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New Solutions to the Tetrahedron Equation Associated with Quantized Six-Vertex Models

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Abstract: We present a family of new solutions to the tetrahedron equation of the form $RLLL = LLLR$, where L operator may be regarded as a quantized six-vertex model whose Boltzmann weights are specific representations of the q-oscillator or q-Weyl algebras. When the three L's are associated with the q-oscillator algebra, R coincides with the known intertwiner of the quantized coordinate ring $A_q(sl_3)$. On the other hand, L's based on the q-Weyl algebra lead to new R's whose elements are either factorized or expressed as a terminating q-hypergeometric type series.

1. Introduction

Tetrahedron equation [24] is a key to integrability for lattice models in statistical mechanics in three dimensions. Among its several versions and formulations, let us focus on the so-called $RLLL$ relation:

$$R_{456}L_{236}L_{135}L_{124} = L_{124}L_{135}L_{236}R_{456}. \tag{1}$$

Indices here specify the tensor components on which the associated operators act nontrivially. When the spaces $4, 5, 6$ are evaluated away appropriately, it reduces to the Yang–Baxter equation $L_{23}L_{13}L_{12} = L_{12}L_{13}L_{23}$ [1]. Thus (1) may be viewed as a quantization of the Yang–Baxter equation along the direction of the auxiliary spaces 4, 5 and 6. It has appeared in several guises and studied from various point of view. See for example [3,12,18,19,21,23] and the references therein. A survey from a quantum group theoretical perspective is available in [14].

In this paper we take the spaces $1, 2, 3$ as $V = \mathbb{C}^2$ and consider the three kinds of L operators:

$$L^Z \in \mathrm{End}(V \otimes V) \otimes \pi_Z(\mathcal{W}_q), \tag{2}$$

$$L^X \in \mathrm{End}(V \otimes V) \otimes \pi_X(\mathcal{W}_q), \tag{3}$$

$$L^O \in \mathrm{End}(V \otimes V) \otimes \pi_O(\mathcal{O}_q). \tag{4}$$

They all have the six-vertex model structure [1], i.e., weight conservation property, with respect to the component $V \otimes V$. The last component is taken from specific representations π_X, π_Z of the q-Weyl algebra $\mathcal{W}_q$ (6) on $F = \oplus_{m \in \mathbb{Z}} \mathbb{C}|m\rangle$ or π_O of the q-oscillator algebra $\mathcal{O}_q$ (10) on $F_+ = \oplus_{m \in \mathbb{Z}_{\geq 0}} \mathbb{C}|m\rangle$. In short, these L operators may be viewed as quantized six-vertex models whose Boltzmann weights are $\mathrm{End}(F)$ or $\mathrm{End}(F_+)$-valued. They naturally lead to the generalizations of (1) to

$$R_{456} L^C_{236} L^B_{135} L^A_{124} = L^A_{124} L^B_{135} L^C_{236} R_{456}, \tag{5}$$

where A, B and C can be any one of Z, X and O. Let us temporarily call it the $RLLL$ relation of type ABC.

The main result of this paper is the explicit solution R for types ZZZ, OZZ, ZZO, ZOZ, OOZ, ZOO, OZO, OOO, XXZ, ZXX and XZX. They turn out to be unique up to normalization in each sector specified by a parity condition in an appropriate sense. Elements of R are either factorized or expressed as a terminating q-hypergeometric type series. See Table 1 in Sect. 6 for a summary. They are new except for type OOO, where the $RLLL$ relation [3] is equivalent (cf. Sect. 5.2 and [14, Lem 3.22]) with the intertwining relation of the quantized coordinate ring $A_q(sl_3)$, and the R coincides with the intertwiner obtained in [9]. We will show a similar link to $A_q(sl_3)$ also for type ZZZ in Proposition 16.

The representations π_Z and π_X of the q-Weyl algebra $XZ = qZX$ are natural ones in which Z and X become diagonal, respectively. See (8) and (9). They are q-analogue of the coordinate and the momentum representations of the canonical commutation relation, which are formally interchanged via a q-difference analogue of the Fourier transformation. The representation π_O is a restriction of the special case of π_X as explained around (12). One of our motivation is to investigate systematically how these L operators, including their mixtures, lead to a variety of solutions R for the associated $RLLL$ relation. The new R's obtained in this paper will be important inputs to many interesting future problems which will be discussed in the last section.

The layout of the paper is as follows. In Sect. 2, the L operators L^Z, L^X associated with the q-Weyl algebra and L^O for the q-oscillator algebra are introduced. L^O is a restriction of L^X, and appeared in the earlier works [3,5,17,18,23]. The $RLLL$ relation is formulated. In Sects. 3 and 4, the solutions R are presented for the choices $L = L^Z, L^O$ and $L = L^Z, L^X$, respectively. Some results in the former case can be reproduced as a limit of the latter. In Sect. 5, a connection to the representation theory of $A_q(sl_3)$ is explained. A new result is Proposition 16. Section 6 contains a summary and discussion on the tetrahedron equation of the form $RRRR = RRRR$. Conjecture 17 is promising. Appendix A provides the list of explicit forms of the $RLLL$ relation for type ZZZ.

2. Quantized Six-Vertex Models

We assume that q is generic throughout the paper.

2.1. q-Weyl algebra $\mathcal{W}_q$ and q-oscillator algebra $\mathcal{O}_q$. Let $\mathcal{W}_q$ be the q-Weyl algebra, which is an associative algebra with generators $X^{\pm 1}, Z^{\pm 1}$ obeying the relation

$$XZ = qZX \tag{6}$$

and those following from the obvious ones $XX^{-1} = X^{-1}X = ZZ^{-1} = Z^{-1}Z = 1$. Introduce the infinite dimensional vector spaces[1]:

$$F = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}|m\rangle, \quad F_+ = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbb{C}|m\rangle. \tag{7}$$

The algebra $\mathcal{W}_q$ has irreducible representations π_Z (resp. π_X) on F where Z (resp. X) is diagonal:

$$\pi_Z : X|m\rangle = |m-1\rangle, \quad X^{-1}|m\rangle = |m+1\rangle, \quad Z|m\rangle = q^m|m\rangle, \quad Z^{-1}|m\rangle = q^{-m}|m\rangle, \tag{8}$$

$$\pi_X : X|m\rangle = q^m|m\rangle, \quad X^{-1}|m\rangle = q^{-m}|m\rangle, \quad Z|m\rangle = |m+1\rangle, \quad Z^{-1}|m\rangle = |m-1\rangle. \tag{9}$$

They are q-analogue of the "coordinate" and the "momentum" representations of the canonical commutation relation.

Let $\mathcal{O}_q$ be the q-oscillator algebra, which is an associative algebra with generators $\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ obeying the relation

$$\mathbf{k}\,\mathbf{a}^+ = q\,\mathbf{a}^+\mathbf{k}, \quad \mathbf{k}\,\mathbf{a}^- = q^{-1}\mathbf{a}^-\mathbf{k}, \quad \mathbf{a}^-\mathbf{a}^+ = 1 - q^2\mathbf{k}^2, \quad \mathbf{a}^+\mathbf{a}^- = 1 - \mathbf{k}^2. \tag{10}$$

There is an embedding $\iota : \mathcal{O}_q \hookrightarrow \mathcal{W}_q$ given by

$$\iota : \mathbf{k} \mapsto X, \quad \mathbf{a}^+ \mapsto Z, \quad \mathbf{a}^- \mapsto Z^{-1}(1 - X^2). \tag{11}$$

The composition $\mathcal{O}_q \overset{\iota}{\hookrightarrow} \mathcal{W}_q \xrightarrow{\pi_X} \mathrm{End}(F)$ yields the representation:

$$\mathbf{k}|m\rangle = q^m|m\rangle, \quad \mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1 - q^{2m})|m-1\rangle. \tag{12}$$

Due to $\mathbf{a}^-|0\rangle = 0$, the subspace $F_+ \subset F$ becomes invariant and irreducible. We let $\pi_O : \mathcal{O}_q \to \mathrm{End}(F_+)$ denote the resulting irreducible representation obtained by restricting (12) to $m \geq 0$.

2.2. 3D L operator. Let $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$ be the two dimensional vector space. We consider q-Weyl algebra-valued L operator

$$\mathcal{L} = \mathcal{L}_{r,s,t,w} = \sum_{a,b,i,j=0,1} E_{ai} \otimes E_{bj} \otimes \mathcal{L}^{ab}_{ij} \in \mathrm{End}(V \otimes V) \otimes \mathcal{W}_q, \tag{13}$$

$$\mathcal{L}^{ab}_{ij} = 0 \text{ unless } a + b = i + j, \tag{14}$$

$$\mathcal{L}^{00}_{00} = r, \quad \mathcal{L}^{11}_{11} = s, \quad \mathcal{L}^{10}_{10} = twX, \quad \mathcal{L}^{01}_{01} = -qtX, \quad \mathcal{L}^{10}_{01} = Z,$$
$$\mathcal{L}^{01}_{10} = Z^{-1}(rs - t^2wX^2). \tag{15}$$

Here r, s, t, w are parameters whose dependence has been suppressed in the notation $\mathcal{L}^{ab}_{ij}$. They are assumed to be generic throughout. The symbol E_{ij} denotes the matrix unit on V acting on the basis as $E_{ij}v_k = \delta_{jk}v_i$. The L operator $\mathcal{L}$ may be viewed as a quantized six-vertex model where the Boltzmann weights are $\mathcal{W}_q$-valued. See Fig. 1 for a graphical representation.

[1] The actual coefficient field will contain many parameters introduced subsequently including q.

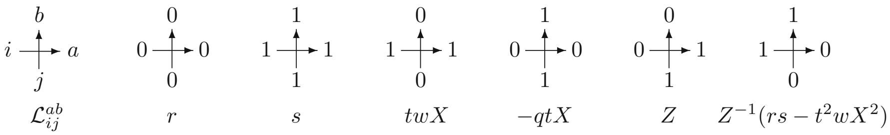

Fig. 1. $\mathcal{L} = \mathcal{L}_{r,s,t,w}$ as a $\mathcal{W}_q$-valued six-vertex model. Assigning another perpendicular arrow corresponding to the $\mathcal{W}_q$-modules leads to a unit of the three dimensional (3D) lattice. In this context, L will also be called the 3D L operator

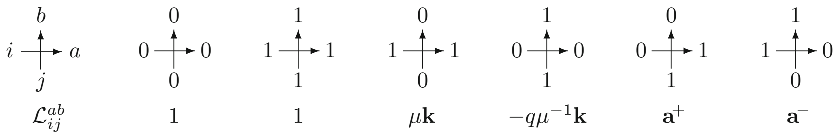

Fig. 2. $\mathcal{L} = \mathcal{L}_{1,1,\mu^{-1},\mu^2}$ as an $\mathcal{O}_q$-valued six-vertex model. The last two relations in (10) is a quantization of the free Fermion condition [1, Fig. 10.1, eq.(10.16.4)$|_{\omega_7=\omega_8=0}$]

Note that $\mathcal{L}$ does not contain X^{-1}, which will be a key in Remark 1 below. Although t can be absorbed into the normalization of X, we keep it for convenience. It is easy to see

$$(\mathcal{L}_{r,s,t,w})^{-1} = (rs)^{-1}\mathcal{L}_{s,r,tw,w^{-1}}. \tag{16}$$

For the special choice of the parameters $(r, s, t, w) = (1, 1, \mu^{-1}, \mu^2)$, $\mathcal{L}$ only contains the combinations appearing in the RHS of (11) which can be pulled back to the q-oscillator algebra. Therefore we regard it as $\mathcal{O}_q$-valued, i.e.,

$$\mathcal{L}_{1,1,\mu^{-1},\mu^2} \in \mathrm{End}(V \otimes V) \otimes \mathcal{O}_q. \tag{17}$$

Its elements are given by

$$\mathcal{L}_{00}^{00} = 1,\ \ \mathcal{L}_{11}^{11} = 1,\ \ \mathcal{L}_{10}^{10} = \mu\mathbf{k},\ \ \mathcal{L}_{01}^{01} = -q\mu^{-1}\mathbf{k},\ \ \ \mathcal{L}_{01}^{10} = \mathbf{a}^+,\ \ \mathcal{L}_{10}^{01} = \mathbf{a}^-. \tag{18}$$

See Fig. 2.

Now we introduce the three types of (represented) L operators:

$$L^Z = L^Z_{r,s,t,w} = (1 \otimes 1 \otimes \pi_Z)(\mathcal{L}_{r,s,t,w}) \in \mathrm{End}(V \otimes V \otimes F), \tag{19}$$
$$L^X = L^X_{r,s,t,w} = (1 \otimes 1 \otimes \pi_X)(\mathcal{L}_{r,s,t,w}) \in \mathrm{End}(V \otimes V \otimes F), \tag{20}$$
$$L^O = L^O_{\mu} = (1 \otimes 1 \otimes \pi_O)(\mathcal{L}_{1,1,\mu^{-1},\mu^2}) \in \mathrm{End}(V \otimes V \otimes F_+). \tag{21}$$

From (16) and (17) we have

$$\begin{aligned}(L^Z_{r,s,t,w})^{-1} = (rs)^{-1}L^Z_{s,r,tw,w^{-1}},\quad (L^X_{r,s,t,w})^{-1} = (rs)^{-1}L^X_{s,r,tw,w^{-1}},\\ (L^O_{\mu})^{-1} = L^O_{\mu^{-1}}.\end{aligned} \tag{22}$$

Remark 1. The operator L^Z in (19) keeps the subspace $V \otimes V \otimes \bigoplus_{m \le n} \mathbb{C}|m\rangle \subset F$ invariant for any $n \in \mathbb{Z}$.

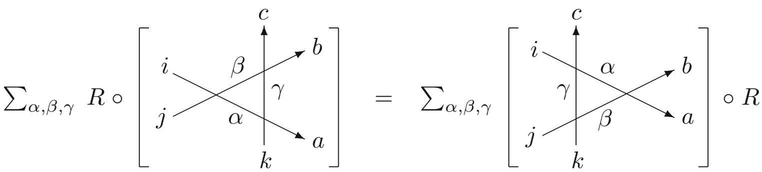

Fig. 3. A pictorial representation of the quantized Yang–Baxter equation (24)

2.3. RLLL relation. Quantized six-vertex model satisfies the quantized Yang–Baxter equation. It is a version of the tetrahedron equation having the form of the Yang–Baxter equation up to conjugation:

$$R_{456}L_{236}L_{135}L_{124} = L_{124}L_{135}L_{236}R_{456}. \tag{23}$$

We also call it $RLLL$ relation. The indices denote the tensor components on which the respective operators act non-trivially. The operator L will be taken as L^Z, L^X or L^O in (19)–(21). The conjugation operator R, which we call 3D R in this paper, will be the main object of study in what follows. In terms of the components of L, (23) reads as

$$R\sum_{\alpha,\beta,\gamma}(\mathcal{L}^{\alpha\beta}_{ij}\otimes\mathcal{L}^{a\gamma}_{\alpha k}\otimes\mathcal{L}^{bc}_{\beta\gamma}) = \sum_{\alpha,\beta,\gamma}(\mathcal{L}^{ab}_{\alpha\beta}\otimes\mathcal{L}^{\alpha c}_{i\gamma}\otimes\mathcal{L}^{\beta\gamma}_{jk})\,R \tag{24}$$

for arbitrary $a, b, c, i, j, k \in \{0, 1\}$. See Fig. 3.

From the conservation condition (14), the Eq. (24) becomes $0 = 0$ unless $a+b+c = i+j+k$. There are 20 choices of $(a, b, c, i, j, k) \in \{0, 1\}^6$ satisfying it. Among them, the cases $(0, 0, 0, 0, 0, 0)$ and $(1, 1, 1, 1, 1, 1)$ yield the trivial relation $R(1\otimes 1\otimes 1) = (1\otimes 1\otimes 1)R$ for any choice of $L = L^Z, L^X, L^O$. Thus there are 18 non-trivial equations on R. By setting[2]

$$R(|i\rangle\otimes|j\rangle\otimes|k\rangle) = \sum_{a,b,c} R^{a,b,c}_{i,j,k}|a\rangle\otimes|b\rangle\otimes|c\rangle, \tag{25}$$

they are translated into linear recursion relations on the matrix elements $R^{a,b,c}_{i,j,k}$. We say that R is *locally finite* if the sum (25) consists of finitely many terms, i.e., $R^{a,b,c}_{i,j,k} = 0$ for all but finitely many (a, b, c) for any given (i, j, k).

3. Solutions of $RLLL$ Relation for $L = L^Z$ and L^O

In this section we treat the cases in which L_{124}, L_{135} and L_{236} are chosen as L^Zor L^O independently. It turns out that they always admit a unique R up to normalization in a sector specified by appropriate parity conditions. Their explicit forms will be presented case by case. We write the characteristic function as $\theta(\text{true}) = 1, \theta(\text{false}) = 0$, $\delta^a_b =$

[2] a, b, c, i, j, k here are labels of the basis of F or F_+ and have different meaning from those in (24) labeling the basis of V.

$\theta(a=b)$ and use the following notation:

$$(z;q)_m = \frac{(z;q)_\infty}{(zq^m;q)_\infty}, \quad (z;q)_\infty = \prod_{n\geq 0}(1-zq^n), \quad \binom{n}{m}_q = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}, \tag{26}$$

$$_2\phi_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};q,z\right) = \sum_{n\geq 0}\frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n}z^n. \tag{27}$$

The above convention for $(z;q)_m$ valid for any $m \in \mathbb{Z}$ is standard and essential in the working below. In particular $1/(q;q)_a = 0$ for $a \in \mathbb{Z}_{<0}$, and we will freely use $(z;q)_m = 1/(zq^m;q)_{-m}$ and $(z;q)_m/(z;q)_n = (zq^n;q)_{m-n}$, etc. The q-binomial $\binom{n}{m}_q$ is zero unless $0 \leq m \leq n$. The q-hypergeometric series will always appear in the terminating situation, i.e., α or $\beta \in q^{\mathbb{Z}_{\leq 0}}$.

3.1. ZZZ type. We consider the $RLLL$ relation

$$R_{456}L^Z_{236}L^Z_{135}L^Z_{124} = L^Z_{124}L^Z_{135}L^Z_{236}R_{456}, \tag{28}$$

where $L^Z_{124}, L^Z_{135}, L^Z_{236}$ are given by (19) with $(r,s,t,w) = (r_1,s_1,t_1,w_1)$, (r_2,s_2,t_2,w_2), (r_3,s_3,t_3,w_3). In this case, $R \in \mathrm{End}(F\otimes F\otimes F)$ and the sum (25) extends over $a,b,c \in \mathbb{Z}$. The equality (28) holds in $\mathrm{End}(V\otimes V\otimes V\otimes F\otimes F\otimes F)$.

The 18 equations (24) corresponding to (28) have been listed in Appendix A. As an illustration consider the cases $(a,b,c,i,j,k) = (0,0,1,0,0,1)$, $(1,0,0,1,0,0)$, $(1,0,0,0,0,1)$, $(1,1,0,0,1,1)$, $(1,0,1,0,1,1)$ and $(1,1,0,1,0,1)$:

$$R(1\otimes X\otimes X) = (1\otimes X\otimes X)R, \quad R(X\otimes X\otimes 1) = (X\otimes X\otimes 1)R, \tag{29}$$

$$-r_1r_3R(1\otimes Z\otimes 1) = (qt_1t_3w_1X\otimes Z\otimes X - r_2Z\otimes 1\otimes Z)R, \tag{30}$$

$$R(-qt_1t_3w_3X\otimes Z\otimes X + s_2Z\otimes 1\otimes Z) = s_1s_3(1\otimes Z\otimes 1)R, \tag{31}$$

$$t_1R(X\otimes Z\otimes Z^{-1}(r_3s_3 - t_3^2w_3X^2) + s_2t_3Z\otimes 1\otimes X) = s_3t_2(Z\otimes X\otimes 1)R, \tag{32}$$

$$R(t_3w_3Z^{-1}(r_1s_1-t_1^2w_1X^2)\otimes Z\otimes X+s_2t_1w_1X\otimes 1\otimes Z)=s_1t_2w_2(1\otimes X\otimes Z)R. \tag{33}$$

Taking their matrix elements for the transition $|i\rangle\otimes|j\rangle\otimes|k\rangle \mapsto |a\rangle\otimes|b\rangle\otimes|c\rangle$, we get the recursion relations for elements of R:

$$R^{a,b,c}_{i,j-1,k-1} = R^{a,b+1,c+1}_{i,j,k}, \quad R^{a,b,c}_{i-1,j-1,k} = R^{a+1,b+1,c}_{i,j,k}, \tag{34}$$

$$(q^{a+c}r_2 - q^jr_1r_3)R^{a,b,c}_{i,j,k} = q^{1+b}t_1t_3w_1R^{a+1,b,c+1}_{i,j,k}, \tag{35}$$

$$(q^{i+k}s_2 - q^bs_1s_3)R^{a,b,c}_{i,j,k} = q^{1+j}t_1t_3w_3R^{a,b,c}_{i-1,j,k-1}, \tag{36}$$

$$q^jr_3s_3t_1R^{a,b,c}_{i-1,j,k} - q^{j+2}t_1t_3^2w_3R^{a,b,c}_{i-1,j,k-2} + q^{i+k}s_2t_3R^{a,b,c}_{i,j,k-1} = q^{a+k}s_3t_2R^{a,b+1,c}_{i,j,k}, \tag{37}$$

$$\begin{aligned}&q^jr_1s_1t_3w_3R^{a,b,c}_{i,j,k-1} - q^{j+2}t_1^2t_3w_1w_3R^{a,b,c}_{i-2,j,k-1}\\ &\quad + q^{i+k}s_2t_1w_1R^{a,b,c}_{i-1,j,k} = q^{c+i}s_1t_2w_2R^{a,b+1,c}_{i,j,k}.\end{aligned} \tag{38}$$

Each recursion relation is actually a collection of infinitely many linear equations on infinitely many $R^{a,b,c}_{i,j,k}$'s depending on the choice of $(a,b,c,i,j,k) \in \mathbb{Z}^6$.

Given two integers d and d', we write the pair $(d \bmod 2, d' \bmod 2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ simply as $(d, d')_{\text{mod } 2}$.

Proposition 2. (i) *Any recursion relation consists of only those* $R^{a,b,c}_{i,j,k}$*'s having the same parity pair* $(d_1, d_2)_{\text{mod } 2}$*, where* $d_1 = a+c-j$ *and* $d_2 = b-i-k$. (ii) *Each subsystem of recursion relations corresponding to a given* $(d_1, d_2)_{\text{mod } 2}$ *allows a solution of dimension at most one.*

Proof. Claim (i) can be checked directly. Let us prove Claim (ii). First, we reduce b, c and k to 0 by using (34) and (36). The result reads

$$R^{a,b,c}_{i,j,k} = q^{(c+i-j)(c-k)} \left(\frac{t_1 t_3 w_3}{s_2}\right)^{-c+k} \frac{1}{(q^{b-i-k}\frac{s_1 s_3}{s_2}; q^2)_{-c+k}} R^{a-b+c,0,0}_{i-k-b+2c,j-b,0}. \quad (39)$$

Applying this to (37) and (38) with $b = c = k = 0$ we get

$$q^j r_3 t_1^2 w_3 R^{a,0,0}_{i-1,j,0} + q^{-j} s_1 (q^{i+1} s_2 - s_1 s_3) R^{a,0,0}_{i+1,j,0} = q^a t_1 t_2 w_3 R^{a-1,0,0}_{i-1,j-1,0}, \quad (40)$$

$$q^2 s_3 t_1^2 w_1 R^{a,0,0}_{i-1,j,0} + r_1 (q^{i+1} s_2 - s_1 s_3) R^{a,0,0}_{i+1,j,0} = q^{i+1} t_1 t_2 w_2 R^{a-1,0,0}_{i-1,j-1,0}. \quad (41)$$

Eliminating $R^{a,0,0}_{i+1,j,0}$ here leads to the recursion relation

$$R^{a,0,0}_{i,j,0} = q^i \frac{t_2 w_2}{s_3 t_1 w_1} \frac{1 - q^{a-i+j-2}\frac{r_1 w_3}{s_1 w_2}}{1 - q^{2j-2}\frac{r_1 r_3 w_3}{s_1 s_3 w_1}} R^{a-1,0,0}_{i,j-1,0}. \quad (42)$$

We remark that combination of (39) and (42) allows one to express $R^{a,b,c}_{i,j,k}$ in terms of $R^{0,0,0}_{i-k-b+2c,j-a-c,0}$ whose indices satisfy $i-k-b+2c \equiv d_2$ and $j-a-c \equiv d_1 \bmod 2$.

Next, consider (35) and (37) again with $a = b = c = k = 0$. Reducing them to the relations among $R^{0,0,0}_{\bullet,\bullet,0}$ by the above remark, and taking a suitable combination, we get

$$R^{0,0,0}_{i,j,0} = q^{2+i} \frac{t_2^2 w_2}{r_2 s_1 s_3} \frac{(1 - q^{j-2+i}\frac{r_3 w_2}{s_3 w_1})(1 - q^{j-2-i}\frac{r_1 w_3}{s_1 w_2})}{(1 - q^j \frac{r_1 r_3}{r_2})(1 - q^{2j-2}\frac{r_1 r_3 w_3}{s_1 s_3 w_1})(1 - q^{2j-4}\frac{r_1 r_3 w_3}{s_1 s_3 w_1})} R^{0,0,0}_{i,j-2,0}, \quad (43)$$

$$R^{0,0,0}_{i,j,0} = q^{-2i+j+2} \frac{s_3 t_1^2 w_1 w_3}{s_1 s_2 w_2} \frac{1 - q^{i+j-2}\frac{r_3 w_2}{s_3 w_1}}{(1 - q^{-i}\frac{s_1 s_3}{s_2})(1 - q^{-i+j}\frac{r_1 w_3}{s_1 w_2})} R^{0,0,0}_{i-2,j,0}. \quad (44)$$

Thus we find any $R^{a,b,c}_{i,j,k}$ is uniquely expressed as $R^{0,0,0}_{p_2,p_1,0}$ times known factors, where $p_1, p_2 \in \{0, 1\}$ are determined by $p_1 \equiv d_1$, $p_2 \equiv d_2 \bmod 2$. □

For $a, b, c, i, j, k \in \mathbb{Z}$ set

$$R^{a,b,c}_{i,j,k} = \left(\frac{r_2}{t_1t_3w_1}\right)^{\frac{d_1}{2}} \left(\frac{s_2}{t_1t_3w_3}\right)^{\frac{d_2}{2}} \left(\frac{t_2}{s_1t_3}\right)^{\frac{d_3}{2}} \left(\frac{t_2w_2}{s_3t_1w_1}\right)^{\frac{d_4}{2}} \times q^{\varphi} \frac{\Phi_{d_2}\left(\frac{s_1s_3}{s_2}\right)\Phi_{d_3}\left(\frac{r_3w_2}{s_3w_1}\right)\Phi_{d_4}\left(\frac{r_1w_3}{s_1w_2}\right)}{\Phi_{-d_1}\left(\frac{q^2r_1r_3}{r_2}\right)\Phi_{d_3+d_4}\left(\frac{r_1r_3w_3}{s_1s_3w_1}\right)}, \tag{45}$$

$$\varphi = \frac{1}{4}\big((d_1-d_2)(d_1+d_2+d_3+d_4)+d_3d_4\big) - d_1, \tag{46}$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} a+c-j \\ b-i-k \end{pmatrix}, \quad \begin{pmatrix} d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} -a-b+c+i+j-k \\ a-b-c-i+j+k \end{pmatrix}, \tag{47}$$

$$\Phi_m(z) = \frac{1}{(zq^m; q^2)_\infty} \quad (m \in \mathbb{Z}), \tag{48}$$

where d_1 and d_2 are the same as those in Proposition 2. It is easy to see $\varphi \in \mathbb{Z} + (d_1 - 1)d_2/2$. The dependence on t_1, t_2, t_3 is actually by the combination $t_1^{-a+i} t_2^{-b+j} t_3^{-c+k}$, which corresponds to a similarity transformation.

By Proposition 2, we know that the solution R of (28), if exists, is unique up to normalization in each sector specified by $(d_1, d_2)_{\text{mod }2}$. The following result establishes the existence together with an explicit form.

Theorem 3. *The 3D R defined by* (45)–(48) *satisfies the RLLL relation* (28).

Proof. From Proposition 2 and $d_3 \equiv d_4 \equiv d_1 + d_2 \mod 2$, the replacement

$$\Phi_m(z) \to \tilde{\Phi}_m(z) = \begin{cases} (z;q^2)_\infty/(zq^m;q^2)_\infty = (z;q^2)_{m/2} & (m \in 2\mathbb{Z}), \\ (zq;q^2)_\infty/(zq^m;q^2)_\infty = (zq;q^2)_{(m-1)/2} & (m \in 2\mathbb{Z}+1) \end{cases} \tag{49}$$

changes the individual recursion relations only by an overall scalar. The results become the relations among finitely many rational functions. To check them is straightforward. □

As the above proof indicates, one may just postulate the property

$$\Phi_{m+2}(z) = (1 - zq^m)\Phi_m(z) \tag{50}$$

instead of specifying $\Phi_m(z)$ concretely as (48). Another option of such sort is to make the replacement

$$1/\Phi_{-d_1}\left(\frac{q^2r_1r_3}{r_2}\right) \to q^{-\frac{d_1^2}{4}+\frac{d_1}{2}} \left(-\frac{r_1r_3}{r_2}\right)^{\frac{d_1}{2}} \Phi_{d_1}\left(\frac{r_2}{r_1r_3}\right), \tag{51}$$

which makes the formula (45) more symmetric with respect to d_1 and d_2 at the cost of the appearance of the factor $(-1)^{d_1/2}$. The R is not locally finite. From (22), its inverse is given by

$$R^{-1} = (\text{scalar})R\,|_{r_i \leftrightarrow s_i, t_i \to t_i w_i, w_i \to w_i^{-1}\ (i=1,2,3)}. \tag{52}$$

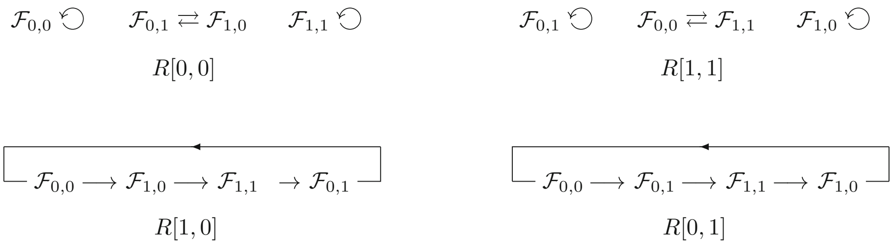

Fig. 4. Action of the four fundamental solutions $R[0,0]$, $R[1,0]$, $R[0,1]$, $R[1,1]$ on the subspaces $\mathcal{F}_{p_1,p_2}$ defined in (53). For example in $R[1,0]$, the condition $(d_1, d_2) = (a+c-j, b-i-k) \equiv (1,0)$ on $|i\rangle \otimes |j\rangle \otimes |k\rangle \mapsto |a\rangle \otimes |b\rangle \otimes |c\rangle$ enforces $R[1,0]\mathcal{F}_{0,0} \subseteq \mathcal{F}_{1,0}$, $R[1,0]\mathcal{F}_{1,0} \subseteq \mathcal{F}_{1,1}$, $R[1,0]\mathcal{F}_{1,1} \subseteq \mathcal{F}_{0,1}$ and $R[1,0]\mathcal{F}_{0,1} \subseteq \mathcal{F}_{0,0}$

The parity condition on (d_1, d_2) mixes the indices i, j, k labeling incoming states and a, b, c concerning outgoing ones. See (25). To illustrate the resulting sectors, we introduce the subspace

$$\mathcal{F}_{p_1,p_2} = \bigoplus_{i+k\equiv p_1,\, j\equiv p_2 \bmod 2} \mathbb{C}|i\rangle \otimes |j\rangle \otimes |k\rangle \subset F^{\otimes 3} \quad (p_1, p_2 = 0, 1). \tag{53}$$

From the proof of Proposition 2, the solution space of R is four dimensional whose basis corresponds to the "initial condition" of the recursion relation taken as $(R^{0,0,0}_{0,0,0}, R^{0,0,0}_{1,0,0}, R^{0,0,0}_{0,1,0}, R^{0,0,0}_{1,1,0}) = (1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$. Call them $R[0,0]$, $R[0,1]$, $R[1,0]$, $R[1,1]$ respectively so that $R[p_1, p_2]$ is the base corresponding to the choice $R^{0,0,0}_{p_2,p_1,0} = 1$ according to the remark after (42).[3] Then they act on (53) as in Fig. 4.

Similar decompositions according to a parity condition also take place in the forthcoming Theorems 9, 10, 11, 13, 14 and 15.

Remark 4. Let $L^{Z\pm}$ be the 3D L operator (19) with π_Z in (8) replaced by

$$X|m\rangle = |m \mp 1\rangle, \quad X^{-1}|m\rangle = |m \pm 1\rangle, \quad Z|m\rangle = q^{\pm m}|m\rangle, \quad Z^{-1}|m\rangle = q^{\mp m}|m\rangle. \tag{54}$$

Theorem 3 is concerned with $L^Z = L^{Z_+}$. Consider a variant of (28) given by

$$R(\varepsilon_1, \varepsilon_2, \varepsilon_3)_{456} L^{Z_{\varepsilon_3}}_{236} L^{Z_{\varepsilon_2}}_{135} L^{Z_{\varepsilon_1}}_{124} = L^{Z_{\varepsilon_1}}_{124} L^{Z_{\varepsilon_2}}_{135} L^{Z_{\varepsilon_3}}_{236} R(\varepsilon_1, \varepsilon_2, \varepsilon_3)_{456} \quad (\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{-1, 1\}). \tag{55}$$

Then elements of $R(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is given by

$$R(\varepsilon_1, \varepsilon_2, \varepsilon_3)^{a,b,c}_{i,j,k} = R^{\varepsilon_1 a,\, \varepsilon_2 b,\, \varepsilon_3 c}_{\varepsilon_1 i,\, \varepsilon_2 j,\, \varepsilon_3 k}, \tag{56}$$

where the RHS is defined by (45)–(48) which corresponds to $R(+,+,+)$.

[3] The formula (45) has not been so normalized.

3.2. OZZ type. We consider the $RLLL$ relation

$$R_{456}L^{Z}_{236}L^{Z}_{135}L^{O}_{124} = L^{O}_{124}L^{Z}_{135}L^{Z}_{236}R_{456}, \tag{57}$$

where L^{Z}_{135} and L^{Z}_{236} are given by (19) with $(r, s, t, w) = (r_2, s_2, t_2, w_2)$ and (r_3, s_3, t_3, w_3), respectively. In this case, $R \in \mathrm{End}(F_+ \otimes F \otimes F)$ and the sum (25) extends over $a \in \mathbb{Z}_{\geq 0}$ and $b, c \in \mathbb{Z}$. The equality (57) holds in $\mathrm{End}(V \otimes V \otimes V \otimes F_+ \otimes F \otimes F)$.

Here are some examples of the $RLLL$ relation (57):

$$R^{a,b,c}_{i,j-1,k-1} = R^{a,b+1,c+1}_{i,j,k}, \quad q^i R^{a,b,c}_{i,j-1,k} = q^a R^{a,b+1,c}_{i,j,k}, \tag{58}$$

$$q^{a+c} r_2 R^{a,b,c}_{i,j,k} - q^k \mu t_2 R^{a,b,c}_{i,j-1,k} + q^b(1-q^{2a+2})\mu t_3 R^{a+1,b,c+1}_{i,j,k} = 0, \tag{59}$$

$$q^c r_2 R^{a-1,b,c}_{i,j,k} - q^j r_3 R^{a,b,c}_{i,j,k} - q^{1+a+b}\mu t_3 R^{a,b,c+1}_{i,j,k} = 0, \tag{60}$$

$$q^{i+j} r_3 s_3 R^{a,b,c}_{i,j,k} + q^k \mu s_2 t_3 R^{a,b,c}_{i+1,j,k-1} - q^k \mu s_3 t_2 R^{a-1,b+1,c}_{i,j,k} - q^{2+i+j} t_3^2 w_3 R^{a,b,c}_{i,j,k-2} = 0. \tag{61}$$

The boundary condition

$$R^{a,b,c}_{i,j,k} = 0 \quad \text{if} \quad \min(a, i) < 0. \tag{62}$$

has to be taken into account. Thus for example when $a = 0$, (60) is to be understood as $q^j r_3 R^{0,b,c}_{i,j,k} + q^{1+b}\mu t_3 R^{0,b,c+1}_{i,j,k} = 0$.

For $a, b, c, i, j, k \in \mathbb{Z}$, set

$$\begin{aligned} R^{a,b,c}_{i,j,k} &= \left(\frac{r_2}{r_3}\right)^a \left(\frac{s_3}{s_2}\right)^i \left(\frac{t_2 w_2}{\mu s_2}\right)^{-b+j} \left(-\frac{\mu t_3}{r_3}\right)^{-c+k} \\ &\quad \times \frac{1}{(q^2;q^2)_a} q^{(a-b+j-1)c-(i-b+j-1)k-aj+bi} \\ &\quad \times \sum_{\beta=0}^{i} q^{\beta(\beta+2j-2b-1)}(-y)^\beta \binom{i}{\beta}_{q^2} \left(xq^{2k-2c-2\beta+2}; q^2\right)_a, \end{aligned} \tag{63}$$

$$x = \frac{\mu^2 s_2}{r_2 w_2}, \quad y = \frac{r_3 w_3}{\mu^2 s_3}, \quad z = xq^{2k-2c+2}. \tag{64}$$

For the convenience of the proof of Theorem 5, we have enlarged the range of the indices a and i from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}$. The property (62) is satisfied thanks to the factor $\binom{i}{\beta}_{q^2}/(q^2;q^2)_a$. The formula (63) is also presented as a terminating q-hypergeometric series:

$$\begin{aligned} R^{a,b,c}_{i,j,k} &= \theta(i \geq 0) \left(\frac{r_2}{r_3}\right)^a \left(\frac{s_3}{s_2}\right)^i \left(\frac{t_2 w_2}{\mu s_2}\right)^{-b+j} \left(-\frac{\mu t_3}{r_3}\right)^{-c+k} \\ &\quad \times \frac{(z;q^2)_a}{(q^2;q^2)_a} q^{(a-b+j-1)c-(i-b+j-1)k-aj+bi} \\ &\quad \times {}_2\phi_1\left(\begin{matrix} q^{-2i}, z^{-1}q^2 \\ z^{-1}q^{-2a+2} \end{matrix}; q^2, yq^{2i+2j-2a-2b}\right). \end{aligned} \tag{65}$$

Theorem 5. *The $RLLL$ relation (57) has a unique solution R up to normalization. It is given by (63)–(65).*

Proof. The first claim, i.e., uniqueness, can be shown by an argument similar to Proposition 2. To prove the second claim, let $S^a_{i,j-b,k-c}(x,y)$ denote the second line of (63). One sees that $S^a_{i,j,k}(x,y) = \sum_{\beta=0}^{i}(-y)^\beta S^a_{i,j,k,\beta}(x)$,

where $S^a_{i,j,k,\beta}(x) = q^{\beta(\beta+2j-1)}\binom{i}{\beta}_{q^2}(xq^{2k-2\beta+2};q^2)_a$ is a polynomial in x and y. The Eq. (57) is reduced to the recursion relations among $S^a_{i,j,k}(x,y)$ with coefficients including $q, q^a, q^i, q^j, q^k, x, y$ only. By picking the coefficients of y^β, they are reduced to the relations containing finitely many $S^a_{i,j,k,\beta}(x)$'s. To check them is straightforward. This proves the recursion relations for generic a and i. This fact together with (62) assure that they are also valid in the vicinity of $a = 0$ and $i = 0$. □

As for the last point of the proof, a similar and more detailed explanation is available in the proof of Theorem 9. The R is not locally finite.

3.3. ZZO type. We consider the $RLLL$ relation

$$R_{456}L^O_{236}L^Z_{135}L^Z_{124} = L^Z_{124}L^Z_{135}L^O_{236}R_{456}, \tag{66}$$

where L^Z_{124} and L^Z_{135} are given by (19) with $(r,s,t,w) = (r_1,s_1,t_1,w_1)$ and (r_2,s_2,t_2,w_2), respectively. In this case, $R \in \mathrm{End}(F\otimes F\otimes F_+)$ and the sum (25) extends over $a,b \in \mathbb{Z}$ and $c \in \mathbb{Z}_{\geq 0}$. The equality (66) holds in $\mathrm{End}(V\otimes V\otimes V\otimes F\otimes F\otimes F_+)$.

Here are some examples of the $RLLL$ relation (66):

$$R^{a,b,c}_{i-1,j-1,k} = R^{a+1,b+1,c}_{i,j,k}, \quad q^k R^{a,b,c}_{i,j-1,k} = q^c R^{a,b+1,c}_{i,j,k}, \tag{67}$$

$$q^{a+c}\mu r_2 R^{a,b,c}_{i,j,k} - q^i t_2 w_2 R^{a,b,c}_{i,j-1,k} + q^b(1-q^{2c+2})t_1 w_1 R^{a+1,b,c+1}_{i,j,k} = 0, \tag{68}$$

$$q^a \mu r_2 R^{a,b,c-1}_{i,j,k} - q^j \mu r_1 R^{a,b,c}_{i,j,k} - q^{1+b+c} t_1 w_1 R^{a+1,b,c}_{i,j,k} = 0, \tag{69}$$

$$\begin{aligned} &q^{j+k}\mu r_1 s_1 R^{a,b,c}_{i,j,k} - q^i s_1 t_2 w_2 R^{a,b+1,c-1}_{i,j,k} - q^{2+j+k}\mu t_1^2 w_1 R^{a,b,c}_{i-2,j,k} \\ &\quad + q^i s_2 t_1 w_1 R^{a,b,c}_{i-1,j,k+1} = 0. \end{aligned} \tag{70}$$

One has the boundary condition analogous to (62):

$$R^{a,b,c}_{i,j,k} = 0 \quad \text{if} \quad \min(c,k) < 0. \tag{71}$$

For $a,b,c,i,j,k \in \mathbb{Z}_{\geq 0}$, set

$$\begin{aligned} R^{a,b,c}_{i,j,k} &= \left(\frac{r_2}{r_1}\right)^c \left(\frac{s_1}{s_2}\right)^k \left(\frac{\mu t_2}{s_2}\right)^{-b+j} \left(-\frac{t_1 w_1}{\mu r_1}\right)^{-a+i} \\ &\times \frac{1}{(q^2;q^2)_c} q^{(c-b+j-1)a-(k-b+j-1)i-cj+bk} \\ &\times \sum_{\beta=0}^{k} q^{\beta(\beta+2j-2b-1)}(-y)^\beta \binom{k}{\beta}_{q^2}\left(xq^{2i-2a-2\beta+2};q^2\right)_c, \end{aligned} \tag{72}$$

$$x = \frac{s_2 w_2}{\mu^2 r_2}, \quad y = \frac{\mu^2 r_1}{s_1 w_1}, \quad z = xq^{2i-2a+2}, \tag{73}$$

where we have redefined x, y, z changing (64). It is also presented as a terminating q-hypergeometric series:

$$R^{a,b,c}_{i,j,k} = \theta(k \geq 0)\left(\frac{r_2}{r_1}\right)^c \left(\frac{s_1}{s_2}\right)^k \left(\frac{\mu t_2}{s_2}\right)^{-b+j} \left(-\frac{t_1 w_1}{\mu r_1}\right)^{-a+i}$$
$$\times \frac{(z;q^2)_c}{(q^2;q^2)_c} q^{(c-b+j-1)a-(k-b+j-1)i-cj+bk}$$
$$\times {}_2\phi_1\left(\begin{matrix} q^{-2k}, z^{-1}q^2 \\ z^{-1}q^{-2c+2} \end{matrix}; q^2, yq^{2j+2k-2b-2c}\right). \tag{74}$$

Theorem 6. *The $RLLL$ relation* (66) *has a unique solution R up to normalization. It is given by* (72)–(74).

The proof is similar to Theorem 5. The R is not locally finite.

3.4. ZOZ type. We consider the $RLLL$ relation

$$R_{456} L^Z_{236} L^O_{135} L^Z_{124} = L^Z_{124} L^O_{135} L^Z_{236} R_{456}, \tag{75}$$

where L^Z_{124} and L^Z_{236} are given by (19) with $(r, s, t, w) = (r_1, s_1, t_1, w_1)$ and (r_3, s_3, t_3, w_3), respectively. In this case, $R \in \mathrm{End}(F \otimes F_+ \otimes F)$ and the sum (25) extends over $a, c \in \mathbb{Z}$ and $b \in \mathbb{Z}_{\geq 0}$. The equality (75) holds in $\mathrm{End}(V \otimes V \otimes V \otimes F \otimes F_+ \otimes F)$.

Here are some examples of the $RLLL$ relation (75):

$$q^j R^{a,b,c}_{i,j,k-1} = q^b R^{a,b,c+1}_{i,j,k}, \quad q^j R^{a,b,c}_{i-1,j,k} = q^b R^{a+1,b,c}_{i,j,k}, \tag{76}$$
$$q^{a+c} R^{a,b,c}_{i,j,k} - r_1 r_3 R^{a,b,c}_{i,j+1,k} - q t_1 t_3 w_1 R^{a+1,b-1,c+1}_{i,j,k} = 0, \tag{77}$$
$$q^{i+k} R^{a,b,c}_{i,j,k} - s_1 s_3 R^{a,b-1,c}_{i,j,k} - q t_1 t_3 w_3 R^{a,b,c}_{i-1,j+1,k-1} = 0, \tag{78}$$
$$q^{a+j+k} r_1 R^{a,b,c}_{i,j,k} - \mu r_1 s_1 t_3 R^{a,b-1,c+1}_{i,j,k} - q^{a+c} \mu t_1 R^{a+1,b,c}_{i,j,k} + \mu t_1^2 t_3 w_1 R^{a+2,b-1,c+1}_{i,j,k} = 0. \tag{79}$$

The boundary condition is given by

$$R^{a,b,c}_{i,j,k} = 0 \quad \text{if} \quad \min(b, j) < 0. \tag{80}$$

For $a, b, c, i, j, k \in \mathbb{Z}_{\geq 0}$, set

$$R^{a,b,c}_{i,j,k} = \theta(j \geq 0)\frac{(s_1 s_3)^b}{(r_1 r_3)^j}\left(\frac{r_1}{\mu t_1}\right)^{a-i} \left(\frac{\mu r_3}{t_3 w_3}\right)^{c-k}$$
$$\times \frac{1}{(q^2;q^2)_b} q^{(j-b)(a+c)+b(a+c-i-k)-(i-a)(k-c)}$$
$$\times \sum_{\beta=0}^{b} q^{\beta(\beta+2i-2a+1)}(-y)^\beta \binom{b}{\beta}_{q^2} \left(q^{2j+2k-2c-2\beta}x^{-1}; q^{-2}\right)_\beta$$
$$\times \left(q^{2k-2c-2\beta+2}x^{-1}; q^2\right)_{b-\beta}, \tag{81}$$

$$x = \frac{\mu^2 s_1}{r_1 w_1}, \quad y = \frac{\mu^2 r_3}{s_3 w_3}. \tag{82}$$

This can also be expressed as a terminating series similar to a generalized q-hypergeometric ${}_3\phi_2$:

$$R^{a,b,c}_{i,j,k} = \theta(j \geq 0)\frac{(s_1s_3)^b}{(r_1r_3)^j}\left(\frac{r_1}{\mu t_1}\right)^{a-i}\left(\frac{\mu r_3}{t_3w_3}\right)^{c-k}$$
$$\times \frac{(q^{2-2c+2k}x^{-1}, q^2)_b}{(q^2;q^2)_b} q^{(j-b)(a+c)+b(a+c-i-k)-(i-a)(k-c)}$$
$$\times \sum_{\beta=0}^{b}(q^{2i+2j-2a-2b+2}y)^\beta \frac{(q^{-2b};q^2)_\beta(q^{2c-2k}x;q^2)_\beta(q^{2c-2j-2k}x;q^2)_{2\beta}}{(q^2;q^2)_\beta(q^{-2b+2c-2k}x;q^2)_{2\beta}(q^{2c-2j-2k}x;q^2)_\beta}. \tag{83}$$

The difference from ${}_3\phi_2$ is the factors $(\bullet;q^2)_{2\beta}$.

Theorem 7. *The $RLLL$ relation (75) has a unique solution R up to normalization. It is given by (81)–(82).*

The proof is similar to Theorem 5. The R is not locally finite.

3.5. OOZ type. We consider the $RLLL$ relation

$$R_{456}L^Z_{236}L^O_{135}L^O_{124} = L^O_{124}L^O_{135}L^Z_{236}R_{456}, \tag{84}$$

where L^O_{124} and L^O_{135} are given by (21) with $\mu = \mu_1$ and μ_2, respectively, and L^Z_{236} is given by (19) with $(r, s, t, w) = (r_3, s_3, t_3, w_3)$. In this case, $R \in \mathrm{End}(F_+ \otimes F_+ \otimes F)$ and the sum (25) extends over $a, b \in \mathbb{Z}_{\geq 0}$ and $c \in \mathbb{Z}$. The equality (84) holds in $\mathrm{End}(V \otimes V \otimes V \otimes F_+ \otimes F_+ \otimes F)$.

Here are some examples of the $RLLL$ relation (84):

$$(q^{i+j} - q^{a+b})R^{a,b,c}_{i,j,k} = 0, \quad q^jR^{a,b,c}_{i,j,k-1} = q^bR^{a,b,c+1}_{i,j,k}, \tag{85}$$
$$(\mu_2q^{b+c} - \mu_1q^{i+k})R^{a,b,c}_{i,j,k} = (1-q^{2i})t_3w_3R^{a,b,c}_{i-1,j+1,k-1}, \tag{86}$$
$$(\mu_2^{-1}q^{j+k} - \mu_1^{-1}q^{a+c})R^{a,b,c}_{i,j,k} = (1-q^{2a+2})t_3R^{a+1,b-1,c+1}_{i,j,k}, \tag{87}$$
$$s_3R^{a,b-1,c}_{i,j,k} - q^kR^{a,b,c}_{i+1,j,k} + q^{i+1}\mu_1^{-1}t_3w_3R^{a,b,c}_{i,j+1,k-1} = 0, \tag{88}$$
$$q^k\mu_1\mu_2t_3R^{a,b,c}_{i+1,j,k-1} + q^i\mu_2r_3s_3R^{a,b,c}_{i,j+1,k} - q^{b+k}\mu_1s_3R^{a-1,b,c}_{i,j,k}$$
$$- q^{2+i}\mu_2t_3^2w_3R^{a,b,c}_{i,j+1,k-2} = 0. \tag{89}$$

As these examples indicate, every recursion relation consists of those $R^{a,b,c}_{i,j,k}$ having the same parity of $a - c + j + k$.

For $a, b, c, i, j, k, d \in \mathbb{Z}$, set

$$R(d)^{a,b,c}_{i,j,k} = \theta(e \in \mathbb{Z})\theta(\min(i,j) \geq 0)\delta^{a+b}_{i+j}\, s_3^i(\mu_2t_3)^{-a}\left(\frac{\mu_2s_3}{t_3w_3}\right)^j\left(\frac{t_3^2w_3}{r_3s_3}\right)^e$$
$$\times q^{cj-bk}\frac{(q^{2+2e-2j};q^2)_j(q^{2a+2};q^2)_{i-a}}{(q^2;q^2)_f(q^{2a-2e};q^2)_{e-a}}, \tag{90}$$
$$e = \frac{1}{2}(a - c + j + k + d), \quad f = \frac{1}{2}(b + c + i - k - d). \tag{91}$$

For the convenience of the proof of Theorem 9, we have defined $R(d)^{a,b,c}_{i,j,k}$ enlarging the range of the indices a, b, i, j from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}$. We note that $R(d)^{a,b,c}_{i,j,k} = q^{bd} R(0)^{a,b,c}_{i,j,k+d}$ and $\theta(e \in \mathbb{Z})\delta^{a+b}_{i+j} = \theta(f \in \mathbb{Z})\delta^{a+b}_{i+j}$ since $e + f = i + j \in \mathbb{Z}$ holds when $a + b = i + j$. The combinations e and f can be either positive or negative.

Lemma 8.

$$R(d)^{a,b,c}_{i,j,k} = 0 \quad \textit{if} \quad \min(a, b, i, j) < 0. \tag{92}$$

Proof. The assertion is obvious if $\min(i, j) < 0$. Thus we are to show that $\min(a, b) < 0$ leads to $R(d)^{a,b,c}_{i,j,k} = 0$ assuming that $\min(i, j) \geq 0$. Suppose $a < 0$. Then (90) indeed vanishes due to $(q^{2a+2}; q^2)_{i-a} = (q^{2a+2}; q^2)_\infty/(q^{2i+2}; q^2)_\infty = 0$. Suppose $b < 0$. We may further concentrate on the non-trivial case $e \geq a$ since otherwise $1/(q^{2a-2e}; q^2)_{e-a} = 0$. Then $1/(q^2; q^2)_f = 0$ because of $f = i + j - e = (a - e) + b < 0$. □

When $a, b, i, j \geq 0$, $R(d)^{a,b,c}_{i,j,k}$ is divergence-free and $R(d)^{a,b,c}_{i,j,k} = 0$ unless $e \geq \max(a, j)$ and $f \geq 0$. From these conditions it follows that

$$R(d)^{a,b,c}_{i,j,k} = 0 \quad \text{unless } |b - i| \leq k - c + d \leq b + i. \tag{93}$$

Theorem 9. *The RLLL relation* (84) *has a non-trivial solution if and only if* $d := \log_q\left(\frac{\mu_1}{\mu_2}\right) \in \mathbb{Z}$. *Up to overall normalization it is given by* $R^{a,b,c}_{i,j,k} = R(d)^{a,b,c}_{i,j,k}$ *specified by* (90) *and* (91).

Proof. The only if part of the first claim can be shown by an argument similar to Proposition 2. To show the rest, one first checks that the formula (90) satisfies the recursion relation when a, b, c, i, j, k are generic, i.e., when $\theta(\min(i, j) \geq 0) = 1$. This can be done easily since (90) is factorized. The remaining task is to verify the boundary condition (92) to assure that the contribution from the "unwanted terms" to the recursion relation is zero. This has been guaranteed by Lemma 8. For example in (88) at $b = 0$, i.e.,

$$s_3 R^{a,-1,c}_{i,j,k} - q^k R^{a,0,c}_{i+1,j,k} + q^{i+1}\mu_1^{-1} t_3 w_3 R^{a,0,c}_{i,j+1,k-1} = 0, \tag{94}$$

the first term is unwanted. □

From (90) and (93), R is locally finite. From (22), its inverse is given by

$$R^{-1} = R\,|_{\mu_i \to \mu_i^{-1}\,(i=1,2),\, r_3 \leftrightarrow s_3, t_3 \to t_3 w_3, w_3 \to w_3^{-1}}, \tag{95}$$

where the normalization has been deduced from $R^{a,b,c}_{0,0,0}(d) = \delta^a_0 \delta^b_0 \delta^c_d$ and $R^{a,b,c}_{0,0,d}(-d) = \delta^a_0 \delta^b_0 \delta^c_0$.

3.6. ZOO type. We consider the $RLLL$ relation

$$R_{456}L^O_{236}L^O_{135}L^Z_{124} = L^Z_{124}L^O_{135}L^O_{236}R_{456}, \tag{96}$$

where L^O_{135} and L^O_{236} are given by (21) with $\mu = \mu_2$ and μ_3, respectively, and L^Z_{124} is given by (19) with $(r, s, t, w) = (r_1, s_1, t_1, w_1)$. In this case, $R \in \mathrm{End}(F \otimes F_+ \otimes F_+)$ and the sum (25) extends over $a \in \mathbb{Z}$ and $b, c \in \mathbb{Z}_{\geq 0}$. The equality (96) holds in $\mathrm{End}(V \otimes V \otimes V \otimes F \otimes F_+ \otimes F_+)$.

Here are some examples of the $RLLL$ relation (96):

$$(q^{j+k} - q^{b+c})R^{a,b,c}_{i,j,k} = 0, \quad q^j R^{a,b,c}_{i-1,j,k} = q^b R^{a+1,b,c}_{i,j,k}, \tag{97}$$

$$(\mu_2^{-1}q^{a+b} - \mu_3^{-1}q^{i+k})R^{a,b,c}_{i,j,k} = (1-q^{2k})t_1 R^{a,b,c}_{i-1,j+1,k-1}, \tag{98}$$

$$(\mu_2 q^{i+j} - \mu_1 q^{a+c})R^{a,b,c}_{i,j,k} = (1-q^{2c+2})t_1 w_1 R^{a+1,b-1,c+1}_{i,j,k}, \tag{99}$$

$$s_1 R^{a,b-1,c}_{i,j,k} - q^i R^{a,b,c}_{i,j,k+1} + q^{k+1}\mu_3 t_1 R^{a,b,c}_{i-1,j+1,k} = 0, \tag{100}$$

$$q^{b+i}\mu_2 s_1 R^{a,b,c-1}_{i,j,k} + q^{2+k}\mu_3 t_1^2 w_1 R^{a,b,c}_{i-2,j+1,k} - q^i t_1 w_1 R^{a,b,c}_{i-1,j,k+1} - q^k \mu_3 r_1 s_1 R^{a,b,c}_{i,j+1,k} = 0. \tag{101}$$

As these examples indicate, every recursion relation consists of those $R^{a,b,c}_{i,j,k}$ having the same parity of $-a+c+i+j$. The boundary condition is given by

$$R^{a,b,c}_{i,j,k} = 0 \quad \text{if} \quad \min(b, c, j, k) < 0. \tag{102}$$

For $a, b, c, i, j, k, d \in \mathbb{Z}$, set

$$R(d)^{a,b,c}_{i,j,k} = \theta(e \in \mathbb{Z})\theta(\min(j,k) \geq 0)\delta^{b+c}_{j+k}\, s_1^k \left(\frac{\mu_2}{t_1 w_1}\right)^c \left(\frac{s_1}{\mu_2 t_1}\right)^j \left(\frac{t_1^2 w_1}{r_1 s_1}\right)^e \times q^{aj-bi}\frac{(q^{2+2e-2j};q^2)_j (q^{2+2c};q^2)_{k-c}}{(q^2;q^2)_f (q^{2c-2e};q^2)_{e-c}}, \tag{103}$$

$$e = \frac{1}{2}(-a+c+i+j-d), \quad f = \frac{1}{2}(a+b-i+k+d). \tag{104}$$

We note that $R(d)^{a,b,c}_{i,j,k} = q^{-bd}R(0)^{a,b,c}_{i-d,j,k}$ and $\theta(e \in \mathbb{Z})\delta^{b+c}_{j+k} = \theta(f \in \mathbb{Z})\delta^{b+c}_{j+k}$ since $e + f = j + k \in \mathbb{Z}$ when $b + c = j + k$. The combinations e and f can be either positive or negative. From $b, c, j, k \geq 0$ and the definition (26), $R^{abc}_{ijk}(d)$ is divergence-free and $R(d)^{a,b,c}_{i,j,k} = 0$ unless $e \geq \max(c, j)$ and $f \geq 0$. From these conditions it follows that

$$R(d)^{a,b,c}_{i,j,k} = 0 \quad \text{unless } |b-k| \leq i-a-d \leq b+k. \tag{105}$$

Theorem 10. *The $RLLL$ relation (96) has a non-trivial solution if and only if $d := \log_q\left(\frac{\mu_3}{\mu_2}\right) \in \mathbb{Z}$. Up to overall normalization it is given by $R^{a,b,c}_{i,j,k} = R(d)^{a,b,c}_{i,j,k}$ specified by (103) and (104).*

The proof is similar to Theorem 9. From (103) and (105), R is locally finite. From (22), its inverse is given by

$$R^{-1} = R\,|_{\mu_i \to \mu_i^{-1}\,(i=2,3),\, r_1 \leftrightarrow s_1, t_1 \to t_1 w_1, w_1 \to w_1^{-1}}, \tag{106}$$

where the normalization has been deduced from $R^{a,b,c}_{0,0,0}(d) = \delta^a_{-d}\delta^b_0\delta^c_0$ and $R^{a,b,c}_{-d,0,0}(-d) = \delta^a_0\delta^b_0\delta^c_0$.

3.7. OZO type. We consider the $RLLL$ relation

$$R_{456} L^O_{236} L^Z_{135} L^O_{124} = L^O_{124} L^Z_{135} L^O_{236} R_{456}, \tag{107}$$

where L^Z_{135} is given by (19) with $(r, s, t, w) = (r_2, s_2, t_2, w_2)$, and L^O_{124} and L^O_{236} are given by (21) with $\mu = \mu_1$ and μ_3, respectively. In this case, $R \in \mathrm{End}(F_+ \otimes F \otimes F_+)$ and the sum (25) extends over $a, c \in \mathbb{Z}_{\geq 0}$ and $b \in \mathbb{Z}$. The equality (107) holds in $\mathrm{End}(V \otimes V \otimes V \otimes F_+ \otimes F \otimes F_+)$.

Here are some examples of the $RLLL$ relation (107):

$$q^k R^{a,b,c}_{i,j-1,k} = q^c R^{a,b+1,c}_{i,j,k}, \quad q^i R^{a,b,c}_{i,j-1,k} = q^a R^{a,b+1,c}_{i,j,k}, \tag{108}$$

$$\mu_3 r_2 R^{a-1,b,c-1}_{i,j,k} = (q^{1+a+b+c}\mu_1 + q^j \mu_3) R^{a,b,c}_{i,j,k}, \tag{109}$$

$$\mu_1 s_2 R^{a,b,c}_{i+1,j,k+1} = (q^{1+i+j+k}\mu_3 + q^b \mu_1) R^{a,b,c}_{i,j,k}, \tag{110}$$

$$q^a \mu_3 r_2 R^{a,b,c-1}_{i,j,k} + q^{b+c}(1 - q^{2a+2})\mu_1 R^{a+1,b,c}_{i,j,k} - t_2 \mu_1 \mu_3 R^{a,b,c}_{i,j-1,k+1} = 0, \tag{111}$$

$$\begin{aligned} &q^{i+j}(1 - q^{2k})\mu_3 r_2 R^{a,b,c}_{i,j,k-1} - q^{2+k}\mu_1 t_2^2 w_2 R^{a,b,c}_{i+1,j-2,k} \\ &\quad + q^k \mu_1 r_2 s_2 R^{a,b,c}_{i+1,j,k} - q^j (1 - q^{2+2c})\mu_1 \mu_3 t_2 R^{a,b+1,c+1}_{i,j,k} = 0. \end{aligned} \tag{112}$$

As these examples indicate, every recursion relation consists of those $R^{a,b,c}_{i,j,k}$ having the same parity of $a + b + c - j$.

$$R^{a,b,c}_{i,j,k} = 0 \quad \text{if} \quad \min(a, c, i, k) < 0. \tag{113}$$

For $a, b, c, i, j, k, d \in \mathbb{Z}$, set

$$\begin{aligned} R(d)^{a,b,c}_{i,j,k} &= \theta(e \in \mathbb{Z})\theta(\min(i,k) \geq 0)\delta^{a-c}_{i-k} r_2^c (\mu_3 t_2)^{-k} \left(\frac{\mu_3 r_2}{t_2 w_2}\right)^i \left(\frac{t_2^2 w_2}{r_2 s_2}\right)^e \\ &\quad \times q^{bk-cj} \frac{(q^{2+2e-2k}; q^2)_k}{(q^2; q^2)_f (q^{2i-2e}; q^2)_{e-i}}, \end{aligned} \tag{114}$$

$$e = \frac{1}{2}(i + j + k - b - d - 1), \quad f = \frac{1}{2}(a + b + c - j + d + 1). \tag{115}$$

We note that $R(d)^{a,b,c}_{i,j,k} = q^{-dk} R(0)^{a,b+d,j}_{i,j,k}$ and $\theta(e \in \mathbb{Z})\delta^{a-c}_{i-k} = \theta(f \in \mathbb{Z})\delta^{a-c}_{i-k}$ since $e + f = c + i$ when $a - c = i - k$. The combinations e and f can be either positive or negative. From $a, c, i, k \geq 0$ and the definition (26), $R(d)^{a,b,c}_{i,j,k}$ is divergence-free and $R(d)^{a,b,c}_{i,j,k} = 0$ unless $e \geq \max(i, k)$ and $f \geq 0$. From these conditions it follows that

$$R(d)^{a,b,c}_{i,j,k} = 0 \quad \text{unless } |a - c| \leq j - b - d - 1 \leq a + c. \tag{116}$$

Theorem 11. *The $RLLL$ relation (107) has a non-trivial solution if and only if $d{:=}\log_q\left(-\frac{\mu_1}{\mu_3}\right) \in \mathbb{Z}$. Up to overall normalization it is given by $R^{a,b,c}_{i,j,k} = R(d)^{a,b,c}_{i,j,k}$ specified by (114) and (115).*

The proof is similar to Theorem 9. R is not locally finite.

3.8. OOO type. We consider the $RLLL$ relation

$$R_{456}L^O_{236}L^O_{135}L^O_{124} = L^O_{124}L^O_{135}L^O_{236}R_{456}, \tag{117}$$

where $L^O_{124}, L^O_{135}, L^O_{236}$ are given by (21) with $\mu = \mu_1, \mu_2, \mu_3$. In this case, $R \in \mathrm{End}(F_+ \otimes F_+ \otimes F_+)$ and the sum (25) extends over $a, b, c \in \mathbb{Z}_{\geq 0}$. The equality (117) holds in $\mathrm{End}(V \otimes V \otimes V \otimes F_+ \otimes F_+ \otimes F_+)$. The problem of finding the solution to (117) was studied in [3,5]. The result has been shown [15, eq.(2.29)] to coincide with the intertwiner of the quantized coordinate ring $A_q(sl_3)$ that had been obtained earlier in [9]. See also the explanation in Sect. 5.2. For μ_i's general, the following formula is valid (cf. [14, eq.(3.85)]):

$$R^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\left(\frac{\mu_3}{\mu_2}\right)^i\left(-\frac{\mu_1}{\mu_3}\right)^b\left(\frac{\mu_2}{\mu_1}\right)^k q^{ik+b(k-i+1)}\binom{a+b}{a}_{q^2} \times {}_2\phi_1\left(\begin{matrix} q^{-2b}, q^{-2i} \\ q^{-2a-2b}\end{matrix}; q^2, q^{-2c}\right). \tag{118}$$

R is obviously locally finite. From (22) and [14, eq.(3.60)], its inverse is given by

$$R^{-1} = R\,|_{\mu_i \to \mu_i^{-1}\,(i=1,2,3)}. \tag{119}$$

Remark 12. Let $\mathcal{R}^{a,b,c}_{i,j,k} = R^{a,b,c}_{i,j,k}\,|_{\mu_i=1 (i=1,2,3)}$ be the parameter-free 3D R of type OOO. It satisfies the tetrahedron equation (164). See Sect. 5.2. It is known ([15, Prop.24], [14, eq.(3.63)]) that $\mathcal{R}^{a,b,c}_{i,j,k}$ is a *polynomial* in q with integer coefficients satisfying

$$\mathcal{R}^{a,b,c}_{i,j,k} = \frac{(q^2;q^2)_i(q^2;q^2)_j(q^2;q^2)_k}{(q^2;q^2)_a(q^2;q^2)_b(q^2;q^2)_c}\mathcal{R}^{i,j,k}_{a,b,c}. \tag{120}$$

When q is a primitive root of unity of odd degree $N \geq 3$, it follows that $\mathcal{R}^{a,b,c}_{i,j,k} = 0$ if $\max(i, j, k) \geq N$ and $\max(a, b, c) < N$. It implies that the subspace $\bigoplus_{i,j,k \geq 0, \max(i,j,k) \geq N} \mathbb{C}|i\rangle \otimes |j\rangle \otimes |k\rangle \subset F_+ \otimes F_+ \otimes F_+$ is invariant under $\mathcal{R}$. This fact was originally shown in [11, Th.2.2.1b] for each tensor component by resorting to the recursion relations, where an important consequence on the quotient was also pointed out in Proposition 2.3.2 therein. The above proof based on (120) is an illuminating simplification and has a natural generalization to the quantized coordinate rings of other types [14, eqs. (5.75), (8.39)].

4. Solutions of $RLLL$ Relation for $L = L^Z$ and L^X

In this section we deal with the $RLLL$ relations which contain L^Z (19) and L^X (20). As mentioned after (15), the parameters r, s, t, w are assumed to be generic, hence the boundary conditions like (62), (71), (80), (92), (102) and (113) need not be considered. We shall only treat the types XXZ, ZXX and XZX, and leave ZZX, XZZ, ZXZ and XXX cases for future study as they are considerably more complicated. Throughout the section, $R \in \mathrm{End}(F \otimes F \otimes F)$ with the sum (25) extending over $a, b, c \in \mathbb{Z}$, and the $RLLL$ relation holds in $\mathrm{End}(V \otimes V \otimes V \otimes F \otimes F \otimes F)$.

4.1. XXZ type. We consider the $RLLL$ relation

$$R_{456}L^Z_{236}L^X_{135}L^X_{124} = L^X_{124}L^X_{135}L^Z_{236}R_{456}, \tag{121}$$

where L^X_{124} and L^X_{135} are given by (20) with $(r, s, t, w) = (r_1, s_1, t_1, w_1)$ and (r_2, s_2, t_2, w_2), respectively, and L^Z_{236} is given by (19) with $(r, s, t, w) = (r_3, s_3, t_3, w_3)$.

Here are some examples of the $RLLL$ relation (24), which are natural extensions of those for the OOZ type:

$$(q^{i+j} - q^{a+b})R^{a,b,c}_{i,j,k} = 0, \quad q^j R^{a,b,c}_{i,j,k-1} = q^b R^{a,b,c+1}_{i,j,k}, \tag{122}$$

$$(s_1t_2w_2q^{b+c} - s_2t_1w_1q^{i+k})R^{a,b,c}_{i,j,k} = t_3w_3(r_1s_1 - t_1^2w_1q^{2i})R^{a,b,c}_{i-1,j+1,k-1}, \tag{123}$$

$$(r_1t_2q^{j+k} - r_2t_1q^{a+c})R^{a,b,c}_{i,j,k} = t_3(r_1s_1 - t_1^2w_1q^{2a+2})R^{a+1,b-1,c+1}_{i,j,k}, \tag{124}$$

$$s_1s_3R^{a,b-1,c}_{i,j,k} - q^k s_2R^{a,b,c}_{i+1,j,k} + q^{i+1}t_1t_3w_3R^{a,b,c}_{i,j+1,k-1} = 0, \tag{125}$$

$$q^b s_3t_2R^{a-1,b,c}_{i,j,k} + q^{2+i-k}t_1t_3^2w_3R^{a,b,c}_{i,j+1,k-2} - q^{i-k}r_3s_3t_1R^{a,b,c}_{i,j+1,k} - s_2t_3R^{a,b,c}_{i+1,j,k-1} = 0. \tag{126}$$

For $a, b, c, i, j, k \in \mathbb{Z}$, set

$$R^{a,b,c}_{i,j,k} = \delta^{a+b}_{i+j}\left(\frac{s_1s_3}{s_2}\right)^i\left(\frac{s_1t_3}{t_2}\right)^{-a}\left(\frac{s_1s_3t_2w_2}{r_1s_2t_3w_3}\right)^j\left(\frac{r_2s_2}{t_2^2w_2}\frac{t_3^2w_3}{r_3s_3}\right)^g q^{cj-bk}$$
$$\times\frac{(q^{b+c+i-k+2}\frac{t_1t_2w_2}{r_1s_2};q^2)_{g-a-b}(q^{h+2}\frac{t_1w_1t_2}{r_2s_1};q^2)_g(q^{2a+2}\frac{t_1^2w_1}{r_1s_1};q^2)_{i-a}}{(q^{-b+c+i-k}\frac{r_2t_1}{r_1t_2};q^2)_{g-a}(q^{h+2}\frac{s_2t_1w_1}{s_1t_2w_2};q^2)_{g-j}}R^{0,0,0}_{0,0,h}, \tag{127}$$

$$2g + h = a - c + j + k \ (g \in \mathbb{Z},\ h = 0, 1), \tag{128}$$

where $R^{0,0,0}_{0,0,0}$ and $R^{0,0,0}_{0,0,1}$ can be taken arbitrarily.

Theorem 13. *Recursion relations derived from (121) consists of only those $R^{a,b,c}_{i,j,k}$'s having the same parity of $a - c + j + k$. Each subsystem specified by h admits a unique solution up to normalization, which is given by (127)–(128).*

Proof. The former assertion on the parity can be verified directly. Solving a partial set of recursion relations already leads to (127)–(128), proving the uniqueness. Then it is straightforward to check that it actually satisfies all the remaining recursion relations.

□

R is not locally finite.

Let us compare the 3D R (127) for XXZ with (90) for OOZ. To fit L^X in (121) to L^O, we specialize the parameters as

$$r_i = s_i = 1, \quad t_i = \mu_i^{-1}, \quad t_i w_i = \mu_i \tag{129}$$

for $i = 1, 2$. Then (127) becomes

$$\begin{aligned} R^{a,b,c}_{i,j,k} &= \delta^{a+b}_{i+j}\, s_3^i (\mu_2 t_3)^{-a} \left(\frac{\mu_2 s_3}{t_3 w_3}\right)^j \left(\frac{t_3^2 w_3}{r_3 s_3}\right)^g \\ &\quad \times q^{cj-bk} \frac{(q^{-b-c+i+k+2}\frac{\mu_1}{\mu_2}; q^2)_j (q^{2a+2}; q^2)_{i-a}}{(q^{-b+c+i-k}\frac{\mu_2}{\mu_1}; q^2)_{b+1}} \left(1 - q^{-h}\frac{\mu_2}{\mu_1}\right) R^{0,0,0}_{0,0,h}. \end{aligned} \tag{130}$$

Using the notation e, f in (91), where $g = e - \frac{1}{2}(h+d)$, and assuming $d \in \mathbb{Z}$ is so chosen that $e \in \mathbb{Z}$, this is rewritten as

$$\begin{aligned} R^{a,b,c}_{i,j,k} &= \delta^{a+b}_{i+j}\, s_3^i (\mu_2 t_3)^{-a} \left(\frac{\mu_2 s_3}{t_3 w_3}\right)^j \left(\frac{t_3^2 w_3}{r_3 s_3}\right)^e \\ &\quad \times q^{cj-bk} \frac{(q^{2+2e-2j-d}\frac{\mu_1}{\mu_2}; q^2)_j (q^{2a+2}; q^2)_{i-a}}{(q^{2+d}\frac{\mu_2}{\mu_1}; q^2)_f (q^{2a-2e+d}\frac{\mu_2}{\mu_1}; q^2)_{e-a}} \\ &\quad \times \frac{1 - q^{-h}\frac{\mu_2}{\mu_1}}{1 - q^d \frac{\mu_2}{\mu_1}} \left(\frac{t_3^2 w_3}{r_3 s_3}\right)^{-\frac{1}{2}(h+d)} R^{0,0,0}_{0,0,h}. \end{aligned} \tag{131}$$

Note that the condition $e \in \mathbb{Z}$ is equivalent to $h + d \in 2\mathbb{Z}$. Therefore, if $R^{0,0,0}_{0,0,h}$ $(h = 0, 1)$ are taken as

$$\frac{1 - q^{-h}\frac{\mu_2}{\mu_1}}{1 - q^d \frac{\mu_2}{\mu_1}} \left(\frac{t_3^2 w_3}{r_3 s_3}\right)^{-\frac{1}{2}(h+d)} R^{0,0,0}_{0,0,h} \quad \to \quad \theta(h + d \in 2\mathbb{Z}) = \theta(e \in \mathbb{Z}) \tag{132}$$

in the limit $\mu_1 \to \mu_2 q^d$, the 3D R (90) for the OOZ case is formally reproduced.

4.2. ZXX type. We consider the $RLLL$ relation

$$R_{456} L^X_{236} L^X_{135} L^Z_{124} = L^Z_{124} L^X_{135} L^X_{236} R_{456}, \tag{133}$$

where L^Z_{124} is given by (19) with $(r, s, t, w) = (r_1, s_1, t_1, w_1)$, and L^X_{135} and L^X_{246} are given by (20) with $(r, s, t, w) = (r_2, s_2, t_2, w_2)$ and (r_3, s_3, t_3, w_3), respectively.

Here are some examples of the $RLLL$ relation (24), which are natural extensions of those for the ZOO type:

$$(q^{j+k} - q^{b+c})R^{a,b,c}_{i,j,k} = 0, \quad q^j R^{a,b,c}_{i-1,j,k} = q^b R^{a+1,b,c}_{i,j,k}, \tag{134}$$

$$(s_3 t_2 q^{a+b} - s_2 t_3 q^{i+k})R^{a,b,c}_{i,j,k} = t_1(r_3 s_3 - t_3^2 w_3 q^{2k})R^{a,b,c}_{i-1,j+1,k-1}, \tag{135}$$

$$(r_3 t_2 w_2 q^{i+j} - r_2 t_3 w_3 q^{a+c})R^{a,b,c}_{i,j,k} = t_1 w_1(r_3 s_3 - t_3^2 w_3 q^{2c+2})R^{a+1,b-1,c+1}_{i,j,k}, \tag{136}$$

$$s_1 s_3 R^{a,b-1,c}_{i,j,k} - q^i s_2 R^{a,b,c}_{i,j,k+1} + q^{k+1} t_1 t_3 w_3 R^{a,b,c}_{i-1,j+1,k} = 0, \tag{137}$$

$$q^b s_1 t_2 w_2 R^{a,b,c-1}_{i,j,k} + q^{2-i+k} t_1^2 t_3 w_1 w_3 R^{a,b,c}_{i-2,j+1,k} - s_2 t_1 w_1 R^{a,b,c}_{i-1,j,k+1} - q^{-i+k} r_1 s_1 t_3 w_3 R^{a,b,c}_{i,j+1,k} = 0. \tag{138}$$

Every recursion relation consists of those $R^{a,b,c}_{i,j,k}$ having the same parity of $-a+c+i+j$.

For $a, b, c, i, j, k \in \mathbb{Z}$, set

$$R^{a,b,c}_{i,j,k} = \delta^{b+c}_{j+k} \left(\frac{s_1 s_3}{s_2}\right)^k \left(\frac{s_3 t_1 w_1}{t_2 w_2}\right)^{-c} \left(\frac{s_1 s_3 t_2}{r_3 s_2 t_1}\right)^j \left(\frac{r_2 s_2}{t_2^2 w_2}\frac{t_1^2 w_1}{r_1 s_1}\right)^g q^{aj-bi} \times \frac{(q^{a+b-i+k+2}\frac{t_2 t_3 w_3}{r_3 s_2}; q^2)_{g-b-c}(q^{h+2}\frac{t_2 t_3 w_2}{r_2 s_3}; q^2)_g (q^{2c+2}\frac{t_3^2 w_3}{r_3 s_3}; q^2)_{k-c}}{(q^{a-b-i+k}\frac{r_2 t_3 w_3}{r_3 t_2 w_2}; q^2)_{g-c}(q^{h+2}\frac{s_2 t_3}{s_3 t_2}; q^2)_{g-j}} R^{0,0,0}_{h,0,0}, \tag{139}$$

$$2g + h = -a + c + i + j \ (g \in \mathbb{Z}, h = 0, 1), \tag{140}$$

where $R^{0,0,0}_{0,0,0}$ and $R^{0,0,0}_{1,0,0}$ can be taken arbitrarily.

Theorem 14. *Recursion relations derived from* (133) *consists of only those $R^{a,b,c}_{i,j,k}$'s having the same parity of $-a+c+i+j$. Each subsystem specified by h admits a unique solution up to normalization, which is given by* (139)–(140)*.*

The proof is similar to Theorem 13. R is not locally finite.

As the XXZ type, by specializing the parameters as (129) with $i = 2, 3$ and taking the limit $\mu_3 \to \mu_2 q^d$ with appropriate tuning of $R^{0,0,0}_{h,0,0}$ $(h = 0, 1)$, one can reproduce the 3D R (103) for ZOO from (139).

4.3. XZX type. We consider the $RLLL$ relation

$$R_{456} L^X_{236} L^Z_{135} L^X_{124} = L^X_{124} L^Z_{135} L^X_{236} R_{456}, \tag{141}$$

where L^X_{124} and L^X_{236} are given by (20) with $(r, s, t, w) = (r_1, s_1, t_1, w_1)$ and (r_3, s_3, t_3, w_3), respectively, and L^Z_{135} is given by (19) with $(r, s, t, w) = (r_2, s_2, t_2, w_2)$.

Here are some examples of the $RLLL$ relation (24), which are natural extensions of those for the OZO type:

$$q^k R^{a,b,c}_{i,j-1,k} = q^c R^{a,b+1,c}_{i,j,k}, \quad q^i R^{a,b,c}_{i,j-1,k} = q^a R^{a,b+1,c}_{i,j,k}, \tag{142}$$

$$r_2 R^{a-1,b,c-1}_{i,j,k} = (t_1 t_3 w_1 q^{1+a+b+c} + r_1 r_3 q^j) R^{a,b,c}_{i,j,k}, \tag{143}$$

$$s_2 R^{a,b,c}_{i+1,j,k+1} = (t_1 t_3 w_3 q^{1+i+j+k} + s_1 s_3 q^b) R^{a,b,c}_{i,j,k}, \tag{144}$$

$$q^a r_2 t_1 R^{a,b,c-1}_{i,j,k} + t_3(r_1 s_1 - t_1^2 w_1 q^{2a+2}) q^{b+c} R^{a+1,b,c}_{i,j,k} - r_1 t_2 R^{a,b,c}_{i,j-1,k+1} = 0, \tag{145}$$

$$\begin{aligned} &q^{-b+c} r_2 s_2 t_3 w_3 R^{a-1,b,c}_{i,j,k} - q^{-b+c} t_2^2 t_3 w_2 w_3 R^{a-1,b+2,c}_{i,j,k} \\ &\quad - s_1 t_2 w_2 (r_3 s_3 - t_3^2 w_3 q^{2k}) R^{a,b,c}_{i,j-1,k-1} + q^a s_2 t_1 w_1 (r_3 s_3 - t_3^2 w_3 q^{2+2c}) R^{a,b,c+1}_{i,j,k} = 0. \end{aligned} \tag{146}$$

For $a, b, c, i, j, k \in \mathbb{Z}$, set

$$\begin{aligned} R^{a,b,c}_{i,j,k} &= \delta^{a-c}_{i-k} \left(\frac{r_2}{r_1 r_3}\right)^c \left(\frac{s_1 t_3}{t_2}\right)^k \left(\frac{r_2 t_3 w_3}{r_3 t_2 w_2}\right)^i \left(\frac{r_3 s_3}{t_3^2 w_3} \frac{t_2^2 w_2}{r_2 s_2}\right)^g q^{bk-cj} \\ &\quad \times \frac{(-q^{h+1}\frac{t_1 t_3 w_3}{s_1 s_3}; q^2)_g}{(-q^{h+1}\frac{r_3 t_1}{s_1 t_3}; q^2)_{g-k}} \frac{(-q^{-h+1}\frac{s_3 t_1 w_1}{r_1 t_3 w_3}; q^2)_{i-g}}{(-q^{-h+3}\frac{t_1 t_3 w_1}{r_1 r_3}; q^2)_{c+i-g}} R^{0,0,0}_{0,h,0} \end{aligned} \tag{147}$$

$$2g + h = -b + i + j + k \; (g \in \mathbb{Z}, h = 0, 1), \tag{148}$$

where $R^{0,0,0}_{0,0,0}$ and $R^{0,0,0}_{0,1,0}$ can be taken arbitrarily.

Theorem 15. *Recursion relations derived from* (141) *consists of only those* $R^{a,b,c}_{i,j,k}$*'s having the same parity of* $-b+i+j+k$*. Each subsystem specified by* h *admits a unique solution up to normalization, which is given by* (147)–(148)*.*

The proof is similar to Theorem 13. R is not locally finite.

Let us compare the 3D R (147) for XZX with (114) for OZO. To fit L^X in (141) to L^O, we specialize the parameters as (129) with $i = 1, 3$. Then (147) becomes

$$\begin{aligned} R^{a,b,c}_{i,j,k} &= \delta^{a-c}_{i-k} r_2^c (\mu_3 t_2)^{-k} \left(\frac{\mu_3 r_2}{t_2 w_2}\right)^i \left(\frac{r_2 s_2}{t_2^2 w_2}\right)^g \\ &\quad \times q^{bk-cj} \frac{(-q^{-b+i+j-k+1}\frac{\mu_3}{\mu_1}; q^2)_k}{(-q^{a+b-c-j+1}\frac{\mu_1}{\mu_3}; q^2)_{c+1}} \left(1 + q^{-h+1}\frac{\mu_1}{\mu_3}\right) R^{0,0,0}_{0,h,0}. \end{aligned} \tag{149}$$

Using the notation e, f in (115), where $g = e - \frac{1}{2}(h - d - 1)$, and assuming $d \in \mathbb{Z}$ is so chosen that $e \in \mathbb{Z}$, this is rewritten as

$$\begin{aligned} R^{a,b,c}_{i,j,k} &= \delta^{a+b}_{i+j} r_2^c (\mu_3 t_2)^{-k} \left(\frac{\mu_3 r_2}{t_2 w_2}\right)^i \left(\frac{t_2^2 w_2}{r_2 s_2}\right)^e \\ &\quad \times q^{bk-cj} \frac{(-q^{2e-2k+d+2}\frac{\mu_3}{\mu_1}; q^2)_k}{(-q^{-d+2}\frac{\mu_1}{\mu_3}; q^2)_f (-q^{-2e+2i-d}\frac{\mu_1}{\mu_3}; q^2)_{e-i}} \\ &\quad \times \frac{1 + q^{-h+1}\frac{\mu_1}{\mu_3}}{1 + q^{-d}\frac{\mu_1}{\mu_3}} \left(\frac{t_2^2 w_2}{r_2 s_2}\right)^{-\frac{1}{2}(h-d-1)} R^{0,0,0}_{0,h,0}. \end{aligned} \tag{150}$$

Note that the condition $e \in \mathbb{Z}$ is equivalent to $h - d - 1 \in 2\mathbb{Z}$. Therefore, if $R^{0,0,0}_{0,h,0}$ ($h = 0, 1$) are taken as

$$\frac{1 + q^{-h+1}\frac{\mu_1}{\mu_3}}{1 + q^{-d}\frac{\mu_1}{\mu_3}} \left(\frac{t_2^2 w_2}{r_2 s_2}\right)^{-\frac{1}{2}(h-d-1)} R^{0,0,0}_{0,h,0} \quad \to \quad \theta(h - d - 1 \in 2\mathbb{Z}) = \theta(e \in \mathbb{Z}) \quad (151)$$

in the limit $\mu_1 \to -\mu_3 q^d$, the 3D R (114) for the OZO case is formally reproduced.

5. Relation to the Representation Theory of the Quantized Coordinate Ring

5.1. Quantized coordinate ring $A_q(sl_3)$. The algebra $A_q(sl_3)$ is a Hopf algebra dual to the quantized universal enveloping algebra $U_q(sl_3)$. See for example [6,8,10,13,20] and the references therein. It is generated by t_{ij} $(1 \le i, j \le 3)$ with the relations

$$[t_{ik}, t_{jl}] = \begin{cases} 0 & (i < j, k > l), \\ (q - q^{-1})t_{jk}t_{il} & (i < j, k < l), \end{cases} \quad (152)$$

$$t_{ik}t_{jk} = q t_{jk}t_{ik} \ (i < j), \quad t_{ki}t_{kj} = q t_{kj}t_{ki} \ (i < j), \quad (153)$$

$$\sum_{\sigma \in \mathfrak{S}_3} (-q)^{l(\sigma)} t_{1\sigma_1}t_{2\sigma_2}t_{3\sigma_3} = 1, \quad (154)$$

where $\mathfrak{S}_3$ denotes the symmetric group of degree 3 and $l(\sigma)$ is the length of the permutation σ. The coproduct $\Delta : A_q(sl_3) \to A_q(sl_3)^{\otimes N}$ is given by the matrix product form $\Delta t_{ij} = \sum_{1 \le i_2, \dots, i_N \le 3} t_{ii_2} \otimes t_{i_2 i_3} \otimes \cdots \otimes t_{i_N j}$.

The following maps define the algebra homomorphisms to the q-Weyl algebra (6):

$$\rho_1 : A_q(sl_3) \to \mathcal{W}_q,$$
$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \mapsto \begin{pmatrix} Z^{-1}(u_1 - g_1 h_1 X^2) & g_1 X & 0 \\ -q h_1 X & Z & 0 \\ 0 & 0 & u_1^{-1} \end{pmatrix}, \quad (155)$$

$$\rho_2 : A_q(sl_3) \to \mathcal{W}_q,$$
$$\begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \mapsto \begin{pmatrix} u_2^{-1} & 0 & 0 \\ 0 & Z^{-1}(u_2 - g_2 h_2 X^2) & g_2 X \\ 0 & -q h_2 X & Z \end{pmatrix}. \quad (156)$$

Here, u_i, g_i, h_i are arbitrary parameters.

We let $\rho_{Z,i} = \pi_Z \circ \rho_i$ and $\rho_{X,i} = \pi_X \circ \rho_i$ denote the representations $A_q(sl_3) \to \mathrm{End}(F)$ obtained by the compositions with π_Z and π_X in (8) and (9).

5.2. 3D R of type OOO as an intertwiner of $\rho_{O,i}$. From the remark after (16), one can restrict $\rho_{X,i}$ with $(u_i, g_i, h_i) = (1, \mu_i, \mu_i^{-1})$ from $\mathrm{End}(F)$ to $\mathrm{End}(F_+)$. The resulting representation will be denoted by $\rho_{O,i} : A_q(sl_3) \to \mathrm{End}(F_+)$.

The representation $\rho_{O,i}$ is irreducible and well-studied [13]. In fact, the isomorphism of the tensor product representations $\rho_{O,1} \otimes \rho_{O,2} \otimes \rho_{O,1} \simeq \rho_{O,2} \otimes \rho_{O,1} \otimes \rho_{O,2}$ is valid, and they turn out to be irreducible. Let $\Phi \in \mathrm{End}(F_+^{\otimes 3})$ be the intertwiner, i.e., the unique solution to the intertwining relation $\Phi \circ (\rho_{O,1} \otimes \rho_{O,2} \otimes \rho_{O,1}) = (\rho_{O,2} \otimes \rho_{O,1} \otimes \rho_{O,2}) \circ \Phi$

up to normalization. Set $\mathcal{R} = \Phi \circ P$ where P is the transposition $P(|i\rangle \otimes |j\rangle \otimes |k\rangle) = |k\rangle \otimes |j\rangle \otimes |i\rangle$. We also call $\mathcal{R}$ the intertwiner. The intertwining relation for the generator t_{lm} reads as

$$\mathcal{R}(\rho_{O,1} \otimes \rho_{O,2} \otimes \rho_{O,1})(\Delta' t_{lm}) = (\rho_{O,2} \otimes \rho_{O,1} \otimes \rho_{O,2})(\Delta t_{lm})\mathcal{R} \quad (1 \le l, m \le 3), \tag{157}$$

where $\Delta' = P \circ \Delta \circ P$, hence $\Delta' t_{lm} = \sum_{j,k} t_{km} \otimes t_{jk} \otimes t_{lj}$. It is known that the set of equations (157) are equivalent to the $RLLL$ relation $(117)|_{\mu_3=\mu_1}$ under the identification $\mathcal{R} = R$. See [15, Sec.2] and [14, Lem.3.22]. As a result, the 3D R (118) with $\mu_3 = \mu_1$ is identified with the intertwiner of the $A_q(sl_3)$ modules.

5.3. 3D R of type ZZZ as an intertwiner of $\rho_{Z,i}$. Consider the equation on $R \in \mathrm{End}(F^{\otimes 3})$ given by

$$R(\rho_{Z,1} \otimes \rho_{Z,2} \otimes \rho_{Z,1})(\Delta' t_{lm}) = (\rho_{Z,2} \otimes \rho_{Z,1} \otimes \rho_{Z,2})(\Delta t_{lm})R \quad (1 \le l, m \le 3), \tag{158}$$

which includes the parameters $u_i, g_i, h_i (i = 1, 2)$. On the other hand, recall that the $RLLL$ relation (28) of ZZZ type contains 18 equations depending on $r_\alpha, s_\alpha, t_\alpha, w_\alpha (\alpha = 1, 2, 3)$. Now we state a result analogous to the OOO case in the previous subsection.

Proposition 16. *The intertwining relation (158) and the $RLLL$ relation (28) are equivalent provided that the parameters in the former obey the constraint $u_1 = u_2 (=: u)$ and $g_1 h_1 = g_2 h_2 (=: p)$, and those in the latter satisfy*

$$\frac{r_1}{t_1} = \frac{r_2}{t_2}, \quad \frac{s_2}{t_2} = \frac{s_3}{t_3}, \quad \frac{r_2}{r_1 r_3} = u, \quad \frac{s_1 s_3}{s_2} = u^2, \quad \frac{t_1^2 w_1}{r_1 s_1} = \frac{t_2^2 w_2}{r_2 s_2} = \frac{t_3^2 w_3}{r_3 s_3} = \frac{p}{u}. \tag{159}$$

Proof. Set $\mathcal{L}^{abc}_{ijk} = \sum_{\alpha,\beta,\gamma} \mathcal{L}^{\alpha\beta}_{ij} \otimes \mathcal{L}^{a\gamma}_{\alpha k} \otimes \mathcal{L}^{bc}_{\beta\gamma}$ and $\tilde{\mathcal{L}}^{abc}_{ijk} = \sum_{\alpha,\beta,\gamma} \mathcal{L}^{ab}_{\alpha\beta} \otimes \mathcal{L}^{\alpha c}_{i\gamma} \otimes \mathcal{L}^{\beta\gamma}_{jk}$ so that (24) reads as $R\mathcal{L}^{abc}_{ijk} = \tilde{\mathcal{L}}^{abc}_{ijk} R$. Then one can directly check that the relations (159) validate the equalities

$$(\rho_{Z,1} \otimes \rho_{Z,2} \otimes \rho_{Z,1})(\Delta' t_{lm}) = A_{lm}\mathcal{L}^{abc}_{ijk} = B_{lm}\left(\tilde{\mathcal{L}}^{a'b'c'}_{i'j'k'}\Big|_{r_\gamma \leftrightarrow s_\gamma, t_\gamma \to t_\gamma w_\gamma, w_\gamma \to w_\gamma^{-1}}\right), \tag{160}$$

$$(\rho_{Z,2} \otimes \rho_{Z,1} \otimes \rho_{Z,2})(\Delta t_{lm}) = A_{lm}\tilde{\mathcal{L}}^{abc}_{ijk} = B_{lm}\left(\mathcal{L}^{a'b'c'}_{i'j'k'}\Big|_{r_\gamma \leftrightarrow s_\gamma, t_\gamma \to t_\gamma w_\gamma, w_\gamma \to w_\gamma^{-1}}\right), \tag{161}$$

where the constants A_{lm}, B_{lm} are given by

$$(A_{lm})_{1 \le l,m \le 3} = \begin{pmatrix} \frac{1}{r_2^2 s_2} & \frac{pu}{h_1 r_2 s_2 t_3} & \frac{p^2}{h_1 h_2 u} \\ -\frac{q h_1 t_3}{p r_2^2 s_2} & -\frac{u}{r_2 s_2} & \frac{p}{h_2 r_2 t_1 u} \\ \frac{q^2 h_1 h_2}{u} & -\frac{q h_2 t_1 u}{p r_2 s_2} & \frac{1}{r_2 u} \end{pmatrix}, \tag{162}$$

$$(B_{lm})_{1 \le l,m \le 3} = \begin{pmatrix} -\frac{1}{r_2^2 s_2} & \frac{pu}{h_1 r_2 s_2 t_3} & \frac{p^2}{h_1 h_2 u} \\ -\frac{q h_1 t_3}{p r_2^2 s_2} & \frac{u}{r_2 s_2} & \frac{p}{h_2 r_2 t_1 u} \\ \frac{q^2 h_1 h_2}{u} & -\frac{q h_2 t_1 u}{p r_2 s_2} & \frac{1}{r_2 u} \end{pmatrix}. \tag{163}$$

Table 1. Type ABC of $R_{456}L^C_{236}L^B_{135}L^A_{124} = L^A_{124}L^B_{135}L^C_{236}R_{456}$ and the basic feature of the solution $R = R^{ABC}$

ABC	♯(Z)	Feature	Locally finiteness	♯(Sector)	Formula
ZZZ	3	Factorized	No	4	(45)
OZZ		$_2\phi_1$	No		(65)
ZZO	2	$_2\phi_1$	No	1	(74)
ZOZ		$_3\phi_2$-like	No		(83)
OOZ		Factorized	Yes		(90)
ZOO	1	Factorized	Yes	1	(103)
OZO		Factorized	No		(114)
OOO	0	$_2\phi_1$	Yes	1	(118)
XXZ		Factorized	No		(127)
ZXX	1	factorized	No	2	(139)
XZX		Factorized	No		(147)

We observe the factorization when the number ♯(Z) of Z in ABC is odd. ♯(sector) is the dimension of the solution space for the recursion relations of $R^{a,b,c}_{i,j,k}$

The correspondence between the indices l, m and $a, b, c, i, j, k, a', b', c', i', j', k'$ is specified as follows:

l	abc	$i'j'k'$
1	001	011
2	010	101
3	100	110

m	ijk	$a'b'c'$
1	100	110
2	010	101
3	001	011

The relations (160) and (161) including A_{lm} enable us to identify (158) with $R\mathcal{L}^{abc}_{ijk} = \tilde{\mathcal{L}}^{abc}_{ijk}R$, covering the case $a+b+c = i+j+k = 1$ of the latter. Let us show the other case $a'+b'+c' = i'+j'+k' = 2$ of the $RLLL$ relation in the form $R^{-1}\tilde{\mathcal{L}}^{a'b'c'}_{i'j'k'} = \mathcal{L}^{a'b'c'}_{i'j'k'}R^{-1}$. Due to (52) it is equivalent to $R\left(\tilde{\mathcal{L}}^{a'b'c'}_{i'j'k'}\Big|_{r_\gamma \leftrightarrow s_\gamma, t_\gamma \to t_\gamma w_\gamma, w_\gamma \to w_\gamma^{-1}}\right) = \left(\mathcal{L}^{a'b'c'}_{i'j'k'}\Big|_{r_\gamma \leftrightarrow s_\gamma, t_\gamma \to t_\gamma w_\gamma, w_\gamma \to w_\gamma^{-1}}\right)R$. This equality follows from (158) by applying the relations (160) and (161) including B_{lm}. □

6. Discussion

6.1. Summary. In this paper we have studied the tetrahedron equation of the form $R_{456}L^C_{236}L^B_{135}L^A_{124} = L^A_{124}L^B_{135}L^C_{236}R_{456}$ for the three kinds of 3D L operators L^Z, L^X, L^O in (19)–(21) which can be regarded as quantized six-vertex models with Boltzmann weights taken from the q-Weyl algebra $\mathcal{W}_q$ (6) or the q-oscillator algebra $\mathcal{O}_q$ (10). In each case the solution R has been obtained explicitly whose elements are factorized or expressed in terms of terminating q-hypergeometric type series as in Table 1. They are new except for the OOO case.

6.2. On tetrahedron equation of the form $RRRR = RRRR$. Let us discuss the tetrahedron equation of the form

$$R_{456}R_{236}R_{135}R_{124} = R_{124}R_{135}R_{236}R_{456}. \tag{164}$$

A standard strategy for the proof is to compare the two maneuvers:

$$\begin{aligned}
&R_{124}R_{135}R_{236}R_{456}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1}\\
&\quad= R_{124}R_{135}R_{236}L_{\beta\gamma4}L_{\alpha\gamma5}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\gamma\delta1}R_{456}\\
&\quad= R_{124}R_{135}L_{\beta\gamma4}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}R_{236}R_{456}\\
&\quad= R_{124}R_{135}L_{\beta\gamma4}L_{\beta\delta2}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\alpha\beta6}R_{236}R_{456}\\
&\quad= R_{124}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}L_{\alpha\delta3}L_{\alpha\gamma5}L_{\alpha\beta6}R_{135}R_{236}R_{456}\\
&\quad= L_{\gamma\delta1}L_{\beta\delta2}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456},\\
&\quad= L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{124}R_{135}R_{236}R_{456},
\end{aligned} \tag{165}$$

$$\begin{aligned}
&R_{456}R_{236}R_{135}R_{124}L_{\alpha\beta6}L_{\alpha\gamma5}\underline{L_{\beta\gamma4}L_{\alpha\delta3}}L_{\beta\delta2}L_{\gamma\delta1}\\
&\quad= R_{456}R_{236}R_{135}R_{124}L_{\alpha\beta6}L_{\alpha\gamma5}L_{\alpha\delta3}\underline{L_{\beta\gamma4}L_{\beta\delta2}L_{\gamma\delta1}}\\
&\quad= R_{456}R_{236}R_{135}L_{\alpha\beta6}\underline{L_{\alpha\gamma5}L_{\alpha\delta3}L_{\gamma\delta1}}L_{\beta\delta2}L_{\beta\gamma4}R_{124}\\
&\quad= R_{456}R_{236}\underline{L_{\alpha\beta6}L_{\gamma\delta1}}L_{\alpha\delta3}\underline{L_{\alpha\gamma5}L_{\beta\delta2}}L_{\beta\gamma4}R_{135}R_{124}\\
&\quad= R_{456}R_{236}L_{\gamma\delta1}\underline{L_{\alpha\beta6}L_{\alpha\delta3}L_{\beta\delta2}}L_{\alpha\gamma5}L_{\beta\gamma4}R_{135}R_{124}\\
&\quad= R_{456}L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}\underline{L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}}R_{236}R_{135}R_{124}\\
&\quad= L_{\gamma\delta1}L_{\beta\delta2}L_{\alpha\delta3}L_{\beta\gamma4}L_{\alpha\gamma5}L_{\alpha\beta6}R_{456}R_{236}R_{135}R_{124}.
\end{aligned} \tag{166}$$

The underlines indicate the components to be rewritten by the $RLLL = LLLR$ relation or trivial commutativity of the operators acting on distinct set of components. The above relations show that the composition $(R_{124}R_{135}R_{236}R_{456})^{-1}R_{456}R_{236}R_{135}R_{124}$ commutes with $L_{\alpha\beta6}L_{\alpha\gamma5}L_{\beta\gamma4}L_{\alpha\delta3}L_{\beta\delta2}L_{\gamma\delta1}$. Therefore if the action of the latter is irreducible, Schur's lemma compels $R_{124}R_{135}R_{236}R_{456} = (\text{scalar})R_{456}R_{236}R_{135}R_{124}$ and the scalar can be fixed by considering the special case.

In this type of argument, $RLLL = LLLR$ serves as an auxiliary linear problem for $RRRR = RRRR$, which is analogous to the quantum group symmetry ensuring the Yang–Baxter equation. It indeed works when all the L's are L^O, where $RLLL = LLLR$ is identified with the intertwining relation of the quantized coordinate ring $A_q(sl_3)$. See Sect. 5.2. The corresponding 3D R of type OOO (118) certainly satisfies the tetrahedron equation $R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}$ [3,9].

The results in this paper suggest a natural generalization where the six L operators in (165) and (166) are taken either as L^Z or L^O (resp. L^Z or L^X) in the context of Sect. 3 (resp. Sect. 4). Let us exhibit them as $L^F_{\alpha\beta6}L^E_{\alpha\gamma5}L^D_{\beta\gamma4}L^C_{\alpha\delta3}L^B_{\beta\delta2}L^A_{\gamma\delta1}$, where A, B, C, D, E and F assume Z, O or X. The corresponding generalization of (164) reads as

$$R^{DEF}_{456}R^{BCF}_{236}R^{ACE}_{135}R^{ABD}_{124} = R^{ABD}_{124}R^{ACE}_{135}R^{BCF}_{236}R^{DEF}_{456}, \tag{167}$$

where R^{ABD}_{124} for example denotes the 3D R of type ABD acting on the tensor components 1, 2 and 4. Let us call (167) the $RRRR$ relation of type ABCDEF. Its proof or disproof is an important future problem. It has been settled only for type OOOOOO as explained in the above. However, the argument employed there does not persist naively when L^Z is involved since the irreducibility no longer holds due to Remark 1. Moreover, the presence of locally non-finite 3D R makes the convergence of the compositions in $RRRR = RRRR$ non-trivial.[4] In spite of such difficulties, we have made promising observations which are reported below.

[4] The convergence also matters when one attempts to perform the *reduction* of the 3D R's to the solutions of the Yang–Baxter equation by the trace [3] and the boundary vectors [16,18].

From Table 1, the tetrahedron equation (167) consisting of only locally finite 3D R's are of type OOOOOO and the following:

Type	Tetrahedron equation	
ZOOOOO :	$R_{456}^{OOO} R_{236}^{OOO} R_{135}^{ZOO} R_{124}^{ZOO} = R_{124}^{ZOO} R_{135}^{ZOO} R_{236}^{OOO} R_{456}^{OOO},$	(168)
OOOZOO :	$R_{456}^{ZOO} R_{236}^{OOO} R_{135}^{OOO} R_{124}^{OOZ} = R_{124}^{OOZ} R_{135}^{OOO} R_{236}^{OOO} R_{456}^{ZOO},$	(169)
OOOOOZ :	$R_{456}^{OOZ} R_{236}^{OOZ} R_{135}^{OOO} R_{124}^{OOO} = R_{124}^{OOO} R_{135}^{OOO} R_{236}^{OOZ} R_{456}^{OOZ},$	(170)
ZOOOOZ :	$R_{456}^{OOZ} R_{236}^{OOZ} R_{135}^{ZOO} R_{124}^{ZOO} = R_{124}^{ZOO} R_{135}^{ZOO} R_{236}^{OOZ} R_{456}^{OOZ}.$	(171)

In these equations, images of any given input vector $|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle \otimes |m\rangle \otimes |n\rangle$ by the two sides are linear combinations of finitely many bases with finite coefficients, so one can compare them directly.

The tetrahedron equation (167) containing *only one* locally non-finite 3D R on each side are the following:

Type	Tetrahedron equation	
OZOOOO :	$R_{456}^{OOO} R_{236}^{ZOO} R_{135}^{OOO} R_{124}^{OZO} = R_{124}^{OZO} R_{135}^{OOO} R_{236}^{ZOO} R_{456}^{OOO},$	(172)
OOOOZO :	$R_{456}^{OZO} R_{236}^{OOO} R_{135}^{OOZ} R_{124}^{OOO} = R_{124}^{OOO} R_{135}^{OOZ} R_{236}^{OOO} R_{456}^{OZO},$	(173)
ZZOOOO :	$R_{456}^{OOO} R_{236}^{ZOO} R_{135}^{ZOO} R_{124}^{ZZO} = R_{124}^{ZZO} R_{135}^{ZOO} R_{236}^{ZOO} R_{456}^{OOO},$	(174)
ZOOZOO :	$R_{456}^{ZOO} R_{236}^{OOO} R_{135}^{ZOO} R_{124}^{ZOZ} = R_{124}^{ZOZ} R_{135}^{ZOO} R_{236}^{OOO} R_{456}^{ZOO},$	(175)
OZOZOO :	$R_{456}^{ZOO} R_{236}^{ZOO} R_{135}^{OOO} R_{124}^{OZZ} = R_{124}^{OZZ} R_{135}^{OOO} R_{236}^{ZOO} R_{456}^{ZOO},$	(176)
OOOZZO :	$R_{456}^{ZZO} R_{236}^{OOO} R_{135}^{OOZ} R_{124}^{OOZ} = R_{124}^{OOZ} R_{135}^{OOZ} R_{236}^{OOO} R_{456}^{ZZO},$	(177)
OOOZOZ :	$R_{456}^{ZOZ} R_{236}^{OOZ} R_{135}^{OOO} R_{124}^{OOZ} = R_{124}^{OOZ} R_{135}^{OOO} R_{236}^{OOZ} R_{456}^{ZOZ},$	(178)
OOOOZZ :	$R_{456}^{OZZ} R_{236}^{OOZ} R_{135}^{OOZ} R_{124}^{OOO} = R_{124}^{OOO} R_{135}^{OOZ} R_{236}^{OOZ} R_{456}^{OZZ},$	(179)
ZZOZOO :	$R_{456}^{ZOO} R_{236}^{ZOO} R_{135}^{ZOO} R_{124}^{ZZZ} = R_{124}^{ZZZ} R_{135}^{ZOO} R_{236}^{ZOO} R_{456}^{ZOO},$	(180)
OOOZZZ :	$R_{456}^{ZZZ} R_{236}^{OOZ} R_{135}^{OOZ} R_{124}^{OOZ} = R_{124}^{OOZ} R_{135}^{OOZ} R_{236}^{OOZ} R_{456}^{ZZZ}.$	(181)

In these equations, transition amplitudes for any *pair* of input and output bases $|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle \otimes |m\rangle \otimes |n\rangle \to |a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle \otimes |e\rangle \otimes |f\rangle$ by the two sides are finite. Explcitly they are the two sides of

$$\sum_{u,v,w,x,y,z} R_{x,y,z}^{d,e,f} R_{v,w,n}^{b,c,z} R_{u,k,m}^{a,w,y} R_{i,j,l}^{u,v,x} = \sum_{u,v,w,x,y,z} R_{u,v,x}^{a,b,d} R_{i,w,y}^{u,c,e} R_{j,k,z}^{v,w,f} R_{l,m,n}^{x,y,z}, \tag{182}$$

where each factor also depends on the type as specified in (167) in general. For instance the leftmost $R_{x,y,z}^{d,e,f}$ is an element of R^{DEF}, whereas the next $R_{v,w,n}^{b,c,z}$ is the one for R^{BCF}, etc.

There are a couple of tetrahedron equations involving more than one locally non-finite 3D R's on each side, which, nevertheless, allow only finitely many quintet (u, v, w, x, y, z)

in (182) thanks to the constraint (116). Such types of ABCDEF are the following:

Type	Tetrahedron equation	
OOZOOO :	$R^{OOO}_{456}R^{OZO}_{236}R^{OZO}_{135}R^{OOO}_{124} = R^{OOO}_{124}R^{OZO}_{135}R^{OZO}_{236}R^{OOO}_{456}$,	(183)
ZOZOOO :	$R^{OOO}_{456}R^{OZO}_{236}R^{ZZO}_{135}R^{ZOO}_{124} = R^{ZOO}_{124}R^{ZZO}_{135}R^{OZO}_{236}R^{OOO}_{456}$,	(184)
ZOOOZO :	$R^{OZO}_{456}R^{OOO}_{236}R^{ZOZ}_{135}R^{ZOO}_{124} = R^{ZOO}_{124}R^{ZOZ}_{135}R^{OOO}_{236}R^{OZO}_{456}$,	(185)
OZZOOO :	$R^{OOO}_{456}R^{ZZO}_{236}R^{OZO}_{135}R^{OZO}_{124} = R^{OZO}_{124}R^{OZO}_{135}R^{ZZO}_{236}R^{OOO}_{456}$,	(186)
OZOOOZ :	$R^{OOZ}_{456}R^{ZOZ}_{236}R^{OOO}_{135}R^{OZO}_{124} = R^{OZO}_{124}R^{OOO}_{135}R^{ZOZ}_{236}R^{OOZ}_{456}$,	(187)
OOZZOO :	$R^{ZOO}_{456}R^{OZO}_{236}R^{OZO}_{135}R^{OOZ}_{124} = R^{OOZ}_{124}R^{OZO}_{135}R^{OZO}_{236}R^{ZOO}_{456}$,	(188)
OOZOZO :	$R^{OZO}_{456}R^{OZO}_{236}R^{OZZ}_{135}R^{OOO}_{124} = R^{OOO}_{124}R^{OZZ}_{135}R^{OZO}_{236}R^{OZO}_{456}$,	(189)
OOZOOZ :	$R^{OOZ}_{456}R^{OZZ}_{236}R^{OZO}_{135}R^{OOO}_{124} = R^{OOO}_{124}R^{OZO}_{135}R^{OZZ}_{236}R^{OOZ}_{456}$,	(190)
ZOZOZO :	$R^{OZO}_{456}R^{OZO}_{236}R^{ZZZ}_{135}R^{ZOO}_{124} = R^{ZOO}_{124}R^{ZZZ}_{135}R^{OZO}_{236}R^{OZO}_{456}$,	(191)
OZZOOZ :	$R^{OOZ}_{456}R^{ZZZ}_{236}R^{OZO}_{135}R^{OZO}_{124} = R^{OZO}_{124}R^{OZO}_{135}R^{ZZZ}_{236}R^{OOZ}_{456}$.	(192)

Note that (186), (189), (191) and (192) contain *three* locally non-finite 3D R's. To summarize so far, type OOOOOO and (168)–(181) and (183)–(192) are the complete list of tetrahedron equations of type ABCDEF $\in$ {O,Z}^{6} which allow finitely many (u, v, w, x, y, z) in (182) enabling us to perform a direct check for various (a, b, c, d, e, f) and (i, j, k, l, m, n).

When doing so, parameters in the 3D R's are to be chosen with care. Let us illustrate it along the example (169). The representations π_O and π_Z carry the parameter μ and the quartet (r, s, t, w), respectively. Thus the tensor components corresponding to 1, 2, 3, 4, 5 and 6 are assigned with the parameters $\mu_1, \mu_2, \mu_3, (r_4, s_4, t_4, w_4), \mu_5$ and μ_6, respectively. R^{OOO}_{236} and R^{OOO}_{135} are given by (118) by replacing (μ_1, μ_2, μ_3) with (μ_2, μ_3, μ_6) and (μ_1, μ_3, μ_5), respectively. In view of the presence of R^{ZOO}_{456} and Theorem 10, we assume $\frac{\mu_6}{\mu_5} = q^d$ for some $d \in \mathbb{Z}$, and take R^{ZOO}_{456} to be (103) with $(r_1, s_1, t_1, w_1) \to (r_4, s_4, t_4, w_4)$, $\mu_2 \to \mu_5$ and $\mu_3 \to \mu_6$. Similarly from R^{OOZ}_{124} and Theorem 9, we postulate $\frac{\mu_1}{\mu_2} = q^{d'}$ for some $d' \in \mathbb{Z}$, and R^{OOZ}_{124} is given by (90) with $(r_3, s_3, t_3, w_3) \to (r_4, s_4, t_4, w_4)$ and $d \to d'$. With these choices, the tetrahedron equation (169) depends on the seven continuous parameters $\mu_1, \mu_3, \mu_5, r_4, s_4, t_4, w_4$ and the two integer parameters d, d' in addition to the ubiquitous q. Parameters in the other tetrahedron equations are to be tuned similarly. They can be arbitrary as long as the relevant 3D R's are non-singular, being free from the vanishing q-shifted factorials in the denominators (if any).

Now we state a conjecture based on computer experiments, indicating a sort of *coherence* prevailing the 3D R's obtained in Sect. 3.

Conjecture 17. *The tetrahedron equations* (168)–(181) *and* (183)–(192) *are valid in full generality of parameters.*

Typically, equalities have been checked for about 10,000 choices of the pairs $((a, b, c, d, e, f), (i, j, k, l, m, n))$.

Let us turn to the tetrahedron equation in which at least one side of (182) becomes a sum over infinitely many (u, v, w, x, y, z)'s. A typical examples is type ZZZZZZ. A possible regularization in such a circumstance is to specialize q to a root of unity and thereby to replace F by a finite dimensional vector space. For R^{XXX}, such a recipe is known [5] to yield the 3D R corresponding to [2], which is closely related with the generalized Chiral Potts models [2,4,7,22]. It is an interesting problem to explore a similar connection for the other 3D R's in this paper. Remark 12 is a key to such studies concerning R^{OOO}.

Finally the whole setting concerning the quantized Yang–Baxter equation $RLLL = LLLR$ in this paper has a natural analogue in the *quantized reflection equation* $K(GLGL) = (LGLG)K$ [17] which is related to the quantized coordinate rings of type B and C. It awaits a discovery of new 3D K's different from the known ones in [15] and [14, Chap. 5 & 6].

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Appendix A: Explicit form of $R_{456}L^{Z}_{236}L^{Z}_{135}L^{Z}_{124} = L^{Z}_{124}L^{Z}_{135}L^{Z}_{236}R_{456}$

We write down (28) explicitly together with the corresponding choice of $(abcijk)$ in (24) or in Fig. 3. As mentioned around (25), there are 18 non-trivial cases. To save the space, we write $Y_\alpha = Z^{-1}(r_\alpha s_\alpha - t_\alpha^2 w_\alpha X^2)$.

$$(001001) : R(1 \otimes X \otimes X) = (1 \otimes X \otimes X)R, \tag{193}$$

$$(001010) : R\big(r_2 t_1 X \otimes 1 \otimes Y_3 + t_3 Z \otimes Y_2 \otimes X\big) = r_1 t_2 \big(1 \otimes X \otimes Y_3\big)R, \tag{194}$$

$$(001100) : R\big(-q t_1 t_3 w_1 X \otimes Y_2 \otimes X + r_2 Y_1 \otimes 1 \otimes Y_3\big) = r_1 r_3 \big(1 \otimes Y_2 \otimes 1\big)R, \tag{195}$$

$$(010001) : r_1 t_2 R(1 \otimes X \otimes Z) = \big(r_2 t_1 X \otimes 1 \otimes Z + t_3 Y_1 \otimes Z \otimes X\big)R, \tag{196}$$

$$\begin{aligned}(010010) : R\big(&q r_2 t_1 t_3 w_3 X \otimes 1 \otimes X - Z \otimes Y_2 \otimes Z\big) \\ &= \big(q r_2 t_1 t_3 w_3 X \otimes 1 \otimes X - Y_1 \otimes Z \otimes Y_3\big)R,\end{aligned} \tag{197}$$

$$(010100) : R\big(t_1 w_1 X \otimes Y_2 \otimes Z + r_2 t_3 w_3 Y_1 \otimes 1 \otimes X\big) = r_3 t_2 w_2 \big(Y_1 \otimes X \otimes 1\big)R, \tag{198}$$

$$(011011) : R(X \otimes X \otimes 1) = (X \otimes X \otimes 1)R, \tag{199}$$

$$(011101) : s_3 t_2 R\big(Y_1 \otimes X \otimes 1\big) = \big(t_1 X \otimes Y_2 \otimes Z + s_2 t_3 Y_1 \otimes 1 \otimes X\big)R, \tag{200}$$

$$(011110) : s_1 s_3 R\big(1 \otimes Y_2 \otimes 1\big) = \big(-q t_1 t_3 w_3 X \otimes Y_2 \otimes X + s_2 Y_1 \otimes 1 \otimes Y_3\big) R, \quad (201)$$

$$(100001) : r_1 r_3 R(1 \otimes Z \otimes 1) = (-q t_1 t_3 w_1 X \otimes Z \otimes X + r_2 Z \otimes 1 \otimes Z) R, \quad (202)$$

$$(100010) : r_3 t_2 w_2 R(Z \otimes X \otimes 1) = \big(t_1 w_1 X \otimes Z \otimes Y_3 + r_2 t_3 w_3 Z \otimes 1 \otimes X\big) R, \quad (203)$$

$$(100100) : R(X \otimes X \otimes 1) = (X \otimes X \otimes 1) R, \quad (204)$$

$$(101011) : R(t_1 X \otimes Z \otimes Y_3 + s_2 t_3 Z \otimes 1 \otimes X) = s_3 t_2 (Z \otimes X \otimes 1) R, \quad (205)$$

$$(101101) : R\big(-q s_2 t_1 t_3 w_1 X \otimes 1 \otimes X + Y_1 \otimes Z \otimes Y_3\big) = \big(-q s_2 t_1 t_3 w_1 X \otimes 1 \otimes X + Z \otimes Y_2 \otimes Z\big) R, \quad (206)$$

$$(101110) : R\big(s_1 t_2 w_2 1 \otimes X \otimes Y_3) = \big(s_2 t_1 w_1 X \otimes 1 \otimes Y_3 + t_3 w_3 Z \otimes Y_2 \otimes X\big) R, \quad (207)$$

$$(110011) : R(-q t_1 t_3 w_3 X \otimes Z \otimes X + s_2 Z \otimes 1 \otimes Z) = s_1 s_3 (1 \otimes Z \otimes 1) R, \quad (208)$$

$$(110101) : R\big(t_3 w_3 Y_1 \otimes Z \otimes X + s_2 t_1 w_1 X \otimes 1 \otimes Z\big) = s_1 t_2 w_2 (1 \otimes X \otimes Z) R, \quad (209)$$

$$(110110) : R(1 \otimes X \otimes X) = (1 \otimes X \otimes X) R. \quad (210)$$

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