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Lyapunov Exponents and Nonadapted Measures for Dispersing Billiards

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Abstract: For hyperbolic systems with singularities, such as dispersing billiards, Pesin theory as developed by Katok and Strelcyn applies to measures that are "adapted" in the sense that they do not give too much weight to neighborhoods of the singularity set. The zero-entropy measures supported on grazing periodic orbits are nonadapted, but it has been an open question whether there are nonadapted measures with positive entropy. We construct such measures for any dispersing billiard with a periodic orbit having a single grazing collision; we then use our construction to show that the thermodynamic formalism for such billiards has a phase transition even when one restricts attention to adapted or to positive entropy measures.

1. Introduction

Sinai introduced dispersing billiards as a mathematically tractable example of a mechanical system where the Boltzmann hypothesis can be verified [S]. Since then, mathematical billiards, and the Lorentz gas in particular, have become central models in mathematical physics. Although the billiard map associated with a Lorentz gas has discontinuities, it preserves a smooth invariant measure, which we denote by μ_{SRB}. Classical results focus on establishing quantitative statistical properties such as the Central Limit Theorem, decay of correlations and large deviation estimates with respect to μ_{SRB}, and they use a variety of techniques, including Markov partitions and sieves [BSC1,BSC2], Young towers [Y], spectral analysis of the transfer operator [DZ] and, most recently, projective cones [DL].

In recent years, there has been significant interest in the existence and properties of other invariant measures for the billiard map. In particular, the existence and uniqueness of equilibrium measures with respect to the geometric family of potentials, $-t \log J^u T$, $t \geq 0$, where $J^u T$ denotes the unstable Jacobian of the billiard map T, are studied in [BD1] (for $t = 0$) and [CWZ,BD2] (for $t > 0$). A renewed interest in the construction of Markov partitions for surface diffeomorphisms stemming from [Sar] has also led to

symbolic codings for more general hyperbolic measures for billiards [LM, ALP]. These constructions in turn lead to more questions regarding properties of these invariant measures, such as possible Lyapunov spectra and phase transitions in the associated pressure functions.

For a class of planar dispersing billiard maps corresponding to a finite horizon Lorentz gas, we present a construction that answers two open questions in the billiards literature. The first relates to the construction of countable Markov partitions for billiards, which relies on Pesin theory as developed by Katok and Strelcyn; this theory applies to invariant probability measures μ that are *adapted*, meaning that

$$\int |\log d(x, \mathcal{S})| \, d\mu < \infty, \tag{1.1}$$

where $\mathcal{S}$ is the singular set for T.[1] Using this theory, Lima and Matheus showed that if μ is adapted and hyperbolic, then there is a countable Markov partition for the billiard map T which provides a symbolic coding for μ-almost every point [LM]. Since measures with large entropy are likely to be hyperbolic (this occurs e.g. when the Ruelle inequality holds), a natural question arises: are there T-invariant *nonadapted* measures with positive entropy? We answer this question in the affirmative.

Theorem 1.1. *There are finite horizon planar dispersing billiards with invariant probability measures of positive entropy that are not adapted.*

The second question is related to the behavior of the pressure function $P(t)$ for the family of geometric potentials $-t \log J^u T$ for t near 0, where $J^u T$ denotes the unstable Jacobian of T. The pressure function is defined as

$$P(t) = \sup_{\mu \in \mathcal{M}} \left\{ h_\mu(T) - t \int \log J^u T \, d\mu \right\}, \tag{1.2}$$

where $\mathcal{M}$ is the set of T–invariant probability measures. The function $P(t)$ is finite and decreasing for $t \geq 0$, and [BD2] proved it is analytic on an interval $(0, t_*)$, for some $t_* > 1$ depending on the billiard table. While $P(0) < \infty$ is the topological entropy of the map[2] [BD1], it is clear that for a table with a grazing periodic orbit, $P(t) = \infty$ for $t < 0$ due to the invariant measure supported on such an orbit. We arrive at the following natural question: does one still see a jump in $P(t)$ at $t = 0$ if one restricts the supremum in (1.2) to a smaller class of measures? For example, letting $N_\varepsilon(\mathcal{S})$ denote the ε–neighborhood of $\mathcal{S}$, we can restrict the supremum to measures satisfying:

(a) There are constants $C, \alpha > 0$ such that $\mu(N_\varepsilon(\mathcal{S})) \leq C\varepsilon^\alpha$ for all $\varepsilon > 0$ (such a measure must be adapted); or
(b) μ has positive entropy.

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[1] See [H, p. 342] for the story of the genesis of this condition, or rather of the slightly stronger condition (a) that appears in [KSLP, P]. The term "adapted" was introduced in [LS], in the context of smooth flows equipped with an invariant measure, as a condition on a Poincaré section (requiring that the boundary $\mathcal{S}$ satisfies (1.1) with respect to the induced measure). It has subsequently been used as a condition on measures in systems with singularities [LM, BD1]. For planar dispersing billiards, it is equivalent to assuming the finiteness of the positive Lyapunov exponent almost everywhere (see the proof of Theorem 1.2).

[2] [BD1] constructs a unique measure of maximal entropy under a mild sparse recurrence condition to the singularity set. (If a conjecture of Balint and Toth holds, then this condition is satisfied generically for finite horizon tables [DK].) Yet even without assuming this condition, [BD1] proves that $P(0) < \infty$. We do not make any assumption of sparse recurrence to singularities in the present paper.

Our construction answers this question in the affirmative for both restricted suprema under condition (a) or (b) whenever a certain type of periodic orbit exists on the billiard table. This implies that, for some classes of dispersing billiards, there is a phase transition at $t = 0$.

Theorem 1.2. *There are finite horizon planar dispersing billiards such that each of them possesses a sequence of periodic orbits* (p_n) *with positive and finite Lyapunov exponents, but tending to infinity as $n \to \infty$. As a consequence, for all $t < 0$ we have*

$$\sup_{\mu \in \mathcal{M}} \left\{ h_\mu(T) - t \int \log J^u T \, d\mu : \exists C, \alpha > 0 \text{ such that } \mu(N_\varepsilon(\mathcal{S})) \leq C\varepsilon^\alpha, \forall \varepsilon > 0 \right\}$$
$$= \infty. \tag{1.3}$$

Moreover, it follows from Theorem 1.1 *that for such dispersing billiards and for all $t < 0$ we have*

$$\sup_{\mu \in \mathcal{M}} \left\{ h_\mu(T) - t \int \log J^u T \, d\mu : \mu \text{ has positive entropy} \right\} = \infty. \tag{1.4}$$

The case of the infinite horizon Lorentz gas has been studied in [CT]. There, the authors show that $P(0) = \infty$ and construct invariant measures with both infinite metric entropy as well as invariant measures with finite entropy, but infinite Lyapunov exponent.

Remark 1.3. We remark that for $t > 0$, restricting to measures that satisfy both *(a)* and *(b)* does not change the pressure $P(t)$; specifically, the equilibrium states produced in [BD2] satisfy both conditions for all $t \in (0, t_*)$. However, it is not known whether the measure of maximal entropy μ_0 (in the case $t = 0$) satisfies *(a)*. Indeed, [BD1] produces the weaker bound $\mu_0(N_\varepsilon(\mathcal{S})) \leq C|\log \varepsilon|^{-\gamma}$ for some $\gamma > 1$; this is sufficient for μ_0 to be adapted and hyperbolic.

We state Theorem 1.2 using the stronger condition *(a)* since it makes for a stronger statement: even restricting to this smaller class of measures, the pressure is infinite when $t < 0$.

2. Setting

Our billiard table is defined by placing finitely many pairwise disjoint closed, convex sets $O_i, i = 1, \ldots, d$, in $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We assume that the boundaries ∂O_i are C^3 curves with strictly positive curvature. The billiard table is then defined as $\mathcal{Q} := \mathbb{T}^2 \setminus (\bigcup_{i=1}^d O_i)$. The billiard flow is induced by the motion of a point particle traveling at unit speed in $\mathcal{Q}$ and reflecting elastically at collisions with $\partial \mathcal{Q}$. The associated billiard map, which we denote by T, is the Poincaré map under the flow with respect to $\partial \mathcal{Q}$.

Parametrizing ∂O_i for each i by arclength r (oriented clockwise), and letting $\varphi \in [-\pi/2, \pi/2]$ denote the angle made by the post-collision velocity vector and the outward normal to ∂O_i at the point of collision, we represent the phase space of our billiard map as $M = \bigcup_{i=1}^d (\partial O_i \times [-\pi/2, \pi/2])$. We denote by $\tau(x)$ the flight time from $x = (r, \varphi) \in M$ to $T(x)$. We assume that the table has *finite horizon*, i.e. $\sup_{x \in M} \tau(x) < \infty$. The finite horizon assumption together with the fact that the scatterers are disjoint implies that, there exist constants $\tau_{\min}, \tau_{\max} \in \mathbb{R}$, such that $0 < \tau_{\min} \leq \tau(x) \leq \tau_{\max} < \infty$ for all $x \in M$.
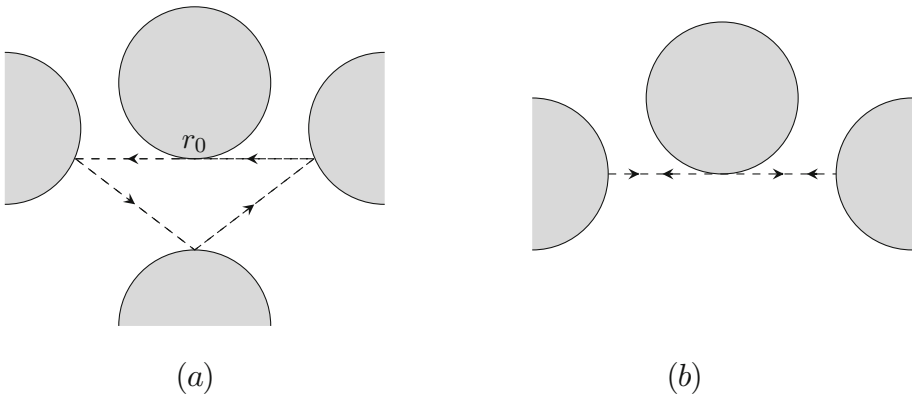
Fig. 1. **a** A periodic orbit of period 4 with grazing collision at r_0. The corresponding periodic point is $x_0 = (r_0, \pi/2)$. Since there is only one grazing collision, this configuration satisfies our assumption. **b** A periodic orbit of period 4 with two grazing collisions. It does not satisfy our assumption. Both figures are local and are not meant to illustrate the finite horizon condition

It is well-known that T preserves a smooth invariant probability measure μ_{SRB}, such that

$$d\mu_{\text{SRB}}(x) = \frac{1}{2|\partial Q|} \cos \varphi \, dr d\varphi.$$

Yet T has discontinuities created by grazing collisions. Set $\mathcal{S}_0 = \left\{ (r, \varphi) \in M : \varphi = \pm\frac{\pi}{2} \right\}$ and denote by $\mathcal{S}_n = \bigcup_{i=0}^n T^{-i} \mathcal{S}_0$ and $\mathcal{S}_{-n} = \bigcup_{i=0}^n T^i \mathcal{S}_0$ the singularity sets for T^n and T^{-n}, respectively. Then T^n is a C^2 diffeomorphism of $M \backslash \mathcal{S}_n$ onto $M \backslash \mathcal{S}_{-n}$. Note that in this notation, $\mathcal{S}_1$ is precisely the same as the set $\mathcal{S}$ discussed in Sect. 1.

2.1. Strategy of proof. The class of billiards tables $\mathcal{Q}$ we will consider admits the following type of periodic orbit with a grazing collision. If we choose a direction and follow the orbit around one full cycle, then either all grazing collisions along the orbit occur on our right, or all grazing collisions along the orbit occur on our left. See Fig. 1a for an example of an orbit that satisfies our assumption, and Fig. 1b for an example of an orbit that does not. As we will see in the next section, this assumption allows us to easily understand how a neighborhood in $\mathcal{S}_0$ of the periodic point is cut under iterations.

The proofs of Theorems 1.1 and 1.2 will both use the existence of a Cantor rectangle defined in a one-sided neighborhood of the periodic point $x_0 \in \mathcal{S}_0$, and it will be divided into the following steps:

1. The point x_0 has a local stable manifold $W^s(x_0)$ and a local unstable manifold $W^u(x_0)$ of positive length that terminate at x_0. Both manifolds are $C^{1+\text{Lip}}$, and the wedge between $W^s(x_0)$ and $W^u(x_0)$ belongs to a single component of $M \setminus (\mathcal{S}_p \cup \mathcal{S}_{-p})$, where p is the period of the orbit.
2. Construct a Cantor rectangle $\mathcal{R}$ formed by intersections of local stable and local unstable manifolds, with $W^s(x_0)$ and $W^u(x_0)$ comprising two boundaries of $\mathcal{R}$, and such that $\mu_{\text{SRB}}(\mathcal{R}) > 0$.
3. Use the product structure of the Cantor set and the mixing property of μ_{SRB} to define a horseshoe K on $\mathcal{R}$.

4. Use the coding provided by the horseshoe K to construct periodic orbits with large period and arbitrarily large Lyapunov exponent.
5. Use the coding for the induced map on the horseshoe K to define a Bernoulli measure whose projection to M is nonadapted.

One of the difficulties involved in the construction is that $DT(x_0)$ is not defined, and $DT(x)$ converges to infinity as x converges to x_0. This means that T has huge expansion near x_0, and it prevents the use of classical techniques, such as the graph transform method. Nevertheless, the assumption on the periodic orbit allows us to bypass this issue.

Notation. Throughout the text, we will use the notation $A \approx B$ if there exists a constant $C > 0$, depending only on the billiard table and the definition of the homogeneity strips (defined in (3.3)), such that $C^{-1}A \leq B \leq CA$.

3. Construction of the Cantor Rectangle $\mathcal{R}$

For ease of the presentation, we assume that the configuration of $x_0 = (r_0, \pi/2)$ is the one depicted in Fig. 1a, i.e. $T^4(x_0) = x_0$ with a single grazing collision in one full cycle.[3] Let $\mathcal{K}$ denote the curvature function of the obstacles. We recall from [CM, Section 4.5] that T has invariant cones:

$$\mathcal{C}_x^s = \{(dr, d\varphi) \in T_x M : -\mathcal{K} - \cos\varphi/\tau \leq d\varphi/dr \leq -\mathcal{K}\}$$
$$\mathcal{C}_x^u = \{(dr, d\varphi) \in T_x M : \mathcal{K} \leq d\varphi/dr \leq \mathcal{K} + \cos\varphi/\tau\}. \tag{3.1}$$

Since the table has finite horizon and the obstacles are disjoint and C^3 with strictly positive curvature, these cones are uniformly transverse and bounded away from the lines $r = $ const and $\varphi = $ const. According to [CM, Eq. (4.19)], vectors in these cones contract and expand uniformly: there exists $C_e > 0$ such that

$$\|DT^n(x)\vec{v}\| \geq C_e \Lambda^n \|\vec{v}\| \text{ for all } \vec{v} \in \mathcal{C}_x^u \text{ and all } n \geq 1, \tag{3.2}$$

where $\Lambda = 1 + 2\mathcal{K}_{\min}\tau_{\min}$. Similar bounds hold for $\vec{v} \in \mathcal{C}_x^s$.

We call a C^1 curve W a stable curve if the tangent vector at each $x \in W$ lies in $\mathcal{C}_x^s$. Unstable curves are defined similarly.

3.1. Existence of stable and unstable manifolds for x_0. In this section, we will construct one-sided stable and unstable manifolds for x_0. The main result of this section is the following proposition.

Proposition 3.1. *Let $x_0 \in \mathcal{S}_0$ be a periodic point with grazing collision as described above. Then x_0 has one-sided local stable and unstable manifolds, which are $C^{1+\text{Lip}}$ curves in M.*

Proposition 3.1 follows from Lemma 3.3.

Let O_1 be the obstacle of the grazing collision and let M_1 denote the corresponding component of the phase space. We assume that the orientation of the trajectory is counter-clockwise, and that the parametrization of the obstacles is clockwise. Consider

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[3] The period 4 is not essential to our construction, yet a second grazing collision along the periodic orbit could complicate the argument and indeed might destroy the horseshoe entirely.
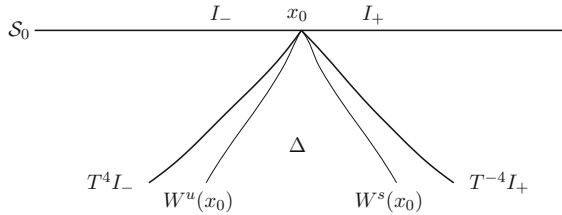
Fig. 2. A δ_0-neighborhood of x_0

a neighborhood $I = [r_1, r_2] \times \{\pi/2\} \subset \mathcal{S}_0$ of x_0 and let $I_- = [r_1, r_0] \times \{\pi/2\}$ and $I_+ = (r_0, r_2] \times \{\pi/2\}$. Since $\mathcal{S}_0$ is a dispersing wave front, we may choose r_1 and r_2 sufficiently close to r_0 such that the following behavior holds for the first four iterates of I:

- If $x \in I_- \setminus \{x_0\}$ then $r(T(x)) > r(T(x_0))$, $r(T^2(x)) < r(T^2(x_0))$, $r(T^3(x)) > r(T^3(x_0))$, so that $T^4(x)$ hits O_1 with $r(T^4(x)) < r(T^4(x_0)) = r_0$.
- If $x \in I_+$ then $r(T(x)) < r(T(x_0))$, $r(T^2(x)) > r(T^2(x_0))$, $r(T^3(x)) < r(T^3(x_0))$, so that $T^4(x)$ *does not* hit O_1 in a neighborhood $N_\varepsilon(x_0)$ of x_0.

This means that I gets cut after four iterations: $T^4(I_-)$ is a curve in M_1 terminating at x_0, and $T^4(I_+) \cap N_\varepsilon(x_0) = \emptyset$ for some $\varepsilon > 0$. Analogous facts hold for I_+ under T^{-1}. Thus we may choose r_1 and r_2 sufficiently close to r_0 so that T^4 is continuous on I_- and T^{-4} is continuous on I_+. See Fig. 2.

Consider the sector $\mathcal{A}$ defined by rotating the horizontal segment $I_- = \{(r, \varphi) : \varphi = \pi/2, r \in [r_1, r_0]\}$ around x_0 by an angle of $\pi/2$ radians, until I_- reaches the vertical segment $V = \{(r, \varphi) : r = r_0, \varphi \in [\pi/2 - (r_0 - r_1), \pi/2]\}$. Both segments I_- and V terminate on x_0, a fixed point for T^4. Moreover, both $T^4(I_-)$ and $T^4(V)$ are unstable curves, tangent at x_0 (the cone at x_0 consists of a single line with slope $\mathcal{K}(r_0)$, see (3.1)), and contained strictly between I_- and V. Since T is orientation preserving, $T^4(I_-)$ comprises the upper boundary of $T^4(\mathcal{A})$ and $T^4(V)$ comprises its lower boundary. Repeating this argument, since the sector $\mathcal{A}$ is strictly contracted by T^4, the intersection $T^8(\mathcal{A}) \cap \mathcal{A}$, which has upper boundary $T^8(I_-) \cap \mathcal{A}$ and lower boundary $T^8(V) \cap \mathcal{A}$, lies strictly inside the region with upper boundary $T^4(I_-)$ and lower boundary $T^4(V)$.

Inductively, we see that for each $n \geq 1$ the curve $T^{4n}(I_-)$ contains an unstable curve $\gamma_n(x_0)$, defined by an equation $\{(r, \varphi_n(r)) : r \in [r_1, r_0]\}$ where $\varphi_n : [r_1, r_0] \to \mathbb{R}$ is an increasing C^2 function with $\varphi_n(r_0) = \pi/2$ and $\varphi_n'(r_0) = \mathcal{K}(r_0)$. Moreover, the sequence (φ_n) is decreasing, and (φ_n') is bounded since it must belong to the unstable cone for each n. Thus, (φ_n) converges to a function $\varphi^u : [r_1, r_0] \to \mathbb{R}$. This function defines an unstable curve, which indeed is the local unstable manifold at x_0, as we will prove below. We need the following auxiliary lemma.

Lemma 3.2. *Suppose $n \geq 1$ and W is an unstable curve passing through x_0 such that $T^{-4k}W$ is an unstable curve for all $1 \leq k \leq n$. The following holds for all $z \in W$:*

(a) $d(T^{-4}z, x_0) \approx d(z, x_0)^2$;
(b) $d(T^{-4n}z, x_0) \leq C_0 d(z, x_0)^{(3/2)^n}$, *for some $C_0 > 0$ independent of W and n.*

The analogous statement holds for stable curves W passing through x_0 such that $T^{4k}W$ is a stable curve for all $1 \leq k \leq n$.

This means that near x_0, the map T^4 exhibits a huge expansion along unstable curves.

Proof. We consider the homogeneity strips

$$H_k = \left\{ x = (r, \varphi) \in M : \varphi \in \left[\tfrac{\pi}{2} - k^{-2}, \tfrac{\pi}{2} - (k+1)^{-2} \right) \right\}, \ k \geq 1. \quad (3.3)$$

Notice that $\cos \varphi(x) \approx k^{-2}$ for $x \in H_k$. Let z and W be as in the statement of the lemma and suppose $z \in H_j$. Let V denote the subcurve of W with endpoints z and x_0, and write $V = \bigcup_{k \geq j} V_k$, where $V_k = V \cap H_k$. Since V_k is uniformly transversal to the boundaries of H_k, we have $|V_k| \approx k^{-3}$, hence

$$|V| = \sum_{k \geq j} |V_k| \approx \sum_{k \geq j} k^{-3} \approx j^{-2}.$$

We will show that $|T^{-4}V| \approx j^{-4}$, which will complete the proof of part (a).

By [CM, Equation (4.20)], for all $x \in H_k$ we have

$$J^u T^4 (T^{-4} x) \approx \frac{1}{\cos \varphi(x)} \approx k^2.$$

Hence $|T^{-4} V_k| \approx |V_k| \cdot k^{-2} \approx k^{-5}$ and so

$$|T^{-4} V| = \sum_{k \geq j} |T^{-4} V_k| \approx \sum_{k \geq j} k^{-5} \approx j^{-4},$$

which concludes the proof of part (a).

To prove (b), note that (a) yields a constant $C > 0$ such that $d(T^{-4}z, x_0) \leq Cd(z, x_0)^2$. Without loss of generality, we may assume that $d(z, x_0) \leq C^{-2}$, as this must occur in a fixed number of iterates due to the uniform expansion given by (3.2). Statement (b) then follows directly by iterating (a) since for such z we have $Cd(T^{-4k}z, x_0)^2 \leq d(T^{-4k}z, x_0)^{3/2}$. $\qquad \square$

Lemma 3.3. *The curve*

$$W^u(x_0) := \{ (r, \varphi^u(r)) : r \in [r_1, r_0] \}$$

is the local unstable manifold at x_0 for T^4, i.e. $d(T^{-4n}x, x_0) \xrightarrow{n \to \infty} 0$ for every $x \in W^u(x_0)$. In particular, $W^u(x_0)$ is a $C^{1+\text{Lip}}$ curve.

By time reversibility of billiard systems, x_0 also has a $C^{1+\text{Lip}}$ local stable manifold $W^s(x_0)$, see Fig. 2.

Proof. It is clear that the function φ^u is Lipschitz and that $W^u(x_0)$ is an unstable curve. Fix $x \in W^u(x_0)$, and consider a stable curve W passing through x. Fix $n \geq 1$. For $1 \leq k \leq n$, recall that $\gamma_k \subset T^{4k} I_-$ denotes the graph of φ_k over the interval $[r_1, r_0]$. Let y be the intersection point of W and γ_n. Consider the pre-images $T^{-4n}x$ and $T^{-4n}y$, and consider the triangle $\alpha_1 \alpha_2 \alpha_3$ formed by the three curves:

$$\alpha_1 = \text{segment of } S_0 \text{ joining } x_0 \text{ and } T^{-4n}y$$
$$\alpha_2 = \text{curve of } T^{-4n}W \text{ joining } T^{-4n}y \text{ and } T^{-4n}x$$
$$\alpha_3 = \text{curve of } W^u(x_0) \text{ joining } T^{-4n}x \text{ and } x_0.$$

Since α_2 is a stable curve and α_3 is an unstable curve, and cones make angles bounded away from zero with respect to horizontal and vertical curves of M, it follows that

$d(T^{-4n}x, x_0) \leq \text{const} \times d(T^{-4n}y, x_0)$. Since γ_k is an unstable curve for each $k \geq 1$ and $T^{4(k-n)}y \in \gamma_k$, by Lemma 3.2(b) we have

$$d(T^{-4n}y, x_0) \leq C d(T^{-4(n-1)}y, x_0) \leq C C_0 d(y, x_0)^{(3/2)^{n-1}},$$

and so $d(T^{-4n}x, x_0) \leq \text{const} \times d(y, x_0)^{(3/2)^{n-1}}$, which converges to 0 as $n \to \infty$.

It remains to prove that φ^u is $C^{1+\text{Lip}}$. According to the first part of the proof, for each $x \in W^u(x_0) \backslash \{x_0\}$, the curve $W^u(x_0)$ is the unstable manifold of x. By [CM, Corollary 4.61], $W^u(x_0)$ is C^1 and its derivative is Lipschitz. $\qquad \square$

3.2. The Cantor rectangle $\mathcal{R}$.

In the last section, we showed the existence of $C^{1+\text{Lip}}$ local stable/unstable manifolds $W^{s/u}(x_0)$ at x_0 for the map T^4. Note that, since these are a stable/unstable curve respectively, they make an angle at x_0 that is in[4] $(0, \pi)$. We denote by Δ the region between these two curves, see Fig. 2. In this section, we will construct a Cantor rectangle of stable and unstable manifolds with x_0 as one corner point of the rectangle.

First we recall some terminology. We call D a solid rectangle in M if ∂D comprises four smooth curves, two local stable and two local unstable manifolds. Given a solid rectangle D, let $\mathfrak{S}^s(D)$ and $\mathfrak{S}^u(D)$ denote the set of local stable and unstable manifolds, respectively, of points in D that do not terminate in the interior of D. Define the Cantor rectangle

$$R(D) = \mathfrak{S}^s(D) \cap \mathfrak{S}^u(D) \cap D.$$

By construction, $R(D)$ has a locally maximal hyperbolic product structure. We will work with such Cantor rectangles R, closed sets formed by locally maximal intersections of stable and unstable manifolds such that $\text{Leb}(R) > 0$. Conversely, given such a Cantor rectangle R, we denote by $D(R)$ the smallest solid rectangle containing R.

Unlike [CM, Section 7.11], we do not restrict ourselves to H–manifolds, i.e. to local stable (respectively unstable) manifolds whose forward (respectively backward) trajectories lie in a single homogeneity strip at each step. We do this because the rectangle we construct, $\mathcal{R}$, will necessarily cross infinitely many homogeneity strips in order to incorporate x_0 in its boundary.

Now consider a segment of $W^u(x_0)$ that lies between homogeneity strips of index k_1 and k_1^2. Call this curve A_1. Since A_1 is a finite union of homogeneous unstable manifolds, almost every point on A_1 has stable manifolds of positive length [CM, Theorem 5.70].

Now consider $T^{-4}A_1 \subset W^u(x_0)$. If W is a local stable manifold of a point in A_1, then $T^{-4}W$ is a local stable manifold of a point in $T^{-4}A_1$, and it has one connected component intersecting Δ. This component cannot be cut until it exits a δ_0–neighborhood of x_0, and it cannot cross $W^s(x_0)$ (since stable manifolds do not intersect). Therefore,

$$|W^s(y)| \geq \min \left\{ \frac{|W^s(T^4 y)|}{J^s T^4(y)}, \delta_0 \right\} \quad \text{for all } y \in T^{-4}A_1.$$

Note that $J^s T^4(y) \approx \cos \varphi(y)$ so that the expansion is large in a δ_0–neighborhood of x_0.

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[4] Indeed, since the slopes are $-\mathcal{K}(x_0)$ and $\mathcal{K}(x_0)$, respectively, they form an angle equal to $2\text{Arctan}(1/\mathcal{K}(x_0))$.

Applying this construction inductively, we are guaranteed the existence of increasingly longer stable manifolds for points in $W^u(x_0)$ as we approach x_0, and so a positive measure of such manifolds extend a minimum distance of $\delta_1 > 0$ inside Δ.

By time-reversal symmetry, this construction also implies that almost every point on $W^s(x_0)$ has an unstable manifold of positive length, and that unstable manifolds of a minimum length δ_1 comprise a positive measure set of points inside $W^s(x_0)$.

Fixing once and for all $\delta_0, \delta_1 > 0$ small enough such that the above constructions hold in the δ_0–neighborhood of x_0, we obtain a (locally maximal) closed, Cantor rectangle $\mathcal{R}$ comprised of the intersection of a positive measure set of stable and unstable manifolds, having a segment of $W^s(x_0)$ along one boundary, a segment of $W^u(x_0)$ along another, and containing the point x_0 as one of its corner points.

3.3. Definition of the horseshoe. We let $G = T^4$, and construct a horseshoe for a power of G. The idea is to have two branches for the horseshoe, which are two stable rectangles in $\mathcal{R}$, one of them containing x_0. We first show that an iterate of $W^u(x_0)$ crosses $W^s(x_0)$, i.e. there is an homoclinic intersection. This is a consequence of standard results in the theory of dispersing billiards.

We need a little more terminology. Given a Cantor rectangle R, we call $S \subset R$ an s–subrectangle of R if for each $x \in S$, $W^s(x) \cap S = W^s(x) \cap R$. Similarly, $U \subset R$ is a u–subrectangle if $W^u(x) \cap U = W^u(x) \cap R$ for all $x \in U$. We say a local unstable manifold W^u *fully crosses* R if $W^u \cap \mathring{D}(R) \neq \emptyset$ and W^u does not terminate in the interior of $D(R)$. A Cantor rectangle R' u–crosses R if every unstable manifold $W^u \in \mathfrak{S}^u(R')$ fully crosses R. Analogous definitions hold for stable manifolds and s–crossings.

Given a Cantor rectangle R, $T^n(R)$ is a finite union of (maximal) Cantor rectangles (recall that stable manifolds cannot be cut by singularities of T). We label them $R_{n,i}$. Then each $T^{-n}(R_{n,i})$ is an s–subrectangle of R.

With these preliminaries, we are ready to proceed with the construction of our horseshoe.

- By [CM, Lemma 7.87], there is a finite collection of rectangles $R_1, \ldots, R_N$ of positive measure each such that any stable and unstable curve of length at least $\delta_1/2$ properly crosses at least one of the rectangles. Without loss of generality, we can assume that $W^u(x_0)$ properly crosses R_1 and $W^s(x_0)$ properly crosses R_2.
- By [CM, Lemma 7.90], there is a 'magnet' rectangle R^* of positive measure, and a 'high density' subset $\mathfrak{P}^* \subset R^*$, satisfying the following property: if $R_{k,n,i}$ is a maximal rectangle in $T^n(R_k)$ and $R_{k,n,i} \cap \mathfrak{P}^* \neq \emptyset$ where n is large enough, then $R_{k,n,i}$ u–crosses R^*.[5] The analogous properties hold for maximal rectangles $R_{k,-n,i}$ of $T^{-n}(R_k)$ and s–crossings of R^*.
- Since μ_{SRB} is mixing, there are $m, n > 0$ such that $G^n(R_1) \cap \mathfrak{P}^* \neq \emptyset$ and $G^{-m}(R_2) \cap \mathfrak{P}^* \neq \emptyset$. Therefore there are an s-subrectangle $R' \subset R_1$ and a u-subrectangle $R'' \subset R_2$ such that $G^n(R')$ u-crosses R^* and $G^{-m}(R'')$ s-crosses R^*. This implies that $G^n(W^u(x_0)) \cap G^{-m}(W^s(x_0)) \neq \emptyset$, proving the desired transverse intersection.

Let $x_1 \in W^u(x_0) \cap G^{-(m+n)}W^s(x_0)$. Take points $z \in W^u(x_0) \cap \mathcal{R}$ and $w \in W^s(x_0) \cap \mathcal{R}$ such that x_1 belongs to the segment of $W^u(x_0)$ joining x_0 and z. The points $x_0, z, [z, w], w$ define a solid rectangle D. Let U_0, U_1 be disjoint s-subrectangles

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[5] Indeed, a slightly stronger property holds: for every $x \in R_{k,n,i}$ its local unstable manifold $W^u(x)$ properly crosses R^*, a proper crossing being a full crossing whose distance from the unstable boundary of R^* is at least a fixed fraction of the unstable diameter of R^*.

of D such that U_0 contains x_0 as a vertex and the solid rectangle of U_1 contains x_1 in the interior of its u–side. If $\ell > 0$ is large, then $G^\ell(U_0)$ u-crosses D. Also, by construction, $G^{m+n}(U_1)$ intersects $W^s(x_0)$ transversally, and so if $\ell_0 > 0$ is large enough then $G^{\ell_0+m+n}(U_1)$ also u-crosses D. Hence, we can fix $\ell_0 > 0$ such that for $\ell = \ell_0 + m + n$ the sets $G^\ell(U_0), G^\ell(U_1)$ both u-cross D. Letting $f = G^\ell$, we obtain a horseshoe for f as the intersection

$$K := \bigcap_{n=-\infty}^{\infty} f^n(U_0 \cup U_1).$$

By construction, K is conjugate to a full shift in two symbols, i.e. if (Σ, σ) is the topological Markov shift with $\Sigma = \{0,1\}^{\mathbb{Z}}$, then there is a measurable bijection $\pi : \Sigma \to K$ such that $\pi \circ \sigma = f \circ \pi$.

4. Proof of Theorem 1.1

The idea to prove Theorem 1.1 is the following: points that spend many iterates near x_0 approach x_0 super-exponentially fast. This is expressed for unstable curves (and analogously for stable curves) by Lemma 3.2. Given $x \in \Delta$, let W^s, W^u be a stable, unstable curve passing through x such that $\{x_s\} = W^u \cap W^s(x_0)$ and $\{x_u\} = W^s \cap W^u(x_0)$ are defined. Hence

$$d(x, x_0) \approx \max\{d(x_s, x_0), d(x_u, x_0)\}.$$

Denote an element of Σ by $\underline{v} = (v_n)_{n \in \mathbb{Z}}$. Let $C = \{\underline{v} \in \Sigma : v_0 = 1\}$, $C_n = \{\underline{v} \in C : v_1 = \cdots = v_{n-1} = 0 \text{ and } v_n = 1\}$ for $n \geq 1$, and $C_\infty = \{\underline{v} \in C : v_n = 0 \text{ for all } n \geq 1\}$. Clearly $C \setminus C_\infty = \bigcup_{n \geq 1} C_n$, and the first return map $\widetilde{\sigma} : C \to C$ of σ to C is a full topological Markov shift in an infinite countable alphabet with $\widetilde{\sigma} = \sigma^n$ on C_n. For $n \geq 1$, let $x = \pi(\underline{v})$ with $\underline{v} \in C_n$.

As in the previous paragraph, let W^u be an unstable curve connecting $f(x)$ and $W^s(x_0)$ and define $\{x_s^1\} = W^u \cap W^s(x_0)$. Let W^s be a stable curve connecting $f^{n-1}(x)$ and $W^u(x_0)$ and define $\{x_u^{n-1}\} = W^s \cap W^u(x_0)$. Since $f^k(x_s^1) \in W^s(x_0)$ and $f^{-k}(x_u^{n-1}) \in W^u(x_0)$ for $k \geq 0$, we may apply Lemma 3.2(b) and obtain that:

- $d(f^{\lfloor \frac{n}{2} \rfloor} x_s^1, x_0) \leq C_0 d(x_s^1, x_0)^{(3/2)^{\lfloor \frac{n}{2} \rfloor}}$.
- $d(f^{-\lfloor \frac{n}{2} \rfloor} x_u^{n-1}, x_0) \leq C_0 d(x_u^{n-1}, x_0)^{(3/2)^{\lfloor \frac{n}{2} \rfloor}}$.

Since W^s is a stable curve, $f^{-\lfloor \frac{n}{2} \rfloor}(W^s)$ is again a stable curve connecting $f^{\lfloor \frac{n}{2} \rfloor}(x)$ with $W^u(x_0)$. Similarly, since W^u is an unstable curve, $f^{\lfloor \frac{n}{2} \rfloor}(W^u)$ is an unstable curve connecting $f^{\lfloor \frac{n}{2} \rfloor}(x)$ with $W^s(x_0)$. In particular, letting $\delta = \text{diam}(D) < 1$, there is a constant $C \geq 1$ such that

$$d(f^{\lfloor \frac{n}{2} \rfloor} x, \mathcal{S}_0) \leq C\delta^{(3/2)^{\lfloor \frac{n}{2} \rfloor}}, \ \forall x \in \pi(C_n). \tag{4.1}$$

Let $b = \sum_{n \geq 1} \left(\frac{3}{2}\right)^{-\lfloor \frac{n}{2} \rfloor} < \infty$. Take $p_n = b^{-1} \left(\frac{3}{2}\right)^{-\lfloor \frac{n}{2} \rfloor}$ for $n \geq 1$, and consider a Bernoulli probability measure $\widetilde{\nu}$ on $C \cong \mathbb{N}^{\mathbb{Z}}$ such that $\widetilde{\nu}(C_n) = p_n$. This measure induces a σ–invariant probability measure ν on Σ, which descends by π to a f–invariant probability measure η on K. Note that ν is supported on the disjoint union $\bigcup_{n \geq 1} \bigcup_{0 \leq k < n} \sigma^k C_n$, and η is supported on the disjoint union $\bigcup_{n \geq 1} \bigcup_{0 \leq k < n} f^k \pi(C_n)$. Applying (4.1),

$$\int |\log d(x, \mathcal{S}_0)| \, d\eta \geq \int_{\bigcup_{n \geq 1} f^{\lfloor \frac{n}{2} \rfloor} \pi(C_n)} |\log d(x, \mathcal{S}_0)| \, d\eta$$

$$= \sum_{n\geq 1} \int_{\pi(C_n)} |\log d(f^{\lfloor \frac{n}{2} \rfloor} x, \mathcal{S}_0)| \, d\eta$$

$$\geq \sum_{n\geq 1} \left| \log \left(C\delta^{(3/2)^{\lfloor \frac{n}{2} \rfloor}} \right) \right| \eta[\pi(C_n)]$$

$$\geq -\log C + |\log \delta| \sum_{n\geq 1} p_n \left(\tfrac{3}{2}\right)^{\lfloor \frac{n}{2} \rfloor} = \infty.$$

Now we estimate $h_\eta(f)$. Let n be the return time function defining $\widetilde{\sigma}$. By the Abramov formula, and using that $\widetilde{\nu}$ is a Bernoulli measure, we have

$$h_\eta(f) = h_\nu(\sigma) = \frac{h_{\widetilde{\nu}}(\widetilde{\sigma})}{\int_C n(\nu) d\widetilde{\nu}(\nu)} = \frac{\displaystyle\sum_{n=1}^{\infty} -p_n \log p_n}{\displaystyle\sum_{n=1}^{\infty} n p_n}$$

$$\approx \frac{\displaystyle\sum_{n=1}^{\infty} \left(\tfrac{3}{2}\right)^{-\lfloor \frac{n}{2} \rfloor} \lfloor \tfrac{n}{2} \rfloor \log(3/2)}{\displaystyle\sum_{n=1}^{\infty} n \left(\tfrac{3}{2}\right)^{-\lfloor \frac{n}{2} \rfloor}} \approx \frac{\log(3/2)}{2} > 0.$$

Finally, let $\mu = \frac{1}{4\ell} \sum_{k=0}^{4\ell-1} \eta \circ T^{-k}$, which is a T–invariant probability measure. We claim that μ is a measure satisfying Theorem 1.1. Firstly, we have

$$\int |\log d(x, \mathcal{S}_0)| \, d\mu = \frac{1}{4\ell} \sum_{k=0}^{4\ell-1} \int |\log d(x, \mathcal{S}_0)| \, d(\eta \circ T^{-k})$$

$$\geq \frac{1}{4\ell} \int |\log d(x, \mathcal{S}_0)| \, d\eta = \infty. \tag{4.2}$$

Since $\mathcal{S}_0 \subset \mathcal{S}_1$, this implies that the measure μ is not adapted.

Now, using that each $\eta \circ T^{-k}$ is f–invariant, it follows again by the Abramov formula and the linearity of the metric entropy that

$$h_\mu(T) = \frac{1}{4\ell} h_\mu(f) = \frac{1}{16\ell^2} \sum_{k=0}^{4\ell-1} h_{\eta \circ T^{-k}}(f) \geq \frac{\log(3/2)}{32\ell^2} > 0.$$

This concludes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

The subset $\pi(C_n)$ contains a periodic orbit of period n for f, equal to $y_n = \pi(\nu)$ where $\nu_0 = 1$ and $\nu_1 = \cdots = \nu_{n-1} = 0$. Due to (4.1), we have $d(f^{\lfloor \frac{n}{2} \rfloor} y_n, \mathcal{S}_0) \leq C\delta^{(3/2)^{\lfloor \frac{n}{2} \rfloor}}$ and so, using again [CM, eq. 4.20],

$$J^u f(f^{\lfloor \frac{n}{2} \rfloor - 1} y_n) \geq (\cos \varphi(f^{\lfloor \frac{n}{2} \rfloor} y_n))^{-1} \approx d(f^{\lfloor \frac{n}{2} \rfloor} y_n, \mathcal{S}_0)^{-1} \geq C^{-1}\delta^{-(3/2)^{\lfloor \frac{n}{2} \rfloor}}.$$

The Lyapunov exponent for the point y_n with respect to f is at least

$$\lim_{j \to \infty} \frac{1}{jn} \log J^u f^{jn}(y_n) \geq \lim_{j \to \infty} \frac{1}{jn} j \log J^u f(f^{\lfloor \frac{n}{2} \rfloor - 1} y_n)$$

$$\geq \frac{1}{n} \log \left(C^{-1} \delta^{-(3/2)\lfloor \frac{n}{2} \rfloor} \right) \approx \frac{\left(\frac{3}{2}\right)^{\lfloor \frac{n}{2} \rfloor} |\log \delta|}{n}.$$

Therefore, the Lyapunov exponent for the point y_n with respect to T is at least of the order of $\left(\frac{3}{2}\right)^{\lfloor \frac{n}{2} \rfloor} / 4\ell n$. We have thus constructed a sequence of periodic orbits with finite, but arbitrarily large Lyapunov exponents. This proves the first statement of Theorem 1.2.

Equation (1.3) follows immediately since each of the above periodic orbits contains no grazing collisions and therefore each corresponding invariant measure is supported outside of an ε–neighborhood of $\mathcal{S}_0$ for some $\varepsilon > 0$.

To prove the final statement of the theorem, equation (1.4), let μ be the T–invariant measure constructed in the proof of Theorem 1.1. Using again [CM, (4.20)], or [CM, eq. (5.36)], we have $J^u T(x) \approx (\cos \varphi(Tx))^{-1}$. By the invariance of μ and estimate (4.2), we get that

$$\int \log J^u T \, d\mu \approx \int -\log \cos(\varphi \circ T) \, d\mu$$

$$= \int -\log \cos \varphi \, d\mu \geq \int |\log d(x, \mathcal{S}_0)| \, d\mu(x) = \infty.$$

Since $h_\mu(T) > 0$, this proves (1.4).

Declarations

Data sharing Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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