



Stability of Rotating Gaseous Stars

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Abstract: We consider stability of rotating gaseous stars modeled by the Euler–Poisson system with general equation of states. When the angular velocity of the star is Rayleigh stable, we proved a sharp stability criterion for axi-symmetric perturbations. We also obtained estimates for the number of unstable modes and exponential trichotomy for the linearized Euler–Poisson system. By using this stability criterion, we proved that for a family of slowly rotating stars parameterized by the center density with fixed angular velocity, the turning point principle is not true. That is, unlike the case of non-rotating stars, the change of stability of the rotating stars does not occur at extrema points of the total mass. By contrast, we proved that the turning point principle is true for the family of slowly rotating stars with fixed angular momentum distribution. When the angular velocity is Rayleigh unstable, we proved linear instability of rotating stars. Moreover, we gave a complete description of the spectra and sharp growth estimates for the linearized Euler–Poisson system.

1. Introduction

Consider a self-gravitating gaseous star modeled by the Euler–Poisson system of compressible fluids

$$\begin{cases} \rho_t + \nabla \cdot (\rho v) = 0, \\ \rho (v_t + v \cdot \nabla v) + \nabla p = -\rho \nabla V, \\ \Delta V = 4\pi\rho, \quad \lim_{|x| \rightarrow \infty} V(t, x) = 0, \end{cases} \quad (1.1)$$

where $x \in \mathbb{R}^3$, $t > 0$, $\rho(x, t) \geq 0$ is the density, $v(x, t) \in \mathbb{R}^3$ is the velocity, $p = P(\rho)$ is the pressure and V is the self-consistent gravitational potential. Assume $P(\rho)$ satisfies:

$$P(s) = C^1(0, \infty), \quad P' > 0, \quad (1.2)$$

and there exists $\gamma_0 \in (\frac{6}{5}, 2)$ such that

$$\lim_{s \rightarrow 0^+} s^{1-\gamma_0} P'(s) = K > 0. \quad (1.3)$$

The assumption (1.3) implies that the pressure $P(\rho) \approx K\rho^{\gamma_0}$ for ρ near 0. For physical equations of states of Newtonian stars such as white dwarf stars and in the limiting case of extreme nonrelativistic, $\gamma_0 = \frac{5}{3}$ (see [6,37]).

The Euler–Poisson system (1.1) has many steady solutions. The simplest one is the spherically symmetric non-rotating star with $(\rho_0, v_0) = (\rho_0(|x|), 0)$. We refer to [29] and references therein for the existence and stability of non-rotating stars. A turning point principle (TPP) was shown in [29] that the stability of the non-rotating stars is entirely determined by the mass-radius curve parameterized by the center density. In particular, the stability of a non-rotating star can only change at extrema (i.e. local maximum or minimum points) of the total mass.

We consider axi-symmetric rotating stars of the form

$$(\rho_0, \vec{v}_0) = (\rho_0(r, z), r\omega_0(r) \mathbf{e}_\theta),$$

where (r, θ, z) are the cylindrical coordinates, $\omega_0(r)$ is the angular velocity and $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$ denote unit vectors along r, θ, z directions. We note that for barotropic equation of states $P = P(\rho)$, it was known as Poincaré-Wavre theorem [45, Section 4.3] that the angular velocity must be independent of z . The existence and stability of rotating stars is a classical problem in astrophysics. For homogeneous (i.e. constant density) rotating stars, it had been extensively investigated since the work of Maclaurin in 1740s, by many people including Dirichlet, Jacobi, Riemann, Poincaré and Chandrasekhar etc. We refer to the books [7,20] for history and results on this topic. The compressible rotating stars are much less understood. From 1920s, Lichtenstein initiated a mathematical study of compressible rotating stars, which was summarized in his monograph [26]. In particular, he showed the existence of slowly rotating stars near non-rotating stars by implicit function theorem. See also [14,17,18,41] for related results. The existence of rotating stars can also be established by variational methods ([2,5,9–11,25,30,32]), or global bifurcation theory [1,42,43]. Compared with the existence theory, there has been relatively few rigorous works on the stability of rotating stars. In this paper, we consider the stability of rotating stars under axi-symmetric perturbations. There are two natural questions to address: 1) Does TPP still hold for a family of rotating stars? 2) How does the rotation affect the stability (instability) of rotating stars?

The answers to these two questions have been disputed in astrophysical literature. Bisnovaty-Kogan and Blinnikov [4] suggested that for a family of rotating stars with fixed angular momentum distribution per unit mass and parameterized by the center density μ , TPP is true (i.e. stability changes at the extrema of the total mass). They used heuristic arguments (so called static method) as in the non-rotating case. Such arguments suppose that at the transition point of stability, there must exist a zero frequency mode which can only be obtained by infinitesimally transforming equilibrium configurations near the given one, without changing the total mass $M(\mu)$. Hence, the transition point is a critical point of the total mass (i.e. $M'(\mu) = 0$). It is reasonable to study the family of rotating stars with fixed angular momentum distribution, which is invariant under the Euler–Poisson dynamics. In [4], they also considered a family of rigidly rotating stars (i.e. ω_0 is constant) for a special equation of state similar to white dwarf stars. By embedding each rigidly rotating star into a family with the same angular momentum distribution and with some numerical help, it was found that the transition of stability

is not at the extrema of mass. In [40], for a family of rotating stars with fixed rotational parameter (i.e. the ratio of rotational energy to gravitational energy), similar arguments as in [4] were used to indicate that TPP is true for this family and their numerical results suggested that instability occurs beyond the first mass extrema. However, up to date there is no rigorous proof or disproof of TPP for different families of rotating stars.

The issue that whether rotation can have a stabilizing effect on rotating stars has long been in debate. For a long time, it was believed that rotation is stabilizing for any angular velocity profile. This conviction was based on conclusions drawn from perturbation analysis near neutral modes of non-rotating stars, which was done by Ledoux [24] for rigidly rotating stars and by Lebovitz [23] for general angular velocities. However, the later works of Sidorov [38,39] and Kähler [21] showed that rotation could be destabilizing. Hazlehurst [13] argued that the advocates of destabilization of rotation had used an argument that is open to criticism and disagreed that rotation could be destabilizing.

In this paper, we answer the above two questions in a rigorous way. To state our results more precisely, we introduce some notations. Let $(\rho_0(r, z), \vec{v}_0 = r\omega_0(r)\mathbf{e}_\theta)$ be an axi-symmetric rotating star solution of (1.1). The support of ρ_0 is denoted by Ω , which is an axi-symmetric bounded domain. The rotating star solutions satisfy

$$\vec{v}_0 \cdot \nabla \vec{v}_0 + \nabla \Phi'(\rho_0) + \nabla V = 0 \text{ in } \Omega, \tag{1.4}$$

$$V = -|x|^{-1} * \rho_0 \text{ in } \mathbb{R}^3, \tag{1.5}$$

equivalently,

$$\Phi'(\rho_0) - |x|^{-1} * \rho_0 - \int_0^r \omega_0^2(s)s \, ds + c_0 = 0 \text{ in } \Omega, \tag{1.6}$$

where $c_0 > 0$ is a constant and the enthalpy $\Phi(\rho) > 0$ is defined by

$$\Phi(0) = \Phi'(0) = 0, \quad \Phi'(\rho) = \int_0^\rho \frac{P'(s)}{s} ds.$$

Let R_0 be the maximum of r such that $(r, z) \in \Omega$. We assume $\omega_0 \in C^1[0, R_0]$, $\partial\Omega$ is C^2 with positive curvature near $(R_0, 0)$, and for any (r, z) near $\partial\Omega$

$$\rho_0(r, z) \approx \text{dist}((r, z), \partial\Omega)^{\frac{1}{\gamma_0-1}}, \tag{1.7}$$

which are satisfied for slowly rotating stars near non-rotating stars as constructed in [14,17,18,41]. Let $X = L^2_{\Phi''(\rho_0)} \times L^2_{\rho_0}$ and $Y = (L^2_{\rho_0})^2$, where $L^2_{\Phi''(\rho_0)}$ and $L^2_{\rho_0}$ are axi-symmetric weighted L^2 spaces in Ω with weights $\Phi''(\rho_0)$ and ρ_0 . Denote $\mathbf{X} := X \times Y$. Define the Rayleigh discriminant $\Upsilon(r) = \frac{\partial_r(\omega_0^2 r^4)}{r^3}$.

For Rayleigh stable angular velocity ω_0 satisfying $\Upsilon(r) > 0$ for $r \in [0, R_0]$, the linearization of the axi-symmetric Euler–Poisson system at (ρ_0, \vec{v}_0) can be written in a Hamiltonian form

$$\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{JL} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{1.8}$$

where $u_1 = (\rho, v_\theta)$ and $u_2 = (v_r, v_z)$, and $\rho, (v_r, v_\theta, v_z)$ are perturbations of density and (r, θ, z) -components of velocity respectively. In the linearized Euler–Poisson system, the steady density ρ_0 and the perturbations $\delta\rho$ have the same support. This is

reasonable because of the underlying Lagrangian formulation of the problem (see the Appendix in [29] for the derivation of the linearized Euler–Poisson system for non-rotating stars). The operators

$$\mathbf{J} := \begin{pmatrix} 0, & B \\ -B', & 0 \end{pmatrix} : \mathbf{X}^* \rightarrow \mathbf{X}, \quad \mathbf{L} := \begin{pmatrix} \mathbb{L}, & 0 \\ 0, & A \end{pmatrix} : \mathbf{X} \rightarrow \mathbf{X}^*, \tag{1.9}$$

are off-diagonal anti-self-dual and diagonal self-dual operators respectively, where

$$\mathbb{L} = \begin{pmatrix} L & 0 \\ 0 & A_1 \end{pmatrix} : X \rightarrow X^*, \tag{1.10}$$

with

$$L = \Phi''(\rho_0) - 4\pi(-\Delta)^{-1}, \tag{1.11}$$

$$B = (B_1, B_2)^T, \quad B_1 = -\nabla \cdot, \quad B_2 = -\frac{\partial_r(\omega_0 r^2)}{r\rho_0} \mathbf{e}_r, \tag{1.12}$$

$A = \rho_0$, and $A_1 = \frac{4\omega_0^2 r^3 \rho_0}{\partial_r(\omega_0^2 r^4)} = \frac{4\omega_0^2 \rho_0}{\Upsilon(r)}$. More precise definition and properties of these operators can be found in Sect. 2.2.

Our main result for the Rayleigh stable case is the following.

Theorem 1.1. *Assume $\omega_0 \in C^1[0, R_0]$, $\Upsilon(r) > 0$, (1.7), $\partial\Omega$ is C^2 and has positive curvature near $(R_0, 0)$. Then the operator \mathbf{JL} defined by (1.9) generates a C^0 group $e^{t\mathbf{JL}}$ of bounded linear operators on $\mathbf{X} = X \times Y$ and there exists a decomposition*

$$\mathbf{X} = E^u \oplus E^c \oplus E^s,$$

of closed subspaces $E^{u,s,c}$ satisfying the following properties:

i) E^c, E^u, E^s are invariant under $e^{t\mathbf{JL}}$.

ii) E^u (E^s) only consists of eigenvectors corresponding to positive (negative) eigenvalues of \mathbf{JL} and

$$\dim E^u = \dim E^s = n^- \left(\mathbb{L}|_{\overline{R(B)}} \right) = n^- \left(\mathcal{K}|_{R(B_1)} \right),$$

where $\langle \mathcal{K} \cdot, \cdot \rangle$ is a bounded bilinear quadratic form on $L^2_{\Phi''(\rho_0)}$ defined by

$$\langle \mathcal{K}\delta\rho, \delta\rho \rangle = \langle L\delta\rho, \delta\rho \rangle + 2\pi \int_0^{R_0} \Upsilon(r) \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds \right)^2}{r \int_{-\infty}^{+\infty} \rho_0(r, z) dz} dr, \tag{1.13}$$

for any $\delta\rho \in L^2_{\Phi''(\rho_0)}$ and $n^- \left(\mathcal{K}|_{R(B_1)} \right)$ denotes the number of negative modes of $\langle \mathcal{K} \cdot, \cdot \rangle$ restricted to the subspace

$$R(B_1) = \left\{ \delta\rho \in L^2_{\Phi''(\rho_0)} \mid \int_{\Omega} \delta\rho dx = 0 \right\}. \tag{1.14}$$

iii) The exponential trichotomy is true in the space \mathbf{X} in the sense of (2.2) and (2.3).

Remark 1.1. The perturbations $\delta\rho(r, z) \in L^2_{\Phi''(\rho_0)}$ and $\delta v_\theta(r, z) \in L^2_{\rho_0}$ imply that $\text{supp}(\delta\rho, \delta v_\theta) \subset \bar{\Omega}$. But for convenience of notations, we use $\int_{-\infty}^{+\infty} \bullet dz$ for $\int_{-Z(r)}^{+Z(r)} \bullet dz$ where $(r, \pm Z(r)) \in \partial\Omega$, “ \bullet ” represents an integrand supported in Ω . Here, it should be understood that the integrand function is extended by 0 outside the support Ω .

Corollary 1.1. *Under the assumptions of Theorem 1.1, the rotating star solution (ρ_0, \vec{v}_0) is spectrally stable to axi-symmetric perturbations if and only if*

$$\langle \mathcal{K}\delta\rho, \delta\rho \rangle \geq 0,$$

for all $\delta\rho \in L^2_{\Phi''(\rho_0)}$ with $\int_{\Omega} \delta\rho dx = 0$.

Theorem 1.1 gives not only a sharp stability criteria for rotating stars with Rayleigh stable angular velocity, but also more detailed information on the spectra of the linearized Euler–Poisson operator and exponential trichotomy estimates for the linearized Euler–Poisson system. These will be useful for the future study of nonlinear dynamics near unstable rotating stars, particularly, the construction of invariant (stable, unstable and center) manifolds for the nonlinear Euler–Poisson system.

The sharp stability criterion in Corollary 1.1 is used to study the stability of two families of slowly rotating stars. For the first family of slowly rotating stars with fixed Rayleigh stable angular velocity and parameterized by the center density, we show that TPP is not true and the transition of stability does not occur at the first mass extrema. More precisely, for fixed $\omega_0(r) \in C^{1,\beta}$ ($\beta \in (0, 1)$), satisfying $\Upsilon(r) > 0$ and κ small enough, by implicit function theorem as in [14, 18, 41], there exists a family of slowly rotating stars $(\rho_{\mu,\kappa}, \kappa r \omega_0(r) \mathbf{e}_\theta)$ parameterized by the center density μ . We show that the transition of stability for this family is not at the first extrema of the total mass $M_{\mu,\kappa}$. In particular, when $\gamma_0 > \frac{4}{3}$, the slowly rotating stars are stable for small center density and remain stable slightly beyond the first mass maximum. This is consistent with the numerical evidence in [4] (Figure 10, p. 400) for the example of rigidly rotating stars and an equation of state with $\gamma_0 = \frac{5}{3}$. It shows that Rayleigh stable rotation is indeed stabilizing for rotating stars. By contrast, for the second family of slowly rotating stars with fixed monotone increasing angular momentum distribution (equivalently Rayleigh stable angular velocity), we show that TPP is indeed true. More precisely, for fixed $j(p, q) \in C^{1,\beta}(\mathbb{R}^+ \times \mathbb{R}^+)$ satisfying $\partial_p(j^2(p, q)) > 0$, $j(0, q) = \partial_p j(0, q) = 0$ and ε sufficiently small, there exists a family of slowly rotating stars $(\rho_{\mu,\varepsilon}, \frac{\varepsilon}{r} j(m_{\rho_{\mu,\varepsilon}}, M_{\mu,\varepsilon}) \mathbf{e}_\theta)$ parameterized by the center density μ , where

$$m_{\rho_{\mu,\varepsilon}}(r) = \int_0^r s \int_{-\infty}^{\infty} \rho_{\mu,\varepsilon}(s, z) ds dz$$

is the mass distribution in the cylinder and $M_{\mu,\varepsilon}$ is the total mass. We show that the transition of stability for this family of rotating stars exactly occurs at the first extrema of the total mass $M_{\mu,\varepsilon}$. This not only confirms the claim in [4] based on heuristic arguments when $j(m, M) = \frac{1}{M} j(\frac{m}{M})$, but also can apply to other examples studied in the literature, including $j(m, M) = j(m)$ (see [2, 18, 30, 31]) and $j(m, M) = j(\frac{m}{M})$ (see [34]).

The issue of TPP is also not so clear for relativistic rotating stars. For relativistic stars, TPP was shown for the secular stability (with dissipation) of a family of rigidly rotating stars ([12]), while numerical results in [44] indicated that the transition of dynamic instability (without dissipation) does not occur at the mass maximum (i.e. TPP

is not true) for such a family. Our approach for the Newtonian case might be useful for studying the relativistic case.

For the Rayleigh stable case, the stability of rotating stars is studied by using the separable Hamiltonian framework as in the non-rotating stars [29]. However, there are fundamental differences between these two cases. For the non-rotating stars, the stability condition is reduced to find $n^- (L|_{R(B_1)})$, that is, the number of negative modes of $\langle L \cdot, \cdot \rangle$ restricted to $R(B_1)$, where L and $R(B_1)$ are defined in (1.11) and (1.14) respectively. We note that the dynamically accessible space $R(B_1)$ (for density perturbation) is one co-dimensional with only the mass constraint. For the rotating stars, by using the separable Hamiltonian formulation (1.8), the stability is reduced to find $n^- (\mathbb{L}|_{\overline{R(B)}})$, where \mathbb{L}, B are defined in (1.10) and (1.12) respectively. Here, the dynamically accessible space $\overline{R(B)}$ (for density and θ -component of velocity) is infinite co-dimensional, which corresponds to perturbations preserving infinitely many generalized total angular momentum (2.11) in the first order. It is hard to compute the negative modes of $\langle \mathbb{L} \cdot, \cdot \rangle$ with such infinitely many constraints. A key point in our proof is to find a reduced functional \mathcal{K} defined in (1.13) for density perturbation such that $n^- (\mathbb{L}|_{\overline{R(B)}}) = n^- (\mathcal{K}|_{R(B_1)})$, where $R(B_1)$ denotes the density perturbations preserving the mass as in the non-rotating case. Therefore, the study of negative modes of $\mathbb{L}|_{\overline{R(B)}}$ with infinitely many constraints is reduced to study $\mathcal{K}|_{R(B_1)}$ with only one mass constraint. This reduced stability criterion in terms of $\mathcal{K}|_{R(B_1)}$ is crucial to prove or disprove TPP for different families of rotating stars.

Next we consider rotating stars with Rayleigh unstable angular velocity $\omega_0(r)$. That is, there exists a point $r_0 \in (0, R_0)$ such that $\Upsilon(r_0) = \frac{\partial_r(\omega_0^2 r^4)}{r^3} \Big|_{r=r_0} < 0$. In this case, we cannot write the linearized Euler–Poisson system as a separable linear Hamiltonian PDEs since $A_1 = \frac{4\omega_0^2 r^3 \rho_0}{\partial_r(\omega_0^2 r^4)}$ is not defined at r_0 . Instead, we use the following second order system for $u_2 = (v_r, v_z)$

$$\partial_{tt} u_2 = -(\mathbb{L}_1 + \mathbb{L}_2)u_2 := -\tilde{\mathbb{L}}u_2, \tag{1.15}$$

where $\tilde{\mathbb{L}} = \mathbb{L}_1 + \mathbb{L}_2$,

$$\begin{aligned} \mathbb{L}_1 u_2 &= \nabla[\Phi''(\rho_0)(\nabla \cdot (\rho_0 u_2)) - 4\pi(-\Delta)^{-1}(\nabla \cdot (\rho_0 u_2))], \\ \mathbb{L}_2 u_2 &= \begin{pmatrix} \Upsilon(r)v_r \\ 0 \end{pmatrix}, \end{aligned}$$

are self-adjoint operators on Y . The following properties of the spectra of $\tilde{\mathbb{L}}$ are obtained in Proposition 4.1: i) $\sigma_{ess}(\tilde{\mathbb{L}}) = range(\Upsilon(r)) = [-a, b]$, where $a > 0, b \geq 0$; ii) There are finitely many negative eigenvalues and infinitely many positive eigenvalues outside the interval $[-a, b]$. In particular, the infimum of $\sigma(\tilde{\mathbb{L}})$ is negative, which might correspond to either discrete or continuous spectrum.

Define the space

$$Z = \left\{ u_2 \in Y \mid \nabla \cdot (\rho_0 u_2) \in L^2_{\Phi''(\rho_0)} \right\},$$

with the norm

$$\|u_2\|_Z = \|u_2\|_Y + \|\nabla \cdot (\rho_0 u_2)\|_{L^2_{\Phi''(\rho_0)}}. \tag{1.16}$$

Theorem 1.2. Assume $\omega_0 \in C^1[0, R_0]$, (1.7) and $\inf_{r \in [0, R_0]} \Upsilon(r) < 0$. Let $\eta_0 \leq -a$ be

the minimum of $\lambda \in \sigma(\tilde{\mathbb{L}})$. Then we have:

i) Equation (1.15) defines a C^0 group $T(t)$, $t \in \mathbb{R}$, on $Z \times Y$. There exists $C > 0$ such that for any $(u_2(0), u_{2t}(0)) \in Z \times Y$,

$$\|u_2(t)\|_Z + \|u_{2t}(t)\|_Y \leq C e^{\sqrt{-\eta_0}t} (\|u_2(0)\|_Z + \|u_{2t}(0)\|_Y), \quad \forall t > 0. \quad (1.17)$$

The flow $T(t)$ conserves the total energy

$$E(u_2, u_{2t}) = \|u_{2t}\|_Y^2 + \langle \tilde{\mathbb{L}}u_2, u_2 \rangle. \quad (1.18)$$

ii) For any $\varepsilon \in (0, -\eta_0)$, there exists initial data $u_2^\varepsilon(0) \in Z, u_{2t}^\varepsilon(0) = 0$ such that

$$\|u_2^\varepsilon(t)\|_Y \gtrsim e^{\sqrt{-\eta_0-\varepsilon}t} \|u_2^\varepsilon(0)\|_Z, \quad \forall t > 0. \quad (1.19)$$

The above theorem shows that rotating stars with Rayleigh unstable angular velocity are always linearly unstable. The maximal growth rate is obtained either by a discrete eigenvalue beyond the range of $\Upsilon(r)$ or by unstable continuous spectrum due to Rayleigh instability (i.e. negative $\Upsilon(r)$). In [23], it was shown that for slowly rotating stars with any angular velocity profile, discrete unstable modes cannot be perturbed from neutral modes of non-rotating stars. However, the unstable continuous spectrum was not considered there.

We briefly mention some recent mathematical works on the stability of rotating gaseous stars. The conditional Lyapunov stability of some rotating star constructed by variational methods had been obtained by Luo and Smoller [30–33] under Rayleigh stability assumption, also called Sölberg stability criterion in their works.

The paper is organized as follows. In Sect. 2, we study rotating stars with Rayleigh stable angular velocity and prove the sharp stability criterion. In Sect. 3, we use the stability criterion to prove/disprove TPP for two families of slowly rotating stars. In Sect. 4, we prove linear instability of rotating stars with Rayleigh unstable angular velocity.

Throughout this paper, for $a, b > 0$ we use $a \lesssim b$ to denote the estimate $a \leq Cb$ for some constant C independent of a, b , $a \approx b$ to denote the estimate $C_1a \leq b \leq C_2b$ for some constants $C_1, C_2 > 0$ and $a \sim b$ to denote $|a - b| < \varepsilon$ for some $\varepsilon > 0$ small enough.

2. Stability Criterion for Rayleigh Stable Case

In this section, we consider rotating stars with Rayleigh stable angular velocity profiles. The linearized Euler–Poisson system is studied by using a framework of separable Hamiltonian systems in [29]. First, we give a summary of the abstract theory in [29].

2.1. *Separable linear Hamiltonian PDEs.* Consider a linear Hamiltonian PDEs of the separable form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & B \\ -B' & 0 \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{JL} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (2.1)$$

where $u \in X, v \in Y$ and X, Y are real Hilbert spaces. We briefly describe the results in [29] about general separable Hamiltonian PDEs (2.1). The triple (L, A, B) is assumed to satisfy assumptions:

- (G1) The operator $B : Y^* \supset D(B) \rightarrow X$ and its dual operator $B' : X^* \supset D(B') \rightarrow Y$ are densely defined and closed (and thus $B'' = B$).
- (G2) The operator $A : Y \rightarrow Y^*$ is bounded and self-dual (i.e. $A' = A$ and thus $\langle Au, v \rangle$ is a bounded symmetric bilinear form on Y). Moreover, there exist $\delta > 0$ such that

$$\langle Au, u \rangle \geq \delta \|u\|_Y^2, \quad \forall u \in Y.$$

- (G3) The operator $L : X \rightarrow X^*$ is bounded and self-dual (i.e. $L' = L$ etc.) and there exists a decomposition of X into the direct sum of three closed subspaces

$$X = X_- \oplus \ker L \oplus X_+, \quad \dim \ker L < \infty, \quad n^-(L) \triangleq \dim X_- < \infty,$$

satisfying

- (G3.a) $\langle Lu, u \rangle < 0$ for all $u \in X_- \setminus \{0\}$;
- (G3.b) there exists $\delta > 0$ such that

$$\langle Lu, u \rangle \geq \delta \|u\|^2, \quad \text{for any } u \in X_+.$$

We note that the assumptions $\dim \ker L < \infty$ and $A > 0$ can be relaxed (see [29]). But these simplified assumptions are enough for the applications to the Euler–Poisson system studied in this section under the Rayleigh stability assumption (i.e. $\Upsilon(r) > 0$ for all $r \in [0, R_0]$).

Theorem 2.1. [29] *Assume (G1–3) for (2.1). The operator \mathbf{JL} generates a C^0 group $e^{t\mathbf{JL}}$ of bounded linear operators on $\mathbf{X} = X \times Y$ and there exists a decomposition*

$$\mathbf{X} = E^u \oplus E^c \oplus E^s,$$

of closed subspaces $E^{u,s,c}$ with the following properties:

- i) E^c, E^u, E^s are invariant under $e^{t\mathbf{JL}}$.
- ii) E^u (E^s) only consists of eigenvectors corresponding to positive(negative) eigenvalues of \mathbf{JL} and

$$\dim E^u = \dim E^s = n^-\left(L|_{\overline{R(B)}}\right),$$

where $n^-\left(L|_{\overline{R(B)}}\right)$ denotes the number of negative modes of $\langle L\cdot, \cdot \rangle|_{\overline{R(B)}}$. If $n^-\left(L|_{\overline{R(B)}}\right) > 0$, then there exists $M > 0$ such that

$$\left|e^{t\mathbf{JL}}|_{E^s}\right| \leq M e^{-\lambda_u t}, \quad t \geq 0; \quad \left|e^{t\mathbf{JL}}|_{E^u}\right| \leq M e^{\lambda_u t}, \quad t \leq 0, \quad (2.2)$$

where $\lambda_u = \min\{\lambda \mid \lambda \in \sigma(\mathbf{JL}|_{E^u})\} > 0$.

- iii) The quadratic form $\langle \mathbf{L}\cdot, \cdot \rangle$ vanishes on $E^{u,s}$, i.e. $\langle \mathbf{L}\mathbf{u}, \mathbf{u} \rangle = 0$ for all $\mathbf{u} \in E^{u,s}$, but is non-degenerate on $E^u \oplus E^s$, and

$$E^c = \{\mathbf{u} \in \mathbf{X} \mid \langle \mathbf{L}\mathbf{u}, \mathbf{v} \rangle = 0, \quad \forall \mathbf{v} \in E^s \oplus E^u\}.$$

There exists $M > 0$ such that

$$\left|e^{t\mathbf{JL}}|_{E^c}\right| \leq M(1 + |t|)^3, \quad \text{for all } t \in \mathbb{R}. \quad (2.3)$$

- iv) Suppose $\langle L\cdot, \cdot \rangle$ is non-degenerate on $\overline{R(B)}$, then $\left|e^{t\mathbf{JL}}|_{E^c}\right| \leq M$ for some $M > 0$. Namely, there is Lyapunov stability on the center space E^c .

Remark 2.1. Above theorem shows that the solutions of (2.9) are spectrally stable if and only if $L|_{\overline{R(B)}} \geq 0$. Moreover, $n^- \left(L|_{\overline{R(B)}} \right)$ equals to the number of unstable modes. The exponential trichotomy estimates (2.2)–(2.3) are important in the study of nonlinear dynamics near an unstable steady state, such as the proof of nonlinear instability or the construction of invariant (stable, unstable and center) manifolds. The exponential trichotomy can be lifted to more regular spaces if the spaces $E^{\mu,s}$ have higher regularity. We refer to Theorem 2.2 in [28] for more precise statements.

2.2. Hamiltonian formulation of linearized EP system. Consider an axi-symmetric rotating star solution $(\rho_0(r, z), \vec{v}_0 = v_0 \mathbf{e}_\theta = r\omega_0(r) \mathbf{e}_\theta)$. The support of density ρ_0 is denoted by Ω , which is an axi-symmetric simply connected bounded domain. Let R_0 be support radius in r , that is, the maximum of r such that $(r, z) \in \Omega$. We choose the coordinate system such that $(R_0, 0) \in \partial\Omega$. We make the following assumptions:

- i) $\omega_0 \in C^1[0, R_0]$ satisfies the Rayleigh stability condition (i.e. $\Upsilon(r) > 0$ for $r \in [0, R_0]$);
- ii) $\partial\Omega$ is C^2 near $(R_0, 0)$ and has positive curvature (equivalently Ω is locally convex) at $(R_0, 0)$;
- iii) ρ_0 satisfies (1.7).

The following lemma will be used later.

Lemma 2.1. *Under Assumptions ii) and iii) above, for $\varepsilon > 0$ small enough we have*

$$\int_{-\infty}^{+\infty} \rho_0^\lambda(r, z) dz \approx (R_0 - r)^{\frac{\lambda}{\gamma_0 - 1} + \frac{1}{2}},$$

for any $\lambda > 0$ and $r \in (R_0 - \varepsilon, R_0)$.

Proof. By (1.7), we have

$$\int_{-\infty}^{+\infty} \rho_0^\lambda(r, z) dz \approx \int_{(r,z) \in \Omega} \text{dist}((r, z), \partial\Omega)^{\frac{\lambda}{\gamma_0 - 1}} dz.$$

First, we consider the case when Ω is the ball $\{r^2 + z^2 < R_0^2\}$. Then for r close to R_0

$$\begin{aligned} \int_{(r,z) \in \Omega} \text{dist}((r, z), \partial\Omega)^{\frac{\lambda}{\gamma_0 - 1}} dz &= 2 \int_0^{\sqrt{R_0^2 - r^2}} (R_0 - \sqrt{r^2 + z^2})^{\frac{\lambda}{\gamma_0 - 1}} dz \quad (2.4) \\ &\approx \int_0^{\sqrt{R_0^2 - r^2}} (R_0^2 - r^2 - z^2)^{\frac{\lambda}{\gamma_0 - 1}} dz \\ &= (R_0^2 - r^2)^{\frac{\lambda}{\gamma_0 - 1} + \frac{1}{2}} \int_0^1 (1 - u^2)^{\frac{\lambda}{\gamma_0 - 1}} du \\ &\approx (R_0 - r)^{\frac{\lambda}{\gamma_0 - 1} + \frac{1}{2}}. \end{aligned}$$

For general Ω , let $\frac{1}{r_0} > 0$ be the curvature of $\partial\Omega$ at $(R_0, 0)$ and

$$\Gamma = \left\{ (r, z) \mid (r - R_0 + r_0)^2 + z^2 = r_0^2 \right\}$$

be the osculating circle at $(R_0, 0)$. Then near $(R_0, 0)$, $\partial\Omega$ is approximated by Γ to the 2nd order. For any $r \in (R_0 - \varepsilon, R_0)$, let $(r, -z_1(r)), (r, z_2(r))$ be the intersection of $\partial\Omega$ with the vertical line $r' = r$, where $z_1(r), z_2(r) > 0$. Then for ε small enough, we have

$$z_1(r), z_2(r) = \sqrt{r_0^2 - (r - R_0 + r_0)^2} + o\left(\sqrt{r_0^2 - (r - R_0 + r_0)^2}\right).$$

For $(r, z) \in \Omega$ with $r \in (R_0 - \varepsilon, R_0)$,

$$\begin{aligned} \text{dist}((r, z), \partial\Omega) &= \text{dist}((r, z), \Gamma) + o(\text{dist}((r, z), \Gamma)) \\ &= \left(r_0 - \sqrt{(r - R_0 + r_0)^2 + z^2}\right) + o\left(\left(r_0 - \sqrt{(r - R_0 + r_0)^2 + z^2}\right)\right). \end{aligned}$$

Then similar to (2.4), we have

$$\int_{-\infty}^{+\infty} \rho_0^\lambda(r, z) dz \approx \left(r_0^2 - (r - R_0 + r_0)^2\right)^{\frac{\lambda}{\gamma_0 - 1} + \frac{1}{2}} \approx (R_0 - r)^{\frac{\lambda}{\gamma_0 - 1} + \frac{1}{2}}.$$

□

Let $X_1 := L^2_{\Phi''(\rho_0)}$, $X_2 := L^2_{\rho_0}$, $X := X_1 \times X_2$, $Y := (L^2_{\rho_0})^2$ and $\mathbf{X} := X \times Y$.

The linearized Euler–Poisson system for axi-symmetric perturbations around the rotating star solution $(\rho_0(r, z), \omega_0(r) r \mathbf{e}_\theta)$ is

$$\begin{cases} \partial_t v_r &= 2\omega_0(r) v_\theta - \partial_r(\Phi''(\rho_0)\rho + V(\rho)), \\ \partial_t v_z &= -\partial_z(\Phi''(\rho_0)\rho + V(\rho)), \\ \partial_t v_\theta &= -\frac{1}{r}\partial_r(\omega_0 r^2)v_r, \\ \partial_t \rho &= -\nabla \cdot (\rho_0 v) = -\nabla \cdot (\rho_0(v_r, 0, v_z)), \end{cases} \tag{2.5}$$

with $\Delta V = 4\pi\rho$. Here, $(\rho, \vec{v} = (v_r, v_\theta, v_z)) \in \mathbf{X}$ are perturbations of density and velocity.

Define the operators

$$L := \Phi''(\rho_0) - 4\pi(-\Delta)^{-1} : X_1 \rightarrow (X_1)^*, \quad A = \rho_0 : Y \rightarrow Y^*,$$

$$A_1 := \frac{4\omega_0^2 r^3 \rho_0}{\partial_r(\omega_0^2 r^4)} = \frac{4\omega_0^2 \rho_0}{\Upsilon(r)} : X_2 \rightarrow (X_2)^*,$$

and

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} : D(B) \subset Y^* \rightarrow X, \quad B' = (B'_1, B'_2) : X^* \supset D(B') \rightarrow Y, \tag{2.6}$$

where

$$B_1 \begin{pmatrix} v_r \\ v_z \end{pmatrix} = -\nabla \cdot (v_r, 0, v_z), \quad B'_1 \rho = \begin{pmatrix} \partial_r \rho \\ \partial_z \rho \end{pmatrix}, \tag{2.7}$$

and

$$B_2 \begin{pmatrix} v_r \\ v_z \end{pmatrix} = -\frac{\partial_r(\omega_0 r^2)}{r\rho_0} v_r, \quad (B_2)' v_\theta = \begin{pmatrix} -\frac{\partial_r(\omega_0 r^2)}{r\rho_0} v_\theta \\ 0 \end{pmatrix}. \tag{2.8}$$

Then the linearized Euler–Poisson system (2.5) can be written in a separable Hamiltonian form

$$\frac{d}{dt} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{JL} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{2.9}$$

where $u_1 = (\rho, v_\theta)$ and $u_2 = (v_r, v_z)$. The operators

$$\mathbf{J} := \begin{pmatrix} 0, & B \\ -B', & 0 \end{pmatrix} : \mathbf{X}^* \rightarrow \mathbf{X}, \quad \mathbf{L} := \begin{pmatrix} \mathbb{L}, & 0 \\ 0, & A \end{pmatrix} : \mathbf{X} \rightarrow \mathbf{X}^*,$$

are off-diagonal anti-self-dual and diagonal self-dual respectively, where

$$\mathbb{L} = \begin{pmatrix} L, & 0 \\ 0, & A_1 \end{pmatrix} : X \rightarrow X^*.$$

First, we check that (\mathbb{L}, A, B) in (2.9) satisfy the assumptions (G1)–(G3) for the abstract theory in Sect. 2.1. The assumptions (G1) and (G2) can be shown by the same arguments as in the proof of Lemma 3.5 in [29] and the fact that B_2 is bounded. The Rayleigh stability condition $\Upsilon(r) > 0$ implies that the operator A_1 is bounded, positive and self-dual. By the same proof of Lemma 3.6 in [29], we have the following lemma.

Lemma 2.2. *There exists a direct sum decomposition $X_1 = X_- \oplus \ker L \oplus X_+$ and $\delta_0 > 0$ such that:*

- i) $\dim(X_-), \dim \ker L < \infty$;
- ii) $L|_{X_-} < 0, L|_{X_+} \geq \delta_0$ and $X_- \perp X_+$ in the inner product of X_1 .

The assumption (G3) readily follows from above lemma. Therefore, we can apply Theorem 2.1 to the linearized Euler–Poisson system (2.9). This proves the conclusions in Theorem 1.1 except for the formula $n^- (\mathbb{L}|_{\overline{R(B)}}) = n^- (\mathcal{K}|_{R(B_1)})$, which will be shown later. Here, $\overline{R(B)}$ is the closure of $R(B)$ in X , and the operators B, B_1 are defined in (2.6)–(2.8).

Remark 2.2. In some literature [30–33], the Rayleigh stability condition is $\Upsilon(r) \geq 0$ for all $r \in [0, R_0]$. Here, we used the stability condition $\Upsilon(r) > 0$ for all $r \in [0, R_0]$ as in the astrophysical literature such as [4, 46]. If $\Upsilon(r) \geq 0$ for all $r \in [0, R_0]$ and $\Upsilon(r) = 0$ only at some isolated points, let $\Lambda(r, z) = \frac{4\omega_0^2 \rho_0}{\Upsilon(r)}$ and the operator $A_1 : L_\Lambda^2 \rightarrow (L_\Lambda^2)^*$ is bounded and positive. The linearized Euler–Poisson system can still be studied in the framework of separable Hamiltonian systems and similar results as in Theorem 1.1 can be obtained.

2.3. *Dynamically accessible perturbations.* By Theorem 1.1, the solutions of (2.9) are spectrally stable (i.e. nonexistence of exponentially growing solution) if and only if $\mathbb{L}|_{\overline{R(B)}} \geq 0$. More precisely, we have

Corollary 2.1. *Assume $\omega_0 \in C^1[0, R_0]$, (1.7), and $\inf_{r \in [0, R_0]} \Upsilon(r) > 0$. The rotating star solution $(\rho_0(r, z), \vec{v}_0 = r\omega_0(r) \mathbf{e}_\theta)$ of Euler–Poisson system is spectrally stable if and only if*

$$\langle L\delta\rho, \delta\rho \rangle + \langle A_1\delta v_\theta, \delta v_\theta \rangle \geq 0 \text{ for all } (\delta\rho, \delta v_\theta) \in \overline{R(B)}. \tag{2.10}$$

In this section, we discuss the physical meaning of above stability criterion by using the variational structure of the rotating stars.

For any solution (ρ, v) of the axi-symmetric Euler–Poisson system (1.1), define the angular momentum $j = v_\theta r$ and the generalized total angular momentum

$$A_g(\rho, v_\theta) = \int_{\mathbb{R}^3} \rho g(v_\theta r) dx, \tag{2.11}$$

for any function $g \in C^1(\mathbb{R})$.

Lemma 2.3. *For any $g \in C^1(\mathbb{R})$, the functional $A_g(\rho, v_\theta)$ is conserved for the Euler–Poisson system (1.1).*

Proof. First, we note that the angular momentum j is an invariant of the particle trajectory under the axi-symmetric force field $-\nabla V - \nabla\Phi'(\rho)$. Let $\varphi(x, t)$ be the flow map of the velocity field v with initial position x , and $J(x, t)$ be the Jacobian of φ . Then $\rho(\varphi(x, t), t) J(x, t) = \rho(x, 0)$ and

$$\begin{aligned} A_g(\rho, v_\theta)(0) &= \int_{\mathbb{R}^3} \rho(x, 0) g(j(x)) dx \\ &= \int_{\mathbb{R}^3} \rho(\varphi(x, t), t) J(x, t) g(j(\varphi(x, t))) dx \\ &= \int_{\mathbb{R}^3} \rho(y, t) g(j(y)) dy = A_g(\rho, v_\theta)(t). \end{aligned}$$

□

The steady state $(\rho_0, \omega_0 r \mathbf{e}_\theta)$ has the following variational structure. By the steady state equation (1.6), we have

$$\frac{1}{2}\omega_0^2 r^2 + \Phi'(\rho_0) - |x|^{-1} * \rho_0 + g_0(\omega_0 r^2) + c_0 = 0 \text{ in } \Omega, \tag{2.12}$$

where $c_0 > 0$ is the constant in (1.6) and $g_0 \in C^1(\mathbb{R})$ satisfies the equation

$$g_0'(\omega_0(r) r^2) = -\omega_0(r), \quad \forall r \in [0, R_0]. \tag{2.13}$$

The existence of g_0 satisfying (2.13) is ensured by the Rayleigh stability condition $\Upsilon(r) > 0$ which implies that $\omega_0(r) r^2$ is monotone to r . The equations (1.6) and (2.12) are equivalent since

$$g_0(\omega_0(r) r^2) = -\frac{1}{2}\omega_0^2 r^2 - \int_0^r \omega_0^2(s) s ds,$$

due to (2.13) and integration by parts. Denote the the total energy by

$$H(\rho, v) = \int_{\mathbb{R}^3} \frac{1}{2} \rho v^2 + \Phi(\rho) - \frac{1}{8\pi} |\nabla V|^2 dx, \quad \Delta V = 4\pi\rho,$$

which is conserved for the Euler–Poisson system (1.1). Define the energy-Casimir functional

$$H_c(\rho, v) = H(\rho, v) + c_0 \int_{\mathbb{R}^3} \rho dx + \int_{\mathbb{R}^3} \rho g_0(v_\theta r) dx,$$

where c_0 and g_0 are as in (2.12). Then $(\rho_0, \omega_0 r \mathbf{e}_\theta)$ is a critical point of $H_c(\rho, v)$, since

$$\begin{aligned} \langle DH_c(\rho_0, \omega_0 r \mathbf{e}_\theta), (\delta\rho, \delta v) \rangle &= \int_{\mathbb{R}^3} \left[\frac{1}{2} \omega_0^2 r^2 + \Phi'(\rho_0) + V(\rho_0) + c_0 + g_0(\omega_0 r^2) \right] \delta\rho dx \\ &\quad + \int_{\mathbb{R}^3} [\rho_0 \omega_0 r + \rho_0 g'_0(\omega_0 r^2) r] \delta v_\theta dx = 0 \end{aligned}$$

by equations (2.12) and (2.13). By direct computations,

$$\begin{aligned} &\langle D^2 H_c(\rho, v)[\rho_0, \omega_0 r \mathbf{e}_\theta](\delta\rho, \delta v), (\delta\rho, \delta v) \rangle \\ &= \int_{\mathbb{R}^3} (\Phi''(\rho_0) (\delta\rho)^2 - 4\pi(-\Delta^{-1} \delta\rho) \delta\rho + \rho_0 (\delta v_r)^2 + \rho_0 (\delta v_z)^2) dx \\ &\quad + \int_{\mathbb{R}^3} \rho_0 (1 + g''_0(\omega_0 r^2) r^2) (\delta v_\theta)^2 dx \\ &= \langle L \delta\rho, \delta\rho \rangle + \langle A_1 \delta v_\theta, \delta v_\theta \rangle + \langle A (\delta v_r, \delta v_z), (\delta v_r, \delta v_z) \rangle, \end{aligned} \tag{2.14}$$

where we used the identity

$$1 + g''_0(\omega_0 r^2) r^2 = 1 - \frac{\omega'_0 r^2}{\frac{d}{dr}(\omega_0 r^2)} = \frac{4\omega_0^2 r^3}{\frac{d}{dr}(\omega_0^2 r^4)} = \frac{4\omega_0^2}{\Upsilon(r)}.$$

The functional (2.14) is a conserved quantity of the linearized Euler–Poisson system due to the Hamiltonian structure (2.9). We note that the number of negative directions of (2.14) is given by $n^-(\mathbb{L})$.

Lemma 2.4. *It holds that*

$$R(B_1) = \overline{R(B_1)} = \left\{ \delta\rho \in L^2_{\Phi''(\rho_0)} \mid \int_{\Omega} \delta\rho dx = 0 \right\}.$$

Proof. Since $\ker B'_1 = \ker \nabla$ is spanned by constant functions, we have

$$\overline{R(B_1)} = (\ker B'_1)^\perp = \left\{ \delta\rho \in L^2_{\Phi''(\rho_0)} \mid \int_{\Omega} \delta\rho dx = 0 \right\}.$$

It remains to show $R(B_1) = \overline{R(B_1)}$ which is equivalent to show $R(B_1 A) = \overline{R(B_1 A)}$. By Lemma 3.15 in [29], we have the orthogonal decomposition

$$L^2_{\rho_0} = \ker(B_1 A) \oplus W,$$

where $W = \{w = \nabla p \in L^2_{\rho_0}\}$. For any $\delta\rho \in R(B_1A)$, by the proof of Lemma 3.15 in [29], there exists a unique gradient field $\nabla p \in L^2_{\rho_0}$ such that

$$B_1A\nabla p = \nabla \cdot (\rho_0\nabla p) = \delta\rho.$$

By Proposition 12 in [19], we have

$$\|\nabla p\|_{L^2_{\rho_0}} \lesssim \|\nabla \cdot (\rho_0\nabla p)\|_{L^2_{\Phi''(\rho_0)}} = \|\delta\rho\|_{L^2_{\Phi''(\rho_0)}}. \tag{2.15}$$

For any $u \in D(B_1A)$, let $v \in W$ be the projection of u to W . Then above estimate (2.15) implies that

$$\text{dist}(u, \ker(B_1A)) = \inf_{z \in \ker(B_1A)} \|u - z\|_{L^2_{\rho_0}} = \|v\|_{L^2_{\rho_0}} \lesssim \|B_1Au\|_{L^2_{\Phi''(\rho_0)}}.$$

By Theorem 5.2 in [22, P. 231], this implies that $R(B_1) = \overline{R(B_1)}$. □

Definition 2.1. The perturbation $(\delta\rho, \delta v_\theta) \in X$ is called dynamically accessible if $(\delta\rho, \delta v_\theta) \in \overline{R(B)}$.

In the next lemma, we give two equivalent characterizations of the dynamically accessible perturbations.

Lemma 2.5. For $(\delta\rho, \delta v_\theta) \in X$, the following statements are equivalent.

- (i) $(\delta\rho, \delta v_\theta) \in \overline{R(B)}$;
- (ii)

$$\int_{\Omega} g(\omega_0 r^2) \delta\rho \, dx + \int_{\Omega} \rho_0 r g'(\omega_0 r^2) \delta v_\theta \, dx = 0, \quad \forall g \in C^1(\mathbb{R}); \tag{2.16}$$

- (iii) $\int_{\Omega} \delta\rho \, dx = 0$ and for any $r \in [0, R_0]$,

$$\int_{-\infty}^{+\infty} \delta v_\theta \rho_0(r, z) \, dz = \frac{\partial_r(\omega_0 r^2)}{r^2} \int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) \, dz \, ds. \tag{2.17}$$

Proof. First, we show (i) and (ii) are equivalent. We have $\overline{R(B)} = (\ker B')^\perp$, where the dual operator $B' : X^* \rightarrow Y$ is defined in (2.7)–(2.8). Let (ρ, v_θ) be a C^1 function in $\ker B'$, then

$$B' \begin{pmatrix} \rho \\ v_\theta \end{pmatrix} = \begin{pmatrix} \partial_r \rho - \frac{\partial_r(\omega_0 r^2)}{r\rho_0} v_\theta \\ \partial_z \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $\partial_z \rho = 0$ and $\omega_0 r^2$ is monotone to r by the Rayleigh stability condition, we can write $\rho = g(\omega_0 r^2)$ for some function $g \in C^1$. Then $\partial_r \rho - \frac{\partial_r(\omega_0 r^2)}{r\rho_0} v_\theta = 0$ implies that $v_\theta = \rho_0 r g'(\omega_0 r^2)$. Thus $\ker B'$ is the closure of the set

$$\left\{ \left(g(\omega_0 r^2), \rho_0 r g'(\omega_0 r^2) \right), g \in C^1(\mathbb{R}) \right\},$$

in X^* . Therefore, $(\delta\rho, \delta v_\theta) \in \overline{R(B)} = (\ker B')^\perp$ if and only if (2.16) is satisfied.

Next, we show (ii) and (iii) are equivalent. If (ii) is satisfied, by choosing $g = 1$ we get $\int \delta\rho \, dx = 0$. Then by (2.16) and integration by parts, we have

$$\int_0^{R_0} \left[r^2 \int_{-\infty}^{+\infty} \delta v_\theta \rho_0(r, z) dz - \partial_r(\omega_0 r^2) \left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds \right) \right] g'(\omega_0 r^2) dr = 0.$$

which implies (2.17) since $g \in C^1(\mathbb{R})$ is arbitrary. On the other hand, by reversing the above computation, (ii) follows from (iii). \square

The statement (ii) above implies that for any $(\delta\rho, \delta v_\theta) \in \overline{R(B)}$, we have

$$\langle DA_g(\rho_0, \omega_0 r), (\delta\rho, \delta v_\theta) \rangle = 0,$$

where the generalized angular momentum A_g is defined in (2.11). That is, a dynamically accessible perturbation $(\delta\rho, \delta v_\theta)$ must lie on the tangent space of the hypersurface $A_g(\rho, v_\theta) = A_g(\rho_0, \omega_0(r)r\mathbf{e})$. Since g is arbitrary, this implies infinite many constraints for dynamically accessible perturbations. The stability criterion (2.10) implies that rotating stars are stable if and only if they are local minimizers of energy functional $H(\rho, v)$ under the constraints of fixed generalized angular momentum A_g for all g . This contrasts significantly with the case of non-rotating stars. It was shown in [29] that non-rotating stars are stable if and only if they are local minimizers of the energy functional under the only constraint of fixed total mass. The stability criterion (2.10) for rotating stars involves infinitely many constraints and is much more difficult to check. In the next section, we give an equivalent stability criterion in terms of a reduced functional (1.13) under only the mass constraint.

Remark 2.3. For non-rotating stars, the dynamically accessible perturbations are given by $R(B_1) = \overline{R(B_1)}$ which consists of the perturbations preserving the mass (see Lemma 2.4). For rotating stars, the dynamically accessible space $\overline{R(B)}$ is different from $R(B)$.

2.4. Reduced functional and the equivalent stability criterion. In this section, we prove the formula $n^-(\mathbb{L}|_{\overline{R(B)}}) = n^-(\mathcal{K}|_{R(B_1)})$ and complete the proof of Theorem 1.1.

Lemma 2.6. For any $\delta\rho \in R(B_1)$, define

$$u_\theta^{\delta\rho} = \frac{\partial_r(\omega_0 r^2) \int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds}{r^2 \int_{-\infty}^{+\infty} \rho_0(r, z) dz}. \tag{2.18}$$

Then $(\delta\rho, u_\theta^{\delta\rho}) \in \overline{R(B)}$ and $\|u_\theta^{\delta\rho}\|_{L^2_{\rho_0}} \lesssim \|\delta\rho\|_{L^2_{\Phi''(\rho_0)}}$.

Proof. We have

$$\begin{aligned} \|u_\theta^{\delta\rho}\|_{L^2_{\rho_0}}^2 &\lesssim \int_\Omega \rho_0 \left(\frac{\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds}{r \int_{-\infty}^{+\infty} \rho_0(r, z) dz} \right)^2 dx = 2\pi \int_0^{R_0} \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds \right)^2}{r \int_{-\infty}^{+\infty} \rho_0(r, z) dz} dr \\ &= 2\pi \int_0^{R_0-\varepsilon} \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds \right)^2}{r \int_{-\infty}^{+\infty} \rho_0(r, z) dz} dr + 2\pi \int_{R_0-\varepsilon}^{R_0} \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds \right)^2}{r \int_{-\infty}^{+\infty} \rho_0(r, z) dz} dr \\ &= I + II, \end{aligned}$$

where $\varepsilon > 0$ is chosen such that Lemma 2.1 holds. Since the function $h_1(r) = \int_{-\infty}^{+\infty} \rho_0(r, z) dz$ has a positive lower bound in $[0, R_0 - \varepsilon]$ and $h_2(r) = \int_{-\infty}^{+\infty} \frac{1}{\Phi''(\rho_0(r, z))} dz$ is bounded, by Hardy's inequality (see Lemma 3.21 in [29] or [3]) we have

$$\begin{aligned} I &\lesssim \int_0^{R_0-\varepsilon} r^{-2} \left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds \right)^2 dr \\ &\lesssim \int_0^{R_0-\varepsilon} r^2 \left(\int_{-\infty}^{+\infty} \delta\rho(r, z) dz \right)^2 dr \\ &\lesssim \int_0^{R_0-\varepsilon} r^2 \left(\int_{-\infty}^{+\infty} \Phi''(\rho_0) (\delta\rho)^2(r, z) dz \right) \left(\int_{-\infty}^{+\infty} \frac{1}{\Phi''(\rho_0(r, z))} dz \right) dr \\ &\lesssim \int_0^{R_0-\varepsilon} r \int_{-\infty}^{+\infty} \Phi''(\rho_0) (\delta\rho)^2(r, z) dz dr \lesssim \|\delta\rho\|_{L^2_{\Phi''(\rho_0)}}^2. \end{aligned}$$

By Hardy's inequality and Lemma 2.1, we have

$$\begin{aligned} II &= 2\pi \int_{R_0-\varepsilon}^{R_0} \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds \right)^2}{r \int_{-\infty}^{+\infty} \rho_0(r, z) dz} dr \\ &\lesssim \int_{R_0-\varepsilon}^{R_0} \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds \right)^2}{(R_0 - r)^{\frac{1}{\gamma_0-1} + \frac{1}{2}}} dr \\ &\lesssim \int_{R_0-\varepsilon}^{R_0} \left(\int_{-\infty}^{+\infty} \delta\rho(r, z) dz \right)^2 (R_0 - r)^{-\frac{1}{\gamma_0-1} + \frac{3}{2}} dr \\ &\lesssim \int_{R_0-\varepsilon}^{R_0} \left(\int_{-\infty}^{+\infty} \Phi''(\rho_0) (\delta\rho)^2 dz \right) \left(\int_{-\infty}^{+\infty} \frac{1}{\Phi''(\rho_0)} dz \right) (R_0 - r)^{-\frac{1}{\gamma_0-1} + \frac{3}{2}} dr \\ &\lesssim \int_{R_0-\varepsilon}^{R_0} \left(\int_{-\infty}^{+\infty} \Phi''(\rho_0) (\delta\rho)^2 dz \right) (R_0 - r) dr \\ &\lesssim \|\delta\rho\|_{L^2_{\Phi''(\rho_0)}}^2, \end{aligned}$$

where we used the estimate

$$\int_{-\infty}^{+\infty} \frac{1}{\Phi''(\rho_0)} dz \approx \int_{-\infty}^{+\infty} \rho_0^{2-\gamma_0} dz \approx (R_0 - r)^{\frac{2-\gamma_0}{\gamma_0-1} + \frac{1}{2}},$$

since $\Phi''(s) \approx s^{\gamma_0-2}$ for s small. This proves $\|u_\theta^{\delta\rho}\|_{L^2_{\rho_0}} \lesssim \|\delta\rho\|_{L^2_{\Phi''(\rho_0)}}$.

The statement $(\delta\rho, u_\theta^{\delta\rho}) \in \overline{R(B)}$ follows from Lemma 2.5 since $\int_\Omega \delta\rho dx = 0$ for $\delta\rho \in R(B_1)$ and $u_\theta^{\delta\rho}$ obviously satisfies (2.17). □

With the help of Lemma 2.6, we can finish the proof of Theorem 1.1.

Proof of Theorem 1.1. We only need to show $n^-(\mathbb{L}|_{\overline{R(B)}}) = n^-(\mathcal{K}|_{R(B_1)})$. First, we have

$$\left\langle \mathbb{L} \begin{pmatrix} \delta\rho \\ \delta v_\theta \end{pmatrix}, \begin{pmatrix} \delta\rho \\ \delta v_\theta \end{pmatrix} \right\rangle \geq \langle \mathcal{K} \delta\rho, \delta\rho \rangle, \quad \forall (\delta\rho, \delta v_\theta) \in \overline{R(B)}, \tag{2.19}$$

since

$$\begin{aligned}
 \langle A_1 \delta v_\theta, \delta v_\theta \rangle &= \int_\Omega \frac{4\omega_0^2}{\Upsilon(r)} \rho_0 (\delta v_\theta)^2 dx = 2\pi \int_0^{R_0} \frac{4\omega_0^2 r}{\Upsilon(r)} \int_{-\infty}^{+\infty} \rho_0 (\delta v_\theta)^2 dz dr \\
 &= 2\pi \int_0^{R_0} \frac{4\omega_0^2 r}{\Upsilon(r)} \int_{-\infty}^{+\infty} \rho_0 \left(u_\theta^{\delta\rho}\right)^2 dz dr \\
 &\quad + 2\pi \int_0^{R_0} \frac{4\omega_0^2 r}{\Upsilon(r)} \int_{-\infty}^{+\infty} \rho_0 \left(\delta v_\theta - u_\theta^{\delta\rho}\right)^2 dz dr \\
 &\geq 2\pi \int_0^{R_0} \frac{4\omega_0^2 r}{\Upsilon(r)} \int_{-\infty}^{+\infty} \rho_0 \left(u_\theta^{\delta\rho}\right)^2 dz dr \\
 &= 2\pi \int_0^{R_0} \Upsilon(r) \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds\right)^2}{r \int_{-\infty}^{+\infty} \rho_0(r, z) dz} dr.
 \end{aligned}$$

In the above, we used the observation that

$$\int_{-\infty}^{+\infty} \rho_0 \left(\delta v_\theta - u_\theta^{\delta\rho}\right) dz = \int_{-\infty}^{+\infty} \rho_0 \delta v_\theta dz - u_\theta^{\delta\rho}(r) \int_{-\infty}^{+\infty} \rho_0 dz = 0,$$

since

$$\int_{-\infty}^{+\infty} \rho_0 \delta v_\theta dz = u_\theta^{\delta\rho}(r) \int_{-\infty}^{+\infty} \rho_0 dz = \frac{\partial_r (\omega_0 r^2)}{r^2} \int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds$$

due to (2.17) and (2.18). Since $\delta\rho \in R(B_1)$, it follows from (2.19) that $n^-(\mathcal{K}|_{R(B_1)}) \geq n^-(\mathbb{L}|_{\overline{R(B)}})$. On the other hand, we also have $n^-(\mathcal{K}|_{R(B_1)}) \leq n^-(\mathbb{L}|_{\overline{R(B)}})$, since

$$\langle \mathcal{K} \delta\rho, \delta\rho \rangle = \left\langle \mathbb{L} \begin{pmatrix} \delta\rho \\ u_\theta^{\delta\rho} \end{pmatrix}, \begin{pmatrix} \delta\rho \\ u_\theta^{\delta\rho} \end{pmatrix} \right\rangle.$$

Thus $n^-(\mathcal{K}|_{R(B_1)}) = n^-(\mathbb{L}|_{\overline{R(B)}})$. This finishes the proof of Theorem 1.1. □

3. TPP for Slowly Rotating Stars

In this section, we use the stability criterion in Theorem 1.1 to study two families of slowly rotating stars parameterized by the center density.

3.1. The case of fixed angular velocity. In this subsection, we consider a family of slowly rotating stars with fixed angular velocity.

Under the assumptions (1.2)–(1.3), for some $\mu_{\max} > 0$, there exists a family of nonrotating stars with radially symmetric density $\rho_\mu(|x|)$ parametrized by the center density $\mu \in (0, \mu_{\max})$. We refer to [29] and references therein for such results. Let R_μ be the support radius of ρ_μ and $B_\mu = B(0, R_\mu)$ be the support of ρ_μ . The radial density ρ_μ satisfies

$$\Delta(\Phi'(\rho_\mu)) + 4\pi\rho_\mu = 0, \quad \text{in } B_\mu,$$

with $\rho_\mu(0) = \mu$. For the general equations of state satisfying (1.2)–(1.3) with $\gamma_0 \geq 4/3$, it was shown in [15] that $\mu_{\max} = +\infty$.

Let $\omega(r) \in C^{1,\beta}[0, \infty)$ be fixed for some $\beta \in (0, 1)$. We consider a family of rotating stars for the Euler–Poisson system with the following form

$$\begin{cases} \rho_0 = \rho_{\mu,\kappa}(r, z) = \rho_\mu(g_{\zeta_{\mu,\kappa}}^{-1}((r, z))), \\ \vec{v}_0 = \kappa r \omega(r) \mathbf{e}_\theta, \end{cases}$$

where the dilating function is

$$g_{\zeta_{\mu,\kappa}} = x \left(1 + \frac{\zeta_{\mu,\kappa}}{|x|^2} \right),$$

and $\zeta_{\mu,\kappa}(x) : B_\mu \rightarrow \mathbb{R}$ is axi-symmetric and even in z .

The governing equation of rotating stars $(\rho_{\mu,\kappa}, \kappa r \omega(r) \mathbf{e}_\theta)$ turns out to be:

$$\begin{aligned} -\kappa^2 \int_0^r \omega^2(s) s ds + \Phi'(\rho_{\mu,\kappa}) + V_{\mu,\kappa} + c_{\mu,\kappa} &= 0 \text{ in } \Omega_{\mu,\kappa}, \\ V_{\mu,\kappa} &= -|x|^{-1} * \rho_{\mu,\kappa} \text{ in } \mathbb{R}^3, \end{aligned} \tag{3.1}$$

where $c_{\mu,\kappa}$ is a constant and $\Omega_{\mu,\kappa} = g_{\zeta_{\mu,\kappa}}(B_\mu)$ is the support of the density $\rho_{\mu,\kappa}$ of the rotating star solution.

Remark 3.1. We consider rotating stars near a family of non-rotating stars parameterized by center density. In the literature, the constructions are mostly near a non-rotating star with a fixed center density [14, 18] or a fixed total mass [41].

By similar arguments as in [14, 18, 41], we can get the following existence theorem.

Theorem 3.1. *Let $\mu \in [\mu_0, \mu_1] \subset (0, \mu_{\max})$, $P(\rho)$ satisfy (1.2)–(1.3), and $\omega(r) \in C^{1,\beta}[0, \infty)$. Then there exist $\tilde{\kappa} > 0$ and solutions $\rho_{\mu,\kappa}$ of (3.1) for all $|\kappa| < \tilde{\kappa}$, satisfying the following properties:*

- 1) $\rho_{\mu,\kappa} \in C_c^{1,\alpha}(\mathbb{R}^3)$, where $\alpha = \min(\frac{2-\gamma_0}{\gamma_0-1}, 1)$.
- 2) $\rho_{\mu,\kappa}$ is axi-symmetric and even in z .
- 3) $\rho_{\mu,\kappa}(0) = \mu$.
- 4) $\rho_{\mu,\kappa} \geq 0$ has compact support $g_{\zeta_{\mu,\kappa}}(B_\mu)$.
- 5) For all $\mu \in [\mu_0, \mu_1]$, the mapping $\kappa \rightarrow \rho_{\mu,\kappa}$ is continuous from $(-\tilde{\kappa}, \tilde{\kappa})$ into $C_c^1(\mathbb{R}^3)$.

When $\kappa = 0$, $\rho_{\mu,0} = \rho_\mu(|x|)$ is the nonrotating star solution with $\rho_\mu(0) = \mu$.

Now we use Theorem 1.1 to study the stability of above rotating star solutions $(\rho_{\mu,\kappa}, \kappa \omega(r) r \mathbf{e}_\theta)$, for $\mu \in [\mu_0, \mu_1]$, κ small enough, and $\omega \in C^{1,\beta}[0, \infty)$ satisfying the Rayleigh condition $\Upsilon(r) := \frac{\partial_r(\omega^2 r^4)}{r^3} > 0$. First, we check the assumptions in Theorem 1.1. Let $R_{\mu,\kappa}$ be the support radius in r for $\Omega_{\mu,\kappa} = g_{\zeta_{\mu,\kappa}}(B_\mu)$. Since $g_{\zeta_{\mu,\kappa}} \in C^2(B_\mu)$ depends continuously on κ , it is easy to check the assumptions on $\Omega_{\mu,\kappa}$ for κ small enough. That is, $\partial\Omega_{\mu,\kappa}$ is C^2 and has positive curvature near $(R_{\mu,\kappa}, 0)$. Next, we check the assumption (1.7). For nonrotating stars, it is known ([6, 16, 27, 29]) that

$$\rho_\mu(r, z) \approx ((R_\mu - \sqrt{r^2 + z^2})^{\frac{1}{\gamma_0-1}}) \text{ for } \sqrt{r^2 + z^2} \sim R_\mu.$$

For κ small enough, by the definition of the dilating function $g_{\zeta_{\mu,\kappa}}$, we have

$$\begin{aligned} \rho_{\mu,\kappa}(r, z) &= \rho_{\mu}(g_{\zeta_{\mu,\kappa}}^{-1}(r, z)) \\ &\approx ((R_{\mu} - |g_{\zeta_{\mu,\kappa}}^{-1}(r, z)|)^{\frac{1}{\gamma_0-1}}) \\ &\approx \text{dist}((r, z), \partial g_{\zeta_{\mu,\kappa}}(B_{\mu}))^{\frac{1}{\gamma_0-1}}, \end{aligned}$$

for (r, z) near $(R_{\mu,\kappa}, 0) = g_{\zeta_{\mu,\kappa}}(R_{\mu}, 0)$.

Below, for rotating stars $(\rho_{\mu,\kappa}, \kappa r \omega(r) \mathbf{e}_{\theta})$ we use $X_{\mu,\kappa}, X_1^{\mu,\kappa}, Y_{\mu,\kappa}, L_{\mu,\kappa}, A_1^{\mu,\kappa}, B_1^{\mu,\kappa}, B_2^{\mu,\kappa}, K_{\mu,\kappa}$, etc., to denote the corresponding spaces X, X_1, Y , and operators $L, A_1, B_1, B_2, \mathcal{K}$ etc. defined in Sect. 2.

By Theorem 1.1, the rotating star $(\rho_{\mu,\kappa}, \kappa \omega(r) r \mathbf{e}_{\theta})$ is spectrally stable if and only if

$$\langle K_{\mu,\kappa} \delta \rho, \delta \rho \rangle = \langle L_{\mu,\kappa} \delta \rho, \delta \rho \rangle + 2\kappa^2 \pi \int_0^{R_{\mu,\kappa}} \Upsilon(r) \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \delta \rho(s, z) dz ds \right)^2}{r \int_{-\infty}^{+\infty} \rho_{\mu,\kappa}(r, z) dz} dr \geq (\mathbf{B}, 2)$$

for all

$$\delta \rho \in R(B_1^{\mu,\kappa}) = \left\{ \delta \rho \in X_1^{\mu,\kappa} \mid \int_{\Omega_{\mu,\kappa}} \delta \rho dx = 0 \right\}.$$

Moreover, the number of unstable modes equals $n^-(K_{\mu,\kappa}|_{R(B_1^{\mu,\kappa})})$. From the stability criterion, we can obtain the following instability result.

Theorem 3.2. (Sufficient condition for instability)

Let $I \subset [\mu_0, \mu_1]$ be an interval such that the non-rotating star $(\rho_{\mu}, 0)$ is unstable for any $\mu \in I$. Then for any $\omega \in C^{1,\beta}[0, \infty)$ satisfies $\Upsilon(r) > 0$, there exists $\kappa_0 > 0$ such that the rotating star $(\rho_{\mu,\kappa}, \kappa \omega(r) r \mathbf{e}_{\theta})$ is unstable for any $0 < \kappa < \kappa_0$ and $\mu \in I$.

Proof. The instability of $(\rho_{\mu}, 0)$ implies that $n^-(L_{\mu,0}|_{R(B_1^{\mu,0})}) > 0$ for $\mu \in I$. Thus there exists some $\epsilon > 0$ (independent of μ) and $\delta \rho_{\mu,0} = \delta \rho_{\mu,0}(|x|) \in R(B_1^{\mu,0})$ such that $\langle L_{\mu,0} \delta \rho_{\mu,0}, \delta \rho_{\mu,0} \rangle = -2\epsilon < 0$ for $\mu \in I$. Let

$$\delta \rho_{\mu,\kappa}(r, z) = \delta \rho_{\mu,0}(g_{\zeta_{\mu,\kappa}}(r, z)) - \frac{\int_{B_{\mu}} \delta \rho_{\mu,0}(|x|) \det Dg_{\zeta_{\mu,\kappa}}(x) dx}{M_{\mu,\kappa}} \rho_{\mu,\kappa}(r, z),$$

then $\delta \rho_{\mu,\kappa}(r, z) \in R(B_1^{\mu,\kappa})$. Noticing that

$$\lim_{\kappa \rightarrow 0} \int_{B_{\mu}} \delta \rho_{\mu,0}(|x|) \det Dg_{\zeta_{\mu,\kappa}}(x) dx = \int_{B_{\mu}} \delta \rho_{\mu,0}(|x|) dx = 0,$$

we have

$$\lim_{\kappa \rightarrow 0} \langle L_{\mu,\kappa} \delta \rho_{\mu,\kappa}, \delta \rho_{\mu,\kappa} \rangle = \langle L_{\mu,0} \delta \rho_{\mu,0}, \delta \rho_{\mu,0} \rangle = -2\epsilon < 0.$$

Thus, there exists $\kappa_0 > 0$ such that when $0 < \kappa < \kappa_0$

$$\begin{aligned} & \langle K_{\mu,\kappa} \delta\rho_{\mu,\kappa}, \delta\rho_{\mu,\kappa} \rangle \\ &= \langle L_{\mu,\kappa} \delta\rho_{\mu,\kappa}, \delta\rho_{\mu,\kappa} \rangle + 2\kappa^2 \pi \int_0^{R_{\mu,\kappa}} \Upsilon(r) \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho_{\mu,\kappa}(s, z) dz ds \right)^2}{r \int_{-\infty}^{+\infty} \rho_{\mu,\kappa}(r, z) dz} dr < -\epsilon < 0. \end{aligned}$$

The linear instability of $(\rho_{\mu,\kappa}, \kappa\omega(r)\mathbf{re}_\theta)$ follows. □

Let $\tilde{\mu}$ be the first critical point of the mass-radius ratio $\frac{M_\mu}{R_\mu}$ for the nonrotating stars and set $\tilde{\mu} = +\infty$ if $\frac{M_\mu}{R_\mu}$ has no critical point. Consider the rotating stars $(\rho_{\mu,\kappa}, \kappa\omega(r)\mathbf{re}_\theta)$ for $\mu \in [\mu_0, \mu_1] \subset (0, \tilde{\mu})$ and κ small. We have the following sufficient condition for stability.

Theorem 3.3. (Sufficient condition for stability)
 Suppose $P(\rho)$ satisfies (1.2)–(1.3), and $\omega \in C^{1,\beta}[0, \infty)$ satisfies $\Upsilon(r) > 0$. For any $\mu \in [\mu_0, \mu_1] \subset (0, \tilde{\mu})$ and κ small enough, if $\frac{dM_{\mu,\kappa}}{d\mu} \geq 0$, then the rotating star $(\rho_{\mu,\kappa}, \kappa\omega\mathbf{re}_\theta)$ is spectrally stable.

For the proof of above Theorem, first we compute $n^-(L_{\mu,\kappa}|_{X_{\mu,\kappa}})$. Let \dot{H}_{ax}^1 and \dot{H}_{ax}^{-1} be the axi-symmetric subspaces of $\dot{H}^1(\mathbb{R}^3)$ and $\dot{H}^{-1}(\mathbb{R}^3)$ respectively. Define the reduced operator $D_{\mu,\kappa} : \dot{H}_{ax}^1 \rightarrow \dot{H}_{ax}^{-1}$ by

$$D_{\mu,\kappa} := -\Delta - \frac{4\pi}{\Phi''(\rho_{\mu,\kappa})}.$$

Then

$$\langle D_{\mu,\kappa} \psi, \psi \rangle = \int_{\mathbb{R}^3} |\nabla \psi|^2 dx - 4\pi \int_{\mathbb{R}^3} \frac{|\psi|^2}{\Phi''(\rho_{\mu,\kappa})} dx, \quad \psi \in \dot{H}_{ax}^1,$$

defines a bounded bilinear symmetric form on \dot{H}_{ax}^1 . By the same proof of Lemma 3.7 in [29], we have

Lemma 3.1. *It holds that $n^-(L_{\mu,\kappa}|_{X_1^{\mu,\kappa}}) = n^-(D_{\mu,\kappa})$ and $\dim \ker L_{\mu,\kappa} = \dim \ker D_{\mu,\kappa}$.*

Since the rotating star solution $(\rho_{\mu,\kappa}, \kappa\omega(r)\mathbf{re}_\theta)$ is even in z , we can compute $n^-(L_{\mu,\kappa}|_{X_1^{\mu,\kappa}})$ and $n^-(D_{\mu,\kappa})$ on the even and odd (in z) subspaces respectively. Define

$$\begin{aligned} X_{od}^{\mu,\kappa} &:= \{\rho \in X_1^{\mu,\kappa} \mid \rho(r, z) = -\rho(r, -z)\}, \quad X_{ev}^{\mu,\kappa} := \{\rho \in X_1^{\mu,\kappa} \mid \rho(r, z) = \rho(r, -z)\}, \\ H^{od} &:= \{\varphi \in \dot{H}_{ax}^1 \mid \varphi(r, z) = -\varphi(r, -z)\}, \quad H^{ev} := \{\varphi \in \dot{H}_{ax}^1 \mid \varphi(r, z) = \varphi(r, -z)\}. \end{aligned} \tag{3.3}$$

Lemma 3.2. *Assume $P(\rho)$ satisfies (1.2)–(1.3), $\omega \in C^{1,\beta}[0, \infty)$ satisfies $\Upsilon(r) > 0$. Then for any $\mu \in [\mu_0, \mu_1] \subset (0, \tilde{\mu})$ and κ small enough, we have $n^-(L_{\mu,\kappa}) = n^-(L_{\mu,0}) = 1$ and $\ker L_{\mu,\kappa} = \text{span}\{\partial_z \rho_{\mu,\kappa}\}$. Moreover, we have the following direct sum decompositions for $X_{ev}^{\mu,\kappa}$ and $X_{od}^{\mu,\kappa}$:*

$$X_{ev}^{\mu,\kappa} = X_{-,ev}^{\mu,\kappa} \oplus X_{+,ev}^{\mu,\kappa}, \quad \dim X_{-,ev}^{\mu,\kappa} = 1,$$

and

$$X_{od}^{\mu,\kappa} = \overline{\text{span}\{\partial_z \rho_{\mu,\kappa}\}} \oplus X_{+,od}^{\mu,\kappa},$$

satisfying: i) $L_{\mu,\kappa}|_{X_{-,ev}^{\mu,\kappa}} < 0$;

ii) there exists $\delta > 0$ such that

$$\langle L_{\mu,\kappa} u, u \rangle \geq \delta \|u\|_{L^2_{\Phi''(\rho_{\mu,\kappa})}}^2 \quad \text{for any } u \in X_{+,ev}^{\mu,\kappa} \oplus X_{+,od}^{\mu,\kappa},$$

where δ is independent of μ and κ .

The same decompositions are also true for $K_{\mu,\kappa}$ on $X_{ev}^{\mu,\kappa}$ and $X_{od}^{\mu,\kappa}$. In addition, for any $\mu \in [\mu_0, \mu_1]$, it holds that $\frac{dV_{\mu,\kappa}(0, Z_{\mu,\kappa})}{d\mu} < 0$ for κ small enough.

Proof. It was showed in [29] that: for any $\mu \in (0, \tilde{\mu})$, we have $n^-(D_{\mu,0}) = 1$ and $\ker D_{\mu,0} = \text{span}\{\partial_z V_\mu\}$ in the axi-symmetric function space. Here, $V_\mu = -|x|^{-1} * \rho_\mu$ is the gravitational potential of the non-rotating star. Since $\partial_z V_\mu$ is odd in z , it follows that for any $\mu \in (0, \tilde{\mu})$: i) on H^{ev} , $n^-(D_{\mu,0}) = 1$, $\ker D_{\mu,0} = \{0\}$; ii) on H^{od} , $\ker D_{\mu,0} = \text{span}\{\partial_z V_\mu\}$ and $n^-(D_{\mu,0}) = 0$. Moreover, for $\mu \in [\mu_0, \mu_1] \subset (0, \tilde{\mu})$, there exists $\delta_0 > 0$ (independent of μ) and decompositions $H^{ev} = H_-^{ev} \oplus H_+^{ev}$ and $H^{od} = \text{span}\{\partial_z V_\mu\} \oplus H_+^{od}$ satisfying that: i) $\dim H_-^{ev} = 1$, $D_{\mu,0}|_{H_-^{ev}} < -\delta_0$; ii) $D_{\mu,0}|_{H_+^{ev} \oplus H_+^{od}} \geq \delta_0$. Since $\partial_z V_{\mu,\kappa} \in H^{od} \cap \ker D_{\mu,\kappa}$ and

$$\begin{aligned} \langle (D_{\mu,\kappa} - D_{\mu,0})\psi, \psi \rangle &= \int \left(\frac{4\pi}{\Phi''(\rho_{\mu,\kappa})} - \frac{4\pi}{\Phi''(\rho_\mu)} \right) \psi^2 dx \\ &\lesssim \left(\int \left(\frac{4\pi}{\Phi''(\rho_{\mu,\kappa})} - \frac{4\pi}{\Phi''(\rho_\mu)} \right)^{\frac{3}{2}} dx \right)^{\frac{2}{3}} \|\psi\|_{L^6}^2 \\ &\lesssim O(\kappa) \|\nabla \psi\|_{L^2}^2 \rightarrow 0, \text{ as } \kappa \rightarrow 0, \end{aligned}$$

by the perturbation arguments (e.g. Corollary 2.19 in [29]) it follows that for $\mu \in [\mu_0, \mu_1]$ and κ sufficiently small, the decompositions $H^{ev} = H_-^{ev} \oplus H_+^{ev}$ and $H^{od} = \text{span}\{\partial_z V_{\mu,\kappa}\} \oplus H_+^{od}$ satisfy: i) $\dim H_-^{ev} = 1$, $D_{\mu,\kappa}|_{H_-^{ev}} < -\frac{1}{2}\delta_0$; ii) $D_{\mu,\kappa}|_{H_+^{ev} \oplus H_+^{od}} \geq \frac{1}{2}\delta_0$.

By the proof of Lemma 3.4 in [29], for any $\rho \in X_1^{\mu,\kappa}$ we have

$$\langle L_{\mu,\kappa} \rho, \rho \rangle = \|\rho\|_{X_1^{\mu,\kappa}}^2 - \frac{1}{4\pi} \|\nabla \psi\|_{L^2}^2 \geq \frac{1}{4\pi} \langle D_{\mu,\kappa} \psi, \psi \rangle, \tag{3.4}$$

where $\psi = \frac{1}{4\pi} \Delta^{-1} \rho$. We note that $\partial_z \rho_{\mu,\kappa} \in \ker L_{\mu,\kappa} \cap X_{od}^{\mu,\kappa}$ and $\partial_z V_{\mu,\kappa} = \frac{1}{4\pi} \Delta^{-1} \rho_{\mu,\kappa}$. The existence of decompositions for $X_{ev}^{\mu,\kappa}$ and $X_{od}^{\mu,\kappa}$ as stated in the lemma follows readily from (3.4) and above decompositions for H^{od} and H^{ev} .

Since

$$\|((L_{\mu,\kappa} - K_{\mu,\kappa}) \rho, \rho)\| \lesssim O(\kappa^2) \|\rho\|_{X_1^{\mu,\kappa}}^2, \quad \forall \rho \in X_1^{\mu,\kappa},$$

and $\partial_z \rho_{\mu,\kappa} \in \ker K_{\mu,\kappa} \cap X_{od}^{\mu,\kappa}$, we have the same decompositions for $K_{\mu,\kappa}$ on $X_{ev}^{\mu,\kappa}$ and $X_{od}^{\mu,\kappa}$.

Since $\gamma_0 \in (6/5, 2)$, it is known that (see [29])

$$\frac{dV_\mu(0, R_\mu)}{d\mu} = -\frac{d}{d\mu} \left(\frac{M_\mu}{R_\mu} \right) < 0,$$

for μ small. Recall that $\tilde{\mu}$ is the first critical point of $\frac{M_\mu}{R_\mu}$. Therefore, when $\mu \in [\mu_0, \mu_1] \subset (0, \tilde{\mu})$, we have $\frac{dV_\mu(0, R_\mu)}{d\mu} < -\epsilon_0$ for some constant $\epsilon_0 > 0$ independent of μ . Since $\left| \frac{dV_{\mu,\kappa}(0, Z_{\mu,\kappa})}{d\mu} - \frac{dV_\mu(0, R_\mu)}{d\mu} \right| = O(\kappa)$, we have $\frac{dV_{\mu,\kappa}(0, Z_{\mu,\kappa})}{d\mu} < 0$ for any $\mu \in [\mu_0, \mu_1]$ and κ small enough. This finishes the proof of the lemma. \square

Proof of Theorem 3.3. The spectral stability of $(\rho_{\mu,\kappa}, \kappa \text{wre}_\theta)$ is equivalent to show $n^-(K_{\mu,\kappa}|_{R(B_1^{\mu,\kappa})}) = 0$. By Lemma 3.2 and the fact that $K_{\mu,\kappa} = L_{\mu,\kappa}$ on $X_{od}^{\mu,\kappa}$, we have

$$n^-(K_{\mu,\kappa}|_{X_{od}^{\mu,\kappa} \cap R(B_1^{\mu,\kappa})}) = n^-(L_{\mu,\kappa}|_{X_{od}^{\mu,\kappa} \cap R(B_1^{\mu,\kappa})}) \leq n^-(L_{\mu,\kappa}|_{X_{od}^{\mu,\kappa}}) = 0.$$

Since $K_{\mu,\kappa} \geq L_{\mu,\kappa}$ on $X_{ev}^{\mu,\kappa}$ due to $\Upsilon(r) > 0$, for spectral stability it suffices to show $n^-(L_{\mu,\kappa}|_{X_{ev}^{\mu,\kappa} \cap R(B_1^{\mu,\kappa})}) = 0$.

Applying $\frac{d}{d\mu}$ to (3.1), we obtain that

$$L_{\mu,\kappa} \frac{d\rho_{\mu,\kappa}}{d\mu} = -\frac{dc_{\mu,\kappa}}{d\mu}.$$

From (3.1) we know that $c_{\mu,\kappa} = -V_{\mu,\kappa}(R_{\mu,\kappa}, 0)$. By Lemma 3.2, $\frac{dc_{\mu,\kappa}}{d\mu} > 0$ for $\mu \in [\mu_0, \mu_1]$ and κ small enough. Therefore,

$$X_{ev}^{\mu,\kappa} \cap R(B_1^{\mu,\kappa}) = \left\{ \delta\rho \in X_{ev}^{\mu,\kappa} \mid \left\langle L_{\mu,\kappa} \frac{d\rho_{\mu,\kappa}}{d\mu}, \delta\rho \right\rangle = 0 \right\},$$

i.e. $\delta\rho$ is orthogonal to $\frac{d\rho_{\mu,\kappa}}{d\mu}$ in $\langle L_{\mu,\kappa} \cdot, \cdot \rangle$.

When $\frac{dM_{\mu,\kappa}}{d\mu} > 0$, we have

$$\left\langle L_{\mu,\kappa} \frac{d\rho_{\mu,\kappa}}{d\mu}, \frac{d\rho_{\mu,\kappa}}{d\mu} \right\rangle = -\frac{dc_{\mu,\kappa}}{d\mu} \int_{g_{\zeta_{\mu,\kappa}}(B_\mu)} \frac{d\rho_{\mu,\kappa}}{d\mu} dx = \frac{dV_{\mu,\kappa}(0, Z_{\mu,\kappa})}{d\mu} \frac{dM_{\mu,\kappa}}{d\mu} < 0.$$

Combining above with Lemma 3.2, we get $n^-(L_{\mu,\kappa}|_{X_{ev}^{\mu,\kappa} \cap R(B_1^{\mu,\kappa})}) = 0$. Hence we get the spectrally stability.

When $\frac{dM_{\mu,\kappa}}{d\mu} = 0$, since

$$\frac{dM_{\mu,\kappa}}{d\mu} = \int_{\mathbb{R}^3} \frac{d\rho_{\mu,\kappa}}{d\mu} dx = 0,$$

we have $\frac{d\rho_{\mu,\kappa}}{d\mu} \in X_{ev}^{\mu,\kappa} \cap R(B_1^{\mu,\kappa})$. Meanwhile, since $\ker L_{\mu,\kappa} = \{0\}$ on $X_{ev}^{\mu,\kappa}$, by the same argument as in the proof of Theorem 1.1 in [29], we have $n^-(L_{\mu,\kappa}|_{X_{ev}^{\mu,\kappa} \cap R(B_1^{\mu,\kappa})}) = 0$. The spectral stability is again true. \square

It is natural to ask if extrema points of the total mass $M_{\mu,\kappa}$ of the rotating stars $(\rho_{\mu,\kappa}, \kappa\omega r \mathbf{e}_\theta)$ are the transition points for stability as in the case of nonrotating stars. Below, we show that this is not true.

First, we give conditions to ensure that the first extrema point of total mass $M_{\mu,\kappa}$ is obtained at a center density μ_*^κ before $\tilde{\mu}$ (the first critical point of M_μ/R_μ). Assume $P(\rho)$ satisfies the following asymptotically polytropic conditions:

H1)

$$P(\rho) = c_- \rho^{\gamma_0} (1 + O(\rho^{a_0})) \text{ when } \rho \rightarrow 0, \tag{3.5}$$

for some $\gamma_0 \in (\frac{4}{3}, 2)$ and $c_-, a_0 > 0$;

H2)

$$P(\rho) = c_+ \rho^{\gamma_\infty} (1 + O(\rho^{-a_\infty})) \text{ when } \rho \rightarrow +\infty, \tag{3.6}$$

for some $\gamma_\infty \in (1, 6/5) \cup (6/5, 4/3)$ and $c_+, a_\infty > 0$.

Under assumptions H1)-H2), it was shown in [15] that the total mass M_μ of the non-rotating stars $\rho_\mu := \rho_{\mu,0}$ has extrema points. Moreover, the first extrema point of M_μ , which is a maximum point denoted by μ_* , must be less than $\tilde{\mu}$ (see Lemma 3.14 in [29]). For any $\mu_0 < \mu_* < \mu_1 < \tilde{\mu}$, we have $M_{\mu,\kappa} \rightarrow M_\mu$ in $C^1[\mu_0, \mu_1]$ when $\kappa \rightarrow 0$. Thus when κ is small enough, the function $M_{\mu,\kappa}$ has the first maximum $\mu_*^\kappa \in (\mu_0, \mu_1)$ and $\lim_{\kappa \rightarrow 0} \mu_*^\kappa = \mu_*$. By Theorem 3.3, the rotating stars $(\rho_{\mu,\kappa}, \kappa\omega(r)r \mathbf{e}_\theta)$ are stable for $\mu \in [\mu_0, \mu_*^\kappa]$. It is shown below that the transition of stability occurs beyond μ_*^κ .

Theorem 3.4. *Suppose $P(\rho)$ satisfies (3.5)–(3.6), $\omega \in C^{1,\beta}[0, \infty)$ satisfies $\Upsilon(r) > 0$. Fixed κ small, let $\hat{\mu}_\kappa$ be the first transition point of stability of the rotating stars $(\rho_{\mu,\kappa}, \kappa\omega(r)r \mathbf{e}_\theta)$. Then for any $\kappa \neq 0$ small enough, we have $\hat{\mu}_\kappa > \mu_*^\kappa$.*

Proof. As in the proof of Theorem 3.3, the spectral stability is equivalent to show $K_{\mu,\kappa} \geq 0$ on $X_{ev}^{\mu,\kappa} \cap R(B_1^{\mu,\kappa})$. Suppose the maxima point μ_*^κ of $M_{\mu,\kappa}$ is the first transition point for stability, then we have

$$\inf_{\rho \in X_{ev}^{\mu_*^\kappa, \kappa} \cap R(B_1^{\mu_*^\kappa, \kappa})} \frac{\langle K_{\mu_*^\kappa, \kappa} \rho, \rho \rangle}{\|\rho\|_{L^2_{\Phi''(\rho_{\mu_*^\kappa, \kappa})}}} = 0. \tag{3.7}$$

By Lemma 3.2, when κ is small enough, we have the decomposition

$$X_{ev}^{\mu_*^\kappa, \kappa} = X_{-,ev}^{\mu_*^\kappa, \kappa} \oplus X_{+,ev}^{\mu_*^\kappa, \kappa}, \quad \dim X_{-,ev}^{\mu_*^\kappa, \kappa} = 1,$$

satisfying: i) $K_{\mu_*^\kappa, \kappa}|_{X_{-,ev}^{\mu_*^\kappa, \kappa}} < 0$; ii) there exists $\delta > 0$ such that

$$\langle K_{\mu_*^\kappa, \kappa} \rho, \rho \rangle \geq \delta \|\rho\|_{L^2_{\Phi''(\rho_{\mu_*^\kappa, \kappa})}}^2 \text{ for any } \rho \in X_{+,ev}^{\mu_*^\kappa, \kappa}.$$

By using above decomposition, it is easy to show that the infimum in (3.7) is obtained by some $\rho^* \in X_{ev}^{\mu_*^\kappa, \kappa} \cap R(B_1^{\mu_*^\kappa, \kappa})$. Then

$$\langle L_{\mu_*^\kappa, \kappa} \rho^*, \rho^* \rangle \leq \langle K_{\mu_*^\kappa, \kappa} \rho^*, \rho^* \rangle = 0.$$

On the other hand, we have

$$\left\langle L_{\mu_*^k, \kappa} \frac{d\rho_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k}, \frac{d\rho_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k} \right\rangle = \frac{dV_{\mu, \kappa}(0, Z_{\mu, \kappa})}{d\mu} \Big|_{\mu=\mu_*^k} \frac{dM_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k} = 0,$$

and

$$\left\langle L_{\mu_*^k, \kappa} \frac{d\rho_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k}, \rho^* \right\rangle = \frac{dV_{\mu, \kappa}(0, Z_{\mu, \kappa})}{d\mu} \Big|_{\mu=\mu_*^k} \int_{\Omega_{\mu_*^k, \kappa}} \rho^* dx = 0.$$

This implies that $\rho^* = c \frac{d\rho_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k}$ for some constant $c \neq 0$. Since otherwise,

$$n^{\leq 0} \left(L_{\mu_*^k, \kappa} \Big|_{X_{ev}^{\mu_*^k, \kappa}} \right) \geq n^{\leq 0} \left(L_{\mu_*^k, \kappa} \Big|_{\text{span} \left\{ \frac{d\rho_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k}, \rho^* \right\}} \right) = 2.$$

which is in contradiction to $n^{\leq 0} \left(L_{\mu_*^k, \kappa} \Big|_{X_{ev}^{\mu_*^k, \kappa}} \right) = 1$. Thus, we have

$$\begin{aligned} 0 &= \left\langle K_{\mu_*^k, \kappa} \frac{d\rho_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k}, \frac{d\rho_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k} \right\rangle \\ &= 2\pi \kappa^2 \int_0^{R_{\mu_*^k, \kappa}} \Upsilon(r) \frac{\left(\int_0^r s \int_{-\infty}^{+\infty} \frac{d\rho_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k}(s, z) dz ds \right)^2}{r \int_{-\infty}^{+\infty} \rho_{\mu_*^k, \kappa}(r, z) dz} dr. \end{aligned}$$

and consequently

$$\int_{-\infty}^{+\infty} \frac{d\rho_{\mu, \kappa}}{d\mu} \Big|_{\mu=\mu_*^k}(r, z) dz = 0, \quad \forall r \in [0, R_{\mu_*^k, \kappa}]. \tag{3.8}$$

Nevertheless, it is not true as shown below.

For non-rotating stars $(\rho_{\mu}(\mathbf{r}), 0)$, we have

$$\Delta V_{\mu} = \frac{1}{\mathbf{r}^2} \left(\mathbf{r}^2 (V_{\mu}(\mathbf{r}))' \right)' = 4\pi \rho_{\mu},$$

where $\mathbf{r} = \sqrt{r^2 + z^2}$ and $V_{\mu}(\mathbf{r})$ is the gravitational potential. Applying $\frac{d}{d\mu}$ to above equation, one has

$$\frac{1}{\mathbf{r}^2} \left(\mathbf{r}^2 \left(\frac{dV_{\mu}(\mathbf{r})}{d\mu} \right)' \right)' = 4\pi \frac{d\rho_{\mu}}{d\mu}.$$

When $\mathbf{r} \geq R_{\mu}$, since $\frac{d\rho_{\mu}}{d\mu}(\mathbf{r}) = 0$ we have

$$\mathbf{r}^2 \left(\frac{dV_{\mu}}{d\mu} \right)'(\mathbf{r}) = R_{\mu}^2 \left(\frac{dV_{\mu}}{d\mu} \right)'(R_{\mu}) = 4\pi \int_0^{R_{\mu}} s^2 \frac{d\rho_{\mu}}{d\mu}(s) ds = \frac{dM_{\mu}}{d\mu},$$

and consequently

$$\frac{dV_{\mu}}{d\mu}(\mathbf{r}) = -\frac{dM_{\mu}}{d\mu} \frac{1}{\mathbf{r}}, \quad \text{for } \mathbf{r} \geq R_{\mu}.$$

Since $\lim_{\kappa \rightarrow 0} \mu_*^\kappa = \mu_*$, we have $\lim_{\kappa \rightarrow 0} \frac{dM_\mu}{d\mu}(\mu_*^\kappa) = \frac{dM_\mu}{d\mu}(\mu_*) = 0$. Thus

$$\frac{dV_\mu}{d\mu}(R_\mu)|_{\mu=\mu_*^\kappa} = -\frac{dM_\mu}{d\mu}(\mu_*^\kappa) \frac{1}{R_{\mu_*^\kappa}} \rightarrow 0, \text{ as } \kappa \rightarrow 0.$$

Define $y_\mu(\mathbf{r}) = V_\mu(R_\mu) - V_\mu(\mathbf{r}) = \Phi'(\rho_\mu)$. Then by Lemma 3.13 in [29], we have

$$\begin{aligned} \frac{dy_\mu}{d\mu}(R_\mu)|_{\mu=\mu_*^\kappa} &= -\frac{d}{d\mu} \left(\frac{M_\mu}{R_\mu} \right) |_{\mu=\mu_*^\kappa} - \frac{dV_\mu}{d\mu}(R_\mu)|_{\mu=\mu_*^\kappa} \\ &\rightarrow -\frac{d}{d\mu} \left(\frac{M_\mu}{R_\mu} \right) |_{\mu=\mu_*} \neq 0, \text{ as } \kappa \rightarrow 0. \end{aligned} \tag{3.9}$$

Thus by (3.9), we obtain

$$\frac{d\rho_\mu}{d\mu}(\mathbf{r}) = \frac{1}{\Phi''(\rho_\mu)} \frac{dy_\mu}{d\mu}(\mathbf{r}) \approx \rho_\mu^{2-\gamma_0} \approx (R_\mu - \mathbf{r})^{\frac{2-\gamma_0}{\gamma_0-1}},$$

for $\mathbf{r} \sim R_\mu$ and $\mu = \mu_*^\kappa$. By (3.36) and (4.78) in [41], we know

$$\left| \frac{dg_{\zeta_{\mu,\kappa}}^{-1}}{d\mu}(y) \right| = \left| \lim_{\mu_1 \rightarrow \mu} \frac{g_{\zeta_{\mu_1,\kappa}}^{-1} - g_{\zeta_{\mu,\kappa}}^{-1}}{\mu_1 - \mu}(y) \right| \leq C\kappa,$$

for some constant C independent of μ and κ . Therefore,

$$\begin{aligned} \frac{d\rho_{\mu,\kappa}}{d\mu}(r, z) &= \frac{d\rho_\mu(g_{\zeta_{\mu,\kappa}}^{-1}(r, z))}{d\mu} = \frac{d\rho_\mu}{d\mu}(g_{\zeta_{\mu,\kappa}}^{-1}(r, z)) + \frac{d\rho_\mu(\mathbf{r})}{d\mathbf{r}}|_{\mathbf{r}=g_{\zeta_{\mu,\kappa}}^{-1}(r, z)} \frac{dg_{\zeta_{\mu,\kappa}}^{-1}}{d\mu} \\ &\approx \rho_\mu(g_{\zeta_{\mu,\kappa}}^{-1}(r, z))^{2-\gamma_0} = \rho_{\mu,\kappa}(r, z)^{2-\gamma_0}, \end{aligned}$$

for $g_{\zeta_{\mu,\kappa}}^{-1}(r, z) \sim R_\mu$ and $\mu = \mu_*^\kappa$. By Lemma 2.1, we have

$$\int_{-\infty}^{+\infty} \frac{d\rho_{\mu,\kappa}}{d\mu}|_{\mu=\mu_*^\kappa}(r, z) dz \approx \int_{-\infty}^{+\infty} \rho_{\mu,\kappa}(r, z)^{2-\gamma_0} dz \approx (R_{\mu_*^\kappa} - r)^{\frac{2-\gamma_0}{\gamma_0-1} + \frac{1}{2}} \neq 0,$$

for $r \sim R_{\mu_*^\kappa}$. This is in contradiction to (3.8) and finishes the proof of the theorem. \square

3.2. *The case of fixed angular momentum distribution.* Let $j(p, q) : \mathbb{R}^2 \mapsto \mathbb{R}$ be a given function satisfying

$$j(p, q) \in C^{1,\beta}(\mathbb{R}^+ \times \mathbb{R}^+) \text{ and } j(0, q) = \partial_p j(0, q) = 0. \tag{3.10}$$

Define $J(p, q) = j^2(p, q)$. We consider a family of rotating stars of the following form

$$\begin{cases} \rho_{\mu,\varepsilon}(r, z) = \rho_\mu(g_{\zeta_{\mu,\varepsilon}}^{-1}((r, z))), \\ \vec{v}_{\mu,\varepsilon} = \varepsilon \frac{j(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon})}{r} \mathbf{e}_\theta, \end{cases}$$

where

$$m_{\rho_{\mu,\varepsilon}}(r) = \int_0^r s \int_{-\infty}^{\infty} \rho_{\mu,\varepsilon}(s, z) ds dz, \quad g_{\zeta_{\mu,\varepsilon}} = x \left(1 + \frac{\zeta_{\mu,\varepsilon}(x)}{|x|^2} \right),$$

and $\zeta_{\mu,\varepsilon}(x) : B_\mu \rightarrow \mathbb{R}$ is axi-symmetric and even in z . The Euler–Poisson system satisfied by $(\rho_{\mu,\varepsilon}, \vec{v}_{\mu,\varepsilon})$ is reduced to the following equations:

$$\Phi'(\rho_{\mu,\varepsilon}) + V_{\mu,\varepsilon} - \varepsilon^2 \int_0^r J(m_{\rho_{\mu,\varepsilon}}(s), M_{\mu,\varepsilon}) s^{-3} ds + c_{\mu,\varepsilon} = 0, \quad \text{in } \Omega_{\mu,\varepsilon}, \tag{3.11}$$

$$V_{\mu,\varepsilon} = -|x|^{-1} * \rho_{\mu,\varepsilon} \text{ in } \mathbb{R}^3, \tag{3.12}$$

where $\Omega_{\mu,\varepsilon} = g_{\zeta_{\mu,\varepsilon}}(B_\mu)$ and $c_{\mu,\varepsilon}$ is a constant.

Although (3.11) is a little different from the steady state equations in [14], the key linearized operator at the point $\varepsilon = 0$ is the same as [14]. By similar arguments as [14,41], we can get the following existence theorem.

Theorem 3.5. *Let $\mu \in [\mu_0, \mu_1] \subset (0, \mu_{\max})$, $P(\rho)$ satisfy (1.2)–(1.3) and $j(p, q)$ satisfy (3.10). Then there exists $\tilde{\varepsilon} > 0$ and solutions $\rho_{\mu,\varepsilon}$ of (3.11) for all $|\varepsilon| < \tilde{\varepsilon}$, with the following properties:*

- 1) $\rho_{\mu,\varepsilon} \in C_c^{1,\alpha}(\mathbb{R}^3)$, where $\alpha = \min(\frac{2-\gamma_0}{\gamma_0-1}, 1)$.
- 2) $\rho_{\mu,\varepsilon}$ is axi-symmetric and even in z .
- 3) $\rho_{\mu,\varepsilon}(0) = \mu$.
- 4) $\rho_{\mu,\varepsilon} \geq 0$ has compact support $g_{\zeta_{\mu,\varepsilon}}(B_\mu)$.
- 5) For all $\mu \in [\mu_0, \mu_1]$, the mapping $\varepsilon \rightarrow \rho_{\mu,\varepsilon}$ is continuous from $(-\tilde{\varepsilon}, \tilde{\varepsilon})$ into $C_c^1(\mathbb{R}^3)$.

When $\varepsilon = 0$, $\rho_{\mu,0}(x) = \rho_\mu(|x|)$ is the nonrotating star solution with $\rho_\mu(0) = \mu$.

Now we use Theorem 1.1 to study the stability of rotating star solutions $(\rho_{\mu,\varepsilon}, \varepsilon j(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon})/r e_\theta)$, where ε is small enough, $j(p, q)$ satisfies (3.10) and the Rayleigh stability condition $\partial_p J(p, q) > 0$ (i.e. $j \partial_p j > 0$). As in Sect. 3.1, the assumptions in Theorem 1.1 can be verified. That is, $\partial \Omega_{\mu,\varepsilon}$ is C^2 and has positive curvature near $(R_{\mu,\varepsilon}, 0)$ and (1.7) holds for any $\mu \in [\mu_0, \mu_1]$ and ε small enough.

Below, for rotating stars $(\rho_{\mu,\varepsilon}, \varepsilon j(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon})/r e_\theta)$ we use $X_{\mu,\varepsilon}, X_1^{\mu,\varepsilon}, Y_{\mu,\varepsilon}, L_{\mu,\varepsilon}, A_1^{\mu,\varepsilon}, B_1^{\mu,\varepsilon}, B_2^{\mu,\varepsilon}, K_{\mu,\varepsilon}$, etc., to denote the corresponding spaces X, X_1, Y , and operators $L, A_1, B_1, B_2, \mathcal{K}$ etc. defined in Sect. 2. Again, we denote $\tilde{\mu}$ to be the first critical point of M_μ/R_μ for non-rotating stars. Define the spaces $X_{ev}^{\mu,\varepsilon}$ and $X_{od}^{\mu,\varepsilon}$ as in (3.3). By the same proof of Lemma 3.2, we have the following.

Lemma 3.3. *Assume $P(\rho)$ satisfies (3.5)–(3.5) and $j(p, q)$ satisfies (3.10) and $\partial_p(j^2(p, q)) > 0$. Then for any $\mu \in [\mu_0, \mu_1] \subset (0, \tilde{\mu})$ and ε small enough, we have $n^-(K_{\mu,\varepsilon}) = 1$ and $\ker K_{\mu,\varepsilon} = \text{span}\{\partial_z \rho_{\mu,\varepsilon}\}$. Moreover, we have the following direct sum decompositions for $X_{ev}^{\mu,\varepsilon}$ and $X_{od}^{\mu,\varepsilon}$:*

$$X_{ev}^{\mu,\varepsilon} = X_{-,ev}^{\mu,\varepsilon} \oplus X_{+,ev}^{\mu,\varepsilon}, \quad \dim X_{-,ev}^{\mu,\varepsilon} = 1,$$

and

$$X_{od}^{\mu,\varepsilon} = \text{span}\{\partial_z \rho_{\mu,\varepsilon}\} \oplus X_{+,od}^{\mu,\varepsilon},$$

satisfying: i) $K_{\mu,\varepsilon}|_{X_{-,ev}^{\mu,\varepsilon}} < 0$;

ii) there exists $\delta > 0$ such that

$$\langle K_{\mu,\varepsilon}u, u \rangle \geq \delta \|u\|_{L^2_{\Phi''(\rho_{\mu,\varepsilon})}}^2 \quad \forall u \in X_{+,ev}^{\mu,\varepsilon} \oplus X_{+,od}^{\mu,\varepsilon},$$

where δ is independent of μ and ε .

In addition, for any $\mu \in [\mu_0, \mu_1]$, it holds that $\frac{dV_{\mu,\varepsilon}(R_{\mu,\varepsilon}, 0)}{d\mu} < 0$ for ε small.

By Theorem 1.1, we get the following necessary and sufficient condition for the stability of rotating stars $(\rho_{\mu,\varepsilon}, \varepsilon j(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon})/r\mathbf{e}_\theta)$:

$$\begin{aligned} \langle K_{\mu,\varepsilon}\delta\rho, \delta\rho \rangle &= \langle L_{\mu,\varepsilon}\delta\rho, \delta\rho \rangle \\ &+ 2\varepsilon^2\pi \int_0^{R_{\mu,\varepsilon}} \frac{\partial_p J(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon})}{r^3} \left(\int_0^r s \int_{-\infty}^{+\infty} \delta\rho(s, z) dz ds \right)^2 dr \geq 0, \end{aligned}$$

for all $\delta\rho \in R(B_1^{\mu,\varepsilon}) = \left\{ \delta\rho \in X_1^{\mu,\varepsilon} \mid \int_{\Omega_{\mu,\varepsilon}} \delta\rho dx = 0 \right\}$.

The following Theorem shows that the stability of this family of rotating stars can only change at the mass extrema.

Theorem 3.6. Assume $P(\rho)$ satisfies (3.5)–(3.6), and $j(p, q)$ satisfy (3.10) and $\partial_p(j^2(p, q)) > 0$. Let $n^u(\mu)$ be the number of unstable modes, namely the total algebraic multiplicities of unstable eigenvalues of the linearized Euler–Poisson system at $(\rho_{\mu,\varepsilon}, \varepsilon j(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon})/r\mathbf{e}_\theta)$. Then for any $\mu \in [\mu_0, \mu_1] \subset (0, \tilde{\mu})$ and ε small enough, we have

$$n^u(\mu) = \begin{cases} 1, & \text{when } \frac{dM_{\mu,\varepsilon}}{d\mu} < 0, \\ 0, & \text{when } \frac{dM_{\mu,\varepsilon}}{d\mu} \geq 0. \end{cases}$$

Proof. By the same arguments in the proof of Theorem 3.3, we have

$$n^u(\mu) = n^- \left(K_{\mu,\varepsilon} \mid_{X_{ev}^{\mu,\varepsilon} \cap R(B_1^{\mu,\varepsilon})} \right).$$

Thus it is reduced to find the number of negative modes of the quadratic form $\langle K_{\mu,\varepsilon} \cdot, \cdot \rangle$ restricted to the even subspace of $R(B_1^{\mu,\varepsilon})$.

Applying $\frac{d}{d\mu}$ to (3.11), we obtain that

$$\begin{aligned} L_{\mu,\varepsilon} \frac{d\rho_{\mu,\varepsilon}}{d\mu} &= \varepsilon^2 \int_0^r \partial_p J(m_{\rho_{\mu,\varepsilon}}(s), M_{\mu,\varepsilon}) \frac{dm_{\rho_{\mu,\varepsilon}}}{d\mu} s^{-3} ds \\ &+ \varepsilon^2 \int_0^r \partial_q J(m_{\rho_{\mu,\varepsilon}}(s), M_{\mu,\varepsilon}) \frac{dM_{\mu,\varepsilon}}{d\mu} s^{-3} ds - \frac{dc_{\mu,\varepsilon}}{d\mu}, \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \frac{dc_{\mu,\varepsilon}}{d\mu} &= \frac{d}{d\mu} \left(-V_{\mu,\varepsilon}(R_{\mu,\varepsilon}, 0) + \varepsilon^2 \int_0^{R_{\mu,\varepsilon}} J(m_{\rho_{\mu,\varepsilon}}(s), M_{\mu,\varepsilon}) s^{-3} ds \right) \\ &= -\frac{dV_{\mu,\varepsilon}(R_{\mu,\varepsilon}, 0)}{d\mu} + \varepsilon^2 \int_0^{R_{\mu,\varepsilon}} \partial_p J(m_{\rho_{\mu,\varepsilon}}(s), M_{\mu,\varepsilon}) \frac{dm_{\rho_{\mu,\varepsilon}}(s)}{d\mu} s^{-3} ds \\ &+ \varepsilon^2 \frac{dM_{\mu,\varepsilon}}{d\mu} h_{\mu,\varepsilon}(R_{\mu,\varepsilon}) + \varepsilon^2 J(M_{\mu,\varepsilon}, M_{\mu,\varepsilon}) R_{\mu,\varepsilon}^{-3} \frac{dR_{\mu,\varepsilon}}{d\mu}. \end{aligned}$$

By integration by parts and (3.13), we obtain that

$$\begin{aligned}
 & 2\pi \int_0^{R_{\mu,\varepsilon}} \varepsilon^2 \left[\partial_p J(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon}) r^{-3} \right] \left(\int_0^r \int_{-\infty}^{\infty} s \frac{d\rho_{\mu,\varepsilon}}{d\mu} dz ds \right) \left(\int_0^r \int_{-\infty}^{\infty} s \varphi dz ds \right) dr \\
 &= \varepsilon^2 \left[\int_0^{R_{\mu,\varepsilon}} \partial_p J(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon}) r^{-3} \frac{dm_{\rho_{\mu,\varepsilon}}}{d\mu} dr \right] \int_{\Omega_{\mu,\varepsilon}} \varphi dx \\
 &\quad - 2\pi \int_0^{R_{\mu,\varepsilon}} \int_{-\infty}^{\infty} \varepsilon^2 \left[\int_0^r \partial_p J(m_{\rho_{\mu,\varepsilon}}(s), M_{\mu,\varepsilon}) \frac{dm_{\rho_{\mu,\varepsilon}}(s)}{d\mu} s^{-3} ds \right] r \varphi dz dr \\
 &= \varepsilon^2 \left[\int_0^{R_{\mu,\varepsilon}} \partial_p J(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon}) r^{-3} \frac{dm_{\rho_{\mu,\varepsilon}}}{d\mu} dr \right] \int_{\Omega_{\mu,\varepsilon}} \varphi dx - \left\langle \frac{dc_0}{d\mu}, \varphi \right\rangle \\
 &\quad - \left\langle L_{\mu,\varepsilon} \frac{d\rho_{\mu,\varepsilon}}{d\mu}, \varphi \right\rangle - \varepsilon^2 \frac{dM_{\mu,\varepsilon}}{d\mu} \left\langle \int_0^r \partial_q J(m_{\rho_{\mu,\varepsilon}}(s), M_{\mu,\varepsilon}) s^{-3} ds, \varphi \right\rangle \\
 &= \left(\frac{dV_{\mu,\varepsilon}(R_{\mu,\varepsilon}, 0)}{d\mu} - \varepsilon^2 J(M_{\mu,\varepsilon}, M_{\mu,\varepsilon}) R_{\mu,\varepsilon}^{-3} \frac{dR_{\mu,\varepsilon}}{d\mu} - \varepsilon^2 \frac{dM_{\mu,\varepsilon}}{d\mu} h_{\mu,\varepsilon}(R_{\mu,\varepsilon}) \right) \int_{\Omega_{\mu,\varepsilon}} \varphi dx \\
 &\quad - \left\langle L_{\mu,\varepsilon} \frac{d\rho_{\mu,\varepsilon}}{d\mu}, \varphi \right\rangle - \varepsilon^2 \frac{dM_{\mu,\varepsilon}}{d\mu} \langle K_{\mu,\varepsilon} g_{\mu,\varepsilon}, \varphi \rangle.
 \end{aligned}$$

Here, in the above we used

$$h_{\mu,\varepsilon}(r) = \int_0^r \partial_q J(m_{\rho_{\mu,\varepsilon}}(s), M_{\mu,\varepsilon}) s^{-3} ds,$$

and $g_{\mu,\varepsilon} = K_{\mu,\varepsilon}^{-1} h_{\mu,\varepsilon}$. The inverse operator

$$K_{\mu,\varepsilon}^{-1} : (X_{ev}^{\mu,\varepsilon})^* \subset L^2_{\frac{1}{\Phi''(\rho_{\mu,\varepsilon})}} \rightarrow X_{ev}^{\mu,\varepsilon}$$

exists and is bounded by Lemma 3.3. Since $\frac{1}{\Phi''(\rho_{\mu,\varepsilon})}$ has compact support and $\Phi''(s) \approx s^{\gamma_0-2}$ for $s \sim 0^+$, we have

$$\left| \int g_{\mu,\varepsilon} dx \right| \lesssim \|g_{\mu,\varepsilon}\|_{L^2_{\Phi''(\rho_{\mu,\varepsilon})}} \lesssim \|K_{\mu,\varepsilon}^{-1}\| \|h_{\mu,\varepsilon}\|_{L^2_{\frac{1}{\Phi''(\rho_{\mu,\varepsilon})}}} \lesssim \left(\int \frac{h_{\mu,\varepsilon}^2}{\Phi''(\rho_{\mu,\varepsilon})} dx \right)^{\frac{1}{2}} < +\infty.$$

Therefore, we have

$$\left\langle K_{\mu,\varepsilon} \left[\frac{d\rho_{\mu,\varepsilon}}{d\mu} + \varepsilon^2 \frac{dM_{\mu,\varepsilon}}{d\mu} g_{\mu,\varepsilon} \right], \varphi \right\rangle = \left(\frac{dV_{\mu,\varepsilon}(R_{\mu,\varepsilon}, 0)}{d\mu} + O(\varepsilon^2) \right) \int_{\Omega_{\mu,\varepsilon}} \varphi dx, \tag{3.14}$$

for any $\varphi \in X_{ev}^{\mu,\varepsilon}$.

By (3.14) and the fact that $\frac{dV_{\mu,\varepsilon}(R_{\mu,\varepsilon}, 0)}{d\mu} + O(\varepsilon^2) < 0$ when $\mu \in [\mu_0, \mu_1]$ and ε is small, we have

$$X_{ev}^{\mu,\varepsilon} \cap R(B_1^{\mu,\varepsilon}) = \left\{ \delta\rho \in X_{ev}^{\mu,\varepsilon} \mid \left\langle K_{\mu,\varepsilon} \left(\frac{d\rho_{\mu,\varepsilon}}{d\mu} + \varepsilon^2 \frac{dM_{\mu,\varepsilon}}{d\mu} g_{\mu,\varepsilon} \right), \delta\rho \right\rangle = 0 \right\}.$$

On the other hand, we have

$$\begin{aligned} & \left\langle K_{\mu,\varepsilon} \left(\frac{d\rho_{\mu,\varepsilon}}{d\mu} + \varepsilon^2 \frac{dM_{\mu,\varepsilon}}{d\mu} g_{\mu,\varepsilon} \right), \left(\frac{d\rho_{\mu,\varepsilon}}{d\mu} + \varepsilon^2 \frac{dM_{\mu,\varepsilon}}{d\mu} g_{\mu,\varepsilon} \right) \right\rangle \\ &= \left(\frac{dV_{\mu,\varepsilon}(R_{\mu,\varepsilon}, 0)}{d\mu} + O(\varepsilon^2) \right) \int_{\mathbb{R}^3} \left(\frac{d\rho_{\mu,\varepsilon}}{d\mu} + \varepsilon^2 \frac{dM_{\mu,\varepsilon}}{d\mu} g_{\mu,\varepsilon} \right) dx \\ &= \left(\frac{dV_{\mu,\varepsilon}(R_{\mu,\varepsilon}, 0)}{d\mu} + O(\varepsilon^2) \right) \frac{dM_{\mu,\varepsilon}}{d\mu}. \end{aligned}$$

By Lemma 3.3, $n^-(K_{\mu,\varepsilon}|_{X_{ev}^{\mu,\varepsilon}}) = 1$ and $\ker K_{\mu,\varepsilon}|_{X_{ev}^{\mu,\varepsilon}} = \{0\}$. We consider two cases:

1) $\frac{dM_{\mu,\varepsilon}}{d\mu} \neq 0$. A combination of above properties immediately yields

$$n^u(\mu) = n^-\left(K_{\mu,\varepsilon}|_{X_{ev}^{\mu,\varepsilon} \cap R(B_1^{\mu,\varepsilon})}\right) = \begin{cases} 1 & \text{when } \frac{dM_{\mu,\varepsilon}}{d\mu} < 0, \\ 0 & \text{when } \frac{dM_{\mu,\varepsilon}}{d\mu} > 0. \end{cases}$$

2) When $\frac{dM_{\mu,\varepsilon}}{d\mu} = 0$, as in the proof of Theorem 3.3, we have

$$n^u(\mu) = n^-\left(K_{\mu,\varepsilon}|_{X_{ev}^{\mu,\varepsilon} \cap R(B_1^{\mu,\varepsilon})}\right) = 0.$$

This finishes the proof of the theorem. □

Remark 3.2. The above theorem implies that for a family of rotating stars with fixed angular momentum distribution $j(m, M)$, the transition of stability occurs at the first extrema of the total mass. That is, the turning point principle (TPP) is true for this family of rotating stars. This contrasts greatly to rotating stars of fixed angular velocity, for which case TPP is shown to be not true (see Theorem 3.4).

In the literature, there are three common choices of $j(m, M)$ in the study of rotating stars.

- i) (Fixed angular momentum distribution) The most common one is $j(m, M) = j(m)$. See for example [2, 18, 30–33];
- ii) (Fixed angular momentum distribution per unit mass) $j(m, M) = j(m/M)$. See for example [34];
- iii) (Fixed angular momentum distribution with given total angular momentum) $j(m, M) = \frac{1}{M}j(m/M)$. See for example [4]. We note that for this case, the total angular momentum given by

$$\int \frac{1}{M} j\left(\frac{m}{M}\right) dm = \int_0^1 j(m') dm' \quad \left(m' = \frac{m}{M}\right),$$

is a constant depending only on j .

In the rest of this subsection, we use Theorem 3.6 to study two examples of rotating stars with mass extrema points.

Example 1. Asymptotically polytropic rotating stars

Assume $P(\rho)$ satisfies assumptions (3.5)–(3.6). By the same arguments as in the case of fixed angular velocity, when ε is small enough and $\mu \in [\mu_0, \mu_1] \subset (0, \tilde{\mu})$, the mass $M_{\mu,\varepsilon}$ of the rotating stars $(\rho_{\mu,\varepsilon}, \varepsilon j(m_{\rho_{\mu,\varepsilon}}(r), M_{\mu,\varepsilon})/re_\theta)$ has the the first maximum $\mu_*^\varepsilon \in (\mu_0, \mu_1)$. Then by Theorem 3.6, the rotating stars are stable when $\mu \in [\mu_0, \mu_*^\varepsilon]$ and unstable when μ goes between μ_*^ε and the next extrema point of $M_{\mu,\varepsilon}$ in $(\mu_*^\varepsilon, \mu_1)$.

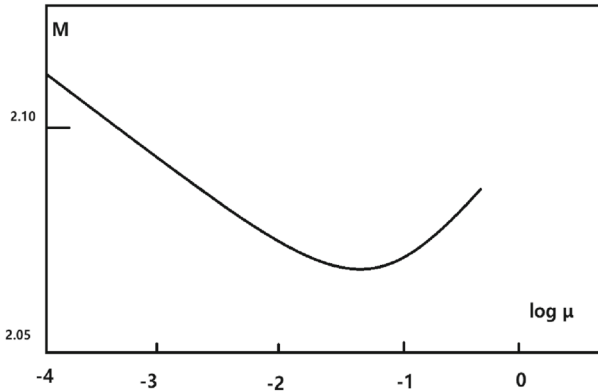


Fig. 1. The dependence of the mass $M(\mu)$ on the center density μ for $\gamma = \frac{4.03}{3.03}$ and the angular momentum distribution $j(m, M) = \frac{1}{M}[1 - (1 - \frac{m}{M})^{2/3}]$. From Bisnovaty-Kogan and Blinnikov [4]

Example 2. Polytropic rotating stars

Consider the polytropic equation of state $P(\rho) = \rho^\gamma$ ($\gamma \in (\frac{6}{5}, 2)$). The non-rotating stars (i.e. Lane-Emden stars) with any center density μ are stable when $\gamma \in (4/3, 2)$ and are unstable when $\gamma \in (6/5, 4/3)$. In particular, $M_\mu = C_\gamma \mu^{\frac{1}{2}(3\gamma-4)}$ is a monotone function when $\gamma \neq \frac{4}{3}$ and there is no transition point of stability.

However, polytropic rotating stars with fixed angular momentum distribution $j(m, M)$ can have mass extrema points, which are also the transition points of stability. One such example was given in [4] for $\gamma = \frac{4.03}{3.03} < \frac{4}{3}$ and $j(m, M) = \frac{1}{M}[1 - (1 - \frac{m}{M})^{2/3}]$. With numerical help, it was found (see Figure 1 below taken from [4]) that there is a mass minimum point μ^* for the total mass $M(\mu)$. This is the first transition point of stability. In particular, rotating stars with center density μ beyond μ^* become stable (Fig. 1).

Remark 3.3. It can also be seen from above Example 2 that the critical index γ^* for the onset of instability of rotating polytropic stars is not $\frac{4}{3}$. Ledoux [24], Chandrasekhar and Lebovitz [8] indicated that the critical index γ^* is reduced from $\frac{4}{3}$ to $\gamma^* = \frac{4}{3} - \frac{2\omega^2 I}{9|W|}$ for small uniform rotating stars, where $I > 0$ is the moment of inertia about the center of mass and W is the gravitational potential energy. For more discussion about the critical index γ^* of rotating stars, see [13, 21, 38, 39].

4. Instability for Rayleigh Unstable Case

Consider an axi-symmetric rotating star $(\rho_0, \vec{v}_0) = (\rho_0(r, z), \omega_0(r)r\mathbf{e}_\theta)$, where the angular velocity $\omega_0(r)$ satisfies the Rayleigh instability condition [35, 45, 46]. That is, there exists a point $r_0 \in (0, R_0)$ such that

$$\Upsilon(r_0) = \frac{\partial_r(\omega_0^2 r^4)}{r^3} \Big|_{r=r_0} < 0. \tag{4.1}$$

For incompressible Euler equation, it is a classical result by Rayleigh in 1880 [36] that condition (4.1) implies linear instability of the rotating flow $\vec{v}_0 = \omega_0(r)r\mathbf{e}_\theta$ under axi-symmetric perturbations. In this section, we will show the axi-symmetric instability of rotating stars with Rayleigh unstable angular velocity.

From the linearized Euler–Poisson system (2.5), we get the following second order equation for $u_2 = \begin{pmatrix} v_r \\ v_z \end{pmatrix}$,

$$\partial_{tt} u_2 = -\tilde{\mathbb{L}}u_2 = -(\mathbb{L}_1 + \mathbb{L}_2)u_2, \tag{4.2}$$

where $\mathbb{L}_1, \mathbb{L}_2$ are operators on $Y = (L^2_{\rho_0})^2$ defined by

$$\mathbb{L}_1 u_2 = B'_1 L B_1 A = \nabla[\Phi''(\rho_0)(\nabla \cdot (\rho_0 u_2)) - 4\pi(-\Delta)^{-1}(\nabla \cdot (\rho_0 u_2))],$$

and

$$\mathbb{L}_2 u_2 = \begin{pmatrix} \Upsilon(r)v_r \\ 0 \end{pmatrix}.$$

Lemma 4.1. $\tilde{\mathbb{L}}$ is a self-adjoint operator on $(Y, [\cdot, \cdot])$ with the equivalent inner product $[\cdot, \cdot] = \langle A\cdot, \cdot \rangle$.

Proof. By Lemma 2.9 in [29], \mathbb{L}_1 is self-adjoint on $(Y, [\cdot, \cdot])$ with the equivalent inner product $[\cdot, \cdot] := \langle A\cdot, \cdot \rangle$. Since \mathbb{L}_2 is a symmetric bounded operator on $(Y, [\cdot, \cdot])$, $\tilde{\mathbb{L}} = \mathbb{L}_1 + \mathbb{L}_2$ is self-adjoint by Kato-Rellich Theorem. \square

The next lemma on the quadratic form of $\tilde{\mathbb{L}}$ will be used later.

Lemma 4.2. There exists constants $m > 0$ such that for any $u_2 \in Y$, we have

$$[\tilde{\mathbb{L}}u_2, u_2] + m \|u_2\|_Y^2 \geq \|\nabla \cdot (\rho_0 u_2)\|_{L^2_{\Phi''(\rho_0)}}^2.$$

Proof. Since

$$[\tilde{\mathbb{L}}u_2, u_2] = [\mathbb{L}_1 u_2, u_2] + [\mathbb{L}_2 u_2, u_2],$$

and obviously $[\mathbb{L}_2 u_2, u_2] \lesssim \|u_2\|_{L^2_{\rho_0}}^2$, it suffices to estimate

$$[\mathbb{L}_1 u_2, u_2] = \langle L B_1 A u_2, B_1 A u_2 \rangle = \|\nabla \cdot (\rho_0 u_2)\|_{L^2_{\Phi''(\rho_0)}}^2 - 4\pi \int_{\mathbb{R}^3} |\nabla V|^2 dx,$$

where $-\Delta V = \nabla \cdot (\rho_0 u_2)$. By integration by parts,

$$\int_{\mathbb{R}^3} |\nabla V|^2 dx = - \int_{\mathbb{R}^3} \rho_0 u_2 \cdot \nabla V dx \lesssim \left(\|u_2\|_Y^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla V|^2 dx\right)^{\frac{1}{2}},$$

which implies that $\int_{\mathbb{R}^3} |\nabla V|^2 dx \lesssim \|u_2\|_Y^2$. This finishes the proof of the lemma. \square

The study of equation (4.2) is reduced to understand the spectra of the self-adjoint operator $\tilde{\mathbb{L}}$. First, we give a Helmholtz type decomposition of vector fields in Y .

Lemma 4.3. *There is a direct sum decomposition $Y = Y_1 \oplus Y_2$, where Y_1 is the closure of*

$$\left\{ u \in Y \mid u = \nabla p, \text{ for some } p \in C^1(\Omega) \right\},$$

in Y and Y_2 is the closure of

$$\left\{ u \in \left(C^1(\Omega) \right)^2 \cap Y \mid \nabla \cdot (\rho_0 u) = 0 \right\},$$

in Y .

The proof of above lemma is similar to that of Lemma 3.15 in [29] and we skip. Denote $\mathbb{P}_1 : Y \mapsto Y_1$ and $\mathbb{P}_2 : Y \mapsto Y_2$ to be the projection operators. Then $\|\mathbb{P}_1\|, \|\mathbb{P}_2\| \leq 1$.

For any $u_2 \in Y$, let $u_2 = v_1 + v_2$ where $v_1 = \mathbb{P}_1 u_2 \in Y_1$ and $v_2 = \mathbb{P}_2 u_2 \in Y_2$. Since

$$\tilde{\mathbb{L}}u_2 = \mathbb{L}_1 v_1 + \mathbb{P}_1 \mathbb{L}_2 v_1 + \mathbb{P}_1 \mathbb{L}_2 v_2 + \mathbb{P}_2 \mathbb{L}_2 v_1 + \mathbb{P}_2 \mathbb{L}_2 v_2,$$

the operator $\tilde{\mathbb{L}} : Y \rightarrow Y$ is equivalent to the following matrix operator on $Y_1 \times Y_2$

$$\begin{aligned} & \begin{pmatrix} \tilde{\mathbb{L}}_1, \mathbb{C} \\ \mathbb{C}^*, \tilde{\mathbb{L}}_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= \left[\begin{pmatrix} \tilde{\mathbb{L}}_1, \mathbb{C} \\ 0, \tilde{\mathbb{L}}_2 \end{pmatrix} + \begin{pmatrix} 0, 0 \\ \mathbb{C}^*, 0 \end{pmatrix} \right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= (T + \mathbb{A})v, \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathbb{L}}_1 &= \mathbb{L}_1 + \mathbb{P}_1 \mathbb{L}_2 \mathbb{P}_1 : Y_1 \rightarrow Y_1, \quad \tilde{\mathbb{L}}_2 = \mathbb{P}_2 \mathbb{L}_2 \mathbb{P}_2 : Y_2 \rightarrow Y_2, \\ \mathbb{C} &= \mathbb{P}_1 \mathbb{L}_2 \mathbb{P}_2 : Y_2 \rightarrow Y_1, \quad \mathbb{C}^* = \mathbb{P}_2 \mathbb{L}_2 \mathbb{P}_1 : Y_1 \rightarrow Y_2, \end{aligned}$$

and

$$T = \begin{pmatrix} \tilde{\mathbb{L}}_1, \mathbb{C} \\ 0, \tilde{\mathbb{L}}_2 \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} 0, 0 \\ \mathbb{C}^*, 0 \end{pmatrix} : Y_1 \times Y_2 \rightarrow Y_1 \times Y_2.$$

Lemma 4.4. *The operator \mathbb{A} is T -compact.*

Proof. For any $v = (v_1, v_2) \in D(T)$, the graph norm $\|v\|_T$ is defined by

$$\begin{aligned} \|v\|_T &= \|v\|_Y + \|Tv\|_Y \\ &\approx \|v\|_Y + \|\tilde{\mathbb{L}}_1 v_1\|_Y \approx \|v\|_Y + \|\mathbb{L}_1 v_1\|_Y. \end{aligned}$$

It is obvious that $D(\mathbb{A}) \supset D(T)$. To prove \mathbb{A} is T -compact, we need to prove $\mathbb{A} : (D(\mathbb{A}), \|\cdot\|_T) \mapsto (Y, \|\cdot\|_Y)$ is compact. By the definition of \mathbb{A} , we notice that $\mathbb{A}v = (0, \mathbb{P}_2 \mathbb{L}_2 v_1) : Y_1 \times Y_2 \mapsto \{0\} \times Y_2$. For $v_1 = \nabla \xi \in Y_1$,

$$\|v_1\|_Z = \|\nabla \cdot (\rho_0 v_1)\|_{L^2_{\Phi''(\rho_0)}} + \|v_1\|_Y = \|\nabla \cdot (\rho_0 \nabla \xi)\|_{L^2_{\Phi''(\rho_0)}} + \|\nabla \xi\|_Y,$$

as defined in (1.16). By the proof of Lemma 4.2, we have

$$\begin{aligned} \|\nabla \cdot (\rho_0 v_1)\|_{L^2_{\Phi''(\rho_0)}}^2 + \|v_1\|_Y^2 &\lesssim \langle \mathbb{L}_1 v_1, v_1 \rangle + 2m \|v_1\|_Y^2 \\ &\lesssim \|\mathbb{L}_1 v_1\|_Y^2 + \|v_1\|_Y^2 \approx \|v\|_T^2. \end{aligned}$$

Thus $\|v_1\|_Z \lesssim \|v\|_T$. Since the embedding $(Y_1, \|\cdot\|_Z) \hookrightarrow (Y_1, \|\cdot\|_Y)$ is compact by Proposition 12 in [19] and $\mathbb{P}_2, \mathbb{L}_2$ are bounded operators, it follows that $\mathbb{A} : (D(\mathbb{A}), \|\cdot\|_T) \mapsto (Y, \|\cdot\|_Y)$ is compact. \square

The above lemma implies that the essential spectra of $\tilde{\mathbb{L}}$ is the same as $\tilde{\mathbb{L}}_2$.

Lemma 4.5. $\sigma_{ess}(\tilde{\mathbb{L}}) = \sigma_{ess}(\tilde{\mathbb{L}}_2)$.

Proof. We have $\sigma_{ess}(\tilde{\mathbb{L}}) = \sigma_{ess}(T + \mathbb{A})$ by the definition of the operator $T + \mathbb{A}$. By Lemma 4.4 and Weyl’s Theorem, we have $\sigma_{ess}(T + \mathbb{A}) = \sigma_{ess}(T)$. By Theorem 2.3 v) in [29] and the compact embedding of $(Y_1, \|\cdot\|_Z) \hookrightarrow (Y_1, \|\cdot\|_Y)$, the spectra of \mathbb{L}_1 on Y_1 are purely discrete and $\sigma_{ess}(\mathbb{L}_1) = \emptyset$. By the same arguments as in the proof of Lemma 4.4, $\tilde{\mathbb{L}}_1$ is relative compact to \mathbb{L}_1 and as a result $\sigma_{ess}(\tilde{\mathbb{L}}_1) = \sigma_{ess}(\mathbb{L}_1) = \emptyset$. Since the matrix operator T is upper triangular, it follows that

$$\sigma_{ess}(T) = \sigma_{ess}(\tilde{\mathbb{L}}_1) \cup \sigma_{ess}(\tilde{\mathbb{L}}_2) = \sigma_{ess}(\tilde{\mathbb{L}}_2).$$

\square

We study the essential spectra of $\tilde{\mathbb{L}}_2$ in the next two lemmas. By the Rayleigh instability condition (4.1) and the fact that $\Upsilon(0) = 4\omega_0(0)^2 \geq 0$, we know that range $(\Upsilon(r)) = [-a, b]$ for some $a > 0, b \geq 0$.

Lemma 4.6. $\sigma_{ess}(\tilde{\mathbb{L}}_2) \supset \text{range}(\Upsilon(r)) = [-a, b]$.

Proof. For any $\lambda \in (-a, b)$, let $r_0 \in (0, R_0)$ be such that $\lambda = \Upsilon(r_0)$. Choose $(r_0, z_0) \in \Omega$ and ε_0 small enough, such that $(r, z) \in \Omega$ when $|r - r_0| \leq \varepsilon_0$ and $|z - z_0| \leq \varepsilon_0^2$. Choose a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0, \varepsilon_0)$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Let $\varphi(r), \psi(z) \in C_0^\infty(-1, 1)$ be two smooth cutoff functions such that $\varphi(0) = \psi(0) = 1$. Define $\delta v^{\varepsilon_n} = (\delta v_r^{\varepsilon_n}, \delta v_z^{\varepsilon_n})$ with

$$\delta v_z^{\varepsilon_n} = -\frac{\varepsilon_n}{A_{\varepsilon_n} \rho_0 r} \varphi' \left(\frac{r - r_0}{\varepsilon_n} \right) \psi \left(\frac{z - z_0}{\varepsilon_n^2} \right),$$

and

$$\delta v_r^{\varepsilon_n} = \frac{1}{A_{\varepsilon_n} \rho_0 r} \varphi \left(\frac{r - r_0}{\varepsilon_n} \right) \psi' \left(\frac{z - z_0}{\varepsilon_n^2} \right),$$

where

$$\begin{aligned} A_{\varepsilon_n}^2 &= \int_{\mathbb{R}^3} \rho_0 \left(\left| \frac{\varepsilon_n}{\rho_0 r} \varphi' \left(\frac{r - r_0}{\varepsilon_n} \right) \psi \left(\frac{z - z_0}{\varepsilon_n^2} \right) \right|^2 + \left| \frac{1}{\rho_0 r} \varphi \left(\frac{r - r_0}{\varepsilon_n} \right) \psi' \left(\frac{z - z_0}{\varepsilon_n^2} \right) \right|^2 \right) dx \\ &= 2\pi \varepsilon_n^3 \int_{-1}^1 \int_{-1}^1 \frac{\left(\varepsilon_n^2 |\varphi'(t)\psi(s)|^2 + |\varphi(t)\psi'(s)|^2 \right)}{\rho_0 r|_{(r,z)=(\varepsilon t+r_0, \varepsilon_n^2 s+z_0)}} dt ds = O \left(\varepsilon_n^3 \right). \end{aligned}$$

Then $\|\delta v^{\varepsilon_n}\|_Y = 1$ and $\delta v^{\varepsilon_n} \in Y_2$ owing to

$$\delta \rho^{\varepsilon_n} = B_1 A \delta v^{\varepsilon_n} = \frac{1}{r} \partial_r (r \rho_0 \delta v_r^{\varepsilon_n}) + \partial_z (\rho_0 \delta v_z^{\varepsilon_n}) = 0.$$

We will show that $\{\delta v^{\varepsilon_n}\}$ is a Weyl’s sequence for the operator $\tilde{\mathbb{L}}_2$ and therefore $\lambda \in \sigma_{ess}(\tilde{\mathbb{L}}_2)$.

First, we check that δv^{ε_n} converge to 0 weakly in Y_2 . For any $\xi \in Y_2$, since δv^{ε_n} is supported in $\Omega_{\varepsilon_n} = \{|r - r_0| \leq \varepsilon_n, |z - z_0| \leq \varepsilon_n^2\}$, we have

$$|\langle \delta v^{\varepsilon_n}, \xi \rangle| \leq \|\delta v^{\varepsilon_n}\|_Y \left(2\pi \int_{r_0 - \varepsilon_n}^{r_0 + \varepsilon_n} \int_{z_0 - \varepsilon_n^2}^{z_0 + \varepsilon_n^2} \rho_0 |\xi|^2 r dr dz \right)^{\frac{1}{2}} \rightarrow 0,$$

when $\varepsilon_n \rightarrow 0$.

Next, we prove that $(\tilde{\mathbb{L}}_2 - \lambda)\delta v^{\varepsilon_n}$ converge to 0 strongly in Y_2 . We write

$$(\tilde{\mathbb{L}}_2 - \lambda)\delta v^{\varepsilon_n} = \mathbb{P}_2 \begin{pmatrix} \Upsilon(r)\delta v_r^{\varepsilon_n} \\ 0 \end{pmatrix} - \lambda \delta v^{\varepsilon_n} = \mathbb{P}_2 \begin{pmatrix} (\Upsilon(r) - \Upsilon(r_0))\delta v_r^{\varepsilon_n} \\ -\Upsilon(r_0)\delta v_z^{\varepsilon_n} \end{pmatrix}.$$

Noticing that $\|\mathbb{P}_2\| \leq 1$ and

$$\|\delta v_z^{\varepsilon_n}\|_Y^2 = \frac{O(\varepsilon_n^5)}{A_{\varepsilon_n}^2} = O(\varepsilon_n^2),$$

then we have

$$\begin{aligned} \|(\tilde{\mathbb{L}}_2 - \lambda)\delta v^{\varepsilon_n}\|_Y^2 &\leq \max_{(r,z) \in \Omega_{\varepsilon_n}} (\Upsilon(r) - \Upsilon(r_0))^2 \|\delta v_r^{\varepsilon_n}\|_Y^2 + \Upsilon(r_0)^2 \|\delta v_z^{\varepsilon_n}\|_Y^2 \\ &\leq \max_{(r,z) \in \Omega_{\varepsilon_n}} (\Upsilon(r) - \Upsilon(r_0))^2 + O(\varepsilon_n^2) \rightarrow 0, \end{aligned}$$

as $\varepsilon_n \rightarrow 0$. This shows that δv^{ε_n} is a Weyl’s sequence for $\tilde{\mathbb{L}}_2$ and $\lambda \in \sigma_{ess}(\tilde{\mathbb{L}}_2)$. Thus $(-a, b) \subset \sigma_{ess}(\tilde{\mathbb{L}}_2)$ which implies $[-a, b] \subset \sigma_{ess}(\tilde{\mathbb{L}}_2)$ since $\sigma_{ess}(\tilde{\mathbb{L}}_2)$ is closed. □

Lemma 4.7. $\sigma(\tilde{\mathbb{L}}_2) = \sigma_{ess}(\tilde{\mathbb{L}}_2) = range(\Upsilon(r)) = [-a, b]$.

Proof. Fix $\lambda \notin [-a, b]$. For any $u = (u_r, u_z) \in Y_2$, we have

$$\begin{aligned} [(\tilde{\mathbb{L}}_2 - \lambda)u, u] &= [(\mathbb{L}_2 - \lambda)u, u] \\ &= [(\Upsilon(r) - \lambda)u_r, u_r] - [\lambda u_z, u_z] \\ &= \int_{\Omega} \rho_0 (\Upsilon(r) - \lambda) u_r^2 dx + \int_{\Omega} (-\lambda) \rho_0 u_z^2 dx. \end{aligned}$$

Since $a > 0, b \geq 0$, we have

$$|[(\tilde{\mathbb{L}}_2 - \lambda)u, u]| \geq c_1 \|u\|_Y^2,$$

where $c_1 = \min\{|\lambda - b|, |a + \lambda|\} > 0$. Thus $\|(\tilde{\mathbb{L}}_2 - \lambda)u\| \geq c_1 \|u\|_Y$, which implies that $(\tilde{\mathbb{L}}_2 - \lambda)^{-1}$ is bounded and $\lambda \in \rho(\tilde{\mathbb{L}}_2)$. Therefore, $\sigma(\tilde{\mathbb{L}}_2) \subset [-a, b]$. This prove the lemma by combining with Lemma 4.6. □

The following proposition gives a complete characterization of the spectra of $\tilde{\mathbb{L}}$.

Proposition 4.1. *Under the Rayleigh instability condition (4.1), it holds:*

- i) $\sigma_{ess}(\tilde{\mathbb{L}}) = \text{range}(\Upsilon(r)) = [-a, b]$.
- ii) $\sigma(\tilde{\mathbb{L}}) \cap (-\infty, -a)$ consists of at most finitely many negative eigenvalues of finite multiplicity.
- iii) $\sigma(\tilde{\mathbb{L}}) \cap (b, +\infty)$ consists of a sequence of positive eigenvalues tending to infinity.

Proof. The conclusion in i) follows from Lemmas 4.5 and 4.7. This implies that any $\lambda \in \sigma(\tilde{\mathbb{L}})$ in $(-\infty, -a)$ or $(b, +\infty)$ must be a discrete eigenvalue of finite multiplicity.

Proof of ii): Suppose otherwise. Then there exists an infinite dimensional eigenspace for negative eigenvalues in $(-\infty, -a)$. We notice that

$$\tilde{\mathbb{L}} + aI = \mathbb{L}_1 + \mathbb{L}_2 + aI \geq \mathbb{L}_1,$$

since $\mathbb{L}_2 + aI$ is nonnegative. It follows that $n^-(\mathbb{L}_1) = \infty$ since $n^-(\tilde{\mathbb{L}} + aI) = \infty$. This is in contradiction to that $n^-(\mathbb{L}_1) \leq n^-(L) < \infty$.

Proof of iii): Suppose otherwise. Then there exists an upper bound of $\sigma(\tilde{\mathbb{L}})$, denoted by $\lambda_{max} \geq b$. Thus $\tilde{\mathbb{L}} \leq \lambda_{max}I$ which implies that

$$\mathbb{L}_1 \leq -\mathbb{L}_2 + \lambda_{max}I \leq (a + \lambda_{max})I.$$

Consequently the eigenvalues of \mathbb{L}_1 cannot exceed $a + \lambda_{max}$. This is in contradiction to the fact that \mathbb{L}_1 has a sequence of positive eigenvalues tending to infinity. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. Denote $\pi_\lambda \in L(X)$ ($\lambda \in \mathbf{R}$) to be the spectral family of the self-adjoint operator $\tilde{\mathbb{L}}$. Let $\{\mu_i\}_{i=1}^\infty$ be the eigenvalues of $\tilde{\mathbb{L}}$ in (b, ∞) . If $\sigma(\tilde{\mathbb{L}}) \cap (-\infty, -a) \neq \emptyset$, we denote the eigenvalues in $(-\infty, -a)$ by $\nu_1 < \dots < \nu_K$ where $K = \dim(R(\pi_{-a}))$. For $1 \leq i < \infty$, $1 \leq j \leq K$, let $P_i^+ = \pi_{\mu_i+} - \pi_{\mu_i-}$ and $P_j^- = \pi_{\nu_j+} - \pi_{\nu_j-}$ be the projections to $\ker(\tilde{\mathbb{L}} - \mu_i I)$ and $\ker(\tilde{\mathbb{L}} - \nu_j I)$ respectively, and $P_0 = \pi_{0+} - \pi_{0-}$ be the projection to $\ker \tilde{\mathbb{L}}$. By Proposition 4.1, we have

$$\tilde{\mathbb{L}} = \int \lambda d\pi_\lambda = \sum_{i=1}^\infty \mu_i P_i^+ + \sum_{j=1}^K \nu_j P_j^- + \int_{-a}^b \lambda d\pi_\lambda.$$

For any initial data $(u_2(0), u_{2t}(0)) \in Z \times Y$, the solution to the second order equation (4.2) can be written as

$$\begin{aligned}
 u_2(t) = & \sum_{i=1}^{\infty} \left[\cos(\sqrt{\mu_i}t) P_i^+ u_2(0) + \frac{1}{\sqrt{\mu_i}} \sin(\sqrt{\mu_i}t) P_i^+ u_{2t}(0) \right] \\
 & + \sum_{j=1}^K \left[\cosh(\sqrt{-\nu_j}t) P_j^- u_2(0) + \frac{1}{\sqrt{-\nu_j}} \sinh(\sqrt{-\nu_j}t) P_j^- u_{2t}(0) \right] \\
 & + \int_0^b \cos(\sqrt{\lambda}t) d\pi_{\lambda} u_2(0) + \int_0^b \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t) d\pi_{\lambda} u_{2t}(0) \\
 & + \int_{-a}^0 \cosh(\sqrt{-\lambda}t) d\pi_{\lambda} u_2(0) + \int_{-a}^0 \frac{1}{\sqrt{-\lambda}} \sinh(\sqrt{-\lambda}t) d\pi_{\lambda} u_{2t}(0) \\
 & + P_0 u_2(0) + t P_0 u_{2t}(0).
 \end{aligned} \tag{4.3}$$

If $\sigma(\tilde{\mathbb{L}}) \cap (-\infty, -a) = \emptyset$, the solution $u_2(t)$ is obtained by removing the second term above.

Denote the minimum of $\lambda \in \sigma(\tilde{\mathbb{L}})$ by η_0 , that is,

$$\begin{aligned}
 \eta_0 = & \min_{\|\psi\|_Y=1} [\tilde{\mathbb{L}}\psi, \psi] \\
 = & \begin{cases} -a, & \text{if } \sigma(\tilde{\mathbb{L}}) \cap (-\infty, -a) = \emptyset, \\ v_1, & \text{if } \sigma(\tilde{\mathbb{L}}) \cap (-\infty, -a) = \{v_1 < \dots < v_K\}. \end{cases}
 \end{aligned}$$

By the formula (4.3), it is easy to see that $\|u_2(t)\|_Y \lesssim e^{\sqrt{-\eta_0}t}$ for $t > 0$. To estimate $\|u_2(t)\|_Z$, we note that by Lemma 4.2

$$\|u_2\|_Z^2 \approx [\tilde{\mathbb{L}}u_2, u_2] + 2m\|u_2\|_Y^2. \tag{4.4}$$

By using (4.3), we have

$$\begin{aligned}
 [\tilde{\mathbb{L}}u_2(t), u_2(t)] & \lesssim \sum_{i=1}^{\infty} \left[\mu_j \|P_i^+ u_2(0)\|_Y^2 + \|P_i^+ u_{2t}(0)\|_Y^2 \right] \\
 & + e^{2\sqrt{-\eta_0}t} \sum_{j=1}^K \left[\|P_j^- u_2(0)\|_Y^2 + \|P_j^- u_{2t}(0)\|_Y^2 \right] \\
 & + \int_0^b d(\pi_{\lambda} u_2(0), u_2(0)) + \int_0^b d(\pi_{\lambda} u_{2t}(0), u_{2t}(0)) \\
 & + e^{2\sqrt{-\eta_0}t} \left[\int_{-a}^0 d(\pi_{\lambda} u_2(0), u_2(0)) + \int_{-a}^0 d(\pi_{\lambda} u_{2t}(0), u_{2t}(0)) \right] \\
 & \lesssim e^{2\sqrt{-\eta_0}t} \left([\tilde{\mathbb{L}}u_2(0), u_2(0)] + m\|u_2(0)\|_Y^2 + \|u_{2t}(0)\|_Y^2 \right) \\
 & \lesssim e^{2\sqrt{-\eta_0}t} \left(\|u_2(0)\|_Z^2 + \|u_{2t}(0)\|_Y^2 \right).
 \end{aligned}$$

This implies

$$\|u_2(t)\|_Z \lesssim e^{\sqrt{-\eta_0}t} (\|u_2(0)\|_Z + \|u_{2t}(0)\|_Y),$$

by using (4.4) and the estimate for $\|u_2(t)\|_Y$. Since

$$\begin{aligned} u_{2t}(t) &= \sum_{i=1}^{\infty} \left[-\sqrt{\mu_i} \sin(\sqrt{\mu_i}t) P_i^+ u_2(0) + \cos(\sqrt{\mu_i}t) P_i^+ u_{2t}(0) \right] \\ &\quad + \sum_{j=1}^K \left[\sqrt{-\nu_j} \sinh(\sqrt{-\nu_j}t) P_j^- u_2(0) + \cosh(\sqrt{-\nu_j}t) P_j^- u_{2t}(0) \right] \\ &\quad + \int_0^b -\sqrt{\lambda} \sin(\sqrt{\lambda}t) d\pi_{\lambda} u_2(0) + \int_0^b \cos(\sqrt{\lambda}t) d\pi_{\lambda} u_{2t}(0) \\ &\quad + \int_{-a}^0 \sqrt{-\lambda} \sinh(\sqrt{-\lambda}t) d\pi_{\lambda} u_2(0) + \int_{-a}^0 \cosh(\sqrt{-\lambda}t) d\pi_{\lambda} u_{2t}(0) + P_0 u_{2t}(0), \end{aligned}$$

by similar estimates as above for $\|u_2(t)\|_Z$, we obtain

$$\|u_{2t}(t)\|_Y \lesssim e^{\sqrt{-\eta_0}t} (\|u_2(0)\|_Z + \|u_{2t}(0)\|_Y).$$

This finishes the proof of the upper bound estimate (1.17). It is straightforward to show that the energy $E(u_2, u_{2t})$ defined in (1.18) is conserved for solutions of (4.2).

Next, we prove the lower bound estimate (1.19) in two cases.

Case 1: $\sigma(\tilde{\mathbb{L}}) \cap (-\infty, -a) \neq \emptyset$. We choose $u_2(0) = \psi_1$ and $u_{2t}(0) = \sqrt{-\nu_1} \psi_1$ where $\psi_1 \in Z$ is the eigenfunction of $\tilde{\mathbb{L}}$ corresponding to the smallest eigenvalue ν_1 in $(-\infty, -a)$. Then

$$(u_2(t), u_{2t}(t)) = \left(e^{\sqrt{-\nu_1}t} \psi_1, \sqrt{-\nu_1} e^{\sqrt{-\nu_1}t} \psi_1 \right),$$

which clearly implies $\|u_2(t)\|_Y \gtrsim e^{\sqrt{-\eta_0}t} \|u_2(0)\|_Z$.

Case 2: $\sigma(\tilde{\mathbb{L}}) \cap (-\infty, -a) = \emptyset$. Since $\sigma_{ess}(\tilde{\mathbb{L}}) = [-a, b]$, for any $\varepsilon > 0$ small there exists a nonzero function $\phi \in R(\pi_{-a+\varepsilon} - \pi_{-a}) \subset Z$. Choose the initial data $u_2(0) = \phi$ and $u_{2t}(0) = 0$. Then the solution $u_2(t)$ for the equation (4.2) is given by

$$u_2(t) = \int_{-a}^{-a+\varepsilon} \cosh(\sqrt{-\lambda}t) d\pi_{\lambda} \phi.$$

Thus

$$\begin{aligned} \|u_2(t)\|_Y^2 &= \int_{-a}^{-a+\varepsilon} \cosh^2(\sqrt{-\lambda}t) d(\pi_{\lambda} \phi, \phi) \gtrsim e^{\sqrt{-\eta_0-\varepsilon}t} \int_{-a}^{-a+\varepsilon} d(\pi_{\lambda} \phi, \phi) \\ &\gtrsim e^{\sqrt{-\eta_0-\varepsilon}t} \|\phi\|_Z. \end{aligned}$$

This finishes the proof of the theorem. □

Remark 4.1. By Theorem 1.2, the maximal growth rate of unstable rotating stars can be due to either discrete or continuous spectrum. Consider a family of slowly rotating stars $(\rho_{\varepsilon}, \vec{v}_{\varepsilon} = \varepsilon r \omega_0(r) \mathbf{e}_{\theta})$ near a non-rotating star $(\rho_0(|x|), \vec{v}_0 = \vec{0})$ with $\omega_0(r)$ satisfying the Rayleigh instability condition (4.1). If the non-rotating star is linearly stable, then for sufficiently small ε , the linear instability of $(\rho_{\varepsilon}, \vec{v}_{\varepsilon})$ is due to the continuous spectrum. On the other hand, if the non-rotating star is linearly unstable, then for sufficiently small ε , $(\rho_{\varepsilon}, \vec{v}_{\varepsilon})$ remains unstable and the maximal growth rate is due to the discrete eigenvalue perturbed from the unstable eigenvalue of the non-rotating star.

Remark 4.2. In [23], Lebovitz indicated that for slowly rotating stars with any angular velocity profile $\omega_0(r)$, discrete unstable modes cannot be perturbed from neutral modes of non-rotating stars. More precisely, Lebovitz showed the stabilizing influence of rotation on the fundamental mode (corresponding to the first eigenvalue of the operator \tilde{L} in (4.2)) even when $\omega_0(r)$ does not satisfy the Rayleigh stability condition. However, this does not imply the stability of the rotating stars since the unstable continuous spectrum was not considered in [23].

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