



Wightman Fields for Two-Dimensional Conformal Field Theories with Pointed Representation Category

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Dedicated to Roberto Longo in occasion of his 70th birthday

Abstract: Two-dimensional full conformal field theories have been studied in various mathematical frameworks, from algebraic, operator-algebraic to categorical. In this work, we focus our attention on theories with chiral components having pointed braided tensor representation subcategories, namely having automorphisms whose equivalence classes necessarily form an abelian group. For such theories, we exhibit the explicit Hilbert space structure and construct primary fields as Wightman fields for the two-dimensional full theory. Given a finite collection of chiral components with automorphism categories with trivial total braiding, we also construct a local extension of their tensor product as a chiral component. We clarify the relations with the Longo–Rehren construction, and illustrate these results with concrete examples including the $U(1)$ -current.

1. Introduction

Two-dimensional conformal field theories (CFTs) have been studied extensively [DMS97] and have attracted the interest of mathematicians for their algebraic, analytic and geometric structures. In particular, the conformal symmetry in two-dimensional Minkowski spacetime is described by the diffeomorphism group of the lightrays, therefore it is infinite-dimensional. This allows to study first the theories that depend only on one of the lightray coordinates (the chiral components) and then their two-dimensional (full) extensions.

From the operator-algebraic point of view (Haag–Kastler axioms), a general quantum field theory can be formulated as a net of von Neumann algebras associated with the open spacetime regions [Haa96]. The relations between the full theory and the chiral components have been obtained, e.g., in [Reh00], and they resulted in classification schemes for certain classes of two-dimensional CFTs, see, e.g., [KL04b, BKL15]. In this course, it was important that from two copies of a single chiral theory and a family

of charged sectors, one can construct two-dimensional theories as extensions of the two-dimensional theory obtained just by taking tensor products of the chiral components, and then extended it by introducing non-chiral (bulk) fields. Among such extensions, the most studied ones are the “diagonal” extensions, obtained by letting the chiral theories act simultaneously on a direct sum of copies of their vacuum representation, and then introducing “charged” fields that mix the different components.

Since [LR95], in the operator-algebraic setting, the (finite index, i.e., “relatively small”) extensions can be equivalently well described by Q-systems (i.e., C^* -Frobenius algebra objects) in the unitary braided tensor representation category of the net that one wants to extend. Locality of the extension can also be characterized by means of a commutativity condition on the associated Q-system. In the operator-algebras context, the Q-system associated with a finite index “diagonal” extension is called a Longo–Rehren Q-system.

It is worth mentioning that the method of Q-systems applies both to chiral and to two-dimensional theories (in fact even in four-dimensions), and that similar ideas (commutative Frobenius algebra objects in tensor categories) emerged independently in other algebraic approaches to CFT, such as vertex operator algebras [HKL15, KO02], both in one and two dimensions, see, e.g., [FRS02, HK07, Kon07, RFFS07]. In the unitary VOA [CKLW18, DL14, Gui22] and unitary tensor category context [GLR85, DR89b, LR97], these notions (Frobenius algebras, C^* -Frobenius algebras, Q-systems, to describe extensions) have been recently shown to be equivalent [CGGH23].

The relationship between chiral and full two-dimensional CFTs (not restricted to a single Minkowski spacetime) can also be cast and studied [BGS22] in the more general categorical/operadic framework of locally covariant AQFT [BFV03, BSW19, BSW21].

Let us note, however, that most of the purely algebraic frameworks, see, e.g., [HK07], are designed particularly for conformal field theories, and do not apply, as they are, to massive theories. On the other hand, it is natural to expect that a two-dimensional full CFT can be described in a more traditional framework for quantum field theory, the Wightman axioms [SW00]. Wightman fields are operator-valued distributions on a Hilbert space, and a reasonable description of such fields and the Hilbert space would be desirable.

Having explicit Wightman fields is not just interesting on its own, but could be a starting point for constructing non-conformal field theory, e.g., by perturbing the dynamics of the CFT by these fields. Such an idea is presented in [Zam89], where certain massive integrable fields are associated with charged fields in CFT. Perturbing the dynamics of the free field on the same Hilbert space has been carried out on the de Sitter space [BJM23]. Therefore, developing a theory of Wightman fields in two dimensions will be a basis for rigorously studying the relations between CFT and massive models [JT23].

In this paper, we study full two-dimensional CFTs whose chiral components admit a (finite or infinite) collection of automorphisms (invertible objects in the language of tensor categories) among their irreducible representations (superselection sectors). We will define and construct both two-dimensional conformal Haag–Kastler nets and two-dimensional conformal Wightman fields explicitly in terms of the Hilbert space of the chiral components and of their charged sectors. We introduce charged primary fields as operators between different charged sectors of the chiral components, and combine them to obtain local bulk fields.

To be more specific, let \mathcal{A}_L and \mathcal{A}_R be chiral conformal nets on S^1 admitting a family of automorphisms (including the defining vacuum representation) among their charged representations (in the sense of Doplicher–Haag–Roberts [DHR69a, DHR69b, DHR71, DHR74], but in one and two dimensions instead of four) parametrized by the

same abelian group G . These automorphisms, respectively denoted by $L(g)$ and $R(g)$, $g \in G$, are defined on the same Hilbert spaces, but we denote them as $\mathcal{H}_L^{L(g)}$, $\mathcal{H}_R^{R(g)}$ to distinguish the representation. On the Hilbert space $\bigoplus_{g \in G} \mathcal{H}_L^{L(g)} \otimes \mathcal{H}_R^{R(g)}$, the chiral observables $\mathcal{A}_L \otimes \mathcal{A}_R$ act diagonally. We will add charged fields that shift the sectors in the direct sum, under some condition on their braiding (which is satisfied in many cases, see, e.g., the end of Sect. 6) and obtain full two-dimensional conformal nets and conformal Wightman fields. Moreover, with a similar technique, we construct some (presumably new) local conformal nets on S^1 by taking the tensor product of a family of conformal nets and then extending it, in such a way that a certain trivial total braiding condition is fulfilled. This generalizes the well-known extensions of a single chiral $U(1)$ -current net [BMT88], and it can be seen as a variation of the “gluing” construction due to [CKM22] for VOAs.

We describe explicitly the fields for the $U(1)$ -current algebra. Charged primary fields of a single chiral component are given as formal series between charged sectors [TZ12, TL97]. We expect that our construction works for loop group nets (for a simply laced, simple, simply connected, compact group) at level 1 as well [Was98, TL97]. With a more involved combination of left and right chiral components, it should also be possible to generalize it to other completely rational nets (when the charged fields are available), in the presence of irreducible representations with statistical dimension greater than 1 (i.e., in the non-pointed tensor category case).

This paper is organized as follows. In Section 2, we recall the fact that two-dimensional conformal field theories extend to the Einstein cylinder, then set out the operator-algebraic formulation of the chiral components and of the full two-dimensional CFTs. We also collect some facts about representations and charged fields of chiral components, in the case of automorphisms. In Section 3, from a family of chiral components equipped with a collection of automorphisms and satisfying certain conditions on the braiding, we construct an extension of their tensor product on S^1 . In Section 4, we take a pair of left and right chiral components with a collection of automorphisms and assume that their braidings cancel in a certain sense, and we construct a full two-dimensional CFT extending the tensor product of the chiral components. In Section 5, we study the case where the charged primary fields are given as formal series. Under similar assumptions on the braiding and assuming energy bounds, we exhibit the Wightman fields responsible for the extensions at the level of nets constructed in the previous sections. In Section 6, we consider the explicit example of the $U(1)$ -current. We exhibit its superselection charge structure, the associated braiding and charged fields, and we show that they fit in the general construction of the previous sections. In Section 7, we summarize our outlook.

Notations

In order to keep uniform notations throughout the paper, we label chiral (one-dimensional) objects, while we use symbols without a label for two-dimensional objects.

- $\mathcal{A}_K, \phi_{K,j}, \psi_K, \mathcal{A}_L, \mathcal{A}_R, \psi_L, \psi_R$: chiral net/chiral fields/chiral charged fields.
- \mathcal{A}_K, ψ_K : chiral extension/chiral charged local fields.

- \mathcal{A} : two-dimensional net.
- $\tilde{\mathcal{A}}, \tilde{\psi}$: two-dimensional net/two-dimensional charged local fields.

2. Preliminaries

2.1. Einstein cylinder. As suggested by [LM75], it is natural to study four-dimensional conformally covariant Wightman fields on the Einstein cylinder¹. Moreover, the conformal group acts on the Einstein cylinder in a natural way and the fields are covariant with respect to its action. In two dimensions, an analogous procedure can be carried out with one more step because the two-dimensional Einstein cylinder is not simply connected. To state these results, let us first discuss the conformal geometry. We follow [KL04b], and start the discussion with the lightrays \mathbb{R} in \mathbb{R}^{1+1} .

The Möbius group $\text{PSL}(2, \mathbb{R}) \cong \text{Möb}$ acts on the one-point compactification S^1 of \mathbb{R} , where \mathbb{R} is identified with $S^1 \setminus \{-1\}$ through the stereographic projection. Hence, by lifting such action, its universal covering group $\overline{\text{Möb}}$ acts on \mathbb{R} , the universal covering of S^1 . The original lightray \mathbb{R} is identified with the interval $(-\pi, \pi)$ in \mathbb{R} as the universal covering of S^1 . For an interval I such that $\bar{I} \subset (-\pi, \pi)$, there is a neighborhood \mathcal{U} of the unit element in Möb such that if $\gamma \in \mathcal{U}$, then $\gamma \cdot I \subset (-\pi, \pi)$. In this sense, the group Möb acts locally on the lightray.

The two-dimensional Minkowski space \mathbb{R}^{1+1} has the metric $(a, b) = a_0b_0 - a_1b_1$, where $a, b \in \mathbb{R}^{1+1}$. With the lightcone coordinates $(a_-, a_+) = \left(\frac{a_0 - a_1}{\sqrt{2}}, \frac{a_0 + a_1}{\sqrt{2}}\right)$, the Minkowski space is the product of two lightrays $\mathbb{R}^{1+1} = \mathbb{R} \times \mathbb{R}$ (in the lightcone coordinates, not the (a_0, a_1) -coordinates) and the metric can be written as $(a, b) = a_-b_+ + a_+b_-$. A conformal transformation of \mathbb{R}^{1+1} is, by definition, a transformation of \mathbb{R}^{1+1} that preserves the metric up to a scalar. From the above expression, it is clear that a product of any pair of orientation-preserving diffeomorphisms of lightrays is a conformal transformation.

The diamond $D_0 = \{(a_-, a_+) : -\pi < a_{\pm} < \pi\}$ can be mapped to the Minkowski space \mathbb{R}^{1+1} by the conformal transformation $(a_-, a_+) \mapsto (\tan(\frac{1}{2}a_-), \tan(\frac{1}{2}a_+))$. Through this transformation, \mathbb{R}^{1+1} can be identified with D_0 . On $D_0 = (-\pi, \pi) \times (-\pi, \pi)$, the group $\text{Möb} \times \text{Möb}$ acts locally in the sense above.

Let R_t be the lift of the rotation by t in Möb . In a two-dimensional conformal field theory, the correlation functions are invariant under the local action of $\overline{\text{Möb}} \times \overline{\text{Möb}}$, and moreover, the spacelike 2π -rotations $\mathfrak{R} := \{R_{2n\pi} \times R_{-2n\pi} : n \in \mathbb{Z}\}$ are often trivial. If this holds, the field theory can be extended to the Einstein cylinder² $\mathcal{E} = \mathbb{R}^{1+1}/\mathfrak{R}$. In the most favorable case, the theory extends to \mathcal{E} and there is also a local action of the conformal group $\mathcal{C} = \overline{\text{Diff}}_+(S^1) \times \overline{\text{Diff}}_+(S^1)/\mathfrak{R}$ (Fig. 1).

¹ More precisely, [LM75] showed that, assuming that there are Wightman fields and their Euclidean n -point functions are invariant under Euclidean conformal group, the representation of the Poincaré group can be extended to $\text{SO}(4, 2)$. See [KQR21, Section 8.3] for a recent review.

² This is topologically equivalent to $S^1 \times \mathbb{R}$, but the product structure is different from the lightray decomposition.

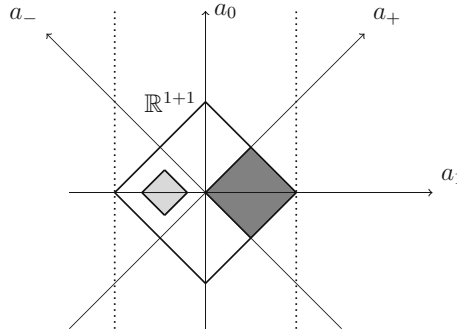


Fig. 1. The Minkowski space \mathbb{R}^{1+1} , depicted as the diamond. When the spacelike rotation $R_{2\pi} \times R_{-2\pi}$ is trivial, the dotted lines are identified and it is a subset of the Einstein cylinder \mathcal{E}

2.2. Chiral components and representation theory.

2.2.1. Conformal net on S^1 . Among two-dimensional conformal fields, there are those that do not depend on one of the lightray coordinates. They are called **chiral fields**, and can be restricted to the lightray, then extended to the circle S^1 by locality. We put the index κ to the objects in this section, although we study a single chiral field theory at a time. Note that we do not use the index κ to distinguish between left and right chiral components. This convention will be useful later when we combine different chiral fields on either component of a two-dimensional conformal field theory, while we denote two-dimensional objects without index.

Let us start with a net on \mathbb{R} , and see how it extends to S^1 . Note that if a unitary projective U_κ representation of $\overline{\text{Diff}_+(S^1)}$ on a Hilbert space \mathcal{H}_κ is restricted to the semisimple subgroup $\overline{\text{Möb}}$, there is a unique true (non-projective) representation whose quotient in $\text{PU}(\mathcal{H}_\kappa)$ coincides with U_κ . In this sense, we can consider the spectrum of U_κ restricted to $\overline{\text{Möb}}$ without ambiguity [Bar54, Theorem 7.1].

We call a triple $(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)$ a **conformal net on \mathbb{R}** if \mathcal{A}_κ assigns to each open non-dense non-empty interval $I \subset \mathbb{R}$ a von Neumann algebra $\mathcal{A}_\kappa(I)$ on a Hilbert space \mathcal{H}_κ , U_κ is a unitary projective representation of $\overline{\text{Diff}_+(S^1)}$ and $\Omega_\kappa \in \mathcal{H}_\kappa$ such that

- (1dCN1) **Isotony:** if $I_1 \subset I_2$, then $\mathcal{A}_\kappa(I_1) \subset \mathcal{A}_\kappa(I_2)$.
- (1dCN2) **Locality:** if I_1 and I_2 are disjoint, then $\mathcal{A}_\kappa(I_1) \subset \mathcal{A}_\kappa(I_2)'$.
- (1dCN3) **Diffeomorphism covariance:** For a bounded interval $I \subset \mathbb{R}$, there is a neighborhood \mathcal{U} of the unit element of $\overline{\text{Diff}_+(S^1)}$ such that if $\gamma \in \mathcal{U}$ then $\gamma \cdot I \subset \mathbb{R}$ and

$$U_\kappa(\gamma)\mathcal{A}_\kappa(I)U_\kappa(\gamma)^* = \mathcal{A}_\kappa(\gamma \cdot I).$$

Furthermore, if $\text{supp } \gamma$ is disjoint from I , then $\text{Ad } U_\kappa(\gamma)(x) = x$ for $x \in \mathcal{A}_\kappa(I)$.

- (1dCN4) **Positivity of energy:** the restriction of U_κ to the translation subgroup $\mathbb{R} \subset \overline{\text{Möb}}$ has the spectrum contained in \mathbb{R}_+ .
- (1dCN5) **Vacuum and the Reeh-Schlieder property:** there exists a unique (up to a phase) vector $\Omega_\kappa \in \mathcal{H}_\kappa$ such that $U_\kappa(g)\Omega_\kappa = \Omega_\kappa$ for $g \in \overline{\text{Möb}}$ and $\overline{\mathcal{A}_\kappa(I)\Omega_\kappa} = \mathcal{H}_\kappa$.

If we assume only the covariance with respect to $\overline{\text{Möb}}$, we call it a **Möbius-covariant net**.

If U_κ factors through $\text{Diff}_+(S^1)$ as a projective representation, that is, if $U_\kappa(R_{2n\pi})$, $n \in \mathbb{Z}$ is a scalar, then \mathcal{A}_κ can be extended to a net defined on the set \mathcal{I} of non-dense, non-trivial open intervals on S^1 and it satisfies the usual axioms of **conformal net on S^1** , that is a triple $(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)$ satisfying isotony, locality, $\text{Diff}_+(S^1)$ -covariance, positivity of energy and the vacuum properties. See, e.g., [KL04a, Section 2.1], [CKLW18, Chapter 3].

We also recall that U_κ can be made into a unitary multiplier representation (rather than projective) of $\overline{\text{Diff}_+(S^1)}$, that is, $U_\kappa(\gamma_1)U_\kappa(\gamma_2) = c(\gamma_1, \gamma_2)U_\kappa(\gamma_1\gamma_2)$ for some $c(\gamma_1, \gamma_2) \in \mathbb{C}$ [Car04, Theorem A.2], [FH05, Proposition 5.1]. Such c is called the **cocycle** of the multiplier representation.

Proposition 2.1. *Let $(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)$ be a conformal net on \mathbb{R} . Assume furthermore that $U_\kappa(R_{2\pi}) = \mathbb{1}$ (in $\text{PU}(\mathcal{H}_\kappa)$), that is, the lift of 2π -rotation is trivial. Then $(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)$ extends to S^1 .*

Proof. If $U_\kappa(R_{2\pi}) = \mathbb{1}$, then its adjoint action is trivial, and the natural extension of \mathcal{A}_κ on $(-\pi, \pi)$ to \mathbb{R} (the universal covering of S^1) is periodic, hence we can regard it as a net on S^1 . It is known that the positivity of energy restricted to the rotation subgroup and that to the translation subgroup are equivalent (see, e.g., [Wei06, Lemma 3.1]). The remaining axioms of [KL04a, Section 2.1] follow easily. \square

The assumption $U_\kappa(R_{2\pi}) = \mathbb{1}$ is necessary, but we do not know whether this follows from more general assumptions (cf. the Bisognano–Wichmann property is necessary and sufficient [GLW98, Theorem 1.4], but we are not aware whether it is automatic for conformal nets on \mathbb{R}).

There is an example of a net on \mathbb{R} with a weaker covariance and not extending to S^1 (U_κ is only a projective representation of the group generated by translations, dilations and diffeomorphisms of \mathbb{R} with compact support): the $U(1)$ -current net with the perturbed stress-energy tensor [BSM90], see also the discussion in [MT19, Section 5.2].

2.2.2. Representations of chiral conformal nets. A **representation** of a conformal net \mathcal{A}_κ (or more precisely of $(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)$) on S^1 is a family of representations $\rho = \{\rho_I\}$ of $\{\mathcal{A}_\kappa(I)\}_{I \in \mathcal{I}}$ on a single Hilbert space \mathcal{H}_κ^ρ , which is compatible in the sense that if $I_1 \subset I_2$, then $\rho_{I_2}|_{\mathcal{A}_\kappa(I_1)} = \rho_{I_1}$. If \mathcal{H}_κ^ρ is equal to \mathcal{H}_κ and $\rho_I(\mathcal{A}_\kappa(I)) = \mathcal{A}_\kappa(I)$, equivalently if the C^* -tensor categorical or statistical dimension d_ρ equals 1, we say that ρ is an **automorphism** of \mathcal{A}_κ .

Let us summarize the general theory of representations in the case of automorphisms (cf. [MTW18, Section 6]):

- Any automorphism is irreducible, hence diffeomorphism covariant in the following sense (cf. [DFK04, Theorem 6] for projective representations, [CDVIT21, Section 3.2] for local multiplier representations): there is a local unitary multiplier representation U_κ^ρ of a neighborhood \mathcal{U} of the unit element of $\text{Diff}_+(S^1)$ with the same cocycle as that of U_κ such that $U_\kappa^\rho(\gamma) = \rho(U_\kappa(\gamma))$ if $\text{supp } \gamma \subset I$ for some I and $\text{Ad } U_\kappa^\rho(\gamma)(\rho(x)) = \rho(\text{Ad } U_\kappa(\gamma)(x))$ for $x \in \mathcal{A}_\kappa(I)$ for some $I, \gamma \in \mathcal{U}$ (the last restriction is why we call it a **local representation**). U_κ^ρ extends to $\text{Diff}_+(S^1)$ as a projective representation because $\text{Diff}_+(S^1)$ is simply connected. Moreover, its restriction to Möb is a true representation and it has automatically positive energy [Wei06, Theorem 3.8]. Therefore, the lowest eigenvalue of the generator of lift rotations L_0^ρ is

uniquely determined, and we denote it by $D_{\kappa,\rho}$. This is called the **conformal dimension**. As the 2π -rotation is trivial, the spectrum of L_0^ρ is contained in $D_{\kappa,\rho} + (\mathbb{N} \cup \{0\})$. By the first part of the proof of [FH05, Theorem 5.1], the representation U_κ^ρ extends to a unitary multiplier representation of $\overline{\text{Diff}_+(S^1)}$.

- If ρ, ρ' are two representations of \mathcal{A}_κ and there is a unitary $V \in \text{U}(\mathcal{H}_\kappa)$ such that $V\rho_I(x) = \rho'_I(x)V$, for every $x \in \mathcal{A}_\kappa(I)$ and for all $I \in \mathcal{I}$ we say that they are unitarily equivalent. We also say that V is a unitary intertwiner between ρ and ρ' .
- An automorphism is said to be **localized in I** if $\rho_{I'} = \text{id}$, where I' is the interior of $S^1 \setminus I$. In this case, ρ_I maps $\mathcal{A}_\kappa(I)$ to itself, i.e., it is an endomorphism (in this case an automorphism) of $\mathcal{A}_\kappa(I)$, by Haag duality on S^1 , [GF93, Theorem 2.19] (i.e., $\mathcal{A}_\kappa(I') = \mathcal{A}_\kappa(I)'$ for every $I \in \mathcal{I}$, cf. [BSM90]). Given ρ localized in I_1 , one can always find a unitarily equivalent automorphism localized in another interval I_2 . (More generally, every representation of a conformal net on a separable Hilbert space can be localized in any given interval, [GF93, Lemma 4.6]). If $I_1 \cup I_2$ is not dense in S^1 , one can take a proper interval I that contains $I_1 \cup I_2$, and by Haag duality on S^1 , the unitary operator V implementing the equivalence belongs to $\mathcal{A}_\kappa(I)$. Such a V is called a **charge transporter**.
- The operator $z_{\kappa,\rho}(\gamma) := U_\kappa(\gamma)U_\kappa^\rho(\gamma)^*$ for $\gamma \in \overline{\text{Diff}_+(S^1)}$ is a charge transporter between ρ and ρ^γ , where $\rho^\gamma := \text{Ad } U_\kappa(\gamma) \circ \rho \circ \text{Ad } U_\kappa(\gamma^{-1})$ is localized in $\gamma \cdot I$ if ρ is localized in I . We call $z_{\kappa,\rho}(\gamma)$ the **covariance cocycle**, or just cocycle, of ρ .
- The **DHR tensor (or monoidal) product** of representations ρ_1 and ρ_2 is defined by the composition of the associated endomorphisms (automorphisms in this case) localized in each I . Note that the tensor product is not given by considering the tensor product Hilbert space, which would not give a well-defined family of representations of the local algebras. In symbols, $(\rho_1)_I \otimes (\rho_2)_I := (\rho_1)_I \circ (\rho_2)_I$, or just $(\rho_1)_I(\rho_2)_I$, for short. The resulting representation is denoted by $\rho_1 \otimes \rho_2$, or just by $\rho_1 \rho_2$. See, e.g., [GF93, Section IV.2]. For strongly additive conformal nets \mathcal{A}_κ on S^1 (i.e., $\mathcal{A}_\kappa(I_1) \vee \mathcal{A}_\kappa(I_2) = \mathcal{A}_\kappa(I)$ for every $I \in \mathcal{I}$ and $I_1, I_2 \in \mathcal{I}$ arising as connected components of $I \setminus \{p\}$, $p \in I$), or equivalently for conformal nets on \mathbb{R} that satisfy Haag duality on \mathbb{R} , the tensor product can be defined globally, see, [KLM01, Appendix B], [GR18, Section 2], which is closer in spirit to the original definition of Doplicher–Haag–Roberts [DHR69a,DHR69b].
- Let ρ_1, ρ_2 be two automorphisms localized in I . In order to define the **DHR braiding** [DHR71,FRS89] between ρ_1 and ρ_2 , let us choose a point on S^1 , identified with the point at infinity by means of the corresponding stereographic projection onto \mathbb{R} . Take $\tilde{\rho}_1, \tilde{\rho}_2$ localized in \tilde{I}_1, \tilde{I}_2 , respectively, such that $\tilde{I}_1 \cap \tilde{I}_2 = \emptyset$, \tilde{I}_1, \tilde{I}_2 are away from infinity, and take charge transporters V_1, V_2 between ρ_1 and $\tilde{\rho}_1, \rho_2$ and $\tilde{\rho}_2$, respectively. The operator $\epsilon_{\rho_1, \rho_2}^\pm := \rho_2(V_1^*)V_2^*V_1\rho_1(V_2)$, where \pm depends on whether I_1 is on the left or right of I_2 , is a unitary intertwiner between $\rho_1\rho_2$ and $\rho_2\rho_1$ with all the properties of a unitary braiding in a unitary tensor category. It does not depend on the choice of $\tilde{\rho}_1, \tilde{\rho}_2$ or V_1, V_2 under the same configuration of \tilde{I}_1, \tilde{I}_2 , or on the chosen point at infinity, see, e.g., [GF93, Section IV.4].
- Let ρ be an automorphism localized in I . The conjugate automorphism $\bar{\rho}$ of ρ is the automorphism localized in I' given by $j \circ \rho \circ j$, where $j = \text{Ad } J_I$ and J_I is the modular conjugation of $\mathcal{A}_\kappa(I)$ with respect to the vacuum. It follows that $\rho \circ \bar{\rho} \cong \text{id}$, see [GL92, Theorem 8.3], [GL96, Theorem 2.11].

Let $(\mathcal{B}_\kappa, U_\kappa, \Omega_\kappa)$ be a conformal net on S^1 and let $\{\mathcal{A}_\kappa\}_{I \in \mathcal{I}}$ be a family of von Neumann subalgebras $\mathcal{A}_\kappa(I) \subset \mathcal{B}_\kappa(I)$ satisfying covariance with respect to U_κ . Then,

on the subspace $\mathcal{H}_{\mathcal{A}_\kappa} = \overline{\bigcup_{I \in \mathcal{I}} \mathcal{A}_\kappa(I)\Omega_\kappa}$, $(\mathcal{A}_\kappa|_{\mathcal{H}_{\mathcal{A}_\kappa}}, U_\kappa|_{\mathcal{H}_{\mathcal{A}_\kappa}}, \Omega_\kappa)$ is a Möbius-covariant net with respect to U_κ because $U_\kappa(\gamma)x\Omega_\kappa = \text{Ad } U_\kappa(\gamma)(x)\Omega_\kappa \in \mathcal{A}(\gamma \cdot I)\Omega$ for $x \in \mathcal{A}_\kappa(I)$, thus $U_\kappa(\gamma)$ preserves $\mathcal{H}_{\mathcal{A}_\kappa}$. The restriction of $U_\kappa|_{\mathcal{H}_{\mathcal{A}_\kappa}}$ to Möb often extends to $\text{Diff}_+(S^1)$ and $\mathcal{A}_\kappa|_{\mathcal{H}_{\mathcal{A}_\kappa}}$ is covariant with respect to it, in which case we say that \mathcal{A}_κ is a (conformal) subnet of \mathcal{B}_κ and write $\mathcal{A}_\kappa \subset \mathcal{B}_\kappa$ for simplicity. In this case, we say that \mathcal{B}_κ is an **extension** of \mathcal{A}_κ .

2.2.3. *Charged fields associated with automorphisms.* Let $\{(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)\}$ be a conformal net on S^1 . In this section, we assume that among its irreducible representations there are non-trivial automorphisms. In the language of tensor categories, the representations of \mathcal{A}_κ contain a (braided) **pointed tensor subcategory**.

The construction of charged fields and extensions performed in this section is inspired by [DHR69b, DR89a, DR90] in the four-dimensional context, cf. [Bau95, Chapter 3], where the unitary tensor categories in question are in addition symmetrically braided (hence dual to a compact group [DR90]). In [DHR69b] the pointed case is investigated, as in our case. See also [Müg01] for comparison with the one and two-dimensional context.

Fix an interval $I \in \mathcal{I}$. Let Δ_κ be a choice of mutually inequivalent automorphisms of \mathcal{A}_κ localized in I , one for each unitary equivalence class, including the trivial automorphism id (the defining vacuum representation) of \mathcal{A}_κ . Under the present assumption, the equivalence classes of automorphisms form a *discrete (finite or infinite) abelian group* G under class multiplication $[\rho][\sigma] = [\rho\sigma]$ (where $\rho\sigma$ is the composition of automorphisms) and inversion $[\rho]^{-1} = [\rho^{-1}]$. Let us assume that G is *finitely generated*, whose elements we denote by g, h, g^{-1}, \dots , and we denote by ι the identity element. Let $\kappa(g) \in \Delta_\kappa$, for every $g \in G$, be the previously made choice of automorphisms of \mathcal{A}_κ . We assume that $\kappa(\iota) = \text{id}$. Without loss of generality, by suitably changing the localization of the automorphisms inside I , we may assume that $\kappa(g)\kappa(h) = \kappa(h)\kappa(g)$ for every $g, h \in G$. Note that we are *not* assuming $\kappa(g)\kappa(h) = \kappa(gh)$, as Δ_κ need *not* be closed under composition and inverses. For every $g, h \in G$, let $V^{g,h}$ be a unitary intertwiner in $\mathcal{A}_\kappa(I)$ between $\kappa(gh)$ and $\kappa(g)\kappa(h)$ realizing the equivalence $[\kappa(gh)] = [\kappa(g)][\kappa(h)]$. Namely, $V^{g,h}\kappa(gh)(x) = \kappa(g)(\kappa(h)(x))V^{g,h}$ for every $x \in \mathcal{A}_\kappa(I)$. We may assume that $V^{g,h} = V^{h,g}$ and that $V^{g,\iota} = V^{\iota,h} = 1$.

For each $g \in G$, we define the operator ψ_κ^g acting on the Hilbert space $\hat{\mathcal{H}}_\kappa := \bigoplus_{g \in G} \mathcal{H}_\kappa^{\kappa(g)}$, where $\mathcal{H}_\kappa^{\kappa(g)} := \mathcal{H}_\kappa$ the vacuum Hilbert space of \mathcal{A}_κ , by setting

$$(\psi_\kappa^g \Psi)_h := V^{g,h}(\Psi)_{gh},$$

for every $\Psi \in \hat{\mathcal{H}}_\kappa$. We call the ψ_κ^g **charged field operators**. They satisfy the following:

- Each ψ_κ^g is unitary on $\hat{\mathcal{H}}_\kappa$ and it maps $\mathcal{H}_\kappa^{\kappa(h)}$ to $\mathcal{H}_\kappa^{\kappa(gh)}$ for every $h \in G$.
- Consider the representation $\hat{\kappa}(x) := \bigoplus_{h \in G} \kappa(h)(x)$ of $x \in \mathcal{A}_\kappa(I)$ on $\hat{\mathcal{H}}_\kappa$. Then

$$\begin{aligned} (\psi_\kappa^g \hat{\kappa}(x)\Psi)_h &= V^{g,h}(\hat{\kappa}(x)\Psi)_{gh} \\ &= V^{g,h}\kappa(gh)(x)(\Psi)_{gh} \\ &= \kappa(g)(\kappa(h)(x))V^{g,h}(\Psi)_{gh} \\ &= \kappa(h)(\kappa(g)(x))V^{g,h}(\Psi)_{gh} \\ &= \kappa(h)(\kappa(g)(x))(\psi_\kappa^g \Psi)_h \\ &= (\hat{\kappa}(\kappa(g)(x))\psi_\kappa^g \Psi)_h, \end{aligned}$$

for every $\Psi \in \hat{\mathcal{H}}_\kappa$, where we used that $\kappa(h)$ and $\kappa(g)$ commute in the 4th equality. Therefore, we have the “charged field intertwiner” property (in the sense of Doplicher–Roberts [DR72]) together with its conjugate (by substituting x with $\kappa(g)^{-1}(x^*)$):

$$\begin{aligned} \psi_\kappa^g \hat{\kappa}(x) &= \hat{\kappa}(\kappa(g)(x)) \psi_\kappa^g, \\ (\psi_\kappa^g)^* \hat{\kappa}(x) &= \hat{\kappa}(\kappa(g)^{-1}(x)) (\psi_\kappa^g)^*, \end{aligned} \tag{2.1}$$

for every $x \in \mathcal{A}_\kappa(I)$, as operators on $\hat{\mathcal{H}}_\kappa$.

- The covariance cocycles $z_{\kappa(g)}(\gamma)$ for $\gamma \in \text{Diff}_+(S^1)$ are unitary charged transporters between $\kappa(g)$ and $\kappa(g)^\gamma$, and they fulfill the following tensoriality property. *Tensoriality of cocycles:* $z_{\kappa(g)}(\gamma)\kappa(g)(z_{\kappa(h)}(\gamma)) = z_{\kappa(g)\kappa(h)}(\gamma)$, or equivalently written as $z_{\kappa(g)}(\gamma) \otimes z_{\kappa(h)}(\gamma) = z_{\kappa(g) \otimes \kappa(h)}(\gamma)$. See [Lon97, Appendix A], [DG18, Section 7], [MTW18, Proposition 6.1].
- Let $\hat{U}_\kappa(\gamma) := \bigoplus_{h \in G} U_\kappa^{\kappa(h)}(\gamma)$ be a representation of $\text{Diff}_+(S^1)$. As in [DG18, Theorem 7.7], [MTW18, Section 6], we get

$$\begin{aligned} (\hat{U}_\kappa(\gamma) \psi_\kappa^g \hat{U}_\kappa(\gamma)^* \Psi)_h &= U_\kappa^{\kappa(h)}(\gamma) (\psi_\kappa^g \hat{U}_\kappa(\gamma)^* \Psi)_h \\ &= U_\kappa^{\kappa(h)}(\gamma) V^{g,h} (\hat{U}_\kappa(\gamma)^* \Psi)_{gh} \\ &= U_\kappa^{\kappa(h)}(\gamma) V^{g,h} U_\kappa^{\kappa(gh)}(\gamma)^* (\Psi)_{gh} \\ &= U_\kappa^{\kappa(h)}(\gamma) U_\kappa^{\kappa(g)\kappa(h)}(\gamma)^* V^{g,h} (\Psi)_{gh} \\ &= U_\kappa^{\kappa(h)}(\gamma) U_\kappa^{\kappa(i)}(\gamma)^* U_\kappa^{\kappa(i)}(\gamma) U_\kappa^{\kappa(g)\kappa(h)}(\gamma)^* V^{g,h} (\Psi)_{gh} \\ &= z_{\kappa(h)}(\gamma)^* z_{\kappa(h)\kappa(g)}(\gamma) V^{g,h} (\Psi)_{gh} \\ &= \kappa(h)(z_{\kappa(g)}(\gamma)) V^{g,h} (\Psi)_{gh} \\ &= (\hat{\kappa}(z_{\kappa(g)}(\gamma)) \psi_\kappa^g \Psi)_h, \end{aligned}$$

for every $\Psi \in \hat{\mathcal{H}}_\kappa$. We used that $U_\kappa^{\kappa(g)}(\gamma) = \kappa(g)(U_\kappa(\gamma))$ if $\text{supp } \gamma \subset I$ in the 4th equality (see Sect. 2.2.2), the definition of $z_{\kappa(g)}(\gamma)$ in the 6th equality, and the tensoriality of the cocycles in the 7th equality.

Therefore, we have the following “covariance property” of the charged fields: for every $\gamma \in \text{Diff}_+(S^1)$, as operators on $\hat{\mathcal{H}}_\kappa$,

$$\begin{aligned} \text{Ad } \hat{U}_\kappa(\gamma) (\psi_\kappa^g) &= \hat{\kappa}(z_{\kappa(g)}(\gamma)) \psi_\kappa^g, \\ \text{Ad } \hat{U}_\kappa(\gamma) ((\psi_\kappa^g)^*) &= \hat{\kappa}(z_{\kappa(g)^{-1}}(\gamma)) (\psi_\kappa^g)^*, \end{aligned} \tag{2.2}$$

where we used that $(\psi_\kappa^g)^* \hat{\kappa}(z_{\kappa(g)}(\gamma)^*) = \hat{\kappa}(\kappa(g)^{-1}(z_{\kappa(g)}(\gamma)^*)) (\psi_\kappa^g)^* = \hat{\kappa}(z_{\kappa(g)^{-1}}(\gamma)) (\psi_\kappa^g)^*$ by the charged field intertwiner property of $(\psi_\kappa^g)^*$ and the tensoriality of cocycle.

Remark 2.2. Note that the charged fields constructed above are highly non-canonical, as they depend, e.g., on the choice of $V^{g,h}$ for every $g, h \in G$ (each $V^{g,h}$ is unique up to unitary equivalence). Note also that we are demanding neither $\psi_\kappa^g \psi_\kappa^h = \psi_\kappa^{gh}$, nor $\psi_\kappa^{g^*} = \psi_\kappa^{g^{-1}}$ for $g, h \in G$, i.e., the charged field operators need not form a group,

cf. [Reh90a, Section 3]. Nevertheless, by the intertwiner and covariance properties, for every choice of $V^{g,h}$, they give rise to *relatively local* (but possibly neither irreducible, nor factorial) extensions of \mathcal{A}_κ .

Later we shall need the condition $\psi_\kappa^g \psi_\kappa^h = \psi_\kappa^h \psi_\kappa^g$ (established in Proposition 2.3 below) on the charged fields just defined, in order to produce local extensions of tensor products by “gluing” conformal nets. Note that $\psi_\kappa^g \psi_\kappa^h = \psi_\kappa^h \psi_\kappa^g$, alone, does not mean locality of the extension. Note also that the group multiplication condition $\psi_\kappa^g \psi_\kappa^h = \psi_\kappa^{g^h}$ would imply the commutativity condition $\psi_\kappa^g \psi_\kappa^h = \psi_\kappa^h \psi_\kappa^g$, as G is abelian.

Proposition 2.3. *In the notation of this section, suppose that $G = \mathbb{Z}_n$ whose elements we label by $\{0, 1, \dots, n - 1\}$ modulo n , and let $\alpha \in \Delta_\kappa$ be an automorphism localized in I such that $\alpha^k \in \Delta_\kappa$ for all $k = 0, 1, \dots, n - 1$ and $[\alpha^n] = [\text{id}]$. Choose a unitary $V \in \mathcal{A}_\kappa(I)$ intertwining id with α^n , i.e., $Vx = \alpha^n(x)V$ for every $x \in \mathcal{A}_\kappa(I)$. Let the $V^{g,h}$, $g, h \in G$, be defined by $V^{g,h} := V$ if $g + h \geq n$, where $g, h \in \{0, 1, \dots, n - 1\}$ are thought of as representatives, and by $V^{g,h} := \mathbb{1}$ if $g + h < n$. Then $\psi_\kappa^g \psi_\kappa^h = \psi_\kappa^h \psi_\kappa^g$ and $(\psi_\kappa^g)^* \psi_\kappa^h = \psi_\kappa^h (\psi_\kappa^g)^*$ for every $g, h \in G$.*

Proof. Only in this proof, we use the additive notation $g + h$ for the group operation.

We first note that with our choice $V^{g,h} \kappa(g + h) = \kappa(g)\kappa(h)V^{g,h}$. By taking the representative $g, h \in \{0, 1, \dots, n - 1\}$, this can be checked by cases depending on whether $g + h \geq n$ or $g + h < n$ and using that $\kappa(g) = \kappa(1)^g$.

From the definition, it is straightforward that ψ_κ^1 (where the upper index 1 is the group element, not the exponent) is the product of a shift on $\bigoplus_{g \in G} \mathcal{H}_\kappa^{\kappa(g)} = \bigoplus_{g \in G} \mathcal{H}_\kappa$ as a Hilbert space followed by 1 or V , and that $\psi_\kappa^g = (\psi_\kappa^1)^g$, again with $g \in \{0, 1, \dots, n - 1\}$. From this it is clear that $\psi_\kappa^g, (\psi_\kappa^g)^*$ and ψ_κ^h commute. \square

Now let G be a finitely generated abelian group. Then it is isomorphic to a finite product of cyclic groups \mathbb{Z}_n or \mathbb{Z} . Let us assume that Δ_κ is a choice of mutually inequivalent automorphisms of \mathcal{A}_κ localized in I and their equivalence classes form G as a group. For each of the finite cyclic groups \mathbb{Z}_n in G , we can choose a generator in Δ_κ localized in a smaller interval $I \subset I$. For each of these generators, we choose $V^{h,g} \in \mathcal{A}(I)$ as in Proposition 2.3, constructing the operator ψ_κ^h for such $h, g \in \mathbb{Z}_n$. We can make this choice for each copy of \mathbb{Z}_n by choosing finitely many mutually disjoint smaller intervals inside I , obtaining commuting operators $\psi_\kappa^h, \psi_\kappa^g$ where g, h are in different finite cyclic groups in G . For each of the copies of the infinite cyclic group \mathbb{Z} , we can take a generator 1 and we may assume that $\kappa(g) = \kappa(1)^g$ for all $g \in \mathbb{Z}$, thus we can set $V^{g,h} := \mathbb{1}$ and $\psi_\kappa^g \psi_\kappa^h = \psi_\kappa^{gh}$ for every $g, h \in \mathbb{Z}$. Altogether, we obtain charged fields ψ_κ^h for every $h \in G$, satisfying $\psi_\kappa^h \psi_\kappa^g = \psi_\kappa^g \psi_\kappa^h$.

Remark 2.4. Even if G is not finitely generated, if there is a choice of Δ_κ such that $\kappa(g)\kappa(h) = \kappa(gh)$, then one can construct ψ_κ^h satisfying $\psi_\kappa^g \psi_\kappa^h = \psi_\kappa^{gh} = \psi_\kappa^h \psi_\kappa^g$ as in [MTW18, Section 6], thus $(\psi^g)^* = \psi^{g^{-1}}$ and the rest of our construction works as well. This happens with the $U(1)$ -current net with $G = \mathbb{Q}$ or \mathbb{R} .

2.3. Two-dimensional conformal nets. In [KL04b], it was shown that any Haag–Kastler net on \mathbb{R}^{1+1} that is locally conformally covariant extends to the Einstein cylinder, using a slight modification of the conformal spin-statistics theorem [GL96], [MT19, Theorem A.5]. To state this result precisely, let us start with a net on \mathbb{R}^{1+1} . We call a triple

(\mathcal{A}, U, Ω) a **conformal net on \mathbb{R}^{1+1}** if \mathcal{A} assigns to each open region $O \subset \mathbb{R}^{1+1}$ a von Neumann algebra $\mathcal{A}(O)$ on a Hilbert space \mathcal{H} , U is a unitary projective representation of $\overline{\text{Diff}_+(S^1)} \times \overline{\text{Diff}_+(S^1)}$ and $\Omega \in \mathcal{H}$ satisfying

- (2dCN1) **Isotony:** if $O_1 \subset O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- (2dCN2) **Locality:** if O_1 and O_2 are spacelike separated, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)'$.
- (2dCN3) **Diffeomorphism covariance:** For a bounded region $O \subset \mathbb{R}^{1+1}$, there is a neighborhood \mathcal{U} of the unit element of $\overline{\text{Diff}_+(S^1)} \times \overline{\text{Diff}_+(S^1)}$ such that if $\gamma \in \mathcal{U}$ then $\gamma \cdot O \subset \mathbb{R}^{1+1}$ and

$$U(\gamma)\mathcal{A}(O)U(\gamma)^* = \mathcal{A}(\gamma \cdot O).$$

Furthermore, if $\text{supp } \gamma$ is disjoint from O , then $\text{Ad } U(\gamma)(x) = x$ for $\mathcal{A}(O)$.

- (2dCN4) **Positivity of energy:** the restriction of U to the translation subgroup $\mathbb{R}^2 \subset \overline{\text{Möb}} \times \overline{\text{Möb}}$ has the joint spectrum contained in the closed forward light cone $\overline{V_+} = \{(a_0, a_1) \in \mathbb{R}^{1+1} : a_0^2 - a_1^2 \geq 0, a_0 \geq 0\}$.
- (2dCN5) **Vacuum and the Reeh-Schlieder property:** there exists a unique (up to a phase) vector $\Omega \in \mathcal{H}$ such that $U(g)\Omega = \Omega$ for $g \in \overline{\text{Möb}}$ and is cyclic for any local algebra, namely $\overline{\mathcal{A}(O)\Omega} = \mathcal{H}$.

If there is a conformal net on \mathbb{R}^{1+1} as above, we can consider it as a net on the diamond $D_0 = (-\pi, \pi) \times (-\pi, \pi)$ as in Sect. 2.1. With this identification, the group $\overline{\text{Diff}_+(S^1)} \times \overline{\text{Diff}_+(S^1)}$ acts on $\mathbb{R} \times \mathbb{R}$, and we can extend the net \mathcal{A} by covariance. However, this extension is not very natural. Indeed, it often happens that the conformal net extends to \mathcal{E} rather than to $\mathbb{R} \times \mathbb{R}$. We say that (\mathcal{A}, U, Ω) is a **conformal net on \mathcal{E}** if \mathcal{A} is defined for regions in \mathcal{E} , covariant with respect to U along the natural action of $\overline{\text{Diff}_+(S^1)} \times \overline{\text{Diff}_+(S^1)}$ and local in the sense that $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute whenever there is a translation γ of the cylinder such that D_0 contains $\gamma \cdot O_1, \gamma \cdot O_2$ and they are spacelike there. A sufficient condition is given in [MT19, Theorem A.5] (the Bisognano–Wichmann property for wedges), however, it is not immediate to check it in the examples we construct. Instead, the following is easier to check and gives immediately the desired extension.

Proposition 2.5. *Let (\mathcal{A}, U, Ω) be a conformal net on \mathbb{R}^{1+1} . Assume furthermore that $U(R_{2\pi} \times R_{-2\pi}) = \mathbb{1}$ (in $\text{PU}(\mathcal{H})$), that is, the spacelike 2π -rotation is trivial. Then (\mathcal{A}, U, Ω) extends to a conformal net on \mathcal{E} .*

Proof. This is parallel to Proposition 2.1.

As $U(R_{2\pi} \times R_{-2\pi}) = \mathbb{1}$, the representation U factors through the group \mathcal{C} (see Sect. 2.1). Furthermore, as $U(R_{2\pi} \times R_{-2\pi}) = \mathbb{1}$, for any region O , $\mathcal{A}(O)$ and $\mathcal{A}(R_{2\pi} \times R_{-2\pi} \cdot O)$ coincide. Therefore, we can identify any point x on $\mathbb{R} \times \mathbb{R}$ and $R_{2\pi} \times R_{-2\pi} \cdot x$, and obtain the Einstein cylinder \mathcal{E} . Covariance follows by definition, and locality in the sense above follows from covariance and locality in D_0 . □

In the situation of Proposition 2.5, our net (\mathcal{A}, U, Ω) is equivalent to a local conformal net in the sense of [KL04b, Section 2].

Rehren introduced the maximal chiral nets of a two-dimensional conformal net (\mathcal{A}, U, Ω) that extends to \mathcal{E} [Reh00]: let $I_L \times I_R$ be a diamond in \mathbb{R}^{1+1} . Define $\mathcal{A}_L^{\max}(I_L) = \mathcal{A}(I_L \times I_R) \cap U(\iota \times \overline{\text{Diff}_+(S^1)})'$, where ι is the unit element of $\overline{\text{Diff}_+(S^1)}$. Then \mathcal{A}_L^{\max} is *a priori* a conformal net on \mathbb{R} defined on the Hilbert subspace $\mathcal{H}_L = \overline{\mathcal{A}_L^{\max}\Omega}$, and satisfies the condition of Proposition 2.1, therefore, it extends to S^1 . Similarly, one can

define \mathcal{A}_R^{\max} on \mathcal{H}_R . The original full net \mathcal{A} contains the tensor product $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max}$ on the subspace $\mathcal{H}_L \otimes \mathcal{H}_R \subset \mathcal{H}$ (more precisely, there is a natural injective homomorphism from $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max}$ into \mathcal{A}). In this sense, a generic two-dimensional conformal net \mathcal{A} is an extension of the tensor product net $\mathcal{A}_L^{\max} \otimes \mathcal{A}_R^{\max}$.

3. One-Dimensional Gluing from Trivial Total Braiding

In this Section, from a family of conformal nets on S^1 satisfying certain conditions, we construct *local* extensions of their tensor products. Let us start with a *finite* collection conformal nets $\{(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)\}_{\kappa \in K}$ on S^1 labelled by $\kappa \in K$, $|K| < \infty$. We fix an interval $I \in \mathcal{I}$. We assume that each \mathcal{A}_κ admits a (not necessarily finite) collection Δ_κ of mutually inequivalent and commuting automorphisms localized in I , containing the trivial automorphism id_κ , and whose fusion rules (up to unitary equivalence) are isomorphic to the *same* abelian group G , as in Sect. 2.2.3. This group G should be either a finitely generated abelian group, or we assume that we can choose elements in Δ_κ and charged fields ψ_κ^h as in Sect. 2.2.3, fulfilling the conclusions of Proposition 2.3 (for each $\kappa \in K$). Denote also by $\kappa : G \rightarrow \Delta_\kappa$ a bijection (not necessarily an isomorphism, as each Δ_κ need not be closed under composition and inverses), such that $\kappa(\iota) = \text{id}_\kappa$, for each $\kappa \in K$. Here κ is used as an index as well as a map, with a slight abuse of notations, hence $\kappa(g)$ is an automorphism of \mathcal{A}_κ . Furthermore, we assume that, for all $g_1, g_2, g \in G$,

- $\prod_\kappa \epsilon_{\kappa(g_1), \kappa(g_2)}^\pm = 1$,
- $\prod_\kappa \epsilon_{\kappa(g_1)^{-1}, \kappa(g_2)}^\pm = 1$,
- $\sum_\kappa D_{\kappa, \kappa(g)} \in \mathbb{Z}$,

where the choice of \pm above is common for all κ , and we denoted by the same symbol $\epsilon_{\kappa(g_1), \kappa(g_2)}^\pm$ the phase multiple of $\mathbb{1}$ (the trivial intertwiner between $\kappa(g_1)\kappa(g_2)$ and itself) associated with the braiding of $\kappa(g_1)$ and $\kappa(g_2)$. This makes sense by the commutativity assumption on Δ_κ , namely $\kappa(g_1)\kappa(g_2) = \kappa(g_2)\kappa(g_1)$.

Now we construct a conformal net \mathcal{A}_K on S^1 as follows.

- The Hilbert space of our net is $\mathcal{H}_K := \bigoplus_{g \in G} \bigotimes_{\kappa \in K} \mathcal{H}_\kappa^{\kappa(g)}$. On this space, we let any operator of the form $\bigotimes_\kappa x_\kappa \in \bigotimes_\kappa \mathcal{A}_\kappa(I)$ act³ as $\bigoplus_{g \in G} \bigotimes_\kappa \kappa(g)(x_\kappa)$. Denote this representation of the tensor product net $\bigotimes_\kappa \mathcal{A}_\kappa$ by K .
- We also consider an auxiliary tensor product space $\hat{\mathcal{H}}_K := \bigotimes_{\kappa \in K} \bigoplus_{g \in G} \mathcal{H}_\kappa^{\kappa(g)} \supset \mathcal{H}_K$. The representation of the element $\bigotimes_\kappa x_\kappa$ on this space is denoted by $\hat{K}(\bigotimes_\kappa x_\kappa)$. We have $\hat{K}(\bigotimes_\kappa x_\kappa)|_{\mathcal{H}_K} = K(\bigotimes_\kappa x_\kappa)$.
- The vacuum vector of the net \mathcal{A}_K will be $\Omega_K := \bigotimes_\kappa \Omega_\kappa \in \bigotimes_\kappa \mathcal{H}_\kappa^{\kappa(\iota)}$, belonging to the ι -th component of \mathcal{H}_K .
- The covariance is given by $U_K(\gamma) := \bigoplus_{g \in G} \bigotimes_\kappa U_\kappa^{\kappa(g)}(\gamma)$, which is a unitary multiplier representation of $\overline{\text{Diff}_+(S^1)}$. By the condition that $\sum_\kappa D_{\kappa, \kappa(g)} \in \mathbb{Z}$, U_K satisfies the assumptions of Proposition 2.1 and is a unitary multiplier representation of $\overline{\text{Diff}_+(S^1)}$. Thus by ignoring the phase, it is a unitary projective representation of $\overline{\text{Diff}_+(S^1)}$. This also extends naturally to \hat{U}_K on $\hat{\mathcal{H}}_K$.

³ To clarify the notation once more, κ in x_κ and Δ_κ works as an index to distinguish different tensor components, while κ in $\kappa(g)$ indicates the choice of automorphism in Δ_κ labelled by $g \in G$. Therefore, $\kappa(g)(x_\kappa)$ is a bounded operator on \mathcal{H}_κ^g . In general, κ works as an index except for $\kappa(g)$.

- For each element $g \in G$, we introduce the charged field operator ψ_K^g as follows. Our Hilbert space $\mathcal{H}_K = \bigoplus_{g \in G} \bigotimes_{\kappa} \mathcal{H}_{\kappa}^{\kappa(g)}$ is a “diagonal” subspace of the auxiliary Hilbert space $\hat{\mathcal{H}}_K = \bigotimes_{\kappa} \hat{\mathcal{H}}_{\kappa} = \bigotimes_{\kappa} \bigoplus_{g \in G} \mathcal{H}_{\kappa}^{\kappa(g)}$ in a natural way, where recall that $\hat{\mathcal{H}}_{\kappa}$ was defined and used in Sect. 2.2.3. Let

$$\psi_K^g := \bigotimes_{\kappa} \psi_{\kappa}^g,$$

where ψ_{κ}^g are charged fields acting as in Sect. 2.2.3 on $\hat{\mathcal{H}}_{\kappa}$, hence ψ_K^g acts on $\bigotimes_{\kappa} \hat{\mathcal{H}}_{\kappa}$ but preserves \mathcal{H}_K . Now, for the ψ_K^g we make the choice of the $V^{g,h}$ leading to the conclusions of Proposition 2.3 and comments thereafter. Namely, we choose charged fields such that $\psi_{\kappa}^g \psi_{\kappa}^h = \psi_{\kappa}^h \psi_{\kappa}^g$ and $(\psi_{\kappa}^g)^* \psi_{\kappa}^h = \psi_{\kappa}^h (\psi_{\kappa}^g)^*$ for every $g, h \in G$.

- From the charged intertwiner property (2.1), it follows that

$$\psi_K^g \hat{K} \left(\bigotimes_{\kappa} x_{\kappa} \right) = \hat{K} \left(\bigotimes_{\kappa} \kappa(g)(x_{\kappa}) \right) \psi_K^g, \tag{3.1}$$

$$(\psi_K^g)^* \hat{K} \left(\bigotimes_{\kappa} x_{\kappa} \right) = \hat{K} \left(\bigotimes_{\kappa} \kappa(g)^{-1}(x_{\kappa}) \right) (\psi_K^g)^*. \tag{3.2}$$

- The local algebras are given as follows. For the interval I fixed above, we set

$$\mathcal{A}_K(I) := \hat{K} \left(\bigotimes_{\kappa} \mathcal{A}_{\kappa}(I) \right) \vee \{ \psi_K^g, (\psi_K^g)^* \}_{g \in G}.$$

For any other interval $\gamma \cdot I \subset \mathbb{R}$ given by the action of a diffeomorphism γ , we set $\mathcal{A}_K(\gamma \cdot I) := \text{Ad } U_K(\gamma)(\mathcal{A}_K(I))$. We will show below that this net is well-defined, diffeomorphism covariant, local, and it extends to S^1 .

Theorem 3.1. *Let $\{(\mathcal{A}_{\kappa}, U_{\kappa}, \Omega_{\kappa})\}_{\kappa \in K}$ as above and assume that $\{\Delta_{\kappa}\}_{\kappa \in K}$, their braiding and $D_{\kappa, \kappa(g)}$ satisfy the conditions stated at the beginning of this section. Then the net $(\mathcal{A}_K, U_K, \Omega_K)$ satisfies (1dCN1)–(1dCN5) and the condition of Proposition 2.1, hence provides a conformal net on S^1 , extending the chiral tensor product net $\bigotimes_{\kappa} \mathcal{A}_{\kappa}$.*

Proof. Although the calculations will be similar to those of [MTW18, Section 6], the setting is different because we construct a chiral net on S^1 , instead of two-dimensional on \mathcal{E} . Let us present the proofs to exhibit why this works for tensor products of $|K|$ factors.

Let us first compute on the auxiliary Hilbert space $\bigotimes_{\kappa} \hat{\mathcal{H}}_{\kappa}$ with $\hat{U}_K(\gamma) := \bigotimes_{\kappa} \hat{U}_{\kappa}(\gamma)$, where $\hat{U}_{\kappa}(\gamma) = \bigoplus_{h \in G} U_{\kappa}^{\kappa(h)}(\gamma)$ was defined and used in Sect. 2.2.3. Using the covariance property of charged fields (2.2) (i.e., the formula for the adjoint action of $\hat{U}_{\kappa}(\gamma)$ on ψ_{κ}^g acting on each $\hat{\mathcal{H}}_{\kappa}$), we get

$$\begin{aligned} \text{Ad } \hat{U}_K(\gamma)(\psi_K^g) &= \hat{K} \left(\bigotimes_{\kappa} z_{\kappa(g)}(\gamma) \right) \psi_K^g, \\ \text{Ad } \hat{U}_K(\gamma)((\psi_K^g)^*) &= \hat{K} \left(\bigotimes_{\kappa} z_{\kappa(g)^{-1}}(\gamma) \right) (\psi_K^g)^*. \end{aligned}$$

Furthermore, we also have $\psi_K^g \psi_K^h = \psi_K^h \psi_K^g$, $(\psi_K^g)^* \psi_K^h = \psi_K^h (\psi_K^g)^*$ by the commutation condition we imposed on the charged fields ψ_K^g , ψ_K^h , and $(\psi_K^g)^*$.

We have to make sure that $\mathcal{A}_K(\gamma \cdot I)$ is well-defined, that is, the definition does not depend on the choice of γ . This question reduces to whether $\text{Ad } U_K(\gamma)(\mathcal{A}_K(I)) = \mathcal{A}_K(I)$ if $\gamma \cdot I = I$. This is true because we can check the inclusion for γ with compact support and generating elements of $\mathcal{A}_K(I)$:

$$\begin{aligned} \text{Ad } U_K(\gamma) \left(K \left(\bigotimes_{\kappa} x_{\kappa} \right) \right) &= K \left(\bigotimes_{\kappa} \text{Ad } U_K(\gamma)(x) \right) \in \mathcal{A}_K(I), \\ \text{Ad } \hat{U}_K(\gamma)(\psi_K^g) &= \hat{K} \left(\bigotimes_{\kappa} z_{\kappa(g)}(\gamma) \right) \psi_K^g \in \mathcal{A}_K(I), \\ \text{Ad } \hat{U}_K(\gamma)((\psi_K^g)^*) &= \hat{K} \left(\bigotimes_{\kappa} z_{\kappa(g)^{-1}}(\gamma) \right) (\psi_K^g)^* \in \mathcal{A}_K(I). \end{aligned}$$

because $z_{\kappa(g)}(\gamma) \in \mathcal{A}_{\kappa}(I)$. As a general γ can be written as a product of diffeomorphisms with compact supports, this gives the inclusion $\text{Ad } U_K(\gamma)(\mathcal{A}_K(I)) \subset \mathcal{A}_K(I)$, and the converse inclusion is obtained by applying this to γ^{-1} .

With this well-definedness, the first part of covariance (1dCN3) follows by definition. Concerning the second part, if $\text{supp } \gamma$ is disjoint from I , then $U_K(\gamma)$ commutes both with $\hat{K} \left(\bigotimes_{\kappa} \mathcal{A}_{\kappa}(I) \right)$ by covariance of \mathcal{A}_{κ} . Furthermore, $U_K(\gamma)$ commutes with ψ_K^g because componentwise ψ_K^g is a product of the shift and an element in $\bigotimes_{\kappa} \mathcal{A}_{\kappa}(I)$, while $U_K(\gamma) \in \hat{K} \left(\bigotimes_{\kappa} \mathcal{A}_{\kappa}(I') \right)$ and this commutes with the shift because the $\kappa(g)$ are localized in I .

Positivity of energy (1dCN4) follows because U_K is a direct sum of positive-energy representations. The vacuum Ω_K is invariant under $U_K|_{\text{Möb}}$ because it is the tensor product of the vacuum vectors of \mathcal{A}_{κ} . It is cyclic for $\mathcal{A}_K(I_1)$, where $I_1 \subset I'$, because $\mathcal{A}_K(I_1)\Omega_K$ spans $\bigotimes_{\kappa} \mathcal{H}_{\kappa}$ due to the cyclicity of the vacua of \mathcal{A}_{κ} , and the whole \mathcal{H}_K , since $\mathcal{A}_K(I_1)$ contains the shifts $\hat{K}(z_g(\gamma))\psi_K^g$ (up to a unitary on the left). From the assumption $\sum_{\kappa} D_{\kappa, \kappa(g)} \in \mathbb{Z}$ for each $g \in G$, we can apply Proposition 2.1.

As for locality, we take γ such that $\gamma \cdot I$ is disjoint from I . We have to show that generating elements $\hat{K} \left(\bigotimes_{\kappa} x_{\kappa} \right)$, $\psi_K^{g_1}$ of $\mathcal{A}_K(I)$ and $\text{Ad } U_K(\gamma) \left(\hat{K} \left(\bigotimes_{\kappa} y_{\kappa} \right) \right)$, $\text{Ad } U_K(\gamma) (\psi_K^{g_2})$ of $\mathcal{A}_K(\gamma \cdot I)$ commute. This is easy except the one involving $\psi_K^{g_1}$ and $\text{Ad } U_K(\gamma)(\psi_K^{g_2})$. As we have seen, $\text{Ad } U_K(\gamma)(\psi_K^g) = \hat{K} \left(\bigotimes_{\kappa} z_{\kappa(g)}(\gamma) \right) \psi_K^g$. Therefore, to compute the commutator,

$$\begin{aligned} \psi_K^{g_1} \text{Ad } U_K(\gamma)(\psi_K^{g_2}) &= \psi_K^{g_1} \hat{K} \left(\bigotimes_{\kappa} z_{\kappa(g_2)}(\gamma) \right) \psi_K^{g_2} \\ &= \hat{K} \left(\bigotimes_{\kappa} \kappa(g_1)(z_{\kappa(g_2)}(\gamma)) \right) \psi_K^{g_1} \psi_K^{g_2}, \end{aligned}$$

and

$$\text{Ad } U_K(\gamma)(\psi_K^{g_2})\psi_K^{g_1} = \hat{K} \left(\bigotimes_{\kappa} z_{\kappa(g_2)}(\gamma) \right) \psi_K^{g_2} \psi_K^{g_1}.$$

Now observe that $z_{\kappa(g_2)}(\gamma)^* \kappa(g_1)(z_{\kappa(g_2)}(\gamma)) = \epsilon_{\kappa(g_1), \kappa(g_2)}^{\pm}$ (the DHR braiding, a scalar in the case of automorphisms $\kappa(g_1)$ and $\kappa(g_2)$) where the \pm sign depends only on whether γ moves I to the left or to the right, and the choice of \pm is common for all κ). Hence the two left hand sides above are equal if and only if $\prod_{\kappa} \epsilon_{\kappa(g_1), \kappa(g_2)}^{\pm} = 1$ and $\psi_K^{g_1} \psi_K^{g_2} = \psi_K^{g_2} \psi_K^{g_1}$ for every $g_1, g_2 \in G$. The commutation between $(\psi_K^{g_1})^*$ and $\text{Ad } U_K(\gamma)(\psi_K^{g_2})$ follows too by replacing $\kappa(g_1)$ by $\kappa(g_1)^{-1}$ and using the condition $\prod_{\kappa} \epsilon_{\kappa(g_1)^{-1}, \kappa(g_2)}^{\pm} = 1$ and $(\psi_K^{g_1})^* \psi_K^{g_2} = \psi_K^{g_2} (\psi_K^{g_1})^*$. Locality for general intervals follows from the previous paragraph and covariance. \square

If all \mathcal{A}_{κ} are *strongly additive*, that is, $\mathcal{A}_{\kappa}(I_3) = \mathcal{A}_{\kappa}(I_1) \vee \mathcal{A}_{\kappa}(I_2)$ where I_1, I_2 is obtained by removing one point from I_3 , then \mathcal{A}_K is strongly additive, too. This can be seen by taking $I_1 = I$, then $\mathcal{A}_K(I_3)$ is generated by the same $\psi_K^g \in \mathcal{A}_{\kappa}(I)$ and $K(\otimes_{\kappa} x_{\kappa})$, $x_{\kappa} \in \mathcal{A}_{\kappa}(I_3) = \mathcal{A}_{\kappa}(I) \vee \mathcal{A}_{\kappa}(I_2)$.

This construction can be carried out even if there is only one index κ , if the condition $\epsilon_{\kappa(g_1), \kappa(g_2)}^{\pm} = 1$ is satisfied. When applied to the $U(1)$ -current net, this gives the extensions considered in [BMT88], cf. Sect. 6, where $G = \mathbb{Z}$, the map κ is a group isomorphism, and $\kappa(g_1), \kappa(g_2)$ are powers of the same fixed automorphism $\kappa(h)$ having $\epsilon_{\kappa(h), \kappa(h)}^{\pm} = 1$ (bosonic automorphism).

4. Two-Dimensional Conformal Field Theory Arising from Braiding–Cancelling Map

With an idea similar to that of Sect. 3, we construct conformal nets on \mathbb{R}^{1+1} . Here we start with left and right chiral components, take their tensor product as a two-dimensional conformal net, then find extensions of it. The problem of constructing extensions on \mathbb{R}^{1+1} by adding “charged fields” associated with suitable subgroups of automorphisms on top of a tensor product theory has also been tackled in [Reh90b, Section 5], see in particular [Reh90b, Proposition 5.5], within the framework of the reduced field bundle [FRS89, FRS92].

4.1. Extensions with pointed tensor category. Let us start with a pair of conformal nets $\mathcal{A}_L, \mathcal{A}_R$ on S^1 . We fix an interval $I \subset S^1$. As in Sect. 3, we assume that \mathcal{A}_L and \mathcal{A}_R admit, respectively, a family Δ_L, Δ_R (finite or infinite) of mutually inequivalent and commuting automorphisms localized in I , containing the trivial automorphism id_L, id_R , and whose fusion rules (up to unitary equivalence) are isomorphic to the same abelian group G , cf. Sect. 2.2.3. This group G should be either finitely generated, or we assume that we can choose elements in Δ_L, Δ_R and charged fields ψ_L^g, ψ_R^g with the commuting property as in the conclusions of Proposition 2.3 (for $\kappa = L, R$) and comments thereafter. In this section, we denote by L, R (instead of κ) the bijections from G to Δ_L, Δ_R , respectively, corresponding to the choice of automorphisms, and by $D_{L(g)}, D_{R(g)}$, where $g \in G$, the conformal dimensions of the associated automorphisms.

We assume that

- Δ_L and Δ_R contain the trivial automorphisms id_L, id_R of \mathcal{A}_L and \mathcal{A}_R , corresponding to the identity element ι of G via $L(\iota) = \text{id}_L, R(\iota) = \text{id}_R$.

- For every $g_1, g_2 \in G$, it holds that $\overline{\epsilon_{L(g_1), L(g_2)}^\pm} = \epsilon_{R(g_1), R(g_2)}^\mp$ and $\overline{\epsilon_{L(g_1), L(g_2)}^\pm}^{-1} = \epsilon_{R(g_1), R(g_2)}^\mp$ (braiding cancellation), and $D_{L(g)} - D_{R(g)} \in \mathbb{Z}$ for every $g \in G$.

We denote by the same symbol $\epsilon_{L(g_1), L(g_2)}^\pm$ the phase multiple of $\mathbb{1}$ (the trivial intertwiner between $L(g_1)L(g_2) = L(g_2)L(g_1)$ and itself) associated with the braiding of $L(g_1)$ and $L(g_2)$. Similarly for $R(g_1)$ and $R(g_2)$.

Remark 4.1. Let $\mathcal{A}_L = \mathcal{A}_R$. If G is cyclic, then the braiding cancellation is automatic if we take $\Delta_L = \Delta_R$ and $L(g) = R(g)^{-1}$ for every $g \in G$. Indeed, in general, for commuting automorphisms, it holds that $\overline{\epsilon_{L(g_1), L(g_2)}^\pm} = \epsilon_{L(g_2)^{-1}, L(g_1)^{-1}}^\mp$. Now, for automorphisms $L(g_1), L(g_2)$ that are powers of the same $L(h)$, namely $L(g_1) = L(h)^n, L(g_2) = L(h)^m$ for $n, m \in \mathbb{Z}$, it holds $\overline{\epsilon_{L(g_1), L(g_2)}^\pm} = (\epsilon_{L(h), L(h)}^\pm)^{nm} = \epsilon_{L(g_2), L(g_1)}^\pm$ (namely, the nm -th power of the statistical phase of $L(h)$). Hence g_1 and g_2 can be exchanged and we have braiding cancellation.

With this input, we construct a conformal net $\tilde{\mathcal{A}}$ on \mathbb{R}^{1+1} as follows.

- The full Hilbert space is $\tilde{\mathcal{H}} := \bigoplus_{g \in G} \mathcal{H}_L^{L(g)} \otimes \mathcal{H}_R^{R(g)}$. On $\tilde{\mathcal{H}}$, any operator of the form $x_L \otimes x_R \in \mathcal{A}_L(I_L) \otimes \mathcal{A}_R(I_R)$ acts as $\bigoplus_{g \in G} L(g)(x_L) \otimes R(g)(x_R)$ (see the footnote 3 for this notation). Denote this representation by $\tilde{\tau}$.
- The vacuum vector of $\tilde{\mathcal{A}}$ is $\Omega_L \otimes \Omega_R \in \mathcal{H}_L^{L(i)} \otimes \mathcal{H}_R^{R(i)}$.
- The covariance is given by $\tilde{U}(\gamma_L \times \gamma_R) := \bigoplus_{g \in G} U_L^{L(g)}(\gamma_L) \otimes U_R^{R(g)}(\gamma_R)$, which is a unitary projective representation of $\text{Diff}_+(S^1) \times \text{Diff}_+(S^1)$. By the assumption that $D_{L(g)} - D_{R(g)} \in \mathbb{Z}$, \tilde{U} satisfies the condition of Proposition 2.5.
- Note that $\tilde{\mathcal{H}}$ can be seen as a subspace of $\hat{\mathcal{H}}_L \otimes \hat{\mathcal{H}}_R := \left(\bigoplus_{g \in G} \mathcal{H}_L^{L(g)} \right) \otimes \left(\bigoplus_{g \in G} \mathcal{H}_R^{R(g)} \right)$ in a natural way. We denote $\hat{L} := \bigoplus_{g \in G} L(g), \hat{R} := \bigoplus_{g \in G} R(g)$.
- Let ψ_L^g, ψ_R^g be as in Sect. 2.2.3, with the $V^{g,h}$ chosen as in Proposition 2.3. For each $g \in G$, we introduce the charge operator $\tilde{\psi}^g$ as follows

$$\tilde{\psi}^g := \psi_L^g \otimes \psi_R^g \in \mathcal{B}(\hat{\mathcal{H}}_L) \otimes \mathcal{B}(\hat{\mathcal{H}}_R). \tag{4.1}$$

It is then clear that $\tilde{\psi}^g$ preserves $\tilde{\mathcal{H}}$.

From this definition and by (2.1), it follows that

$$\tilde{\psi}^g \hat{L}(x_L) \otimes \hat{R}(x_R) = \hat{L}(L(g)(x_L)) \otimes \hat{R}(R(g)(x_R)) \tilde{\psi}^g, \tag{4.2}$$

just as in (3.1).

- The local algebras of $\tilde{\mathcal{A}}$ are given as follows: For the diamond $I \times I$, we set $\tilde{\mathcal{A}}(I \times I) := \tilde{\tau}(\mathcal{A}_L(I) \otimes \mathcal{A}_R(I)) \vee \{\psi^g, (\psi^g)^*\}_{g \in G}$. For any other diamond $\gamma_L \cdot I \times \gamma_R \cdot I$, we set $\tilde{\mathcal{A}}(\gamma_L \cdot I \times \gamma_R \cdot I) := \text{Ad } \tilde{U}(\gamma_L \times \gamma_R)(\tilde{\mathcal{A}}(I \times I))$.

Theorem 4.2. *The net $(\tilde{\mathcal{A}}, \tilde{U}, \tilde{\Omega})$ satisfies (2dCN1)–(2dCN5) and the condition of Proposition 2.5, hence provides a conformal net on \mathcal{E} , extending the two-dimensional conformal net $\mathcal{A}_L \otimes \mathcal{A}_R$.*

Proof. Most of the properties, well-definedness of $\tilde{\mathcal{A}}$, (2dCN1)–(2dCN5) except for (2dCN2) and the condition of Proposition 2.5 can be verified as in [MTW18, Section 6], using the assumption $D_{L(g)} - D_{R(g)} \in \mathbb{Z}$. As for locality, in

the proof of [MTW18, Section 6], it is only important that $\overline{\epsilon_{L(g_1),L(g_2)}^\pm} = \epsilon_{R(g_1),R(g_2)}^\mp$ as scalars, which we assume. The commutation between $(\tilde{\psi}^{g_1})^*$ and $\text{Ad } U(\gamma_L \times \gamma_R)(\psi^{g_2})$ follows from $\overline{\epsilon_{L(g_1)^{-1},L(g_2)}^\pm} = \epsilon_{R(g_1)^{-1},R(g_2)}^\mp$. Therefore, the proof of locality (2dCN2) works as well. \square

4.2. Comparison with Longo–Rehren extensions. In this section, we review the so-called Longo–Rehren extensions introduced in [LR95, Proposition 4.10], see also [Mas00], [KLM01, Appendix A], [BKLR15, Section 3.3] and references therein, and we interpret them as “generalized shift constructions”. Later, we shall relate them to the extensions presented in the previous sections.

Let \mathcal{N} be an infinite factor (e.g., $\mathcal{N} = \mathcal{A}_\kappa(I)$ a local algebra of a conformal net on S^1). Let \mathcal{C} be a **unitary fusion category** (not necessarily braided, for the moment) realized as a full subcategory of $\text{End}(\mathcal{N})$, the set of normal injective unital $*$ -endomorphisms of \mathcal{N} . For the preliminaries on unitary fusion categories (realized, without loss of generality, as endomorphisms of von Neumann algebras) we refer, e.g., to [BKLR15, EGNO15]. We use the notation $\text{Hom}_{\mathcal{C}}(\rho, \sigma)$ for the spaces of intertwiners $V \in \mathcal{N}$ between ρ and σ objects in \mathcal{C} , namely $V\rho(x) = \sigma(x)V$ for every $x \in \mathcal{N}$, the arrows in our subcategory of endomorphism.

For every unitary equivalence class of irreducible objects in \mathcal{C} , choose one representative $\rho_i, i = 0, \dots, n$, with $\rho_0 = \text{id}$, and denote $\Delta := \{\rho_0, \dots, \rho_n\}$. Note that, if $\mathcal{C}, \mathcal{D} \subset \text{End}(\mathcal{N})$ are two unitary fusion categories, then the Deligne tensor product $\mathcal{C} \boxtimes \mathcal{D}$ [BK01, Definition 1.1.15] can be realized in $\text{End}(\mathcal{N} \bar{\otimes} \mathcal{N})$, where $\mathcal{N} \bar{\otimes} \mathcal{N}$ is the spatial tensor product⁴ von Neumann algebra. Next, let $\mathcal{J} : \mathcal{N} \rightarrow \mathcal{N}^{\mathcal{J}}$ be an anti-linear isomorphism of von Neumann algebras, where $\mathcal{N}^{\mathcal{J}} := \mathcal{J}(\mathcal{N})$, e.g., the natural involution $\mathcal{J} : x \mapsto x^*$, where $\mathcal{N}^{\mathcal{J}} \cong \mathcal{N}^{\text{opp}}$. Consider the unitary fusion category $\mathcal{C}^{\mathcal{J}} \subset \text{End}(\mathcal{N}^{\mathcal{J}})$ with objects $\rho^{\mathcal{J}} := \mathcal{J} \circ \rho \circ \mathcal{J}^{-1} \in \text{End}(\mathcal{N}^{\mathcal{J}})$, where ρ is an object in \mathcal{C} , and with arrows $t^{\mathcal{J}} := \mathcal{J}(t)$, where t is an arrow in \mathcal{C} .

The **Longo–Rehren extension** of the spatial tensor product von Neumann algebra $\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}}$ (an irreducible finite index unital inclusion of factors $\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}} \subset \mathcal{M}$) is specified by the irreducible Q-system $(\Theta_{\text{LR}}, W_{\text{LR}}, X_{\text{LR}})$ in $\mathcal{C} \boxtimes \mathcal{C}^{\mathcal{J}} \subset \text{End}(\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}})$ defined as follows. Let

$$\Theta_{\text{LR}} := \bigoplus_{i=0, \dots, n} \rho_i \bar{\otimes} \rho_i^{\mathcal{J}} \in \text{End}(\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}}).$$

The direct sum is defined by choosing a family of $n + 1$ isometries $T_{i,i} \in \mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}}, i = 0, \dots, n$, “mutually orthogonal” in the sense that $T_{i,i}^* T_{i',i'} = \delta_{i,i'} \mathbb{1}$, and “complete” in the sense that $\sum_{i=0, \dots, n} T_{i,i} T_{i,i}^* = \mathbb{1}$, and letting $\Theta_{\text{LR}}(x) := \sum_{i=0, \dots, n} T_{i,i} \rho_i \bar{\otimes} \rho_i^{\mathcal{J}}(x) T_{i,i}^*$ for every $x \in \mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}}$. Hence, by definition, $T_{i,i} \in \text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\mathcal{J}}}(\rho_i \bar{\otimes} \rho_i^{\mathcal{J}}, \Theta_{\text{LR}})$ for every $i = 0, \dots, n$. Note that the endomorphism Θ_{LR} is not a “simple tensor” in $\mathcal{C} \boxtimes \mathcal{C}^{\mathcal{J}}$, unless $n = 0$.

Let $W_{\text{LR}} := T_{0,0} \in \text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\mathcal{J}}}(\text{id} \bar{\otimes} \text{id}, \Theta_{\text{LR}})$ be the unit of the Q-system and let

$$X_{\text{LR}} := \sum_{i,j,k=0, \dots, n} \sqrt{\frac{d(\rho_i)d(\rho_j)}{d(\rho_k)}} (T_{i,i} \otimes T_{j,j}) \left(\sum_V V \bar{\otimes} V^{\mathcal{J}} \right) T_{k,k}^* \in \text{Hom}_{\mathcal{C} \boxtimes \mathcal{C}^{\mathcal{J}}}(\Theta_{\text{LR}}, \Theta_{\text{LR}}^2)$$

⁴ In this section, we denote by $\bar{\otimes}$ the spatial tensor product of operators, in order to distinguish it from the categorical tensor product functor \otimes , which we shall employ below on intertwiners.

be the comultiplication of the Q-system, where $V \in \mathcal{N}$ runs in a chosen orthonormal basis of $N_{i,j}^k$ (= the multiplicity of ρ_k in $\rho_i \rho_j$, possibly 0) isometries of $\text{Hom}_{\mathcal{C}}(\rho_k, \rho_i \rho_j)$ (whose dimension indeed equals $N_{i,j}^k$) for every fixed $i, j, k = 0, \dots, n$, with respect to the inner product $V^*V' = \delta_{V,V'}$. The definition of X_{LR} is *independent* of this choice because \mathcal{J} is antilinear. Note that $T_{i,i} \otimes T_{j,j}$ denotes the tensor product of arrows in the category of endomorphisms, and that both X_{LR} and W_{LR} belong to $\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}}$. The normalizations read $W_{\text{LR}}^* W_{\text{LR}} = \mathbb{1}$ and $X_{\text{LR}}^* X_{\text{LR}} = (\sum_i d(\rho_i)^2) \mathbb{1}$. Being $(\Theta_{\text{LR}}, W_{\text{LR}}, X_{\text{LR}})$ a Q-system, see [LR95, Proposition 4.10] for the proof and cf. [BKLR15, Definition 3.8], [BKLR15, Proposition 3.19], then by [Lon94] $\Theta_{\text{LR}} \in \text{End}(\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}})$ is a dual canonical endomorphism of an irreducible finite index unital extension $\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}} \subset \mathcal{M}$ realized on the Hilbert space $\mathcal{H} \bar{\otimes} \mathcal{H}^{\mathcal{J}}$ if \mathcal{N} and $\mathcal{N}^{\mathcal{J}}$ are respectively realized on \mathcal{H} and $\mathcal{H}^{\mathcal{J}}$. The beginning of the Jones/canonical tunnel, see [LR95, Section 2.5], reads

$$\mathcal{M}_{-2} \subset \mathcal{M}_{-1} \subset \mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}} \subset \mathcal{M},$$

where $\mathcal{M}_{-2} := \Theta_{\text{LR}}(\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}})$ is the image of Θ_{LR} , and $\mathcal{M}_{-1} := \langle \Theta_{\text{LR}}(\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}}), X_{\text{LR}} \rangle$ is the von Neumann algebra generated together with X_{LR} . The two subfactors $\mathcal{M}_{-2} \subset \mathcal{M}_{-1}$ and $\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}} \subset \mathcal{M}$ are spatially isomorphic. The Q-system $(\Theta_{\text{LR}}, W_{\text{LR}}, X_{\text{LR}})$ also specifies a (unique, by irreducibility, normal faithful) conditional expectation E from \mathcal{M} onto $\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}}$ with Jones projection $e \in \mathcal{M}_1$ (the Jones extension of \mathcal{M} with respect to E), or equivalently $E_{-2}(\cdot) := \Theta_{\text{LR}}(W_{\text{LR}}^* \cdot W_{\text{LR}})$ from \mathcal{M}_{-1} onto \mathcal{M}_{-2} with Jones projection $e_{-2} := W_{\text{LR}} W_{\text{LR}}^*$.

Remark 4.3. Choosing a state ω on $\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}}$ (e.g., the vacuum state in the QFT context) and working in the GNS Hilbert space of $(\mathcal{M}, \omega \circ E)$ yields the more common description in QFT of the extension $\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}} \subset \mathcal{M}$ with a vacuum vector Ω (cyclic and separating for \mathcal{M} and inducing an E -invariant state) such that $e = [\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}} \Omega]$, cf. the proof of [LR95, Theorem 4.9].

Let $\psi_{j,j} := \Theta_{\text{LR}}(T_{j,j}^*) X_{\text{LR}} \in \mathcal{M}_{-1}$, for every $j = 0, \dots, n$. Each $\psi_{j,j}$ has the “charged field intertwiner” property on $\mathcal{M}_{-2} \subset \mathcal{M}_{-1}$ in the sense of Doplicher–Roberts [DR72], i.e.,

$$\psi_{j,j} \Theta_{\text{LR}}(x) = \Theta_{\text{LR}}(\rho_j \bar{\otimes} \rho_j^{\mathcal{J}}(x)) \psi_{j,j}, \quad x \in \mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}}.$$

We refer to the $\psi_{j,j}$ as the **charged fields** of the Longo–Rehren Q-system $(\Theta_{\text{LR}}, W_{\text{LR}}, X_{\text{LR}})$. Such charged fields can be defined for every other Q-system of endomorphisms.

As $X_{\text{LR}} = \sum_{j=0, \dots, n} \Theta_{\text{LR}}(T_{j,j}) \psi_{j,j}$, because $\sum_{j=0, \dots, n} T_{j,j} T_{j,j}^* = \mathbb{1}$, it holds

$$\mathcal{M}_{-1} = \langle \mathcal{M}_{-2}, \psi_{0,0}, \dots, \psi_{n,n} \rangle.$$

Note that $\psi_{0,0} = \Theta_{\text{LR}}(T_{0,0}^*) X_{\text{LR}} = \Theta_{\text{LR}}(W_{\text{LR}}^*) X_{\text{LR}} = \mathbb{1}$.

Setting $\mathcal{H}_{j,j} := T_{j,j} T_{j,j}^* \mathcal{H} \bar{\otimes} \mathcal{H}^{\mathcal{J}}$, for every $j = 0, \dots, n$, yields an orthogonal decomposition

$$\mathcal{H} \bar{\otimes} \mathcal{H}^{\mathcal{J}} = \bigoplus_{j=0, \dots, n} \mathcal{H}_{j,j}. \tag{4.3}$$

Note that $\mathcal{H}_{0,0}$ is the range of the Jones projection $e_{-2} = T_{0,0} T_{0,0}^* = W_{\text{LR}} W_{\text{LR}}^*$.

Moreover, by definition of X_{LR} and $T_{i,i} \otimes T_{j,j} = \Theta_{LR}(T_{j,j})T_{i,i}$, it follows

$$\psi_{j,j} = \sum_{i,k=0,\dots,n} \sqrt{\frac{d(\rho_i)d(\rho_j)}{d(\rho_k)}} T_{i,i} \left(\sum_V V \bar{\otimes} V^{\mathcal{J}} \right) T_{k,k}^*$$

where the first sum runs over all i, k such that $\text{Hom}(\rho_k, \rho_i \rho_j) \neq \langle 0 \rangle$ for fixed j , and the second sum runs over V in the chosen orthonormal basis of isometries in $\text{Hom}_{\mathcal{C}}(\rho_k, \rho_i \rho_j)$. Note that, while \mathcal{M}_{-2} acts diagonally on $\mathcal{H} \bar{\otimes} \mathcal{H}^{\mathcal{J}}$ by the very definition of Θ_{LR} , the charged fields $\psi_{j,j}$'s mix different components.

Remark 4.4. In the special case of **pointed** unitary fusion categories \mathcal{C} (in particular, $d(\rho_i) = d(\rho_j) = d(\rho_k) = 1$ and the fusion ring is a finite group), then all vector spaces $\text{Hom}_{\mathcal{C}}(\rho_k, \rho_i \rho_j)$ have either dimension 1 or 0. In this case each $\psi_{j,j}$ operates as a “right j^{-1} shift operator” on the grading of $\mathcal{H} \bar{\otimes} \mathcal{H}^{\mathcal{J}}$, mapping each subspace $\mathcal{H}_{k,k}$, $k = 0, \dots, n$, to the subspace $\mathcal{H}_{i,i}$ such that $[\rho_k] = [\rho_i \rho_j]$, i.e., $[\rho_k \rho_j^{-1}] = [\rho_i]$. Moreover, for every $j, j' = 0, \dots, n$,

$$\psi_{j,j} \psi_{j',j'} = \sum_{i,k=0,\dots,n} T_{i,i} (V \bar{\otimes} V^{\mathcal{J}}) T_{k,k}^* \sum_{i',k'=0,\dots,n} T_{i',i'} (V' \bar{\otimes} V'^{\mathcal{J}}) T_{k',k'}^*$$

where i, k, i', k' are such that $\dim(\text{Hom}_{\mathcal{C}}(\rho_k, \rho_i \rho_j)) = 1$ and $\dim(\text{Hom}_{\mathcal{C}}(\rho_{k'}, \rho_{i'} \rho_{j'})) = 1$. Note that, as Δ is fixed, V, V' are both unique up to a phase factor, and \mathcal{J} is antilinear, hence the irrelevance of their choice is immediately evident in the pointed case. If we could choose Δ to be a group (i.e., closed under multiplication, as $\rho_0 = \text{id}$), then using $T_{k,k}^* T_{i',i'} = \delta_{k,i'} \mathbb{1}$ and $V V' \in \text{Hom}(\rho_{k'}, \rho_i \rho_j \rho_{j'}) = \text{Hom}(\rho_{k'}, \rho_i \rho_h)$ for some $\rho_h \in \Delta$ such that $\rho_j \rho_{j'} = \rho_h$, then it would follow that $\psi_{j,j} \psi_{j',j'} = \psi_{h,h}$. If we chose the inverse (which is a conjugate in the case of automorphisms) of some ρ_j to be some $\rho_l \in \Delta$, together with $\rho_0 = \text{id}$, then $\psi_{j,j}^* = \psi_{l,l}$.

If \mathcal{C} is in addition **braided**, not necessarily pointed, and $\mathcal{C}^{\mathcal{J}}$ is endowed with the braiding induced by \mathcal{J} , then the Longo–Rehren Q-system is *commutative*. See the discussion following the proof of [LR95, Proposition 4.10] and cf. [BKLR15, Proposition 4.21]. Let $\epsilon_{\rho_i, \rho_j}^{\pm}$ be the braiding in \mathcal{C} and its opposite, then the commutativity condition reads $\epsilon_{\Theta_{LR}, \Theta_{LR}}^{\pm} X_{LR} = X_{LR}$, i.e., for every $i, j = 0, \dots, n$,

$$\epsilon_{\rho_i, \rho_j}^{\pm} \bar{\otimes} \epsilon_{\rho_i, \rho_j}^{\pm, \mathcal{J}} \psi_{i,i} \psi_{j,j} = \psi_{j,j} \psi_{i,i}, \tag{4.4}$$

where $\epsilon_{\rho_i, \rho_j}^{\pm} \bar{\otimes} \epsilon_{\rho_i, \rho_j}^{\pm, \mathcal{J}}$ is the braiding between $\rho_i \bar{\otimes} \rho_i^{\mathcal{J}}$ and $\rho_j \bar{\otimes} \rho_j^{\mathcal{J}}$ in $\mathcal{C} \boxtimes \mathcal{C}^{\mathcal{J}}$, the irreducible summands of Θ_{LR} .

Remark 4.5. If \mathcal{C} is braided and pointed, then each braiding $\epsilon_{\rho_i, \rho_j}^{\pm}$ is a phase multiple of any fixed unitary intertwiner $U \in \text{Hom}_{\mathcal{C}}(\rho_i \rho_j, \rho_j \rho_i)$ (both $\rho_i \rho_j$ and $\rho_j \rho_i$ are automorphisms hence irreducible). Thus $\epsilon_{\rho_i, \rho_i}^{\pm} \bar{\otimes} \epsilon_{\rho_i, \rho_i}^{\pm, \mathcal{J}} = \mathbb{1} \bar{\otimes} \mathbb{1}$ for $i = j$, by taking $U = \mathbb{1}$ and by the antilinearity of \mathcal{J} . Equivalently, the statistical phase of each $\rho_i \bar{\otimes} \rho_i^{\mathcal{J}}$ is +1 (all bosonic automorphisms in $\mathcal{C} \boxtimes \mathcal{C}^{\mathcal{J}}$) being equal to the (phase associated with the) self-braiding in the case of automorphisms. If, in addition, $\rho_i \rho_j = \rho_j \rho_i$ for every $i \neq j$ as automorphisms of \mathcal{N} (e.g., by spacelike localization in the case of DHR automorphisms and $\mathcal{N} = \mathcal{A}_{\kappa}(I)$, or if ρ_i and ρ_j are different powers of the same automorphism as for

cyclic groups), then one can take $U = \mathbb{1}$ as well, and $\epsilon_{\rho_i, \rho_j}^\pm \bar{\otimes} \epsilon_{\rho_i, \rho_j}^{\pm, \mathcal{J}} = \mathbb{1} \bar{\otimes} \mathbb{1}$ for every $i, j = 0, \dots, n$. In the latter case of commuting automorphisms, the commutativity of the Longo–Rehren Q-system for braided pointed fusion categories reads

$$\psi_{i,i} \psi_{j,j} = \psi_{j,j} \psi_{i,i}.$$

If we take \mathcal{C} to be the representation category of a (completely) rational conformal net on \mathbb{R} or S^1 , then \mathcal{C} is necessarily unitary fusion (finitely many sectors) and braided (also modular, see, e.g., [BK01]), equipped with the DHR braiding, by a result of [KLM01]. Then the Longo–Rehren Q-system provides an irreducible finite index *local* (by the commutativity property (4.4)) and “diagonal” extension $\tilde{\mathcal{A}}_{\text{LR}}$ of the tensor product conformal net $\mathcal{A}_\kappa \bar{\otimes} \mathcal{A}_\kappa^{\mathcal{J}}$ on \mathbb{R}^{1+1} (with equal left and right chiral components), by choosing $\mathcal{N} \bar{\otimes} \mathcal{N}^{\mathcal{J}} = \mathcal{A}_\kappa(I) \bar{\otimes} \mathcal{A}_\kappa(I)^{\mathcal{J}}$, where I is an interval on \mathbb{R} symmetric around the origin, and \mathcal{J} the modular conjugation with respect to $\mathcal{A}_\kappa(\mathbb{R}_+)$, Ω_κ (with this choice, $\rho_i^{\mathcal{J}} \cong \bar{\rho}_i$, hence in particular $\rho_i^{\mathcal{J}} \cong \rho_i^{-1}$ in the pointed case). See [LR95, Theorem 4.9], cf. [DG18, Theorem 6.8], for the general construction of extensions from arbitrary Q-systems in the representation category of a local net, and the comments after the proof of [LR95, Proposition 4.10] for the Longo–Rehren Q-system.

5. Conformal Wightman Fields from Charged Primary Fields

5.1. Conformal Wightman axioms. Here we show that to some of the extensions we discussed in Sects. 3 and 4 we can associate conformal Wightman fields, as we define below. It is natural to expect that such Wightman fields on \mathbb{R}^{1+1} also extend to \mathcal{E} as conformal nets do, and hence we formulate Wightman fields on \mathcal{E} .

For simplicity, let us start with chiral fields on S^1 . Let us see $S^1 \subset \mathbb{C}$ and the counterclockwise direction as the lightlike direction when S^1 is seen as the one-point compactification of \mathbb{R} as in Sect. 2.1.

A conformal Wightman field on S^1 with **conformal dimension** D is a quantum field that transforms as the tensor field $f \frac{d}{d\theta} \otimes^{D-1}$ under diffeomorphisms as (1dW2) below, and such a field is called a **primary field**: For $D \in \mathbb{N}$, $\gamma \in \text{Diff}_+(S^1)$ and $f \in C^\infty(S^1)$, we set (cf. [RTT22])

$$\begin{aligned} X_\gamma(e^{i\theta}) &:= -i \frac{d}{d\theta} \log(\gamma(e^{i\theta})), \\ (\beta_D(\gamma)f)(z) &:= (X_\gamma(\gamma^{-1}(z)))^{D-1} f(\gamma^{-1}(z)). \end{aligned}$$

Note that γ is orientation-preserving, hence X is strictly positive and $(X_\gamma(\gamma^{-1}(z)))^{D-1}$ is bounded below by some positive number for a fixed γ .

A **conformal Wightman field theory on S^1** is a family of operator-valued distributions $\{\phi_{\kappa, j}\}$ (κ is a fixed label, while j indexes the family) on S^1 , closed under the conjugate ($\phi_{\kappa, j}^\dagger$ is also in the family), with corresponding conformal dimensions $\{D_{\kappa, j}\}$ defined on a common invariant dense domain $\mathcal{D} \subset \mathcal{H}_\kappa$, a unitary projective representation U_κ of $\text{Diff}_+(S^1)$ and a vector $\Omega_\kappa \in \mathcal{H}_\kappa$ such that

(1dW1) **Locality**: for $f, g \in C^\infty(S^1)$ with disjoint supports, $[\phi_{\kappa, j_1}(f), \phi_{\kappa, j_2}(g)] = 0$ on \mathcal{D} .

(1dW2) **Diffeomorphism covariance:** $U_\kappa(\gamma)$ preserves \mathcal{D} and it holds that

$$\text{Ad } U_\kappa(\gamma)(\phi_{\kappa,j}(f)) = \phi_{\kappa,j}(\beta_{D_{\kappa,j}}(\gamma)f), \quad \text{for } \gamma \in \text{Diff}_+(S^1).$$

(1dW3) **Positivity of energy:** the spectrum of rotations of U_κ is contained in $\mathbb{N} \cup \{0\}$.

(1dW4) **Vacuum and the Reeh-Schlieder property:** there exists a unique (up to a phase) vector $\Omega_\kappa \in \mathcal{H}_\kappa$ such that $U_\kappa(\gamma)\Omega_\kappa = \Omega_\kappa$ for $\gamma \in \text{Möb}$ and vectors of the form $\phi_{\kappa,j_1}(f_1) \cdots \phi_{\kappa,j_\ell}(f_\ell)\Omega_\kappa$, where $f_1, \dots, f_\ell \in C^\infty(S^1)$, are total in \mathcal{H}_κ .

As fields $\phi_{\kappa,j}$ are operator-valued distributions, they can be smeared with functions $\epsilon_n(\theta) = e^{in\theta}$ to give their Fourier components $\phi_{\kappa,j,n} = \phi_{\kappa,j}(\epsilon_n)$. Let $L_{\kappa,0}$ be the generator of $U_\kappa(R_t)$, where $R_t \in \text{Möb}$ are rotations. We assume that $\mathcal{D} = C^\infty(L_{\kappa,0}) := \bigcap_{\ell \in \mathbb{N}} \text{Dom}(L_{\kappa,0}^\ell)$. Furthermore,

(1dW5) **Polynomial energy bounds:** there are $r_j, p_j, C > 0$ such that, for $\Psi \in C^\infty(L_{\kappa,0})$,

$$\|\phi_{\kappa,j,n}\Psi\| \leq C(1 + |n|)^{r_j} \|(L_{\kappa,0} + \mathbb{1})^{p_j}\Psi\|$$

According to [CKLW18, Section 6], polynomial energy bounds allow one to define the smeared fields. For $f \in C^\infty(S^1)$ whose Fourier components are $f_n = \frac{1}{2\pi} \int_{-\pi}^\pi f(e^{-in\theta})d\theta$, we define

$$\phi_{\kappa,j}(f) := \sum_{n \in \mathbb{Z}} f_n \phi_{\kappa,j,n}.$$

This is convergent on any vector $\Psi \in C^\infty(L_{\kappa,0})$ and $\phi_{\kappa,j}(f)$ preserves the domain $C^\infty(L_{\kappa,0})$.

We know from (the proofs of) [CKLW18, Proposition 6.4, Theorem 8.3] the following:

Lemma 5.1. *Let $\{\phi_{\kappa,j}\}$ satisfy polynomial energy bounds with $p_j \leq 1$. Then it holds that $\text{Ad } U(\gamma)(\phi_{\kappa,j}(f)) = \phi_{\kappa,j}(\beta_{D_{\kappa,j}}(\gamma)(f))$ and $\phi_{\kappa,j_1}(f)$ and $\phi_{\kappa,j_2}(g)$ commute strongly for f, g with disjoint supports.*

A general two-dimensional conformal field is not chiral and depends on both light-like variables. There are fields called primary fields and they are distinguished by the conformal dimensions (D_L, D_R) , where $D_L, D_R > 0$ and can be non-integer. We set $\beta_{D_L, D_R}(\gamma) = \beta_{D_L}(\gamma_L)\beta_{D_R}(\gamma_R)$ for $\gamma = \gamma_L \times \gamma_R \in \text{Diff}_+(S^1) \times \text{Diff}_+(S^1)$. This can be extended to \mathcal{C} if $D_L - D_R \in \mathbb{Z}$, which we always assume (cf. Section 4.1). Let H be the generator of $\{R_t \times R_t : t \in \mathbb{R}\}$ in U , which plays a similar role to that of L_0 in chiral fields.

A **conformal Wightman field theory** on \mathcal{E} is a family of operator-valued distributions $\{\tilde{\psi}_j\}$ on \mathcal{E} , closed under conjugation, with corresponding conformal dimensions $\{(D_{L,j}, D_{R,j})\}$ defined on a common invariant domain $\mathcal{D} \subset \mathcal{H}$ consisting of vectors of the form $\tilde{\psi}_{j_1}(f_1) \cdots \tilde{\psi}_{j_n}(f_n)\Omega$, a unitary projective representation U of \mathcal{C} and a vector $\Omega \in \mathcal{H}$ such that

(2dW1) **Locality:** for $f, g \in C^\infty(\mathcal{E})$ with spacelike separated supports, it holds that $[\tilde{\psi}_{j_1}(f), \tilde{\psi}_{j_2}(g)] = 0$.

(2dW2) **Diffeomorphism covariance:** $U(\gamma)$ preserves \mathcal{D} and it holds that

$$\text{Ad } U(\gamma)(\tilde{\psi}_j(f)) = \tilde{\psi}_j(\beta_{D_{L,j}, D_{R,j}}(\gamma)f), \quad \text{for } \gamma \in \mathcal{C}$$

(2dW3) **Positivity of energy:** the joint spectrum of the translation subgroup of \mathbb{R}^{1+1} in U is contained in the closed forward light cone $\overline{V}_+ = \{(a_0, a_1) \in \mathbb{R}^{1+1} : a_0^2 - a_1^2 \geq 0, a_0 \geq 0\}$.

(2dW4) **Vacuum and the Reeh-Schlieder property:** there exists a unique (up to a phase) vector $\Omega \in \mathcal{H}$ such that $U(\gamma)\Omega = \Omega$ for $\gamma \in \overline{\text{Möb}} \times \overline{\text{Möb}}/\mathfrak{R}$ and \mathcal{H} is spanned by vectors of the form $\tilde{\psi}_{j_1}(f_1) \cdots \tilde{\psi}_{j_k}(f_k)\Omega$.

(2dW5) **Polynomial energy bounds:** it holds that $\mathcal{D} = C^\infty(H)$ and there are $p_j > 0$ such that, for $\Psi \in C^\infty(H)$,

$$\|\tilde{\psi}_j(f)\Psi\| \leq C_f \|H^{p_j}\Psi\|,$$

for some $C_f > 0$ depending only on f .

As is well-known, linear energy bounds ($p_j = 1$) assure that the conformal Wightman fields commute strongly.

Lemma 5.2. *Let $\{\tilde{\psi}_j\}$ satisfy the bound $\|\tilde{\psi}_j(f)\Psi\| \leq C_{f,j}\|H\Psi\|$. Then $\tilde{\psi}_{j_1}(f)$ and $\tilde{\psi}_{j_2}(g)$ commute strongly for f, g with spacelike supports.*

Proof. As H is the generator of the one-parameter group $R_t \times R_t$ in $\overline{\text{Möb}} \times \overline{\text{Möb}}$, we have $[H, \tilde{\psi}_j(f)] = i\tilde{\psi}_j(f')$ as in [CDVIT21, (3.7)], where f' is the derivative of f with respect to translations $R_t \times R_t$ of the cylinder \mathcal{E} . Furthermore, we have $\|[H, \tilde{\psi}_j(f)]\Psi\| = \|\tilde{\psi}_j(f')\Psi\| \leq C_{f',j}\|H\Psi\|$. By applying commutator with H , $\|\delta^k(\tilde{\psi}_j(f))\Psi\| = \|\tilde{\psi}_j(f^{(k)})\Psi\| \leq C_{f^{(k)},j}\|H\Psi\|$, $k = 2, 3$, where $\delta(A) = i[H, A]$. Therefore, $\tilde{\psi}_1(f)$ and $\tilde{\psi}_2(g)$ satisfy the assumption of Theorem A.2 with H as the reference operator and they strongly commute. \square

5.2. Formal series of operators. Let A_s be a family of operators parametrized by $s \in \mathbb{R}$. By a formal series we mean a symbol of the form $\sum_s A_s z^s$, where the summation actually has no meaning. We refer to A_s as **Fourier components** of a formal series. One can consider sums and scalar multiples of such formal series: $\sum_s A_s z^s + \sum_s B_s z^s = \sum_s (A_s + B_s)z^s$, $c \sum_s A_s z^s = \sum_s cA_s z^s$. The product of two formal series is not always defined, but in some cases there is a natural product. We define the product of $\sum_s A_s z^s$ and $\sum_s B_s z^s$ as the formal series $\sum_s C_s z^s$ with the coefficient $C_s = \sum_{t \in \mathbb{R}} A_{s-t} B_t$, whenever these sums make sense.

Similarly, we consider formal series in two or more variables. If we have two formal series in two different variables z, w , the product $\sum_s A_s z^s \sum_t B_t w^t = \sum_{s,t} A_s B_t z^s w^t$ makes always sense. A product of such formal series in two variables is defined similarly as above.

A typical case we use in this paper is when the vector spaces are graded as $V^j = \bigoplus_{t \in \mathbb{N} + h_j} V_t^j$, where $j = 1, 2$, $h_j \in \mathbb{R}$ and A_s are operators $V_t^1 \rightarrow V_{t-s}^2$ and they are nonzero only for $s \in \mathbb{Z} + D$ for some $D \in \mathbb{R}$ and $t - s \in \mathbb{N} + h_2$. We also consider the situation where $V = \bigoplus_j V^j$ and $\sum_s A_s$ is a formal series on V , that has the form for each pair V^{j_1}, V^{j_2} as above (and D_{j_1, j_2} may depend on j_1, j_2).

Let $\sum_s A_s z^s, \sum_s B_s z^s$ such two formal series with $[A_{s_1}, B_{s_2}] = 0$ for all $s_1, s_2 \in \mathbb{R}$. In the situation of the previous paragraph, we can define the product $\sum_s A_s z^s \sum_t B_s z^s =$

$\sum_s z^s \sum_t A_{s-t} B_t$ (this is a special case of normal product, see [CTW23, Section 2.1]): indeed, for each vector $\Psi \in V$ the sum over t is finite because either $A_{s-t}\Psi$ or $B_t\Psi$ vanishes for large $|t|$. Therefore, the sum $\sum_t A_{s-t} B_t$ defines an operator on V and this is a formal series on V .

5.3. Charged primary fields. We continue studying a single chiral component, but we omit κ from certain parameters that do not appear later. We assume that a conformal net $(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)$ on S^1 is generated by conformal Wightman fields $\{\phi_{\kappa,j}\}$ which commute strongly when smeared with test functions with disjoint supports.

Unitary operators $e^{i\phi_{\kappa,j}(f)}$ with compactly supported f are represented in a representation ρ as a unitary operator, and the question arises whether the operator $\rho(\phi_{\kappa,j}(f))$ makes sense and whether it is the generator of $\rho(e^{i\phi_{\kappa,j}(f)})$. This property is called *strong integrability*, and verifying it in examples is a non-trivial problem (although it holds in interesting cases, like the $U(1)$ -current [CWX, Gui20]). Furthermore, we can also consider localized unbounded intertwiners between representations and there is the problem whether the commutation relations between them hold strongly (*strong intertwining property* and *strong braiding*) [Gui20], see also [Ten19b, Ten19a]. These properties should be useful in comparing the extension of nets and Wightman fields, see Remark 5.8.

Let $\kappa(g)$ be an automorphism of the conformal net \mathcal{A}_κ for $g \in G$ as in Sect. 2.2. We consider the Fourier components of chiral fields and assume that they have corresponding operators $\phi_{\kappa,j,n}^{\kappa(g)}$ on the representation spaces $\mathcal{H}_\kappa^{\kappa(g),\text{fin}}$, where \mathcal{V}^{fin} denotes the linear span of eigenspaces $\mathcal{V}_t = \ker(L_{\kappa,0} - t)$ of $L_{\kappa,0}$, where $L_{\kappa,0}$ is represented on \mathcal{V} as an unbounded operator. We denote the algebraic direct sum of the $\phi_{\kappa,j,n}^{\kappa(g)}$ over $g \in G$ by $\hat{\phi}_{\kappa,j,n}$ defined on the (algebraic) linear span $\bigoplus_{g \in G}^{\text{alg}} \mathcal{H}_\kappa^{\kappa(g),\text{fin}}$ in $\bigoplus_{g \in G} \mathcal{H}_\kappa^{\kappa(g),\text{fin}}$. Similarly, we denote $\hat{L}_{\kappa,n} = \bigoplus_{g \in G} L_{\kappa,n}^{\kappa(g)}$, where $L_{\kappa,n}^{\kappa(g)}$ is a representation of the Virasoro algebra on $\mathcal{H}_\kappa^{\kappa(g),\text{fin}}$.

Let $(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)$ be a conformal net on S^1 generated by fields $\{\phi_{\kappa,j}\}$ with conformal dimensions $\{D_j\}$ as above and a collection Δ_κ of automorphisms to which there is a bijective map κ from a group G . A **charged primary field** $\{\psi_\kappa^h(\xi, s)\}_{s \in \mathbb{R}}$ is a family of operators (labelled by $h \in G$ and ξ in some index set⁵ Ξ depending on κ and h), acting on the domain $\bigoplus_{g \in G}^{\text{alg}} \mathcal{H}_\kappa^{\kappa(g),\text{fin}}$ with the associated formal series $\psi_\kappa^h(\xi, z) = \sum_{s \in \mathbb{R}} \psi_\kappa^h(\xi, s) z^{-s-D_h}$, where $D_h > 0$ is the conformal dimension and z is the formal variable, such that

- $\psi_\kappa^h(\xi, s)$ maps $\mathcal{H}_\kappa^{\kappa(g),\text{fin}}$ to $\mathcal{H}_\kappa^{\kappa(hg),\text{fin}}$, and there is $\ell_{h,g} \in \mathbb{R}$ such that $\psi_\kappa^h(\xi, s) \neq 0$ only for $s \in \ell_{h,g} + \mathbb{Z}$.
- (primaryity) $[\hat{L}_{\kappa,m}, \psi_\kappa^h(\xi, s)] = ((D_h - 1)m - s)\psi_\kappa^h(\xi, m + s)$.
- (relative locality) $[\hat{\phi}_{\kappa,j,m}, \psi_\kappa^h(\xi, s)] = \sum_{\xi'} X_{j,\xi'}^\xi \psi_\kappa^h(\xi', m + s)$, where $X_{j,\xi'}^\xi \in \mathbb{C}$.

⁵ Examples where we need all these labels are the WZW models associated with a simply laced, simply connected, simple, connected compact group G (different from the group above) at level 1. The sectors are parametrized by the center $Z(G)$, which can be identified with Λ_W/Λ_R , the weight lattice quotiented by the root lattice [TL97, Lemma I.2.1.1]. The primary fields are of the form $\Phi_\mu(z)$, where μ is a minimal weight of G [TL97, Section V.5.3]. In this example, h is $\mu + \Lambda_R \in \Lambda_W/\Lambda_R$, while μ is one of minimal weights, playing the role of ξ in our notations. See [TL97, Theorem V.5.3.2] for relative locality of these fields.

- As formal series, the braiding relation holds:

$$(1 - \frac{z}{w})^{\alpha(h_1, h_2)} w^{\alpha(h_1, h_2)} \psi_{\kappa}^{h_1}(\xi_1, w) \psi_{\kappa}^{h_2}(\xi_2, z) = (1 - \frac{w}{z})^{\alpha(h_1, h_2)} z^{\alpha(h_1, h_2)} \psi_{\kappa}^{h_2}(\xi_1, z) \psi_{\kappa}^{h_1}(\xi_2, w)$$

where $\alpha(h_1, h_2) \in \mathbb{R}$. Recall that $(1 - u)^{\beta} = \sum_{n \geq 0} \binom{\beta}{n} (-u)^n$ as a formal series, where $\binom{\beta}{n} = \frac{\beta(\beta-1)\dots(\beta-n+1)}{n!}$.

- We assume that the family of charged primary fields is closed under conjugation, in the sense that for given h, ξ there are $\bar{h}, \bar{\xi}$ such that $\psi_{\kappa}^h(\xi, z)^* = \psi_{\kappa}^{\bar{h}}(\bar{\xi}, z)$, with the convention $z^* = z^{-1}$ in the formal series expansion.

We further assume polynomial energy bounds:

- for each h , there are $r_{\xi}, p_{\xi}, C_{\xi} \geq 0$ such that $\|\psi_{\kappa}^h(\xi, s)\Psi\| \leq C_{\xi}(1 + |s|)^{r_{\xi}} \|(L_{\kappa,0} + \mathbb{1})^{p_{\xi}}\Psi\|$ for $\Psi \in C^{\infty}(L_{\kappa,0})$.

This allows us to define smeared charged primary fields: For $f \in C^{\infty}(S^1 \setminus \{-1\})$, we put $f_s = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})e^{-is\theta} d\theta$ and

$$\psi_{\kappa}^h(\xi, f) := \sum_{s \in \mathbb{R}} f_s \psi_{\kappa}^h(\xi, s),$$

and this defines operators on $C^{\infty}(\hat{L}_{\kappa,0})$ (the sum makes sense because $\psi_{\kappa}^h(\xi, s) = 0$ except for countable s on each $\mathcal{H}_{\kappa,t}^{\kappa(g)}$).

Let $\epsilon_{\kappa(h_1), \kappa(h_2)}^+ := \lim_{\Im \zeta > 0, \zeta \rightarrow -1} \zeta^{\alpha(h_1, h_2)}$, where $\Im \zeta$ is the imaginary part of ζ , $w^{\alpha(h_1, h_2)} = e^{\alpha(h_1, h_2) \log w}$ and we take the branch of $\log w$ on $\mathbb{C} \setminus (-\infty, 0]$ such that $\log w \in \mathbb{R}$ on $(0, \infty)$.

Lemma 5.3. *Let $f, g \in C^{\infty}(S^1 \setminus \{-1\})$ such that $\arg \text{supp } f < \arg \text{supp } g$, that is, they are in the counterclockwise order on $S^1 \setminus \{-1\}$. Then it holds that $\psi_{\kappa}^{h_1}(\xi_1, f) \psi_{\kappa}^{h_2}(\xi_2, g) = \epsilon_{\kappa(h_1), \kappa(h_2)}^+ \psi_{\kappa}^{h_2}(\xi_2, g) \psi_{\kappa}^{h_1}(\xi_1, f)$.*

Proof. Let $z_0 \in S^1, z_0 \neq -1$ such that $\arg \text{supp } f < \arg z_0 < \arg \text{supp } g$, and I_+ be the interval on S^1 from z_0 to -1 (counterclockwise) while I_- be the interval on S^1 from -1 to z_0 , hence we have $\text{supp } f \subset I_-, \text{supp } g \subset I_+$ (see Fig. 2), and put $f_s = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})e^{-is\theta} d\theta, g_t = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\theta})e^{-it\theta} d\theta$. As f, g are smooth, these coefficients are rapidly decaying.

The product $\psi_{\kappa}^{h_1}(\xi_1, w) \psi_{\kappa}^{h_2}(\xi_2, z)$ is a formal series in w, z . For finite-energy vectors Ψ_1, Ψ_2 in some $\mathcal{H}_{\kappa}^{\kappa(h_3), \text{fin}}, \mathcal{H}_{\kappa}^{\kappa(h_4), \text{fin}}$, the scalar product $\varphi_1(w, z) := \langle \Psi_1, \psi_{\kappa}^{h_1}(\xi_1, w) \psi_{\kappa}^{h_2}(\xi_2, z) \Psi_2 \rangle$ can be considered as a formal series in w, z with coefficients in \mathbb{C} , and by the first assumption on charged primary fields, it is just a countable sum. Furthermore, these coefficients of $w^s z^t$ vanish for s large and t small, because each $\mathcal{H}_{\kappa}^{\kappa(h_j), \text{fin}}$ is a positive-energy representation of the Virasoro algebra. By polynomial energy bounds, it can be seen as a distribution in w, z , which we denote again by $\varphi_1(w, z)$. Then it holds that $\varphi_1(f, g) = \sum \varphi_1(s, t) f_s g_t$, where $\varphi_1(s, t)$ is the coefficient of the formal series $\varphi_1(w, z)$ of $w^s z^t$ (with a slight abuse of notation).

⁶ We use the same notation for the braiding of the charged fields and the braiding of sectors in Sect. 2.2. We plan to show that they are indeed equal under assumptions such as strong braiding [Gui20] in a separate publication.

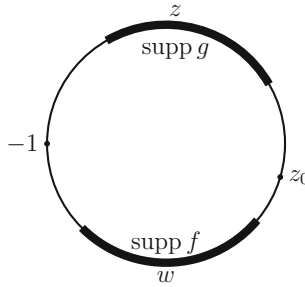


Fig. 2. The circle, the point of infinity (-1) and two functions with disjoint supports

Let $a < 1$ and $b > 1$. We introduce $\varphi_1^{a,b}(z, w) := \langle \Psi_1, \psi_\kappa^{h_1}(\xi_1, bw) \psi_\kappa^{h_2}(\xi_2, az) \Psi_2 \rangle$, then it has only finitely many terms with negative powers in z and those with positive powers in w , therefore, it is a convergent series for $|z| = |w| = 1$ again by the polynomial energy bounds and the choice $a < 1, b > 1$. Now it can be seen as a function on $I_- \times I_+$, thus defines a distribution and $\varphi_1^{a,b}(f, g) = \sum \varphi_1^{a,b}(s, t) f_s g_t$, in the sense above. Furthermore, as $a \rightarrow 1, b \rightarrow 1$, this converges to $\varphi_1(f, g)$.

The formal series $(1 - \frac{z}{w})^{\alpha(h_1, h_2)}$ has only negative powers in w and positive powers in z , therefore, the product

$$\langle \Psi_1, (1 - \frac{z}{w})^{\alpha(h_1, h_2)} \psi_\kappa^{h_1}(\xi_1, w) \psi_\kappa^{h_2}(\xi_2, z) \Psi_2 \rangle$$

makes sense again as a formal series (see the remark on formal series above). Again by polynomial energy bounds, these coefficients grow only polynomially in s, t . Therefore, it can be seen as a distribution with two variables $w \in I_-, z \in I_+$, which we denote by $\varphi_2(w, z)$. Define also $\varphi_2^{a,b}(w, z) = \varphi_2(bw, az)$. As $a \rightarrow 1, b \rightarrow 1$, each coefficients converge, and $\varphi_2^{a,b}(f, g)$ converge as distributions to $\varphi_2(f, g)$ by polynomial energy bounds.

Note that the expansion $(1 - u)^\beta = \sum_{n \geq 0} \binom{\beta}{n} (-u)^n$ converges for $|u| < 1$, and can be analytically continued to $\mathbb{C} \setminus [1, \infty)$. This expansion coincides with the definition $(1 - u)^\beta = e^{\beta \log(1-u)}$ as an analytic function in $u \in \mathbb{C} \setminus [1, \infty)$, where $\log w$ is defined on $\mathbb{C} \setminus (-\infty, 0]$ and $\log w \in \mathbb{R}$ when $w \in (0, \infty)$ (the last condition fixes a branch of $\log w$ uniquely). We use the same continuation for $\log w$ when $w^\beta = e^{\beta \log w}$ is considered as a function. If we consider $(1 - \frac{w}{z})^{\alpha(h_1, h_2)}$ as a function in w, z , we can multiply it to the above distribution φ_1 and obtain a distribution in $I_- \times I_+$, which we denote by $\varphi_3(w, z)$. We also introduce $\varphi_3^{a,b}(w, z) = \varphi_3(bw, az)$. Then $\varphi_3^{a,b}(f, g)$ converges to $\varphi_3(f, g)$ as $a \rightarrow 1, b \rightarrow 1$ because both $\varphi_1^{a,b}(w, z)$ and $(1 - \frac{bw}{az})^{\alpha(h_1, h_2)}$ converge as distributions and smooth functions on $I_- \times I_+$, respectively.

This shows that $\varphi_2(f, g) = \varphi_3(f, g)$. That is, $(1 - \frac{z}{w})^{\alpha(h_1, h_2)} \psi_\kappa^{h_1}(\xi_1, w) \psi_\kappa^{h_2}(\xi_2, z)$ can be seen as the operator-valued distribution multiplied by a smooth function $(1 - \frac{z}{w})^{\alpha(h_1, h_2)}$ on $I_- \times I_+$.

Similarly, $(1 - \frac{w}{z})^{\alpha(h_1, h_2)} \psi_\kappa^{h_2}(\xi_1, z) \psi_\kappa^{h_1}(\xi_2, w)$ can be seen as an operator-valued distribution. Therefore, by the braiding relation of the primary fields, we have the equality

$$(1 - \frac{z}{w})^{\alpha(h_1, h_2)} w^{\alpha(h_1, h_2)} \psi_\kappa^{h_1}(\xi_1, w) \psi_\kappa^{h_2}(\xi_2, z) = (1 - \frac{w}{z})^{\alpha(h_1, h_2)} z^{\alpha(h_1, h_2)} \psi_\kappa^{h_2}(\xi_1, z) \psi_\kappa^{h_1}(\xi_2, w)$$

in the sense of operator-valued distributions, under the restriction on the supports of f and g . Seen as functions, the quotient of $(1 - \frac{w}{z})^{\alpha(h_1, h_2)} z^{\alpha(h_1, h_2)}$ by $(1 - \frac{z}{w})^{\alpha(h_1, h_2)} w^{\alpha(h_1, h_2)}$ is exactly $\lim_{\zeta \rightarrow -1} \zeta^{\alpha(h_1, h_2)}$. Therefore, by dividing the equation by the former factor, we obtain

$$\psi_\kappa^{h_1}(\xi_1, w) \psi_\kappa^{h_2}(\xi_2, z) = \epsilon_{\kappa(h_1), \kappa(h_2)}^+ \psi_\kappa^{h_2}(\xi_1, z) \psi_\kappa^{h_1}(\xi_2, w)$$

as desired, as operator-valued distributions. □

The following is essentially due to [CKL08, Section 6.3], where the case $D = \frac{1}{2}$ is treated, see also [CKLW18, Proposition 6.4], [TL99].

Lemma 5.4. *Assume that a charged primary field ψ_κ^h has conformal dimension D and satisfies polynomial energy bounds as above (we omit the dependence on ξ). Then it is diffeomorphism covariant, that is, with \hat{U}_κ the projective unitary representation of $\overline{\text{Diff}_+(S^1)}$, $\text{Ad } \hat{U}_\kappa(\gamma)(\psi_\kappa^h(f)) = \psi_\kappa^h(\beta_D(\gamma)(f))$.*

Proof. Let us sketch the arguments of [CKL08, Section 6.3]. Note that $\hat{L}_{\kappa, n}$ is a representation of the Virasoro algebra on the dense domain $\hat{\mathcal{H}}_\kappa^{\text{fin}}$ in the Hilbert space $\hat{\mathcal{H}}_\kappa$. By polynomial energy bounds they extend to $C^\infty(\hat{L}_{\kappa, 0})$. Furthermore, on this domain we have the commutation relations

$$i[\hat{L}_\kappa(f_1), \psi_\kappa^h(f_2)] = \psi_\kappa^h(Df_1' f_2 - f_1 f_2').$$

From this, the following relation follows:

$$\frac{d}{dt} \psi_\kappa^h(\beta_D(\exp(t f_1)) f_2)|_{t=0} = i[\hat{L}_\kappa(f_1), \psi_\kappa^h(f_2)].$$

Again by energy bounds for $\hat{L}_{\kappa, n}$, it holds that $U_\kappa(\exp(t f))$ preserves the domain $C^\infty(L_{\kappa, 0})$.

For $\Psi \in C^\infty(\hat{L}_{\kappa, 0})$, we can consider two vector-valued funtions of $t \in \mathbb{R}$

$$\begin{aligned} \Psi_1(t) &= \psi_\kappa^h(\beta_D(\exp(t f_1)) f_2) \hat{U}_\kappa(\exp(t f_1)) \Psi, \\ \Psi_2(t) &= \hat{U}_\kappa(\exp(t f_1)) \psi_\kappa^h(f_2) \Psi, \end{aligned}$$

and they both satisfy the differential equation $\Psi'(t) = i\hat{L}_\kappa(f)\Psi(t)$. As they satisfy the same initial condition $\Psi_1(0) = \Psi_2(0) = \psi_\kappa^h(f_2)\Psi$, they must coincide. Therefore, we have $\text{Ad } \hat{U}_\kappa(\exp(t f_1))(\psi_\kappa^h(f_2)) = \psi_\kappa^h(\beta_D(\exp(t f_1)) f_2)$.

Recall that $\text{Diff}_+(S^1)$ is algebraically simple [Thu74] while the subgroup generated by one-parameter groups is a normal subgroup of $\text{Diff}_+(S^1)$. Therefore, they must coincide. This implies that any element in $\overline{\text{Diff}_+(S^1)}$ is a product of elements in some one-parameter group and an element in the center $\{R_{2\pi n}\}$.

Now the relation holds for all $\gamma \in \overline{\text{Diff}_+(S^1)}$ which is a product of elements in one-parameter groups, and it is straightforward to verify it for $R_{2\pi}$, therefore, we have the desired covariance. □

5.4. *One-dimensional Wightman fields arising from trivial total braiding.* Let us first assume that there are two conformal nets $\mathcal{A}_1, \mathcal{A}_2$ on S^1 and charged primary fields ψ_1, ψ_2 of these nets (here we omit the dependence on $h \in G$ and $\xi \in \Xi$).

Lemma 5.5. *Let $\psi_1(z) \otimes \mathbb{1}, \mathbb{1} \otimes \psi_2(z)$ be charged primary fields satisfying polynomial energy bounds with respect to an operator $\hat{L}_0 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_0$. Then $\psi_1(z) \otimes \psi_2(z)$ satisfies polynomial energy bounds.*

Proof. Let us assume polynomial energy bounds: there are $r_j, p_j, C > 0$, for $j = 1, 2$ such that

$$\|\psi_j(s)\Psi\| \leq C(1 + |s|)^{r_j} \|(\hat{L}_0 + \mathbb{1})^{p_j} \Psi\|$$

for $\Psi \in C^\infty(\hat{L}_0)$. Note that

$$\begin{aligned} (\hat{L}_0 + \mathbb{1})\psi_2(t - s) &= [\hat{L}_0, \psi_2(t - s)] + \psi_2(t - s)\hat{L}_0 + \psi_2(t - s) \\ &= (s - t + 1)\psi_2(t - s) + \psi_2(t - s)\hat{L}_0 \\ &= \psi_2(t - s)(\hat{L}_0 + (s - t + 1)), \end{aligned}$$

hence $(\hat{L}_0 + \mathbb{1})^q \psi_2(t - s) = \psi_2(t - s)(\hat{L}_0 + (s - t + 1)\mathbb{1})^q$. Then, for a fixed s and Ψ such that $H\Psi = \ell\Psi$ where $H = \hat{L}_0 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_0$,

$$\begin{aligned} \left\| \sum_t \psi_1(s - t) \otimes \psi_2(t) \Psi \right\| &\leq \sum_t C(1 + |s - t|)^{r_1} \|(H + \mathbb{1})^{p_1} (\mathbb{1} \otimes \psi_2(t)) \Psi\| \\ &\leq \sum_t C(1 + |s - t|)^{r_1} \|(H + \mathbb{1})^{\lceil p_1 \rceil} (\mathbb{1} \otimes \psi_2(t)) \Psi\| \\ &\leq \sum_t C(1 + |s - t|)^{r_1} \|(\mathbb{1} \otimes \psi_2(t))(H + (t + 1)\mathbb{1})^{\lceil p_1 \rceil} \Psi\| \\ &= \sum_t C(1 + |s - t|)^{r_1} (\ell + (t + 1))^{\lceil p_1 \rceil} \|(\mathbb{1} \otimes \psi_2(t)) \Psi\| \\ &\leq \sum_t C^2(1 + |s - t|)^{r_1} (1 + |t|)^{r_2} (\ell + (t + 1))^{\lceil p_1 \rceil} \|(H + \mathbb{1})^{p_2} \Psi\| \\ &\leq C^2(1 + |s|)^{2r_1 + r_2 + \lceil p_1 \rceil} \|(H + \mathbb{1})^{r_1 + r_2 + 2\lceil p_1 \rceil + p_2} \Psi\|, \end{aligned}$$

where $\lceil p \rceil$ is the smallest integer larger or equal to p , in the last line we estimated $1 + |s - t| \leq 1 + |s| + \ell \leq (1 + |s|)(1 + \ell)$ because $(\psi_1(s - t) \otimes \mathbb{1})\Psi = 0$ when $t < -s - \ell$ or $(\mathbb{1} \otimes \psi_2(t))\Psi = 0$ when $t > \ell$.

For a general $\Psi = \sum_\ell \Psi_\ell$, the $\sum_t \psi_1(s - t) \otimes \psi_2(t) \Psi_\ell$ are orthogonal for different ℓ . Therefore, a polynomial energy bound follows for $\psi_1(z) \otimes \psi_2(z)$. \square

Clearly, this estimate is not optimal. It can happen, as we will see with concrete examples, that a product of two charged primary fields whose Fourier components are bounded $\|\psi(s)\| \leq C$ has again bounded Fourier components.

Lemma 5.6. *Let $\psi_1(z), \psi_2(z)$ be charged primary fields for nets $(\mathcal{A}_1, U_1, \Omega_1), (\mathcal{A}_2, U_2, \Omega_2)$ with polynomial energy bounds. Then $\psi_1(z) \otimes \psi_2(z)$ is diffeomorphism covariant with respect to $\tilde{U}_1 \otimes \tilde{U}_2$.*

Proof. By Lemma 5.5, $\psi_1(z) \otimes \psi_2(z)$ satisfies polynomial energy bounds with respect to $H = \hat{L}_{1,0} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_{2,0}$. Then the diffeomorphism covariance follows from Lemma 5.4. \square

As in Sect. 3, let $\{(\mathcal{A}_\kappa, U_\kappa, \Omega_\kappa)\}_{\kappa \in K}$ be a finite family of conformal nets on S^1 generated by fields $\{\phi_{\kappa,j}\}_{\kappa \in K}$, with a collection Δ_κ of localized automorphisms in some interval, with a common (finitely generated) abelian group structure G and a family of bijections $\kappa : G \rightarrow \Delta_\kappa$ for every index $\kappa \in K$. Let $\{\psi_\kappa^h(\xi, z)\}_{\kappa \in K, h \in G, \xi \in \Xi_\kappa}$ be a family of charged primary fields satisfying the conditions of Sect. 5.3.

We assume that $\prod_\kappa \epsilon_{\kappa(h_1), \kappa(h_2)}^+ = 1$ for all pairs $h_1, h_2 \in G$, as in Sect. 3. Let us pick ξ_κ for each κ . Then we consider the formal series $\psi_K^h(\xi_K, z) := \bigotimes_\kappa \psi_\kappa^h(\xi_\kappa, z)$ acting on the auxiliary space $\bigotimes_\kappa \hat{\mathcal{H}}_\kappa$, where $\xi_K = (\xi_1, \dots, \xi_K)$ is an index to label the formal series for the tensor product of K factors. This is a (normal) product of two commuting formal series, hence this makes sense as a formal series, see Sect. 5.2. It follows from Lemma 5.5 that it makes sense as an operator-valued distribution if each of the fields satisfies polynomial energy bounds.

Theorem 5.7. *The fields $\psi_K^h, h \in G$, together with chiral fields $\phi_{\kappa,j}$, is a conformal Wightman field theory acting on the Hilbert space $\mathcal{H}_K = \bigoplus_{g \in G} \bigotimes_\kappa \mathcal{H}_\kappa^{\kappa(g)}$. If, in addition, ψ_K^h satisfies linear energy bounds with respect to $\sum_\kappa \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes \hat{L}_0 \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}_{\kappa-th}$ for all $h \in G$, then $\{\psi_K^h : h \in G\}$ generates a conformal net on S^1 .*

Proof. The formal series ψ_K^h is defined on the Hilbert space $\bigotimes_\kappa \hat{\mathcal{H}}_\kappa$. It generates vectors in $\mathcal{H}_K = \bigoplus_{g \in G} \bigotimes_\kappa \mathcal{H}_\kappa^{\kappa(g)}$ from the vacuum. Then the $\phi_{\kappa,j}(f)$ generate a dense set of vectors in each of these summands. On the other hand, no other vector is generated from ψ_K^h and $\phi_{\kappa,j}$ acting on the vacuum.

Let us prove locality. For two test functions f_1, f_2 with $\arg \text{supp } f_1 < \arg \text{supp } f_2$, we have by Lemma 5.3 that

$$\begin{aligned} \psi_K^{h_1}(f_1) \psi_K^{h_2}(f_2) &= \prod_\kappa \epsilon_{\kappa(h_1), \kappa(h_2)}^+ \psi_K^{h_2}(f_2) \psi_K^{h_1}(f_1) \\ &= \psi_K^{h_2}(f_2) \psi_K^{h_1}(f_1), \end{aligned}$$

giving locality as operator-valued distributions. The fields ψ_K^h and $\phi_{\kappa,j}$ are relatively local. The representation U_K is a tensor product of positive-energy representations, therefore it has itself positive energy.

If we assume linear energy bounds, then the fields strongly commute and generate a conformal net on S^1 by Theorem A.2. □

Remark 5.8. We expect that the conformal net on S^1 in Theorem 5.7 generated by the conformal Wightman field is unitarily equivalent to the net extension \mathcal{A}_K constructed in Theorem 3.1. This, at least for $|K| = 2$, should follow by a similar argument as in the proof of Theorem 5.9. More generally, assuming properties that assures that the VOA intertwiners intertwine DHR sectors (strong integrability, strong intertwining property and strong braiding [CWX,Gui20]), it should be possible to show that the two nets coincide. We will address this question in a future publication.

5.5. Two-dimensional Wightman fields through braiding-cancelling map. From a pair of charged primary fields with the braiding satisfying certain conditions, we can construct a two-dimensional conformal Wightman field. This is in particular the case if we take two copies of the same conformal net $\mathcal{A}_L = \mathcal{A}_R$, with a pointed braided fusion subcategory

of automorphisms whose fusion ring is isomorphic to a *finite* abelian group G , and we take the tensor product of the left charged fields ψ_L^h and $\psi_L^{h^{-1}}$, where $h \in G$, similarly to (4.1), namely $\tilde{\psi}^h := \psi_L^h \otimes \psi_L^{h^{-1}}$ acting on $\mathcal{H}_L \otimes \mathcal{H}_L$.

If we assume that the braidings cancel in the sense⁷ that $\epsilon_{L(g),L(h)}^+ = \overline{\epsilon_{R(g),R(h)}^-}$, we can prove locality of the combined Wightman field. We show that this corresponds to a (finite index) Longo–Rehren extension with respect to a pointed braided fusion category, [LR95], see Sect. 4.2. The basic idea behind these constructions is also present in the two-dimensional quantum field theory context, e.g., in [Reh97], see in particular [Reh97, Section 2.3].

Let $\{(\mathcal{A}_{L/R}, U_{L/R}, \Omega_{L/R})\}$ be a pair of conformal nets on S^1 generated by fields $\{\phi_{L/R,j}\}$, with a collection $\Delta_{L/R}$ of (mutually commuting inequivalent) localized automorphisms in some interval (and closed under composition and inverses up to unitary equivalence), as in Sect. 4, with a common finite abelian group structure G and bijections $L/R : G \rightarrow \Delta_{L/R}$. Let $\{\psi_{L/R}^h(\xi, z)\}_{h \in G, \xi \in \Xi_{L/R}}$ be a family of primary fields satisfying the conditions of Sect. 5.3. Denote $\epsilon^+ := \epsilon_{L(g),L(h)}^+$ and $\epsilon^- := \epsilon_{R(g),R(h)}^-$, for short. Then our braiding cancellation assumption reads $\epsilon^+ = \overline{\epsilon^-}$. For $w_1, w_2 \in S^1 \setminus \{-1\}$, $\arg w_1 < \arg w_2$, we have by Lemma 5.3 that

$$\begin{aligned} \psi_L^g(\xi_L, w_1)\psi_L^h(\xi_L, w_2) &= \epsilon^+ \psi_L^h(\xi_L, w_2)\psi_L^g(\xi_L, w_1), \\ \psi_R^g(\xi_R, w_2)\psi_R^h(\xi_R, w_1) &= \epsilon^- \psi_R^h(\xi_R, w_1)\psi_R^g(\xi_R, w_2) \\ &= \overline{\epsilon^+} \psi_R^h(\xi_R, w_1)\psi_R^g(\xi_R, w_2). \end{aligned}$$

For fixed ξ_L and ξ_R , let us introduce a two-dimensional formal power series by

$$\tilde{\psi}^h(w, z) := \psi_L^h(\xi_L, w) \otimes \psi_R^h(\xi_R, z).$$

Theorem 5.9. *The field $\tilde{\psi}^h$ is a two-dimensional conformal Wightman field on the Hilbert space $\tilde{\mathcal{H}} = \bigoplus_{h \in G} \mathcal{H}_L^{L(h)} \otimes \mathcal{H}_R^{R(h)}$. If the field satisfies a linear energy bound with respect to $\hat{L}_0 \otimes \mathbb{1} + \mathbb{1} \otimes \hat{L}_0$, then it generates a two-dimensional conformal net. If G is finite, the net coincides (up to unitary equivalence) with $\tilde{\mathcal{A}}$ constructed in Theorem 4.2 in the case $\mathcal{A}_L = \mathcal{A}_R$.*

Proof. The field $\tilde{\psi}^h$ satisfies two-dimensional locality: the point (w_1, z_1) is spacelike to (w_2, z_2) if $\arg w_1 < \arg w_2$ and $\arg z_1 > \arg z_2$ (or the reversed relations, and in this case, with $\epsilon^+ := \epsilon_{L(h),L(g)}^+$),

$$\begin{aligned} \tilde{\psi}^h(w_1, z_1)\tilde{\psi}^g(w_2, z_2) &= \psi_L^h(\xi_L, w_1) \otimes \psi_R^h(\xi_R, z_1) \cdot \psi_L^g(\xi_L, w_2) \otimes \psi_R^g(\xi_R, z_2) \\ &= \epsilon^+ \overline{\epsilon^+} \psi_L^g(\xi_L, w_2) \otimes \psi_R^g(\xi_R, z_2) \cdot \psi_L^h(\xi_L, w_1) \otimes \psi_R^h(\xi_R, z_1) \\ &= \psi_L^g(\xi_L, w_2) \otimes \psi_R^g(\xi_R, z_2) \cdot \psi_L^h(\xi_L, w_1) \otimes \psi_R^h(\xi_R, z_1) \\ &= \tilde{\psi}^{h_2}(w_2, z_2)\tilde{\psi}^{h_1}(w_1, z_1). \end{aligned}$$

⁷ Here, instead of assuming additionally that $\epsilon_{L(g)^{-1},L(h)}^+ = \overline{\epsilon_{R(g)^{-1},R(h)}^-}$ as we did in Sect. 4, we assume that the set of charged primary fields is closed under conjugate and each of them satisfies the braiding relation.

It is a two-dimensional operator-valued distribution because it is a tensor product of two one-dimensional operator-valued distributions. Other axioms, positivity of energy, diffeomorphism covariance, the cyclicity of vacuum in $\tilde{\mathcal{H}}$, can be proven as in Theorem 5.7. If we assume linear energy bounds, strong commutativity of smeared fields follows.

Let G be finite. As the decomposition of the vacuum representation $\tilde{\mathcal{H}} = \bigoplus_{h \in G} \mathcal{H}_L^{L(h)} \otimes \mathcal{H}_R^{R(h)}$ with respect to the action of $\mathcal{A}_L \otimes \mathcal{A}_R$ is the same as the decomposition of the vacuum representation of the Longo–Rehren extension $\tilde{\mathcal{A}}_{LR}$ (4.3), and the former determines the dual canonical endomorphism of the extension we constructed, by [LR95, Proposition 3.4], the latter endomorphism must be unitarily equivalent to the Longo–Rehren endomorphism Θ_{LR} . As we are in the braided pointed fusion case with finite abelian group G , the uniqueness of the associated Longo–Rehren extension (among finite index local extensions) provided by [KL04b, Section 4, Example 4.5] proves the last claim. \square

6. Examples: The U(1)-Current

We construct examples of a conformal net on S^1 , collections of sectors and charged primary fields associated with the U(1)-current. To the best of our knowledge, most of the material of this section is scattered in several places, see, e.g., [BMT88, TL97, TZ12], or it is known as folklore. We collect some useful facts for the sake of the reader to appreciate the construction of Wightman fields through charged primary fields in the special case of extensions of the U(1)-current, cf. [Reh97] for the two-dimensional case.

Here we use the symbols $\mathcal{H}, \mathcal{A}, U, \Omega, L_n, J_n, Y_{\alpha,n}$ for a chiral component, differently from the previous sections where we used the index κ .

6.1. The field and the net. In literature, several constructions of the U(1)-current appeared. One can use the lowest weight representation with the lowest weight 1 of $\text{PSL}(2, \mathbb{R})$, and the associated net standard subspaces, then by second quantization the local algebras of the U(1)-current net are generated by the Weyl operators, cf. [BGL93].

The construction of the U(1)-current that we illustrate here relies on the so-called current algebra, namely the complex Lie algebra generated as \mathbb{C} -linear space by the family of symbols $\{J_n : J_n : n \in \mathbb{Z}\}$ and a central element c that verify $[J_m, J_n] = m\delta_{n+m}c$, see [BMT88]. The vacuum representation of the current algebra is given as a Verma module (with the lowest weight 0, hence $J_0\Omega = 0$), which is the linear span of the symbols

$$J_{-i_1} \dots J_{-i_m} \Omega, \quad 0 < i_1 \leq i_2 \leq \dots \leq i_m \in \mathbb{N}, m \in \mathbb{N}$$

where, by abuse of notation, J_n are interpreted as operators with the Lie brackets given for an associative algebra, $J_k\Omega = 0$ for $k \geq 0$ and one defines the actions of J_n for $n \in \mathbb{Z}$ by using the commutation relations. There is a unique inner product satisfying $J_n^* = J_{-n}$ and

$$\langle J_{-i_1} \dots J_{-i_m} \Omega, J_{-l_1} \dots J_{-l_s} \Omega \rangle := \langle \Omega, J_{i_m} \dots J_{i_1} J_{-l_1} \dots J_{-l_s} \Omega \rangle, \quad (6.1)$$

and we extend it by linearity. Its completion is isomorphic to the Bosonic Fock space with the one-particle space spanned by $\{J_{-n}\Omega : n \in \mathbb{N}\}$, see, e.g., [KR87, Proposition 2.1].

Using Sugawara’s formula to define new generators L_n as

$$L_n := \frac{1}{2} \sum_{k \in \mathbb{Z}} : J_{k+n} J_k :, \quad n \in \mathbb{Z}$$

where $: \cdot :$ is the Wick product, one defines a projective unitary representation of the Virasoro algebra with the central charge $c = 1$. In particular, the **conformal Hamiltonian** L_0 acts as

$$L_0 J_{-i_1} \dots J_{-i_m} \Omega = (i_1 + \dots + i_m) J_{-i_1} \dots J_{-i_m} \Omega,$$

for $0 \leq i_1 \leq \dots \leq i_m \in \mathbb{N}$. Therefore, L_0 extends to a positive self-adjoint operator. Moreover, as J_n can be regarded as an annihilation operator for $n > 0$, we have the following bound: for every $n \in \mathbb{Z}$ and every Ψ in the Verma module

$$\begin{aligned} \|J_n \Psi\| &\leq (n + 1) \|(L_0 + 1)^{\frac{1}{2}} \Psi\|, \\ \|L_n \Psi\| &\leq \sqrt{\frac{13}{12}} (1 + |n|^{\frac{3}{2}}) \|(L_0 + 1) \Psi\|. \end{aligned}$$

From this, the representation $\{L_n\}$ of the Virasoro algebra integrates to a projective unitary representation U of $\text{Diff}^+(S^1)$.

Moreover, the current

$$J(f) := \sum_{n \in \mathbb{Z}} f_n J_n,$$

for a test function $f \in C^\infty(S^1)$ is an essentially self-adjoint operator on $\mathcal{D}(L_0)$. Therefore, $J(f)$ for $f \in C^\infty(S^1)$ is well-defined on the dense Verma module and $J(f)\Psi \in \mathcal{D}(L_0)$. We have

$$\|[L_0, J(f)]\Psi\| = \|J(f')\Psi\| \leq \alpha_{f'} \|(L_0 + 1)^{\frac{1}{2}} \Psi\|,$$

so by Theorem A.2 is essentially self-adjoint and it can be defined on $\mathcal{D}(L_0)$. In this way, if $f, g \in C^\infty(S^1)$, then the commutator $[J(f), J(g)]$ is well-defined on $\mathcal{D}(L_0)$ and

$$[J(f), J(g)]\Psi = \int_{S^1} f'(t)g(t)dt \cdot \Psi, \quad \Psi \in \mathcal{D}(L_0).$$

Moreover, by Theorem A.2, if the support of f and g is disjoint, then $J(f)$ and $J(g)$ commute strongly, i.e., $[e^{itJ(f)}, e^{itJ(g)}] = 0$. One can also show that $\text{Ad } U(\varphi)(e^{iJ(f)}) = e^{iJ(f \circ \varphi^{-1})}$.

For any $\lambda \in \mathbb{R}$, and denoting the lowest weight vector by Ω_λ , one can construct a Verma module such that the representation J_n^λ is of the form explained above, except that $J_0^\lambda \Omega_\lambda = \lambda \Omega_\lambda$. The same commutation relations and similar estimates hold for $\lambda \in \mathbb{R}$.

To every non-empty non-dense open interval $I \subset S^1$, one associates the following von Neumann algebra

$$\mathcal{A}_{U(1)}(I) := \{e^{iJ(f)} : \text{supp } f \subset I\}''.$$

The triple $(\mathcal{A}_{U(1)}, U, \Omega)$ is a conformal net on the circle S^1 .

6.2. *Representations of the U(1)-current.* Let us fix an open non-empty non-dense interval $I \subset S^1$ and a function $h \in C^\infty(S^1)$ with $\text{supp } h \subset I$. Then, we define a map on the Weyl operators $W(f) := e^{iJ(f)}$ for $f \in C^\infty(S^1)$ supported in I in the following way

$$\sigma_{h,I}(W(f)) := e^{i \int_{S^1} f(t)h(t)dt} W(f).$$

The map $\sigma_{h,I}$ can be extended to a representation of the local algebra $\mathcal{A}_{U(1)}(I)$ on the Fock space. Indeed, for every interval I_1 , σ_{h,I_1} is implemented by $\text{Ad } W(H)$, where $H' = h$ on I_1 and supported on another interval \tilde{I}_1 . Therefore, σ_{h,I_1} extends to a representation of the local algebra $\mathcal{A}_{U(1)}(I_1)$. If $I_2 \subset S^1$ is another interval such that $I_1 \subset I_2$, then a function H supported in an interval $\tilde{I}_2 \supset \bar{I}_2$ such that $H' = h$ on I_2 and $\sigma_{h,I_1}(W(f)) = \text{Ad } W(H)(W(f))$. Therefore, one can define a representation σ_h of the net $\mathcal{A}_{U(1)}$ as the family $\{\sigma_{h,I_1}\}_{I_1}$ for $I_1 \subset S^1$ as above, verifying the compatibility condition $\sigma_{h,I_2}|_{\mathcal{A}_{U(1)}(I_1)} = \sigma_{h,I_1}$ whenever $I_1 \subset I_2$ are two non-empty non-dense open intervals of S^1 .

The unitary equivalence class of the representations is determined by the value $\int_{S^1} h(t)dt$. In particular, when $\int_{S^1} h(t)dt = 0$, then σ_h is unitary equivalent to the vacuum representation. Let $\alpha = \int_{S^1} h(t)dt$. Although the representation σ_h is defined on the vacuum Hilbert space, we denote it \mathcal{H}_α in order to distinguish the representation. Analogously, the lowest weight vector in \mathcal{H}_α is denoted by Ω_α . See [BMT88] for these results.

6.3. *Fusion relations and braiding.* Let $h_1, h_2 \in C^\infty(S^1)$. Then the maps $\sigma_{h_1}, \sigma_{h_2}$ are automorphisms of the U(1)-current net $\mathcal{A}_{U(1)}$. Let $I_0 \in \mathcal{I}$ and consider the local algebra $\mathcal{A}_{U(1)}(I_0)$. Then, $\sigma_{h_1,I_0}, \sigma_{h_2,I_0}$ are representations of the local algebra $\mathcal{A}_{U(1)}(I_0)$ such that $\sigma_{h_1,I_0} \circ \sigma_{h_2,I_0} = \sigma_{h_1+h_2,I_0}$. Since the interval I_0 is arbitrary, we have

$$\sigma_{h_1} \circ \sigma_{h_2} = \sigma_{h_1+h_2}.$$

We now compute the braiding for the U(1)-current, which we believe is well-known, but for which we couldn't find any reference.

Let $I \in \mathcal{I}$ be fixed. Consider two functions $h_1, h_2 \in C^\infty(S^1)$ with $\text{supp}(h_1), \text{supp}(h_2) \subset I$ and let σ_{h_1} and σ_{h_2} be two automorphisms of the net $\mathcal{A}_{U(1)}$. Next, let $I_0 \in \mathcal{I}$ such that $I \subset I_0$ and choose $I_1, I_2 \subset I_0$ in \mathcal{I} such that I_2 stays in the future of I_1 , and I_1, I_2, I are pairwise disjoint.

Let $i = 1, 2$ be fixed, and let $g_i \in C^\infty(S^1)$ such that $\text{supp } g_i \subset I_i$ and $\int_{S^1} g_i(t)dt = \int_{S^1} h_i(t)dt$. Then there exist $H_i \in C^\infty(S^1)$ supported in I_0 such that $\text{Ad } W(H_i)\sigma_{h_i} = \sigma_{g_i}$. Moreover, for the mutual position of I_1 and I_2 , H_1 and H_2 have to be of the form $H'_1 = -h_1, H'_2 = -h_2$ on I , $H'_i = g_i$ on I_i and a constant elsewhere. As the conditions on h_i, g_i show, H_i are piecewise constant in the union of intervals $(I \cup I_i)^c \cap I_0$ (see Fig. 3). In particular, if the intervals are between I_0^c and I or between I_0^c and I_c , then H_i is defined as 0, since H_i is continuous. On the rest, and for the same reason, we set H_i to $\int_I h_i(t)dt = \int_{I_i} g_i(t)dt := \sqrt{2\pi}\alpha_i$, where α_i is the so called **charge**.

The braiding is given by the following formula

$$\epsilon(\sigma_{h_1}, \sigma_{h_2}) := \epsilon_{\sigma_{h_1}, \sigma_{h_2}} = \sigma_{h_2, I_0}(W(H_1)^*)W(H_2)^*W(H_1)\sigma_{h_1, I_0}(W(H_2)).$$

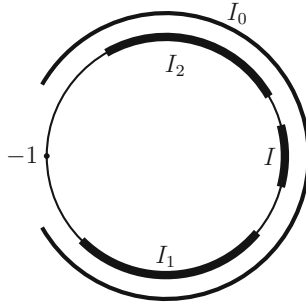


Fig. 3. An example configuration for the intervals I, I_0, I_1 and I_0 . The choice of “future” and “past” is fixed by considering the point -1 as the point of infinity for the lightray, and choosing I_0 that does not contain it

Therefore, we have

$$\begin{aligned}
 \epsilon(\sigma_{h_1}, \sigma_{h_2}) &= \sigma_{h_2, I_0}(W(H_1)^*)W(H_2)^*W(H_1)\sigma_{h_1, I_0}(W(H_2)) \\
 &= e^{-i \int_{S^1} H_1(t)h_2(t)dt} W(H_1)^*W(H_2)^*W(H_1)e^{i \int_{S^1} h_1(t)H_2(t)dt} W(H_2) \\
 &= e^{-i \int_I H_1(t)h_2(t)dt} W(H_1)^*W(H_2)^*W(H_1)e^{i \int_I h_1(t)H_2(t)dt} W(H_2) \\
 &= e^{i \int_I [h_1(t)H_2(t) - H_1(t)h_2(t)]dt} W(H_1)^*W(H_2)^*W(H_1)W(H_2) \\
 &= e^{i \int_I [h_1(t)H_2(t) - H_1(t)h_2(t)]dt} e^{-\frac{i}{2} \text{Im}\langle H_1, H_2 \rangle} W(-H_1 - H_2)e^{-\frac{i}{2} \text{Im}\langle H_1, H_2 \rangle} W(H_1 + H_2) \\
 &= e^{i \int_I [h_1(t)H_2(t) - H_1(t)h_2(t)]dt} e^{i \text{Im}\langle H_1, H_2 \rangle}.
 \end{aligned}$$

If one recalls that $\text{Im}\langle H_1, H_2 \rangle = \int_{S^1} H_1(t)H_2'(t)dt$, then

$$\int_{S^1} H_1(t)H_2'(t)dt = \sqrt{2\pi}\alpha_1 \int_{I_2} g_2(t)dt + \int_I H_1(t)h_2(t)dt = 2\pi\alpha_1\alpha_2 + \int_{I \cup I_1} h_1(t)H_2(t)dt.$$

Therefore, the braiding is

$$\begin{aligned}
 \epsilon(\sigma_{h_1}, \sigma_{h_2}) &= e^{i \int_I [h_1(t)H_2(t) - H_1(t)h_2(t)]dt} e^{i \text{Im}\langle H_1, H_2 \rangle} \\
 &= e^{i \int_I [h_1(t)H_2(t) - H_1(t)h_2(t)]dt} e^{i2\pi\alpha_1\alpha_2 + i \int_{I \cup I_1} h_1(t)H_2(t)dt} \\
 &= e^{i \int_I h_1(t)H_2(t)dt} e^{i2\pi\alpha_1\alpha_2}.
 \end{aligned}$$

If we take $h_i = \sqrt{2\pi}\alpha_i h$ for a single function h such that $\int_{S^1} h(t)dt = 1$, then H_2 is proportional to a function H on I such that $H' = -h$ on I . Therefore, by integration by parts,

$$\int_I h_1(t)H_2(t)dt = 2\pi\alpha_1\alpha_2 \int_I h(t)H(t)dt = 2\pi\alpha_1\alpha_2 \left(\left[-H^2(t) \right]_I - \int_I h(t)H(t)dt \right)$$

implying

$$\int_I h_1(t)H_2(t)dt = -\pi\alpha_1\alpha_2.$$

Hence we conclude that in this case

$$\epsilon(\sigma_{\alpha_1 h}, \sigma_{\alpha_2 h}) = e^{i\pi\alpha_1\alpha_2},$$

with the configuration above. In particular, it holds that $\epsilon(\sigma_{\alpha_1 h}, \sigma_{\alpha_2 h}) = \epsilon(\sigma_{\alpha_2 h}, \sigma_{\alpha_1 h})$.

6.4. *The charged field.* The $U(1)$ -current net has charged primary fields in the sense of Sect. 5.3. We equip \mathbb{R} with the scalar product $\langle \alpha, \beta \rangle = \alpha\beta$. In the sequel, we check that the formal series $Y_\alpha(z) = \sum_{s \in \mathbb{R}} Y_{\alpha,s} z^{-s-D}$ are charged primary fields, where $D = \langle \alpha, \alpha \rangle / 2$ and each coefficient $Y_{\alpha,s} \in \mathbb{R}$ is a map $\mathcal{H}_\beta^{\text{fin}} \rightarrow \mathcal{H}_{\beta+\alpha}^{\text{fin}}$ (on each $\mathcal{H}_\beta^{\text{fin}}$, $\beta \in \mathbb{R}$, only $Y_{\alpha,s}$ with $s \in \mathbb{Z} - \langle \alpha, \beta \rangle - D$ are non-zero). Explicitly, let c_α be the unitary charge shift operator $\mathcal{H}_\beta^{\text{fin}} \rightarrow \mathcal{H}_{\beta+\alpha}^{\text{fin}}$ defined by $c_\alpha J_{-n_1} \cdots J_{-n_k} \Omega_\beta = J_{-n_1} \cdots J_{-n_k} \Omega_{\beta+\alpha}$, $n_j > 0$. Following [TZ12] ($\alpha(n)$ there is identified with αJ_n in our notation, cf. [TL97, Chapter V (3.2.1)]), we define, as formal series,

$$E^\pm(\alpha, z) = \exp\left(\mp \sum_{n>0} \frac{\alpha J_{\pm n}}{n} z^{\mp n}\right),$$

$$Y_\alpha(z) = c_\alpha E^-(\alpha, z) E^+(\alpha, z) z^{\alpha J_0}, \tag{6.2}$$

where $\alpha J_0 = \langle \alpha, \beta \rangle$ is a scalar on each \mathcal{H}_β . The formal series of the exponential in $E^\pm(\alpha, z)$ is defined without problems, because, when expanding it into a Taylor series the coefficients of z^n are finite sums of operators on $\mathcal{H}_0^{\text{fin}}$. As for $Y_\alpha(z)$, the coefficients of the product $E^-(\alpha, z) E^+(\alpha, z)$ are infinite sums, but they still make sense as operators. Indeed, for each n , the coefficient $E_{\alpha,n}^+$ of z^{-n} in $E^+(\alpha, z)$ is a linear combination of $J_{k_1} \cdots J_{k_j}$ such that $\sum_{m=1}^j k_m = n$, $k_m > 0$, and there are only finitely many such combinations. The series $E^-(\alpha, z)$ has a similar structure. As any vector Ψ in $\mathcal{H}_0^{\text{fin}}$ has finite energy M and J_m lowers the energy by m , Ψ is annihilated by any product $J_{k_1} \cdots J_{k_j}$ if $\sum_m k_m > M$, that is, Ψ is annihilated by $E_{\alpha,n}^+$ if $n > M$. The coefficient of z^n in $E^-(\alpha, z) E^+(\alpha, z)$ is the sum $\sum_{j \in \mathbb{Z}_+} E_{\alpha,n-j}^- E_{\alpha,j}^+$. Therefore, on each Ψ only finitely many terms contribute, and this sum defines an operator on $\mathcal{H}_0^{\text{fin}}$. Its restriction $Y_\alpha(z)|_{\mathcal{H}_\beta}$ has the form

$$Y_\alpha(z)|_{\mathcal{H}_\beta} = \sum_{n \in \mathbb{Z}} Y_{\alpha,n-\alpha\beta-D}|_{\mathcal{H}_\beta} z^{-n-D}.$$

Braiding By [TL97, Chapter VI (1.2.2)], it holds that

$$\left(1 - \frac{z}{w}\right)^{-\langle \alpha, \beta \rangle} E^+(\alpha, w) E^-(\beta, z) = E^-(\beta, z) E^+(\alpha, w)$$

where $(1-u)^a = \sum_{n \geq 0} \binom{a}{n} (-u)^n$. Note that the left-hand side makes sense, because $(1 - \frac{z}{w})^{-\langle \alpha, \beta \rangle}$ has positive powers in z and negative powers in w , while $E^-(\beta, z)$ has positive powers in z and $E^+(\alpha, w)$ has negative powers in w . Similarly,

$$E^-(\alpha, w) E^+(\beta, z) = \left(1 - \frac{w}{z}\right)^{-\langle \alpha, \beta \rangle} E^+(\beta, z) E^-(\alpha, w).$$

Therefore, if we introduce the pre-vertex operators $\underline{Y}_\alpha(z) = E^-(\alpha, z) E^+(\alpha, z) z^{\alpha J_0}$ (it is $Y_\alpha(z)$ without c_α), and using that $E^\pm(\alpha, z)$ and $E^\pm(\beta, w)$ commute (when \pm coincide), we obtain

$$\left(1 - \frac{z}{w}\right)^{-\langle \alpha, \beta \rangle} \underline{Y}_\alpha(w) \underline{Y}_\beta(z) = \left(1 - \frac{w}{z}\right)^{-\langle \alpha, \beta \rangle} \underline{Y}_\beta(z) \underline{Y}_\alpha(w). \tag{6.3}$$

Here again, the equality is understood as the equality between the coefficients of $z^n \zeta^m$ on each fixed vector $\Psi \in \mathcal{H}_0^{\text{fin}}$.

Now we take $\bigoplus_{\lambda \in \mathbb{R}} \mathcal{H}_\lambda^{\text{fin}}$ (algebraic direct sum). Using $c_{\lambda_1} z^{\lambda_2 J_0} = z^{-(\lambda_1, \lambda_2) + \lambda_2 J_0} c_{\lambda_1}$, and the fact that c_{λ_1} commute with $E^+(\alpha, z)$, $E^-(\alpha, w)$, we obtain

$$\left(1 - \frac{z}{w}\right)^{-\langle \alpha, \beta \rangle} w^{-\langle \alpha, \beta \rangle} Y_\alpha(w) Y_\beta(z) = \left(1 - \frac{w}{z}\right)^{-\langle \alpha, \beta \rangle} z^{-\langle \alpha, \beta \rangle} Y_\beta(z) Y_\alpha(w). \tag{6.4}$$

Relative locality Here again we consider $Y_\alpha(z)$ as a formal series on $\bigoplus_{\lambda \in \mathbb{R}} \mathcal{H}_\lambda^{\text{fin}}$. On each of these summands, L_m and J_m act naturally as a representation. We denote their direct sum on $\bigoplus_{\lambda \in \mathbb{R}} \mathcal{H}_\lambda^{\text{fin}}$ by \hat{L}_m, \hat{J}_m , respectively.

The commutation relation $[\hat{J}_m, Y_\alpha(z)] = \alpha Y_\alpha(z) z^m$ can be checked easily from the definition of $Y_\alpha(z)$. This is equivalent to $[\hat{J}_m, Y_{\alpha, s}] = \alpha Y_{\alpha, m+s}$.

Primarity As we will see, the conformal dimension of Y_α should be $D = \frac{\alpha^2}{2}$. Note that

$$\partial Y_\alpha(z) = z^{-1} Y_\alpha(z) \alpha \hat{J}_0 + \sum_{j < 0} z^{-j-1} \alpha Y_\alpha(z) \hat{J}_j + \sum_{j > 0} z^{-j-1} \alpha \hat{J}_j Y_\alpha(z).$$

On the other hand, Y_α being primary is equivalent to $[\hat{L}_m, Y_\alpha(z)] = \partial Y_\alpha(z) z^{m+1} + D(m+1) Y_\alpha(z)$. Therefore, we need to show

$$\begin{aligned} [\hat{L}_m, Y_\alpha(z)] &= (z^{-1} Y_\alpha(z) \alpha \hat{J}_0 + \sum_{j < 0} z^{-j-1} \alpha Y_\alpha(z) \hat{J}_j + \sum_{j > 0} z^{-j-1} \alpha \hat{J}_j Y_\alpha(z)) z^{m+1} \\ &\quad + D(m+1) Y_\alpha(z) z^m. \end{aligned} \tag{6.5}$$

It holds that $\hat{J}_j c_\alpha = c_\alpha (\hat{J}_j + \alpha \delta_j)$. From the Sugawara formula $\hat{L}_m = \frac{1}{2} \sum_k : \hat{J}_k \hat{J}_{m-k} :$, we have $\hat{L}_m c_\alpha = c_\alpha (\hat{L}_m + \alpha \hat{J}_m)$ for $m \neq 0$ and $\hat{L}_0 c_\alpha = c_\alpha (\hat{L}_0 + \alpha \hat{J}_0 + \frac{\alpha^2}{2})$.

We want to verify directly that $Y_\alpha(z)$ is primary in the sense above (this should also follow by using VOA modules, see [CGP21, (36.1)] and references therein, in particular, [FHL93, Section 5.4]). To do this, recall the commutation relations

$$[\hat{L}_m, \hat{J}_j] = -j \hat{J}_{j+m}.$$

In the definition of E^\pm , the order of product does not matter, hence

$$E^+(\alpha, z) = \prod_{j > 0} \exp\left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right), \quad E^-(\alpha, z) = \prod_{j < 0} \exp\left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right).$$

Using the formula $[A, BC] = [A, B]C + B[A, C]$, we calculate

$$\begin{aligned} [\hat{L}_m, Y_\alpha(z)] &= [\hat{L}_m, c_\alpha] E^-(\alpha, z) E^+(\alpha, z) z^{\alpha \hat{J}_0} \\ &\quad + c_\alpha [\hat{L}_m, E^-(\alpha, z)] E^+(\alpha, z) z^{\alpha \hat{J}_0} + c_\alpha E^-(\alpha, z) [\hat{L}_m, E^+(\alpha, z)] z^{\alpha \hat{J}_0}. \end{aligned} \tag{6.6}$$

Let us consider the three cases separately. We illustrate our strategy for the simpler case of $m = 0$.

- $m = 0$. In this case, (6.5) becomes the following and thus we have to show

$$\begin{aligned}
 [\hat{L}_0, Y_\alpha(z)] &= (z^{-1}Y_\alpha(z)\alpha\hat{J}_0 + \sum_{j>0} z^{-j-1}\alpha\hat{J}_jY_\alpha(z) + \sum_{j<0} z^{-j-1}\alpha Y_\alpha(z)\hat{J}_j)z + DY_\alpha(z) \\
 &= Y_\alpha(z)\alpha\hat{J}_0 + \sum_{j>0} z^{-j}\alpha\hat{J}_jY_\alpha(z) + \sum_{j<0} z^{-j}\alpha Y_\alpha(z)\hat{J}_j + DY_\alpha(z). \tag{6.7}
 \end{aligned}$$

As $[\hat{L}_0, c_\alpha] = c_\alpha(D + \hat{J}_0)$, the first and the last terms in (6.7) are obtained. By noting $[\hat{L}_0, \hat{J}_j] = -j\hat{J}_j$, it is straightforward that

$$\left[\hat{L}_0, \exp\left(-\frac{\alpha\hat{J}_j}{j}z^{-j}\right) \right] = \alpha\hat{J}_jz^{-j} \exp\left(-\frac{\alpha\hat{J}_j}{j}z^{-j}\right)$$

and hence $[\hat{L}_0, E^\pm(\alpha, z)] = \sum_{\pm j>0} z^{-j}\alpha\hat{J}_jE^\pm(\alpha, z)$. Inserting them in (6.6), they yield the second and the third terms of (6.7).

- $m > 0$, odd. In this case, there is no j such that $-2j = m$. Using $[\hat{L}_m, \hat{J}_j] = -j\hat{J}_{m+j}$, we have

$$\begin{aligned}
 \left[\hat{L}_m, \exp\left(-\frac{\alpha\hat{J}_j}{j}z^{-j}\right) \right] &= \sum_{k=0}^{\infty} \frac{1}{k!} \left[\hat{L}_m, \left(-\frac{\alpha\hat{J}_j}{j}z^{-j}\right)^k \right] \\
 &= \sum_{k=1}^{\infty} \frac{1}{k!} \frac{kj\alpha\hat{J}_{m+j}}{j} z^{-j} \left(-\frac{\alpha\hat{J}_j}{j}z^{-j}\right)^{k-1} \\
 &= \alpha\hat{J}_{m+j}z^{-j} \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \left(-\frac{\alpha\hat{J}_j}{j}z^{-j}\right)^{k-1} \\
 &= \alpha\hat{J}_{m+j}z^{-j} \exp\left(-\frac{\alpha\hat{J}_j}{j}z^{-j}\right)
 \end{aligned}$$

and hence $[\hat{L}_m, E^\pm(\alpha, z)] = \sum_{\pm j>0} z^{-j}\alpha\hat{J}_{m+j}E^\pm(\alpha, z)$. The contribution from the case $j = -m$ in (6.6) gives exactly the term $z^{-1}Y_\alpha(z)\alpha\hat{J}_0z^{m+1}$ in (6.5).

If $m + j < -j$, we need to bring the factor \hat{J}_{m+j} past $\exp(-\frac{\alpha\hat{J}_{-m-j}}{-m-j}z^{m+j})$. Recalling that $[\hat{J}_j, \hat{J}_{-j}] = j$, and hence $[\exp(c\hat{J}_j), \hat{J}_{-j}] = \sum_{k\geq 0} \frac{1}{k!}c^k[\hat{J}_j^k, \hat{J}_{-j}] = cj \exp(c\hat{J}_j)$, we get a contribution $\alpha^2z^mY_\alpha(z)$ for each such j . There are $\frac{m-1}{2}$ such cases and it is $(m-1)\frac{\alpha^2}{2}z^mY_\alpha(z)$.

Finally, $[\hat{L}_m, c_\alpha]E^-(\alpha, z)E^+(\alpha, z)z^{\alpha\hat{J}_0} = \alpha\hat{J}_mY_\alpha(z) = z^{-m-1}\alpha\hat{J}_mY_\alpha(z)z^{m+1}$. As $m > 0$, we need to bring the factor \hat{J}_m through $E^-(\alpha, z)$, from which there is an additional contribution $\alpha^2z^mY_\alpha(z)$. Altogether, we obtain a contribution of $(m+1)\frac{\alpha^2}{2}z^mY_\alpha(z)$ which is the last term of (6.5) and the term corresponding to $j = m$. This completes the proof of the case $m > 0$ odd.

- $m > 0$, even. In this case, there is j such that $-2j = m$. We calculate the commutator $[\hat{L}_{-2j}, \exp(-\frac{\alpha\hat{J}_j}{j}z^{-j})]$ slightly modifying the argument above for $m > 0$ odd

$$\begin{aligned}
 \left[\hat{L}_{-2j}, \exp\left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right) \right] &= \sum_{k=0}^{\infty} \frac{1}{k!} \left[\hat{L}_{-2j}, \left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right)^k \right] \\
 &= -\frac{\alpha z^{-j}}{j} [\hat{L}_{-2j}, J_j] + \sum_{k=2}^{\infty} \frac{1}{k!} \left[\hat{L}_{-2j}, \left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right)^k \right] \\
 &= \alpha z^{-j} J_{-j} + \sum_{k=2}^{\infty} \frac{1}{k!} \frac{(-1)^k \alpha^k z^{-jk}}{j^k} \frac{j^{2k(k-1)}}{2} J_j^{k-2} \\
 &\quad - \sum_{k=2}^{\infty} \frac{1}{k!} \frac{(-1)^k \alpha^k z^{-jk}}{j^k} k j J_j^{k-1} J_{-j} \\
 &= \alpha z^{-j} J_{-j} + \frac{\alpha^2}{2} z^{-2j} \sum_{k=2}^{\infty} \frac{1}{(k-2)!} \frac{(-1)^k \alpha^{k-2} z^{-j(k-2)}}{j^{k-2}} J_j^{k-2} \\
 &\quad - \sum_{k=2}^{\infty} \frac{1}{k!} \frac{(-1)^k \alpha^k z^{-jk}}{j^k} k j J_j^{k-1} J_{-j} \\
 &= \frac{\alpha^2}{2} z^{-2j} \exp\left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right) + \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \frac{(-1)^{k-1} \alpha^{k-1} z^{-j(k-1)}}{j^{k-1}} J_j^{k-1} J_{-j} \\
 &= \frac{\alpha^2}{2} z^{-2j} \exp\left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right) + \exp\left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right) J_{-j}
 \end{aligned}$$

and then moving $\hat{J}_{j+m} = \hat{J}_{-j}$, we obtain an additional term of

$$\frac{\alpha^2}{2} z^{-2j} \exp\left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right) = \frac{\alpha^2}{2} z^m \exp\left(-\frac{\alpha \hat{J}_j}{j} z^{-j}\right)$$

and this gives the contribution $\frac{\alpha^2}{2} z^m Y_\alpha(z)$.

From j with $m + j < -j$, we get a contribution in (6.5) $\alpha^2 z^m Y_\alpha(z)$ for each such j as before, and as m is even there are $\frac{m-2}{2}$ such cases, and together with the contribution from the previous paragraph we obtain $\frac{m-1}{2} \alpha^2 z^m Y_\alpha(z)$ as before.

The rest is the same as in the case where m is odd.

- The case $m < 0$ is obtained by taking the conjugate and substituting α by $-\alpha$.

Energy bounds By [TL97, Proposition VI.1.2.1], $\|Y_{\alpha,n}\| \leq 1$ if $\langle \alpha, \alpha \rangle \leq 1$. In particular, Y_α satisfies the linear energy bound for such α . It is also shown that for any α , $Y_\alpha(z)$ satisfies a polynomial energy bound.

Wightman fields Let G be a subgroup of \mathbb{R} . Then, for a fixed function h with $\text{supp } h \subset I$, we can choose a family of automorphisms σ_α parametrized by $\alpha \in G$ such that, by omitting the dependence on h , $\sigma_\alpha \circ \sigma_\beta = \sigma_{\alpha+\beta}$. Their braidings satisfy $\epsilon^\pm(\sigma_\alpha, \sigma_\beta) = \epsilon^\pm(\sigma_\beta, \sigma_\alpha) = \epsilon^\pm(\sigma_{-\alpha}, \sigma_{-\beta})$. Therefore, if we take $\kappa_L(\alpha) = \sigma_\alpha$, $\kappa_R(\alpha) = \sigma_{-\alpha}$, the objects satisfy the conditions of Sect. 4. We can also take $\kappa_L(\alpha) = \sigma_\alpha$, $\kappa_R(\alpha) = \sigma_{\pm\sqrt{\alpha^2+2\ell}}$ for any $\ell \in \mathbb{Z}$ such that $\alpha^2 + 2\ell > 0$.

Instead, let $G = \mathbb{Z}$ and $K = \{1, 2, \dots, |K|\}$ be a finite set. For $j \in K$, we take α_j such that $\sum_{j \in K} \alpha_j^2/2 = 1$, then $\kappa_j(n) = \sigma_{n\alpha_j}$ satisfy the conditions of Sect. 3, obtaining extensions of $(\mathcal{A}_{U(1)})^{\otimes j}$ as conformal nets on S^1 .

On the other hand, we have checked that the charged primary fields Y_α satisfy the conditions of Sect. 5.3 (without the index ξ). Moreover, if there is $\alpha \in G$, $|\alpha| \leq 1$, then $Y_{\alpha,s}$ are bounded, and hence the two-dimensional Wightman field as constructed

in Sect. 5.5 satisfies a linear energy bound, and generate the conformal net on \mathcal{E} . If G is a subgroup without such α , the extension given by G can be embedded with a larger net, where the field satisfy a linear energy bound. From this, it follows that the fields for $\alpha > 1$ strongly commute when smeared with spacelike separated test functions by an analogue of [CTW22, Lemma 3.6].

7. Outlook

This construction should apply also to loop group nets at level 1 [Was98, TL97]. The bosonic construction of charged primary fields of [TL97] should give two-dimensional Wightman fields by our construction.

There are a few works on two-dimensional extension of CFT in a language similar to that of vertex operator algebras, e.g., [HK07, CKM22, Mor23]. We plan to investigate the construction problems of Wightman fields with more generality, in particular, for loop group nets with higher levels and for Virasoro nets [KL04b]. Deforming CFT by a pair of currents [Mor23] should also have a similar realization in the Wightman setting, and it would be interesting to see whether it has a dynamical meaning, at least in the sense of perturbation theory, cf. [CRV22].

In principle, our fields should be Wick-rotated to an Euclidean theory, that should correspond to the works above. It would be interesting to understand these Euclidean models in terms of Hilbert spaces and operators, cf. [FFK89].

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A On Strong Locality

Let us prove a simple criterion for strong commutativity of operators satisfying linear energy bounds which follows from [GJ87, Theorem 19.4.4]. The arguments here are due to Sebastiano Carpi.

Let H be a positive self-adjoint operator, A a symmetric operator on $\text{Dom}(H)$ and assume that, $\|A\Psi\| \leq C\|H\psi\|$, $\|[H, A]\Psi\| \leq C\|H\psi\|$. We denote $R(\lambda) = (H + (\lambda + 1)\mathbb{1})^{-1}$, $R = R(0) = (H + \mathbb{1})^{-1}$, $\delta(A) = i[H, A]$, $\delta^k(A) = \delta(\underbrace{\delta(\cdots(\delta(A)\cdots)})}_{k\text{-times}})$. By the proof of [GJ87, Theorem 19.4.1], we have

$$\begin{aligned}
 R^{\frac{1}{2}} &= \frac{1}{\pi} \int_0^\infty R(\lambda)\lambda^{-\frac{1}{2}} d\lambda, \\
 [A, R^{\frac{1}{2}}] &= \frac{1}{\pi} \int_0^\infty R(\lambda)((H + (\lambda + 1)\mathbb{1})A - A(H + (\lambda + 1)\mathbb{1}))R(\lambda)\lambda^{-\frac{1}{2}} d\lambda \\
 &= -\frac{i}{\pi} \int_0^\infty R(\lambda)\delta(A)R(\lambda)\lambda^{-\frac{1}{2}} d\lambda.
 \end{aligned}$$

Note also that $\|(H + \mathbb{1})R(\lambda)\| = \|R(\lambda)(H + \mathbb{1})\| \leq 1$, $\|R(\lambda)\| \leq \frac{1}{1+\lambda}$ and $\|R^{-1}R(\lambda)\| = \|\frac{H+\mathbb{1}}{H+(\lambda+1)\mathbb{1}}\| \leq \frac{1}{\lambda+1} \leq 1$. Therefore, we find that

$$\begin{aligned}
 \|[A, R^{\frac{1}{2}}]\| &\leq \frac{1}{\pi} \int_0^\infty \frac{\lambda^{-\frac{1}{2}}}{1+\lambda} \|\delta(A)R(\lambda)\| d\lambda \\
 &\leq \frac{1}{\pi} \int_0^\infty \frac{\lambda^{-\frac{1}{2}}}{1+\lambda} \|\delta(A)R\| d\lambda \\
 &= \|\delta(A)R\|.
 \end{aligned}$$

This implies that

$$\|AR - R^{\frac{1}{2}}AR^{\frac{1}{2}}\| \leq \|[A, R^{\frac{1}{2}}]\| \cdot \|R^{\frac{1}{2}}\| \leq \|\delta(A)R\|.$$

In particular, if $\|AR\|$ is bounded, it follows that $\|R^{\frac{1}{2}}AR^{\frac{1}{2}}\| \leq \|AR\| + \|\delta(A)R\|$. If we apply this to $\delta(A)$ and $\delta^2(A) = -[H, [H, A]]$ instead of A , we obtain $\|R^{\frac{1}{2}}\delta(A)R^{\frac{1}{2}}\| \leq \|\delta(A)R\| + \|\delta^2(A)R\|$ and $\|R^{\frac{1}{2}}\delta^2(A)R^{\frac{1}{2}}\| \leq \|\delta^2(A)R\| + \|\delta^3(A)R\|$, respectively.

Let us cite [GJ87, Theorem 19.4.3] with $n = 1$ and [GJ87, Theorem 19.4.4]:

Theorem A.1. *The following hold.*

- Let H, A, R as above, and suppose that $R^{\frac{1}{2}}\delta(A)R^{\frac{1}{2}}$ and AR are bounded. Then A is essentially self-adjoint on any core of H .
- Let H, A, B, R as above, and suppose that $R^{\frac{1}{2}}\delta(A)R^{\frac{1}{2}}, R^{\frac{1}{2}}\delta(B)R^{\frac{1}{2}}, AR, BR, R^{\frac{1}{2}}\delta^2(A)R^{\frac{1}{2}}, R^{\frac{1}{2}}\delta^2(B)R^{\frac{1}{2}}, \delta(A)R, \delta(B)R$ are bounded. Suppose furthermore that AB, BA are defined on $\text{Dom}(H)$ and $AB = BA$. Then A, B are essentially self-adjoint on any core of H and their closures commute strongly.

Combining these observations, we have the following result.

Theorem A.2. *The following hold.*

- Let H, A as above, and suppose that the operators $\|A\Psi\| \leq C\|(H + \mathbb{1})\Psi\|$, $\|[H, A]\Psi\| \leq C\|(H + \mathbb{1})\Psi\|$, $\|[H, [H, A]]\Psi\| \leq C\|(H + \mathbb{1})\Psi\|$. Then A is essentially self-adjoint on any core of H .
- Let H, A, B, R as above, and suppose that the operators
 - $A, \delta(A) = i[H, A], \delta^2(A) = -[H, [H, A]], \delta^3(A) = -i[H, [H, [H, A]]]$
 - $B, \delta(B) = i[H, B], \delta^2(B) = -[H, [H, B]], \delta^3(B) = -i[H, [H, [H, B]]]$
 are defined on $\text{Dom}(H)$ and
 - $\|A\Psi\| \leq C\|(H + \mathbb{1})\Psi\|, \|\delta^k(A)\Psi\| \leq C\|(H + \mathbb{1})\Psi\|, k = 1, 2, 3$
 - $\|B\Psi\| \leq C\|(H + \mathbb{1})\Psi\|, \|\delta^k(B)\Psi\| \leq C\|(H + \mathbb{1})\Psi\|, k = 1, 2, 3$
 Suppose furthermore that AB, BA are defined on $\text{Dom}(H)$ and $AB = BA$. Then A, B are essentially self-adjoint on any core of H and their closures commute strongly.

Proof. By the hypothesis $\|A\Psi\| \leq C\|(H + \mathbb{1})\Psi\|, \|\delta(A)\Psi\| \leq C\|(H + \mathbb{1})\Psi\|$, we have that $\|AR\| \leq C, \|\delta(A)R\| \leq C$, therefore, by the observation above, $\|R^{\frac{1}{2}}AR^{\frac{1}{2}}\| \leq 2C$. Applying the same argument to $\delta(A)$ with the hypothesis $\|\delta^2(A)\Psi\| \leq C\|(H + \mathbb{1})\Psi\|$, we obtain $\|R^{\frac{1}{2}}\delta(A)R^{\frac{1}{2}}\| \leq 2C$. This and the first assertion of Theorem A.1 (with C replaced by $2C$) complete the proof of the first assertion.

As for the second assertion, we use $\|\delta^3(A)R\| \leq C$ to infer that the boundedness of $R^{\frac{1}{2}}\delta^2(A)R^{\frac{1}{2}}$, which is in the assumption of the second assertion of Theorem A.1. We have analogously the bounds for operators involving B , therefore, the second assertion of Theorem A.1 applies. \square

This should be compared with some other formulations of the commutator theorem, e.g., [RS75, Theorem X.37] which assumes that $\|A\psi\| \leq C\|H\psi\|$ and $|\langle\psi, [H, A]\psi\rangle| \leq C\langle\psi, H\psi\rangle$ (these assumptions are very similar to that of Theorem A.1, the first assertion) and proves that A is essentially self-adjoint on any core of H . In general, for a closable operator $B, \|B\psi\| \leq C\|H\psi\|$ for all ψ does not imply⁸ $|\langle\psi, B\psi\rangle| \leq C\langle\psi, H\psi\rangle$, therefore, we cannot infer the essential self-adjointness of A from just from $\|A\Psi\| \leq C\|H\Psi\|, \|[H, A]\Psi\| \leq C\|(H + \mathbb{1})\Psi\|$, but we need a bound on the higher commutators.

It is also possible to have weaker assumptions on the commutators, for example assuming bounds as quadratic forms [DF77], but then one must be careful with the domains, see [Tan16, Appendix C].

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⁸ A counterexample on \mathbb{C}^2 is $H = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 4 \\ -1 & 4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} H, \psi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

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