Commun. Math. Phys. 401, 1627–1663 (2023)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-023-04673-1

Communications in
Mathematical Physics

Partition Functions of Determinantal and Pfaffian Coulomb Gases with Radially Symmetric Potentials

Sung-Soo Byun[1], Nam-Gyu Kang[2], Seong-Mi Seo[3]

[1] Center for Mathematical Challenges, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea. E-mail: sungsoobyun@kias.re.kr
[2] School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea. E-mail: namgyu@kias.re.kr
[3] Department of Mathematics, Chungnam National University, 99 Daehak-ro, Yuseong-gu, Daejeon 34134, Republic of Korea. E-mail: smseo@cnu.ac.kr

Received: 10 November 2022 / Accepted: 11 February 2023
Published online: 8 March 2023 –

Abstract: We consider random normal matrix and planar symplectic ensembles, which can be interpreted as two-dimensional Coulomb gases having determinantal and Pfaffian structures, respectively. For a class of radially symmetric potentials with soft edges, we derive the asymptotic expansions of the log-partition functions up to and including the $O(1)$-terms as the number N of particles increases. Notably, our findings stress that the formulas of the $O(\log N)$- and $O(1)$-terms in these expansions depend on the connectivity of the droplet. For random normal matrix ensembles, our formulas agree with the predictions proposed by Zabrodin and Wiegmann up to an additive constant depending on N but not on the background potential. For planar symplectic ensembles, the expansions contain a new kind of ingredient in the $O(N)$-terms, the logarithmic potential evaluated at the origin in addition to the entropy of the ensembles.

1. Introduction and Main Results

The Coulomb gas ensemble in the complex plane is governed by the law

$$dP_N^{(\beta)}(z_1,\dots,z_N) := \frac{1}{Z_N^{(\beta)}} \prod_{j>k=1}^{N} |z_j - z_k|^{\beta} \prod_{j=1}^{N} e^{-\frac{\beta N}{2} Q(z_j)}\, dA(z_j), \qquad (1.1)$$

where N is the number of particles, β is the inverse temperature and $dA(z) := d^2z/\pi$ is the area measure. Here, $Q : \mathbb{C} \to \mathbb{R}$ is called the confining/external potential that satisfies suitable potential theoretic conditions. We refer to [38,56,61] and references

Sung-Soo Byun and Nam-Gyu Kang were partially supported by Samsung Science and Technology Foundation (SSTF-BA1401-51) and by the National Research Foundation of Korea (NRF-2019R1A5A1028324). Sung-Soo Byun was partially supported by a KIAS Individual Grant (SP083201) via the Center for Mathematical Challenges at Korea Institute for Advanced Study. Nam-Gyu Kang was partially supported by a KIAS Individual Grant (MG058103) at Korea Institute for Advanced Study. Seong-Mi Seo was partially supported by the National Research Foundation of Korea (NRF-2019R1A5A1028324, NRF-2022R1I1A1A01072052).

therein for recent developments of two-dimensional Coulomb gases. Contrary to (1.1), the configurational canonical Coulomb gas ensemble in the upper-half plane [39,50] (cf. [20, Appendix A]) has an additional complex conjugation symmetry (i.e. the particles come in complex conjugate pairs) and is governed by the law

$$d\widetilde{P}_N^{(\beta)}(z_1,\dots,z_N) := \frac{1}{\widetilde{Z}_N^{(\beta)}} \prod_{j>k=1}^{N} |z_j - z_k|^\beta |z_j - \bar{z}_k|^\beta \times \prod_{j=1}^{N} |z_j - \bar{z}_j|^\beta e^{-\beta N Q(z_j)}\, dA(z_j). \tag{1.2}$$

In (1.1) and (1.2), the normalization constants

$$Z_N^{(\beta)} := \int_{\mathbb{C}^N} \prod_{j>k=1}^{N} |z_j - z_k|^\beta \prod_{j=1}^{N} e^{-\frac{\beta N}{2} Q(z_j)}\, dA(z_j), \tag{1.3}$$

$$\widetilde{Z}_N^{(\beta)} := \int_{\mathbb{C}^N} \prod_{j>k=1}^{N} |z_j - z_k|^\beta |z_j - \bar{z}_k|^\beta \prod_{j=1}^{N} |z_j - \bar{z}_j|^\beta e^{-\beta N Q(z_j)}\, dA(z_j) \tag{1.4}$$

that make (1.1) and (1.2) probability measures are called partition functions. Furthermore, the logarithm of a partition function (divided by N^2) is often called free energy.

For the special value $\beta = 2$, (1.1) and (1.2) represent joint probability distributions of the random normal matrix and planar symplectic ensembles, respectively. In particular, if $Q(z) = |z|^2$, these correspond to the complex and symplectic Ginibre ensembles [42]. An important feature of this special value $\beta = 2$ is that, due to the factors identified in terms of Vandermonde determinants, the ensembles (1.1) and (1.2) form *determinantal* and *Pfaffian* point processes in the plane [38], respectively. In other words, all their correlation functions can be expressed in terms of the (pre-)kernel of planar (skew-)orthogonal polynomials. We refer the reader to [19,21] for recent reviews on these models. In the sequel, for $\beta = 2$, we omit the superscript (β) in (1.3) and (1.4), and simply write $Z_N \equiv Z_N^{(2)}$ and $\widetilde{Z}_N \equiv \widetilde{Z}_N^{(2)}$.

We mention that the definition of partition functions (1.3) and (1.4) is more common in the statistical physics community. On the other hand, in the random matrix theory community, another widely used convention for the (canonical) partition functions is

$$\mathcal{Z}_N := \frac{1}{N!} Z_N, \qquad \widetilde{\mathcal{Z}}_N := \frac{1}{N!} \widetilde{Z}_N, \tag{1.5}$$

see e.g. [38, Section 1.4]. The prefactor $1/N!$ in (1.5) allows writing $\mathcal{Z}_N$ and $\widetilde{\mathcal{Z}}_N$ in terms of a structured determinant and Pfaffian, respectively.

In this work, we study the asymptotic expansions of Z_N and $\widetilde{Z}_N$ as $N \to \infty$.

1.1. Summary of previous results. Before introducing our results, let us summarize some known results on the asymptotics of $Z_N^{(\beta)}$ for general β and Q. Cf. the literature on $\widetilde{Z}_N^{(\beta)}$ is much more limited.

- **(Zabrodin–Wiegmann prediction)** In [67], it was predicted that the partition function $Z_N^{(\beta)}$ has an asymptotic expansion of the form

$$\log Z_N^{(\beta)} = C_0 N^2 + C_1 N \log N + C_2 N + C_3 \log N + C_4 + O(\frac{1}{N}). \tag{1.6}$$

Furthermore, they proposed explicit formulas for the constants $C_j \equiv C_j(\beta, Q)$ $(j = 0, \dots, 4)$ depending on β and Q, cf. (1.28). Incidentally, the formulas for C_3 and C_4 in [67] have been controversial as pointed out for instance in [59,62]. (See also [23,47,63] for a similar prediction, which contains non-trivial $O(\sqrt{N})$-terms for $\beta \neq 2$.)

- **(Asymptotic of the leading order $O(N^2)$-term)** It was shown in [44, Theorem 2.11] and [24, Theorem 1.1] (among others) that as $N \to \infty$,

$$\log Z_N^{(\beta)} = -\frac{\beta}{2} N^2 I_Q[\mu_Q] + o(N^2).$$

Here μ_Q is Frostman's equilibrium measure [58], a unique probability measure that minimizes the weighted logarithmic energy

$$I_Q[\mu] := -\iint_{\mathbb{C}^2} \log|z-w| \, d\mu(z) \, d\mu(w) + \int_{\mathbb{C}} Q \, d\mu. \tag{1.7}$$

- **(Asymptotic up to the $O(N)$-term)** It was shown by Leblé and Serfaty [53, Corollary 1.1] that as $N \to \infty$,

$$\log Z_N^{(\beta)} = -\frac{\beta}{2} N^2 I_Q[\mu_Q] + \frac{\beta}{4} N \log N - \Big(C(\beta) + \Big(1 - \frac{\beta}{4}\Big) E_Q[\mu_Q] \Big) N + o(N), \tag{1.8}$$

where $C(\beta)$ is a constant independent of the potential Q and

$$E_Q[\mu_Q] := \int_{\mathbb{C}} \log(\Delta Q) \, d\mu_Q \tag{1.9}$$

is the entropy associated with μ_Q.[1] Here, $\Delta := \partial\bar{\partial}$ is the quarter of the usual Laplacian. The precise assumptions on the potential Q can be found in [53, Section 2.1]. Notably, it is assumed that ΔQ is bounded above in the droplet. We also refer the reader to [14,62] for the expansion (1.8) with quantitative error bounds.

Beyond the general cases mentioned above, for $\beta = 2$ with a specific (and fundamental from the random matrix theory viewpoint) potential, there have been several works on the precise asymptotic expansion of the partition functions, see e.g. [27,28] and references therein. This type of potential usually contains certain singularities. As a result, the asymptotic expansions of the associated partition functions are more complicated (for instance, some non-trivial $O(\sqrt{N})$ terms appear as well). Several topics in this direction will be discussed in a separate remark at the end of the next subsection.

[1] We mention that the physical entropy is $-\int_{\mathbb{C}} \log(\Delta Q/\pi) \, d\mu_Q$.

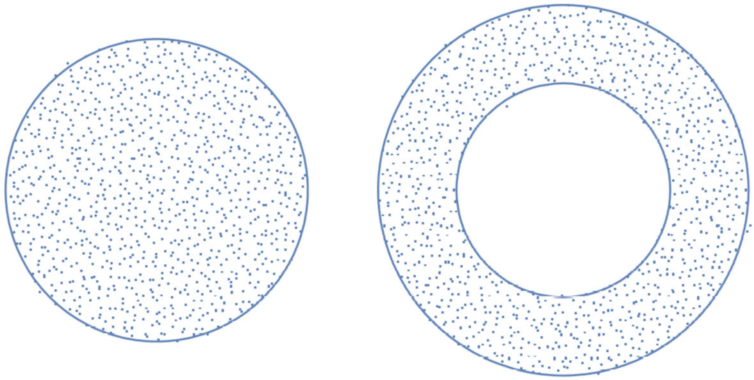

Fig. 1. Eigenvalues of complex Ginibre (left) and complex induced Ginibre (right) matrices where $N = 1000$, i.e. the model (1.1) with $\beta = 2$ and $Q(\zeta) = |\zeta|^2 - 2c \log |\zeta|$, where $c = 0$ (left) and $c > 0$ (right)

1.2. Main results. We study asymptotic behaviors of Z_N and $\widetilde{Z}_N$ for the exactly-solvable case where Q is radially symmetric. Our main findings are summarized as follows.

(i) We derive the large-N expansions of $\log Z_N$ and $\log \widetilde{Z}_N$ up to and including the $O(1)$-terms.

(ii) In the large-N expansions, the formulas of the $O(\log N)$- and $O(1)$-terms depend on whether the limiting spectrum is an *annulus* or a *disc*, see Theorems 1.1 and 1.2, respectively, cf. Fig. 1. This distinction is crucial in the asymptotic analysis but seems not considered in [67]. Nonetheless, a precise prediction of the $\log N$ term given in terms of the Euler index of the droplet was made in the earlier work [47] of Jancovici, Manificat, and Pisani. We refer the reader to [19, Section 4.1] for a review and more references.

(iii) For the partition function Z_N of random normal matrix ensembles, our expansions (1.17) and (1.23) up to the $O(N)$-terms agree with the formula (1.8) with $\beta = 2$. Furthermore, we verify from (1.23) that the asymptotic formula given in [67, Eqs.(1.2), (C.7)] holds up to an additive constant (1.34). Here, the meaning of constant is with respect to the background potential Q, not with respect to N. Thus the prediction (1.6) is faulty at the level of C_3.

(iv) For the partition function $\widetilde{Z}_N$ of planar symplectic ensembles, the asymptotic formulas (1.18) and (1.24) are new to the best of our knowledge. Contrary to (1.8), the $O(N)$-terms in these expansions contain not only the entropy but also the logarithmic potential (1.14).

Let us be more precise in introducing our results. It is well known [15,44] that under some mild assumptions on Q, as $N \to \infty$, the empirical measures $\frac{1}{N}\sum_{j=1}^{N} \delta_{z_j}$ of (1.1) and (1.2) weakly converge to μ_Q, which takes the form

$$d\mu_Q = \Delta Q \cdot \mathbb{1}_S \, dA. \tag{1.10}$$

Here $S \equiv S_Q$ is a certain compact subset of $\mathbb{C}$ called the *droplet*, see Fig. 1.

We consider the case where the external potential Q is radially symmetric, i.e. $Q(z) = q(|z|)$ for some function q defined in $[0, \infty)$. Throughout this paper, we focus on the case Q is independent of N. We assume the basic growth condition

$$\liminf_{|z|\to\infty} \frac{Q(z)}{2\log|z|} > 1, \tag{1.11}$$

which guarantees that $Z_N, \widetilde{Z}_N < +\infty$. Furthermore, we assume that Q is C^∞-smooth in a neighborhood of the droplet, subharmonic in $\mathbb{C}$, and strictly subharmonic in a neighborhood of the droplet. We mention that away from the origin, the latter conditions can be written as the requirements that $rq'(r)$ is increasing on $(0, \infty)$, and strictly increasing in a neighborhood of the droplet, cf. (2.3). Under the above assumptions, the droplet is given by

$$S = \mathbb{A}_{r_0,r_1} := \{z \in \mathbb{C} : r_0 \le |z| \le r_1\}, \tag{1.12}$$

where r_0 is the largest solution to $rq'(r) = 0$ and r_1 is the smallest solution to $rq'(r) = 2$, see [58, Section IV.6]. (We mention that the annular droplets often appear in non-Hermitian random matrix theory, see e.g. [43].) In particular, if $r_0 = 0$, we denote $\mathbb{D}_{r_1} = \mathbb{A}_{0,r_1}$. Henceforth, we keep the assumptions on Q described above. For instance, we cover the case $Q(z) = |z|^{2\lambda} - 2c \log |z|$ for general $\lambda > 0$ and $c > 0$, see Sect. 4.1. However, our result does not cover the case $Q(z) = |z|^{2\lambda}$ with $\lambda \neq 1$ since it is not strictly subharmonic at the origin, which is inside the droplet.

For a radially symmetric potential Q, by using (1.10) and (1.12), one can show that the energy $I_Q[\mu_Q]$ in (1.7) is given by

$$I_Q[\mu_Q] = q(r_1) - \log r_1 - \frac{1}{4}\int_{r_0}^{r_1} rq'(r)^2\, dr. \tag{1.13}$$

Similarly, in terms of the logarithmic potential

$$U_\mu(z) = \int \log \frac{1}{|z-w|}\, d\mu(w), \tag{1.14}$$

we have

$$U_{\mu_Q}(0) = -\int_S \log |w|\, d\mu_Q(w) = -\log r_1 + \frac{q(r_1) - q(r_0)}{2}. \tag{1.15}$$

See [58, Section IV.6] for more details.

For the annular droplet case, we have the following.

Theorem 1.1. (Large-N expansion of the partition functions: annular droplet case) *Suppose that $r_0 > 0$, i.e. the droplet S in (1.12) is an annulus. Let*

$$\begin{aligned} F_Q[\mathbb{A}_{r_0,r_1}] := \frac{1}{12}\log\Big(\frac{r_0^2\Delta Q(r_0)}{r_1^2\Delta Q(r_1)}\Big) - \frac{1}{16}\Big(r_1\frac{(\partial_r\Delta Q)(r_1)}{\Delta Q(r_1)} - r_0\frac{(\partial_r\Delta Q)(r_0)}{\Delta Q(r_0)}\Big) \\ + \frac{1}{24}\int_{r_0}^{r_1}\Big(\frac{\partial_r\Delta Q(r)}{\Delta Q(r)}\Big)^2 r\, dr. \end{aligned} \tag{1.16}$$

Then as $N \to \infty$, the following holds.

(i) **(Random normal matrix ensemble)** *We have*

$$\begin{aligned} \log Z_N = -N^2 I_Q[\mu_Q] + \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - 1 - \frac{1}{2}E_Q[\mu_Q]\Big)N \\ + \frac{1}{2}\log N + \frac{\log(2\pi)}{2} + F_Q[\mathbb{A}_{r_0,r_1}] + O(N^{-1}). \end{aligned} \tag{1.17}$$

(ii) **(Planar symplectic ensemble)** *We have*

$$\log \widetilde{Z}_N = -2N^2 I_Q[\mu_Q] + \frac{1}{2}N\log N + \Big(\frac{\log(4\pi)}{2} - 1 - U_{\mu_Q}(0) - \frac{1}{2}E_Q[\mu_Q]\Big)N + \frac{1}{2}\log N + \frac{\log(2\pi)}{2} + \frac{1}{2}F_Q[\mathbb{A}_{r_0,r_1}] + \frac{1}{8}\log\Big(\frac{\Delta Q(r_0)}{\Delta Q(r_1)}\Big) + O(N^{-1}). \tag{1.18}$$

Using the convention (1.5) together with (2.22), our result can also be rewritten as

$$\log \mathcal{Z}_N = -N^2 I_Q[\mu_Q] - \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - \frac{1}{2}E_Q[\mu_Q]\Big)N + F_Q[\mathbb{A}_{r_0,r_1}] + O(N^{-1}) \tag{1.19}$$

and

$$\log \widetilde{\mathcal{Z}}_N = -2N^2 I_Q[\mu_Q] - \frac{1}{2}N\log N + \Big(\frac{\log(4\pi)}{2} - U_{\mu_Q}(0) - \frac{1}{2}E_Q[\mu_Q]\Big)N + \frac{1}{2}F_Q[\mathbb{A}_{r_0,r_1}] + \frac{1}{8}\log\Big(\frac{\Delta Q(r_0)}{\Delta Q(r_1)}\Big) + O(N^{-1}). \tag{1.20}$$

We mention that these formulas (1.19) and (1.20) as well as the formulas (1.25) and (1.26) below are more convenient to compare with some asymptotic results in the previous literature [27,28].

We mention that the term $\log(r_1/r_0)$ is the extremal length of the annulus (1.12), see e.g. [40, p.142].

Remark (Renormalized energy). It is worth pointing out that a characteristic difference between the expansions (1.17) and (1.18) is the appearance of the logarithmic potential $U_{\mu_Q}(0)$ in the $O(N)$-term of (1.18). This additional term can be rewritten as

$$U_{\mu_Q}(0) = -\int_S \log|w-\bar{w}|\,d\mu_Q(w). \tag{1.21}$$

To see this, we use the polar coordinate to rewrite the right-hand side of (1.21) as

$$\begin{aligned}
&-\int_0^{2\pi}\int_{r_0}^{r_1} r\log|2r\sin\theta|\,\Delta Q(r)\,dr\,d\theta \\
&= U_{\mu_Q}(0) - \int_0^{2\pi}\log|2\sin\theta|\int_{r_0}^{r_1} r\,\Delta Q(r)\,dr\,d\theta \\
&= U_{\mu_Q}(0) - \frac{1}{2}\int_0^{2\pi}\log|2\sin\theta|\,d\theta = U_{\mu_Q}(0).
\end{aligned}$$

Here, the last identity $\int_0^{2\pi}\log|2\sin\theta|\,d\theta = 0$ is an elementary exercise in complex analysis. The interpretation (1.21) is natural from the perspective of the repulsion term $|z_j - \bar{z}_j|^\beta$ in (1.2) and is closely related to the notion of the next-order energy, see e.g. [54]. (We thank T. Leblé for pointing out this.)

In Sect. 4.1 we present an example of Theorem 1.1 for the Mittag-Leffler ensembles from which we expect that the error terms $O(N^{-1})$ are optimal.

For the disc droplet case, we have the following.

Theorem 1.2. (Large-N expansion of the partition functions: disc droplet case) *Suppose that $r_0 = 0$, i.e. the droplet S in (1.12) is a disc. Let*

$$F_Q[\mathbb{D}_{r_1}] := \frac{1}{12}\log\Big(\frac{1}{r_1^2\Delta Q(r_1)}\Big) - \frac{1}{16}r_1\frac{(\partial_r\Delta Q)(r_1)}{\Delta Q(r_1)} + \frac{1}{24}\int_0^{r_1}\Big(\frac{\partial_r\Delta Q(r)}{\Delta Q(r)}\Big)^2 r\,dr. \tag{1.22}$$

Then as $N\to\infty$, the following holds.

(i) ***(Random normal matrix ensemble)*** *We have*

$$\log Z_N = -N^2 I_Q[\mu_Q] + \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - 1 - \frac{1}{2}E_Q[\mu_Q]\Big)N + \frac{5}{12}\log N + \frac{\log(2\pi)}{2} + \zeta'(-1) + F_Q[\mathbb{D}_{r_1}] + O(N^{-\frac{1}{12}}(\log N)^3). \tag{1.23}$$

(ii) ***(Planar symplectic ensemble)*** *We have*

$$\begin{aligned}\log \widetilde{Z}_N = &-2N^2 I_Q[\mu_Q] + \frac{1}{2}N\log N \\ &+ \Big(\frac{\log(4\pi)}{2} - 1 - U_{\mu_Q}(0) - \frac{1}{2}E_Q[\mu_Q]\Big)N + \frac{11}{24}\log N \\ &+ \frac{\log(2\pi)}{2} + \frac{1}{2}\zeta'(-1) + \frac{1}{2}F_Q[\mathbb{D}_{r_1}] + \frac{5}{24}\log 2 + \frac{1}{8}\log\Big(\frac{\Delta Q(0)}{\Delta Q(r_1)}\Big) \\ &+ O(N^{-\frac{1}{12}}(\log N)^3).\end{aligned} \tag{1.24}$$

Here ζ is the Riemann zeta function.

Again, using the convention (1.5), we have

$$\log \mathcal{Z}_N = -N^2 I_Q[\mu_Q] - \frac{1}{2}N\log N + \Big(\frac{\log(2\pi)}{2} - \frac{1}{2}E_Q[\mu_Q]\Big)N - \frac{1}{12}\log N + \zeta'(-1) + F_Q[\mathbb{D}_{r_1}] + O(N^{-\frac{1}{12}}(\log N)^3) \tag{1.25}$$

and

$$\begin{aligned}\log \widetilde{\mathcal{Z}}_N = &-2N^2 I_Q[\mu_Q] - \frac{1}{2}N\log N \\ &+ \Big(\frac{\log(4\pi)}{2} - U_{\mu_Q}(0) - \frac{1}{2}E_Q[\mu_Q]\Big)N - \frac{1}{24}\log N \\ &+ \frac{1}{2}\zeta'(-1) + \frac{1}{2}F_Q[\mathbb{D}_{r_1}] + \frac{5}{24}\log 2 + \frac{1}{8}\log\Big(\frac{\Delta Q(0)}{\Delta Q(r_1)}\Big) + O(N^{-\frac{1}{12}}(\log N)^3).\end{aligned} \tag{1.26}$$

In Sect. 4.2, we provide an example of Theorem 1.2 for truncated unitary ensembles. Contrary to Theorem 1.1, the error terms in Theorem 1.2 do not coincide with the expected optimal orders $O(N^{-1})$. Our error bounds originate from a decomposition of the analytic expressions of Z_N, $\widetilde{Z}_N$ (see Sect. 1.3), which depends on sufficiently large but seemingly arbitrary number $m_N > 0$. (Such a decomposition was not necessary for the proof of Theorem 1.1.) Later, we choose $m_N = N^{1/6}$ that gives rise to the control of the total error bounds presented in Theorem 1.2. We mention that such error estimates also naturally appeared in similar computations, see e.g. [17,27,28]. Nevertheless, we expect that the estimates can be improved with more effort.

In terms of the function $\chi := \frac{1}{2}\log \Delta Q$, one can rewrite (1.22) as

$$F_Q[\mathbb{D}_{r_1}] = \frac{1}{12}\log\Big(\frac{1}{r_1^2}\Big) - \frac{1}{12\pi}\oint_{\partial S}\kappa\,\chi\,ds - \frac{1}{4}\int_S \Delta\chi\,dA + \frac{1}{12}\int_S |\nabla\chi|^2\,dA, \tag{1.27}$$

where $S = \mathbb{D}_{r_1}$ and $\kappa = 1/r_1$ is the curvature of the boundary, see [67, p.8960] and (1.33). Here, the third term $\int_S \Delta\chi\,dA$ on the right-hand side of (1.27) is known as a "zero mode" of the loop operator (cf. [67, Eq.(5.26)]), whereas the fourth term corresponds to the Dirichlet energy of χ.

We end this subsection by giving some crucial remarks on our theorems.

Remark (Comparison with Zabrodin-Wiegmann formula). We compare our formula (1.23) with the prediction by Zabrodin and Wiegmann. For $\beta = 2$ and a radially symmetric Q associated with a disc droplet of radius r_1, the asymptotic formula (1.6) is written in [67, Eqs.(1.2),(C.7)] as

$$\begin{aligned}\log Z_N &= F_0N^2 + F_{1/2}N + F_1 + c(N) + O(N^{-1}), \quad \Big(c(N) := \log N! - \frac{1}{2}N\log N + \tilde{\gamma}N\Big)\\ &= F_0N^2 + \frac{1}{2}N\log N + \Big(\tilde{\gamma} - 1 + F_{1/2}\Big)N + \frac{1}{2}\log N + \frac{\log(2\pi)}{2} + F_1 + O(N^{-1}),\end{aligned} \tag{1.28}$$

where $\tilde{\gamma}$ is a "numerical" constant [67, p.8938] (that is not explicitly presented). (For reader's convenience, let us mention that in [67], the authors use a different convention for β so that $\beta = 2$ in our case corresponds to $\beta = 1$ in [67]. Furthermore, the Planck constant $\hbar$ in [67] is identified as $1/N$.) The coefficients F_0, $F_{1/2}$ and F_1 in (1.28) are given by

$$F_0 := \pi\int_0^{r_1^2}(W_{\rm rad}(x) - W'_{\rm rad}(x)x\log x)\sigma_{\rm rad}(x)\,dx, \tag{1.29}$$

$$F_{1/2} := -\frac{\pi}{2}\int_0^{r_1^2}\sigma_{\rm rad}(x)\log(\pi\sigma_{\rm rad}(x))\,dx, \tag{1.30}$$

$$\begin{aligned}F_1 := &\frac{1}{12}\log\Big(\frac{1}{r_1^2}\Big) - \frac{1}{6}\chi_{\rm rad}(r_1^2) - \frac{1}{4}r_1^2\chi'_{\rm rad}(r_1^2)\\ &+ \frac{1}{3}\int_0^{r_1^2}x(\chi'_{\rm rad}(x))^2\,dx,\end{aligned} \tag{1.31}$$

where

$$W_{\rm rad}(r) := -Q(\sqrt{r}), \qquad \sigma_{\rm rad}(r) := \frac{1}{\pi}\Delta Q(\sqrt{r}), \qquad \chi_{\rm rad}(r) := \frac{1}{2}\log\Delta Q(\sqrt{r}). \tag{1.32}$$

Using (1.32), it is straightforward to check that the formulas (1.29), (1.30) and (1.31) can be identified as

$$F_0 = -I_Q[\mu_Q], \qquad F_{1/2} = -\frac{1}{2} E_Q[\mu_Q], \qquad F_1 = F_Q[\mathbb{D}_{r_1}], \tag{1.33}$$

where I_Q, E_Q and $F_Q[\mathbb{D}_{r_1}]$ are given by (1.13), (1.9) and (1.22). (Cf. the identification of F_0 follows from the computation (2.16) below.) Then by letting $\tilde{\gamma} = \log(2\pi)/2$, one can deduce from (1.33) that the asymptotic formula (1.28) agrees with our result (1.23) up to the additive terms

$$-\frac{1}{12} \log N + \zeta'(-1). \tag{1.34}$$

We remark that the asymptotic expansion of the partition function of the complex Ginibre ensemble was presented in [23,65], where the universal coefficient for the $\log N$ in the case of disc geometry is also exhibited, see also [19, Section 4.1] for further references. Furthermore, one can also observe the term $\zeta'(-1)$ in (1.34). Indeed, the term $\zeta'(-1)$ and its generalizations have appeared in similar situations in the Hermitian matrix theory, see [31, Remark 1.3, Proposition 1.4], [26, Theorem 1.1], [29, Theorem 1.2]. Interestingly, the coefficients of the $\zeta'(-1)$ term depend on the connectivity of the droplet (i.e. the number of disjoint intervals in this case) and the number of hard edges.

Remark (Non-triviality of the limit $r_0 \to 0$). The formulas (1.23) and (1.24) cannot be recovered by simply taking the limit $r_0 \to 0$ of (1.17) and (1.18). Namely, it is obvious that as $r_0 \to 0$, the terms

$$-\frac{1}{12} \log\Big(\frac{1}{r_0^2 \Delta Q(r_0)}\Big) + \frac{1}{16} r_0 \frac{(\partial_r \Delta Q)(r_0)}{\Delta Q(r_0)} \tag{1.35}$$

do not correspond to the terms (1.34). (One may however notice that for the standard microscale $r_0 = O(1/\sqrt{N})$, at least the $-\frac{1}{12}\log N$ term in (1.34) follows.)

From the viewpoint of the proof, the origin of (1.34) and (1.35) is essentially similar in the sense that these terms arise from the asymptotic behaviors of the summand in (1.37) of *lower degrees*. Nevertheless, it is essential (but seems not discussed in [67]) that these asymptotic behaviors depend on whether the droplet is contractible or not, i.e. for the radially symmetric potentials, disc or annulus. We remark that the contractible case requires considerably more analysis than the other case, see the following subsection for more discussion.

Remark (Invariance of the $O(1)$-terms under the dilation). For $a > 0$, let $Q_a(z) := Q(z/a)$. Then the droplet associated with Q_a is given by $\{z \in \mathbb{C} : ar_0 \le |z| \le ar_1\}$, where r_0 and r_1 are given in (1.12). Then it follows from (1.16) and (1.22) that

$$F_Q[\mathbb{A}_{r_0,r_1}] = F_{Q_a}[\mathbb{A}_{ar_0,ar_1}], \qquad F_Q[\mathbb{D}_{r_1}] = F_{Q_a}[\mathbb{D}_{ar_1}]. \tag{1.36}$$

This in turn means that the $O(1)$-terms in the expansions in Theorems 1.1 and 1.2 are invariant under the dilation $\{z_j\} \mapsto \{a \cdot z_j\}$. The property (1.36) can be expected from the analytic expression (1.37) below. More precisely, by the change of variables when computing the orthogonal norms, the asymptotic expansions of the partition functions associated with Q_a and Q should differ only up to the $O(N)$-term, see [19, below Eq.(5.13)] for a similar discussion.

Remark (Weight function with singularities and classical problems in random matrix theory). In Theorems 1.1 and 1.2, we focus on the weight function e^{-NQ} without any kind of singularities. In contrast, if a specific singularity is allowed for the weight function, the problems of deriving asymptotic expansions of the associated partition function are (when combined with Theorems 1.1 and 1.2) equivalent to several classical problems in random matrix theory.

To be more concrete, we list various problems in this direction. If the weight function has a *hard-edge* inside the droplet, the associated partition function provides the large gap (hole) probability, see [1,2,8,18,27,37,41,46] and references therein. The weight function with a *jump-type singularity* gives rise to the moment generating function of the disc counting function. It has been extensively studied in recent years [4,11,17,25,28,30]. (We also refer to [35,51,52,64] for physical motivations of these problems from the counting statistics of rotating free fermions.) Finally, a *root-type singularity* arises in the study of the log-characteristic polynomials [17,33,66].

We stress that the literature mentioned above is limited mainly to a particular model, such as the Ginibre ensemble, when deriving precise asymptotic results or to the leading order asymptotic when considering general potentials. We expect that Theorems 1.1 and 1.2 provide the building blocks for obtaining precise asymptotic results on the problems mentioned above with general radially symmetric potentials.

Remark (Planar point processes with a general external potential Q). For a general potential Q beyond a radially symmetric one, the asymptotic behaviors of planar orthogonal polynomials (of sufficiently large degrees) with respect to $e^{-NQ}\,dA$ were recently obtained in [45]. We expect that this will be helpful to extend Theorem 1.1 (i) to a general potential Q associated with a "non-contractible" droplet. On the other hand, for the extension of Theorem 1.2 (i), it is required to derive asymptotics of orthogonal polynomials of lower degrees as well.

Such a generalization of Theorems 1.1 and 1.2 (ii) for planar symplectic ensembles seems at present far from being solved. More precisely, in order to obtain an analytic expression of $\widetilde{Z}_N$, it is required to construct the associated skew-orthogonal polynomial. However, for a non-radially symmetric potential, this construction has been known only in a few special cases [3,48] (cf. see [7] for a possible generality).

Remark (Multi-component ensembles). For a general Q, as a consequence of the associated equilibrium measure problem, it is possible that the droplet consists of several disconnected components, see e.g. [5,13,16,32,55] and references therein. In relation with the models (1.1), such multi-component ensembles have recently gained a particular interest due to their special statistical properties at the boundaries of the droplets as well as some theta function oscillations in various statistics, see e.g. [9,10,22,27]. For these models, it would also be interesting to investigate the precise asymptotic behaviours of the partition functions, for which it is expected that the coefficient of the $\log N$-term is again related to the Euler characteristics of the droplets and that certain theta function behaviours appear.

1.3. Outline of the proof. In this subsection, we outline the proofs of our main results. Using the determinantal (resp., Pfaffian) structure and de Bruijn's type formulas, one can express Z_N (resp., $\widetilde{Z}_N$) in terms of the (skew-)orthogonal norms. Consequently, since

Q is radially symmetric, we find

$$\log Z_N = \log N! + \sum_{j=0}^{N-1} \log h_j, \qquad \log \widetilde{Z}_N = \log N! + \sum_{j=0}^{N-1} \log(2\widetilde{h}_{2j+1}), \tag{1.37}$$

where

$$h_j := \int_{\mathbb{C}} |z|^{2j} e^{-NQ(z)} \, dA(z), \qquad \widetilde{h}_j := \int_{\mathbb{C}} |z|^{2j} e^{-2NQ(z)} \, dA(z). \tag{1.38}$$

These formulas can be found for instance in [28, Lemma 1.9] and [7, Remark 2.5]. In particular, for planar symplectic ensembles, we have used the explicit construction of skew-orthogonal polynomials associated with radially symmetric potentials, see [7, Corollary 3.3].

In order to obtain the large-N expansions of partition functions up to the $O(1)$-terms, we need to derive asymptotic behaviors of h_j and $\widetilde{h}_j$ up to the first subleading terms, for which we apply Laplace's method. For this purpose, let r_τ be a unique number r_τ such that $r_\tau q'(r_\tau) = 2\tau$ for $0 \le \tau \le 1$. Such a function $\tau \mapsto r_\tau$ plays an important role in Laplace's method, and we defer more explanations to Sect. 2.1. In the asymptotic expansions of h_j and $\widetilde{h}_j$, one should distinguish the following two cases depending on a small constant $\varepsilon > 0$.

- **Case 1:** $r_{j/N} \gg N^{-\varepsilon}$. For the annular droplet case where $r_0 > 0$, this case covers all $j = 0, 1, \dots, N-1$ (Lemma 2.1). On the other hand, for the disc droplet case where $r_0 = 0$, this case covers only $j = m_N, m_N + 1, \dots, N-1$ for $m_N = N^\epsilon$ with some $\epsilon > 0$ (Lemma 3.2).
- **Case 2:** $r_{j/N} \ll N^{-\varepsilon}$. This covers the remaining disc droplet case with $j = 0, 1, \dots, m_N - 1$ (Lemma 3.1). Notably, the asymptotic expansion involves gamma functions in this case.

Furthermore, we apply the Euler–Maclaurin formula (see e.g. [57, Section 2.10]) to precisely analyze the summations in (1.37)

$$\sum_{j=m}^{n} f(j) = \int_m^n f(x)\, dx + \frac{f(m)+f(n)}{2} + \sum_{k=1}^{l-1} \frac{B_{2k}}{(2k)!}\Big(f^{(2k-1)}(n) - f^{(2k-1)}(m) \Big) + R_l, \tag{1.39}$$

where B_k is the Bernoulli number defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{1.40}$$

Here, the error term R_l is given by [57, Eq.(2.10.1)]

$$R_l = \int_m^n \frac{B_{2l} - B_{2l}(x - \lfloor x \rfloor)}{(2l)!} f^{(2l)}(x)\, dx,$$

where B_l is the Bernoulli polynomial [57, Chapter 24]. Using the inequality

$$|B_{2l} - B_{2l}(x)| \le (2 - 2^{1-2l})|B_{2l}| = 4(1-4^{-l}) \frac{(2l)!}{(2\pi)^{2l}} \zeta(2l)$$

(see [57, Eqs.(24.9.2),(25.6.2)]), one can notice that the error term R_l satisfies the estimate

$$|R_l| \leq \frac{4\,\zeta(2l)}{(2\pi)^{2l}} \int_m^n |f^{(2l)}(x)|\,dx.$$

Here ζ is the Riemann zeta function. In particular, for the disc droplet case, in the summation of lower degrees $j = 0, 1, \ldots, m_N - 1$, we consider the Barnes G-function [57, Section 5.17]

$$G(z+1) = (2\pi)^{z/2} e^{-(z+z^2(1+\gamma))/2} \prod_{n=0}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z+z^2/(2n)}$$

(here γ is Euler's constant), which can also be defined recursively by

$$G(z+1) = \Gamma(z)G(z), \qquad G(1) = 1. \tag{1.41}$$

We then use its asymptotic expansion [57, Eq.(5.17.5)]:

$$\log G(z+1) = \frac{z^2 \log z}{2} - \frac{3}{4}z^2 + \frac{\log(2\pi)z}{2} - \frac{\log z}{12} + \zeta'(-1) + O\left(\frac{1}{z^2}\right), \qquad (z \to \infty). \tag{1.42}$$

This asymptotic expansion leads to the appearance of the Riemann zeta function in Theorem 1.2.

Plan of the paper. The rest of this paper is organized as follows. In Sect. 2, we prove Theorem 1.1. Section 2.1 is devoted to deriving asymptotic behaviors of h_j and $\widetilde{h}_j$ using Laplace's method. Then we show Theorem 1.1 (i) on random normal matrices in Sect. 2.2 and Theorem 1.1 (ii) on planar symplectic ensembles in Sect. 2.3. Section 3 is structured in parallel with a goal to show Theorem 1.2 albeit it requires considerably more computations compared to those in Sect. 2. In Sect. 4, we present examples of Theorems 1.1 and 1.2 for the Mittag-Leffler and truncated unitary ensembles whose partition functions can be explicitly expressed in terms of well-known special functions.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Throughout this section, we assume that $r_0 > 0$.

2.1. Asymptotics of the orthogonal norm. We first introduce an auxiliary function V_τ in $(0, \infty)$

$$V_\tau(r) := q(r) - 2\tau \log r. \tag{2.1}$$

With the following choices of $\tau = \tau(j), \widetilde{\tau}(j)$

$$\tau(j) := \frac{j}{N}, \qquad \widetilde{\tau}(j) := \frac{j}{2N}, \tag{2.2}$$

the integrands in $h_j, \widetilde{h}_j$ (1.38) can be expressed in terms of V_τ :

$$r^{2j} e^{-Nq(r)} = e^{-NV_{\tau(j)}(r)}, \qquad r^{2j} e^{-2Nq(r)} = e^{-NV_{\widetilde{\tau}(j)}(r)}.$$

For a radially symmetric potential Q, we represent ΔQ in terms of q as

$$4\Delta Q(z)|_{z=r} = \frac{1}{r}(rq'(r))' = \frac{q'(r)}{r} + q''(r). \tag{2.3}$$

Differentiating (2.1), we have

$$\begin{aligned} V_\tau'(r) &= q'(r) - \frac{2\tau}{r}, \qquad V_\tau''(r) = 4\Delta Q(r) - \frac{1}{r} V_\tau'(r), \\ V_\tau^{(3)}(r) &= 4\partial_r \Delta Q(r) - \frac{4}{r}\Delta Q(r) + \frac{2}{r^2} V_\tau'(r), \\ V_\tau^{(4)}(r) &= 4\partial_r^2 \Delta Q(r) + \frac{12}{r^2}\Delta Q(r) - \frac{4}{r}\partial_r \Delta Q(r) - \frac{6}{r^3} V_\tau'(r). \end{aligned} \tag{2.4}$$

We now set the stage to apply Laplace's method. Since $rq'(r)$ is strictly increasing inside the droplet, for $0 \le \tau \le 1$, there exists a unique number $r(\tau)$ such that

$$V_\tau'(r(\tau)) = 0, \qquad V_\tau''(r(\tau)) > 0. \tag{2.5}$$

Moreover, by (2.3) and the relation (2.5), it follows that

$$\frac{dr(\tau)}{d\tau} = \frac{2}{(rq'(r))'}\Big|_{r=r(\tau)} = \frac{1}{2r(\tau)\Delta Q(r(\tau))} > 0. \tag{2.6}$$

Thus $r(\tau)$ is an increasing function of τ. On the other hand, $r(1) = r_1$, $r(0) = r_0$, where r_0, r_1 are given in (1.12). Therefore, we denote $r_\tau = r(\tau)$ making the notation consistent with (1.12). We also mention here that r_τ corresponds to the outer radius of the so-called "τ-droplet" [45]. By (2.4) and (2.5), r_τ satisfies

$$r_\tau q'(r_\tau) = 2\tau. \tag{2.7}$$

In particular, $q'(r_0) = 0$ and $r_1 q'(r_1) = 2$.

Let

$$\mathfrak{B}_1(r) := -\frac{1}{32}\frac{\partial_r^2 \Delta Q(r)}{(\Delta Q(r))^2} - \frac{19}{96r}\frac{\partial_r \Delta Q(r)}{(\Delta Q(r))^2} + \frac{5}{96}\frac{(\partial_r \Delta Q(r))^2}{\Delta Q(r)^3} + \frac{1}{12r^2}\frac{1}{\Delta Q(r)}. \tag{2.8}$$

Here, the subscript 1 is added to $\mathfrak{B}$ to emphasize that this function appears as the first subleading term of the asymptotic expansion of orthonormal polynomials, see Lemma 2.1 below. Indeed, function $\mathfrak{B}_1$ is closely related to function $\mathfrak{B}_{\tau,1}$ in [45, Theorem 1.3].

Lemma 2.1. *As $N \to \infty$, the following holds.*

- *For each j with $0 \le j \le N-1$,*

$$h_j = N^{-\frac{1}{2}} e^{-NV_{\tau(j)}(r_{\tau(j)})} \Big(\frac{2\pi r_{\tau(j)}^2}{\Delta Q(r_{\tau(j)})}\Big)^{\frac{1}{2}} \Big(1 + \frac{1}{N}\mathfrak{B}_1(r_{\tau(j)}) + O(N^{-2})\Big).$$

- *For each j with $0 \le j \le 2N-1$,*

$$\widetilde{h}_j = (2N)^{-\frac{1}{2}} e^{-2NV_{\widetilde{\tau}(j)}(r_{\widetilde{\tau}(j)})} \Big(\frac{2\pi r_{\widetilde{\tau}(j)}^2}{\Delta Q(r_{\widetilde{\tau}(j)})}\Big)^{\frac{1}{2}} \Big(1 + \frac{1}{2N}\mathfrak{B}_1(r_{\widetilde{\tau}(j)}) + O(N^{-2})\Big).$$

Here, the error terms are uniform for j.

Proof. It suffices to show the first assertion as the second one follows by replacing N with $2N$.

Write $\delta_N := \log N/\sqrt{N}$. As seen in (2.1), (2.5) and (2.6), the function V_τ has a global minimum at $r = r_\tau$ and $r_{\tau(j+\frac{1}{2})} - r_{\tau(j)} = O(N^{-1})$. As $N \to \infty$, uniformly for j with $0 \le j \le N-1$ we have

$$\begin{aligned} V_{\tau(j+\frac{1}{2})}(r) &\ge V_{\tau(j+\frac{1}{2})}(r_{\tau(j)} + \delta_N) \\ &= V_{\tau(j+\frac{1}{2})}(r_{\tau(j+\frac{1}{2})}) + \frac{1}{2} V''_{\tau(j+\frac{1}{2})}(r_{\tau(j+\frac{1}{2})})(r_{\tau(j)} + \delta_N - r_{\tau(j+\frac{1}{2})})^2 + O(\delta_N^3) \end{aligned}$$

for all r with $r > r_{\tau(j)} + \delta_N$ since $rq'(r)$ is increasing in $(0,\infty)$. A similar estimate holds for $r < r_{\tau(j)} - \delta_N$. Thus we deduce from the estimate

$$V_{\tau(j+\frac{1}{2})}(r_{\tau(j+\frac{1}{2})}) - V_{\tau(j)}(r_{\tau(j)}) = O(N^{-1})$$

that there exists a positive number $c > 0$ such that for all $j \in \{0, 1, \cdots, N-1\}$ and r with $|r - r_{\tau(j)}| > \delta_N$

$$V_{\tau(j+\frac{1}{2})}(r) - V_{\tau(j)}(r_{\tau(j)}) \ge c\,\delta_N^2. \tag{2.9}$$

We split h_j in (1.38) into two integrals

$$\begin{aligned} h_j &= \int_0^\infty e^{-NV_{\tau(j)}(r)} 2r\,dr \\ &= \int_{|r-r_{\tau(j)}|<\delta_N} e^{-NV_{\tau(j)}(r)} 2r\,dr + \int_{|r-r_{\tau(j)}|>\delta_N} e^{-NV_{\tau(j)}(r)} 2r\,dr. \end{aligned}$$

Using (1.11), we choose sufficiently large $M > 0$ such that

$$\int_0^\infty \Big(r^2 e^{-q(r)}\Big)^M r\,dr < +\infty. \tag{2.10}$$

We then use (2.9) and (2.10) to find an error estimate for the second integral

$$\begin{aligned} &\int_{|r-r_{\tau(j)}|>\delta_N} e^{-NV_{\tau(j)}(r)} 2r\,dr \\ &= e^{-NV_{\tau(j)}(r_{\tau(j)})} \int_{|r-r_{\tau(j)}|>\delta_N} e^{-N\big(V_{\tau(j)}(r) - V_{\tau(j)}(r_{\tau(j)})\big)}\, 2r\,dr \\ &\le e^{-NV_{\tau(j)}(r_{\tau(j)})} e^{-c(N-M)\delta_N^2} \int_{|r-r_{\tau(j)}|>\delta_N} e^{-M\big(V_{\tau(j)}(r) - V_{\tau(j)}(r_{\tau(j)})\big)}\, 2r\,dr \\ &= e^{-NV_{\tau(j)}(r_{\tau(j)})} \epsilon_N, \end{aligned} \tag{2.11}$$

where $\epsilon_N = O(e^{-c(\log N)^2})$ for some $c > 0$ and the O-constants are bounded uniformly for all j with $0 \le j \le N-1$. We deduce from the asymptotic expansion of $V_\tau(r)$ near the critical point r_τ and (2.11) that

$$h_j = e^{-NV_{\tau(j)}(r_{\tau(j)})} \times \Big(\int_{-\delta_N}^{\delta_N} e^{-2N\Delta Q(r_{\tau(j)})r^2} e^{-N(\frac{1}{3!}V^{(3)}_{\tau(j)}(r_{\tau(j)})r^3 + \frac{1}{4!}V^{(4)}_{\tau(j)}(r_{\tau(j)})r^4 + O(r^5))} 2(r_{\tau(j)} + r)\, dr + \epsilon_N \Big).$$

A change of variables gives that

$$e^{NV_{\tau(j)}(r_{\tau(j)})}h_j = \frac{2\,r_{\tau(j)}}{\sqrt{N}} \int_{-\sqrt{N}\delta_N}^{\sqrt{N}\delta_N} e^{-2\Delta Q(r_{\tau(j)})r^2 - \frac{1}{\sqrt{N}}\frac{1}{3!}V^{(3)}_{\tau(j)}(r_{\tau(j)})r^3 - \frac{1}{N}\frac{1}{4!}V^{(4)}_{\tau(j)}(r_{\tau(j)})r^4 + O(N^{-\frac{3}{2}}r^5)} \Big(1 + \frac{r}{r_{\tau(j)}\sqrt{N}}\Big)\, dr + \epsilon_N.$$

Using the Taylor series expansion of the function

$$e^{-\frac{1}{\sqrt{N}}\frac{1}{3!}V^{(3)}_{\tau(j)}(r_{\tau(j)})r^3 - \frac{1}{N}\frac{1}{4!}V^{(4)}_{\tau(j)}(r_{\tau(j)})r^4},$$

we have the asymptotic expansion

$$\frac{\sqrt{N}}{2r_{\tau(j)}}e^{NV_{\tau(j)}(r_{\tau(j)})}h_j = \int_{-\infty}^{\infty} e^{-2\Delta Q(r_{\tau(j)})r^2}\Big[1 - \frac{1}{N}\Big(\frac{1}{6r_{\tau(j)}}V^{(3)}_{\tau(j)}(r_{\tau(j)}) + \frac{1}{24}V^{(4)}_{\tau(j)}(r_{\tau(j)})\Big)r^4 + \frac{1}{N}\frac{1}{72}(V^{(3)}_{\tau(j)}(r_{\tau(j)}))^2 r^6\Big]\, dr + O(N^{-2})$$

since the odd terms vanish in the integral, leaving only the even terms. Note that the O-terms are uniform for $j \in \{0, 1, \cdots, N-1\}$.

Combining (2.4) with the elementary Gaussian integrals

$$\int_{\mathbb{R}} e^{-2ar^2}\, dr = \sqrt{\frac{\pi}{2}}\frac{1}{a^{1/2}}, \qquad \int_{\mathbb{R}} e^{-2ar^2} r^4\, dr = \sqrt{\frac{\pi}{2}}\frac{3}{16}\frac{1}{a^{5/2}},$$
$$\int_{\mathbb{R}} e^{-2ar^2} r^6\, dr = \sqrt{\frac{\pi}{2}}\frac{15}{64}\frac{1}{a^{7/2}},$$

we obtain the desired asymptotic behavior after some straightforward computations. □

The following elementary integration will be helpful later.

Lemma 2.2. *We have*

$$\int_S \mathfrak{B}_1\, d\mu_Q = F_Q[\mathbb{A}_{r_0,r_1}] - \frac{1}{4}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(r_0)}\Big) + \frac{1}{3}\log\Big(\frac{r_1}{r_0}\Big),$$

where $F_Q[\mathbb{A}_{r_0,r_1}]$ *is given in* (1.16).

Proof. By (2.8) and (1.10), we have

$$\int_S \mathfrak{B}_1 \, d\mu_Q = \frac{1}{6}\log\Big(\frac{r_1}{r_0}\Big) - \frac{19}{48}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(r_0)}\Big) - \frac{1}{16}\int_{r_0}^{r_1}\Big[\frac{\partial_r^2 \Delta Q(r)}{\Delta Q(r)} - \frac{5}{3}\Big(\frac{\partial_r \Delta Q(r)}{\Delta Q(r)}\Big)^2\Big]\, r\, dr.$$

Then the lemma follows using integration by parts

$$\int_{r_0}^{r_1}\Big[\frac{\partial_r^2 \Delta Q(r)}{\Delta Q(r)} - \frac{5}{3}\Big(\frac{\partial_r \Delta Q(r)}{\Delta Q(r)}\Big)^2\Big]\, r\, dr = r_1\,\frac{(\partial_r \Delta Q)(r_1)}{\Delta Q(r_1)} - r_0\,\frac{(\partial_r \Delta Q)(r_0)}{\Delta Q(r_0)} + \log\Big(\frac{\Delta Q(r_0)}{\Delta Q(r_1)}\Big) - \frac{2}{3}\int_{r_0}^{r_1}\Big(\frac{\partial_r \Delta Q(r)}{\Delta Q(r)}\Big)^2 r\, dr. \tag{2.12}$$

□

2.2. Random normal matrix ensemble. In this subsection, we prove Theorem 1.1 (i). By Lemma 2.1, we have

$$\log h_j = -N V_{\tau(j)}(r_{\tau(j)}) + \frac{1}{2}\Big(\log(2\pi r_{\tau(j)}^2) - \log N - \log \Delta Q(r_{\tau(j)})\Big) + \frac{1}{N}\mathfrak{B}_1(r_{\tau(j)}) + O(N^{-2}) \tag{2.13}$$

as $N \to \infty$ uniformly for $j \in \{0, 1, \cdots, N-1\}$. In the following lemmas, we analyze the asymptotic behavior of the partial sum of each term in (2.13).

Lemma 2.3. *As $N \to \infty$, we have*

$$\sum_{j=0}^{N-1} V_{\tau(j)}(r_{\tau(j)}) = N I_Q[\mu_Q] - U_{\mu_Q}(0) + \frac{1}{6N}\log\Big(\frac{r_0}{r_1}\Big) + O(N^{-3}).$$

Proof. The sequence τ in (2.2) can be extended to the function on $[0, N]$: $\tau(t) = t/N$. Using the Euler–Maclaurin formula (1.39) and (2.2), we have

$$\sum_{j=0}^{N-1} V_{\tau(j)}(r_{\tau(j)}) = \int_0^N V_{\tau(t)}(r_{\tau(t)})\, dt - \frac{1}{2}\Big(V_{\tau(N)}(r_{\tau(N)}) - V_{\tau(0)}(r_{\tau(0)})\Big) + \frac{1}{12}\Big[\partial_t V_{\tau(t)}(r_{\tau(t)})\Big|_{t=N} - \partial_t V_{\tau(t)}(r_{\tau(t)})\Big|_{t=0}\Big] + O(N^{-3}). \tag{2.14}$$

Here, we also used the second Bernoulli number $B_2 = 1/6$, which can be easily seen from the definition (1.40). For the first term on the right-hand side of (2.14), the change of variables $s = r_{\tau(t)}$, the definition (2.1) of V_τ, the definition (2.2) of τ, the formula (2.6) of $dr/d\tau$, and the eq (2.7) $r_\tau q'(r_\tau) = 2\tau$ for r_τ give that

$$\frac{1}{N}\int_0^N V_{\tau(t)}(r_{\tau(t)})\, dt = 2\int_{r_0}^{r_1}\big(q(s) - s q'(s)\log s\big) s\,\Delta Q(s)\, ds. \tag{2.15}$$

We use the polar coordinate system to represent the first term on the right-hand side of the above equation as

$$2\int_{r_0}^{r_1} sq(s)\Delta Q(s)\,ds = \int_S Q\cdot\Delta Q\,dA.$$

By (2.3), the method of integration by parts, and the relation $r_1 q'(r_1)=2$, $q'(r_0)=0$ (see (2.7)), the second term in (2.15) is simplified to

$$\begin{aligned}-2\int_{r_0}^{r_1} s^2 q'(s)\log s\,\Delta Q(s)\,ds &= -\frac{1}{4}\int_{r_0}^{r_1}\log s\cdot((sq'(s))^2)'\,ds\\ &= -\log r_1 + \frac{1}{4}\int_{r_0}^{r_1} s(q'(s))^2\,ds.\end{aligned}$$

Applying the method of integration by parts again to the last integral,

$$\frac{1}{4}\int_{r_0}^{r_1} s(q'(s))^2\,ds = \frac{1}{2}q(r_1) - \frac{1}{4}\int_{r_0}^{r_1} q(s)(sq'(s))'\,ds.$$

Using the formula (1.13) of $I_Q[\mu_Q]$, the representation (2.3) of ΔQ in terms of q, and the equation (2.7) $r_\tau q'(r_\tau)=2\tau$ for r_τ, we have

$$\begin{aligned}\frac{1}{N}\int_0^N V_{\tau(t)}(r_{\tau(t)})\,dt &= \int_S Q\cdot\Delta Q\,dA - \log r_1 + \frac{1}{2}q(r_1) - \frac{1}{4}\int_{r_0}^{r_1} q(s)(sq'(s))'\,ds\\ &= \frac{1}{2}\int_S Q\cdot\Delta Q\,dA - \log r_1 + \frac{1}{2}q(r_1) = I_Q[\mu_Q].\end{aligned} \tag{2.16}$$

For the next term on the right-hand side of (2.14), we observe that

$$V_{\tau(N)}(r_{\tau(N)}) - V_{\tau(0)}(r_{\tau(0)}) = V_1(r_1) - V_0(r_0) = q(r_1) - q(r_0) - 2\log r_1 = 2U_{\mu_Q}(0), \tag{2.17}$$

where we have used in (1.15). To analyze the remaining term in (2.14), we use the Leibniz rule and obtain

$$\begin{aligned}\partial_t V_{\tau(t)}(r_{\tau(t)})\big|_{t=N} &= \frac{1}{N}\Big[\partial_\tau(q(r_\tau) - 2\log r_\tau)\big|_{\tau=1} - 2\log r_1\Big],\\ \partial_t V_{\tau(t)}(r_{\tau(t)})\big|_{t=0} &= \frac{1}{N}\Big[\partial_\tau q(r_\tau)\big|_{\tau=0} - 2\log r_0\Big].\end{aligned}$$

It follows from $r_1 q'(r_1)=2$, $r_0 q'(r_0)=0$ and the formula (2.6) of $dr/d\tau$ that

$$\begin{aligned}&\partial_t V_{\tau(t)}(r_{\tau(t)})\big|_{t=N} - \partial_t V_{\tau(t)}(r_{\tau(t)})\big|_{t=0}\\ &\quad = \frac{1}{N}\Big[\frac{q'(r_1)}{2r_1\Delta Q(r_1)} - 2\log r_1 - \frac{1}{r_1^2\Delta Q(r_1)} - \frac{q'(r_0)}{2r_0\Delta Q(r_0)} + 2\log r_0\Big] = \frac{2}{N}\log\Big(\frac{r_0}{r_1}\Big).\end{aligned} \tag{2.18}$$

Combining (2.14), (2.16), (2.17), and (2.18), the proof is complete. □

Lemma 2.4. *As $N \to \infty$, we have*

$$\sum_{j=0}^{N-1} \log \Delta Q(r_{\tau(j)}) = N E_Q[\mu_Q] - \frac{1}{2} \log\Big(\frac{\Delta Q(r_1)}{\Delta Q(r_0)}\Big) + O(N^{-1}), \tag{2.19}$$

and

$$\sum_{j=0}^{N-1} \log r_{\tau(j)} = -N U_{\mu_Q}(0) - \frac{1}{2} \log\Big(\frac{r_1}{r_0}\Big) + O(N^{-1}). \tag{2.20}$$

Proof. As in Lemma 2.3, we apply the Euler–Maclaurin formula (1.39) and obtain

$$\begin{aligned}\sum_{j=0}^{N-1} \log \Delta Q(r_{\tau(j)}) &= \int_0^N \log \Delta Q(r_{\tau(t)})\, dt - \frac{1}{2} \log\Big(\frac{\Delta Q(r_1)}{\Delta Q(r_0)}\Big) \\ &\quad + \frac{1}{12}\Big[\partial_t \log \Delta Q(r_{\tau(t)})\big|_{t=N} - \partial_t \log \Delta Q(r_{\tau(t)})\big|_{t=0}\Big] + O(N^{-3}).\end{aligned} \tag{2.21}$$

It follows from a change of variables and (2.6) that the first term on the right-hand side of (2.21) gives the entropy term

$$\int_0^N \log \Delta Q(r_{\tau(t)})\, dt = N \int_{r_0}^{r_1} \log \Delta Q(s) 2s \Delta Q(s)\, ds = N \int_S \log \Delta Q\, d\mu_Q.$$

By the chain rule and the formula (2.6) of $dr/d\tau$ again, we also observe that

$$\begin{aligned}\partial_t \log \Delta Q(r_{\tau(t)})\big|_{t=N} - \partial_t \log \Delta Q(r_{\tau(t)})\big|_{t=0} &= \frac{1}{2N}\Big(\frac{(\partial_r \Delta Q)(r_1)}{r_1(\Delta Q(r_1))^2} - \frac{(\partial_r \Delta Q)(r_0)}{r_0(\Delta Q(r_0))^2}\Big) \\ &= O(N^{-1}).\end{aligned}$$

Combining all of the above, we obtain (2.19). The equation (2.20) follows similarly. □

We are now ready to prove the first assertion of Theorem 1.1.

Proof of Theorem 1.1 (i). Combining Lemmas 2.3 and 2.4 with (2.13), we obtain

$$\begin{aligned}\sum_{j=0}^{N-1} \log h_j &= -N^2 I_Q[\mu_Q] - \frac{N}{2} \log N + N\Big(\frac{\log(2\pi)}{2} - \frac{1}{2} E_Q[\mu_Q]\Big) \\ &\quad - \frac{1}{3} \log\Big(\frac{r_1}{r_0}\Big) + \frac{1}{4} \log\Big(\frac{\Delta Q(r_1)}{\Delta Q(r_0)}\Big) + \int_S \mathfrak{B}_1\, d\mu_Q + O(N^{-1}),\end{aligned}$$

where $\mathfrak{B}_1$ is given by (2.8). Then the desired asymptotic expansion (1.17) follows from

$$\log N! = N \log N - N + \frac{1}{2} \log N + \frac{1}{2} \log(2\pi) + O(\frac{1}{N}), \qquad (N \to \infty) \tag{2.22}$$

(see e.g. [57, Eq.(5.11.1)]) and Lemma 2.2. □

2.3. Planar symplectic ensemble. In this subsection, we prove Theorem 1.1 (ii).
Recall that by Lemma 2.1,

$$\begin{aligned}\log \widetilde{h}_j =& -2NV_{\widetilde{\tau}(j)}(r_{\widetilde{\tau}(j)}) + \frac{1}{2}\Big(\log(\pi r^2_{\widetilde{\tau}(j)}) - \log N - \log \Delta Q(r_{\widetilde{\tau}(j)})\Big) \\ &+ \frac{1}{2N}\mathfrak{B}_1(r_{\widetilde{\tau}(j)}) + O(N^{-2})\end{aligned} \tag{2.23}$$

as $N \to \infty$ uniformly for $j \in \{0, 1, \cdots, N-1\}$. In the following lemmas, we derive asymptotic expansions for the partial sum of each term on the right-hand side of (2.23). We obtain the following as a counterpart of Lemma 2.3.

Lemma 2.5. *As $N \to \infty$, we have*

$$\sum_{j=0}^{N-1} V_{\widetilde{\tau}(2j+1)}(r_{\widetilde{\tau}(2j+1)}) = NI_Q[\mu_Q] - \frac{1}{12N}\log\Big(\frac{r_0}{r_1}\Big) + O(N^{-3}). \tag{2.24}$$

Proof. Applying Lemma 2.3 by replacing N with $2N$, we have

$$\sum_{j=0}^{2N-1} V_{\widetilde{\tau}(j)}(r_{\widetilde{\tau}(j)}) = 2NI_Q[\mu_Q] - U_{\mu_Q}(0) + \frac{1}{12N}\log\Big(\frac{r_0}{r_1}\Big) + O(N^{-3}). \tag{2.25}$$

It follows from $V_{\widetilde{\tau}(2j)}(r_{\widetilde{\tau}(2j)}) = V_{\tau(j)}(r_{\tau(j)})$ and Lemma 2.3 that

$$\sum_{j=0}^{N-1} V_{\widetilde{\tau}(2j)}(r_{\widetilde{\tau}(2j)}) = NI_Q[\mu_Q] - U_{\mu_Q}(0) + \frac{1}{6N}\log\Big(\frac{r_0}{r_1}\Big) + O(N^{-3}). \tag{2.26}$$

Then (2.24) follows from (2.25) and (2.26). □

Lemma 2.6. *As $N \to \infty$, we have*

$$\sum_{j=0}^{N-1} \log \Delta Q(r_{\widetilde{\tau}(2j+1)}) = NE_Q[\mu_Q] + O(N^{-1}),$$

and

$$\sum_{j=0}^{N-1} \log r_{\widetilde{\tau}(2j+1)} = -NU_{\mu_Q}(0) + O(N^{-1}).$$

Proof. By Lemma 2.4, we have

$$\sum_{j=0}^{2N-1} \log \Delta Q(r_{\widetilde{\tau}(j)}) = 2NE_Q[\mu_Q] - \frac{1}{2}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(r_0)}\Big) + O(N^{-1}),$$

$$\sum_{j=0}^{2N-1} \log r_{\widetilde{\tau}(j)} = -2NU_{\mu_Q}(0) - \frac{1}{2}\log\Big(\frac{r_1}{r_0}\Big) + O(N^{-1}).$$

Along the lines of Lemma 2.4, one can also show that

$$\sum_{j=0}^{N-1} \log \Delta Q(r_{\tilde\tau(2j)}) = N E_Q[\mu_Q] - \frac{1}{2}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(r_0)}\Big) + O(N^{-1}),$$
$$\sum_{j=0}^{N-1} \log r_{\tilde\tau(2j)} = -N U_{\mu_Q}(0) - \frac{1}{2}\log\Big(\frac{r_1}{r_0}\Big) + O(N^{-1}).$$

This completes the proof. □

We now prove the second assertion of Theorem 1.1.

Proof of Theorem 1.1 (ii). By Lemmas 2.5 and 2.6, we have

$$\sum_{j=0}^{N-1} \log(2\widetilde{h}_{2j+1}) = -2N^2 I_Q[\mu_Q] - \frac{N}{2}\log N + \frac{N}{2}\log 2 + \frac{N}{2}\log(2\pi)$$
$$- N U_{\mu_Q}(0) - \frac{N}{2} E_Q[\mu_Q] + \frac{1}{2}\int_S \mathfrak{B}_1\, d\mu_Q + \frac{1}{6}\log\Big(\frac{r_0}{r_1}\Big) + O(N^{-1}).$$

Combining (1.37), (2.22), and Lemma 2.2 completes the proof. □

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Throughout this section, we let $r_0 = 0$.

3.1. Asymptotics of the orthogonal norm. Let $\delta_N = N^{-1/2}\log N$ and $m_N = N^{\epsilon}$ for $0 < \epsilon < 1/5$. As explained in Sect. 1.3, for the disc droplet case, the asymptotic behaviors of h_j and $\widetilde{h}_j$ depend on whether the degree j is sufficiently small or not, see Lemmas 3.1 and 3.2, respectively.

Lemma 3.1. *As $N \to \infty$, the following holds.*

- *For $j = 0, 1, \dots, m_N - 1$, we have*

$$\log h_j = -Nq(0) - (j+1)\log\Big(\frac{N}{2}q''(0)\Big) + \log j! + O\big(N^{-\frac{1}{2}}(j+1)^{\frac{3}{2}}(\log N)^3\big).$$

- *For $j = 0, 1, \dots, 2m_N - 1$, we have*

$$\log \widetilde{h}_j = -2Nq(0) - (j+1)\log(Nq''(0)) + \log j! + O\big(N^{-\frac{1}{2}}(j+1)^{\frac{3}{2}}(\log N)^3\big).$$

Here, the O-constants are uniformly bounded for j.

Proof. The second assertion is an immediate consequence of the first one. Recall that $\tau(j) = j/N$. Let

$$r_\tau^* := r_\tau \cdot \log N,$$

where r_τ is given in (2.5) or (2.7), $r_\tau q'(r_\tau) = 2\tau$. We consider the decomposition

$$h_j = \int_0^\infty 2r^{2j+1} e^{-Nq(r)}\, dr = \int_0^{r^*_{\tau(j+\frac{1}{2})}} 2r^{2j+1} e^{-Nq(r)}\, dr + \int_{r^*_{\tau(j+\frac{1}{2})}}^\infty 2r^{2j+1} e^{-Nq(r)}\, dr.$$

Due to strict subharmonicity of Q in a neighborhood of the droplet, r_τ defined in (2.5) satisfies $r_\tau = O(\tau^{\frac{1}{2}})$ as $\tau \to 0$. Since the function V_τ in (2.1) has a global minimum at $r = r_\tau$ and increases in (r_τ, ∞), for all $r > r^*_{\tau(j+\frac{1}{2})}$, we have

$$\begin{aligned} V_{\tau(j+\frac{1}{2})}(r) \geq V_{\tau(j+\frac{1}{2})}(r^*_{\tau(j+\frac{1}{2})}) &= q(r^*_{\tau(j+\frac{1}{2})}) - \frac{2j+1}{N} \log(r^*_{\tau(j+\frac{1}{2})}) \\ &= q(0) + \frac{1}{2} q''(0) (r^*_{\tau(j+\frac{1}{2})})^2 - \frac{2j+1}{N} \log(r^*_{\tau(j+\frac{1}{2})}) + O(r^*_{\tau(j+\frac{1}{2})})^3, \end{aligned} \tag{3.1}$$

where the O-constants are uniform for $j \in \{0, 1, \cdots, m_N - 1\}$. Here, we also have used $q'(0) = 0$, which follows from (2.3) and the fact that $\Delta Q(z) \in (0, \infty)$ near the origin. Using (3.1), it follows that

$$\begin{aligned} e^{Nq(0)} \int_{r^*_{\tau(j+\frac{1}{2})}}^\infty 2r^{2j+1} e^{-Nq(r)}\, dr &= \int_{r^*_{\tau(j+\frac{1}{2})}}^\infty 2\, e^{-N(V_{\tau(j+\frac{1}{2})}(r) - q(0))}\, dr \\ &\leq e^{-c_1 (N-M) (r^*_{\tau(j+\frac{1}{2})})^2} (r^*_{\tau(j+\frac{1}{2})})^{(2j+1)\frac{N-M}{N}} \int_{r^*_{\tau(j+\frac{1}{2})}}^\infty 2\, e^{-M(V_{\tau(j+\frac{1}{2})}(r) - q(0))}\, dr =: \epsilon_N(j) \end{aligned}$$

for some $c_1 > 0$. Here, M is given by (2.10). Since $r_\tau = O(\tau^{\frac{1}{2}})$, there exists $c_2 > 0$ such that

$$\begin{aligned} \epsilon_N(j) &= O\Big((r^*_{\tau(j+\frac{1}{2})})^{(2j+1)\frac{N-M}{N}} \cdot e^{-c_2 (j+\frac{1}{2}) (\log N)^2} \Big) \\ &= O\Big(\big(\tfrac{2j+1}{2N} (\log N)^2 \big)^{(2j+1)\frac{N-M}{2N}} e^{-c_2 (j+\frac{1}{2}) (\log N)^2} \Big). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} h_j &= \int_0^\infty 2r^{2j+1} e^{-Nq(r)}\, dr = \int_0^{r^*_{\tau(j+\frac{1}{2})}} 2r^{2j+1} e^{-Nq(r)}\, dr + e^{-Nq(0)} \epsilon_N(j) \\ &= \int_0^{r^*_{\tau(j+\frac{1}{2})}} 2r^{2j+1} e^{-N(q(0) + \frac{1}{2} q''(0) r^2) + O(N (r^*_{\tau(j+\frac{1}{2})})^3)}\, dr + e^{-Nq(0)} \epsilon_N(j) \\ &= e^{-Nq(0)} \Big[N^{-(j+1)} \int_0^\infty 2r^{2j+1} e^{-\frac{1}{2} q''(0) r^2}\, dr\, \big(1 + O(N (r^*_{\tau(j+\frac{1}{2})})^3)\big) + \epsilon_N(j) \Big] \\ &= e^{-Nq(0)} \Big[N^{-(j+1)} \Gamma(j+1) \Big(\frac{1}{2} q''(0) \Big)^{-(j+1)} \Big(1 + O\Big(N^{-\frac{1}{2}} (j + \tfrac{1}{2})^{\frac{3}{2}} (\log N)^3 \Big) \Big) + \epsilon_N(j) \Big], \end{aligned}$$

where the O-constants are uniformly bounded for j and $\epsilon_N(j)$ is negligible. This completes the proof. □

Recall that the sequences $\tau(j)$ and $\widetilde{\tau}(j)$ are given by (2.2): $\tau(j) := j/N$, $\widetilde{\tau}(j) := j/(2N)$ and the function $\mathfrak{B}_1$ is given by (2.8):

$$\mathfrak{B}_1(r) := -\frac{1}{32}\frac{\partial_r^2 \Delta Q(r)}{(\Delta Q(r))^2} - \frac{19}{96r}\frac{\partial_r \Delta Q(r)}{(\Delta Q(r))^2} + \frac{5}{96}\frac{(\partial_r \Delta Q(r))^2}{\Delta Q(r)^3} + \frac{1}{12r^2}\frac{1}{\Delta Q(r)}.$$

Recall also that V_τ is given by (2.1) $V_\tau(r) := q(r) - 2\tau \log r$ and r_τ is given by (2.5) or (2.7) $r_\tau q'(r_\tau) = 2\tau$. As a counterpart of Lemma 2.1, we show the following lemma.

Lemma 3.2. *As $N \to \infty$, the following holds.*

- *For $j = m_N, m_N + 1, \dots, N - 1$, we have*

$$\log h_j = -NV_{\tau(j)}(r_{\tau(j)}) + \frac{1}{2}\log\Big(\frac{2\pi r_{\tau(j)}^2}{N\Delta Q(r_{\tau(j)})}\Big) + \frac{1}{N}\mathfrak{B}_1(r_{\tau(j)}) + O(j^{-\frac{3}{2}}(\log N)^\alpha).$$

- *For $j = 2m_N, 2m_N + 1, \dots, 2N - 1$, we have*

$$\log \widetilde{h}_j = -2NV_{\widetilde{\tau}(j)}(r_{\widetilde{\tau}(j)}) + \frac{1}{2}\log\Big(\frac{\pi r_{\widetilde{\tau}(j)}^2}{N\Delta Q(r_{\widetilde{\tau}(j)})}\Big) + \frac{1}{2N}\mathfrak{B}_1(r_{\widetilde{\tau}(j)}) + O(j^{-\frac{3}{2}}(\log N)^\alpha).$$

Here the O-constants are uniformly bounded for j and $\alpha > 0$ is a small constant.

Proof. This lemma can be shown in a similar way to Lemma 2.1. Recall that r_τ satisfies $r_\tau = O(\tau^{\frac{1}{2}})$ as $\tau \to 0$. Note that for $d \geq 1$

$$|V_\tau^{(d)}(r_\tau)| = |q^{(d)}(r_\tau) + (-1)^d\, 2\tau\,(d-1)!\, r_\tau^{-d}| \leq C_1(1 + \tau^{-\frac{d}{2}+1}),$$

where $C_1 > 0$ is a constant that can be taken uniformly for all τ. We now split the integral for h_j by

$$\begin{aligned} h_j &= \int_0^\infty 2r^{2j+1}e^{-Nq(r)}\,dr = \int_0^\infty 2re^{-NV_{\tau(j)}(r)}\,dr \\ &= \int_{r_{\tau(j)}-\delta_N}^{r_{\tau(j)}+\delta_N} 2re^{-NV_{\tau(j)}(r)}\,dr + \int_{|r-r_{\tau(j)}|>\delta_N} 2re^{-NV_{\tau(j)}(r)}\,dr \end{aligned}$$

and first compute the integral over the outer region. For $m_N \leq j < N$, we have

$$r_{\tau(j+\frac{1}{2})} - r_{\tau(j)} = O((jN)^{-\frac{1}{2}}).$$

Since V_τ is increasing in (r_τ, ∞), the Taylor series expansion for V_τ gives

$$\begin{aligned} V_{\tau(j+\frac{1}{2})}(r) &\geq V_{\tau(j+\frac{1}{2})}(r_{\tau(j)} + \delta_N) \\ &= V_{\tau(j+\frac{1}{2})}(r_{\tau(j+\frac{1}{2})}) + \frac{1}{2}V''_{\tau(j+\frac{1}{2})}(r_{\tau(j+\frac{1}{2})})\big(r_{\tau(j)} + \delta_N - r_{\tau(j+\frac{1}{2})}\big)^2 \\ &\quad + O(\tau(j)^{-\frac{1}{2}}\delta_N^3) \end{aligned} \tag{3.2}$$

for all $r > r_{\tau(j)} + \delta_N$. Here, $O(\tau(j)^{-\frac{1}{2}}\delta_N^3) = O(N^{-1}j^{-\frac{1}{2}}(\log N)^3)$ and the O-constants are uniformly bounded for $j \in \{m_N, \cdots, N-1\}$. Similarly, since V_τ is decreasing in $(0, r_\tau)$, we have

$$\begin{aligned} V_{\tau(j+\frac{1}{2})}(r) &\geq V_{\tau(j+\frac{1}{2})}(r_{\tau(j)} - \delta_N) \\ &= V_{\tau(j+\frac{1}{2})}(r_{\tau(j+\frac{1}{2})}) + \frac{1}{2}V''_{\tau(j+\frac{1}{2})}(r_{\tau(j+\frac{1}{2})})\big(r_{\tau(j)} - \delta_N - r_{\tau(j+\frac{1}{2})}\big)^2 \\ &\quad + O(\tau(j)^{-\frac{1}{2}}\delta_N^3) \end{aligned} \tag{3.3}$$

for all $r < r_{\tau(j)} - \delta_N$. Using the Taylor series for V_τ again, we have

$$\begin{aligned} V_{\tau(j+\frac{1}{2})}(r_{\tau(j+\frac{1}{2})}) - V_{\tau(j)}(r_{\tau(j)}) &= V_{\tau(j)}(r_{\tau(j+\frac{1}{2})}) - V_{\tau(j)}(r_{\tau(j)}) - \frac{1}{N}\log r_{\tau(j+\frac{1}{2})} \\ &= O((jN)^{-1}) - \frac{1}{N}\log r_{\tau(j+\frac{1}{2})}, \end{aligned} \tag{3.4}$$

where the error term is uniform for j. Thus, it follows from (3.2), (3.3), and (3.4) that

$$\begin{aligned} &e^{NV_{\tau(j)}(r_{\tau(j)})}\int_{|r-r_{\tau(j)}|>\delta_N} 2r^{2j+1}e^{-Nq(r)}\,dr = \int_{|r-r_{\tau(j)}|>\delta_N} 2e^{-N\left(V_{\tau(j+\frac{1}{2})}(r)-V_{\tau(j)}(r_{\tau(j)})\right)}\,dr \\ &\leq e^{-c_1(\log N)^2}e^{\frac{N-M}{N}\log r_{\tau(j+\frac{1}{2})}}\int_{|r-r_{\tau(j)}|>\delta_N} 2e^{-M\left(V_{\tau(j+\frac{1}{2})}(r)-V_{\tau(j)}(r_{\tau(j)})\right)}\,dr \\ &= O(e^{-c_2(\log N)^2}) \end{aligned}$$

for some $c_1, c_2 > 0$. Here M is given by (2.10). For the integral near the critical point $r_{\tau(j)}$, we use the Taylor series expansion to obtain

$$\begin{aligned} &\int_{r_{\tau(j)}-\delta_N}^{r_{\tau(j)}+\delta_N} 2re^{-NV_{\tau(j)}(r)}\,dr = e^{-NV_{\tau(j)}(r_{\tau(j)})} \\ &\quad\int_{-\delta_N}^{\delta_N} 2(r_{\tau(j)}+t)\,e^{-N(\frac{1}{2}V''_{\tau(j)}(r_{\tau(j)})t^2+\frac{1}{6}V^{(3)}_{\tau(j)}(r_{\tau(j)})t^3+\frac{1}{24}V^{(4)}_{\tau(j)}(r_{\tau(j)})t^4+O(\tau(j)^{-\frac{3}{2}}|t|^5))}dt \\ &= e^{-NV_{\tau(j)}(r_{\tau(j)})}\frac{1}{\sqrt{N}}\int_{-\sqrt{N}\delta_N}^{\sqrt{N}\delta_N} 2e^{-2\Delta Q(r_{\tau(j)})t^2}\left(r_{\tau(j)} + \frac{t}{\sqrt{N}}\right) \\ &\quad\left(1 - \frac{V^{(3)}_{\tau(j)}(r_{\tau(j)})}{6\sqrt{N}}t^3 - \frac{V^{(4)}_{\tau(j)}(r_{\tau(j)})}{24N}t^4 + \frac{(V^{(3)}_{\tau(j)}(r_{\tau(j)}))^2}{72N}t^6 + \epsilon'_{N,1}\right)dt \end{aligned}$$

where $\epsilon'_{N,1} = O(j^{-\frac{3}{2}}(\log N)^\alpha)$ for some $\alpha > 0$ and the O-constant is uniformly bounded for $j \in \{m_N, \cdots, N-1\}$. Combining the all of the above, we obtain

$$h_j = e^{-NV_{\tau(j)}(r_{\tau(j)})}\left[\frac{1}{\sqrt{N}}\left(\frac{2\pi r_{\tau(j)}^2}{\Delta Q(r_{\tau(j)})}\right)^{\frac{1}{2}}\left(1 + \frac{1}{N}\mathfrak{B}_1(r_{\tau(j)}) + \epsilon_{N,1}\right) + \epsilon_{N,2}\right]$$

where $\epsilon_{N,1} = O(j^{-\frac{3}{2}}(\log N)^\alpha)$ and $\epsilon_{N,2} = O(e^{-c_2(\log N)^2})$. □

3.2. Random normal matrix ensemble. In this subsection, we show Theorem 1.2 (i). According to the asymptotic expansions of h_j given in Lemmas 3.1 and 3.2, we analyze the summation in (1.37) through the decomposition

$$\sum_{j=0}^{N-1} \log h_j = \sum_{j=0}^{m_N-1} \log h_j + \sum_{j=m_N}^{N-1} \log h_j. \tag{3.5}$$

The asymptotic behaviors of each summation on the right-hand side of (3.5) are given in Lemma 3.3 and 3.7, respectively.

Lemma 3.3. *As $N \to \infty$, we have*

$$\sum_{j=0}^{m_N-1} \log h_j = -m_N N q(0) - \frac{m_N(m_N+1)}{2}\Big(\log N + \log\big(\frac{1}{2}q''(0)\big)\Big) + \frac{1}{2}m_N^2 \log m_N - \frac{3}{4}m_N^2 + \frac{\log(2\pi)}{2}m_N - \frac{1}{12}\log m_N + \zeta'(-1) + O(m_N^{-2} + N^{-\frac{1}{2}(1-5\epsilon)}(\log N)^3).$$

Proof. By Lemma 3.1, we have

$$\sum_{j=0}^{m_N-1} \log h_j = -m_N N q(0) - \frac{m_N(m_N+1)}{2}\Big(\log N + \log\big(\frac{1}{2}q''(0)\big)\Big) + \log G(m_N+1) + O(N^{-\frac{1}{2}(1-5\epsilon)}(\log N)^3),$$

where G is the Barnes G-function (1.41). Now the lemma follows from (1.42). □

Lemma 3.4. *As $N \to \infty$, we have*

$$\sum_{j=m_N}^{N-1} V_{\tau(j)}(r_{\tau(j)}) = N I_Q[\mu_Q] - U_{\mu_Q}(0) - \frac{1}{6N}\log r_1 - m_N q(0) - \frac{3}{4}\frac{m_N^2}{N} + \frac{1}{2N}\Big(m_N^2 - m_N + \frac{1}{6}\Big)\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + \frac{m_N}{2N} + O(N^{-\frac{1}{2}(3-5\epsilon)}).$$

Proof. By applying the Euler–Maclaurin formula (1.39), we have

$$\sum_{j=m_N}^{N-1} V_{\tau(j)}(r_{\tau(j)}) = \int_{m_N}^{N} V_{\tau(t)}(r_{\tau(t)})\,dt - \frac{1}{2}\Big(V_{\tau(N)}(r_{\tau(N)}) - V_{\tau(m_N)}(r_{\tau(m_N)})\Big) + \frac{1}{12}\Big(\partial_t(V_{\tau(t)}(r_{\tau(t)}))\big|_{t=N} - \partial_t(V_{\tau(t)}(r_{\tau(t)}))\big|_{t=m_N}\Big) + O(N^{-1-2\epsilon}). \tag{3.6}$$

Here we have used $\partial_t^3(V_{\tau(t)}(r_{\tau(t)}))\big|_{t=m_N} = O(N^{-3}(\tau(m_N))^{-2}) = O(m_N^{-2}N^{-1})$ and $B_2 = 1/6$. By the change of variables $s = r_{\tau(t)}$ and the formula (1.13) of $I_Q[\mu_Q]$, we obtain

$$\begin{aligned}\int_{m_N}^{N} V_{\tau(t)}(r_{\tau(t)})\,dt &= 2N\int_{r_{\tau(m_N)}}^{r_1}(q(s)-sq'(s)\log s)s\Delta Q(s)\,ds\\ &= N\Big(\frac{1}{2}\int_{S\setminus S_{\tau(m_N)}} Q\cdot\Delta Q\,dA - \log r_1 + (\tau(m_N))^2\log r_{\tau(m_N)}\\ &\quad+\frac{1}{2}(q(r_1)-\tau(m_N)\cdot q(r_{\tau(m_N)}))\Big)\\ &= NI_Q[\mu_Q] - \frac{N}{2}\int_{S_{\tau(m_N)}} Q\cdot\Delta Q dA + \frac{m_N^2}{N}\log r_{\tau(m_N)} - \frac{m_N}{2}q(r_{\tau(m_N)}).\end{aligned}$$

Observe here that

$$r_\tau = \Big(\frac{2\tau}{q''(0)}\Big)^{\frac{1}{2}} + O(\tau) = \Big(\frac{\tau}{\Delta Q(0)}\Big)^{\frac{1}{2}} + O(\tau) \qquad \text{as } \tau\to 0. \tag{3.7}$$

Thus we have

$$\log r_{\tau(m_N)} = \frac{1}{2}\log\Big(\frac{\tau(m_N)}{\Delta Q(0)}\Big) + O(\tau(m_N)^{\frac{1}{2}}) = \frac{1}{2}\log\Big(\frac{\tau(m_N)}{\Delta Q(0)}\Big) + O(N^{-\frac{1}{2}(1-\epsilon)}), \tag{3.8}$$

$$q(r_{\tau(m_N)}) = q(0) + \frac{1}{2}q''(0)(r_{\tau(m_N)})^2 + O((\tau(m_N))^{\frac{3}{2}}) = q(0) + \tau(m_N) + O(N^{-\frac{3}{2}(1-\epsilon)}), \tag{3.9}$$

and

$$\begin{aligned}\frac{1}{2}\int_{S_{\tau(m_N)}} Q\cdot\Delta Q\,dA &= \frac{1}{4}\int_0^{r_{\tau(m_N)}} q(s)\cdot(sq'(s))'\,ds\\ &= \frac{1}{2}\tau(m_N)\cdot q(r_{\tau(m_N)}) - \frac{1}{4}\int_0^{r_{\tau(m_N)}} s(q'(s))^2\,ds\\ &= \frac{1}{2}\tau(m_N)\cdot q(r_{\tau(m_N)}) - \frac{1}{4}(q''(0))^2\int_0^{r_{\tau(m_N)}} s^3\,ds + O(r_{\tau(m_N)}^5)\\ &= \frac{1}{2}\tau(m_N)\cdot q(r_{\tau(m_N)}) - \frac{1}{4}\tau(m_N)^2 + O(N^{-\frac{5}{2}(1-\epsilon)}).\end{aligned}$$

Combining all of the above asymptotic expansions, we obtain

$$\begin{aligned}&\int_{m_N}^{N} V_{\tau(t)}(r_{\tau(t)})\,dt\\ &= NI_Q[\mu_Q] - m_N q(r_{\tau(m_N)}) + \frac{1}{4}\frac{m_N^2}{N} + \frac{m_N^2}{N}\log r_{\tau(m_N)} + O(N^{-\frac{1}{2}(3-5\epsilon)})\\ &= NI_Q[\mu_Q] - m_N q(0) - \frac{3}{4}\frac{m_N^2}{N} + \frac{m_N^2}{2N}\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + O(N^{-\frac{1}{2}(3-5\epsilon)}).\end{aligned} \tag{3.10}$$

Furthermore, it follows from the formula (1.15) of $U_{\mu_Q}(0)$, (3.8) and (3.9) that

$$\begin{aligned} V_{\tau(N)}(r_{\tau(N)}) - V_{\tau(m_N)}(r_{\tau(m_N)}) &= q(r_1) - 2\log r_1 - q(r_{\tau(m_N)}) + 2\tau(m_N)\cdot \log r_{\tau(m_N)} \\ &= q(r_1) - 2\log r_1 - q(0) - \frac{m_N}{N} + \frac{m_N}{N}\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + O(N^{-\frac{3}{2}(1-\epsilon)}) \\ &= 2U_{\mu_Q}(0) - \frac{m_N}{N} + \frac{m_N}{N}\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + O(N^{-\frac{3}{2}(1-\epsilon)}). \end{aligned} \tag{3.11}$$

Similarly, we have

$$\begin{aligned} \partial_t(V_{\tau(t)}(r_{\tau(t)}))\big|_{t=N} - \partial_t(V_{\tau(t)}(r_{\tau(t)}))\big|_{t=m_N} &= \frac{2}{N}(\log r_{\tau(m_N)} - \log r_1) \\ &= \frac{1}{N}\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) - \frac{2}{N}\log r_1 + O(N^{-\frac{1}{2}(3-\epsilon)}). \end{aligned} \tag{3.12}$$

Now the lemma follows from (3.6), (3.10), (3.11) and (3.12). □

Lemma 3.5. *As $N\to\infty$, we have*

$$\sum_{j=m_N}^{N-1}\log\Delta Q(r_{\tau(j)}) = NE_Q[\mu_Q] - \frac{1}{2}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(0)}\Big) - m_N\log\Delta Q(0) + O(N^{-\frac{1}{2}(1-\epsilon)}),$$

and

$$\sum_{j=m_N}^{N-1}\log r_{\tau(j)} = -NU_{\mu_Q}(0) + \frac{m_N}{2} - \frac{1}{2}\log r_1 - \Big(\frac{m_N}{2} - \frac{1}{4}\Big)\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + O(N^{-\epsilon}).$$

Proof. Using the Euler–Maclaurin formula (1.39),

$$\begin{aligned} \sum_{j=m_N}^{N-1}\log\Delta Q(r_{\tau(j)}) =& N\int_{S\setminus S_{\tau(m_N)}}\log\Delta Q\, d\mu_Q - \frac{1}{2}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(r_{\tau(m_N)})}\Big) \\ &+ \frac{1}{12}\Big(\partial_t\log\Delta Q(r_{\tau(t)}))\big|_{t=N} - \partial_t\log\Delta Q(r_{\tau(t)}))\big|_{t=m_N}\Big) \\ &+ o(N^{-1}\tau(m_N)^{-\frac{1}{2}}). \end{aligned}$$

We verify from (3.7) that

$$\begin{aligned} \int_{S_{\tau(m_N)}}\log\Delta Q\, d\mu_Q &= \Delta Q(0)\cdot\log\Delta Q(0)\cdot r_{\tau(m_N)}^2 + O(\tau(m_N)^3) \\ &= \frac{m_N}{N}\log\Delta Q(0) + O(N^{-3(1-\epsilon)}) \end{aligned}$$

and

$$\partial_t\log\Delta Q(r_{\tau(t)})\big|_{t=m_N} = \frac{1}{2N}\frac{\partial_r\Delta Q(r_{\tau(m_N)})}{r_{\tau(m_N)}(\Delta Q(r_{\tau(m_N)}))^2} = O(N^{-1}\tau(m_N)^{-\frac{1}{2}}) = O(N^{-\frac{1}{2}(1+\epsilon)}).$$

Observe that $\log \Delta Q(r_{\tau(m_N)}) = \log \Delta Q(0) + O(r_{\tau(m_N)})$. Combining above equations, we obtain the first assertion. Similarly, by using the Euler–Maclaurin formula (1.39), (3.8), and (3.9), we have

$$\begin{aligned}\sum_{j=m_N}^{N} \log r_{\tau(j)} &= N \int_{S\setminus S_{\tau(m_N)}} \log|z|\, d\mu_Q - \frac{1}{2}\log\Big(\frac{r_1}{r_{\tau(m_N)}}\Big) + O(m_N^{-1}) \\ &= \frac{N}{2}\Big(2\log r_1 - 2\tau(m_N)\log r_{\tau(m_N)} - q(r_1) + q(r_{\tau(m_N)})\Big) \\ &\quad - \frac{1}{2}\log\Big(\frac{r_1}{r_{\tau(m_N)}}\Big) \\ &= N\Big(\log r_1 - \frac{q(r_1)-q(0)}{2} + \frac{1}{2}\tau(m_N)\Big) - \frac{1}{2}\log r_1 \\ &\quad - \Big(\frac{m_N}{2} - \frac{1}{4}\Big)\log\Big(\frac{\tau(m_N)}{\Delta Q(0)}\Big) + O(N^{-\epsilon}),\end{aligned}$$

which completes the proof. □

Lemma 3.6. *As $N \to \infty$, we have*

$$\frac{1}{N}\sum_{j=m_N}^{N-1} \mathfrak{B}_1(r_{\tau(j)}) = F_Q[\mathbb{D}_{r_1}] + \frac{1}{3}\log r_1 - \frac{1}{12}\log\Big(\frac{m_N}{N}\Big) - \frac{1}{4}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(0)}\Big) + \frac{1}{6}\log \Delta Q(0) + O(N^{-\epsilon} + N^{-\frac{1}{2}(1-\epsilon)}),$$

where $F_Q[\mathbb{D}_{r_1}]$ is given in (1.22).

Proof. Observe that

$$\sum_{j=m_N}^{N-1} \mathfrak{B}_1(r_{\tau(j)}) = N\int_{S\setminus S_{\tau(m_N)}} \mathfrak{B}_1\, d\mu_Q + O(\tau(m_N)^{-1}).$$

By (3.7) and (3.8),

$$\begin{aligned}\int_{S\setminus S_{\tau(m_N)}} \mathfrak{B}_1\, d\mu_Q &= \frac{1}{6}\log\Big(\frac{r_1}{r_{\tau(m_N)}}\Big) - \frac{19}{48}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(r_{\tau(m_N)})}\Big) \\ &\quad - \frac{1}{16}\int_{r_{m_N}}^{r_1}\Big[\frac{\partial_r^2 \Delta Q(r)}{\Delta Q(r)} - \frac{5}{3}\Big(\frac{\partial_r \Delta Q(r)}{\Delta Q(r)}\Big)^2\Big] r\, dr \\ &= \frac{1}{6}\log r_1 - \frac{1}{12}\log\Big(\frac{\tau(m_N)}{\Delta Q(0)}\Big) - \frac{19}{48}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(0)}\Big) \\ &\quad - \frac{1}{16}\int_0^{r_1}\Big[\frac{\partial_r^2 \Delta Q(r)}{\Delta Q(r)} - \frac{5}{3}\Big(\frac{\partial_r \Delta Q(r)}{\Delta Q(r)}\Big)^2\Big] r\, dr + O(N^{-\frac{1}{2}(1-\epsilon)}).\end{aligned}$$

Thus we have

$$\begin{aligned}\frac{1}{N}\sum_{j=m_N}^{N-1} \mathfrak{B}_1(r_{\tau(j)}) &= \frac{1}{6}\log r_1 - \frac{1}{12}\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) - \frac{19}{48}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(0)}\Big) \\ &\quad - \frac{1}{16}\int_0^{r_1}\Big[\frac{\partial_r^2 \Delta Q(r)}{\Delta Q(r)} - \frac{5}{3}\Big(\frac{\partial_r \Delta Q(r)}{\Delta Q(r)}\Big)^2\Big] r\, dr + O(N^{-\epsilon} + N^{-\frac{1}{2}(1-\epsilon)}).\end{aligned}$$

Now the lemma follows from (2.12). □

Lemma 3.7. *As $N \to \infty$, we have*

$$\sum_{j=m_N}^{N-1} \log h_j = -N^2 I_Q[\mu_Q] + \frac{N-m_N}{2} \log\Big(\frac{2\pi}{N}\Big) + Nm_N q(0) + \frac{3}{4}m_N^2 - \frac{1}{2}NE_Q[\mu_Q] - \frac{1}{2}m_N(m_N+1)\log\Big(\frac{1}{\Delta Q(0)}\Big) + F_Q[\mathbb{D}_{r_1}] - \frac{1}{2}\Big(m_N^2 - \frac{1}{6}\Big)\log\Big(\frac{m_N}{N}\Big) + O(N^{-\frac{1}{12}}(\log N)^3).$$

Proof. By Lemma 3.2, we have

$$\sum_{j=m_N}^{N-1} \log h_j = \sum_{j=m_N}^{N-1} \Big(-NV_{\tau(j)}(r_{\tau(j)}) + \log r_{\tau(j)} - \frac{1}{2}\log \Delta Q(r_{\tau(j)}) + \frac{1}{N}\mathfrak{B}_1(r_{\tau(j)})\Big) + \frac{(N-m_N)}{2}\log\Big(\frac{2\pi}{N}\Big) + O(m_N^{-\frac{1}{2}}(\log N)^{\alpha}) + O(N^{-\frac{1}{2}(1-5\epsilon)}).$$

Now Lemmas 3.2, 3.4, 3.5 and 3.6 complete the proof. Here, for the error term, we take $\epsilon = 1/6$ so that $\epsilon/2 = (1-5\epsilon)/2 = 1/12$. □

We are now ready to prove the first assertion of Theorem 1.1.

Proof of Theorem 1.2 (i). By combining Lemmas 3.3, 3.7 and (3.5), we obtain

$$\sum_{j=0}^{N-1} \log h_j = -N^2 I_Q[\mu_Q] - \frac{1}{2}N\log N - \frac{N}{2}\Big(E_Q[\mu_Q] - \log(2\pi)\Big) - \frac{\log N}{12} + \zeta'(-1) + F_Q[\mathbb{D}_{r_1}] + O(N^{-\frac{1}{12}}(\log N)^3).$$

Note here that all the terms involving m_N in Lemmas 3.3 and 3.7 vanish. Then the desired asymptotic expansion (1.23) follows from (1.37) and (2.22). This completes the proof. □

3.3. Planar symplectic ensemble. In this subsection, we prove the second assertion of Theorem 1.2.

As a counterpart of Lemma 3.3, we have the following.

Lemma 3.8. *As $N \to \infty$, we have*

$$\sum_{j=0}^{m_N-1} \log \widetilde{h}_{2j+1} = -2m_N Nq(0) + m_N^2 \log m_N + \Big(\log 2 - \frac{3}{2} - \log(Nq''(0))\Big)m_N^2 + \frac{1}{2}m_N\log m_N + \Big(\log 2 - \frac{1}{2} + \frac{1}{2}\log\pi - \log(Nq''(0))\Big)m_N - \frac{1}{24}\log m_N + \frac{5}{24}\log 2 + \frac{1}{2}\zeta'(-1) + O(m_N^{-1} + N^{-\frac{1}{2}(1-5\epsilon)}(\log N)^3).$$

Proof. By Lemma 3.1 and (4.6), we have

$$\log \widetilde{h}_{2j+1} = -2Nq(0) - (2j+2)\log(Nq''(0)) + \log\Big(\frac{2^{2j+1}}{\sqrt{\pi}}\Gamma(j+1)\Gamma(j+\tfrac{3}{2})\Big) + O(N^{-\frac{1}{2}(1-3\epsilon)}).$$

Thus we have

$$\sum_{j=0}^{m_N-1} \log \widetilde{h}_{2j+1} = -2m_N Nq(0) - m_N(m_N+1)\log(Nq''(0)) + m_N^2\log 2 - \frac{m_N}{2}\log\pi + \log\Big(G(m_N+1)\frac{G(m_N+\frac{3}{2})}{G(\frac{3}{2})}\Big) + O(N^{-\frac{1}{2}(1-5\epsilon)}).$$

Now the lemma follows from the asymptotic expansion (1.42) of the Barnes G function and

$$G(\tfrac{1}{2}) = 2^{\frac{1}{24}}\exp\Big(\frac{3}{2}\zeta'(-1)\Big)\pi^{-\frac{1}{4}}, \qquad G(\tfrac{3}{2}) = G(\tfrac{1}{2})\Gamma(\tfrac{1}{2}) = G(\tfrac{1}{2})\sqrt{\pi}.$$

□

Lemma 3.9. *As $N \to \infty$, we have*

$$\sum_{j=m_N}^{N-1} V_{\widetilde{\tau}(2j+1)}(r_{\widetilde{\tau}(2j+1)}) = NI_Q[\mu_Q] + \frac{1}{12N}\log r_1 - m_N q(0) - \frac{3}{4}\frac{m_N^2}{N} + \frac{1}{4N}\Big(2m_N^2 - \frac{1}{6}\Big)\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + O(N^{-\frac{1}{2}(3-5\epsilon)}).$$

Proof. By using Lemma 3.4 with $N \to 2N$, we have

$$\sum_{j=2m_N}^{2N-1} V_{\widetilde{\tau}(j)}(r_{\widetilde{\tau}(j)}) = 2NI_Q[\mu_Q] - U_{\mu_Q}(0) - \frac{1}{12N}\log r_1 - 2m_N q(0) - \frac{3}{2}\frac{m_N^2}{N} + \frac{1}{4N}\Big(4m_N^2 - 2m_N + \frac{1}{6}\Big)\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + \frac{m_N}{2N} + O(N^{-\frac{1}{2}(3-5\epsilon)}).$$

On the other hand, by the Euler–Maclaurin formula (1.39), we have

$$\sum_{j=m_N}^{N-1} V_{\widetilde{\tau}(2j)}(r_{\widetilde{\tau}(2j)}) = \frac{1}{2}\int_{2m_N}^{2N} V_{\widetilde{\tau}(t)}(r_{\widetilde{\tau}(t)})\,dt - \frac{1}{2}\big(V_{\widetilde{\tau}(2N)}(r_{\widetilde{\tau}(2N)}) - V_{\widetilde{\tau}(2m_N)}(r_{\widetilde{\tau}(2m_N)})\big) + \frac{1}{12}\Big(\partial_t(V_{\widetilde{\tau}(2t)}(r_{\widetilde{\tau}(2t)}))\big|_{t=N} - \partial_t(V_{\widetilde{\tau}(2t)}(r_{\widetilde{\tau}(2t)}))\big|_{t=m_N}\Big) + O(N^{-1-2\epsilon}).$$

Following the proof of Lemma 3.4, we have

$$\sum_{j=m_N}^{N-1} V_{\widetilde{\tau}(2j)}(r_{\widetilde{\tau}(2j)}) = N I_Q[\mu_Q] - U_{\mu_Q}(0) - \frac{1}{6N}\log r_1 - m_N q(0) - \frac{3}{4}\frac{m_N^2}{N} + \frac{1}{2N}(m_N^2 - m_N + \frac{1}{6})\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + \frac{m_N}{2N} + O(N^{-\frac{1}{2}(3-5\epsilon)}),$$

which completes the proof. □

Lemma 3.10. *As $N \to \infty$, we have*

$$\sum_{j=m_N}^{N-1} \log \Delta Q(r_{\widetilde{\tau}(2j+1)}) = N E_Q[\mu_Q] - m_N \log \Delta Q(0) + O(N^{-\frac{1}{2}(1-\epsilon)}),$$

and

$$\sum_{j=m_N}^{N-1} \log r_{\widetilde{\tau}(2j+1)} = -N U_{\mu_Q}(0) + \frac{m_N}{2} - \frac{m_N}{2}\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + O(N^{-\epsilon}).$$

Proof. By Lemma 3.5 with $N \to 2N$, we have

$$\sum_{j=2m_N}^{2N-1} \log \Delta Q(r_{\widetilde{\tau}(j)}) = 2N E_Q[\mu_Q] - \frac{1}{2}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(0)}\Big) - 2m_N \log \Delta Q(0) + O(N^{-\frac{1}{2}(1-\epsilon)}),$$

$$\sum_{j=2m_N}^{2N-1} \log r_{\widetilde{\tau}(j)} = -2N U_{\mu_Q}(0) + m_N - \frac{1}{2}\log r_1 - \Big(m_N - \frac{1}{4}\Big)\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + O(N^{-\epsilon}).$$

Following the proof of Lemma 3.5, we also have

$$\sum_{j=m_N}^{N-1} \log \Delta Q(r_{\widetilde{\tau}(2j)}) = N E_Q[\mu_Q] - \frac{1}{2}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(0)}\Big) - 2m_N \log \Delta Q(0) + O(N^{-\frac{1}{2}(1-\epsilon)}),$$

$$\sum_{j=m_N}^{N-1} \log r_{\widetilde{\tau}(2j)} = -N U_{\mu_Q}(0) + m_N - \frac{1}{2}\log r_1 - \Big(m_N - \frac{1}{4}\Big)\log\Big(\frac{m_N}{N\Delta Q(0)}\Big) + O(N^{-\epsilon}).$$

This completes the proof. □

Lemma 3.11. *As $N \to \infty$, we have*

$$\frac{1}{2N}\sum_{j=m_N}^{N-1}\mathfrak{B}_1(r_{\widetilde{\tau}(2j+1)}) = \frac{1}{2}F_Q[\mathbb{D}_{r_1}] + \frac{1}{6}\log r_1 - \frac{1}{24}\log\Big(\frac{m_N}{N}\Big) - \frac{1}{8}\log\Big(\frac{\Delta Q(r_1)}{\Delta Q(0)}\Big) + \frac{1}{12}\log\Delta Q(0) + O(m_N^{-1}).$$

Proof. This lemma follows along the same lines of Lemma 3.6. □

Lemma 3.12. *As $N \to \infty$, we have*

$$\begin{aligned}\sum_{j=m_N}^{N-1}\log\widetilde{h}_{2j+1} &= -2N^2 I_Q[\mu_Q] - \frac{N}{2}\log N + N\Big(\frac{\log\pi}{2} - U_{\mu_Q}(0) - \frac{1}{2}E_Q[\mu_Q]\Big)\\ &\quad + \frac{3}{2}m_N^2 - \Big(m_N^2 - \frac{1}{24}\Big)\log\Big(\frac{m_N}{N}\Big) + m_N\Big(2Nq(0) + \frac{1}{2} + \frac{1}{2}\log\Big(\frac{m_N}{\pi}\Big)\Big)\\ &\quad - \Big(m_N^2 + \frac{1}{8}\Big)\log\Big(\frac{1}{\Delta Q(0)}\Big) + \frac{1}{2}F_Q[\mathbb{D}_{r_1}] + \frac{1}{8}\log\Big(\frac{1}{\Delta Q(r_1)}\Big)\\ &\quad + O(m_N^{-\frac{1}{2}}(\log N)^{\alpha}).\end{aligned}$$

Proof. Note that by Lemma 3.2, we have

$$\begin{aligned}\sum_{j=m_N}^{N-1}\log\widetilde{h}_{2j+1} &= \sum_{j=m_N}^{N-1}\Big(-2NV_{\widetilde{\tau}(2j+1)}(r_{\widetilde{\tau}(2j+1)}) + \log r_{\widetilde{\tau}(2j+1)}\\ &\quad - \frac{1}{2}\log\Delta Q(r_{\widetilde{\tau}(2j+1)}) + \frac{1}{2N}\mathfrak{B}_1(r_{\widetilde{\tau}(2j+1)})\Big)\\ &\quad + \frac{(N-m_N)}{2}\log\Big(\frac{\pi}{N}\Big) + O(m_N^{-\frac{1}{2}}(\log N)^{\alpha}).\end{aligned}$$

The lemma now follows from Lemmas 3.9, 3.10 and 3.11. □

We now finish the proof of Theorem 1.2.

Proof of Theorem 1.2 (ii). Combining Lemmas 3.8 and 3.12, after long but straightforward simplifications, we obtain

$$\begin{aligned}&\sum_{j=0}^{N-1}\log(2\widetilde{h}_{2j+1}) = N\log 2 + \sum_{j=0}^{m_N-1}\log\widetilde{h}_{2j+1} + \sum_{j=m_N}^{N-1}\log\widetilde{h}_{2j+1}\\ &= -2N^2 I_Q[\mu_Q] - \frac{1}{2}N\log N + N\Big(\frac{\log(4\pi)}{2} - U_{\mu_Q}(0) - \frac{1}{2}E_Q[\mu_Q]\Big) - \frac{1}{24}\log N\\ &\quad + \frac{5}{24}\log 2 + \frac{1}{2}\zeta'(-1) + \frac{1}{2}F_Q[\mathbb{D}_{r_1}] + \frac{1}{8}\log\Big(\frac{\Delta Q(0)}{\Delta Q(r_1)}\Big) + O(m_N^{-1} + N^{-\frac{1}{2}(1-5\epsilon)}(\log N)^3).\end{aligned}$$

Again, it is noteworthy that all the terms in Lemmas 3.8 and 3.12 involving m_N cancel each other. Now the asymptotic behavior (1.24) follows from (1.37) and the asymptotic expansion (2.22) of $\log N!$ with $\epsilon = 1/6$. □

4. Examples: Mittag–Leffler Ensemble and Truncated Unitary Ensemble

This section presents examples of our Theorems 1.1 and 1.2 for some well-known planar point processes. We also refer to [36, Section 4] for further examples in the context of the induced spherical ensembles.

4.1. Mittag–Leffler ensemble. Let us consider the potential

$$Q(z) = |z|^{2\lambda} - 2c \log |z|, \qquad \lambda, c > 0. \tag{4.1}$$

The models (1.1) and (1.2) associated with the potential (4.1) are known as the Mittag-Leffler ensemble [12]. We refer to [17,27,30] and [6] for recent studies on complex and symplectic Mittag-Leffler ensembles, respectively. Using (1.12), we have

$$r_0 = \Big(\frac{c}{\lambda}\Big)^{\frac{1}{2\lambda}}, \qquad r_1 = \Big(\frac{1+c}{\lambda}\Big)^{\frac{1}{2\lambda}}, \qquad \Delta Q(z) = \lambda^2 |z|^{2\lambda-2}. \tag{4.2}$$

In particular, by (4.2), the Mittag-Leffler ensemble (4.1) falls into the class considered in Theorem 1.1. Let us recall that Q is required to be C^∞ in a neighborhood of S.

By direct computations using (1.9), (1.13) and (1.15), we have

$$I_Q[\mu_Q] = \frac{1}{2\lambda} \log\Big(\frac{c^{c^2}}{(1+c)^{(1+c)^2}}\Big) + \frac{1+2c}{2\lambda}\Big(\log\lambda + \frac{3}{2}\Big), \tag{4.3}$$

$$E_Q[\mu_Q] = \frac{1+\lambda}{\lambda}\log\lambda + \frac{1-\lambda}{\lambda}\Big(1+\log\Big(\frac{c^c}{(1+c)^{1+c}}\Big)\Big). \tag{4.4}$$

It also follows from (1.16) and

$$r\frac{(\partial_r\Delta)Q(r)}{\Delta Q(r)} = 2\lambda - 2, \qquad r^2\Delta Q(r) = \lambda^2 r^{2\lambda}$$

that

$$F_Q[\mathbb{A}_{r_0,r_1}] = \Big(\frac{\lambda}{6} - \frac{(\lambda-1)^2}{6}\Big)\log\Big(\frac{r_0}{r_1}\Big) = -\frac{\lambda^2 - 3\lambda + 1}{12\lambda}\log\Big(\frac{c}{1+c}\Big). \tag{4.5}$$

On the other hand, by using (1.37),

$$h_j = 2\int_0^\infty r^{2j+1+2cN} e^{-Nr^{2\lambda}}\, dr = \frac{1}{\lambda} N^{-\frac{j+1+cN}{\lambda}} \Gamma\Big(\frac{j+Nc+1}{\lambda}\Big),$$

and the analogous formula for $\widetilde{h}_j$, we have

$$Z_N = \frac{N!}{\lambda^N} N^{-\frac{(2c+1)N^2+N}{2\lambda}} \prod_{j=0}^{N-1} \Gamma\Big(\frac{j+Nc+1}{\lambda}\Big),$$

$$\widetilde{Z}_N = \frac{N!}{(\lambda/2)^N} (2N)^{-\frac{(2c+1)N^2+N}{\lambda}} \prod_{j=0}^{N-1} \Gamma\Big(\frac{j+Nc+1}{\lambda/2}\Big).$$

Furthermore, using the multiplication theorem of gamma function ([57, Eq. (5.5.6)])

$$\Gamma(nz) = (2\pi)^{\frac{1-n}{2}} n^{nz-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma(z + \tfrac{k}{n}) \tag{4.6}$$

and the characteristic property (1.41) of the Barnes G-function, we have that for $\frac{1}{\lambda} \in \mathbb{N}$,

$$\begin{aligned} Z_N = N!\, N^{-\frac{(2c+1)N^2+N}{2\lambda}} (2\pi)^{(\frac{1}{2}-\frac{1}{2\lambda})N} \Big(\frac{1}{\lambda}\Big)^{(\frac{1}{2\lambda}+\frac{c}{\lambda})N^2+(\frac{1}{\lambda}+\frac{1}{2}-\frac{1}{2\lambda})N} \\ \times \prod_{k=0}^{\frac{1}{\lambda}-1} \frac{G(N+Nc+1+\lambda k)}{G(Nc+1+\lambda k)}. \end{aligned} \tag{4.7}$$

Similarly, for $\frac{2}{\lambda} \in \mathbb{N}$, we have

$$\begin{aligned} \widetilde{Z}_N = N!\, (2N)^{-\frac{(2c+1)N^2+N}{\lambda}} (2\pi)^{(\frac{1}{2}-\frac{1}{\lambda})N} \Big(\frac{2}{\lambda}\Big)^{(\frac{1}{\lambda}+\frac{2c}{\lambda})N^2+(\frac{2}{\lambda}+\frac{1}{2}-\frac{1}{\lambda})N} \\ \times \prod_{k=0}^{\frac{2}{\lambda}-1} \frac{G(N+Nc+1+\frac{\lambda}{2}k)}{G(Nc+1+\frac{\lambda}{2}k)}. \end{aligned} \tag{4.8}$$

Then by using (1.42), one can directly check that the partition functions (4.7) and (4.8) satisfy the expansions (1.17) and (1.18) with (4.3), (4.4) and (4.5).

4.2. Truncated unitary ensemble. We now consider the potential

$$Q(z) = \begin{cases} -\alpha \log\Big(1 - \dfrac{|z|^2}{R^2(1+\alpha)}\Big) & \text{if } |z| \le R\sqrt{1+\alpha}, \\ \infty & \text{otherwise,} \end{cases} \qquad \alpha, R > 0. \tag{4.9}$$

The models associated with (4.9) correspond to the truncated unitary ensembles at strong non-unitarity [49,68]. These models provide a one-parameter generalization of the Ginibre ensembles that can be recovered in the extremal case, i.e. $\lim_{\alpha\to\infty} Q(z) = |z|^2/R^2$. (See [34,60] and references therein for recent works on these models.) In this case, we have

$$r_0 = 0, \qquad r_1 = R, \qquad \Delta Q(z) = \frac{R^2\,\alpha(1+\alpha)}{(R^2(1+\alpha)-|z|^2)^2}. \tag{4.10}$$

From (4.10), we see that the truncated unitary ensembles are contained in the class covered in Theorem 1.2. (The hard edge condition imposed in (4.9) outside the droplet does not harm the proof of Theorem 1.2.)

It is easy to verify from (1.9), (1.13) and (1.15) that

$$I_Q[\mu_Q] = -\frac{\alpha}{2} - \frac{\alpha(2+\alpha)}{2} \log\Big(\frac{\alpha}{1+\alpha}\Big) - \log R, \tag{4.11}$$

$$E_Q[\mu_Q] = -2 - (1+2\alpha)\log\Big(\frac{\alpha}{1+\alpha}\Big) - 2\log R. \tag{4.12}$$

Since

$$r\,\frac{(\partial_r\Delta)Q(r)}{\Delta Q(r)}\Big|_{r=R}=\frac{4r^2}{R^2(1+\alpha)-r^2}\Big|_{r=R}=\frac{4}{\alpha},\qquad R^2\Delta Q(R)=\frac{1+\alpha}{\alpha},$$

we deduce from (1.22) that

$$F_Q[\mathbb{D}_R]=\frac{1}{12}\log\Big(\frac{\alpha}{1+\alpha}\Big)-\frac{1}{4\alpha}+\frac{1}{3}\Big(\frac{1}{\alpha}+\log\Big(\frac{\alpha}{1+\alpha}\Big)\Big)=\frac{1}{12}\Big(\frac{1}{\alpha}+5\log\Big(\frac{\alpha}{1+\alpha}\Big)\Big). \tag{4.13}$$

Notice here that $F_Q[\mathbb{D}_R]$ is independent of R, which is consistent with the invariance of the $O(1)$-terms of (1.23) and (1.24) under the dilation, see the remark below Theorem 1.2.

Using the Euler's beta integral

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 t^{a-1}(1-t)^{b-1}\,dt,\qquad (a,b>0),$$

the orthogonal norm h_j is computed as

$$h_j=\int_0^{R\sqrt{1+\alpha}} r^{2j+1}\Big(1-\frac{r^2}{R^2(1+\alpha)}\Big)^{\alpha N}dr=R^{2j+2}(1+\alpha)^{j+1}\frac{\Gamma(\alpha N+1)\Gamma(j+1)}{\Gamma(\alpha N+j+2)}.$$

Then by (1.37) and (1.41), we have

$$Z_N=N!\,R^{N(N+1)}(1+\alpha)^{\frac{N^2}{2}+\frac{N}{2}}\,\Gamma(\alpha N+1)^N\,\frac{G(N+1)G(\alpha N+2)}{G(\alpha N+N+2)}. \tag{4.14}$$

Similarly, by (1.37) and the duplication formula of the gamma function (i.e. (4.6) with $n=2$),

$$\begin{aligned}\widetilde{Z}_N&=N!\,R^{2N(N+1)}2^{-2\alpha N^2}(1+\alpha)^{N^2+N}\,\Gamma(2\alpha N+1)^N\\&\quad\times G(N+1)\frac{G(N+\frac{3}{2})}{G(\frac{3}{2})}\,\frac{G(\alpha N+2)}{G(\alpha N+N+2)}\,\frac{G(\alpha N+\frac{3}{2})}{G(\alpha N+N+\frac{3}{2})}.\end{aligned} \tag{4.15}$$

Then by using (1.42), it is again straightforward to check that the partition functions (4.14) and (4.15) satisfy the expansions (1.23) and (1.24) with (4.11), (4.12) and (4.13). In the extremal case where $\alpha\to\infty$, the expansion of the partition function Z_N of the complex Ginibre ensemble appears in [65].

Acknowledgements. We thank Gernot Akemann, Christophe Charlier, Peter Forrester and Thomas Leblé for helpful discussions.

Data availibility statement There is no data associated to this work.

Declarations

Conflict of interest The authors have no conflicts of interest to disclose.

References

1. Adhikari, K.: Hole probabilities for β-ensembles and determinantal point processes in the complex plane. Electron. J. Probab. **23**, 1–21 (2018)
2. Adhikari, K., Reddy, N.K.: Hole probabilities for finite and infinite Ginibre ensembles. Int. Math. Res. Not. **12**, 694–6730 (2016)
3. Akemann, G.: The complex Laguerre symplectic ensemble of non-Hermitian matrices. Nuclear Phys. B **730**(3), 253–299 (2005)
4. Akemann, G., Byun, S.-S., Ebke, M.: Universality of the number variance in rotational invariant two-dimensional Coulomb gases. J. Stat. Phys. **190**(9), 1–34 (2023)
5. Akemann, G., Byun, S.-S., Kang, N.-G.: A non-Hermitian generalisation of the Marchenko–Pastur distribution: from the circular law to multi-criticality. Ann. Henri Poincaré **22**(4), 1035–1068 (2021)
6. Akemann, G., Byun, S.-S., Kang, N.-G.: Scaling limits of planar symplectic ensembles. SIGMA Symmet. Integrabil. Geom. Methods Appl. **18**, 40 (2022)
7. Akemann, G., Ebke, M., Parra, I.: Skew-orthogonal polynomials in the complex plane and their Bergman-like kernels. Commun. Math. Phys. **389**(1), 621–659 (2022)
8. Akemann, G., Phillips, M., Shifrin, L.: Gap probabilities in non-Hermitian random matrix theory. J. Math. Phys. **50**(6), 063504 (2009)
9. Ameur, Y., Charlier, C., Cronvall, J.: The two-dimensional Coulomb gas: fluctuations through a spectral gap (2022). Preprint arXiv:2210.13959
10. Ameur, Y., Charlier, C., Cronvall, J., Lenells, J.: Disk counting statistics near hard edges of random normal matrices: the multi-component regime (2022). Preprint arXiv:2210.13962
11. Ameur, Y., Charlier, C., Cronvall, J., Lenells, J.: Exponential moments for disk counting statistics at the hard edge of random normal matrices (2022). Preprint arXiv:2207.11092
12. Ameur, Y., Kang, N.-G., Seo, S.-M.: The random normal matrix model: insertion of a point charge. Potential Anal. (online) (2021)
13. Balogh, F., Merzi, D.: Equilibrium measures for a class of potentials with discrete rotational symmetries. Constr. Approx. **42**(3), 399–424 (2015)
14. Bauerschmidt, R., Bourgade, P., Nikula, M., Yau, H.-T.: The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem. Adv. Theor. Math. Phys. **23**, 841–1002 (2019)
15. Benaych-Georges, F., Chapon, F.: Random right eigenvalues of Gaussian quaternionic matrices. Random Matrices Theory Appl. **1**(2), 1150009 (2012)
16. Byun, S.-S.: Planar equilibrium measure problem in the quadratic fields with a point charge (2023). Preprint arXiv:2301.00324
17. Byun, S.-S., Charlier, C.: On the characteristic polynomial of the eigenvalue moduli of random normal matrices (2022). Preprint arXiv:2205.04298
18. Byun, S.-S., Charlier, C.: On the almost-circular symplectic induced Ginibre ensemble. Stud. Appl. Math. **150**, 184–217 (2023)
19. Byun, S.-S., Forrester, P.J.: Progress on the study of the Ginibre ensembles I: GinUE (2022). Preprint arXiv:2211.16223
20. Byun, S.-S., Forrester, P.J.: Spherical induced ensembles with symplectic symmetry (2022). Preprint arXiv:2209.01934
21. Byun, S.-S., Forrester, P.J.: Progress on the study of the Ginibre ensembles II: GinOE and GinSE (2023). Preprint arXiv:2301.05022
22. Byun, S.-S., Yang, M.: Determinantal Coulomb gas ensembles with a class of discrete rotational symmetric potentials (2022). Preprint arXiv:2210.04019
23. Can, T., Forrester, P., Téllez, G., Wiegmann, P.: Exact and asymptotic features of the edge density profile for the one component plasma in two dimensions. J. Stat. Phys. **158**(5), 1147–1180 (2015)
24. Chafaï, D., Gozlan, N., Zitt, P.-A.: First-order global asymptotics for confined particles with singular pair repulsion. Ann. Appl. Probab. **24**(6), 2371–2413 (2014)
25. Charles, L., Estienne, B.: Entanglement entropy and Berezin–Toeplitz operators. Commun. Math. Phys. **376**(1), 521–554 (2020)
26. Charlier, C.: Asymptotics of Hankel determinants with a one-cut regular potential and Fisher–Hartwig singularities. Int. Math. Res. Not. IMRN **24**, 7515–7576 (2019)
27. Charlier, C.: Large gap asymptotics on annuli in the random normal matrix model (2021). Preprint arXiv:2110.06908
28. Charlier, C.: Asymptotics of determinants with a rotation-invariant weight and discontinuities along circles. Adv. Math. **408**, 108600 (2022)
29. Charlier, C., Gharakhloo, R.: Asymptotics of Hankel determinants with a Laguerre-type or Jacobi-type potential and Fisher–Hartwig singularities. Adv. Math. **383**, 107672 (2021)
30. Charlier, C., Lenells, J.: Exponential moments for disk counting statistics of random normal matrices in the critical regime (2022). Preprint arXiv:2205.00721

31. Charlier, C., Lenells, J., Mauersberger, J.: Higher order large gap asymptotics at the hard edge for Muttalib–Borodin ensembles. Commun. Math. Phys. **384**(2), 829–907 (2021)
32. Criado del Rey, J.G., Kuijlaars, A.B.: A vector equilibrium problem for symmetrically located point charges on a sphere. Constr. Approx. 1–53 (2022)
33. Deaño, A., Simm, N.: Characteristic polynomials of complex random matrices and Painlevé transcendents. Int. Math. Res. Not. **22**, 210–264 (2022)
34. Dubach, G.: On eigenvector statistics in the spherical and truncated unitary ensembles. Electron. J. Probab. **26**, 1–29 (2021)
35. Fenzl, M., Lambert, G.: Precise deviations for disk counting statistics of invariant determinantal processes. Int. Math. Res. Not. **2022**(10), 7420–7494 (2022)
36. Fischmann, J., Forrester, P.J.: One-component plasma on a spherical annulus and a random matrix ensemble. J. Stat. Mech. Theory Exp. **2011**(10), P10003 (2011)
37. Forrester, P.J.: Some statistical properties of the eigenvalues of complex random matrices. Phys. Lett. A **169**(1–2), 21–24 (1992)
38. Forrester, P.J.: Log-Gases and Random Matrices (LMS-34). Princeton University Press, Princeton (2010)
39. Forrester, P.J.: Analogies between random matrix ensembles and the one-component plasma in two-dimensions. Nucl. Phys. B **904**, 253–281 (2016)
40. Garnett, J.B., Marshall, D.E.: Harmonic Measure, vol. 2. Cambridge University Press, Cambridge (2005)
41. Ghosh, S., Nishry, A.: Point processes, hole events, and large deviations: random complex zeros and Coulomb gases. Constr. Approx. **48**(1), 101–136 (2018)
42. Ginibre, J.: Statistical ensembles of complex, quaternion, and real matrices. J. Math. Phys. **6**(3), 440–449 (1965)
43. Guionnet, A., Krishnapur, M., Zeitouni, O.: The single ring theorem. Ann. Math. **174**(2), 1189–1217 (2011)
44. Hedenmalm, H., Makarov, N.: Coulomb gas ensembles and Laplacian growth. Proc. Lond. Math. Soc. **106**(4), 859–907 (2013)
45. Hedenmalm, H., Wennman, A.: Planar orthogogonal polynomials and boundary universality in the random normal matrix model. Acta Math. **227**(2), 309–406 (2021)
46. Jancovici, B., Lebowitz, J.L., Manificat, G.: Large charge fluctuations in classical Coulomb systems. J. Stat. Phys. **72**(3), 773–787 (1993)
47. Jancovici, B., Manificat, G., Pisani, C.: Coulomb systems seen as critical systems: finite-size effects in two dimensions. J. Stat. Phys. **76**(1), 307–329 (1994)
48. Kanzieper, E.: Eigenvalue correlations in non-Hermitean symplectic random matrices. J. Phys. A **35**(31), 6631–6644 (2002)
49. Khoruzhenko, B.A., Lysychkin, S.: Truncations of random symplectic unitary matrices (2021). Preprint arXiv:2111.02381
50. Kiessling, M.K.-H., Spohn, H.: A note on the eigenvalue density of random matrices. Commun. Math. Phys. **199**(3), 683–695 (1999)
51. Lacroix-A-Chez-Toine, B., Garzón, J.A.M., Calva, C.S.H., Castillo, I.P., Kundu, A., Majumdar, S.N., Schehr, G.: Intermediate deviation regime for the full eigenvalue statistics in the complex Ginibre ensemble. Phys. Rev. E **100**(1), 012137 (2019)
52. Lacroix-A-Chez-Toine, B., Majumdar, S.N., Schehr, G.: Rotating trapped fermions in two dimensions and the complex Ginibre ensemble: exact results for the entanglement entropy and number variance. Phys. Rev. A **99**(2), 021602 (2019)
53. Leblé, T., Serfaty, S.: Large deviation principle for empirical fields of log and Riesz gases. Invent. Math. **210**(3), 645–757 (2017)
54. Leblé, T., Serfaty, S.: Fluctuations of two dimensional coulomb gases. Geom. Funct. Anal. **28**(2), 443–508 (2018)
55. Lee, S.-Y., Makarov, N.G.: Topology of quadrature domains. J. Am. Math. Soc. **29**(2), 333–369 (2016)
56. Lewin, M.: Coulomb and Riesz gases: the known and the unknown. J. Math. Phys. **63**(6), 061101 (2022)
57. Olver, F.W., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds.): NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge (2010)
58. Saff, E.B., Totik, V.: Logarithmic potentials with external fields, volume 316 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Berlin (1997). [Appendix B by Thomas Bloom]
59. Sandier, E., Serfaty, S.: 2D Coulomb gases and the renormalized energy. Ann. Probab. **43**(4), 2026–2083 (2015)
60. Serebryakov, A., Simm, N.: Characteristic polynomials of random truncations: moments, duality and asymptotics. Random Matrices Theory Appl. (2022). [With an appendix by G. Dubach]
61. Serfaty, S.: Microscopic description of Log and Coulomb gases. In Random matrices, volume 26 of IAS/Park City Mathematics Series, pp. 341–387. Amer. Math. Soc., Providence (2019)

62. Serfaty, S.: Gaussian fluctuations and free energy expansion for Coulomb gases at any temperature. Annales Institute Henri Poincaré Probabilities Statistiques (to appear) (2020). arXiv:2003.11704
63. Shakirov, S.: Exact solution for mean energy of 2d Dyson gas at $\beta = 1$. Phys. Lett. A **375**(6), 984–989 (2011)
64. Smith, N.R., Doussal, P.L., Majumdar, S.N., Schehr, G.: Counting statistics for non-interacting fermions in a rotating trap. Phys. Rev. A **105**, 043315 (2022)
65. Téllez, G., Forrester, P.J.: Exact finite-size study of the 2D OCP at $\Gamma = 4$ and $\Gamma = 6$. J. Stat. Phys. **97**(3), 489–521 (1999)
66. Webb, C., Wong, M.D.: On the moments of the characteristic polynomial of a Ginibre random matrix. Proc. Lond. Math. Soc. **118**(5), 1017–1056 (2019)
67. Zabrodin, A., Wiegmann, P.: Large-N expansion for the 2D Dyson gas. J. Phys. A **39**(28), 8933–8963 (2006)
68. Życzkowski, K., Sommers, H.-J.: Truncations of random unitary matrices. J. Phys. A **33**(10), 2045–2057 (2000)

Communicated by L. Erdos