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Existence and Uniqueness of Local Regular Solution to the Schrödinger Flow from a Bounded Domain in $\mathbb{R}^3$ into $\mathbb{S}^2$

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Abstract: In this paper, we show the existence and uniqueness of local regular solutions to the initial-Neumann boundary value problem of the Schrödinger flow from a smooth bounded domain $\Omega \subset \mathbb{R}^3$ into $\mathbb{S}^2$(namely Landau–Lifshitz equation without dissipation). The proof is built on a parabolic perturbation method, an intrinsic geometric energy argument, the symmetric (algebraic) properties of $\mathbb{S}^2$ and some observations on the behaviors of some geometric quantities on the boundary of the domain manifold.It is based on methods from Ding and Wang (one of the authors of this paper) for the Schrödinger flows of maps from a closed Riemannian manifold into a Kähler manifold as well as on methods by Carbou and Jizzini for solutions of the Landau–Lifshitz equation.

1. Introduction

In this paper, we are concerned with the existence and uniqueness of strong solutions to the initial-Neumann boundary value problem of the following Schrödinger flow from a smooth bounded domain $\Omega \subset \mathbb{R}^3$ into $\mathbb{S}^2$

$$\partial_t u = u \times \Delta u,$$

where "$\times$" denotes the cross product in $\mathbb{R}^3$. It is well-known that this equation is also called the Landau–Lifshitz equation with a long history (see [28]). The solvability of such a Landau–Lifshitz equation with natural boundary condition is an important and challenging problem, and by the knowledge of authors there were few results on its well-posedness in the previous literature if the dimension of the domain Ω denoted by $\dim(\Omega)$ is greater than one.

Bo Chen and Youde Wang have contributed equally to this work.

1.1. Definitions and background. In physics, the Landau–Lifshitz (LL) equation is a fundamental evolution equation for the ferromagnetic spin chain and was proposed on the phenomenological ground in studying the dispersive theory of magnetization of ferromagnets. It was first deduced by Landau and Lifshitz in [28], and then proposed by Gilbert in [22] with dissipation. It is well-known that the Landau and Lifshitz system is of fundamental importance in theory of magnetization and ferromagnets, and has extensive applications in physics.

In fact, this equation describes the Hamiltonian dynamics corresponding to the Landau–Lifshitz energy, which is defined as follows. Consider a ferromagnetic body occupying a bounded, possibly multi-connected domain Ω of the Euclidean space $\mathbb{R}^3$. We neglect mechanical effects due to magnetization (magnetostriction) and assume the temperature to be constant and lower than Curie's temperatures. Let u, denoting magnetization vector, be a mapping from Ω into a unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. The Landau–Lifshitz energy functional of a map $u : \Omega \to \mathbb{S}^2$ is defined by

$$\mathcal{E}(u) := \int_\Omega \Phi(u)\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Omega h_d \cdot u\,dx.$$

Here ∇ denote the gradient operator and dx is the volume element of $\mathbb{R}^3$.

For the above Landau–Lifshitz functional, the first and second terms are the anisotropy and exchange energies, respectively, and $\Phi(u)$ is a smooth function on $\mathbb{S}^2$. The last term is the self-induced energy, and $h_d(u)$ is the demagnetizing field, which has the following form

$$h_d(u)(x) = \nabla \int_\Omega \nabla_y N(x-y)u(y)dy,$$

where $N(x) = -\frac{1}{4\pi|x|}$ is the Newtonian potential in $\mathbb{R}^3$.

The Landau–Lifshitz (for short LL) equation without dissipation can be written as

$$\partial_t u = -u \times h \tag{1.1}$$

where the local field h of $\mathcal{E}(u)$ can be derived as

$$h := -\frac{\delta \mathcal{E}(u)}{\delta u} = \Delta u + h_d - \nabla_u \Phi.$$

In this paper we want to consider the existence of regular solution to equation (1.1). Since the anisotropy term Φ, non-local field $h_d(u)$ and the negative sign"−" in equation (1.1) do not affect on our analysis and main conclusions, for the sake of convenience and simplicity, we only consider the classical Schrödinger flow into $\mathbb{S}^2$ (Landau–Lifshitz)

$$\partial_t u = u \times \Delta u.$$

Intrinsicly, "$u\times$" can be considered as a complex structure J on $\mathbb{S}^2$, which rotates anticlockwise the tangent space at u by an angle of $\frac{\pi}{2}$ degrees. Therefore, we can write the above equation as

$$\partial_t u = J(u)P(u)(\Delta u),$$

where $P(u) : \mathbb{R}^3 \to T_u\mathbb{S}^2$ is a standard projection operator.

From the viewpoint of infinite dimensional symplectic geometry, Ding-Wang [18] proposed to consider the so-called Schrödinger flows for maps from a Riemannian manifold into a symplectic manifold, which can be regarded as an extension of LL equation (1.1) and was also independently introduced from the viewpoint of integrable systems by Terng and Uhlenbeck in [47]. Namely, suppose (M, g) is a Riemannian manifold, (N, J, ω) is a symplectic manifold, the Schrödinger flow is a time-dependent map $u : M \times \mathbb{R}^+ \to N \hookrightarrow \mathbb{R}^{n+k}$ satisfying

$$\partial_t u = J(u)\tau(u)$$

where

$$\tau(u) = \Delta_g u + A(u)(\nabla u, \nabla u)$$

is the tension field of u, where $A(u)(\cdot, \cdot)$ is the second fundamental form of N in $\mathbb{R}^{n+k}$. Here we always embed isometrically (N, J, ω) in an Euclidean space $\mathbb{R}^{n+k}$ where $n = \dim(N)$.

A great deal of effort has been devoted to the study of Landau–Lifshitz equation defined on an Euclidean spaces or a flat torus (closed manifold) in the last five decades. One has made great progress in the PDE aspects of the Schrödinger flows containing the existence, uniqueness and regularities of various kinds of solutions. Now, we recall some known results which are closely related to our work in the present paper.

In 1986, P.L. Sulem, C. Sulem and C. Bardos in [45] employed difference method to prove that the Schrödinger flow for maps from $\mathbb{R}^n$ into $\mathbb{S}^2$ admits a global weak solution or a smooth local solution under suitable initial value conditions. Moreover, they also addressed the existence of global smooth solution if the initial value map is small. In 1998, Y.D. Wang (the second named author) [49] obtained the global existence of weak solution to the Schrödinger flow for maps from a closed Riemannian manifold or a bounded domain in $\mathbb{R}^n$ into $\mathbb{S}^2$ by adopting a more geometric approximation equation than the Ginzburg-Landau penalized equation used for the LLG equation in [1,7,46]. Later, Z.L. Jia and Y.D. Wang [25,26] employed a method originated from [20,49] to achieve the global weak solutions to a large class of generalized Schrödinger flows in more general setting, where the base manifold is a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ or a compact Riemannian manifold M^n and the target space is $\mathbb{S}^2$ or the unit sphere $\mathbb{S}^n_{\mathfrak{g}}$ in a compact Lie algebra $\mathfrak{g}$. However, the existence of global weak solution for the Schrödinger flows between manifolds are still open.

The local existence of the Schrödinger flow from a general closed Riemannian manifold into a Kähler manifold was first obtained by Ding and the second named author of this paper in [18]. By using a parabolic approximation and the intrinsic geometric energy method, they proved that, if M is an m dimensional compact Riemannian manifold or the Euclidean space $\mathbb{R}^m$ and the initial map $u_0 \in W^{k,2}(M, N)$ with $k \geq [m/2] + 2$, then there exists a local solution $u \in L^\infty([0, T), W^{k,2}(M, N))$. The local regular(smooth) solution to the Schrödinger flow from $\mathbb{R}^n$ into a Kähler manifold was also addressed by Ding and Wang in [19] (Later, Kenig, Lamm, Pollack, Staffilani and Toro in [27] also provided another different approach). Furthermore, they also obtained the persistence of regularity results, in that the solution always stays as regular as the initial data (as measured in Sobolev norms), provided that one is within the time of existence guaranteed by the local existence theorem. In proving its well-posedness, the heart of the matter is resolved by estimating multi-linear forms of some intrinsic geometric quantities.

For low-regularity initial data, the initial value problem for Schrödinger flow from an Euclidean space into $\mathbb{S}^2$ has been studied indirectly using the “modified Schrödinger map equations" and certain enhanced energy methods, for instance, A.R. Nahmod, A. Stefanov and K. K. Uhlenbeck [35] have ever used the standard technique of Picard iteration in some suitable function spaces of the Schrödinger equation to obtain a near-optimal (but conditional) local well-posedness result for the Schrödinger map flow equation from two dimensions into a Riemann surface X, in the model cases of the standard sphere $X = \mathbb{S}^2$ or hyperbolic space $X = \mathbb{H}^2$. In proving its well-posedness, the heart of the matter is resolved by considering truly quatrilinear forms of weighted L^2-functions.

For one dimensional global existence for Schrödinger flow from $\mathbb{S}^1$ or $\mathbb{R}^1$ into a Kähler manifold, we refer to [13,36,39,52] and references therein. The global well-posedness result for the Schrödinger flow from $\mathbb{R}^n$ ($n \geq 3$) into $\mathbb{S}^2$ in critical Besov spaces was proved by Ionescu and Kenig in [24], and independently by Bejanaru in [3], and then was improved to global regularity for small data in the critical Sobolev spaces in dimensions $n \geq 4$ in [4]. Finally, in [5] the global well-posedness result for the Schrödinger flow for small data in the critical Sobolev spaces in dimensions $n \geq 2$ was addressed. Recently, Z. Li in [29,30] proved that the Schrödinger flow from $\mathbb{R}^n$ with $n \geq 2$ to compact Kähler manifold with small initial data in critical Sobolev spaces is also global.

On the contrary, F. Merle, P. Raphaël and I. Rodnianski [33] also considered the energy critical Schrödinger flow problem with the 2-sphere target for equivariant initial data of homotopy index $k = 1$. They showed the existence of a codimension one set of smooth well localized initial data arbitrarily close to the ground state harmonic map in the energy critical norm, which generates finite time blowup solutions, and gave a sharp description of the corresponding singularity formation which occurs by concentration of a universal bubble of energy. One also found some self-similar solutions to Schrödinger flow from $\mathbb{C}^n$ into $\mathbb{C}P^n$ with local bounded energy which blow up at finite time, for more details we refer to [17,21,34].

As for some travelling wave solutions with vortex structures, F. Lin and J. Wei [31] employed perturbation method to consider such solutions for the Schrödinger map flow equation with easy-axis and proved the existence of smooth travelling waves with bounded energy if the velocity of travelling wave is small enough. Moreover, they showed the travelling wave solution has exactly two vortices. Later, J. Wei and J. Yang [50] considered the same Schrödinger map flow equation as in [31], i.e. the Landau–Lifshitz equation describing the planar ferromagnets. They constructed a travelling wave solution possessing vortex helix structures for this equation. Using the perturbation approach, they give a complete characterization of the asymptotic behaviour of the solution.

On the other hand, since the seventies of the 20th century magnetic domains have been the object of a considerable research from the applicative viewpoint (e.g., see [42,48]), especially because of invention of "magnetic bubbles" devices and their use in computer hardware. In the literature, physicists and mathematicians are always interested in the Landau–Lifshitz-Gilbert system with Neumann boundary conditions(see [10,40]), for instance, Carbou and Jizzini considered a model of ferromagnetic material subject to an electric current, and proved the local in time existence of very regular solutions for this model in the scale of H^k spaces. In particular, they described in detail the compatibility conditions at the boundary for the initial data, for details we refer to [11]. Roughly speaking, Carbou and Jizzini showed that

$$\begin{cases} \partial_t u = -u \times (u \times \Delta u) + \alpha u \times \Delta u, & \text{(x,t)} \in \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} = 0, & \text{(x,t)} \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to \mathbb{S}^2, \end{cases}$$

admits a very regular solution if u_0 meets some compatibility conditions at the boundary, where α is a real number.

A natural problem is if the following

$$\begin{cases} \partial_t u = u \times \Delta u, & \text{(x,t)} \in \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} = 0, & \text{(x,t)} \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to \mathbb{S}^2, \end{cases}$$

where ν is the outer normal vector on Ω and u_0 is the initial value map, admits a strong or regular solution? The results on the existence of global weak solutions proved by Wang in [49] hints us that the initial-Neumann boundary value problem of the Schrödinger flow should be posed as follows

$$\begin{cases} \partial_t u = u \times \Delta u, & \text{(x,t)} \in \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} = 0, & \text{(x,t)} \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to \mathbb{S}^2, \quad \frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0. \end{cases} \tag{1.2}$$

Our goal of this paper is to prove the above problem (1.2) admits a local in time strong (regular) solution. The above Schrödinger flow with starting manifold is a bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ is a challenging problem, there is few results on the well-posedness of initial-Neumann boundary value problem (1.2) in the literature.

1.2. Strategy and main results. In the present paper, we are intend to studying the local well-posedness of the above equation (1.2). However, the method involving harmonic analysis in $\mathbb{R}^n$ used in [2–6,24] may not be effective for the equation in a bounded domain or a complete compact manifold. And hence, we still apply a similar parabolic perturbation approximation and intrinsic geometric energy method with that in [18] since Carbou and Jizzini [11] have shown that the corresponding problem of the approximating equation is well-posed. More precisely, we will employ the following parabolic perturbation equation to approximate (1.2)

$$\begin{cases} \partial_t u_\varepsilon = \varepsilon(\Delta u_\varepsilon + |\nabla u_\varepsilon|^2 u_\varepsilon) + u_\varepsilon \times \Delta u_\varepsilon, & \text{(x,t)} \in \Omega \times \mathbb{R}^+, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0, & \text{(x,t)} \in \partial\Omega \times \mathbb{R}^+, \\ u_\varepsilon(x,0) = u_0 : \Omega \to \mathbb{S}^2, \quad \frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0 \end{cases} \tag{1.3}$$

where $\varepsilon \in (0,1)$ is the perturbation constant, since the perturbed equation and the Schrödinger flow into $\mathbb{S}^2$ (Landau–Lifshitz equation without dissipative term) share the same compatible conditions of the initial data.

Because of the space of test functions associated to equation (1.2) is much smaller than that for the setting in [18], it is more difficult for us to get the desired geometric energy estimates on the solutions of (1.2). We need to overcome some new essential difficulties caused by the boundary of domain manifold.

Our main conclusions can be presented as follows.

Theorem 1.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Assume that the initial value maps $u_0 \in H^3(\Omega, \mathbb{S}^2)$ with $\frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0$. Then, there exists a positive number $T_0 > 0$ depending only on $\|u_0\|_{H^3}$ and the geometry of Ω such that the equation (1.2) admits a unique regular solution $u \in L^\infty([0,T_0], H^3(\Omega,\mathbb{S}^2)) \cap C^0([0,T_0], H^2(\Omega,\mathbb{S}^2))$ with initial value u_0.*

Remark 1.2. 1. Theorem 1.1 still holds for the case of Ω being a smooth bounded domain in $\mathbb{R}^2$ (also see [37,38]).

2. If the initial map u_0 satisfies some furthermore compatibility and applicable regularity conditions, we also prove (1.2) admits a unique smooth solution u with initial value u_0. We will present these results in another paper [15].
3. In forthcoming papers we will extend the results in Theorem 1.1 to the case the starting manifold of the Schrödinger flow is a 2 or 3 dimensional compact Riemannian maniflod with smooth boundary and the target manifold is a compact Kähler manifold.
4. It is still open for the case the dimension of the starting manifold is greater than three.

On the other hand, we recall that the self-induced vector field is defined by

$$h_d(u) := \nabla \int_\Omega \nabla N(x-y)u(y)\,dy$$

in the sense of distributions. Hence, the following estimates of h_d is a fundamental result in theory of singular integral operators, its proof can be found in [10,11,14].

Proposition 1.3. *Let $p \in (1, \infty)$ and Ω be a smooth bounded domain in $\mathbb{R}^3$. Assume that $u \in W^{k,p}(\Omega, \mathbb{R}^3)$ for $k \in \mathbb{N}$. Then, the restriction of $h_d(u)$ to Ω belongs to $W^{k,p}(\Omega, \mathbb{R}^3)$. Moreover, there exists constants $C_{k,p}$, which is independent of u, such that*

$$\|h_d(u)\|_{W^{k,p}(\Omega)} \le C_{k,p}\|u\|_{W^{k,p}(\Omega)}.$$

In fact, $h_d : W^{k,p}(\Omega, \mathbb{R}^3) \to W^{k,p}(\Omega, \mathbb{R}^3)$ is a bounded linear operator.

With Proposition 1.3 at hand, we take an almost same argument as that in the proof of Theorem 1.1 to conclude the following result.

Theorem 1.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Assume that the initial value maps $u_0 \in H^3(\Omega, \mathbb{S}^2)$ with $\frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0$, and $\Phi \in C^\infty(\mathbb{S}^2)$. Then, there exists a positive number $T_0 > 0$ depending only on $\|u_0\|_{H^3}$ and the geometry of Ω such that the initial-Neumann boundary value problem of equation (1.1) admits a unique regular solution $u \in L^\infty([0, T_0], H^3(\Omega, \mathbb{S}^2)) \cap C^0([0, T_0], H^2(\Omega, \mathbb{S}^2))$ with initial value u_0.*

To prove the above conclusions we need to overcome two essential difficulties. One is how to find test functions which are compatible with Neumann boundary condition (i.e., these test functions produce vanishing boundary terms when integrations by parts are taken) and the other is how to get a uniform H^3 Sobolev norm estimate of u_ε ("uniform" means "independent of $\varepsilon \in (0, 1)$"). We will make full use of the geometric structures of the Schrödinger flow to overcome the two difficulties. Our proofs provided here are based on the following two crucial observations:

1. By L^2-estimate of Laplacian operator with Neumann boundary condition (i.e. $\frac{\partial u_\varepsilon}{\partial \nu}|_{\partial\Omega} = 0$) and the equation (1.3), we can infer the following critical equivalent norm estimate:

$$\|u_\varepsilon\|^2_{H^3} \le C(1 + \|u_\varepsilon\|^2_{H^2} + \|\partial_t u_\varepsilon\|^2_{H^1})^3,$$

where C is independent of ε, namely Lemma 3.6. In order to show uniform H^3 norm estimate, the above equivalent norm inequality implies that it is enough to get a uniform bound of

$$\|u_\varepsilon\|^2_{H^2} + \|\partial_t u_\varepsilon\|^2_{H^1}.$$

2. By the orthogonality of $\tau(u_\varepsilon)$ and $u_\varepsilon \times \Delta u_\varepsilon$, we can see that $\partial_t u_\varepsilon$ and $\tau(u_\varepsilon)$ satisfy the Neumann boundary conditions in the sense of distribution, which are induced by

$\frac{\partial u_\varepsilon}{\partial \nu}|_{\partial\Omega} = 0$ (i.e. Lemma 3.5). This implies that $\partial_t u_\varepsilon$, $\Delta\partial_t u_\varepsilon$ and $\Delta\tau(u_\varepsilon)$ are suitable candidates for test functions compatible with Neumann boundary condition.

Therefore, in order to derive required energy estimates on the solutions to Schrödinger flow as $\varepsilon \to 0$, we should consider the equation satisfied by $\partial_t u_\varepsilon$ and choose $\partial_t u_\varepsilon$, $\Delta\partial_t u_\varepsilon$ and $\Delta\tau(u_\varepsilon)$ as the test functions of the equation satisfied by $\partial_t u_\epsilon$ such as (3.7) and (3.8). We will see that one can use the facts $|u_\varepsilon| = 1$, $(\mathbb{R}^3, \times)$ is a Lie algebra and $J(u) = u\times : T_u\mathbb{S}^2 \to T_u\mathbb{S}^2$ is a integrable complex structure on $\mathbb{S}^2$, i.e. $\nabla^{\mathbb{S}^2} J = 0$, to infer a concise intrinsic equation of $\partial_t u_\varepsilon$ (i.e. Formula (1.4)), from which it is not difficult to show $\partial_t u_\varepsilon$ satisfies an extrinsic parabolic-type equation (i.e. Formula (3.7)) and another key fourth order differential equation (i.e. Formula (3.8)).

Concretely, the proof of Theorem 1.1 is divided into three steps. In the first step, we consider the parabolic perturbation approximation equation (1.3). Recall that the local well-posedness for this parabolic system is established recently in [11], which can be formulated as the following proposition (also see our recent work [14]).

Proposition 1.5. *Suppose that $u_0 \in H^3(\Omega, \mathbb{S}^2)$, there exists a positive number T_ε depending only on ε and $\|u_0\|_{H^2}$ such that the equation (1.3) admits a unique regular solution u_ε on $\Omega \times [0, T_\varepsilon)$ which satisfies for any $T < T_\varepsilon$ that*

1. $|u_\varepsilon(x,t)| = 1$ *for all* $(x,t) \in \Omega \times [0, T]$;
2. $u_\varepsilon \in L^\infty([0,T], H^3(\Omega)) \cap L^2([0,T], H^4(\Omega))$.

In Sect. 3, we will extend T_ε stated in this proposition to the maximal existence time of solution u_ε satisfying the above properties.

In the second step, we get the uniform H^3-energy estimates of u_ε with respect to ϵ. Our basic idea is to make full use of the integrality of complex structure to get higher geometric energy estimates of solution u_ε to

$$\partial_t u = \varepsilon\tau(u) + J(u)\tau(u)$$

as in [18]. Inspired by this idea and the above two new observations, we consider the corresponding equation of $\partial_t u_\varepsilon$ from the viewpoint of intrinsic geometry. For simplicity we let M be a bounded domain $\Omega \subset \mathbb{R}^n$ and choose the natural coordinates $x = \{x_1, \ldots, x_n\}$ on Ω. Noting that $\nabla J = 0$, we take a simple calculation to show

$$\begin{aligned}
&\nabla_{\partial t}\partial_t u_\varepsilon + (1-\varepsilon^2)\nabla_i\nabla_i(\tau(u_\varepsilon)) - 2\varepsilon J(u_\varepsilon)\nabla_i\nabla_i(\tau(u_\varepsilon)) \\
&\quad = \varepsilon^2 R^N(\tau(u_\varepsilon), \nabla_i u_\varepsilon)\nabla_i u_\varepsilon + \varepsilon J(u_\varepsilon)R^N(\tau(u_\varepsilon), \nabla_i u_\varepsilon)\nabla_i u_\varepsilon \\
&\qquad + \varepsilon R^N(J(u_\varepsilon)\tau(u_\varepsilon), \nabla_i u_\varepsilon)\nabla_i u_\varepsilon + J(u_\varepsilon)R^N(J(u_\varepsilon)\tau(u_\varepsilon), \nabla_i u_\varepsilon)\nabla_i u_\varepsilon. \qquad (1.4)
\end{aligned}$$

Here, $\nabla_i = \nabla^N_{\frac{\partial u}{\partial x_i}}$ and R^N is the Riemannian curvature tensor of N.

In particular, if the target manifold $N = \mathbb{S}^2$ with the complex structure $J(u) = u\times$ for $u \in \mathbb{S}^2$, then it is not difficult to show that u_ε satisfies the following extrinsic formula (to see Formula (3.8) for precise form)

$$\begin{aligned}
&\frac{\partial^2 u_\varepsilon}{\partial t^2} + (1-\varepsilon^2)\Delta\tau(u_\varepsilon) - 2\varepsilon\Delta(u_\varepsilon \times \Delta u_\varepsilon) \\
&\quad = \varepsilon\{\frac{\partial}{\partial t}(|\nabla u_\varepsilon|^2 u_\varepsilon) - 2\mathrm{div}(\nabla u_\varepsilon \dot{\times} \nabla^2 u_\varepsilon) + u_\varepsilon \times \Delta(|\nabla u_\varepsilon|^2 u_\varepsilon)\} \\
&\qquad + \varepsilon|\nabla u_\varepsilon|^2 u_\varepsilon \times \Delta u_\varepsilon + \Delta(|\nabla u_\varepsilon|^2 u_\varepsilon) - 2\mathrm{div}^2((\nabla u_\varepsilon \dot{\otimes} \nabla u_\varepsilon))u_\varepsilon \\
&\qquad - 2\left\langle \nabla|\nabla u_\varepsilon|^2, \nabla u_\varepsilon\right\rangle - 2\langle\Delta u_\varepsilon, \nabla u_\varepsilon\rangle \cdot \nabla u_\varepsilon - |\nabla u_\varepsilon|^2\Delta u_\varepsilon \qquad (1.5)
\end{aligned}$$

by using the fact $|u| = 1$ and

$$R^{\mathbb{S}^2}(X, Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$$

for any X, Y and Z in $\Gamma(T\mathbb{S}^2)$.

Let $\varepsilon = 0$. After contracting some terms in the above extrinsic formula, we obtain again the nice equation in [45] (also to see Formula (3.14))

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} + \Delta^2 u = &- 2\mathrm{div}^2((\nabla u\dot{\otimes}\nabla u)u) + 2\mathrm{div}((\nabla u\dot{\otimes}\nabla u)\cdot\nabla u)\\ &- \mathrm{div}(|\nabla u|^2\nabla u).\end{aligned} \tag{1.6}$$

Then, by taking $\partial_t u_\varepsilon$ and $\partial_t \Delta u_\varepsilon$ as test functions to the extrinsic equation (1.5) respectively, we can obtain the desired H^3-estimates of u_ε on a uniform time interval $[0, T_0]$ with $0 < T_0 < T_\varepsilon$ by a delicate complicated computation and by using the following essential geometric information from the target manifold $(\mathbb{S}^2, J = u\times) \hookrightarrow (\mathbb{R}^3, \times)$:

(1) The cross product $(\mathbb{R}^3, \times)$ additionally satisfies the Lagrangain formula

$$a \times (b \times c) = \langle a, c\rangle b - \langle a, b\rangle c$$

for any vectors $a, b, c \in \mathbb{R}^3$.

(2) Any $u \in \mathbb{S}^2 \subset \mathbb{R}^3$ can be considered as a vector in $T_u^{\perp}\mathbb{S}^2$. Namely, for any $X \in T_u\mathbb{S}^2$, we have

$$\langle u, X\rangle = 0.$$

It is worthy to point out that we use the geometric and algebraic structures (i.e., the antisymmetry of $J = u\times$, the above properties (1) and (2) on Lie algebraic structure) associated with this system to eliminate some terms involving high order derivatives of u_ε in the process of getting uniform H^3-estimates, hence the remaining terms can be well-controlled by applying Lemmas 2.1 and Lemma 3.6 on equivalent Sobolev norms. For more details we refer to Sects. 3.1 and 3.2.

Therefore, by letting $\varepsilon \to 0$, the existence part of Theorem 1.1 is proved.

In the last step, we show the uniqueness of the solution u we obtained by adopting the intrinsic energy method introduced in [32,44].

1.3. A related problem. Recently, Chern et al [16] described a new approach for the purely Eulerian simulation of incompressible fluids. In their setting, the fluid state is represented by a $\mathbb{C}^2$-valued wave function evolving under the Schrödinger equation subject to incompressibility constraints. The underlying dynamical system is Hamiltonian and governed by the kinetic energy of the fluid together with an energy of Landau–Lifshitz type. They deduced the following

$$\partial_t u + \mathcal{L}_v u = \tilde{\alpha}(u \times \Delta u),$$

where $\tilde{\alpha}$ is a real number, $u : \Omega \times [0, T) \to \mathbb{S}^2$ and $\mathcal{L}_v$ is the Lie derivative with respect to the field v on Ω with $\mathrm{div}(v) \equiv 0$. They called this dynamical system as incompressible Schrödinger flow.

If $\Omega \subset \mathbb{R}^3$ is a smooth bounded domain, $\mathcal{L}_v$ is just the ∇_v, and hence the above can be written as

$$\partial_t u + \nabla_v u = \tilde{\alpha}(u \times \Delta u),$$

with $\mathrm{div}(v) \equiv 0$ on Ω.

For simplicity, we let $\tilde{\alpha} = 1$ and consider the following initial-Neumann boundary value problem of incompressible Schrödinger flow

$$\begin{cases} \partial_t u + \nabla_v u = u \times \Delta u, & \text{(x,t)} \in \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} = 0, & \text{(x,t)} \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to \mathbb{S}^2, \quad \frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0. \end{cases} \tag{1.7}$$

Here we always suppose that the field v is smooth enough and $\mathrm{div}(v) \equiv 0$ on Ω.

In order to prove the local in time well-posedness, the first thing that needs to be done is to establish the local existence to the following approximation problem

$$\begin{cases} \partial_t u = \varepsilon \tau_v(u) + u \times (\Delta u + u \times \nabla_v u), & \text{(x,t)} \in \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} = 0, & \text{(x,t)} \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to \mathbb{S}^2, \quad \frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0, \end{cases} \tag{1.8}$$

where we denote

$$\tau_v(u) = \tau(u) + u \times \nabla_v u = \Delta u + |\nabla u|^2 u + u \times \nabla_v u.$$

It should be noted that the two terms in the right hand side of the above equation are orthogonal.

Then, by letting $\varepsilon \to 0$ we can also obtain some similar results as that for (1.2) and the proof goes almost the same as that stated in Theorem 1.1 except for that we need to treat the term $\nabla_v u$. Since $\mathrm{div}(v) \equiv 0$ on Ω, and if we additionally provide that v satisfies the boundary compatibility condition

$$\langle v, \nu \rangle |_{\partial\Omega \times \mathbb{R}^+} \equiv 0,$$

the term $\nabla_v u$ does not cause any essential difficulties. In a forthcoming paper we will study this problem and elucidate the details.

It is worthy to point out that Huang [23] has ever considered the coupled system of Navier–Stokes equation and incompressible Schrödinger flow defined on a closed manifold or $\mathbb{R}^n$, and shown the local in time existence of the initial value problem of this system in some suitable Sobolev spaces.

The rest of our paper is organized as follows. In Sect. 2, we introduce the basic notations on Sobolev space and some critical preliminary lemmas. In Sect. 3 and Sect. 4, we give the proof of local existence of regular solution to (1.2) stated in Theorem 1.1. The uniqueness will built up in Sect. 5. We close with two "Appendices". First, this is the locally regular estimates of the approximate solution u_ε. Secondly, it is the characterisation and formulation of the Schrödinger flow in moving frame and parallel transport.

2. Preliminary

In this section, we first recall some notations on Sobolev spaces, which will be used in whole context. Let $u = (u_1, u_2, u_3) : \Omega \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ be a map. We set

$$H^k(\Omega, \mathbb{S}^2) = \{u \in W^{k,2}(\Omega, \mathbb{R}^3) : |u| = 1 \text{ for a.e. x} \in \Omega\}.$$

Moreover, let $(B, \|.\|_B)$ be a Banach space and $f : [0, T] \to B$ be a map. For any $p > 0$ and $T > 0$, we define

$$\|f\|_{L^p([0,T],B)} := \left(\int_0^T \|f\|_B^p dt\right)^{\frac{1}{p}},$$

and

$$L^p([0, T], B) := \{f : [0, T] \to B : \|f\|_{L^p([0,T],B)} < \infty\}.$$

In particular, we denote

$$\begin{aligned}&L^p([0, T], H^k(\Omega, \mathbb{S}^2))\\ &\quad = \{u \in L^p([0, T], H^k(\Omega, \mathbb{R}^3)) : |u| = 1 \text{ for a.e. (x,t)} \in \Omega \times [0, T]\},\end{aligned}$$

where $k, l \in \mathbb{N}$ and $p \geq 1$.

Next, we need to recall some crucial preliminary lemmas which we will use later. The L^2 theory of Laplace operator with Neumann boundary condition implies the following Lemma of equivalent norms, to see [51].

Lemma 2.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^m$ and $k \in \mathbb{N}$. There exists a constant $C_{k,m}$ such that, for all $u \in H^{k+2}(\Omega)$ with $\frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$,*

$$\|u\|_{H^{2+k}(\Omega)} \leq C_{k,m}(\|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^k(\Omega)}). \tag{2.1}$$

Here, for simplicity we denote $H^0(\Omega) := L^2(\Omega)$.

In particular, the above lemma implies that we can define the H^{k+2}-norm of u as follows

$$\|u\|_{H^{k+2}(\Omega)} := \|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^k(\Omega)}.$$

In order to show the uniform estimates and the convergence of solutions to the approximate equation constructed in the coming sections, we also need to use the Gronwall inequality and the classical compactness results in [8,41].

Lemma 2.2. *Let $f : [0, \infty) \to [0, \infty)$ be a nondecreasing continuous function such that $f > 0$ on $(0, \infty)$ and $\int_1^\infty \frac{1}{f} dx < \infty$. Let y be a continuous function which is nonnegative on $[0, T')$ and let g be a nonnegative function in $L^1_{loc}(\mathbb{R}^+)$. We assume that there exists a positive number $y_0 > 0$ such that for all $t \geq 0$, we have the inequality*

$$y(t) \leq y_0 + \int_0^t g(s)ds + \int_0^t f(y(s))ds.$$

Then, there exists a positive number T^* and a constant $C(T^*)$, which both depend only on y_0, g and f, such that for all $T < \min\{T', T^*\}$, there holds true

$$\sup_{0\leq t\leq T} y(t) \leq C(T^*).$$

Lemma 2.3 (Aubin-Lions-Simon compact Lemma, see Theorem II.5.16 in [8] or [41]). *Let $X \subset B \subset Y$ be Banach spaces with compact embedding $X \hookrightarrow B$. Let $1 \le p, q, r \le \infty$. For $T > 0$, we define*

$$E_{p,r} = \{f \in L^p((0,T), X), \frac{df}{dt} \in L^r((0,T), Y)\},$$

then the following properties holds

1. *If $p < \infty$ and $p < q$, the embedding $E_{p,r} \cap L^q((0,T), B)$ in $L^s((0,T), B)$ is compact for all $1 \le s < q$.*
2. *If $p = \infty$ and $r > 1$, the embedding of $E_{p,r}$ in $C^0([0,T], B)$ is compact.*

Lemma 2.4 (Theorem II.5.14 in [8]). *Let $k \in \mathbb{N}$, then the space*

$$E_{2,2} = \{f \in L^2((0,T), H^{k+2}(\Omega)), \frac{\partial f}{\partial t} \in L^2((0,T), H^k(\Omega))\}$$

is continuously embedded in $C^0([0,T], H^{k+1}(\Omega))$.

To end this section, we briefly introduce the notations of Galerkin basis and Galerkin projection. Let Ω be a smooth bounded domain in $\mathbb{R}^m$, λ_i be the i^{th} eigenvalue of the operator $\Delta - I$ with Neumann boundary condition, whose corresponding eigenfunction is f_i. That is,

$$(\Delta - I) f_i = -\lambda_i f_i \qquad \text{with} \qquad \frac{\partial f_i}{\partial \nu}|_{\partial\Omega} = 0.$$

Without loss of generality, we assume $\{f_i\}_{i=1}^{\infty}$ are completely standard orthonormal basis of $L^2(\Omega, \mathbb{R}^1)$. Let $H_n = span\{f_1, \dots f_n\}$ be a finite subspace of L^2, $P_n : L^2(\Omega, \mathbb{R}^1) \to H_n$ be the canonical projection. In fact, for any $f \in L^2$, we define

$$f^n = P_n f = \sum_{1}^{n} \langle f, f_i \rangle_{L^2} f_i,$$

then,

$$\lim_{n \to \infty} \|f - f_n\|_{L^2} = 0.$$

3. Parabolic Perturbation to the Schrödinger Flow

In this section, we consider the parabolic perturbation to the Schrödinger flow (1.2):

$$\begin{cases} \partial_t u_\varepsilon = \varepsilon(\Delta u_\varepsilon + |\nabla u_\varepsilon|^2 u_\varepsilon) + u_\varepsilon \times \Delta u_\varepsilon, & \text{(x,t)} \in \Omega \times \mathbb{R}^+, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0, & \text{(x,t)} \in \partial\Omega \times \mathbb{R}^+, \\ u_\varepsilon(x,0) = u_0 : \Omega \to \mathbb{S}^2, \quad \frac{\partial u_0}{\partial \nu}|_{\partial\Omega} = 0. & \end{cases} \tag{3.1}$$

Here $\varepsilon \in (0,1)$ is the perturbation constant. The local in time regular solution to equation (3.1) is established in [10,11,14] by virtue of Galerkin approximation method in the following theorem, the proof of which can also be found in [14].

Theorem 3.1. *Suppose that $u_0 \in H^3(\Omega)$. Then there exists a maximal existence time T_ε depending on $\|u_0\|_{H^2}$ such that equation (3.1) admits a unique regular solution u_ε on $\Omega \times [0, T_\varepsilon)$ which satisfies that for any $T < T_\varepsilon$*

1. *$|u_\varepsilon(x,t)| = 1$ for all $(x,t) \in [0,T] \times \Omega$;*
2. *$u_\varepsilon \in L^\infty([0,T], H^3(\Omega)) \cap L^2([0,T], H^4(\Omega))$;*
3. *$\frac{\partial u_\varepsilon}{\partial t} \in L^\infty([0,T], H^1(\Omega)) \cap L^2([0,T], H^2(\Omega))$ and $\frac{\partial^2 u_\varepsilon}{\partial t^2} \in L^2([0,T], L^2(\Omega))$.*

Moreover, there exists a constant $C(T,\varepsilon) > 0$ such that

$$\sup_{t\le T}(\|u_\varepsilon\|^2_{H^3(\Omega)} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1(\Omega)}) + \int_0^T (\|u_\varepsilon\|^2_{H^4(\Omega)} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^2(\Omega)} + \|\frac{\partial^2 u_\varepsilon}{\partial t^2}\|^2_{L^2(\Omega)})dt \le C(T,\varepsilon). \tag{3.2}$$

Proof. Let u_ε^n be the solution to the Galerkin approximation equation to (3.1):

$$\begin{cases} \partial_t u_\varepsilon^n = \varepsilon \Delta u_\varepsilon + \varepsilon P_n(|\nabla u_\varepsilon^n|^2 u_\varepsilon^n) + P_n(u_\varepsilon^n \times \Delta u_\varepsilon^n), & \text{(x,t)} \in \Omega \times \mathbb{R}^+, \\ u_\varepsilon^n(x,0) = \sum_{i=1}^n \int_\Omega \langle u_0, f_i\rangle\, dx f_i(x), & \text{x} \in \Omega. \end{cases} \tag{3.3}$$

By establishing some delicate energy estimates and taking a process of convergence for u_ε^n as $n \to \infty$, we can obtain a solution u_1 to equation (3.1) on $\Omega \times [0, T_1)$ for some $T_1 > 0$ depending only on ε and $\|u_0\|_{H^2(\Omega)}$, which satisfies estimate (3.2) for all $T < T_1$ (cf. [11,14]).

Without loss of generality, we assume that T_1 is not the maximal existence time, then the solution u_1 satisfies

$$\sup_{t<T_1}(\|u_1\|^2_{H^3(\Omega)} + \|\frac{\partial u_1}{\partial t}\|^2_{H^1(\Omega)}) + \int_0^{T_1} (\|u_1\|^2_{H^4(\Omega)} + \|\frac{\partial u_1}{\partial t}\|^2_{H^2(\Omega)} + \|\frac{\partial^2 u_1}{\partial t^2}\|^2_{L^2(\Omega)})dt < \infty.$$

It implies

$$u_1 \in C^0([0,T_1], H^3(\Omega)), \tag{3.4}$$

$$\frac{\partial u_1}{\partial t} \in C^0([0,T_1], H^1(\Omega)), \tag{3.5}$$

by applying Lemma 2.4. Then, there exist maps $u_{T_1}(x) \in H^3(\Omega)$ and $v_{T_1}(x) \in H^1(\Omega)$ such that

$$\lim_{T\to T_1} \|u_1(\cdot,t) - u_{T_1}\|_{H^3(\Omega)} = 0,$$

and

$$\lim_{T\to T_1} \|\partial_t u_1(\cdot,t) - v_{T_1}\|_{H^1(\Omega)} = 0.$$

Next, we need to show that $u_{T_1}(x)$ satisfies the Neumann boundary condition, i.e.,

$$\frac{\partial u_{T_1}}{\partial \nu}|_{\partial\Omega} = 0.$$

Since $\frac{\partial u_1(x,t)}{\partial \nu}|_{\partial\Omega\times[0,T_1)} = 0$, it follows that

$$\int_0^{T_1}\int_\Omega \langle \Delta u_1, \phi\rangle\, dxdt + \int_0^{T_1}\int_\Omega \langle \nabla u_1, \nabla\phi\rangle\, dxdt = 0,$$

for all $\phi \in C^\infty(\bar{\Omega}\times[0,T_1))$. In particular, if we choose $\phi(x,t) = f(x)\eta(t)$ for all $f \in C^\infty(\bar{\Omega})$ and $\eta \in C^\infty[0,T_1)$, and denote $g(t) = \int_\Omega \langle \Delta u_1, f\rangle + \langle \nabla u_1, \nabla f\rangle\, dx$, then $g(t) \in C^0[0,T_1]$ and there holds

$$\int_0^{T_1} g(t)\eta(t)dt = 0,$$

which follows $g(t) \equiv 0$, that is

$$\int_\Omega \langle \Delta u_1, f\rangle + \langle \nabla u_1, \nabla f\rangle\, dx \equiv 0.$$

And hence we get

$$\frac{\partial u_{T_1}}{\partial \nu}|_{\partial\Omega} = 0$$

as $t \to T_1$. Here we have used the fact (3.4).

Therefore, by taking the same argument as that in above, we conclude that there exists a regular solution w_1 to equation (3.1) on $\Omega\times[T_1,T_2)$ by replacing u_0 with u_{T_1}. Thus, it is not difficult to show the map

$$u_2(x,t) = \begin{cases} u_1(x,t) & (x,t)\in\Omega\times[0,T_1), \\ w_1(x,t) & (x,t)\in\Omega\times[T_1,T_2), \end{cases}$$

is a solution to equation (3.1) on $\Omega\times[0,T_2)$ satisfying Estimates (3.2) on $\Omega\times[0,T_2)$, since we have the compactness (3.4 and 3.5). By repeating this process, we can get a maximal solution u_ε on $\Omega\times[0,T_\varepsilon)$, which satisfies Estimate (3.2).

Finally, as in [11,14] we can finish the proof of uniqueness by considering the equation of the difference of two solutions and applying a direct energy method. □

Remark 3.2. 1. Since T_ε is the maximal existence time, if $T_\varepsilon < \infty$, then we have

$$\sup_{t<T_\varepsilon}(\|u_\varepsilon\|^2_{H^3(\Omega)} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1(\Omega)}) + \lim_{T\to T_\varepsilon}\{\int_0^T (\|u_\varepsilon\|^2_{H^4(\Omega)} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^2(\Omega)} + \|\frac{\partial^2 u_\varepsilon}{\partial t^2}\|^2_{L^2(\Omega)})dt\} = \infty.$$

2. Let $\{T_i\}$ be the existence times of Galerkin approximation solution constructed in the above Theorem 3.1. If we set

$$S := \{T_1, T_2, \ldots, T_i, \ldots\},$$

then for all $[T,T'] \subset [0,T_\varepsilon)\setminus S$, the estimates in [11,14] also imply the Galerkin approximation solution u^n_ε satisfies

$$\sup_{T\le t\le T'}(\|u^n_\varepsilon\|^2_{H^2(\Omega)} + \|\frac{\partial u^n_\varepsilon}{\partial t}\|^2_{H^1(\Omega)}) + \int_T^{T'}(\|u^n_\varepsilon\|^2_{H^3(\Omega)} + \|\frac{\partial u^n_\varepsilon}{\partial t}\|^2_{H^2(\Omega)})dt < \infty.$$

Next, we show the uniform energy estimates of the solution u_ε, which is independent of ε, and hence obtain a regular solution to the Schrödinger flow (1.2) by taking limit of the sequence of approximation solutions $\{u_\varepsilon\}$ as $\varepsilon \to 0$.

First of all, by choosing u_ε and $-\Delta u_\varepsilon$ as test functions for equation (3.1), we can show the uniform H^1-estimates as follows.

$$\frac{\partial}{\partial t}\int_\Omega (|u_\varepsilon|^2 + |\nabla u_\varepsilon|^2)dx + 2\varepsilon\int_\Omega |u_\varepsilon \times \Delta u_\varepsilon|^2 dx = 0. \tag{3.6}$$

3.1. Uniform H^2-estimates. However, to show directly the further uniform H^2-estimate on u_ε by usual energy estimates seems difficult, because of the spin term of

$$u_\varepsilon \times \Delta u_\varepsilon.$$

To proceed, we need to show the following formulas to parabolic perturbation of the Schrödinger flow in the below lemma, which is mentioned in Sect. 1.

Lemma 3.3. *Let u_ε be the regular solution to equation (3.1) obtained in the above. Then the following properties hold true*

1. *For a.e. $(x,t) \in \Omega \times [0, T_\varepsilon)$, we have*

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} = \varepsilon(\Delta \frac{\partial u_\varepsilon}{\partial t} + \frac{\partial}{\partial t}(|\nabla u_\varepsilon|^2 u_\varepsilon)) + \frac{\partial u_\varepsilon}{\partial t} \times \Delta u_\varepsilon + u_\varepsilon \times \Delta \frac{\partial u_\varepsilon}{\partial t}; \tag{3.7}$$

2. *For a.e. $(x,t) \in \Omega \times [0, T_\varepsilon)$, we have*

$$\begin{aligned}
&\frac{\partial^2 u_\varepsilon}{\partial t^2} + (1-\varepsilon^2)\Delta(\tau(u_\varepsilon)) - 2\varepsilon\Delta(u_\varepsilon \times \Delta u_\varepsilon)\\
&= \varepsilon\{\frac{\partial}{\partial t}(|\nabla u_\varepsilon|^2 u_\varepsilon) - 2\mathrm{div}(\nabla u_\varepsilon \dot{\times} \nabla^2 u_\varepsilon) + u_\varepsilon \times \Delta(|\nabla u_\varepsilon|^2 u_\varepsilon)\}\\
&+ \varepsilon|\nabla u_\varepsilon|^2 u_\varepsilon \times \Delta u_\varepsilon + \Delta(|\nabla u_\varepsilon|^2 u_\varepsilon) - 2\mathrm{div}^2((\nabla u_\varepsilon \dot{\otimes} \nabla u_\varepsilon))u_\varepsilon\\
&- 2\left\langle \nabla|\nabla u_\varepsilon|^2, \nabla u_\varepsilon\right\rangle - 2\left\langle \Delta u_\varepsilon, \nabla u_\varepsilon\right\rangle \cdot \nabla u_\varepsilon - |\nabla u_\varepsilon|^2 \Delta u_\varepsilon,
\end{aligned} \tag{3.8}$$

where

$$div(\nabla u_\varepsilon \dot{\times} \nabla^2 u_\varepsilon) := \sum_{i,j=1}^{3} \partial_i(\partial_j u_\varepsilon \times \partial_{ij} u_\varepsilon),$$

and

$$div^2((\nabla u_\varepsilon \dot{\otimes} \nabla u_\varepsilon))u_\varepsilon := \sum_{i,j=1}^{3} \partial_{ij}((\partial_i u_\varepsilon \cdot \partial_j u_\varepsilon))u_\varepsilon.$$

Proof. The formula (3.7) is given directly by equation (3.1). We need only to show (3.8) presented in the above. A simple calculation gives

$$
\begin{aligned}
&\frac{\partial^2 u_\varepsilon}{\partial t^2} - \varepsilon^2 \Delta(\Delta u_\varepsilon + |\nabla u_\varepsilon|^2 u_\varepsilon) \\
&= \varepsilon\{\frac{\partial}{\partial t}(|\nabla u_\varepsilon|^2 u_\varepsilon) + \Delta(u_\varepsilon \times \Delta u_\varepsilon) + u_\varepsilon \times \Delta^2 u_\varepsilon\} \\
&+\varepsilon\{u_\varepsilon \times \Delta(|\nabla u_\varepsilon|^2 u_\varepsilon) + |\nabla u_\varepsilon|^2 u_\varepsilon \times \Delta u_\varepsilon\} \\
&+\{(u_\varepsilon \times \Delta u_\varepsilon) \times \Delta u_\varepsilon + u_\varepsilon \times (u_\varepsilon \times \Delta^2 u_\varepsilon) + 2u_\varepsilon \times (\nabla u_\varepsilon \dot{\times} \nabla \Delta u_\varepsilon)\} \\
&= \varepsilon I + II
\end{aligned} \tag{3.9}
$$

For the term $u_\varepsilon \times \Delta^2 u_\varepsilon$ in part I, there holds

$$
u_\varepsilon \times \Delta^2 u_\varepsilon = \Delta(u_\varepsilon \times \Delta u_\varepsilon) - 2\mathrm{div}(\nabla u_\varepsilon \dot{\times} \nabla^2 u_\varepsilon).
$$

Next, we turn to presenting the calculation of II. By applying the Lagrangian formula

$$
a \times (b \times c) = \langle a, c\rangle\, b - \langle a, b\rangle\, c
$$

and

$$
(a \times b) \times c = \langle a, c\rangle\, b - \langle b, c\rangle\, a
$$

for any vectors a, b, c in $\mathbb{R}^3$, we have

$$
\begin{aligned}
(u_\varepsilon \times \Delta u_\varepsilon) \times \Delta u_\varepsilon &= \langle u_\varepsilon, \Delta u_\varepsilon\rangle\, \Delta u_\varepsilon - |\Delta u_\varepsilon|^2 u_\varepsilon \\
&= -|\nabla u_\varepsilon|^2 \Delta u_\varepsilon - |\Delta u_\varepsilon|^2 u_\varepsilon, \qquad (3.10) \\
u_\varepsilon \times (u_\varepsilon \times \Delta^2 u_\varepsilon) &= \left\langle \Delta^2 u_\varepsilon, u_\varepsilon \right\rangle u_\varepsilon - \Delta^2 u_\varepsilon \\
&= -(|\Delta u_\varepsilon|^2 + 2|\nabla^2 u_\varepsilon|^2 + 4\langle \nabla \Delta u_\varepsilon, \nabla u_\varepsilon\rangle) u_\varepsilon - \Delta^2 u_\varepsilon, \qquad (3.11)
\end{aligned}
$$

and

$$
\begin{aligned}
u_\varepsilon \times (\nabla u_\varepsilon \dot{\times} \nabla \Delta u_\varepsilon) &= \sum_{i=1}^{3} u_\varepsilon \times (\partial_i u_\varepsilon \times \partial_i \Delta u_\varepsilon) \\
&= \sum_{i=1}^{3} \langle \partial_i \Delta u_\varepsilon, u_\varepsilon\rangle\, \partial_i u_\varepsilon - \langle \partial_i u_\varepsilon, u_\varepsilon\rangle\, \partial_i \Delta u_\varepsilon \\
&= \langle \nabla \Delta u_\varepsilon, u_\varepsilon\rangle\, \nabla u_\varepsilon \\
&= -\left\langle \nabla|\nabla u_\varepsilon|^2, \nabla u_\varepsilon \right\rangle - \langle \Delta u_\varepsilon, \nabla u_\varepsilon\rangle \cdot \nabla u_\varepsilon.
\end{aligned} \tag{3.12}
$$

Here, we have used the fact $|u_\varepsilon| \equiv 1$.

Moreover, we have

$$
\begin{aligned}
\left\langle \Delta^2 u_\varepsilon, u_\varepsilon \right\rangle - |\Delta u_\varepsilon|^2 &= -2(|\Delta u_\varepsilon|^2 + |\nabla^2 u_\varepsilon|^2 + 2\langle \nabla \Delta u_\varepsilon, \nabla u_\varepsilon\rangle) \\
&= -2\mathrm{div}^2(\nabla u_\varepsilon \dot{\otimes} \nabla u_\varepsilon).
\end{aligned} \tag{3.13}
$$

By combining the above equations (3.10)-(3.13) with (3.9), we get the desired formula (3.8). □

Remark 3.4. Let $\varepsilon = 0$, we have

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u = -2\mathrm{div}^2((\nabla u \dot{\otimes} \nabla u))u - 2\left\langle \nabla|\nabla u|^2, \nabla u \right\rangle - 2\langle \Delta u, \nabla u\rangle \cdot \nabla u - |\nabla u|^2 \Delta u.$$

And hence, a tedious but direct calculation again gives the fourth order differential formula of the Schrödinger flow in Sect. 1:

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u = -2\mathrm{div}^2((\nabla u \dot{\otimes} \nabla u)u) + 2\mathrm{div}((\nabla u \dot{\otimes} \nabla u)\cdot \nabla u) - \mathrm{div}(|\nabla u|^2 \nabla u). \tag{3.14}$$

Here

$$\mathrm{div}^2((\nabla u \dot{\otimes} \nabla u)u) := \sum_{i,j=1}^{3} \partial_{ij}((\partial_i u \cdot \partial_j u)u)$$

and

$$\mathrm{div}((\nabla u \dot{\otimes} \nabla u)\cdot \nabla u) := \sum_{i,j=1}^{3} \partial_i((\partial_i u \cdot \partial_j u)\partial_j u).$$

In the following context, we will need to use the following compatibility conditions on the parabolic boundary $\partial\Omega \times [0, T_\varepsilon)$ which is satisfied by u_ε in the sense trace, and to adopt an equivalent H^3-norm of the solution u_ε to equation (3.1) which is related to the H^1-norm of $\frac{\partial u_\varepsilon}{\partial t}$.

Lemma 3.5. *The solution u_ε satisfies the following compatibility conditions on the boundary:*

$$\begin{cases} \frac{\partial}{\partial \nu}\frac{\partial u_\varepsilon}{\partial t}|_{\partial\Omega\times[0,T_\varepsilon)} = 0, \\ \frac{\partial \tau(u_\varepsilon)}{\partial \nu}|_{\partial\Omega\times[0,T_\varepsilon)} = 0 \end{cases} \tag{3.15}$$

in the sense of trace.

Proof. By using the results in Theorem 3.1 and the estimates in Remark 3.2, without loss of generality, we assume the Galerkin approximation solution u^n_ε satisfies

$$\frac{\partial u^n_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u_\varepsilon}{\partial t} \quad \text{weakly in} \quad L^2([T, T'], H^2(\Omega)),$$

for any $[T, T'] \subset [0, T_\varepsilon) \setminus S$.

Let ϕ be a given function in $C^\infty(\bar{\Omega} \times [0, T_\varepsilon))$. Since

$$\frac{\partial}{\partial \nu}\frac{\partial u^n_\varepsilon}{\partial t}|_{\partial\Omega\times[T,T']} = 0,$$

it follows that there holds

$$\int_T^{T'}\int_\Omega \left\langle \Delta \frac{\partial u^n_\varepsilon}{\partial t}, \phi \right\rangle dxdt + \int_T^{T'}\int_\Omega \left\langle \nabla \frac{\partial u^n_\varepsilon}{\partial t}, \nabla\phi \right\rangle dxdt = 0.$$

Letting $n \to \infty$, we can see immediately that

$$\int_T^{T'} \int_\Omega \left\langle \Delta \frac{\partial u_\varepsilon}{\partial t}, \phi \right\rangle dxdt + \int_T^{T'} \int_\Omega \left\langle \nabla \frac{\partial u_\varepsilon}{\partial t}, \nabla \phi \right\rangle dxdt = 0.$$

Since $\frac{\partial u_\varepsilon}{\partial t} \in L^2_{loc}([0, T_\varepsilon), H^2(\Omega))$, it follows for any $0 < T < T_\varepsilon$, there holds

$$\int_0^T \int_\Omega \left\langle \Delta \frac{\partial u_\varepsilon}{\partial t}, \phi \right\rangle dxdt = -\int_0^T \int_\Omega \left\langle \nabla \frac{\partial u_\varepsilon}{\partial t}, \nabla \phi \right\rangle dxdt, \tag{3.16}$$

that is

$$\frac{\partial}{\partial \nu} \frac{\partial u_\varepsilon}{\partial t} |_{\partial\Omega \times [0, T_\varepsilon)} = 0.$$

In consideration of the fact that $\nabla \tau(u_\varepsilon)$ is orthogonal to $u_\varepsilon \times \nabla \tau(u_\varepsilon)$, we can see that the equation

$$\frac{\partial u_\varepsilon}{\partial t} = \varepsilon \tau(u_\varepsilon) + u_\varepsilon \times \tau(u_\varepsilon)$$

implies $\frac{\partial \tau(u_\varepsilon)}{\partial \nu}|_{\partial\Omega \times [0, T_\varepsilon)} = 0$ in the sense of trace, since $\frac{\partial u_\varepsilon}{\partial \nu}|_{\partial\Omega \times [0, T_\varepsilon)} = 0$.

Now we provide the details of the proof for $\frac{\partial \tau(u_\varepsilon)}{\partial \nu}|_{\partial\Omega \times [0, T_\varepsilon)} = 0$. By the equation

$$\frac{\partial u_\varepsilon}{\partial t} = \varepsilon \tau(u_\varepsilon) + u_\varepsilon \times \tau(u_\varepsilon)$$

we can show that

$$\begin{aligned}
\text{LHS of (3.16)} &= \varepsilon \int_0^T \int_\Omega \langle \Delta \tau(u_\varepsilon), \phi \rangle \, dxdt + \int_0^T \int_\Omega \langle \Delta (u_\varepsilon \times \tau(u_\varepsilon)), \phi \rangle \, dxdt \\
&= \int_0^T \int_\Omega \langle \varepsilon \Delta \tau(u_\varepsilon) + \Delta u_\varepsilon \times \tau(u_\varepsilon), \phi \rangle \, dxdt \\
&\quad + \int_0^T \int_\Omega \langle 2\nabla u_\varepsilon \times \nabla \tau(u_\varepsilon) + u_\varepsilon \times \Delta \tau(u_\varepsilon), \phi \rangle \, dxdt
\end{aligned}$$

and

$$\begin{aligned}
&\text{RHS of (3.16)} \\
&= \varepsilon \int_0^T \int_\Omega \langle \Delta \tau(u_\varepsilon), \phi \rangle \, dxdt - \varepsilon \int_0^T \int_{\partial\Omega} \left\langle \frac{\partial \tau(u_\varepsilon)}{\partial \nu}, \phi \right\rangle dxdt \\
&\quad - \int_0^T \int_\Omega \langle \nabla u_\varepsilon, \nabla(\tau(u_\varepsilon) \times \phi) \rangle \, dxdt + \int_0^T \int_\Omega \langle \nabla u_\varepsilon, \nabla \tau(u_\varepsilon) \times \phi \rangle \, dxdt \\
&\quad + \int_0^T \int_\Omega \langle \operatorname{div}(u_\varepsilon \times \nabla \tau(u_\varepsilon)), \phi) \rangle \, dxdt - \int_0^T \int_{\partial\Omega} \left\langle u_\varepsilon \times \frac{\partial \tau(u_\varepsilon)}{\partial \nu}, \phi) \right\rangle dxdt \\
&= \varepsilon \int_0^T \int_\Omega \langle \Delta \tau(u_\varepsilon), \phi \rangle \, dxdt - \int_0^T \int_{\partial\Omega} \left\langle \varepsilon \frac{\partial \tau(u_\varepsilon)}{\partial \nu} + u_\varepsilon \times \frac{\partial \tau(u_\varepsilon)}{\partial \nu}, \phi \right\rangle dxdt \\
&\quad + \int_0^T \int_\Omega \langle \Delta u_\varepsilon \times \tau(u_\varepsilon) + 2\nabla u_\varepsilon \times \nabla \tau(u_\varepsilon) + u_\varepsilon \times \Delta \tau(u_\varepsilon), \phi \rangle \, dxdt.
\end{aligned}$$

Then, the fact “$LHS = RHS$” leads to

$$\int_0^T \int_{\partial\Omega} \left\langle \varepsilon \frac{\partial \tau(u_\varepsilon)}{\partial \nu} + u_\varepsilon \times \frac{\partial \tau(u_\varepsilon)}{\partial \nu}, \phi \right\rangle dxdt = 0$$

in the sense of trace. □

Lemma 3.6. *The solution u_ε has the following properties:*

1. $\Delta u_\varepsilon = \frac{1}{1+\varepsilon^2}(\varepsilon \frac{\partial u_\varepsilon}{\partial t} - u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t}) - |\nabla u_\varepsilon|^2 u_\varepsilon$ *for a.e.* $(x, t) \in \Omega \times [0, T_\varepsilon)$*;*
2. *There exists a constant C which is independent of ε such that there holds*

$$\int_\Omega |\nabla \Delta u_\varepsilon|^2 dx \le C(1 + \frac{1}{1+\varepsilon^2})(1 + \|u_\varepsilon\|_{H^2}^2 + \|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2)^3,$$

and hence we have

$$\|u_\varepsilon\|_{H^3}^2 \le C(1 + \|u_\varepsilon\|_{H^2}^2 + \|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2)^3$$

where C does not depend on $\varepsilon \in (0, 1]$.

Proof. Since u_ε satisfies the perturbation equation (3.1):

$$\frac{\partial u_\varepsilon}{\partial t} = \varepsilon \tau(u) + u_\varepsilon \times \tau(u_\varepsilon),$$

a direct calculation shows

$$\tau(u_\varepsilon) = \frac{1}{1+\varepsilon^2}(\varepsilon \frac{\partial u_\varepsilon}{\partial t} - u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t}).$$

Thus, the proof of the first claim of the lemma is finished. And hence, there holds

$$\begin{aligned}\nabla \Delta u_\varepsilon =& \frac{1}{1+\varepsilon^2}(\varepsilon \nabla \frac{\partial u_\varepsilon}{\partial t} - \nabla u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t} - u_\varepsilon \times \nabla \frac{\partial u_\varepsilon}{\partial t}) \\ &- |\nabla u_\varepsilon|^2 \nabla u_\varepsilon - 2\left\langle \nabla^2 u_\varepsilon, \nabla u_\varepsilon \right\rangle u_\varepsilon.\end{aligned}$$

Immediately it follows

$$\begin{aligned}\int_\Omega |\nabla \Delta u_\varepsilon|^2 dx \le& \frac{C}{(1+\varepsilon^2)^2} \int_\Omega \left((1+\varepsilon^2)|\nabla \frac{\partial u_\varepsilon}{\partial t}|^2 + |\nabla u_\varepsilon|^2 |\frac{\partial u_\varepsilon}{\partial t}|^2 \right) dx \\ &+ C \int_\Omega \left(|\nabla u_\varepsilon|^6 + |\nabla^2 u_\varepsilon|^2 |\nabla u_\varepsilon|^2 \right) dx \\ \le& \frac{C}{1+\varepsilon^2} \left(\|\nabla \frac{\partial u_\varepsilon}{\partial t}\|_{L^2}^2 + \frac{1}{1+\varepsilon^2} I \right) + C(\|u_\varepsilon\|_{H^2}^6 + II).\end{aligned}$$

Here,

$$\begin{aligned}I =& \int_\Omega |\nabla u_\varepsilon|^2 |\frac{\partial u_\varepsilon}{\partial t}|^2 dx \\ \le& C\|\nabla u_\varepsilon\|_{L^4}^2 \|\frac{\partial u_\varepsilon}{\partial t}\|_{L^4}^2 \le C(\|u_\varepsilon\|_{H^2}^4 + \|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^4),\end{aligned}$$

and

$$\begin{aligned}II=&\int_\Omega|\nabla^2u_\varepsilon|^2|\nabla u_\varepsilon|^2dx\\ \le&\|\nabla^2u_\varepsilon\|_{L^3}^2\|\nabla u_\varepsilon\|_{L^6}^2\le\|\nabla^2u_\varepsilon\|_{L^2}\|\nabla^2u_\varepsilon\|_{L^6}\|\nabla u_\varepsilon\|_{L^6}^2\\ \le&C\|u_\varepsilon\|_{H^2}^3(\|u_\varepsilon\|_{H^2}+\|\nabla\Delta u_\varepsilon\|_{L^2})\\ \le&C(\delta)(\|u_\varepsilon\|_{H^2}^4+\|u_\varepsilon\|_{H^2}^6)+\delta\|\nabla\Delta u_\varepsilon\|_{L^2}^2.\end{aligned}$$

To get the above estimate of II, we have used the interpolation inequality

$$\|\nabla^2u_\varepsilon\|_{L^3}\le\|\nabla^2u_\varepsilon\|_{L^2}^{\frac12}\|\nabla^2u_\varepsilon\|_{L^6}^{\frac12}$$

and Lemma 2.1. Thus, in view of the estimates of I and II, we finish the proof of the estimate in the second assertion of the lemma. □

Now, we are in the position to show the uniform H^2-estimates of solution u_ε. By choosing $\frac{\partial u_\varepsilon}{\partial t}$ as a test function to Formula (3.8), we obtain

$$\begin{aligned}&\int_\Omega\left\langle\frac{\partial^2u_\varepsilon}{\partial t^2},\frac{\partial u_\varepsilon}{\partial t}\right\rangle dx+(1-\varepsilon^2)\int_\Omega\left\langle\Delta(\Delta u_\varepsilon+|\nabla u_\varepsilon|^2u_\varepsilon),\frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\\ &=\varepsilon\left\{\int_\Omega\left\langle2\Delta(u_\varepsilon\times\Delta u_\varepsilon)+\frac{\partial}{\partial t}(|\nabla u_\varepsilon|^2u_\varepsilon)-2\mathrm{div}(\nabla u_\varepsilon\dot{\times}\nabla^2u_\varepsilon),\frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right\}\\ &\quad+\varepsilon\left\{\int_\Omega\left\langle u_\varepsilon\times\Delta(|\nabla u_\varepsilon|^2u_\varepsilon)+|\nabla u_\varepsilon|^2u_\varepsilon\times\Delta u_\varepsilon,\frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right\}\\ &\quad+\int_\Omega\left\langle\Delta(|\nabla u_\varepsilon|^2u_\varepsilon)-2\mathrm{div}^2((\nabla u_\varepsilon\dot{\otimes}\nabla u_\varepsilon))u_\varepsilon-2\langle\Delta u_\varepsilon,\nabla u_\varepsilon\rangle\cdot\nabla u_\varepsilon,\frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\\ &\quad-\int_\Omega\left\langle2\left\langle\nabla|\nabla u_\varepsilon|^2,\nabla u_\varepsilon\right\rangle+|\nabla u_\varepsilon|^2\Delta u_\varepsilon,\frac{\partial u_\varepsilon}{\partial t}\right\rangle dx.\end{aligned}$$

For the sake of convenience, we rewrite the above identity as the following

$$\begin{aligned}&\int_\Omega\left\langle\frac{\partial^2u_\varepsilon}{\partial t^2},\frac{\partial u_\varepsilon}{\partial t}\right\rangle dx+(1-\varepsilon^2)\int_\Omega\left\langle\Delta(\Delta u_\varepsilon+|\nabla u_\varepsilon|^2u_\varepsilon),\frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\\ &=\varepsilon(I_1+I_2+I_3+I_4+I_5)+(II_1+II_2+II_3+II_4+II_5).\end{aligned}\tag{3.17}$$

First of all, we estimate the left hand side of the above identity:

$$\begin{aligned}LHS=&\frac12\frac{\partial}{\partial t}\int_\Omega|\frac{\partial u_\varepsilon}{\partial t}|^2dx+\frac{1-\varepsilon^2}{2}\frac{\partial}{\partial t}\int_\Omega|\Delta u_\varepsilon|^2dx\\ &-(1-\varepsilon^2)\int_\Omega\left\langle\nabla(|\nabla u_\varepsilon|^2u_\varepsilon),\nabla\frac{\partial u_\varepsilon}{\partial t}\right\rangle dx.\end{aligned}$$

Here we have used the compatibility boundary conditions in Lemma 3.5.

For the last term in the above identity we have the following estimate:

$$\begin{aligned}
&(1-\varepsilon^2)\left|\int_\Omega \left\langle \nabla(|\nabla u_\varepsilon|^2 u_\varepsilon), \nabla \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right| \\
&\le C\int_\Omega |\nabla u_\varepsilon|^6 + |\nabla u_\varepsilon|^2|\nabla^2 u_\varepsilon|^2 + |\nabla \frac{\partial u_\varepsilon}{\partial t}|^2 dx \\
&\le C(\|u_\varepsilon\|^6_{H^2} + \|u_\varepsilon\|^2_{H^3}\|u_\varepsilon\|^2_{H^2} + \|\nabla \frac{\partial u_\varepsilon}{\partial t}\|^2_{L^2}) \\
&\le C(1+\|u_\varepsilon\|^2_{H^2} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1})^4,
\end{aligned} \tag{3.18}$$

where we have used the estimates in Lemma 3.6.

Next, we estimate the nine terms on the right hand side step by steps.

$$\begin{aligned}
I_1 =& 2\int_\Omega \left\langle \Delta(u_\varepsilon \times \Delta u_\varepsilon), \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx \\
=& -2\int_\Omega \left\langle \Delta(\varepsilon(\Delta u_\varepsilon + |\nabla u_\varepsilon|^2 u_\varepsilon) - \frac{\partial u_\varepsilon}{\partial t}), \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx \\
=& -2\varepsilon\int_\Omega \left\langle \Delta u_\varepsilon + |\nabla u_\varepsilon|^2 u_\varepsilon, \Delta \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx \\
& -2\int_\Omega \left\langle \nabla \frac{\partial u_\varepsilon}{\partial t}, \nabla \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx \\
=& -\varepsilon \frac{\partial}{\partial t}\int_\Omega |\Delta u_\varepsilon|^2 dx - 2\|\nabla \frac{\partial u_\varepsilon}{\partial t}\|^2_{L^2} \\
& + 2\varepsilon\int_\Omega \left\langle \nabla(|\nabla u_\varepsilon|^2 u_\varepsilon), \nabla \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx \\
\le& -\varepsilon \frac{\partial}{\partial t}\int_\Omega |\Delta u_\varepsilon|^2 dx - 2\|\nabla \frac{\partial u_\varepsilon}{\partial t}\|^2_{L^2} \\
& + C\varepsilon(1+\|u_\varepsilon\|^2_{H^2} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1})^4.
\end{aligned} \tag{3.19}$$

Here, we have used the compatibility conditions in Lemma 3.5 in the line 3 of the above inequality (3.19), and applied Lemma 3.6 in the last line.

$$\begin{aligned}
|I_2| =& \left|\int_\Omega \left\langle \frac{\partial}{\partial t}(|\nabla u_\varepsilon|^2 u_\varepsilon), \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right| \le \int_\Omega |\nabla u_\varepsilon|^2 |\frac{\partial u_\varepsilon}{\partial t}|^2 dx \\
\le& C\|u_\varepsilon\|^2_{H^2}\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1} \le C(\|u_\varepsilon\|^2_{H^2} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1})^2.
\end{aligned} \tag{3.20}$$

Here we have used the fact $\left\langle u_\varepsilon, \frac{\partial u_\varepsilon}{\partial t}\right\rangle = 0$.

$$\begin{aligned}
|I_3| =& \left|\int_\Omega \left\langle \operatorname{div}(\nabla u_\varepsilon \dot{\times} \nabla^2 u_\varepsilon), \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right| \\
=& \left|\int_\Omega \sum_{j=1}^3 \left\langle \partial_j u_\varepsilon \times \partial_j \Delta u_\varepsilon, \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq C\|\nabla u_\varepsilon\|_{L^6}\|\nabla\Delta u_\varepsilon\|_{L^2}\|\frac{\partial u_\varepsilon}{\partial t}\|_{L^3}\\
&\leq C(\|\nabla\Delta u_\varepsilon\|_{L^2}^2+\|u_\varepsilon\|_{H^2}^2\|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2)\\
&\leq C(1+\|u_\varepsilon\|_{H^2}^2+\|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2)^3. \qquad (3.21)
\end{aligned}
$$

$$
\begin{aligned}
|I_4| =& \left|\int_\Omega \left\langle u_\varepsilon \times \Delta(|\nabla u_\varepsilon|^2 u_\varepsilon), \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right| \leq C\int_\Omega |\nabla^2 u_\varepsilon||\nabla u_\varepsilon|^2|\frac{\partial u_\varepsilon}{\partial t}|dx\\
&\leq C\|\nabla^2 u_\varepsilon\|_{L^2}\|\nabla u_\varepsilon\|_{L^6}^2\|\frac{\partial u_\varepsilon}{\partial t}\|_{L^6} \leq C(1+\|u_\varepsilon\|_{H^2}^6+\|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2). \qquad (3.22)
\end{aligned}
$$

For the term

$$
|I_5| = \left|\int_\Omega \left\langle |\nabla u_\varepsilon|^2 u_\varepsilon \times \Delta u_\varepsilon, \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right|
$$

and the term

$$
|II_1| = \left|\int_\Omega \left\langle \Delta(|\nabla u_\varepsilon|^2 u_\varepsilon), \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right|,
$$

we can derive the same estimates as that of term I_4.

On the other hand, a simple calculation leads to

$$
\begin{aligned}
|II_2+II_3+II_4+II_5| \leq\, & 2\left|\int_\Omega \left\langle \mathrm{div}^2((\nabla u_\varepsilon \dot{\otimes} \nabla u_\varepsilon))u_\varepsilon, \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right|\\
&+2\left|\int_\Omega \left\langle \langle \Delta u_\varepsilon, \nabla u_\varepsilon\rangle \cdot \nabla u_\varepsilon, \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right|\\
&+2\left|\int_\Omega \left\langle (\nabla|\nabla u_\varepsilon|^2 \cdot \nabla u_\varepsilon), \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right|\\
&+\left|\int_\Omega \left\langle |\nabla u_\varepsilon|^2 \Delta u_\varepsilon, \frac{\partial u_\varepsilon}{\partial t}\right\rangle dx\right|\\
\leq\, & C\int_\Omega |\nabla u_\varepsilon|^2|\nabla^2 u_\varepsilon||\frac{\partial u_\varepsilon}{\partial t}|dx\\
\leq\, & C(1+\|u_\varepsilon\|_{H^2}^6+\|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2). \qquad (3.23)
\end{aligned}
$$

Therefore, by combining inequalities (3.18)-(3.23) with equation (3.17), we get the uniform H^2-estimates of u_ε as follows:

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |\frac{\partial u_\varepsilon}{\partial t}|^2 dx + \frac{1+\varepsilon^2}{2}\frac{\partial}{\partial t}\int_\Omega |\Delta u_\varepsilon|^2 dx\\
&\leq C(1+\varepsilon)(1+\|u_\varepsilon\|_{H^2}^2+\|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2)^4, \qquad (3.24)
\end{aligned}
$$

where the constant C is independent of ε $(0<\varepsilon<1)$ and u_ε.

3.2. Uniform H^3-estimates. In this subsection, we show the uniform H^3-estimates of solution u_ε. By a similar argument with that in the above subsection, we choose $-\Delta\frac{\partial u_\varepsilon}{\partial t}$ as a test function of equation (3.7)(or (3.8)). However, it seems that it is not easy to get energy estimates directly, since the regularity of $\frac{\partial^2 u_\varepsilon}{\partial t^2} \in L^2(\Omega \times [0, T])$ and $\Delta^2 u_\varepsilon \in L^2(\Omega \times [0, T])$ for all $T < T_\varepsilon$ established in Theorem 3.1 is not high enough, and hence integration by parts does not make sense. To proceed, first of all, we enhance the local regularity of solution such that $\frac{\partial u_\varepsilon}{\partial t} \in L^2_{loc}((0, T_\varepsilon), H^3(\Omega))$ by using the L^2-estimates of parabolic equation, since $\frac{\partial u_\varepsilon}{\partial t}$ satisfies equation (3.7):

$$\begin{cases} \frac{\partial v}{\partial t} - \varepsilon\Delta v - u_\varepsilon \times \Delta v = f(u_\varepsilon, \nabla v, v), \\ v(x, 0) = \frac{\partial u_\varepsilon}{\partial t}|_{t=0}, \quad \frac{\partial v}{\partial \nu}|_{\partial\Omega\times(0,T_\varepsilon)} = 0, \end{cases} \tag{3.25}$$

which is linear and uniformly parabolic equation when $\varepsilon > 0$. Here

$$f(u_\varepsilon, \nabla v, v) = v \times \Delta u_\varepsilon + 2\varepsilon\,\langle \nabla u_\varepsilon, \nabla v\rangle\, u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^2 v \in L^2_{loc}([0, T_\varepsilon), H^1(\Omega)),$$

and

$$\frac{\partial u_\varepsilon}{\partial t}|_{t=0} = \varepsilon(\Delta u_0 + |\nabla u_0|^2 u_0) + u_0 \times \Delta u_0.$$

Moreover, since $u_\varepsilon \in L^\infty([0, T], H^3(\Omega))$ and $\frac{\partial u_\varepsilon}{\partial t} \in L^2([0, T], L^2(\Omega))$ for all $T < T_\varepsilon$, then Lemma 2.3 implies

$$u_\varepsilon \in C^0([0, T], H^2(\Omega)).$$

Immediately it follows that $u_\varepsilon \in C^0(\Omega \times [0, T])$. Indeed, for any (x, t) and (x_0, t_0) we have

$$\begin{aligned} &|u_\varepsilon(x, t) - u_\varepsilon(x_0, t_0)| \\ &\quad\le |u_\varepsilon(x, t) - u_\varepsilon(x_0, t)| + |u_\varepsilon(x_0, t) - u_\varepsilon(x_0, t_0)| \\ &\quad\le \sup_\Omega |\nabla u_\varepsilon|(\cdot, t)|x - x_0| + |u_\varepsilon(x_0, t) - u_\varepsilon(x_0, t_0)| \\ &\quad\le C\|u_\varepsilon\|_{L^\infty([0,T],H^3(\Omega))}|x - x_0| + C\|u_\varepsilon(\cdot, t) - u_\varepsilon(\cdot, t_0)\|_{H^2(\Omega)}. \end{aligned} \tag{3.26}$$

and hence it implies $u_\varepsilon \in C^0(\Omega \times [0, T])$. Hence, the L^2-theory of parabolic equation tells us that

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} \in L^2_{loc}((0, T_\varepsilon), H^1(\Omega)),$$

which guarantees the integration by parts in the process of energy estimates makes sense. For the fluency and shortness of this article, we give the above process of improving regularity of $\frac{\partial u_\varepsilon}{\partial t}$ in “Appendix A”.

Now, we turn back to show the uniform H^3-estimates of u_ε. To this end, we choose $-\Delta\frac{\partial u_\varepsilon}{\partial t}$ as a test function of (3.7) and take a simple calculation to obtain

$$\begin{aligned}
&\frac{1}{2}\left(\int_\Omega |\nabla \frac{\partial u_\varepsilon(T)}{\partial t}|^2 dx - \int_\Omega |\nabla \frac{\partial u_\varepsilon(0)}{\partial t}|^2 dx\right) + \varepsilon \int_0^T \int_\Omega |\Delta \frac{\partial u_\varepsilon}{\partial t}|^2 dxdt \\
&= -\int_0^T \int_\Omega \left\langle \frac{\partial u_\varepsilon}{\partial t} \times \Delta u_\varepsilon, \Delta \frac{\partial u_\varepsilon}{\partial t} \right\rangle dxdt \\
&\quad -\varepsilon \int_0^T \int_\Omega \left\langle \frac{\partial}{\partial t}(|\nabla u_\varepsilon|^2 u_\varepsilon), \Delta \frac{\partial u_\varepsilon}{\partial t} \right\rangle dxdt \\
&= V + W,
\end{aligned} \tag{3.27}$$

since $\frac{\partial u_\varepsilon}{\partial t} \in C^0([0, T], H^1(\Omega))$ for $T < T_\varepsilon$.

In order to get the desired estimates of $\frac{\partial u_\varepsilon}{\partial t}$, we turn to estimating the terms V and W.

$$\begin{aligned}
|V| &= \left| \int_0^T \int_\Omega \left\langle \frac{\partial u_\varepsilon}{\partial t} \times \nabla \Delta u_\varepsilon, \nabla \frac{\partial u_\varepsilon}{\partial t} \right\rangle dxdt \right| \\
&\le \left| \int_0^T \int_\Omega \left\langle \frac{\partial u_\varepsilon}{\partial t} \times (\nabla u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon \times \nabla \frac{\partial u_\varepsilon}{\partial t}), \nabla \frac{\partial u_\varepsilon}{\partial t} \right\rangle dxdt \right| \\
&\quad + \left| \int_0^T \int_\Omega \left\langle \frac{\partial u_\varepsilon}{\partial t} \times \nabla(|\nabla u_\varepsilon|^2 u_\varepsilon), \nabla \frac{\partial u_\varepsilon}{\partial t} \right\rangle dxdt \right|.
\end{aligned} \tag{3.28}$$

Here we have used the fact:

$$\Delta u_\varepsilon = \frac{1}{1+\varepsilon^2}(\varepsilon \frac{\partial u_\varepsilon}{\partial t} - u_\varepsilon \times \frac{\partial u_\varepsilon}{\partial t}) - |\nabla u_\varepsilon|^2 u_\varepsilon.$$

Using again the Lagrangian formula, there holds

$$\begin{aligned}
&\frac{\partial u_\varepsilon}{\partial t} \times (u_\varepsilon \times \nabla \frac{\partial u_\varepsilon}{\partial t}) \\
&\quad = \left\langle \nabla \frac{\partial u_\varepsilon}{\partial t}, \frac{\partial u_\varepsilon}{\partial t} \right\rangle u_\varepsilon - \left\langle \frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon \right\rangle \nabla \frac{\partial u_\varepsilon}{\partial t} \\
&\quad = \left\langle \nabla \frac{\partial u_\varepsilon}{\partial t}, \frac{\partial u_\varepsilon}{\partial t} \right\rangle u_\varepsilon.
\end{aligned}$$

Noting $|u_\varepsilon| = 1$ implies

$$\left\langle \nabla \frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon \right\rangle = -\left\langle \frac{\partial u_\varepsilon}{\partial t}, \nabla u_\varepsilon \right\rangle,$$

we have that

$$\begin{aligned}
\left\langle \frac{\partial u_\varepsilon}{\partial t} \times (u_\varepsilon \times \nabla \frac{\partial u_\varepsilon}{\partial t}), \nabla \frac{\partial u_\varepsilon}{\partial t} \right\rangle &= \left\langle \nabla \frac{\partial u_\varepsilon}{\partial t}, \frac{\partial u_\varepsilon}{\partial t} \right\rangle \cdot \left\langle u_\varepsilon, \nabla \frac{\partial u_\varepsilon}{\partial t} \right\rangle \\
&= -\left\langle \nabla \frac{\partial u_\varepsilon}{\partial t}, \frac{\partial u_\varepsilon}{\partial t} \right\rangle \cdot \left\langle \nabla u_\varepsilon, \frac{\partial u_\varepsilon}{\partial t} \right\rangle.
\end{aligned}$$

Then, it follows that there holds

$$
\begin{aligned}
|V| \le & 2\int_0^T\int_\Omega |\frac{\partial u_\varepsilon}{\partial t}|^2|\nabla u_\varepsilon||\nabla\frac{\partial u_\varepsilon}{\partial t}|dxdt \\
& + 2\int_0^T\int_\Omega (|\nabla u_\varepsilon|^3 + |\nabla^2 u_\varepsilon||\nabla u_\varepsilon|)|\frac{\partial u_\varepsilon}{\partial t}||\nabla\frac{\partial u_\varepsilon}{\partial t}|dxdt \\
\le & C\int_0^T \|\frac{\partial u_\varepsilon}{\partial t}\|_{L^6}^2\|\nabla u_\varepsilon\|_{L^6}\|\nabla\frac{\partial u_\varepsilon}{\partial t}\|_{L^2}dt \\
& + C\int_0^T \|\nabla u_\varepsilon\|_{L^\infty}^3\|\frac{\partial u_\varepsilon}{\partial t}\|_{L^2}\|\nabla\frac{\partial u_\varepsilon}{\partial t}\|_{L^2}dt \\
& + C\int_0^T \|\nabla u_\varepsilon\|_{L^\infty}\|\nabla^2 u_\varepsilon\|_{L^3}\|\frac{\partial u_\varepsilon}{\partial t}\|_{L^6}\|\nabla\frac{\partial u_\varepsilon}{\partial t}\|_{L^2}dt \\
\le & C\int_0^T (\|u_\varepsilon\|_{H^2}^2 + \|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^6)dt \\
& + C\int_0^T (\|u_\varepsilon\|_{H^3}^3 + \|u_\varepsilon\|_{H^3}^2)\|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2 dt \\
\le & C\int_0^T (1 + \|u_\varepsilon\|_{H^2}^2 + \|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2)^6 dt.
\end{aligned}
\tag{3.29}
$$

Here we have used the estimate in Lemma 3.6, and used the Sobolov embedding inequality

$$
\|\nabla u_\varepsilon\|_{L^\infty} \le C\|u_\varepsilon\|_{H^3} \le C(1 + \|u_\varepsilon\|_{H^2}^2 + \|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2)^{3/2}.
$$

Next, we give the estimates of W as follows.

$$
\begin{aligned}
|W| \le & \varepsilon\int_0^T\int_\Omega |\frac{\partial}{\partial t}(|\nabla u_\varepsilon|^2 u_\varepsilon)||\Delta\frac{\partial u_\varepsilon}{\partial t}|dxdt \\
\le & C\varepsilon\int_0^T\int_\Omega |\frac{\partial u_\varepsilon}{\partial t}|^2|\nabla u_\varepsilon|^4 + |\nabla u_\varepsilon|^2|\nabla\frac{\partial u_\varepsilon}{\partial t}|^2 dxdt \\
& + \frac{\varepsilon}{2}\int_0^T\int_\Omega |\Delta\frac{\partial u_\varepsilon}{\partial t}|^2 dxdt \\
\le & C\varepsilon\int_0^T (1 + \|u_\varepsilon\|_{H^2}^2 + \|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2)^4 dt + \frac{\varepsilon}{2}\int_0^T\int_\Omega |\Delta\frac{\partial u_\varepsilon}{\partial t}|^2 dxdt.
\end{aligned}
\tag{3.30}
$$

Here we have used the fact:

$$
\int_\Omega |\frac{\partial u_\varepsilon}{\partial t}|^2|\nabla u_\varepsilon|^4 dx \le \|\frac{\partial u_\varepsilon}{\partial t}\|_{L^6}^2\|\nabla u_\varepsilon\|_{L^6}^4,
$$

and

$$
\|\nabla u_\varepsilon\|_{L^\infty} \le C\|u_\varepsilon\|_{H^3} \le C(1 + \|u_\varepsilon\|_{H^2}^2 + \|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1}^2)^{3/2}.
$$

Therefore, we concludes

$$\frac{1}{2}\int_{\Omega}|\nabla\frac{\partial u_\varepsilon}{\partial t}|^2dx|_{t=T}+\frac{\varepsilon}{2}\int_0^T\int_{\Omega}|\Delta\frac{\partial u_\varepsilon}{\partial t}|^2dxdt$$
$$\leq\frac{1}{2}\int_{\Omega}|\nabla\frac{\partial u_\varepsilon}{\partial t}|^2dx|_{t=0}+C\int_0^T(1+\|u_\varepsilon\|^2_{H^2}+\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1})^6dt. \tag{3.31}$$

By combining the estimates (3.6),(3.24) with (3.31), we can derive that there holds

$$((1+\varepsilon^2)\|u_\varepsilon\|^2_{H^2}(T)+\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1}(T))+\varepsilon\int_0^T\int_{\Omega}|\Delta\frac{\partial u_\varepsilon}{\partial t}|^2dxdt$$
$$\leq((1+\varepsilon^2)\|u_0\|_{H^2}+\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1}|_{t=0})+C\int_0^T(1+\|u_\varepsilon\|^2_{H^2}+\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1})^6dt \tag{3.32}$$

for all $0<T<T_\varepsilon$.

By using the Gronwall-type inequality in Lemma 2.2, the desired estimates of approximated solution u_ε are derived from (3.32), we formulated the estimates in the following lemma.

Lemma 3.7. *There exists a positive number T_0 and a constant $C(T_0)$, which both depend only on $\|u_0\|_{H^3}$, such that for all $T<\min\{T_0,T_\varepsilon\}$, the solution u_ε obtained in Theorem (3.1) satisfies the following uniform estimate:*

$$\sup_{0<t\leq T}(\|u_\varepsilon\|^2_{H^2}+\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1})\leq C(T_0).$$

Proof. Let

$$y(t)=((1+\varepsilon^2)\|u_\varepsilon\|^2_{H^2}+\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1})(t).$$

For any $T<T_\varepsilon$, since $u_\varepsilon\in L^2([0,T],H^4(\Omega))$, $\frac{\partial u_\varepsilon}{\partial t}\in L^2([0,T],H^2(\Omega))$ and $\frac{\partial^2u_\varepsilon}{\partial t^2}\in L^2([0,T],L^2(\Omega))$, the embedding Lemma 2.4 implies

$$y(t)=((1+\varepsilon^2)\|u_\varepsilon\|^2_{H^2}+\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1})(t)\in C^0([0,T_\varepsilon)),$$

and hence

$$\frac{\partial u_\varepsilon}{\partial t}|_{t=0}=\varepsilon(\Delta u_0+|\nabla u_0|^2u_0)+u_0\times\Delta u_0,$$

and

$$\nabla\frac{\partial u_\varepsilon}{\partial t}|_{t=0}=\nabla(\varepsilon\Delta u_0+\varepsilon|\nabla u_0|^2u_0+u_0\times\Delta u_0).$$

Thus, a direct calculation shows

$$\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1}|_{t=0}\leq C(1+\varepsilon^2)(1+\|u_0\|^6_{H^3}).$$

□

Let $y_0 = C(1 + \|u_0\|^6_{H^3})$ and $f(y) = C(1 + y)^6$. Then, the function $y(t)$ satisfies the following inequality

$$y(t) \leq y_0 + \int_0^t f(y(s))ds.$$

Then, the Gronwall-type inequality in Lemma 2.2 implies that there exists a positive number $T_0 > 0$ and a constant $C(T_0)$, which both depend only on y_0, such that for all $T < \min\{T_0, T_\varepsilon\}$ there holds

$$\sup_{0<t\leq T} y(t) \leq C(T_0).$$

Thus, the proof is completed.

4. Regular Solution to the Schrödinger Flow

In this section, we prove the local existence of strong solutions to (1.2) in Theorem 1.1. To this end, we need to give an uniform lower bound of existence times T_ε and the compactness of the approximation solution u_ε to (3.1). Consequently, we can claim that the limit map u of sequence $\{u_\varepsilon\}$ is a strong solution to (1.2).

Theorem 4.1. *There exists a positive time T_0 depending only on $\|u_0\|_{H^3(\Omega)}$ such that the equation (1.2) admits a local regular solution on $[0, T_0]$, which satisfies*

$$u \in L^\infty([0, T_0], H^3(\Omega)) \cap C^0([0, T_0], H^2(\Omega, \mathbb{S}^2)).$$

Proof. We divide the proof into three steps.
Step 1: The uniform positive lower bound of T_ε.
We claim $T_0 < T_\varepsilon$, where T_0 was obtained in Lemma 3.7. Suppose that $T_0 \geq T_\varepsilon$, then Lemma 3.7 and Lemma 3.6 tell us that there holds

$$\sup_{0<t<T_\varepsilon} (\|u_\varepsilon\|^2_{H^3} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1}) \leq C(T_0).$$

Since $v = \frac{\partial u_\varepsilon}{\partial t}$ satisfies the following equation

$$\begin{cases} \frac{\partial v}{\partial t} - \varepsilon\Delta v - u_\varepsilon \times \Delta v = f(u_\varepsilon, \nabla v, v), \\ v(x, 0) = \frac{\partial u_\varepsilon}{\partial t}|_{t=0}, \quad \frac{\partial v}{\partial t}|_{\partial\Omega\times(0,T_\varepsilon)} = 0, \end{cases}$$

where

$$f(u_\varepsilon, \nabla v, v) = v \times \Delta u_\varepsilon + 2\varepsilon \langle \nabla u_\varepsilon, \nabla v \rangle u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^2 v,$$

and we can easily to verify that the homogeneous term $f \in L^2([0, T_\varepsilon], L^2(\Omega))$, it follows from the L^2-estimates in Theorem A.1 in "Appendix A" that

$$\frac{\partial u_\varepsilon}{\partial t} \in L^2([\delta, T_\varepsilon], H^2(\Omega)),$$

and

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} \in L^2([\delta, T_\varepsilon], L^2(\Omega))$$

for any small $0 < \delta < T_\varepsilon$. By equation (3.1) and then applying Theorem 3.1, we know that there holds

$$\sup_{0<t<T_\varepsilon}(\|u_\varepsilon\|^2_{H^3} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^1}) + \int_0^{T_\varepsilon}(\|u_\varepsilon\|^2_{H^4} + \|\frac{\partial u_\varepsilon}{\partial t}\|^2_{H^2} + \|\frac{\partial^2 u_\varepsilon}{\partial t^2}\|^2_{L^2})dt < \infty.$$

Thus, it implies that the existence interval $[0, T_\varepsilon]$ can be extended, which is a contradiction with the definition of T_ε. Therefore, we have

$$T_0 < T_\varepsilon.$$

Step 2: The compactness of u_ε.

Lemma 3.7 tells us there exists a constant $C(T_0)$, which is independent of ε, such that there holds true

$$\sup_{0<t\le T_0}\|u_\varepsilon\|_{H^3} + \sup_{0<t\le T_0}\|\frac{\partial u_\varepsilon}{\partial t}\|_{H^1} \le C(T_0).$$

Without loss of generality, we assume there exists a map in $u \in L^\infty([0, T_0], H^3(\Omega))$ such that

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{weakly* in} \quad u \in L^\infty([0, T_0], H^3(\Omega)),\\ \frac{\partial u_\varepsilon}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly* in} \quad L^\infty([0, T_0], H^1(\Omega)),\\ \frac{\partial u_\varepsilon}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly in} \quad L^2([0, T_0], H^1(\Omega)). \end{aligned}$$

Let $X = H^3(\Omega)$, $B = H^2(\Omega)$ and $Y = L^2$, by Lemma 2.3 we have

$$u_\varepsilon \to u \quad \text{strongly in} \quad C^0([0, T_0], H^2(\Omega)),$$

and hence

$$u_\varepsilon \to u \quad \text{a.e. (x,t)} \in \Omega \times [0, T_0].$$

It follows immediately that $|u| = 1$ for a.e. $(x, t) \in [0, T_0] \times \Omega$.

Step 3: The regular solution to (1.2).

Since u_ε is a strong solution to (3.1), there holds

$$\int_0^{T_0}\int_\Omega\left\langle\frac{\partial u_\varepsilon}{\partial t}, \phi\right\rangle dxdt - \varepsilon\int_0^{T_0}\int_\Omega\left\langle\Delta u_\varepsilon + |\nabla u_\varepsilon|^2 u_\varepsilon, \phi\right\rangle dxdt = \int_0^{T_0}\int_\Omega\langle u_\varepsilon \times \Delta u_\varepsilon, \phi\rangle\, dxdt,$$

for all $\phi \in C^\infty(\bar{\Omega} \times [0, T])$. By using the convergence results on u_ε in Step 2, it is easy to show directly u is a strong solution to (1.2) by letting $\varepsilon \to 0$.

To complete the proof, we need to check u satisfies the Neumann boundary condition, that is $\frac{\partial u}{\partial \nu}|_{\partial\Omega\times[0,T_0]} = 0$. Since for any $\xi \in C^\infty(\bar{\Omega} \times [0, T_0])$, there holds

$$\int_0^{T_0}\int_\Omega\langle\Delta u_\varepsilon, \xi\rangle\, dxdt = -\int_0^{T_0}\int_\Omega\langle\nabla u_\varepsilon, \nabla\xi\rangle\, dxdt.$$

Letting $n \to \infty$, we have

$$\int_0^{T_0} \int_\Omega \langle \Delta u, \xi \rangle \, dxdt = -\int_0^{T_0} \int_\Omega \langle \nabla u, \nabla \xi \rangle \, dxdt.$$

This means that there holds true

$$\frac{\partial u}{\partial \nu}|_{\partial\Omega \times [0, T_0]} = 0.$$

□

5. The Uniqueness of Solutions to the Schrödinger Flow

In this section, we show the uniqueness of the solution u to (1.2)(or (B1)) in the space

$$\mathscr{S} = \{u \,|u \in L^\infty([0, T], H^3(\Omega)) \text{ and } \frac{\partial u}{\partial t} \in L^\infty([0, T], H^1(\Omega))\}.$$

We will only give the sketches of the proof for the uniqueness in Theorem 1.1, since the arguments go almost the same as that of the proof of uniqueness for the Schrödinger flows from a general compact Riemannian manifold to a Kähler manifold in [44], and need only to modify some of their treatments to match the Neumann boundary conditions such that integration by parts holds true. In their proof the following intrinsic energy:

$$Q_1 = \int_\Omega d^2(u_1, u_2) dx$$

and

$$Q_2 = \int_\Omega |\mathcal{P}\nabla_2 u_2 - \nabla_1 u_1|^2 dx = \int_\Omega |\Phi|^2 dx$$

was adopted and the geometric energy method was used to achieve a Gronwall-type inequality for $Q_1 + Q_2$, from which the uniqueness of solutions follows. For the sake of simplicity and fluency, some notations about moving frame and parallel transportation as well as some critical lemmas are given in "Appendix B", for more details we refer to [44]. Now, we turn to presenting the proof.

The proof of the uniqueness in Theorem 1.1. Let $u_1, u_2 : \Omega \times [0, T] \to \mathbb{S}^2$ be two solution to (1.2) in space $\mathscr{S}$. An important fact is that the uniqueness is a local property. Namely, once we know $u_1 = u_2$ on a small time interval $[0, T']$, then we can prove $u_1 = u_2$ on the whole interval $[0, T]$ by repeating the argument. Therefore, we only need to prove the uniqueness in a small interval $[0, T']$. Our proof is divided into the following three steps.

Step 1: Estimate of the distance $d(u_1, u_2)$.

Since $u_l \in \mathscr{S}$ with $l = 1, 2$, the embedding Lemma 2.3 implies $u_l \in C^0([0, T], H^2(\Omega))$, and hence $u_l \in C^0(\bar{\Omega} \times [0, T])$. Therefore,

$$\|u_l - u_0\|_{C^0(\bar{\Omega})}(t) \le C\|u_l - u_0\|_{H^2(\Omega)}(t) \to 0,$$

as $t \to 0$.

By using the fact $\mathbb{S}^2$ is of bounded geometry and taking $t \leq T'$ with T' small enough, then one can see that there exists a constant C such that

$$d(u_l, u_0)(t) \leq C\|u_l - u_0\|_{C^0(\bar{\Omega})}(t) < \frac{\pi}{4},$$

for $0 < t \leq T'$ and $l = 1, 2$. This guarantees the parallel transportation between two solutions u_1 and u_2 can be well-defined (to see Sect. B.2).

Step 2: Estimate of Q_1.

Let $d : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ be the distance function on $\mathbb{S}^2$, and $\tilde{\nabla} = \nabla^{\mathbb{S}^2} \otimes \nabla^{\mathbb{S}^2}$ be the product connection on $\mathbb{S}^2 \times \mathbb{S}^2$. Suppose $\{x_1, x_2, x_3\}$ is the coordinates of Ω, and $\nabla_i = \nabla_{\frac{\partial}{\partial x_i}}$ is the bull-back connection on $u_1^* T\mathbb{S}^2$ (or $u_2^* T\mathbb{S}^2$). Then a direct calculation shows

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_\Omega d^2(u_1, u_2)dx\\
&= \int_\Omega \left\langle \tilde{\nabla} d^2, (\nabla_i(J\nabla_i u_1), \nabla_i(J\nabla_i u_2))\right\rangle dx\\
&= \int_\Omega \left\langle \tilde{\nabla} d^2(\cdot, u_2), \nabla_i(J\nabla_i u_1)\right\rangle dx + \int_\Omega \left\langle \tilde{\nabla} d^2(u_1, \cdot), \nabla_i(J\nabla_i u_2)\right\rangle dx\\
&= \int_\Omega \frac{\partial}{\partial x_i}\left\langle \tilde{\nabla} d^2(\cdot, u_2), J\nabla_i u_1\right\rangle dx - \int_\Omega \left\langle \nabla_i\tilde{\nabla} d^2(\cdot, u_2), J\nabla_i u_1\right\rangle dx\\
&\quad + \int_\Omega \frac{\partial}{\partial x_i}\left\langle \tilde{\nabla} d^2(u_1, \cdot), J\nabla_i u_2\right\rangle dx - \int_\Omega \left\langle \nabla_i\tilde{\nabla} d^2(u_1, \cdot), J\nabla_i u_2\right\rangle dx\\
&= -\left(\int_\Omega \left\langle \nabla_i\tilde{\nabla} d^2(\cdot, u_2), J\nabla_i u_1\right\rangle dx + \int_\Omega \left\langle \nabla_i\tilde{\nabla} d^2(u_1, \cdot), J\nabla_i u_2\right\rangle dx\right)\\
&= -\int_\Omega \left\langle \nabla\tilde{\nabla} d^2, (J\nabla u_1, J\nabla u_2)\right\rangle dx\\
&= -\int_\Omega \tilde{\nabla}^2 d^2(X, Y)dx,
\end{aligned} \tag{5.1}$$

where $X = (\nabla u_1, \nabla u_2)$ and $Y = (J\nabla u_1, J\nabla u_2)$. Moreover, here we have used the Stokes formula:

$$\int_\Omega \frac{\partial}{\partial x_i}\left\langle \tilde{\nabla} d^2(\cdot, u_2), J\nabla_i u_1\right\rangle dx = \int_{\partial\Omega} \left\langle \tilde{\nabla} d^2(\cdot, u_2), J(\nabla_i u_1 \cdot \nu_i)\right\rangle ds = 0,$$

and

$$\int_\Omega \frac{\partial}{\partial x_i}\left\langle \tilde{\nabla} d^2(u_1, \cdot), J\nabla_i u_2\right\rangle dx = \int_{\partial\Omega} \left\langle \tilde{\nabla} d^2(u_1, \cdot), J(\nabla_i u_2 \cdot \nu_i)\right\rangle ds = 0,$$

since the Neumann boundary conditions are satisfied: $\sum_{i=1}^3 \nabla_i u_l \cdot \nu_i = 0$ for $l = 1, 2$, where ν is the normal outer vector of $\partial\Omega$ and ds is the area element of $\partial\Omega$.

Therefore, Lemma B.1 gives

$$\frac{1}{2}\frac{\partial}{\partial t}Q_1 \leq Q_2 + C(\|u_1\|^2_{H^3(\Omega)} + \|u_2\|^2_{H^3(\Omega)})Q_1.$$

Step 3: Estimate of Q_2.

Let $\{e_\alpha\}_{\alpha=1}^2$ be a local frame of the pull-back bundle $u_1^*T\mathbb{S}^2$ over $\Omega\times[0,T]$, such that the complex structure J in this frame is reduced to $J_0=\sqrt{-1}$. Denote the parallel transportation by $\mathcal{P}$, let $\nabla_l=u_l^*\nabla^{\mathbb{S}^2}$ and

$$\Phi=\mathcal{P}\nabla_2 u_2-\nabla_1 u_1=\phi_2-\phi_1$$

be defined as in "Appendix B". Then, a direct calculation shows

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega|\mathcal{P}\nabla_2u_2-\nabla_1u_1|^2dx&=\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega|\Phi|^2dx\\&=\int_\Omega\left\langle\Phi,\nabla_{1,t}\Phi\right\rangle dx=\int_\Omega\left\langle\Phi,(\nabla_{1,t}-\nabla_{2,t})\phi_2\right\rangle dx\\&\quad-\int_\Omega\left\langle\Phi,(\nabla_1\nabla_{1,i}-\nabla_2\nabla_{2,i})J_0\phi_{2,i}\right\rangle dx+\int_\Omega\left\langle\Phi,\nabla_1(J_0\nabla_{1,i}\Phi_i)\right\rangle dx\\&=I+II+III.\end{aligned}\tag{5.2}$$

Next, we estimate the above three terms step by steps.

$$\begin{aligned}|I|\le&Q_2+C\int_\Omega|(\nabla_{1,t}-\nabla_{2,t})\phi_2|^2dx\\\le&Q_2+C\|\nabla_2u_2\|^2_{L^\infty(\Omega)}\int_\Omega d^2(u_1,u_2)(|\nabla_tu_2|^2+|\nabla_tu_1|^2)dx\\\le&Q_2+C\|u_2\|^2_{H^3(\Omega)}\|d(u_1,u_2)\|^2_{L^4}(\|\frac{\partial u_1}{\partial t}\|^2_{L^4}+\|\frac{\partial u_2}{\partial t}\|^2_{L^4})\\\le&Q_2+C\|u_2\|^2_{H^3(\Omega)}(\|\frac{\partial u_1}{\partial t}\|^2_{H^1}+\|\frac{\partial u_2}{\partial t}\|^2_{H^1})\|d(u_1,u_2)\|^2_{H^1}\\\le&C\{\|u_2\|^2_{H^3(\Omega)}(\|\frac{\partial u_1}{\partial t}\|^2_{H^1}+\|\frac{\partial u_2}{\partial t}\|^2_{H^1})+1\}(Q_1+Q_2).\end{aligned}$$

Here, in the second line and the last line of the above inequality, we have used Lemma B.2 and the fact

$$|\nabla d(u_1,u_2)|\le|\Phi|=|\mathcal{P}\nabla_2u_2-\nabla_1u_1|$$

respectively. By taking an analogous argument to that for term I and using Lemma B.2, we have

$$|II|\le CC_1(\|u_1\|^4_{H^3}+\|u_2\|^4_{H^3}+1)(Q_1+Q_2),$$

where C_1 comes from Lemma B.2.

For the term III, a simple calculation shows

$$III=\int_\Omega\operatorname{div}\langle\Phi,J_0\nabla_1\cdot\Phi\rangle\,dx-\int_\Omega\langle\nabla_1\cdot\Phi,J_0\nabla_1\cdot\Phi\rangle\,dx=0.$$

Here we denote $\nabla_1\cdot\Phi=\nabla_{1,i}\Phi_i$ and have used the Stokes formula:

$$\int_\Omega\operatorname{div}\langle\Phi,J_0\nabla_1\cdot\Phi\rangle\,dx$$

$$= \int_{\partial\Omega} \left\langle \mathcal{P}\nabla_{2,i}u_2 \cdot \nu_i - \nabla_1 u_{1,i} \cdot \nu_i, J_0\nabla_1 \cdot \Phi \right\rangle ds = 0$$

since we have the Neumann boundary condition

$$\sum_{i=1}^{3} \nabla_i u_l \cdot \nu_i = 0, \quad l = 1, 2,$$

and $\nabla \cdot \Phi \in W^{1,2}(\Omega)$.

Therefore, by combining the estimates in Step 2 and Step 3, we have that for all $0 < t \le T'$ there holds true

$$\frac{\partial}{\partial t}(Q_1 + Q_2) \le \tilde{C}(Q_1 + Q_2),$$

which implies the uniqueness of solutions. Thus, the proof is completed. □

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Declarations

Conflict of interest The authors certify that this manuscript describes original work and has not been submitted elsewhere for publication. The authors also declare that no conflict of interest exists in this article.

Appendix A Locally Regular Estimates of u_ε

In this section, we establish the regular estimates of the solution $v : \Omega \times [0, T] \to \mathbb{R}^3$ to the following uniform parabolic equation:

$$\begin{cases} \frac{\partial v}{\partial t} - \varepsilon \Delta v - u \times \Delta v = f(x,t), \quad (x,t) \in \Omega \times [0,T], \\ v(x,0) = v_0 : \Omega \to \mathbb{R}^3, \quad \frac{\partial v}{\partial \nu}|_{\partial\Omega \times [0,T]} = 0. \end{cases} \tag{A1}$$

where

$$u \in L^\infty([0,T], H^3(\Omega)) \cap C^0(\Omega \times [0,T]) \tag{A2}$$

and

$$f(x,t) \in L^2([0,T], H^1(\Omega)). \tag{A3}$$

Our main result on locally regular estimates of solution v to the above equation (A1) is as follows.

Theorem A.1. *Let* $v \in W_2^{2,1}(\Omega\times[0,T])$ *is a strong solution to (A1) satisfying conditions (A2) and (A3). Then* $v \in L^\infty_{loc}((0,T], H^2(\Omega)) \cap L^2_{loc}((0,T], H^3(\Omega))$. *Moreover, for any* $\delta > 0$, *there exists a positive constant* $C(\delta)$ *depending on* $\|u\|_{L^\infty([0,T],H^3(\Omega))}$ *such that there holds*

$$\begin{aligned}&\|v\|_{L^2([\delta,T],H^2(\Omega))} + \|\frac{\partial v}{\partial t}\|_{L^2([\delta,T],L^2(\Omega))}\\&\le C(\delta)\left(\|v\|_{L^2([0,T]\times\Omega)} + \|f\|_{L^2(\Omega\times[0,T])}\right)\end{aligned}$$

and

$$\begin{aligned}&\|v\|_{L^2([\delta,T],H^3(\Omega))} + \|\frac{\partial v}{\partial t}\|_{L^2([\delta,T],H^1(\Omega))}\\&\le C(\delta)\left(\|v\|_{L^2([0,T],H^1(\Omega))} + \|f\|_{L^2([0,T],H^1(\Omega))}\right).\end{aligned}$$

Proof. Let $\eta(t)$ be a smooth cut-off function such that $\operatorname{supp}\eta \subset (0,T]$ and $\eta \equiv 1$ on $[\delta, T]$ for any $\delta > 0$. Then ηv is a strong solution of the following equation

$$\begin{cases}\frac{\partial\omega}{\partial t} - \varepsilon\Delta\omega - u\times\Delta\omega = \tilde f, & \text{(x,t)} \in \Omega\times[0,T],\\ \frac{\partial\omega}{\partial\nu}|_{\partial\Omega\times[0,T]} = 0, \quad \omega(x,0) = 0, & \text{x} \in \Omega.\end{cases} \tag{A4}$$

Here

$$\tilde f = \eta f + \frac{\partial\eta}{\partial t}v,$$

it is easy to see that the assumptions in Theorem A.1 imply

$$\tilde f \in L^2([0,T], H^1(\Omega)).$$

By the Galerkin approximation method, we claim that there exists a solution $w \in L^\infty([0,T], H^2(\Omega)) \cap L^2([0,T], H^3(\Omega))$ to the above equation (A4). Namely, we consider the following Galerkin approximation equation to (A4)

$$\begin{cases}\frac{\partial\omega^n}{\partial t} - \varepsilon\Delta\omega^n - P_n(u\times\Delta\omega^n) = P_n(\tilde f), & \text{(x,t)} \in \Omega\times[0,T],\\ \omega^n(x,0) = 0, & \text{x} \in \Omega,\end{cases} \tag{A5}$$

where the Galerkin projection P_n is defined in Sect. 2. By the assumptions satisfied by u and $\tilde f$, one can show there exists a unique solution $w^n(x,t) = \sum_{i=1}^n g_i(t)f_i(x)$ to (A5) on $\Omega\times[0,T]$ (cf. [9]).

Then by taking w^n, Δw^n and $\Delta^2 w^n$ as test functions to (A5), a simple calculation shows

$$\begin{aligned}&\frac{\partial}{\partial t}\int_\Omega |\omega^n|^2 dx + \varepsilon\int_\Omega |\nabla\omega^n|^2 dx\\&\le C(\varepsilon)\|u\|_{L^\infty([0,T],H^3)}\int_\Omega |\omega^n|^2 dx + C(\varepsilon)\int_\Omega |\tilde f|^2 dx,\\&\frac{\partial}{\partial t}\int_\Omega |\nabla\omega^n|^2 dx + \varepsilon\int_\Omega |\Delta\omega^n|^2 dx \le C(\varepsilon)\int_\Omega |\tilde f|^2 dx,\\&\frac{\partial}{\partial t}\int_\Omega |\Delta\omega^n|^2 dx + \varepsilon\int_\Omega |\nabla\Delta\omega^n|^2 dx\end{aligned}$$

$$\leq C(\varepsilon)\|u\|_{L^\infty([0,T],H^3)} \int_\Omega |\Delta\omega^n|^2 dx + C(\varepsilon) \int_\Omega |\nabla \tilde{f}|^2 dx.$$

The Gronwall inequality gives the following inequalities

$$\begin{aligned} &\sup_{0\leq t\leq T} \|w^n\|_{H^1}^2 + \varepsilon \int_0^T \int_\Omega |\Delta\omega^n|^2 dxdt \\ &\quad\leq C(\varepsilon, \|u\|_{L^\infty([0,T],H^3)}, T) \int_0^T \int_\Omega |\tilde{f}|^2 dxdt, \end{aligned}$$

and

$$\begin{aligned} &\sup_{0\leq t\leq T} \int_\Omega |\Delta\omega^n|^2 dx + \varepsilon \int_0^T \int_\Omega |\nabla\Delta\omega^n|^2 dxdt \\ &\quad\leq C(\varepsilon, \|u\|_{L^\infty([0,T],H^3)}, T) \int_0^T \int_\Omega |\nabla \tilde{f}|^2 dxdt. \end{aligned}$$

By using Equation (A5) again and then applying Lemma 2.1, one obtains

$$\begin{aligned} &\sup_{0\leq t\leq T} \|w^n\|_{H^1}^2 + \varepsilon \left(\int_0^T \int_\Omega |\frac{\partial w^n}{\partial t}|^2 dxdt + \int_0^T \|w^n\|_{H^2}^2 dt \right) \\ &\quad\leq C(\varepsilon, \|u\|_{L^\infty([0,T],H^3)}, T) \int_0^T \int_\Omega |\tilde{f}|^2 dxdt, \\ &\sup_{0\leq t\leq T} \|w^n\|_{H^2}^2 + \varepsilon \left(\int_0^T \int_\Omega |\nabla\frac{\partial w^n}{\partial t}|^2 dxdt + \int_0^T \|w^n\|_{H^3}^2 dt \right) \\ &\quad\leq C(\varepsilon, \|u\|_{L^\infty([0,T],H^3)}, T) \int_0^T \int_\Omega |\tilde{f}|^2 + |\nabla \tilde{f}|^2 dxdt. \end{aligned}$$

Therefore by letting $n \to \infty$, w^n converges to a solution w of (A4) as we claimed, which satisfies

$$\begin{aligned} &\|\omega\|_{L^\infty([0,T],H^1(\Omega))} + \|\omega\|_{L^2([0,T],H^2(\Omega))} + \|\frac{\partial\omega}{\partial t}\|_{L^2([0,T],L^2(\Omega))} \\ &\quad\leq C(\varepsilon)\|\tilde{f}\|_{L^2([0,T]\times\Omega)}, \\ &\|\omega\|_{L^\infty([0,T],H^2(\Omega))} + \|\omega\|_{L^2([0,T],H^3(\Omega))} + \|\frac{\partial\omega}{\partial t}\|_{L^2([0,T],H^1(\Omega))} \\ &\quad\leq C(\varepsilon)\|\tilde{f}\|_{L^2([0,T],H^1(\Omega))}. \end{aligned}$$

Then the uniqueness of strong solutions to (A4) implies $\eta v = \omega$. Therefore the desired result is proved. □

Now, we apply the above Theorem A.1 to show the local estimates of solution $\frac{\partial u_\varepsilon}{\partial t}$ to (3.7) which can be summarized as the following theorem.

Theorem A.2. *The solution u_ε to (3.1) obtained in Theorem 3.1 satisfies*

$$\frac{\partial u_\varepsilon}{\partial t} \in L^2_{loc}((0, T_\varepsilon), H^3(\Omega))$$

and

$$\frac{\partial^2 u_\varepsilon}{\partial t^2} \in L^2_{loc}((0, T_\varepsilon), H^1(\Omega)).$$

Proof. Since $\frac{\partial u_\varepsilon}{\partial t}$ satisfies the following equation:

$$\begin{cases} \frac{\partial v}{\partial t} - \varepsilon\Delta v - u_\varepsilon \times \Delta v = f(u_\varepsilon, \nabla v, v), \\ v(x,0) = \frac{\partial u_\varepsilon}{\partial t}|_{t=0}, \quad \frac{\partial v}{\partial t}|_{\partial\Omega\times(0,T_\varepsilon)} = 0, \end{cases} \tag{A6}$$

where

$$f(u_\varepsilon, \nabla v, v) = v \times \Delta u_\varepsilon + 2\varepsilon \langle \nabla u_\varepsilon, \nabla v\rangle u_\varepsilon + \varepsilon|\nabla u_\varepsilon|^2 v,$$

then, by the above Theorem A.1 we need only to check that u_ε and $f(u_\varepsilon, \nabla v, v)$ satisfies the conditions (A2) and (A3) respectively.

Since $u_\varepsilon \in L^\infty([0,T], H^3(\Omega))$ and $\frac{\partial u_\varepsilon}{\partial t} \in L^2([0,T], L^2(\Omega))$ for all $T < T_\varepsilon$, then (2) of Lemma 2.3 implies

$$u_\varepsilon \in C^0([0,T], H^2(\Omega)).$$

Indeed, for any (x,t) and (x_0, t_0) we have

$$\begin{aligned} &|u_\varepsilon(x,t) - u_\varepsilon(x_0,t_0)| \\ &\quad\le |u_\varepsilon(x,t) - u_\varepsilon(x_0,t)| + |u_\varepsilon(x_0,t) - u_\varepsilon(x_0,t_0)| \\ &\quad\le \sup_\Omega |\nabla u_\varepsilon|(\cdot,t)|x - x_0| + |u_\varepsilon(x_0,t) - u_\varepsilon(x_0,t_0)| \\ &\quad\le C\|u_\varepsilon\|_{L^\infty([0,T],H^3(\Omega))}|x - x_0| + C\|u_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,t_0)\|_{H^2(\Omega)}. \end{aligned} \tag{A7}$$

This implies $u_\varepsilon \in C^0(\Omega \times [0,T])$.

Next, we want to show $f(u_\varepsilon, \nabla v, v) \in L^2([0,T], H^1(\Omega))$. A simple calculations shows

$$\int_0^T \int_\Omega |f|^2 dxdt \le C(1+\varepsilon)(\|u_\varepsilon\|^4_{L^\infty([0,T],H^3)} + 1)\|\frac{\partial u_\varepsilon}{\partial t}\|^2_{L^2([0,T],H^1)} \le C(T),$$

and

$$\begin{aligned} \nabla f =& \nabla\frac{\partial u_\varepsilon}{\partial t} \times \Delta u_\varepsilon + \frac{\partial u_\varepsilon}{\partial t} \times \nabla\Delta u_\varepsilon \\ &+ \nabla^2\frac{\partial u_\varepsilon}{\partial t}\#\nabla u_\varepsilon\# u_\varepsilon + \nabla\frac{\partial u_\varepsilon}{\partial t}\#\nabla^2 u_\varepsilon\# u_\varepsilon \\ &+ \nabla\frac{\partial u_\varepsilon}{\partial t}\#\nabla u_\varepsilon\#\nabla u_\varepsilon + \frac{\partial u_\varepsilon}{\partial t}\#\nabla^2 u_\varepsilon\#\nabla u_\varepsilon, \end{aligned}$$

where "#" denotes the linear contraction. Thus, we have

$$\int_0^T \int_\Omega |\nabla f|^2 dxdt \le C(\|u_\varepsilon\|^4_{L^\infty([0,T],H^3)} + 1)\|\frac{\partial u_\varepsilon}{\partial t}\|_{L^2([0,T],H^2)} \le C(T).$$

Here, we have used the estimate (3.2). Therefore, from Theorem A.1 we can obtain the desired results. □

Appendix B The Schrödinger Flow in Moving Frame and Parallel Transportation

B.1 The Schrödinger flow in moving frame. Let Ω be a smooth bounded domain in $\mathbb{R}^3$. Suppose that $u : \Omega \times [0, T] \to \mathbb{S}^2$ is a solution to the Schrödinger flow

$$\begin{cases} \partial_t u = J(u)\tau(u), & \text{(x,t)} \in \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \nu} = 0, & \text{(x,t)} \in \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 : \Omega \to \mathbb{S}^2, \end{cases} \tag{B1}$$

where $J(u) = u\times$. We are going to rewrite the above equation in a chosen gauge of the pull-back bundle $u^*T\mathbb{S}^2$ over $\Omega \times [0, T]$.

Let $\nabla^{\mathbb{S}^2}$ be the connection on $\mathbb{S}^2$ and $\nabla = u^*\nabla^{\mathbb{S}^2}$ be the pull-back connection on $u^*T\mathbb{S}^2$. Let $\{x_1, x_2, x_3, t\}$ be the canonical coordinates on $\Omega \times [0, T]$, denote $\nabla_t = \nabla_{\frac{\partial}{\partial t}}$ and $\nabla_i = \nabla_{\frac{\partial}{\partial x_1}}$ for $i = 1, 2, 3$. Recall that the tension field $\tau(u) = \operatorname{tr} \nabla^2 u = \nabla_i \nabla_i u$, we can write the equation (B1) in the form

$$\nabla_t u = J(u)\nabla_i \nabla_i u = \nabla_i (J(u)\nabla_i u).$$

Furthermore, let $\{e_\alpha\}_{\alpha=1}^2$ be a local frame of the pull-back bundle $u^*T\mathbb{S}^2$ over $\Omega \times [0, T]$, such that the complex structure J in this frame is reduced to $J_0 = \sqrt{-1}$. If we denote $\phi := \nabla u = u_i^\alpha e_\alpha \otimes dx^i$, where $i = 1, 2, 3$, then

$$\nabla_t u = J_0 \nabla_i \phi_i \quad \text{and} \quad \nabla_t \phi = \nabla(J_0 \nabla_i \phi_i),$$

The Neumann condition on boundary in (B1) is equivalent to

$$\sum_{i=1}^{3} \phi_i \cdot \nu_i|_{\partial\Omega\times[0,T]} = 0,$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the outer normal vector of $\partial\Omega$.

B.2 Parallel transportation and some lemmas. Let $B \subset \mathbb{S}^2$ be an open geodesic ball with radius $< \frac{\pi}{2}$. Then for any $y_1,\ y_2 \in B$, there exists a unique minimizing geodesic $\gamma(s) : [0, 1] \to \mathbb{S}^2$ connecting y_1 and y_2, and let $\mathcal{P} : T_{y_2}\mathbb{S}^2 \to T_{y_1}\mathbb{S}^2$ be the linear map given by parallel transport along γ. Let $\{e_1(s), e_2(s)\}$ be the frame gotten by parallel transport along γ, set $e_\alpha(y_1) = e_\alpha(0)$ and $e_\alpha(y_2) = e_\alpha(1)$. Then, for any $X = X^\alpha(y_2)e_\alpha(1) \in T_{y_2}\mathbb{S}^2$, the above linear map $\mathcal{P}$ has the following formula

$$\mathcal{P}X = X^\alpha(y_2)e_\alpha(0).$$

Let $d : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ be the distance function on $\mathbb{S}^2$, and $\tilde{\nabla} = \nabla^{\mathbb{S}^2} \otimes \nabla^{\mathbb{S}^2}$ be the product connection on $\mathbb{S}^2 \times \mathbb{S}^2$. We have the following estimates for gradient and Hessian of the distance function, whose proof can be found in [12,43,44].

Lemma B.1. *Suppose that $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$ are two vectors in $T_{y_1}\mathbb{S}^2 \times T_{y_2}\mathbb{S}^2$ where $d(y_1, y_2) < \frac{\pi}{2}$. Then, there hold true*

1. $\frac{1}{2}\tilde{\nabla} d^2(X) = \langle \gamma'(0), \mathcal{P}X_2 - X_1 \rangle,$

2. $\frac{1}{2}|\tilde{\nabla}^2 d^2(X,Y)| \leq |\mathcal{P}X_2 - X_1||\mathcal{P}Y_2 - Y_1| + Cd^2(y_1,y_2)(|X_1|+|X_2|)(|Y_1|+|Y_2|)$.

On the other hand, let $u_l : \Omega \times [0,T] \to \mathbb{S}^2$, $l = 1,2$, with

$$\sup_{\Omega\times[0,T]} d(u_1(x,t), u_2(x,t)) < \frac{\pi}{2}$$

and denote $\nabla_l = u_l^* \nabla^{\mathbb{S}^2}$. Then, for any $(x,t) \in \Omega \times [0,T]$ there exists a unique minimizing geodesic $\gamma_{(x,t)}(s) : [0,1] \to \mathbb{S}^2$ connecting $u_1(x,t)$ and $u_2(x,t)$. More precisely, we define a map $U : \Omega \times [0,T] \times [0,1] \to \mathbb{S}^2$ such that $U(x,t,s) = \gamma_{(x,t)}(s)$, then $u_l^* T\mathbb{S}^2 = U^* T\mathbb{S}^2|_{s=l-1}$ and $\nabla_l = U^*\nabla^{\mathbb{S}^2}|_{s=l-1}$. Therefore, we can define a global bundle isomorphism $\mathcal{P} : u_2^* T\mathbb{S}^2 \to u_1^* T\mathbb{S}^2$ by the parallel transportation along each geodesic. And hence, $\mathcal{P}$ can be extended naturally to a bundle isomorphism from $u_2^* T\mathbb{S}^2 \otimes T^*\Omega$ to $u_1^* T\mathbb{S}^2 \otimes T^*\Omega$.

Let $\{e_1, e_2\}$ be a fixed local frame of bundle $u_1^* T\mathbb{S}^2$ such that $J(u_1) = \sqrt{-1}$. For each point (x,t), we transport parallel this frame to get a moving frame $\{e_1(s), e_2(s)\}$ along the geodesic $\gamma_{(x,t)}(s)$, and set $e_{1,\alpha} = e_\alpha(0)$ and $e_{2,\alpha} = e_\alpha(1)$ for $\alpha = 1,2$. Under this local frame $\{e_1(1), e_2(1)\}$ of $u_2^* T\mathbb{S}^2$, we still have

$$J(u_2) = \sqrt{-1},$$

since $\nabla_{\frac{\partial\gamma}{\partial s}} J(\gamma) = \frac{\partial}{\partial s} J \circ \gamma = 0$ and $J \circ \gamma(0,x,t) = \sqrt{-1}$.

On the other hand, if we denote $\nabla_l u_l = u_{l,i}^\alpha e_{l,\alpha} \otimes dx^i$ and set $\phi_l = u_{l,i}^\alpha e_{1,\alpha} \otimes dx^i$, then

$$\mathcal{P}\nabla_2 u_2 = \mathcal{P} u_{2,i}^\alpha e_{2,\alpha} \otimes dx^i = u_{2,i}^\alpha e_{1,\alpha} \otimes dx^i = \phi_2,$$

and hence

$$\Phi := \mathcal{P}\nabla_2 u_2 - \nabla_1 u_1 = (u_{2,i}^\alpha - u_{1,i}^\alpha) e_{1,\alpha} \otimes dx^i = \phi_2 - \phi_1.$$

Denote the difference of the two connections by

$$B = \nabla_2 - \nabla_1 = (\langle \nabla_2 e_\alpha(1), e_\beta(1)\rangle - \langle \nabla_1 e_\alpha(0), e_\beta(0)\rangle) e_\beta(0),$$

which is a tensor. The following estimates for the difference of connections is essential to control the energy $\int_\Omega |\Phi|^2 dx$ in the proof of the uniqueness, whose proof can be found in [44].

Lemma B.2. *The exists constant C independing on u_1 and u_2, such that the following estimates hold true.*

1. $|B_t| = |\nabla_{2,t} - \nabla_{1,t}| \leq C(|\nabla_t u_1| + |\nabla_t u_2|) d(u_1,u_2)$,
2. $|B_i| = |\nabla_{2,i} - \nabla_{1,i}| \leq C(|\nabla_i u_1| + |\nabla_i u_2|) d(u_1,u_2)$,

where $i = 1,2,3$. Moreover, for any $i,k = 1,2,3$, we have

$$|(\nabla_{2,k}\nabla_{2,i} - \nabla_{1,k}\nabla_{1,i}) J_0 \phi_{2,i}| \leq C_1(|\Phi| + (|\nabla_1^2 u_1| + |\nabla_2^2 u_2| + 1) d(u_1,u_2)),$$

where C_1 depends only on $\|u_1\|_{H^3(\Omega)}$ and $\|u_2\|_{H^3(\Omega)}$.

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