t)$ satisfying $\|T_N(t)\| \le Me^{\omega t}$ (for the same M and ω).

Consistency for every $x \in X$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$

$$\left\|(I-\lambda L_N)^{-1}x-(I-\lambda L)^{-1}x\right\| \to 0 \quad \text{as} \quad N\to\infty.$$

Then, for every $x \in X$ and $t \geq 0$,

$$\|T_N(t)x - T(t)x\| \to 0 \quad \text{as} \quad N\to\infty.$$

Moreover, the convergence is uniform on bounded intervals of t.

Remark 7.1. The classical formulation of the Trotter & Kato result does not include convergence rates. In the special case of Zeitlin's model, the convergence rate is $O(\hbar_N)$. In Appendix A we have included a proof of Theorem 10 with this convergence rate.

7.1. Well-posedness of the continuous system. Here we prove that $\mathcal{L}$ is the infinitesimal generator of a C^0 semigroup with respect to the L^2 norm.

Proposition 11. *The operator $\mathcal{L}\colon C^\infty(\mathbb{S}^2\times\mathbb{S}^2)\to L^2(\mathbb{S}^2\times\mathbb{S}^2)$ is the generator of a C^0 semigroup $\mathcal{T}(t)$ with*

$$\|\mathcal{T}(t)\|_{L^2} \leq \exp t\sqrt{\left(\frac{1}{4}+\frac{1}{2}\|\omega_0\|^2_{H^1}\right)}.$$

Proof. We split $\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2$, where

$$\mathcal{L}_1=\begin{bmatrix}\{\Delta^{-1}\omega_0,\ \cdot\ \} & 0\\ 0 & \{\Delta^{-1}\omega_0,\ \cdot\ \}\end{bmatrix} \quad \text{and} \quad \mathcal{L}_2=\begin{bmatrix}0 & \Delta^{-1}\\ 0 & \{\omega_0,\Delta^{-1}\ \cdot\ \}\end{bmatrix}.$$

Now, $\mathcal{L}_1$ is the generator for the semigroup $\mathcal{T}_1(t)$ given explicitly by

$$\mathcal{T}_1(t)\boldsymbol{\xi}=\boldsymbol{\xi}\circ\eta_t^{-1}$$

where η_t is the diffeomorphism on $\mathbb{S}^2$ generated by the finite-dimensional Hamiltonian vector field $X_{\Delta^{-1}\omega_0}$ (the solution exists for all times since $X_{\Delta^{-1}\omega_0}$ is a smooth vector field on a compact domain). Notice that $\mathcal{T}_1(t)$ is bounded with operator norm 1, since

$$\|\mathcal{T}_1(t)\boldsymbol{\xi}\|^2_{L^2}=\left\|\upsilon\circ\eta_t^{-1}\right\|^2_{L^2}+\left\|\zeta\circ\eta_t^{-1}\right\|^2_{L^2}=\|\upsilon\|^2_{L^2}+\|\zeta\|^2_{L^2}=\|\boldsymbol{\xi}\|^2_{L^2}.$$

For the second part, $\mathcal{L}_2$ is a bounded operator, since

$$\begin{aligned}\|\mathcal{L}_2\boldsymbol{\xi}\|^2_{L^2}&=\left\|\Delta^{-1}\zeta\right\|^2_{L^2}+\left\|\{\omega_0,\Delta^{-1}\zeta\}\right\|^2_{L^2}\\&\leq\frac{1}{4}\|\zeta\|^2_{L^2}+\|\omega_0\|^2_{H^1}\left\|\Delta^{-1}\zeta\right\|^2_{H^1}\leq\left(\frac{1}{4}+\frac{1}{2}\|\omega_0\|^2_{H^1}\right)\|\zeta\|^2_{L^2}.\end{aligned}$$

Thus, $\mathcal{L}_2$ generates a C^0 semigroup $\mathcal{T}_2(t)$. By the Lie–Trotter formula we then obtain the C^0 semigroup $\mathcal{T}(t)$ via

$$\mathcal{T}(t)=\lim_{n\to\infty}\left(\mathcal{T}_1(t/n)\mathcal{T}_2(t/n)\right)^n,$$

which is generated by $\mathcal{L}$. The operator norm estimate also follows from the Lie–Trotter formula. □

7.2. Stability of the semi-discrete method. We now prove that the quantized system (21) is stable, which is the semi-discrete correspondence to Proposition 11.

Proposition 12. *For any $N \geq 0$, the operator $\mathcal{L}_N : C^\infty(\mathbb{S}^2 \times \mathbb{S}^2) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ is the generator of a C^0 semigroup $\mathcal{T}_N(t)$ with*

$$\|\mathcal{T}_N(t)\|_{L^2} \leq \exp t\sqrt{\left(\frac{1}{4} + \frac{1}{2}\|\omega_0\|_{H^1}^2\right)}.$$

Proof. The proof is a direct analogue of the proof of Proposition 11. For the operator splitting $\mathcal{L}_N = \mathcal{L}_{N,1} + \mathcal{L}_{N,2}$, the semigroup for the generator $\mathcal{L}_{N,1}$ is

$$\mathcal{T}_{N,1}(t)\boldsymbol{\xi} = \iota_N \begin{bmatrix} E_t Y E_t^\dagger \\ E_t Z E_t^\dagger \end{bmatrix},$$

where $E_t = \exp(t\Delta_N^{-1} W_0/\hbar_N)$, $Y = p_N \upsilon$, and $Z = p_N \zeta$. We then have

$$\left\|\mathcal{T}_{N,1}(t)\boldsymbol{\xi}\right\|_{L^2}^2 = \left\|E_t Y E_t^\dagger\right\|_{L_N^2}^2 + \left\|E_t Z E_t^\dagger\right\|_{L_N^2}^2 = \|Y\|_{L_N^2}^2 + \|Z\|_{L_N^2}^2 \leq \|\boldsymbol{\xi}\|_{L^2}^2 .$$

The estimate for $\mathcal{L}_{N,2}$ follows as in the proof of Proposition 11, using the estimates in Lemma 6. □

7.3. Consistency of the semi-discrete method. First, let us prove the following lemma.

Lemma 13. *There exist $\alpha > 0$ independent of $\omega_0, \hbar_N$ and $\boldsymbol{\xi} = \begin{bmatrix} f \\ g \end{bmatrix}$, such that*

$$\|\Lambda_N p_N \boldsymbol{\xi} - p_N \mathcal{L}\boldsymbol{\xi}\|_{L_N^2} \leq \alpha\hbar_N \|\omega_0\|_{H^5} \|\boldsymbol{\xi}\|_{H^5}$$

Proof. Using estimates from Sect. 5, we have

$$\begin{aligned}
&\left\|\Lambda_N \begin{bmatrix} p_N f \\ p_N g \end{bmatrix} - p_N \mathcal{L} \begin{bmatrix} f \\ g \end{bmatrix}\right\|_{L_N^2}^2 = \\
&\quad \left\|\frac{1}{\hbar_N}\left[\Delta_N^{-1} p_N \omega_0, p_N f\right] - p_N\left\{\Delta^{-1}\omega_0, f\right\}\right\|_{L_N^2}^2 \\
&\quad + \left\|\frac{1}{\hbar_N}\left[\Delta_N^{-1} p_N \omega_0, p_N g\right] + \frac{1}{\hbar_N}\left[\Delta_N^{-1} p_N g, p_N \omega_0\right] - p_N\left\{\Delta^{-1}\omega_0, g\right\} - p_N\left\{\Delta^{-1} g, \omega_0\right\}\right\|_{L_N^2}^2 \\
&\quad \leq \alpha^2\hbar_N^2 (\left\|\Delta^{-1}\omega_0\right\|_{H^5}^2 \|f\|_{H^5}^2 + \left\|\Delta^{-1}\omega_0\right\|_{H^5}^2 \|g\|_{H^5}^2 + \|\omega_0\|_{H^5}^2 \left\|\Delta^{-1} g\right\|_{H^5}^2) \\
&\quad \leq \alpha^2\hbar_N^2 \|\omega_0\|_{H^5}^2 \left\|\begin{bmatrix} f \\ g \end{bmatrix}\right\|_{H^5}^2
\end{aligned}$$

where $\alpha > 0$ is a constant. □

Then we can prove consistency, namely the following result.

Proposition 14. *For every $\boldsymbol{\xi} \in C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$ and $\lambda > c := \sqrt{\frac{1}{4} + \frac{1}{2}\|\omega_0\|^2_{H^1}}$,*

$$\left\|(I-\lambda\mathcal{L}_N)^{-1}\boldsymbol{\xi} - (I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi}\right\|_{L^2} \le \frac{1}{\lambda - c}\hbar_N\left(\alpha\|\omega_0\|_{H^5}\left\|\tilde{\boldsymbol{\xi}}\right\|_{H^5} + \sqrt{2}\left\|\mathcal{L}\tilde{\boldsymbol{\xi}}\right\|_{H^2}\right)$$

where $\tilde{\boldsymbol{\xi}} = (I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi}$ and $\alpha > 0$ a constant as in Lemma 13.

Proof. Recall from Proposition 12 that $\mathcal{L}_N$ is the infinitesimal generator of a C_0 semi-group $\mathcal{T}_N(t)$ satisfying $\|\mathcal{T}_N(t)\|_{L^2} \le e^{ct}$. The Hille–Yosida theorem (see, e.g., Pazy [24, Thm. 1.5.3]) then implies that

$$\left\|(I-\lambda\mathcal{L}_N)^{-1}\right\|_{L^2} \le \frac{1}{\lambda - c}.$$

Thus, we can proceed to the following estimates:

$$\begin{aligned}
&\left\|(I-\lambda\mathcal{L}_N)^{-1}\boldsymbol{\xi} - (I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi}\right\|_{L^2}\\
&\quad= \left\|(I-\lambda\mathcal{L}_N)^{-1}(I-\lambda\mathcal{L}_N)\left((I-\lambda\mathcal{L}_N)^{-1}\boldsymbol{\xi} - (I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi}\right)\right\|_{L^2}\\
&\quad\le \frac{1}{\lambda-c}\left\|(I-\lambda\mathcal{L}_N)\left((I-\lambda\mathcal{L}_N)^{-1}\boldsymbol{\xi} - (I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi}\right)\right\|_{L^2}\\
&\quad\le \frac{1}{\lambda-c}\left\|\boldsymbol{\xi} - (I-\lambda\mathcal{L}_N)(I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi}\right\|_{L^2}\\
&\quad\le \frac{1}{\lambda-c}\left\|(I-\lambda\mathcal{L})(I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi} - (I-\lambda\mathcal{L}_N)(I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi}\right\|_{L^2}\\
&\quad\le \frac{1}{\lambda-c}\left\|(\lambda\mathcal{L}_N - \lambda\mathcal{L})(I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi}\right\|_{L^2}.
\end{aligned}$$

Let $\tilde{\boldsymbol{\xi}} = (I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi}$. We then have from Lemma 13 that

$$\begin{aligned}
\left\|(\mathcal{L}_N - \mathcal{L})\tilde{\boldsymbol{\xi}}\right\|_{L^2} &\le \underbrace{\left\|\pi_N(\mathcal{L}_N - \mathcal{L})\tilde{\boldsymbol{\xi}}\right\|_{L^2}}_{\|\Lambda_N\boldsymbol{\xi} - p_N\mathcal{L}\boldsymbol{\xi}\|_{L^2_N}} + \left\|(I-\pi_N)(\mathcal{L}_N - \mathcal{L})\tilde{\boldsymbol{\xi}}\right\|_{L^2}\\
&\le \alpha\hbar_N\|\omega_0\|_{H^5}\left\|\tilde{\boldsymbol{\xi}}\right\|_{H^5} + \left\|(I-\pi_N)(\mathcal{L}_N - \mathcal{L})\tilde{\boldsymbol{\xi}}\right\|_{L^2}.
\end{aligned}$$

Recall that $\mathcal{L}_N = \iota_N\Lambda_N p_N$, so $\mathcal{L}_N\tilde{\boldsymbol{\xi}}$ is a finite sum of spherical harmonics with $l \le N-1$. Thus, $\pi_N\mathcal{L}_N = \mathcal{L}_N$, and

$$\left\|(I-\pi_N)(\mathcal{L}_N - \mathcal{L})\tilde{\boldsymbol{\xi}}\right\|_{L^2} = \left\|\pi_N\mathcal{L}\tilde{\boldsymbol{\xi}} - \mathcal{L}\tilde{\boldsymbol{\xi}}\right\|_{L^2}.$$

As $\boldsymbol{\xi} \in C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$, we have $\mathcal{L}\tilde{\boldsymbol{\xi}} = \mathcal{L}(I-\lambda\mathcal{L})^{-1}\boldsymbol{\xi} \in C^\infty(\mathbb{S}^2 \times \mathbb{S}^2) \subset H^2(\mathbb{S}^2 \times \mathbb{S}^2)$. Thus, by Lemma 5,

$$\left\|\pi_N\mathcal{L}\tilde{\boldsymbol{\xi}} - \mathcal{L}\tilde{\boldsymbol{\xi}}\right\|_{L^2} \le \sqrt{2}\hbar_N\left\|\mathcal{L}\tilde{\boldsymbol{\xi}}\right\|_{H^2}.$$

□

We now have everything to apply Theorem 10 above: well-posedness from Proposition 11; stability from Proposition 12; consistency from Proposition 14. This concludes the proof of our second main result, stated in Theorem 2 above.

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Declarations

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Appendix A. Trotter and Kato theorem with Convergence Rate

In this section, we adapt the proof of the Trotter–Kato Theorem 10 from Pazy [24]. Let us denote $\mathcal{R}_\lambda := (\lambda I - \mathcal{L})^{-1}$ and $\mathcal{R}_{\lambda,N} := (\lambda I - \mathcal{L}_N)^{-1}$. Fix $\boldsymbol{\xi} \in D(\mathcal{L})$ and an interval $[0, T]$. For any $t \in [0, T]$:

$$\begin{aligned}\|(\mathcal{T}_N(t) - \mathcal{T}(t))\boldsymbol{\xi}\|_{L^2} \leq\ & \underbrace{\left\|\mathcal{T}_N(t)(\mathcal{R}_\lambda - \mathcal{R}_{\lambda,N})(\mathcal{R}_\lambda^{-1}\boldsymbol{\xi})\right\|_{L^2}}_{N_1} \\ & + \underbrace{\left\|\mathcal{R}_{\lambda,N}(\mathcal{T}_N(t) - \mathcal{T}(t))(\mathcal{R}_\lambda^{-1}\boldsymbol{\xi})\right\|_{L^2}}_{N_2} \\ & + \underbrace{\left\|(\mathcal{R}_\lambda - \mathcal{R}_{\lambda,N})\mathcal{T}(t)(\mathcal{R}_\lambda^{-1}\boldsymbol{\xi})\right\|_{L^2}}_{N_3}\end{aligned}$$

Using the stability of $\mathcal{L}_N$ (Proposition 12) and the consistency estimate (Proposition 14) we have

$$N_1 \leq e^{cT}\frac{1}{\lambda - c}\hbar_N\left(\alpha\,\|\omega_0\|_{H^5}\,\|\boldsymbol{\xi}\|_{H^5} + \sqrt{2}\,\|\mathcal{L}\boldsymbol{\xi}\|_{H^2}\right)$$

and

$$N_3 \leq \frac{1}{\lambda - c}\hbar_N\left(\alpha\,\|\omega_0\|_{H^5}\sup_{t\in[0,T]}\|\mathcal{T}(t)\boldsymbol{\xi}\|_{H^5} + \sqrt{2}\sup_{t\in[0,T]}\|\mathcal{L}\mathcal{T}(t)\boldsymbol{\xi}\|_{H^2}\right)$$

where we have used the fact that $\mathcal{R}_\lambda \mathcal{T}(t) = \mathcal{T}(t)\mathcal{R}_\lambda$. For the term N_2, we first use the identity (e.g. Pazy, [24, Lemma 3.4.1])

$$\mathcal{R}_{\lambda,N}(\mathcal{T}(t) - \mathcal{T}_N(t))\mathcal{R}_\lambda \tilde{\boldsymbol{\xi}} = \int_0^t \mathcal{T}_N(t-s)\left(\mathcal{R}_\lambda - \mathcal{R}_{\lambda,N}\right)\mathcal{T}(s)\tilde{\boldsymbol{\xi}}\, ds.$$

Thus, we have

$$\begin{aligned} N_2 &= \left\| \mathcal{R}_{\lambda,N}(\mathcal{T}_N - \mathcal{T}(t))\mathcal{R}_\lambda(\mathcal{R}_\lambda^{-2}\boldsymbol{\xi}) \right\|_{L^2} \\ &\leq \int_0^T \|\mathcal{T}_N(t-s)\|_{L^2} \left\| (\mathcal{R}_\lambda - \mathcal{R}_{\lambda,N})\mathcal{T}(s)(\mathcal{R}_\lambda^{-2}\boldsymbol{\xi}) \right\|_{L^2} \\ &\leq T e^{cT} \frac{1}{\lambda - c} \hbar_N \left(\alpha \|\omega_0\|_{H^5} \sup_{t\in[0,T]} \left\| \mathcal{T}(t)(\mathcal{R}_\lambda^{-1}\boldsymbol{\xi}) \right\|_{H^5} \right. \\ &\quad \left. + \sqrt{2} \sup_{t\in[0,T]} \left\| \mathcal{L}\mathcal{T}(t)(\mathcal{R}_\lambda^{-1}\boldsymbol{\xi}) \right\|_{H^2} \right). \end{aligned}$$

Then it follows from the above estimates that for any $\boldsymbol{\xi} \in D(\mathcal{L})$:

$$\|\mathcal{T}_N(t)\boldsymbol{\xi} - \mathcal{T}(t)\boldsymbol{\xi}\|_{L^2} \to 0 \quad \text{as} \quad N \to \infty$$

uniformly on $[0, T]$. Since the Hille–Yosida theorem implies that $D(\mathcal{L})$ is dense in $C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$, it follows that the previous statement holds for every $\boldsymbol{\xi} \in C^\infty(\mathbb{S}^2 \times \mathbb{S}^2)$. Moreover, it is clear that regarding only the N dependency, the convergence is $O(\hbar_N)$.

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