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Dynamics of Two Interacting Kinks for the ϕ^6 Model

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Abstract: We consider the nonlinear wave equation known as the ϕ^6 model in dimension 1+1. We describe the long-time behavior of this model's solutions close to a sum of two kinks with energy slightly larger than twice the minimum energy of non-constant stationary solutions. We prove orbital stability of two moving kinks. We show for low energy excess ϵ that these solutions can be described for a long time of order $-\ln(\epsilon)\epsilon^{-\frac{1}{2}}$ as the sum of two moving kinks such that each kink's center is close to an explicit function which is a solution of an ordinary differential system. We give an optimal estimate in the energy norm of the remainder and we prove that this estimate is achieved during a finite instant t of order $-\ln(\epsilon)\epsilon^{-\frac{1}{2}}$.

1. Introduction

1.1. Background. We consider a nonlinear wave equation known as the ϕ^6 model. For the potential function $U(\phi) = \phi^2(1-\phi^2)^2$ and $\dot{U}(\phi) = 2\phi - 8\phi^3 + 6\phi^5$, the equation is written as

$$\partial_t^2\phi(t,x) - \partial_x^2\phi(t,x) + \dot{U}(\phi(t,x)) = 0, \, (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{1}$$

The potential energy E_{pot}, the kinetic energy E_{kin} and total energy E_{total} associated to the Eq. (1) are given by

$$\begin{aligned} E_{pot}(\phi(t)) &= \frac{1}{2}\int_{\mathbb{R}} \partial_x\phi(t,x)^2\,dx + \int_{\mathbb{R}} \phi(t,x)^2(1-\phi(t,x)^2)^2\,dx,\\ E_{kin}(\phi(t)) &= \frac{1}{2}\int_{\mathbb{R}} \partial_t\phi(t,x)^2\,dx,\\ E_{total}(\phi(t),\partial_t\phi(t)) &= \frac{1}{2}\int_{\mathbb{R}} \left[\partial_x\phi(t,x)^2 + \partial_t\phi(t,x)^2\right]dx \end{aligned}$$

$$+\int_{\mathbb{R}}\phi(t,x)^2(1-\phi(t,x)^2)^2\,dx.$$

The vacuum set $\mathcal{V}$ of the potential function U is the set $U^{-1}\{0\}=\{0,1,-1\}$. We say that if a solution $\phi(t,x)$ of the integral equation associated to (1) has $E_{total}(\phi,\partial_t\phi)<+\infty$, then it is in the energy space. The solutions of (1) in the energy space have constant total energy $E_{total}(\phi(t),\partial_t\phi(t))$.

From standard energy estimate techniques, the Cauchy Problem associated to (1) is locally well-posed in the energy space. Moreover, if $E_{total}(\phi(0),\partial_t\phi(0))=E_0<+\infty$, then there exists $M(E_0)>0$ such that $\|\phi(0,x)\|_{L^\infty(\mathbb{R})}<M(E_0)$, otherwise the facts that $U\in C^\infty(\mathbb{R})$ and $\lim_{\phi\to\pm\infty}U(\phi)=+\infty$ would imply that $\int_{\mathbb{R}}U(\phi(0,x))\,dx>E_0$. Therefore, similarly to the proof of Theorem 6.1 from the book [27] of Shatah and Struwe, we can verify that the partial differential equation (1) is globally well-posed in the energy space since U is a Lipschitz function when restricted to the space of real functions ϕ satisfying $\|\phi\|_{L^\infty(\mathbb{R})}<K_0$ for some positive number K_0.

The stationary solutions of (1) are the critical points of the potential energy. The unique constant solutions of (1) in the energy space are the functions $\phi\equiv v$, for any $v\in\mathcal{V}$. The only non-constant stationary solutions of (1) with finite total energy are the topological solitons called kinks and anti-kinks, for more details see chapter 5 of [19]. Each topological soliton H connects different numbers $v_1,\ v_2\in\mathcal{V}$, more precisely,

$$\lim_{x\to-\infty}H(x)=v_1,\ \ \lim_{x\to+\infty}H(x)=v_2,\ \ \mathcal{V}\cap\{H(x)|\,x\in\mathbb{R}\}=\emptyset.$$

The kinks of (1) are given by

$$H_{0,1}(x-a)=\frac{e^{\sqrt{2}(x-a)}}{(1+e^{2\sqrt{2}(x-a)})^{\frac{1}{2}}},\ \ H_{-1,0}(x-a)=-H_{0,1}(-x+a),$$

for any real a. The anti-kinks of (1) are given by $-H_{0,1}(x-a),\ H_{0,1}(-x+a)$ for any $a\in\mathbb{R}$.

In the article [21], for the ϕ^6 model, Manton did approximate computations to verify that the force between two static kinks is repulsive and the force between a kink and anti-kink is attractive. Furthermore, it was also obtained by approximate computations in [21] that the force of interaction between two topological solitons of the ϕ^6 model has an exponential decay with the distance between the solitons.

The study of kink and multi-kinks solutions of nonlinear wave equations has applications in many domains of mathematical physics. More precisely, the model (1) that we study has applications in condensed matter physics [2] and cosmology [9,12,31].

It is well known that the set of solutions in energy space of (1) for any potential U is invariant under space translation, time translation, and space reflection. Moreover, if H is a stationary solution of (1) and $-1<v<1$, then the function

$$\phi(t,x)=H\left(\frac{x-vt}{(1-v^2)^{\frac{1}{2}}}\right),$$

which is denominated the Lorentz transformation of H, is also a solution of the partial differential equation (1).

The problem of stability of multi-kinks is of great interest in mathematical physics, see for example [6,8]. For the integrable model *mKdV*, Muñoz proved in [23] the H^1 stability and asymptotic stability of multi-kinks. However, for many non-integrable models such as the ϕ^6 nonlinear wave equation, the asymptotic and long-time dynamics of multi-kinks after the instant where the collision or interaction happens are still unknown, even though there are numerical studies of kink-kink collision for the ϕ^6 model, see [8], which motivate our research on the topic of the description of long time behavior of a kink-kink pair.

For one-dimensional nonlinear wave equation models, results of stability of a single kink were obtained, for example, asymptotic stability under odd perturbations of a single kink of ϕ^4 model was proved in [16] and the study of the decay rate of this odd perturbation during a long time was studied in [5]. Also, in [17], Martel, Muñoz, Kowalczyk, and Van Den Bosch proved asymptotic stability of a single kink for a general class of nonlinear wave equations, including the model which we study here.

The main purpose of our paper is to describe the long time behavior of solutions $\phi(t, x)$ of (1) in the energy space such that

$$\lim_{x\to+\infty} \phi(t, x) = 1,$$
$$\lim_{x\to-\infty} \phi(t, x) = -1,$$

with total energy equal to $2E_{pot}(H_{01}) + \epsilon$, for $0 < \epsilon \ll 1$. More precisely, we proved orbital stability for a sum of two moving kinks with total energy $2E_{pot}(H_{0,1}) + \epsilon$ and we verified that the remainder has a better estimate during a long time interval which goes to $\mathbb{R}$ as $\epsilon \to 0$, indeed we proved that the estimate of the remainder during this long time interval is optimal. Also, we prove that the dynamics of the kinks' movement is very close to two explicit functions $d_j : \mathbb{R} \to \mathbb{R}$ defined in Theorem 4 during a long time interval. This result is very important to understand the behavior of two kinks after the instant of collision, which happens when the kinetic energy is minimal, indeed, our main results Theorems 2 and 4 describe the dynamics of the kinks before and after the collision instant for a long time interval. The numerical study of interaction and collision between kinks for the ϕ^6 model was done in [8], in which it was verified that the collision of kinks is close to an elastic collision when the speed of each kink is low and smaller than a critical speed v_c.

For nonlinear wave equation models in dimension 2 + 1, there are similar results obtained in the dynamics of topological multi-solitons. For the Higgs Model, there are results in the description of the dynamics of multi-vortices in [28] obtained by Stuart and in [11] obtained by Gustafson and Sigal. Indeed, we took inspiration from the proof and statement of Theorem 2 of [11] to construct our main results. Also, in [29], Stuart described the dynamics of monopole solutions for the Yang–Mills–Higgs equation. For more references, see also [7,10,20,30].

In [1], Bethuel, Orlandi, and Smets described the asymptotic behavior of solutions of a parabolic Ginzburg–Landau equation closed to multi-vortices in the initial instant. For more references, see also [14,26].

There are also results in the dynamics of multi-vortices for nonlinear Schrödinger equation, for example, the description of the dynamics of multi-vortices for the Gross–Pitaevski equation was obtained in [24] by Ovchinnikov and Sigal and results in the dynamics of vortices for the Ginzburg–Landau–Schrödinger equations were proved in [4] by Colliander and Jerrard, see also [15] for more information about Gross–Pitaevski equation.

1.2. Main results. We recall that the objective of this paper is to show orbital stability for the solutions of the Eq. (1) which are close to a sum of two interacting kinks in an initial instant and estimate the size of the time interval where better stability properties hold. The main techniques of the proof are modulation techniques adapted from [13,22,25] and a refined energy estimate method to control the size of the remainder term.

Notation 1. For any $D \subset \mathbb{R}$, any non-negative real function $f : D \subset \mathbb{R} \to \mathbb{R}$, a real function g with domain D is in $O(f(x))$ if and only if there is a uniform constant $C > 0$ such that $0 \leq |g(x)| \leq Cf(x)$. We denote that two real non-negative functions $f, g : D \subset \mathbb{R} \to \mathbb{R}_{\geq 0}$ satisfy

$$f \lesssim g,$$

if there is a constant $C > 0$ such that

$$f(x) \leq Cg(x), \text{ for all } x \in D.$$

If $f \lesssim g$ and $g \lesssim f$, we denote that $f \cong g$. We use the notation $(x)_+ := \max(x, 0)$. If $g(t, x) \in C^1(\mathbb{R}, L^2(\mathbb{R})) \cap C(\mathbb{R}, H^1(\mathbb{R}))$, then we define $\overrightarrow{g(t)} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ by

$$\overrightarrow{g(t)} = (g(t), \partial_t g(t)),$$

and we also denote the energy norm of the remainder $\overrightarrow{g(t)}$ as

$$\left\|\overrightarrow{g(t)}\right\| = \|g(t)\|_{H^1} + \|\partial_t g(t)\|_{L^2}$$

to simplify our notation in the text, where the norms $\|\cdot\|_{H^1}$, $\|\cdot\|_{L^2}$ are defined, respectively,

$$\|f_1\|_{H^1}^2 = \int_{\mathbb{R}} \frac{df_1(x)}{dx}^2 + f_1(x)^2\, dx, \quad \|f_2\|_{L^2}^2 = \int_{\mathbb{R}} f_2(x)^2\, dx,$$

for any $f_1 \in H^1(\mathbb{R})$ and any $f_2 \in L^2(\mathbb{R})$. Finally, we consider the hyperbolic functions $\operatorname{sech}, \cosh : \mathbb{R} \to \mathbb{R}$ and we are going to use the following notations

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \ \operatorname{sech}(x) = (\cosh(x))^{-1}, \text{ for every } x \in \mathbb{R}.$$

Definition 1. We define S as the set $g \in L^\infty(\mathbb{R})$ such that

$$\left\|g(x) - H_{0,1}(x) - H_{-1,0}(x)\right\|_{H^1} < +\infty.$$

From the observations made about the local well-posedness of partial differential equation (1) in the energy space and, since $1, -1$ are in $\mathcal{V}$, we have that (1) is also locally well-posed in the affine space $S \times L^2(\mathbb{R})$. Motivated by the proof and computations that we are going to present, we also consider:

Definition 2. We define for $x_1, x_2 \in \mathbb{R}$

$$H_{0,1}^{x_2}(x) := H_{0,1}(x - x_2) \text{ and } H_{-1,0}^{x_1}(x) := H_{-1,0}(x - x_1),$$

and we say that x_2 is the kink center of $H_{0,1}^{x_2}(x)$ and x_1 is the kink center of $H_{-1,0}^{x_1}(x)$.

There are also non-stationary solutions $(\phi(t,x), \partial_t\phi(t,x))$ of (1) with finite total energy $E_{total}(\phi(t), \partial_t\phi(t))$ that satisfy for all $t \in \mathbb{R}$

$$\lim_{x\to+\infty} \phi(t,x) = 1, \quad \lim_{x\to-\infty} \phi(t,x) = 0. \tag{2}$$

But, for any $a \in \mathbb{R}$, the kinks $H_{0,1}(x-a)$ are the unique functions that minimize the potential energy in the set of functions $\phi(x)$ satisfying condition (2), the proof of this fact follows from the Bogomolny identity, see [19] or section 2 of [13]. By similar reasoning, we can verify that all functions $\phi(x) \in S$ have $E_{pot}(\phi) > 2E_{pot}(H_{0,1})$.

Definition 3. We define the energy excess ϵ of a solution $(\phi(t), \partial_t\phi(t)) \in S \times L^2(\mathbb{R})$ as the following value

$$\epsilon = E_{total}(\phi(t), \partial_t\phi(t)) - 2E_{pot}(H_{0,1}).$$

We recall the notation $(x)_+ := \max(x, 0)$. It's not difficult to verify the following inequalities

(D1) $\left|H_{0,1}(x)\right| \leq e^{-\sqrt{2}(-x)_+}$,
(D2) $\left|H_{-1,0}(x)\right| \leq e^{-\sqrt{2}(x)_+}$,
(D3) $\left|\dot{H}_{0,1}(x)\right| \leq \sqrt{2}e^{-\sqrt{2}(-x)_+}$,
(D4) $\left|\dot{H}_{-1,0}(x)\right| \leq \sqrt{2}e^{-\sqrt{2}(x)_+}$.

Moreover, since

$$\ddot{H}_{0,1}(x) = \dot{U}(H_{0,1}(x)), \tag{3}$$

we can verify by induction the following estimate

$$\left|\frac{d^k H_{0,1}(x)}{dx^k}\right| \lesssim_k \min\left(e^{-2\sqrt{2}x}, e^{\sqrt{2}x}\right) \tag{4}$$

for all $k \in \mathbb{N}\backslash\{0\}$. The following result is crucial in the framework of this manuscript:

Lemma 1 (Modulation Lemma). *There exist $C_0, \delta_0 > 0$, such that if $0 < \delta \leq \delta_0$, x_1, x_2 are real numbers with $x_2 - x_1 \geq \frac{1}{\delta}$ and $g \in H^1(\mathbb{R})$ satisfies $\|g\|_{H^1} \leq \delta$, then for $\phi(x) = H_{-1,0}(x-x_1) + H_{0,1}(x-x_2) + g(x)$, there exist unique y_1, y_2 such that for*

$$g_1(x) = \phi(x) - H_{-1,0}(x-y_1) - H_{0,1}(x-y_2),$$

the four following statements are true

1 $\langle g_1, \partial_x H_{-1,0}(x-y_1)\rangle_{L^2} = 0$,
2 $\langle g_1, \partial_x H_{0,1}(x-y_2)\rangle_{L^2} = 0$,
3 $\|g_1\|_{H^1} \leq C_0\delta$,
4 $|y_2 - x_2| + |y_1 - x_1| \leq C_0\delta$.

We will refer to the first and second statements as the orthogonality conditions of the Modulation Lemma.

Proof. The proof follows from the implicit function theorem for Banach spaces. □

Now, our main results are the following:

Theorem 2. *There exist* $C, \delta_0 > 0$*, such that if* $\epsilon < \delta_0$ *and*

$$(\phi(0), \partial_t\phi(0)) \in S \times L^2(\mathbb{R})$$

with $E_{total}(\phi(0), \partial_t\phi(0)) = 2E_{pot}(H_{0,1}) + \epsilon$*, then there exist functions* $x_1, x_2 \in C^2(\mathbb{R})$ *such that, for all* $t \in \mathbb{R}$*, the unique global time solution* $\phi(t, x)$ *of (1) is given by*

$$\phi(t) = H_{0,1}(x - x_2(t)) + H_{-1,0}(x - x_1(t)) + g(t), \tag{5}$$

with $g(t)$ *satisfying, for any* $t \in \mathbb{R}$*, the orthogonality conditions of the Modulation Lemma and*

$$e^{-\sqrt{2}(x_2(t)-x_1(t))} + \max_{j\in\{1,2\}} \left|\ddot{x}_j(t)\right| + \max_{j\in\{1,2\}} \dot{x}_j(t)^2 + \left\|\overrightarrow{g(t)}\right\|^2_{H^1\times L^2} \leq C\epsilon. \tag{6}$$

Furthermore, we have that

$$\|(g(t), \partial_t g(t))\|^2_{H^1\times L^2} \leq C \min\left(\epsilon, \left[\left\|\overrightarrow{g(0)}\right\|^2 + \epsilon^2\right] \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right) \text{ for all } t \in \mathbb{R}. \tag{7}$$

Remark 1. In notation of the statement of Theorem 2, for any $\delta > 0$, there exists $K(\delta) \in (0, 1)$ such that if $0 < \epsilon < K(\delta)$, $E_{total}(\phi(0), \partial_t\phi(0)) = 2E_{pot}(H_{0,1}) + \epsilon$, then we have that $\|(g(0), \partial_t g(0))\|_{H^1\times L^2} < \delta$ and $x_2(0) - x_1(0) > \frac{1}{\delta}$, for the proof see Lemma 21 and Corollary 22 in the Appendix Section A.

Theorem 3. *In notation of Theorem 2, there exist constants* $\delta, \kappa > 0$ *such that if* $0 < \epsilon < \delta$*, then* $\frac{\epsilon}{\kappa+1} \leq \left\|\overrightarrow{g(T)}\right\|$ *for some* $T \in \mathbb{R}$ *satisfying* $0 \leq T \leq (\kappa + 1)\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

Proof. See the Appendix Section B. □

Remark 2. Theorem 3 implies that estimate (7) is relevant in a time interval $(-T, T)$ for a $T > 0$ of order $-\epsilon^{-\frac{1}{2}}\ln(\epsilon)$. More precisely, for any function $r : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{h\to 0} r(h) = 0$, there is a positive value $\delta(r)$ such that if $0 < \epsilon < \delta(r)$ and $\left\|\overrightarrow{g(0)}\right\| \leq r(\epsilon)\epsilon$, then $\epsilon \lesssim \left\|\overrightarrow{g(t)}\right\|$ for some $0 < t = O\left(\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}\right)$.

Remark 3. Theorem 3 also implies the existence of a $\delta_0 > 0$ such that if $0 < \epsilon < \delta_0$, then, for any $(\phi(0, x), \partial_t\phi(0, x)) \in S \times L^2(\mathbb{R})$ with $E_{total}(\phi(0), \partial_t\phi(0))$ equals to $2E_{pot}(H_{0,1}) + \epsilon$, $g(t, x)$ defined in identity (5) satisfies $\epsilon \lesssim \limsup_{t\to+\infty}\left\|\overrightarrow{g(t)}\right\|$, similarly we have that $\epsilon \lesssim \limsup_{t\to-\infty}\left\|\overrightarrow{g(t)}\right\|$.

Theorem 4. *Let* ϕ *satisfy the assumptions in Theorem 2 and* x_1, x_2*, and* g *be as in the conclusion of this theorem. Let the functions* d_1, d_2 *be defined for any* $t \in \mathbb{R}$ *by*

$$d_1(t) = a + bt - \frac{1}{2\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right), \tag{8}$$

$$d_2(t) = a + bt + \frac{1}{2\sqrt{2}}\ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right), \tag{9}$$

where $a, b, c \in \mathbb{R}$ *and* $v \in (0,1)$ *are the unique real values satisfying* $d_j(0) = x_j(0),\ \dot{d}_j(0) = \dot{x}_j(0)$ *for* $j \in \{1, 2\}$. *Let* $d(t) = d_2(t) - d_1(t),\ z(t) = x_2(t) - x_1(t)$. *Then, for all* $t \in \mathbb{R}$, *we have*

$$|z(t) - d(t)| \leq C \min(\epsilon^{\frac{1}{2}}|t|, \epsilon t^2),\ \ |\dot{z}(t) - \dot{d}(t)| \leq C\epsilon|t|.$$

Furthermore, for any $t \in \mathbb{R}$,

$$\epsilon \max_{j\in\{1,2\}} \left|d_j(t) - x_j(t)\right| = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\right)^2 \left(\ln\frac{1}{\epsilon}\right)^{11} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right), \tag{10}$$

$$\epsilon^{\frac{1}{2}} \max_{j\in\{1,2\}} \left|\dot{d}_j(t) - \dot{x}_j(t)\right| = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\right)^2 \left(\ln\frac{1}{\epsilon}\right)^{11} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right). \tag{11}$$

Remark 4. If $\left\|\overrightarrow{g(0)}\right\| = O(\epsilon)$, then the estimates (10) and (11) imply that the functions $x_j(t),\ \dot{x}_j(t)$ are very close to $d_j(t),\ \dot{d}_j(t)$ during a time interval of order $-\ln(\epsilon)\epsilon^{-\frac{1}{2}}$.

Remark 5. The proof of Theorems 2 and 4 for $t \leq 0$ is analogous to the proof for $t \geq 0$, so we will only prove them for $t \geq 0$.

Theorem 4 describes the repulsive behavior of the kinks. More precisely, if the kinetic energy of the kinks and the energy norm of the remainder g are small enough in the initial instant $t = 0$, then the kinks will move away with displacement $z(t) \cong \epsilon^{\frac{1}{2}}t + \ln\frac{1}{\epsilon}$ when $t > 0$ is big enough belonging to a large time interval.

Furthermore, using Theorem 4, we can also deduce the following corollary.

Corollary 5. *With the same hypotheses as in Theorem 4, we have that*

$$\max_{j\in\{1,2\}} \left|\ddot{d}_j(t) - \ddot{x}_j(t)\right| = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\right)\epsilon^{\frac{1}{2}} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right) + O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\right)^2 \left(\ln\frac{1}{\epsilon}\right)^{11} \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right)\right).$$

Proof of Corollary 5. It follows directly from Theorem 4 and from Lemma 19 presented in the Appendix Section A. □

1.3. Resume of the proof. In this subsection, we present how the article is organized and explain briefly the content of each section.

Section 2. In this section, we prove orbital stability of a perturbation of a sum of two kinks. Moreover, we prove that if the initial data $(\phi(0,x), \partial_t\phi(0,x))$ satisfies the hypotheses of Theorem 2, then there are real functions $x_1,\ x_2$ of class C^2 such that for all $t \geq 0$

$$\left\|\phi(t,x) - H_{0,1}^{x_2(t)} - H_{-1,0}^{x_1(t)}\right\|_{H^1} \lesssim \epsilon^{\frac{1}{2}},$$

$$\left\|\partial_t\left(\phi(t,x) - H_{0,1}^{x_2(t)} - H_{-1,0}^{x_1(t)}\right)\right\|_{L^2} \lesssim \epsilon^{\frac{1}{2}}.$$

First, for every $z > 0$, we are going to demonstrate the following estimate

$$E_{pot}\left(H_{0,1}(x-z)+H_{-1,0}(x)\right)=2E_{pot}\left(H_{0,1}\right)+2\sqrt{2}e^{-\sqrt{2}z}+O\left((z+1)e^{-2\sqrt{2}z}\right). \tag{12}$$

The proof of this inequality is similar to the demonstration of Lemma 2.7 of [13] and it follows using the Fundamental Theorem of Calculus.

The proof of the orbital stability will follow from studying the expression

$$E_{pot}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g)-E_{pot}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}),$$

using the fact that the kinks are critical points of E_{pot} and the spectral properties of the operator $D^2E_{pot}\left(H_{0,1}\right)$, which is also non-negative. Moreover, from the modulation lemma, we will introduce the functions $x_2,\ x_1$ that will guarantee the following coercivity property

$$\left\|\overrightarrow{g(t)}\right\|^2\lesssim E_{pot}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g)-E_{pot}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}).$$

Therefore, the estimate above and (12) will imply that

$$e^{-\sqrt{2}(x_2(t)-x_1(t))}+\left\|\overrightarrow{g(t)}\right\|^2\lesssim\epsilon. \tag{13}$$

From the orthogonality conditions of the Modulation Lemma and standard ordinary differential equation techniques, see Chapter 1 of [3], we also obtain uniform bounds for $\left\|\dot{x}_j(t)\right\|_{L^\infty(\mathbb{R})},\ \left\|\ddot{x}_j(t)\right\|_{L^\infty(\mathbb{R})}$ for $j\in\{1,\ 2\}$. More precisely, the modulation parameters x_1 and x_2 are going to satisfying the following estimate

$$\max_{j\in\{1,2\}}\left\|\dot{x}_j(t)\right\|_{L^\infty(\mathbb{R})}^2+\left\|\ddot{x}_j(t)\right\|_{L^\infty(\mathbb{R})}\lesssim\epsilon. \tag{14}$$

The main techniques of this section are an adaption of sections 2 and 3 of [13].

Section 3. In this section, we study the long-time behavior of $\dot{x}_j(t),\ x_j(t)$ for $j\in\{1,\ 2\}$. More precisely, we prove that the parameters x_1 and x_2 satisfy the following system of differential inequalities

$$\dot{x}_j(t)=p_j(t)+O\left(\zeta(t)\right), \tag{15}$$

$$\dot{p}_j(t)=(-1)^{j+1}\frac{1}{\left\|\dot{H}_{0,1}\right\|_{L^2}^2}\frac{d}{dz}\Big|_{z=x_2(t)-x_1(t)}E_{pot}\left(H_{0,1}^z+H_{-1,0}\right)+O\left(\alpha(t)\right), \tag{16}$$

for $j\in\{1,2\}$, where $\alpha(t),\zeta(t)$ are non-negative functions depending only on the functions $\left(x_j(t)\right)_{j\in\{1,2\}},\ \left(\dot{x}_j(t)\right)_{j\in\{1,2\}},\ \left\|\overrightarrow{g(t)}\right\|$ and satisfying

$$\alpha(t)\lesssim\frac{\epsilon}{\ln\ln\frac{1}{\epsilon}},\ \zeta(t)\lesssim\epsilon\ln\frac{1}{\epsilon},\text{ for all } t\in\mathbb{R}, \tag{17}$$

because of the estimates (13) and (14). However, the estimates (17) can be improved during a large time interval if we could use the estimate (7) in the place of $\left\|\overrightarrow{g(t)}\right\|=O(\epsilon^{\frac{1}{2}})$.

Our proof of estimates (15), (16) is based on the proof of Lemma 3.5 from [13]. First, for each $j \in \{1, 2\}$, the estimate (15) is obtained from the time derivative of the equations

$$\left\langle \phi(t,x) - H_{-1,0}(x - x_1(t)) - H_{0,1}\left(x - x_2(t)\right), \partial_x H_{0,1}(x - x_2(t))\right\rangle_{L^2} = 0,$$
$$\left\langle \phi(t,x) - H_{-1,0}(x - x_1(t)) - H_{0,1}\left(x - x_2(t)\right), \partial_x H_{-1,0}(x - x_1(t))\right\rangle_{L^2} = 0,$$

which are the orthogonality conditions of the Modulation Lemma. Indeed, we are going to obtain that

$$\dot{x}_1(t) = -\frac{\left\langle \partial_t \phi(t,x), \partial_x H^{x_1(t)}_{-1,0}(x)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}} + O\left(\zeta(t)\right),$$
$$\dot{x}_2(t) = -\frac{\left\langle \partial_t \phi(t,x), \partial_x H^{x_2(t)}_{0,1}(x)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}} + O\left(\zeta(t)\right).$$

Next, we are going to construct a smooth cut-off function $0 \leq \chi \leq 1$ satisfying

$$\chi(x) = \begin{cases} 1, \text{ if } x \leq \theta(1-\gamma), \\ 0, \text{ if } x \geq \theta, \end{cases}$$

where $0 < \gamma, \theta < 1$ are parameters that will be chosen later with the objective of minimizing the modulus of the time derivative of

$$p_1(t) = -\frac{\left\langle \partial_t \phi(t),\ \partial_x H^{x_1(t)}_{-1,0}(x) + \partial_x\left(\chi\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right) g(t)\right)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}},$$
$$p_2(t) = -\frac{\left\langle \partial_t \phi(t),\ \partial_x H^{x_2(t)}_{0,1}(x) + \partial_x\left(\left[1 - \chi\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right)\right] g(t)\right)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}},$$

from which with the second time derivative of the orthogonality conditions of Modulation Lemma and the partial differential equation (1), we will deduce the estimate (16) for $j \in \{1, 2\}$.

Section 4. In Section 4, we introduce a function $F(t)$ with the objective of controlling $\left\|\overrightarrow{g(t)}\right\|$ for a long time interval. More precisely, we show that the function $F(t)$ satisfies for a constant $K > 0$ the global estimate $\left\|\overrightarrow{g(t)}\right\|^2 \lesssim F(t) + K\epsilon^2$ and we show that $|\dot{F}(t)|$ is small enough for a long time interval. We start the function from the quadratic part of the total energy of $\phi(t)$, more precisely with

$$D(t) = \|\partial_t g(t,x)\|^2_{L^2} + \|\partial_x g(t,x)\|^2_{L^2} + \int_{\mathbb{R}} \ddot{U}(H^{x_2(t)}_{0,1}(x) + H^{x_1(t)}_{-1,0}(x)) g(t,x)^2\, dx.$$

However, we obtain that the terms of worst decay that appear in the computation of $\dot{D}(t)$ are of the form

$$\int_{\mathbb{R}} \left[\partial_t\left(g(t,x)^k\right)\right] J(x_1, x_2, \dot{x}_1, \dot{x}_2, x)\, dx, \tag{18}$$

where $k \in \{1, 2, 3\}$ and the function J satisfies for some $l \in \mathbb{Q}_{\geq 0}$ the following estimates

$$\sup_{t\in\mathbb{R}} \max_{j\in\{1,2\}} \left\| \frac{\partial}{\partial x_j} J(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x) \right\|_{L^2} \lesssim \epsilon^l,$$
$$\sup_{t\in\mathbb{R}} \max_{j\in\{1,2\}} \left\| \frac{\partial}{\partial \dot{x}_j} J(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x) \right\|_{L^2} \lesssim \epsilon^{l-\frac{1}{2}},$$

and

$$\sup_{t\in\mathbb{R}} \| J\,(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x)\|_{L^2} \lesssim \epsilon^l \text{ if } k = 1, \text{ otherwise}$$
$$\sup_{t\in\mathbb{R}} \| J\,(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x)\|_{L^\infty_x(\mathbb{R})} \lesssim \epsilon^l \text{ when } k \in \{2, 3\}.$$

But, we can cancel these bad terms after we add to the function $D(t)$ correction terms of the form

$$-\int_{\mathbb{R}} \left(g(t, x)^k\right) J(x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t), x)\, dx, \tag{19}$$

and now, in the time derivative of the sum of $D(t)$ with these correction terms, we obtain an expression with a size of order $\epsilon^{l+\frac{1}{2}} \left\|\overrightarrow{g(t)}\right\|^k$ which is much smaller than $\epsilon^l \left\|\overrightarrow{g(t)}\right\|^k$ because of inequality (14) obtained in Section 2 of this manuscript. Next, we consider a smooth cut-off function $0 \leq \omega \leq 1$ satisfying

$$\omega(x) = \begin{cases} 1, \text{ if } x \leq \frac{1}{2}, \\ 0, \text{ if } x \geq \frac{3}{4}, \end{cases}$$

and $\omega_1(t, x) = \omega\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right)$. Based on the argument in the proof of Lemma 4.2 of [13], we aggregate the last correction term

$$2\int_{\mathbb{R}} \partial_t g(t, x)\partial_x g(t, x)\, [\dot{x}_1(t)\omega_1(t, x) + \dot{x}_2(t)\,(1 - \omega_1(t, x))]\, dx,$$

whose time derivative will cancel with the term

$$-\int_{\mathbb{R}} U^{(3)}(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x))(\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)} + \dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)})g(t, x)^2\, dx,$$

which comes from $\dot{D}(t)$, since we cannot remove this expression using the correction terms similar to (19). Finally, we evaluate the time derivative of the function $F(t)$ obtained from the sum $D(t)$ with all the correction terms described above.

Remaining Sections. In the remaining part of this paper, we prove our main results, the estimate (7) of Theorem 2 is a consequence of the energy estimate obtained in Section 4 and the estimates with high precision of the modulation parameters $x_1(t)$, $x_2(t)$ which are obtained in Section 5. In Section 5, we prove the result of Theorem 4, where we study the evolution of the precision of the modulation parameters estimates by comparing it with a solution of a system of ordinary differential equations. Complementary information is given in Appendix Section A and the proof of Theorem 3 is in the Appendix Section B.

2. Global Stability of Two Moving Kinks

Before the presentation of the proofs of the main theorems, we define a function to study the potential energy of a sum of two kinks.

Definition 4. The function $A : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$A(z) := E_{pot}(H^z_{0,1}(x) + H_{-1,0}(x)). \tag{20}$$

The study of the function A is essential to obtain global control of the norm of the remainder g and the lower bound of $x_2(t) - x_1(t)$ in Theorem 2.

Remark 6. It is easy to verify that $E_{pot}(H_{0,1}(x-x_2)+H_{-1,0}(x-x_1)) = E_{pot}(H_{0,1}(x-(x_2-x_1)) + H_{-1,0}(x))$.

We will use several times the following elementary estimate from the Lemma 2.5 of [13] given by:

Lemma 6. *For any real numbers x_2, x_1, such that $x_2 - x_1 > 0$ and $\alpha,\ \beta > 0$ with $\alpha \neq \beta$ the following bound holds:*

$$\int_{\mathbb{R}} e^{-\alpha(x-x_1)_+} e^{-\beta(x_2-x)_+}\, dx \lesssim_{\alpha,\beta} e^{-\min(\alpha,\beta)(x_2-x_1)},$$

For any $\alpha > 0$, the following bound holds

$$\int_{\mathbb{R}} e^{-\alpha(x-x_1)_+} e^{-\alpha(x_2-x)_+}\, dx \lesssim_{\alpha} (1 + (x_2 - x_1))e^{-\alpha(x_2-x_1)}.$$

The main result of this section is the following

Lemma 7. *The function A is of class C^2 and there is a constant $C > 0$, such that*

1. $\left|\ddot{A}(z) - 4\sqrt{2}e^{-\sqrt{2}z}\right| \leq C(z+1)e^{-2\sqrt{2}z}$,
2. $\left|\dot{A}(z) + 4e^{-\sqrt{2}z}\right| \leq C(z+1)e^{-2\sqrt{2}z}$,
3. $\left|A(z) - 2E_{pot}(H_{0,1}) - 2\sqrt{2}e^{-\sqrt{2}z}\right| \leq C(z+1)e^{-2\sqrt{2}z}$.

Proof. By the definition of A, it's clear that

$$\begin{aligned} A(z) &= \frac{1}{2}\int_{\mathbb{R}} \left(\partial_x\left[H^z_{0,1}(x) + H_{-1,0}(x)\right]\right)^2 dx + \int_{\mathbb{R}} U(H^z_{0,1}(x) + H_{-1,0}(x))\, dx \\ &= \left\|\partial_x H_{0,1}\right\|^2_{L^2} + \int_{\mathbb{R}} \partial_x H^z_{0,1}(x)\partial_x H_{-1,0}(x)\, dx + \int_{\mathbb{R}} U(H^z_{0,1}(x) + H_{-1,0}(x))\, dx. \end{aligned}$$

Since the functions U and $H_{0,1}$ are smooth and $\partial_x H_{0,1}(x)$ has exponential decay when $|x| \to +\infty$, it is possible to differentiate $A(z)$ in z. More precisely, we obtain

$$\begin{aligned} \dot{A}(z) &= -\int_{\mathbb{R}} \partial_x^2 H^z_{0,1}(x)\partial_x H_{-1,0}(x)\, dx - \int_{\mathbb{R}} \dot{U}(H^z_{0,1}(x) + H_{-1,0}(x))\partial_x H^z_{0,1}(x)\, dx \\ &= \int_{\mathbb{R}} \partial_x H^z_{0,1}(x)\left[\dot{U}(H_{-1,0})(x) - \dot{U}(H_{-1,0}(x) + H^z_{0,1}(x))\right] dx. \end{aligned}$$

For similar reasons, it is always possible to differentiate $A(z)$ twice, precisely, we obtain

$$\ddot{A}(z) = \int_{\mathbb{R}} \partial_x H^z_{0,1}(x)^2 \ddot{U}(H_{-1,0}(x) + H^z_{0,1}(x)) \\ - \partial_x^2 H^z_{0,1}(x)\big[\dot{U}(H_{-1,0}(x)) - \dot{U}\big(H_{-1,0}(x) + H^z_{0,1}(x)\big)\big]\, dx. \tag{21}$$

Then, integrating by parts, we obtain

$$\ddot{A}(z) = \int_{\mathbb{R}} \partial_x H^z_{0,1}(x) \partial_x H_{-1,0}(x)\big[\ddot{U}(H_{-1,0}(x)) - \ddot{U}(H_{-1,0}(x) + H^z_{0,1}(x))\big]\, dx. \tag{22}$$

Now, we consider the function

$$B(z) = \int_{\mathbb{R}} \partial_x H_{0,1}(x) \partial_x H_{-1,0}(x+z)\big[\ddot{U}(0) - \ddot{U}(H_{0,1}(x))\big]\, dx. \tag{23}$$

Then, we have

$$\ddot{A}(z) - B(z) = \int_{\mathbb{R}} \partial_x H^z_{0,1}(x) \partial_x H_{-1,0}(x)\Big[\ddot{U}(H_{-1,0}(x)) - \ddot{U}(H_{-1,0}(x) + H^z_{0,1}(x))\Big]\, dx \\ - \int_{\mathbb{R}} \partial_x H^z_{0,1}(x) \partial_x H_{-1,0}(x)\Big[\ddot{U}(0) - \ddot{U}(H^z_{0,1}(x))\Big] dx. \tag{24}$$

Also, it is not difficult to verify the following identity

$$\big[\ddot{U}(H_{-1,0}(x)) - \ddot{U}(H_{-1,0}(x) + H^z_{0,1}(x))\big] - \big[\ddot{U}(0) - \ddot{U}(H^z_{0,1}(x))\big] \\ = - \int_0^{H_{-1,0}(x)} \int_0^{H^z_{0,1}(x)} U^{(4)}(\omega_1 + \omega_2)\, d\omega_1\, d\omega_2. \tag{25}$$

So, the identities (25) and (24) imply the following inequality

$$\big|\ddot{A}(z) - B(z)\big| \\ \leq \int_{\mathbb{R}} \left|\partial_x H^z_{0,1}(x) \partial_x H_{-1,0}(x)\right| \left|\int_0^{H_{-1,0}(x)} \int_0^{H^z_{0,1}(x)} U^{(4)}(\omega_1 + \omega_2)\, d\omega_1\, d\omega_2\right|\, dx.$$

Since U is smooth and $\left\|H_{0,1}\right\|_{L^\infty} = 1$, we have that there is a constant $C > 0$ such that

$$\big|\ddot{A}(z) - B(z)\big| \leq C \int_{\mathbb{R}} \left|\partial_x H^z_{0,1}(x) \partial_x H_{-1,0}(x) H_{-1,0}(x) H^z_{0,1}(x)\right|\, dx. \tag{26}$$

Now, using the inequalities from (D1) to (D4) and Lemma 6 to inequality (26), we obtain that there exists a constant C_1 independent of z such that

$$\big|\ddot{A}(z) - B(z)\big| \leq C_1(z+1)e^{-2\sqrt{2}z}. \tag{27}$$

Also, it is not difficult to verify that the estimate

$$\left|\partial_x H_{-1,0}(x) - \sqrt{2}e^{-\sqrt{2}x}\right| \leq C \min(e^{-3\sqrt{2}x}, e^{-\sqrt{2}x}), \tag{28}$$

and the identity (23) imply the inequality

$$\left|B(z)-\sqrt{2}e^{-\sqrt{2}z}\int_{\mathbb{R}}e^{-\sqrt{2}x}\partial_x H_{0,1}(x)(\ddot{U}(0)-\ddot{U}(H_{0,1}(x)))\,dx\right|$$
$$\lesssim \int_{\mathbb{R}} H_{0,1}(x)\partial_x H_{0,1}(x)\min\left(e^{-3\sqrt{2}(x+z)},e^{-\sqrt{2}(x+z)}\right)dx$$
$$\lesssim \int_{\mathbb{R}} e^{-2\sqrt{2}(-x)_+}\min\left(e^{-3\sqrt{2}(x+z)},e^{-\sqrt{2}(x+z)}\right)dx$$
$$\lesssim \int_{-\infty}^{0} e^{-2\sqrt{2}(z-x)_+}e^{-\sqrt{2}x}\,dx+\int_{0}^{+\infty} e^{-2\sqrt{2}(z-x)_+}e^{-3\sqrt{2}(x)_+}\,dx.$$

Since we have the following identity and estimate from Lemma 6

$$\int_{-\infty}^{0} e^{-2\sqrt{2}(z-x)}e^{-\sqrt{2}x}\,dx=\frac{e^{-2\sqrt{2}z}}{\sqrt{2}}, \tag{29}$$

$$\int_{0}^{+\infty} e^{-2\sqrt{2}(z-x)_+}e^{-3\sqrt{2}(x)_+}\lesssim e^{-2\sqrt{2}z}, \tag{30}$$

we obtain, then:

$$\left|B(z)-\sqrt{2}e^{-\sqrt{2}z}\int_{\mathbb{R}}e^{-\sqrt{2}x}\partial_x H_{0,1}(x)\big[\ddot{U}(0)-\ddot{U}(H_{0,1}(x))\big]\,dx\right|\lesssim e^{-2\sqrt{2}z}, \tag{31}$$

which clearly implies with (27) the inequality

$$\left|\ddot{A}(z)-\sqrt{2}e^{-\sqrt{2}z}\int_{\mathbb{R}}e^{-\sqrt{2}x}\partial_x H_{0,1}(x)\big[\ddot{U}(0)-\ddot{U}(H_{0,1}(x))\big]\,dx\right|\lesssim (z+1)e^{-2\sqrt{2}z}. \tag{32}$$

Also, we have the identity

$$\int_{\mathbb{R}}\left(8(H_{0,1}(x))^3-6(H_{0,1}(x))^5\right)e^{-\sqrt{2}x}\,dx=2\sqrt{2}, \tag{33}$$

for the proof see the end of Appendix A. Since we have the identity $\ddot{U}(0)-\ddot{U}(\phi)=24\phi^2-30\phi^4$, by integration by parts, we obtain

$$\int_{\mathbb{R}}\frac{e^{-\sqrt{2}x}}{\sqrt{2}}\partial_x H_{0,1}(x)\big[\ddot{U}(0)-\ddot{U}(H_{0,1}(x))\big]\,dx$$
$$=\int_{\mathbb{R}}\left(8(H_{0,1}(x))^3-6(H_{0,1}(x))^5\right)e^{-\sqrt{2}x}\,dx.$$

In conclusion, inequality (32) is equivalent to $\left|\ddot{A}(z)-4\sqrt{2}e^{-\sqrt{2}z}\right|\lesssim (z+1)e^{-2\sqrt{2}z}$.

The identities

$$\dot{U}(\phi)+\dot{U}(\theta)-\dot{U}(\phi+\theta)=24\phi\theta(\phi+\theta)-6\Big(\sum_{j=1}^{4}\binom{5}{j}\phi^j\omega^{5-j}\Big),$$

$$\dot{A}(z) = -\int_{\mathbb{R}} \partial_x H^z_{0,1}(x)\big[\dot{U}(H^z_{0,1}(x) + H_{-1,0}(x)) + \dot{U}(H_{-1,0}(x)) - \dot{U}(H^z_{0,1}(x))\big]\,dx$$

and Lemma 6 imply the following estimate for $z > 0$

$$\left|\dot{A}(z)\right| \lesssim e^{-\sqrt{2}z},$$

so $\lim_{|z|\to+\infty}\left|\dot{A}(z)\right| = 0$. In conclusion, integrating inequality $\left|\ddot{A}(z) - 4\sqrt{2}e^{-\sqrt{2}z}\right| \lesssim (z+1)e^{-2\sqrt{2}z}$ from z to $+\infty$ we obtain the second result of the lemma

$$\left|\dot{A}(z) + 4e^{-\sqrt{2}z}\right| \lesssim (z+1)e^{-2\sqrt{2}z}. \tag{34}$$

Finally, from the fact that $\lim_{z\to+\infty} E_{pot}(H_{-1,0}+H^z_{0,1}(x)) = 2E_{pot}(H_{0,1})$, we obtain the last estimate integrating inequality (34) from z to $+\infty$, which is

$$\left|2E_{pot}(H_{0,1}) + 2\sqrt{2}e^{-\sqrt{2}z} - A(z)\right| \lesssim (z+1)e^{-2\sqrt{2}z}.$$

□

It is not difficult to verify that the Fréchet derivative of E_{pot} as a linear functional from $H^1(\mathbb{R})$ to $\mathbb{R}$ is given by

$$(DE_{pot}(\phi))(v) := \int_{\mathbb{R}} \partial_x\phi(x)\partial_x v(x) + \dot{U}(\phi(x))v(x)\,dx. \tag{35}$$

Also, for any $v,\ w \in H^1(\mathbb{R})$, it is not difficult to verify that

$$\left\langle D^2E_{pot}(\phi)v,\ w\right\rangle_{L^2} = \int_{\mathbb{R}} \partial_x v(x)\partial_x w(x)\,dx + \int_{\mathbb{R}} \ddot{U}(\phi(x))v(x)w(x)\,dx. \tag{36}$$

Moreover, the operator $D^2E_{pot}\left(H_{0,1}\right) : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$ satisfies the following property.

Lemma 8. *The operator $D^2E_{pot}\left(H_{0,1}\right)$ satisfies:*

$$\ker\left(D^2E_{pot}\left(H_{0,1}\right)\right) = \{c\partial_x H_{0,1}(x) | \, c \in \mathbb{R}\},$$

$$\left\langle D^2E_{pot}\left(H_{0,1}\right)g, g\right\rangle_{L^2} \geq c\left[\|g\|^2_{L^2} - \langle g, \partial_x H_{0,1}\rangle^2_{L^2}\frac{1}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}}\right],$$

for a constant $c > 0$ and any $g \in H^1(\mathbb{R})$.

Proof. See Proposition 2.2 from [13], see also [18]. □

Lemma 9 (Coercivity Lemma). *There exist $C, c,\ \delta > 0$, such that if $x_2 - x_1 \geq \frac{1}{\delta}$, then for any $g \in H^1(\mathbb{R})$ we have*

$$\left\langle D^2E_{pot}(H^{x_2}_{0,1} + H^{x_1}_{-1,0})g,\ g\right\rangle_{L^2} \geq c\,\|g\|^2_{H^1} - C\left[\langle g,\ \partial_x H^{x_1}_{-1,0}\rangle^2_{L^2} + \langle g,\ \partial_x H^{x_2}_{0,1}\rangle^2_{L^2}\right]. \tag{37}$$

Proof of Coercivity Lemma. The proof of this Lemma is analogous to the proof of Lemma 2.4 in [13]. □

Lemma 10. *There is a constant C_2, such that if $x_2 - x_1 > 0$, then*

$$\left\| DE_{pot}(H^{x_2}_{0,1} + H^{x_1}_{-1,0}) \right\|_{L^2(\mathbb{R})} \leq C_2 e^{-\sqrt{2}(x_2 - x_1)}. \tag{38}$$

Proof. By the definition of the potential energy, the equation (3), and the exponential decay of the two kinks functions, we have that

$$DE_{pot}(H^{x_2}_{0,1} + H^{x_1}_{-1,0}) = \dot{U}(H^{x_2}_{0,1} + H^{x_1}_{-1,0}) - \dot{U}(H^{x_2}_{0,1}) - \dot{U}(H^{x_1}_{-1,0})$$

as a bounded linear operator from $L^2(\mathbb{R})$ to $\mathbb{C}$. So, we have that

$$DE_{pot}(H^{x_2}_{0,1} + H^{x_1}_{-1,0}) = -24 H^{x_2}_{0,1} H^{x_1}_{-1,0}\left[H^{x_2}_{0,1} + H^{x_1}_{-1,0}\right] \\ +6\Big[\sum_{j=1}^{4} \binom{5}{j} (H^{x_1}_{-1,0})^j (H^{x_2}_{0,1})^{5-j}\Big],$$

and, then, the conclusion follows directly from Lemma 6, (D1) and (D2). □

Theorem 11 (Orbital Stability of a sum of two moving kinks). *There exists $\delta_0 > 0$ such that if the solution ϕ of (1) satisfies $(\phi(0), \partial_t\phi(0)) \in S \times L^2(\mathbb{R})$ and the energy excess $\epsilon = E_{total}(\phi) - 2E_{pot}(H_{0,1})$ is smaller than δ_0, then there exist $x_1, x_2 : \mathbb{R} \to \mathbb{R}$ functions of class C^2, such that for all $t \in \mathbb{R}$ denoting $g(t) = \phi(t) - H_{0,1}(x - x_2(t)) - H_{-1,0}(x - x_1(t))$ and $z(t) = x_2(t) - x_1(t)$, we have:*

1. $\|g(t)\|_{H^1} = O(\epsilon^{\frac{1}{2}})$,
2. $z(t) \geq \frac{1}{\sqrt{2}}\left[\ln\frac{1}{\epsilon} + \ln 2\right]$,
3. $\|\partial_t\phi(t)\|^2_{L^2} \leq 2\epsilon$,
4. $\max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|^2 + \max_{j\in\{1,2\}} \left|\ddot{x}_j(t)\right| = O(\epsilon)$.

Proof. First, from the fact that $E_{total}(\phi(x)) > 2E_{pot}(H_{0,1})$, we deduce, from the conservation of total energy, the estimate

$$\|\partial_t\phi(t)\|^2_{L^2} \leq 2\epsilon. \tag{39}$$

From Remark 1, we can assume if $\epsilon \ll 1$ that there exist $w_1, w_2 \in \mathbb{R}$ such that

$$\phi(0, x) = H_{0,1}(x - w_2) + H_{-1,0}(x - w_1) + g_1(x),$$

and

$$\|g_1\|_{H^1} < \delta,\ w_2 - w_1 > \frac{1}{\delta},$$

for a small constant $\delta > 0$. Since the Eq. (1) is locally well-posed in the space $S \times L^2(\mathbb{R})$, we conclude that there is a $\delta_1 > 0$ depending only on δ and ϵ such that if $-\delta_1 \leq t \leq \delta_1$, then

$$\left\|\phi(t, x) - H_{0,1}(x - w_2) - H_{-1,0}(x - w_1)\right\|_{H^1} \leq 2\delta. \tag{40}$$

If $\delta, \epsilon > 0$ are small enough, then, from the inequality (40) and the Modulation Lemma, we obtain in the time interval $[-\delta_1, \delta_1]$ the existence of modulation parameters $x_1(t),\ x_2(t)$ such that for

$$g(t) = \phi(t) - H_{0,1}(x - x_2(t)) - H_{-1,0}(x - x_1(t)),$$

we have

$$\big\langle g(t),\ \partial_x H_{0,1}(x - x_2(t))\big\rangle_{L^2} = \big\langle g(t),\ \partial_x H_{-1,0}(x - x_1(t))\big\rangle_{L^2} = 0, \tag{41}$$

$$\frac{1}{|x_2(t) - x_1(t)|} + \|g(t)\|_{H^1} \lesssim \delta. \tag{42}$$

From now on, we denote $z(t) = x_2(t) - x_1(t)$. From the conservation of the total energy, we have for $-\delta_1 \leq t \leq \delta_1$ that

$$\begin{aligned} E_{total}(\phi(t)) = {} & \frac{\|\partial_t\phi(t)\|_{L^2}^2}{2} + E_{pot}\big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\big) \\ & + \big\langle DE_{pot}\big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\big),\ g(t)\big\rangle_{L^2} \\ & + \frac{\big\langle D^2E_{pot}\big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\big)g(t),\ g(t)\big\rangle_{L^2}}{2} + O(\|g(t)\|_{H^1}^3). \end{aligned}$$

From Lemma 7 and (42), the above identity implies that

$$\begin{aligned} \epsilon = {} & \frac{\|\partial_t\phi(t)\|_{L^2}^2}{2} + 2\sqrt{2}e^{-\sqrt{2}z(t)} + \big\langle DE_{pot}\big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\big),\ g(t)\big\rangle_{L^2} \\ & + \frac{\big\langle D^2E_{pot}\big(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\big)g(t),\ g(t)\big\rangle_{L^2}}{2} + O\Big(\|g(t)\|_{H^1}^3 + z(t)e^{-2\sqrt{2}z(t)}\Big) \end{aligned} \tag{43}$$

for any $t \in [-\delta_1, \delta_1]$. From (38), it is not difficult to verify that $\Big|\langle DE_{pot}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}),\ g(t)\rangle_{L^2(\mathbb{R})}\Big| \leq C_2 e^{-\sqrt{2}z(t)}\,\|g(t)\|_{H^1(\mathbb{R})}$. So, the Eq. (43) and the Coercivity Lemma imply, while $-\delta_1 \leq t \leq \delta_1$, the following inequality

$$\begin{aligned} \epsilon + C_2 e^{-\sqrt{2}z(t)}\,\|g(t)\|_{H^1} \geq {} & \frac{\|\partial_t\phi(t)\|_{L^2}^2}{2} + 2\sqrt{2}e^{-\sqrt{2}z(t)} + \frac{c\,\|g(t)\|_{H^1}^2}{2} \\ & + O\left(\|g(t)\|_{H^1}^3 + z(t)e^{-2\sqrt{2}z(t)}\right). \end{aligned} \tag{44}$$

Finally, applying the Young inequality in the term $C_2 e^{-\sqrt{2}z(t)}\,\|g(t)\|_{H^1(\mathbb{R})}$, we obtain that the inequality (44) can be rewritten in the form

$$\epsilon \geq \frac{\|\partial_t\phi(t)\|_{L^2}^2}{2} + 2\sqrt{2}e^{-\sqrt{2}z(t)} + \frac{c\,\|g(t)\|_{H^1}^2}{4} + O\left(\|g(t)\|_{H^1}^3 + (z(t)+1)e^{-2\sqrt{2}z(t)}\right). \tag{45}$$

Then, the estimates (45), (42) imply for $\delta > 0$ small enough the following inequality

$$\epsilon \geq \frac{\|\partial_t\phi(t)\|_{L^2}^2}{2} + 2e^{-\sqrt{2}z(t)} + \frac{c\,\|g(t)\|_{H^1}^2}{8}. \tag{46}$$

So, the inequality (46) implies the estimates

$$e^{-\sqrt{2}z(t)} < \frac{\epsilon}{2},\ \|g(t)\|_{H^1}^2 \lesssim \epsilon, \tag{47}$$

for $t \in [-\delta_1, \delta_1]$. In conclusion, if $\frac{1}{\delta} \lesssim \ln\left(\frac{1}{\epsilon}\right)^{\frac{1}{2}}$, we can conclude by a bootstrap argument that the inequalities (39), (47) are true for all $t \in \mathbb{R}$. More precisely, we study the set

$$C = \left\{ b \in \mathbb{R}_{>0} | \, \epsilon \geq \frac{\|\partial_t \phi(t)\|_{L^2}^2}{2} + 2e^{-\sqrt{2}z(t)} + \frac{c\,\|g(t)\|_{H^1}^2}{8}, \text{ if } |t| \leq b. \right\}$$

and prove that $M = \sup_{b\in C} b = +\infty$. We already have checked that C is not empty, also C is closed by its definition. Now from the previous argument, we can verify that C is open. So, by connectivity, we obtain that $C = \mathbb{R}_{>0}$.

In conclusion, it remains to prove that the modulation parameters $x_1(t),\ x_2(t)$ are of class C^2 and that the fourth item of the statement of Theorem 11 is true.

(Proof of the C^2 regularity of $x_1,\ x_2$, and of the fourth item.)

For $\delta_0 > 0$ small enough, we denote $(y_1(t),\ y_2(t))$ to be the solution of the following system of ordinary differential equations, with the function $g_1(t) = \phi(t, x) - H_{0,1}^{y_2(t)}(x) - H_{-1,0}^{y_1(t)}(x)$,

$$\begin{aligned}
&\left(\left\|\partial_x H_{0,1}\right\|_{L^2}^2 - \left\langle g_1(t),\ \partial_x^2 H_{-1,0}^{y_1(t)} \right\rangle_{L^2} \right) \dot{y}_1(t) + \left(\left\langle \partial_x H_{0,1}^{y_2(t)},\ \partial_x H_{-1,0}^{y_1(t)} \right\rangle_{L^2} \right) \dot{y}_2(t) \\
&\quad = -\left\langle \partial_t \phi(t),\ \partial_x H_{-1,0}^{y_1(t)}(x) \right\rangle_{L^2}, \qquad (48)\\
&\left(\left\langle \partial_x H_{0,1}^{y_2(t)},\ \partial_x H_{-1,0}^{y_1(t)} \right\rangle_{L^2} \right) \dot{y}_1(t) + \left(\left\|\partial_x H_{0,1}(t)\right\|_{L^2}^2 - \left\langle g_1(t),\ \partial_x^2 H_{0,1}^{y_2} \right\rangle_{L^2} \right) \dot{y}_2(t) \\
&\quad = -\left\langle \partial_t \phi(t),\ \partial_x H_{0,1}^{y_2(t)}(x) \right\rangle_{L^2}, \qquad (49)
\end{aligned}$$

with initial condition $(y_2(0), y_1(0)) = (x_2(0), x_1(0))$. This system of ordinary differential equations is motivated by the time derivative of the orthogonality conditions of the Modulation Lemma.

Since we have the estimate $\ln\left(\frac{1}{\epsilon}\right) \lesssim x_2(0) - x_1(0)$ and $g_1(0) = g(0)$, Lemma 6 and the inequalities in (47) imply that the matrix

$$\begin{bmatrix} \left\|\partial_x H_{0,1}\right\|_{L^2}^2 - \left\langle g_1(0),\ \partial_x^2 H_{-1,0}^{y_1(0)} \right\rangle_{L^2} & \left\langle \partial_x H_{0,1}^{y_2(0)},\ \partial_x H_{-1,0}^{y_1(0)} \right\rangle_{L^2} \\ \left\langle \partial_x H_{0,1}^{y_2(0)},\ \partial_x H_{-1,0}^{y_1(0)} \right\rangle_{L^2} & \left\|\partial_x H_{0,1}\right\|_{L^2}^2 - \left\langle g_1(0),\ \partial_x^2 H_{0,1}^{y_2} \right\rangle_{L^2} \end{bmatrix} \tag{50}$$

is positive, so we have from Picard–Lindelöf Theorem that $y_2(t),\ y_1(t)$ are of class C^1 for some interval $[-\delta, \delta]$, with $\delta > 0$ depending on $|x_2(0) - x_1(0)|$ and ϵ. From the fact that $(y_2(0), y_1(0)) = (x_2(0), x_1(0))$, we obtain, from the Eqs. (48) and (49), that $(y_2(t), y_1(t))$ also satisfies the orthogonality conditions of Modulation Lemma for $t \in [-\delta, \delta]$. In conclusion, the uniqueness of Modulation Lemma implies that $(y_2(t), y_1(t)) = (x_2(t), x_1(t))$ for $t \in [-\delta, \delta]$. From this argument, we also have for $t \in [-\delta, \delta]$ that $e^{-\sqrt{2}(y_2(t)-y_1(t))} \leq \frac{\epsilon}{2\sqrt{2}}$. By bootstrap, we can show, repeating the argument above, that

$$\sup\left\{C > 0 |\ (y_2(t), y_1(t)) = (x_2(t), x_1(t)), \text{for } t \in [-C, C]\right\} = +\infty. \tag{51}$$

Also, the argument above implies that if $(y_1(t), y_2(t)) = (x_1(t), x_2(t))$ in an instant t, then $y_1,\ y_2$ are of class C^1 in a neighborhood of t. In conclusion, $x_1,\ x_2$ are functions in $C^1(\mathbb{R})$. Finally, since $\|g(t)\|_{H^1} = O(\epsilon^{\frac{1}{2}})$ and $e^{-\sqrt{2}z(t)} = O(\epsilon)$, the following matrix

$$M(t) := \begin{bmatrix} \|\partial_x H_{0,1}\|_{L^2}^2 - \left\langle g(t),\ \partial_x^2 H_{-1,0}^{x_1(t)} \right\rangle_{L^2} & \left\langle \partial_x H_{0,1}^{x_2(t)},\ \partial_x H_{-1,0}^{x_1(t)} \right\rangle_{L^2} \\ \left\langle \partial_x H_{0,1}^{x_2(t)},\ \partial_x H_{-1,0}^{x_1(t)} \right\rangle_{L^2} & \|\partial_x H_{0,1}\|_{L^2}^2 - \left\langle g(t),\ \partial_x^2 H_{0,1}^{x_2(t)} \right\rangle_{L^2} \end{bmatrix} \tag{52}$$

is uniformly positive for all $t \in \mathbb{R}$. So, from the estimate $\|\partial_t \phi(t)\|_{L^2(\mathbb{R})} = O(\epsilon^{\frac{1}{2}})$, the identities $x_j(t) = y_j(t)$ for $j = 1, 2$ and the Eqs. (48) and (49), we obtain

$$\max_{j \in \{1,2\}} \left|\dot{x}_j(t)\right| = O(\epsilon^{\frac{1}{2}}). \tag{53}$$

Since the matrix $M(t)$ is invertible for any $t \in \mathbb{R}$, we can obtain from the Eqs. (48), (49) that the functions $\dot{x}_1(t), \dot{x}_2(t)$ are given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = M(t)^{-1} \begin{bmatrix} -\left\langle \partial_t \phi(t),\ \partial_x H_{-1,0}^{x_1(t)}(x) \right\rangle_{L^2} \\ -\left\langle \partial_t \phi(t),\ \partial_x H_{0,1}^{x_2(t)}(x) \right\rangle_{L^2} \end{bmatrix}. \tag{54}$$

Now, since we have that $(\phi(t), \partial_t \phi(t)) \in C(\mathbb{R}, S \times L^2(\mathbb{R}))$ and $x_1(t),\ x_2(t)$ are of class C^1, we can deduce that $(g(t), \partial_t g(t)) \in C(\mathbb{R}, H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. So, by definition, we can verify that $M(t) \in C^1(\mathbb{R}, \mathbb{R}^4)$.

Also, since $\phi(t, x)$ is the solution in distributional sense of (1), we have that for any $y_1,\ y_2 \in \mathbb{R}$ the following identities hold

$$\left\langle \partial_x H_{0,1}^{y_2},\ \partial_t^2 \phi(t) \right\rangle_{L^2} = -\left\langle \partial_x^2 H_{0,1}^{y_2},\ \partial_x \phi(t) \right\rangle_{L^2} - \left\langle \partial_x H_{0,1}^{y_2(t)},\ \dot{U}(\phi(t)) \right\rangle_{L^2},$$
$$\left\langle \partial_x H_{-1,0}^{y_1},\ \partial_t^2 \phi(t) \right\rangle_{L^2} = -\left\langle \partial_x^2 H_{-1,0}^{y_1},\ \partial_x \phi(t) \right\rangle_{L^2} - \left\langle \partial_x H_{-1,0}^{y_1},\ \dot{U}(\phi(t)) \right\rangle_{L^2}.$$

Since (1) is locally well-posed in $S \times L^2(\mathbb{R})$, we obtain from the identities above that the following functions $h(t, y) := \left\langle \partial_x H_{0,1}^{y},\ \partial_t^2 \phi(t) \right\rangle_{L^2}$ and $l(t, y) := \left\langle \partial_x H_{-1,0}^{y},\ \partial_t^2 \phi(t) \right\rangle_{L^2}$ are continuous in the domain $\mathbb{R} \times \mathbb{R}$.

So, from the continuity of the functions $h(t, y),\ l(t, y)$ and from the fact that $x_1,\ x_2 \in C^1(\mathbb{R})$, we obtain that the functions

$$h_1(t) := -\left\langle \partial_t \phi(t),\ \partial_x H_{-1,0}^{x_1(t)}(x) \right\rangle_{L^2},\ h_2(t) := -\left\langle \partial_t \phi(t),\ \partial_x H_{0,1}^{x_2(t)}(x) \right\rangle_{L^2}$$

are of class C^1. In conclusion, from the Eq. (54), by chain rule and product rule, we verify that $x_1,\ x_2$ are in $C^2(\mathbb{R})$.

Now, since $x_1,\ x_2 \in C^2(\mathbb{R})$ and $\dot{x}_1,\ \dot{x}_2$ satisfy (54), we deduce after differentiate in time the function

$$M(t) \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}$$

the following equations

$$\begin{aligned}
&\ddot{x}_1(t)\Big(\left\|\partial_x H_{0,1}\right\|_{L^2}^2+\left\langle\partial_x g(t),\,\partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2}\Big)+\ddot{x}_2(t)\left\langle\partial_x H_{-1,0}^{x_1(t)},\,\partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2}\\
&\quad=\dot{x}_1(t)^2\left\langle\partial_x^2 H_{-1,0}^{x_1(t)},\,\partial_x g(t)\right\rangle_{L^2}+\dot{x}_1(t)\left\langle\partial_x^2 H_{-1,0}^{x_1(t)},\,\partial_t g(t)\right\rangle_{L^2}\\
&\qquad+\dot{x}_2(t)^2\left\langle\partial_x H_{-1,0}^{x_1(t)},\,\partial_x^2 H_{0,1}^{x_2(t)}\right\rangle_{L^2}+\dot{x}_1(t)\dot{x}_2(t)\left\langle\partial_x^2 H_{-1,0}^{x_1(t)},\,\partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2}\\
&\qquad+\dot{x}_1(t)\left\langle\partial_x^2 H_{-1,0}^{x_1(t)},\,\partial_t\phi(t)\right\rangle_{L^2}-\left\langle\partial_x H_{-1,0}^{x_1(t)},\,\partial_t^2\phi(t)\right\rangle_{L^2}, \qquad (55)\\
&\ddot{x}_2(t)\Big(\left\|\partial_x H_{0,1}\right\|_{L^2}^2+\left\langle\partial_x g(t),\,\partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2}\Big)+\ddot{x}_1(t)\langle\partial_x H_{-1,0}^{x_1(t)},\,\partial_x H_{0,1}^{x_2(t)}\rangle_{L^2}\\
&\quad=\dot{x}_2(t)^2\left\langle\partial_x^2 H_{0,1}^{x_2(t)},\,\partial_x g(t)\right\rangle_{L^2}+\dot{x}_2(t)\left\langle\partial_x^2 H_{0,1}^{x_2(t)},\,\partial_t g(t)\right\rangle_{L^2}\\
&\qquad+\dot{x}_1(t)\dot{x}_2(t)\left\langle\partial_x H_{-1,0}^{x_1(t)},\,\partial_x^2 H_{0,1}^{x_2(t)}\right\rangle_{L^2}+\dot{x}_1(t)^2\left\langle\partial_x H_{0,1}^{x_2(t)},\,\partial_x^2 H_{-1,0}^{x_1(t)}\right\rangle_{L^2}\\
&\qquad+\dot{x}_2(t)\left\langle\partial_x^2 H_{0,1}^{x_2(t)},\,\partial_t\phi(t)\right\rangle_{L^2}-\left\langle\partial_x H_{0,1}^{x_2(t)},\,\partial_t^2\phi(t)\right\rangle_{L^2}. \qquad (56)
\end{aligned}$$

Also, from the identity $g(t)=\phi(t)-H_{-1,0}^{x_1(t)}-H_{0,1}^{x_2(t)}$, we obtain that $\partial_t g(t)=\partial_t\phi(t,x)+\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}$, so, from the estimates (39) and (53), we obtain that

$$\|\partial_t g(t)\|_{L^2}=O(\epsilon^{\frac{1}{2}}). \qquad (57)$$

Now, since $\phi(t)$ is a distributional solution of (1), we also have, from the global equality $\phi(t)=H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}+g(t)$, the following identity

$$\begin{aligned}
&\left\langle\partial_x H_{-1,0}^{x_1(t)},\,\partial_t^2\phi(t)\right\rangle_{L^2}\\
&\quad=\left\langle\partial_x H_{-1,0}^{x_1(t)},\,\partial_x^2 g(t)-\ddot{U}\left(H_{-1,0}^{x_1(t)}\right)g(t)\right\rangle_{L^2}\\
&\qquad-\left\langle\partial_x H_{-1,0}^{x_1(t)},\left[\ddot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)-\ddot{U}\left(H_{-1,0}^{x_1(t)}\right)\right]g(t)\right\rangle_{L^2}\\
&\qquad+\left\langle\partial_x H_{-1,0}^{x_1(t)},\,\dot{U}\left(H_{-1,0}^{x_1(t)}\right)+\dot{U}\left(H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right\rangle_{L^2}\\
&\qquad-\left\langle\partial_x H_{-1,0}^{x_1(t)},\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}+g(t)\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right\rangle_{L^2}\\
&\qquad+\left\langle\partial_x H_{-1,0}^{x_1(t)},\ddot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)g(t)\right\rangle_{L^2}.
\end{aligned}$$

Since $\partial_x H_{-1,0}^{x_1(t)}\in\ker\left(D^2E_{pot}\left(H_{-1,0}^{x_1(t)}\right)\right)$, we have by integration by parts that $\left\langle\partial_x H_{-1,0}^{x_1(t)},\partial_x^2 g(t)-\ddot{U}\left(H_{-1,0}^{x_1(t)}\right)g(t)\right\rangle_{L^2}=0$. Since we have

$$\begin{aligned}
&\dot{U}\left(H_{-1,0}^{x_1(t)}\right)+\dot{U}\left(H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\\
&\quad=24H_{-1,0}^{x_1(t)}H_{0,1}^{x_2(t)}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)-6\sum_{j=1}^{4}\binom{5}{j}\left(H_{-1,0}^{x_1(t)}\right)^j\left(H_{0,1}^{x_2(t)}\right)^{5-j}, \qquad (58)
\end{aligned}$$

Lemma 6 implies $\left\langle \partial_x H^{x_1(t)}_{-1,0},\, \dot{U}\left(H^{x_1(t)}_{-1,0}\right) + \dot{U}\left(H^{x_2(t)}_{0,1}\right) - \dot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\right\rangle_{L^2}$ $= O\left(e^{-\sqrt{2}(z(t))}\right)$. Also, from Taylor's Expansion Theorem, we have the estimate

$$\left\langle \partial_x H^{x_1(t)}_{-1,0}, \dot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1} + g(t)\right) - \dot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\right\rangle_{L^2}$$
$$-\left\langle \partial_x H^{x_1(t)}_{-1,0}, \ddot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) g(t)\right\rangle_{L^2} = O(\|g(t)\|^2_{H^1}).$$

From Lemma 6, the fact that U is a smooth function and $H_{0,1} \in L^\infty(\mathbb{R})$, we can obtain

$$\begin{aligned}&\left\langle \partial_x H^{x_1(t)}_{-1,0}, \left[\ddot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right) - \ddot{U}\left(H^{x_1(t)}_{-1,0}\right)\right] g(t)\right\rangle_{L^2}\\ &= O\left(\int_{\mathbb{R}} \partial_x H^{x_1(t)}_{-1,0} H^{x_2(t)}_{0,1}\, |g(t)|\, dx\right)\\ &= O\left(e^{-\sqrt{2}z(t)}\, \|g(t)\|_{H^1}\, z(t)^{\frac{1}{2}}\right).\end{aligned}$$

In conclusion, we have

$$\left\langle \partial_x H^{x_1(t)}_{-1,0},\, \partial_t^2 \phi(t)\right\rangle_{L^2} = O\left(\|g(t)\|^2_{H^1} + e^{-\sqrt{2}z(t)}\right), \tag{59}$$

and by similar arguments, we have

$$\left\langle \partial_x H^{x_2(t)}_{0,1},\, \partial_t^2 \phi(t)\right\rangle_{L^2} = O\left(\|g(t)\|^2_{H^1} + e^{-\sqrt{2}z(t)}\right). \tag{60}$$

Also, the Eqs. (55) and (56) form a linear system with $\ddot{x}_1(t),\ \ddot{x}_2(t)$. Recalling that the Matrix $M(t)$ is uniformly positive, we obtain from the estimates (47), (53), (57), (59) and (60) that

$$\max_{j \in \{1,2\}} \left|\ddot{x}_j(t)\right| = O(\epsilon). \tag{61}$$

□

The Theorem 11 can also be improved when the kinetic energy of the solution is included in the computation and additional conditions are added, more precisely:

Theorem 12. *There exist* $C,\ c,\ \delta_0 > 0$, *such that if* $0 < \epsilon \leq \delta_0$, $(\phi(0,x), \partial_t\phi(0,x)) \in S \times L^2(\mathbb{R})$ *and* $E_{total}((\phi(0,x), \partial_t\phi(0,x))) = 2E_{pot}(H_{0,1}) + \epsilon$, *then there are* $x_2,\ x_1 \in C^2(\mathbb{R})$ *such that* $g(t,x) = \phi(t,x) - H^{x_2(t)}_{0,1}(x) - H^{x_1(t)}_{-1,0}(x)$ *satisfies*

$$\left\langle g(t,x), \partial_x H^{x_2(t)}_{0,1}(x)\right\rangle_{L^2} = 0,\ \left\langle g(t,x), \partial_x H^{x_1(t)}_{-1,0}(x)\right\rangle_{L^2} = 0,$$

and, for all $t \in \mathbb{R}$,

$$c\epsilon \leq e^{-\sqrt{2}(x_2(t) - x_1(t))} + \|(g(t), \partial_t g(t))\|^2_{H^1 \times L^2} + |\dot{x}_1(t)|^2 + |\dot{x}_2(t)|^2 \leq C\epsilon. \tag{62}$$

Proof. From Modulation Lemma and Theorem 11, we can rewrite the solution $\phi(t)$ in the form

$$\phi(t,x) = H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x) + g(t,x)$$

with $x_1(t),\ x_2(t),\ g(t)$ satisfying the conclusion of Theorem 11. First, we denote

$$\phi_\sigma(t) = \left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x),\ -\dot{x}_1(t)\partial_x H^{x_1(t)}_{-1,0} - \dot{x}_2(t)\partial_x H^{x_2(t)}_{0,1}\right) \in S \times L^2(\mathbb{R}), \tag{63}$$

then we apply Taylor's Expansion Theorem in $E(\phi(t))$ around $\phi_\sigma(t)$. More precisely, for $R_\sigma(t)$ the residue of quadratic order of Taylor's Expansion of $E(\phi(t), \partial_t\phi(t))$ around $\phi_\sigma(t)$, we have:

$$\begin{aligned} 2E_{pot}(H_{0,1}) + \epsilon &= E_{total}(\phi_\sigma(t)) + \langle DE_{total}(\phi_\sigma(t)),\ (g(t), \partial_t g(t))\rangle_{L^2\times L^2} \\ &\quad + \frac{\big\langle D^2E_{total}(\phi_\sigma(t))\big(g(t), \partial_t g(t)\big),\ \big(g(t), \partial_t g(t)\big)\big\rangle_{L^2\times L^2}}{2} + R_\sigma(t), \end{aligned} \tag{64}$$

such that for $(\nu_1, \nu_2) \in S \times L^2(\mathbb{R})$ and $(v_1, v_2) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, we have the identities

$$\begin{aligned} E_{total}(\nu_1, \nu_2) &= \frac{\|\partial_x \nu_1\|^2_{L^2} + \|\nu_2\|^2_{L^2}}{2} + \int_{\mathbb{R}} U(\nu_1(x))\,dx, \\ \langle DE_{total}(\nu_1, \nu_2), (v_1, v_2)\rangle_{L^2\times L^2} &= \int_{\mathbb{R}} \partial_x \nu_1(x)\partial_x v_1(x) + \dot{U}(\nu_1)v_1 + \nu_2(x)v_2(x)\,dx, \end{aligned} \tag{65}$$

$$D^2E_{total}(\nu_1, \nu_2) = \begin{bmatrix} -\partial_x^2 + \ddot{U}(\nu_1) & 0 \\ 0 & \mathbb{I} \end{bmatrix} \tag{66}$$

with $D^2E_{total}(\nu_1, \nu_2)$ defined as a linear operator from $H^2(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

So, from identities (65) and (66), it is not difficult to verify that

$$\begin{aligned} R_\sigma(t) = &\int_{\mathbb{R}} U\left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x) + g(t,x)\right) - U\left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x)\right)dx \\ &- \int_{\mathbb{R}} \dot{U}\left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x)\right) g(t,x)\,dx \\ &- \int_{\mathbb{R}} \frac{\ddot{U}\left(H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x)\right) g(t,x)^2}{2}\,dx, \end{aligned}$$

and, so,

$$|R_\sigma(t)| = O\left(\|g(t)\|^3_{H^1}\right). \tag{67}$$

Also, we have

$$\begin{aligned} &\langle DE_{total}(\phi_\sigma(t)),\ (g(t), \partial_t g(t))\rangle_{L^2\times L^2} \\ &= \left\langle DE_{pot}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right),\ g(t)\right\rangle_{L^2} \end{aligned}$$

$$-\left\langle \dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)},\ \partial_t g(t)\right\rangle_{L^2}. \tag{68}$$

The orthogonality conditions satisfied by $g(t)$ also imply for all $t \in \mathbb{R}$ that

$$\left\langle \partial_t g(t),\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2} = \dot{x}_1(t)\left\langle g(t),\ \partial_x^2 H_{-1,0}^{x_1(t)}\right\rangle_{L^2}, \tag{69}$$

$$\left\langle \partial_t g(t),\ \partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2} = \dot{x}_2(t)\left\langle g(t),\ \partial_x^2 H_{0,1}^{x_2(t)}\right\rangle_{L^2}. \tag{70}$$

So, the inequality (38) and the identities (68), (69), (70) imply that

$$\left|\langle DE_{total}(\phi_\sigma(t)),\ (g(t), \partial_t g(t))\rangle_{L^2\times L^2}\right| = O\Big(\|g(t)\|_{H^1} \sup_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|^2\Big) + O\Big(\|g(t)\|_{H^1}\, e^{-\sqrt{2}z(t)}\Big). \tag{71}$$

From the Coercivity Lemma and the definition of $D^2E_{total}(\phi_\sigma(t))$, we have that

$$\left\langle D^2E_{total}(\phi_\sigma(t))(g(t), \partial_t g(t)),\ (g(t), \partial_t g(t))\right\rangle_{L^2\times L^2} \cong \|(g(t), \partial_t g(t))\|_{H^1\times L^2}^2. \tag{72}$$

Finally, there is the identity

$$\begin{aligned}&\left\|\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}(x) + \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}(x)\right\|_{L^2}^2\\ &= 2\dot{x}_1(t)\dot{x}_2(t)\left\langle \partial_x H_{0,1}^{z(t)},\ \partial_x H_{-1,0}\right\rangle_{L^2} + \dot{x}_1(t)^2\left\|\partial_x H_{0,1}\right\|_{L^2}^2\\ &\quad + \dot{x}_2(t)^2\left\|\partial_x H_{0,1}\right\|_{L^2}^2.\end{aligned} \tag{73}$$

From Lemma 6, we have that $\left|\langle \partial_x H_{0,1}^z,\ \partial_x H_{-1,0}\rangle_{L^2}\right| = O\big(ze^{-\sqrt{2}z}\big)$ for z big enough. Then, it is not difficult to verify that Lemma 7, (67), (71), (72) and (73) imply directly the statement of the Theorem 12 which finishes the proof. □

Remark 7. Theorem 12 implies that it is possible to have a solution ϕ of the Eq. (1) with energy excess $\epsilon > 0$ small enough to satisfy all the hypotheses of Theorem 2. More precisely, in notation of Theorem 2, if $\|(g(0, x), \partial_t g(0, x))\|_{H^1\times L^2} \ll \epsilon^{\frac{1}{2}}$ and

$$e^{-\sqrt{2}z(0)} + \dot{x}_1(0)^2 + \dot{x}_2(0)^2 \cong \epsilon,$$

then we would have that $E_{total}(\phi(0), \partial_t\phi(0)) - 2E_{pot}(H_{0,1}) \cong \epsilon$.

3. Long Time Behavior of Modulation Parameters

Even though Theorem 11 implies the orbital stability of a sum of two kinks with low energy excess, this theorem does not explain the movement of the kinks' centers $x_2(t),\ x_1(t)$ and their speed for a long time. More precisely, we still don't know if there is an explicit smooth real function $d(t)$, such that $(z(t), \dot{z}(t))$ is close to $(d(t), \dot{d}(t))$ in a large time interval.

But, the global estimates on the modulus of the first and second derivatives of $x_1(t),\ x_2(t)$ obtained in Theorem 11 will be very useful to estimate with high precision the functions $x_1(t),\ x_2(t)$ during a very large time interval. Moreover, we first have the following auxiliary lemma.

Lemma 13. *Let* $0 < \theta,\ \gamma < 1$. *We recall the function*

$$A(z) = E_{pot}(H^{z}_{0,1} + H_{-1,0})$$

for any $z > 0$. *We assume all the hypotheses of Theorem 11 and let* $\chi(x)$ *be a smooth function satisfying*

$$\chi(x) = \begin{cases} 1, & \text{if } x \leq \theta(1-\gamma), \\ 0, & \text{if } x \geq \theta, \end{cases} \tag{74}$$

and $0 \leq \chi(x) \leq 1$ *for all* $x \in \mathbb{R}$. *In notation of Theorem 11, we denote*

$$\chi_0(t,x) = \chi\Big(\frac{x - x_1(t)}{z(t)}\Big),\ \overrightarrow{g(t)} = (g(t), \partial_t g(t)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$$

and $\left\|\overrightarrow{g(t)}\right\| = \|(g(t), \partial_t g(t))\|_{H^1(\mathbb{R})\times L^2(\mathbb{R})}$,

$$\begin{aligned}
\alpha(t) = {} & \left\|\overrightarrow{g(t)}\right\| \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| \Big[1 + \frac{1}{z(t)\gamma} + \frac{1}{z(t)^2\gamma^2} \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|\Big]\Big(e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big) \\
& + \max_{j\in\{1,2\}} \dot{x}_j(t)^2 z(t) e^{-\sqrt{2}z(t)} + \frac{\max_{j\in\{1,2\}} \dot{x}_j(t)^2}{z(t)\gamma}\Big(e^{-2\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big) \\
& + \left\|\overrightarrow{g(t)}\right\|^2 \Big[\frac{1}{\gamma^2 z(t)^2} + \frac{1}{\gamma z(t)} + \Big(e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\Big)\Big] + \left\|\overrightarrow{g(t)}\right\| e^{-\sqrt{2}z(t)}\Big[1 + \frac{1}{\gamma z(t)}\Big].
\end{aligned} \tag{75}$$

Then, for $\theta = \frac{1-\gamma}{2-\gamma}$ *and the correction terms*

$$\begin{aligned}
p_1(t) = & - \frac{\left\langle \partial_t \phi(t),\ \partial_x H^{x_1(t)}_{-1,0}(x) + \partial_x(\chi_0(t,x)g(t))\right\rangle}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}}, \\
p_2(t) = & - \frac{\left\langle \partial_t \phi(t),\ \partial_x H^{x_2(t)}_{0,1}(x) + \partial_x([1-\chi_0(t,x)]g(t))\right\rangle}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}},
\end{aligned}$$

we have the following estimates, for $j \in \{1, 2\}$,

$$\begin{aligned}
\left|\dot{x}_j(t) - p_j(t)\right| \lesssim {} & \Big[1 + \frac{\|\dot{\chi}\|_{L^\infty}}{z(t)}\Big]\Big(\max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| \left\|\overrightarrow{g(t)}\right\| + \left\|\overrightarrow{g(t)}\right\|^2\Big) \\
& + \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| z(t) e^{-\sqrt{2}z(t)},
\end{aligned} \tag{76}$$

$$\left|\dot{p}_j(t) + (-1)^j \frac{\dot{A}(z(t))}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}}\right| \lesssim \alpha(t). \tag{77}$$

Remark 8. We will take $\gamma = \frac{\ln\ln(\frac{1}{\epsilon})}{\ln(\frac{1}{\epsilon})}$. With this value of γ and the estimates of Theorem 11, we will see in Lemma 16 that $\exists C > 0$ such that

$$\alpha(t) \lesssim \frac{\Big(\|(g_0, g_1)\|_{H^1\times L^2} + \epsilon \ln\frac{1}{\epsilon}\Big)^2}{\ln\ln(\frac{1}{\epsilon})} \exp\Big(\frac{2C\,|t|\,\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\Big).$$

Proof. For $\gamma \ll 1$ enough and from the definition of $\chi(x)$, it is not difficult to verify that

$$\|\dot{\chi}\|_{L^\infty(\mathbb{R})} \lesssim \frac{1}{\gamma},\ \|\ddot{\chi}\|_{L^\infty(\mathbb{R})} \lesssim \frac{1}{\gamma^2}. \tag{78}$$

We will only do the proof of the estimates (76) and (77) for $j = 1$, the proof for the case $j = 2$ is completely analogous. From the proof of Theorem 11, we know that $\dot{x}_1(t),\ \dot{x}_2(t)$ solve the linear system

$$M(t)\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}-\langle \partial_t\phi(t),\ \partial_x H_{-1,0}^{x_1(t)}\rangle\\ -\langle \partial_t\phi(t),\ \partial_x H_{0,1}^{x_2(t)}\rangle\end{bmatrix},$$

where $M(t)$ is the matrix defined by (52). Then, from Cramer's rule, we obtain that

$$\begin{aligned}\dot{x}_1(t) = {} & \frac{-\left\langle \partial_t\phi(t),\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2}\left(\left\langle \partial_x H_{0,1}^{x_2(t)},\ \partial_x g(t)\right\rangle_{L^2} + \|\partial_x H_{0,1}\|_{L^2}^2\right)}{\det(M(t))}\\ & + \frac{\left\langle \partial_t\phi(t),\ \partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2}\left\langle \partial_x H_{0,1}^{x_2(t)},\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2}}{\det(M(t))}.\end{aligned} \tag{79}$$

Using the definition (52) of the matrix $M(t)$, $\left\|\overrightarrow{g(t)}\right\| = O(\epsilon^{\frac{1}{2}})$ and Lemma 6 which implies the following estimate

$$\left\langle \partial_x H_{0,1}^{x_2(t)},\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2} = O\left(z(t)e^{-\sqrt{2}z(t)}\right), \tag{80}$$

we obtain that

$$\left|\det(M(t)) - \|\partial_x H_{0,1}\|_{L^2}^4\right| = O\Big(\left\|\overrightarrow{g(t)}\right\| + z(t)^2 e^{-2\sqrt{2}z(t)}\Big) = O(\epsilon^{\frac{1}{2}}). \tag{81}$$

So, from the estimate (81) and the identity (79), we obtain that

$$\begin{aligned}&\left|\dot{x}_1(t) + \frac{\left\langle \partial_t\phi(t),\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2}}{\|\partial_x H_{0,1}\|_{L^2(\mathbb{R})}^2}\right|\\ &\quad = O\Big(\left|\left\langle \partial_x H_{-1,0}^{x_1(t)},\ \partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2}\left\langle \partial_t\phi(t),\ \partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2}\right|\Big)\\ &\qquad + O\left(\left|\left\langle \partial_t\phi(t),\ \partial_x H_{-1,0}^{x_1(t)}(x)\right\rangle_{L^2}\right|\left[\left\|\overrightarrow{g(t)}\right\| + z(t)^2 e^{-2\sqrt{2}z(t)}\right]\right).\end{aligned} \tag{82}$$

Finally, from the definition of $g(t,x)$ in Theorem 11 we know that

$$\partial_t\phi(t,x) = -\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}(x) - \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}(x) + \partial_t g(t,x),$$

from the Modulation Lemma, we also have verified that

$$\begin{aligned}\left\langle \partial_t g(t),\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2} &= O\Big(\left\|\overrightarrow{g(t)}\right\| |\dot{x}_1(t)|\Big),\\ \left\langle \partial_t g(t),\ \partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2} &= O\Big(\left\|\overrightarrow{g(t)}\right\| |\dot{x}_2(t)|\Big),\end{aligned}$$

and from Theorem 11 we have that $\left\|\overrightarrow{g(t)}\right\| + \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| \ll 1$. In conclusion, we can rewrite the estimate (82) as

$$\left|\dot{x}_1(t) + \frac{\left\langle \partial_t\phi(t),\ \partial_x H^{x_1(t)}_{-1,0}\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2(\mathbb{R})}}\right| = O\Big(\max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| \left\|\overrightarrow{g(t)}\right\| + \left\|\overrightarrow{g(t)}\right\|^2\Big) + O\Big(z(t)e^{-\sqrt{2}z(t)} \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|\Big). \tag{83}$$

By similar reasoning, we can also deduce that

$$\left|\dot{x}_2(t) + \frac{\left\langle \partial_t\phi(t),\ \partial_x H^{x_2(t)}_{0,1}\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2(\mathbb{R})}}\right| = O\Big(\max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| \left\|\overrightarrow{g(t)}\right\| + \left\|\overrightarrow{g(t)}\right\|^2\Big) + O\Big(z(t)e^{-\sqrt{2}z(t)} \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|\Big). \tag{84}$$

Following the reasoning of Lemma 3.5 of [13], we will use the terms $p_1(t),\ p_2(t)$ with the objective of obtaining the estimates (77), which have high precision and will be useful later to approximate $x_j(t),\ \dot{x}_j(t)$ by explicit smooth functions during a long time interval.

First, it is not difficult to verify that

$$\langle \partial_t\phi(t),\ \partial_x(\chi_0(t)g(t))\rangle_{L^2} = O\Big(\Big[1 + \frac{\|\dot{\chi}\|_{L^\infty}}{z(t)}\Big] \left\|\overrightarrow{g(t)}\right\|^2 + \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| \left\|\overrightarrow{g(t)}\right\|\Big),$$

which clearly implies with estimate (83) the inequality (76) for $j = 1$. The proof of inequality (76) for $j = 2$ is completely analogous.

Now, the demonstration of the inequality (77) is similar to the proof of the second inequality of Lemma 3.5 of [13]. First, we have

$$\begin{aligned}
\dot{p}_1(t) = &- \frac{\left\langle \partial_t\phi(t),\ \partial_t\left(\partial_x H^{x_1(t)}_{-1,0}(x)\right)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}} - \frac{\left\langle \partial_t\phi(t),\ \partial_x\left(\partial_t\chi_0(t)g(t)\right)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}} \\
&- \frac{\left\langle \partial_x\left(\chi_0(t)\partial_t g(t)\right),\ \partial_t\phi(t)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}} - \frac{\left\langle \partial_x H^{x_1(t)}_{-1,0},\ \partial_t^2\phi(t)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}} \\
&- \frac{\left\langle \partial_x\chi_0(t)g(t),\ \partial_t^2\phi(t)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}} - \frac{\left\langle \chi_0(t)\partial_x g(t),\ \partial_t^2\phi(t)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}} \qquad (85)\\
= &\ I + II + III + IV + V + VI, \qquad (86)
\end{aligned}$$

and we will estimate each term one by one. More precisely, from now on, we will work with a general cut-off function $\chi(x)$, that is a smooth function $0 \leq \chi \leq 1$ satisfying

$$\chi(x) = \begin{cases} 1, & \text{if } x \leq \theta(1-\gamma), \\ 0, & \text{if } x \geq \theta. \end{cases} \tag{87}$$

with $0<\theta,\ \gamma<1$ and

$$\chi_0(t,x)=\chi\Big(\frac{x-x_1(t)}{z(t)}\Big). \tag{88}$$

The reason for this notation is to improve the precision of the estimate of $\dot{p}_1(t)$ by the searching of the $\gamma,\ \theta$ which minimize $\alpha(t)$.

Step 1. (Estimate of I) We will only use the identity $I=\dot{x}_1(t)\frac{\big\langle\partial_t\phi(t),\,\partial_x^2 H_{-1,0}^{x_1(t)}\big\rangle_{L^2}}{\|\partial_x H_{0,1}\|_{L^2}^2}$.

Step 2. (Estimate of II.) We have, by chain rule and definition of χ_0, that

$$\begin{aligned}II=&-\frac{\big\langle\partial_t\phi(t),\ \partial_x\big(\partial_t\chi_0(t)g(t)\big)\big\rangle_{L^2}}{\big\|\partial_x H_{0,1}\big\|_{L^2}^2}\\=&-\frac{\Big\langle\partial_t\phi(t),\ \partial_x\Big(\dot{\chi}\Big(\frac{x-x_1(t)}{z(t)}\Big)\frac{d}{dt}\Big[\frac{x-x_1(t)}{z(t)}\Big]g(t)\Big)\Big\rangle_{L^2}}{\big\|\partial_x H_{0,1}\big\|_{L^2}^2}\\=&\frac{\Big\langle\partial_t\phi(t),\ \partial_x\Big(\dot{\chi}\Big(\frac{x-x_1(t)}{z(t)}\Big)\Big[\frac{\dot{x}_1(t)z(t)+(x-x_1(t))\dot{z}(t)}{z(t)^2}\Big]g(t)\Big)\Big\rangle_{L^2}}{\big\|\partial_x H_{0,1}\big\|_{L^2}^2}.\end{aligned}$$

So, we obtain that

$$\begin{aligned}II=&\frac{\Big\langle\partial_t\phi(t),\ \ddot{\chi}\Big(\frac{x-x_1(t)}{z(t)}\Big)\Big[\frac{\dot{x}_1(t)}{z(t)}+\frac{(x-x_1(t))\dot{z}(t)}{z(t)^2}\Big]g(t)\Big\rangle_{L^2}}{z(t)\,\big\|\partial_x H_{0,1}\big\|_{L^2}^2}\\&+\frac{\Big\langle\partial_t\phi(t),\ \dot{\chi}\Big(\frac{x-x_1(t)}{z(t)}\Big)\frac{\dot{z}(t)}{z(t)^2}g(t)\Big\rangle_{L^2}}{\big\|\partial_x H_{0,1}\big\|_{L^2}^2}\\&+\frac{\Big\langle\partial_t\phi(t),\ \dot{\chi}\Big(\frac{x-x_1(t)}{z(t)}\Big)\Big[\frac{\dot{x}_1(t)}{z(t)}+\frac{(x-x_1(t))\dot{z}(t)}{z(t)^2}\Big]\partial_x g(t)\Big\rangle_{L^2}}{\big\|\partial_x H_{0,1}\big\|_{L^2}^2}.\end{aligned} \tag{89}$$

First, since the support of $\dot{\chi}$ is contained in $[\theta(1-\gamma),\ \theta]$, from the estimates (D3) and (D4) we obtain that

$$\Big\|\partial_x H_{-1,0}^{x_1(t)}\Big\|_{L_x^2\big(\operatorname{supp}\partial_x\chi_0(t,x)\big)}^2=O\Big(e^{-2\sqrt{2}\theta(1-\gamma)z(t)}\Big), \tag{90}$$

$$\Big\|\partial_x H_{0,1}^{x_2(t)}\Big\|_{L_x^2\big(\operatorname{supp}\partial_x\chi_0(t,x)\big)}^2=O\Big(e^{-2\sqrt{2}(1-\theta)z(t)}\Big), \tag{91}$$

Now, we recall the identity $\partial_t\phi(t,x)=-\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}-\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}+\partial_t g(t)$, by using the estimates (90), (91) in the identity (89), we deduce that

$$\begin{aligned}II=O\Bigg(&\|\dot{\chi}\|_{L^\infty(\mathbb{R})}\frac{\max_{j\in\{1,2\}}\big|\dot{x}_j(t)\big|}{z(t)}\Big\|\overrightarrow{g(t)}\Big\|^2\\&+\|\ddot{\chi}\|_{L^\infty(\mathbb{R})}\Big\|\overrightarrow{g(t)}\Big\|^2\frac{\max_{j\in\{1,2\}}\big|\dot{x}_j(t)\big|}{z(t)^2}\end{aligned}$$

$$+ e^{-\sqrt{2}z(t)\min((1-\theta),\theta(1-\gamma))} \|\ddot{\chi}\|_{L^\infty(\mathbb{R})} \frac{\max_{j\in\{1,2\}} \dot{x}_j(t)^2}{z(t)^2} \left\|\overrightarrow{g(t)}\right\|$$
$$+ \left\|\overrightarrow{g(t)}\right\| e^{-\sqrt{2}z(t)\min((1-\theta),\theta(1-\gamma))} \left[\frac{\|\ddot{\chi}\|_{L^\infty(\mathbb{R})}}{z(t)^2} + \frac{\|\dot{\chi}\|_{L^\infty(\mathbb{R})}}{z(t)}\right] \max_{j\in\{1,2\}} \dot{x}_j(t)^2\Bigg). \tag{92}$$

Since $\frac{1-\gamma}{2-\gamma} \leq \max((1-\theta), \theta(1-\gamma))$ for $0 < \gamma, \theta < 1$, we have that the estimate (92) is minimal when $\theta = \frac{1-\gamma}{2-\gamma}$. So, from now on, we consider

$$\theta = \frac{1-\gamma}{2-\gamma}, \tag{93}$$

which implies with (78) and (92) that $II = O(\alpha(t))$.
Step 3. (Estimate of III.) We deduce from the identity

$$III = -\frac{\langle \partial_x(\chi_0(t)\partial_t g(t)),\ \partial_t\phi(t)\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|_{L^2}^2}$$

that

$$\begin{aligned} III =& -\frac{\left\langle \dot{\chi}\left(\frac{x-x_1(t)}{z(t)}\right)\partial_t g(t),\ -\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} - \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)} + \partial_t g(t)\right\rangle_{L^2}}{z(t)\left\|\partial_x H_{0,1}\right\|_{L^2}^2} \\ & - \frac{\left\langle \chi_0(t,x)\partial_{t,x}^2 g(t),\ -\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} - \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)} + \partial_t g(t,x)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|_{L^2}^2} \\ =& III.1 + III.2. \end{aligned} \tag{94}$$

The identity (93) and the estimates (78), (90) and (91) imply by Cauchy–Schwarz inequality that

$$III.1 = O\left(\frac{\max_{j\in\{1,2\}}\left|\dot{x}_j(t)\right| e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}}{\gamma z(t)}\left\|\overrightarrow{g(t)}\right\| + \frac{1}{z(t)\gamma}\left\|\overrightarrow{g(t)}\right\|^2\right). \tag{95}$$

In conclusion, we have estimated that $III.1 = O(\alpha(t))$.

Also, from condition (87) and the estimate (4), we can deduce that

$$\left\|(1-\chi_0(t))\partial_x^2 H_{-1,0}^{x_1(t)}\right\|_{L^2} + \left\|\chi_0(t)\partial_x^2 H_{0,1}^{x_2(t)}\right\|_{L^2} = O\left(e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\right). \tag{96}$$

Additionally, we have that

$$III.2 = -\frac{\left\langle \chi_0(t,x)\left[\partial_{t,x}^2\phi(t) + \dot{x}_1(t)\partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\partial_x^2 H_{0,1}^{x_2(t)}\right],\ \partial_t\phi(t)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|_{L^2}^2}. \tag{97}$$

By integration by parts, we have that

$$\left|\left\langle \chi\left(\frac{x-x_1(t)}{z(t)}\right)\partial_{t,x}^2\phi(t,x),\ \partial_t\phi(t,x)\right\rangle_{L^2}\right| = O\left(\frac{1}{\gamma z(t)}\left\|\partial_t\phi(t)\right\|_{L_x^2(\operatorname{supp}\partial_x\chi_0(t))}^2\right).$$

In conclusion, from the estimates (78), (90), (91) and identity (93), we obtain that

$$\left|\left\langle \chi\left(\frac{x-x_1(t)}{z(t)}\right)\partial^2_{t,x}\phi(t,x),\ \partial_t\phi(t,x)\right\rangle_{L^2}\right|$$
$$= O\left(\frac{1}{\gamma z(t)}\left\|\overrightarrow{g(t)}\right\|^2 + \max_{j\in\{1,2\}}\frac{\dot{x}_j(t)^2}{\gamma z(t)}\left[e^{-2\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\right]\right). \tag{98}$$

Also, from Lemma (6), the estimate (4) and the fact of $0\leq\chi_0\leq 1$, we deduce that

$$\left|\left\langle \chi_0(t,x)\partial_x^2 H_{0,1}^{x_2(t)},\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2}\right| = O\left(z(t)e^{-\sqrt{2}z(t)}\right), \tag{99}$$

$$\left|\left\langle (1-\chi_0(t,x))\partial_x^2 H_{-1,0}^{x_1(t)},\ \partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2}\right| = O\left(z(t)e^{-\sqrt{2}z(t)}\right). \tag{100}$$

From the estimates (90), (91) and identity (93), we can verify by integration by parts the following estimates

$$\left\langle (1-\chi_0(t))\dot{x}_1(t)\partial_x^2 H_{-1,0}^{x_1(t)},\ \dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2} = O\left(\frac{\dot{x}_1(t)^2}{\gamma z(t)}e^{-2\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\right), \tag{101}$$

$$\left\langle \chi_0(t)\dot{x}_2(t)\partial_x^2 H_{0,1}^{x_2(t)},\ \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2} = O\left(\frac{\dot{x}_2(t)^2}{\gamma z(t)}e^{-2\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\right). \tag{102}$$

Finally, from Cauchy–Schwarz inequality and the estimate (96) we obtain that

$$\left\langle (1-\chi_0(t))\dot{x}_1(t)\partial_x^2 H_{-1,0}^{x_1(t)},\ \partial_t g(t)\right\rangle_{L^2} = O\left(|\dot{x}_1(t)|\left\|\overrightarrow{g(t)}\right\| e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\right), \tag{103}$$

$$\left\langle \chi_0(t)\dot{x}_1(t)\partial_x^2 H_{0,1}^{x_2(t)},\ \partial_t g(t)\right\rangle_{L^2} = O\left(|\dot{x}_2(t)|\left\|\overrightarrow{g(t)}\right\| e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\right). \tag{104}$$

In conclusion, we obtain from the estimates (99), (100), (101), (102) (103) and (104) that

$$III.2 = -\dot{x}_1(t)\frac{\left\langle \partial_x^2 H_{-1,0}^{x_1(t)},\ \partial_t\phi(t)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2} + O(\alpha(t)). \tag{105}$$

This estimate of $III.2$ and the estimate (95) of $III.1$ imply

$$III = -\dot{x}_1(t)\frac{\left\langle \partial_x^2 H_{-1,0}^{x_1(t)},\ \partial_t\phi(t)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|^2} + O(\alpha(t)). \tag{106}$$

In conclusion, from the estimates $II = O(\alpha(t))$, (106) and the definition of I, we have that $I + II + III = O(\alpha(t))$.

Step 4. (Estimate of V.) We recall that $V = -\frac{\langle \partial_x\chi_0(t)g(t),\,\partial_t^2\phi(t)\rangle_{L^2}}{\|\partial_x H_{0,1}\|_{L^2}^2}$, and that

$$\partial_t^2\phi(t) = \partial_x^2 g(t) + \left[\dot{U}\left(H_{-1,0}^{x_1(t)}\right) + \dot{U}\left(H_{0,1}^{x_2(t)}\right) - \dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\right]$$
$$+\left[\dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) - \dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)} + g(t)\right)\right]. \tag{107}$$

First, by integration by parts, using estimate (78), we have the following estimate

$$-\frac{1}{\left\|\partial_x H_{0,1}\right\|_{L^2}^2}\langle \partial_x \chi_0(t)\partial_x^2 g(t),\ g(t)\rangle_{L^2} = O\Big(\Big[\frac{1}{\gamma z(t)}+\frac{1}{\gamma^2 z(t)^2}\Big]\left\|\overrightarrow{g(t)}\right\|^2\Big) \\ = O(\alpha(t)). \tag{108}$$

Second, since U is smooth and $\|g(t)\|_{L^\infty} = O\big(\epsilon^{\frac{1}{2}}\big)$ for all $t \in \mathbb{R}$, we deduce that

$$\left|\left\langle \dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}+g(t)\right),\partial_x \chi_0(t) g(t)\right\rangle_{L^2}\right| \\ \lesssim \frac{\left\|\overrightarrow{g(t)}\right\|^2}{z(t)\gamma} = O(\alpha(t)). \tag{109}$$

Next, from Eq. (58) and Lemma 6, we have that

$$\left\|\dot{U}\left(H_{-1,0}^{x_1(t)}\right)+\dot{U}\left(H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right\|_{L^2} = O(e^{-\sqrt{2}z(t)}), \tag{110}$$

then, by Hölder inequality we have that

$$\left\langle \dot{U}\left(H_{-1,0}^{x_1(t)}\right)+\dot{U}\left(H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right),\partial_x \chi_0(t)\partial_x g(t)\right\rangle_{L^2} \\ \lesssim \frac{\left\|\overrightarrow{g(t)}\right\|}{\gamma z(t)} e^{-\sqrt{2}z(t)} = O(\alpha(t)). \tag{111}$$

Clearly, the estimates (108), (109) and (111) imply that $V = O(\alpha(t))$.
Step 5. (Estimate of VI.) We know that

$$VI = -\frac{\left\langle \partial_x g(t)\chi_0(t),\ \partial_t^2\phi(t)\right\rangle_{L^2}}{\left\|\partial_x H_{0,1}\right\|_{L^2}^2}.$$

We recall the Eq. (107) which implies that

$$\left\|\partial_x H_{0,1}\right\|_{L^2}^2 VI \\ = \left\langle \partial_x g(t)\chi_0(t),\ \dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}+g(t)\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right\rangle_{L^2} \\ + \left\langle \partial_x g(t)\chi_0(t),\ \dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}\right)-\dot{U}\left(H_{0,1}^{x_2(t)}\right)\right\rangle_{L^2} \\ - \left\langle \partial_x g(t)\chi_0(t),\ \partial_x^2 g(t)\right\rangle_{L^2}.$$

By integration by parts, we have from estimate (78) that

$$\langle \partial_x g(t,x)\chi_0(t,x),\ \partial_x^2 g(t,x)\rangle_{L^2} = O\Big(\frac{1}{\gamma z(t)}\left\|\overrightarrow{g(t)}\right\|^2\Big). \tag{112}$$

From the estimate (110) and Cauchy–Schwarz inequality, we can obtain the following estimate

$$\left\langle \partial_x g(t)\chi_0(t),\ \dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}\right)-\dot{U}\left(H_{0,1}^{x_2(t)}\right)\right\rangle_{L^2}$$

$$= O\left(e^{-\sqrt{2}z(t)}\left\|\overrightarrow{g(t)}\right\|\right). \tag{113}$$

Then, to conclude the estimate of VI we just need to study the following term $C(t) := \left\langle \partial_x g(t)\chi_0(t),\ \dot{U}(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1} + g(t)) - \dot{U}(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1})\right\rangle_{L^2}$. Since we have from Taylor's theorem that

$$\dot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1} + g(t)\right) - \dot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)$$
$$= \sum_{k=2}^{6} U^{(k)}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\frac{g(t)^{k-1}}{(k-1)!},$$

from estimate (78), we can deduce using integration by parts that

$$C(t) + \left\langle \chi_0(t)\partial_x\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right),\ \sum_{k=3}^{6} U^{(k)}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\frac{g(t)^{k-1}}{(k-1)!}\right\rangle_{L^2}$$
$$= O(\alpha(t)).$$

Since

$$\left\|\chi_0(t)\partial_x H^{x_2(t)}_{0,1}\right\|_{L^\infty} + \left\|(1-\chi_0(t))\partial_x H^{x_1(t)}_{-1,0}\right\|_{L^\infty} = O\left(e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\right),$$

we obtain that

$$C(t) = -\left\langle \partial_x H^{x_1(t)}_{-1,0},\ \sum_{k=3}^{6} U^{(k)}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\frac{g(t)^{k-1}}{(k-1)!}\right\rangle_{L^2}$$
$$+ O\left(\frac{1}{\gamma z(t)}\left\|\overrightarrow{g(t)}\right\|^2 + e^{-\sqrt{2}z(t)(\frac{1-\gamma}{2-\gamma})}\left\|\overrightarrow{g(t)}\right\|^2\right).$$

Also, from Lemma 6 and the fact that $\|g(t)\|_{L^\infty} \lesssim \left\|\overrightarrow{g(t)}\right\|$, we deduce that

$$\left\langle \partial_x H^{x_1}_{-1,0},\left[\ddot{U}\left(H^{x_1(t)}_{-1,0}\right) - \ddot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\right]g(t)\right\rangle_{L^2} = O\left(e^{-\sqrt{2}z(t)}\left\|\overrightarrow{g(t)}\right\|\right). \tag{114}$$

In conclusion, we obtain that

$$C(t) = -\int_{\mathbb{R}} \partial_x H^{x_1(t)}_{-1,0}\left(\dot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1} + g(t)\right) - \dot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\right)dx$$
$$+ \int_{\mathbb{R}} \partial_x H^{x_1(t)}_{-1,0}\ddot{U}\left(H^{x_1(t)}_{-1,0}\right)g(t,x)\,dx + O(\alpha(t)). \tag{115}$$

So

$$VI = \frac{-\int_{\mathbb{R}} \partial_x H^{x_1(t)}_{-1,0}\left(\dot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1} + g(t)\right) - \dot{U}\left(H^{x_1(t)}_{-1,0} + H^{x_2(t)}_{0,1}\right)\right)dx}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}}$$
$$+ \frac{\int_{\mathbb{R}} \partial_x H^{x_1(t)}_{-1,0}\ddot{U}\left(H^{x_1(t)}_{-1,0}\right)g(t,x)\,dx}{\left\|\partial_x H_{0,1}\right\|^2_{L^2}} + O(\alpha(t)). \tag{116}$$

Step 6. (Sum of $IV,\ VI$.) From the identities (107) and

$$IV = -\frac{\left\langle \partial_x H_{-1,0}^{x_1(t)},\ \partial_t^2 \phi(t)\right\rangle_{L^2}}{\left\| \partial_x H_{0,1}\right\|_{L^2}^2},$$

we obtain that

$$\begin{aligned} IV =& -\frac{\left\langle \dot{U}\left(H_{-1,0}^{x_1(t)}\right) + \dot{U}\left(H_{0,1}^{x_2(t)}\right) - \dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right),\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2}}{\left\| \partial_x H_{0,1}\right\|_{L^2}^2} \\ & - \frac{\left\langle \partial_x^2 g(t) - \left(\dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)} + g(t)\right) - \dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\right),\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2}}{\left\| \partial_x H_{0,1}\right\|_{L^2}^2}. \end{aligned} \tag{117}$$

In conclusion, from the identity

$$\left[\partial_x^2 - \ddot{U}\left(H_{-1,0}^{x_1(t)}\right)\right]\partial_x H_{-1,0}^{x_1(t)} = 0$$

and by integration by parts, we have that

$$\begin{aligned} IV + VI = & -\frac{\left\langle \dot{U}\left(H_{-1,0}^{x_1(t)}\right) + \dot{U}\left(H_{0,1}^{x_2(t)}\right) - \dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right),\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2}}{\left\| \partial_x H_{0,1}\right\|_{L^2}^2} \\ & + O(\alpha(t)). \end{aligned}$$

From our previous results, we conclude that

$$\begin{aligned} & I + II + III + IV + V + VI \\ & \quad = -\frac{\left\langle \dot{U}\left(H_{-1,0}^{x_1(t)}\right) + \dot{U}\left(H_{0,1}^{x_2(t)}\right) - \dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right),\ \partial_x H_{-1,0}^{x_1(t)}\right\rangle_{L^2}}{\left\| \partial_x H_{0,1}\right\|_{L^2}^2} \\ & \quad + O(\alpha(t)). \end{aligned} \tag{118}$$

The conclusion of the lemma follows from estimate (118) with identity

$$\dot{A}(z(t)) = -\left\langle \dot{U}\left(H_{-1,0}\right) + \dot{U}\left(H_{0,1}^{z(t)}\right) - \dot{U}\left(H_{-1,0} + H_{0,1}^{z(t)}\right),\ \partial_x H_{-1,0}\right\rangle_{L^2},$$

which can be obtained from (21) by integration by parts with the fact that

$$\left\langle \dot{U}\left(H_{-1,0} + H_{0,1}^{z(t)}\right),\ \partial_x H_{-1,0} + \partial_x H_{0,1}^{z(t)}\right\rangle_{L^2} = 0.$$

□

Remark 9. Since, we know from Lemma 6 that

$$\left|\dot{A}(z(t)) + 4e^{-\sqrt{2}z(t)}\right| \lesssim z(t)e^{-2\sqrt{2}z(t)},$$

and, by elementary calculus with change of variables, that $\left\|\partial_x H_{0,1}\right\|_{L^2}^2 = \frac{1}{2\sqrt{2}}$, then the estimates (76) and (77) obtained in Lemma 13 motivate us to study the following ordinary differential equation

$$\ddot{d}(t) = 16\sqrt{2}e^{-\sqrt{2}d(t)}. \tag{119}$$

Clearly, the solution of (119) satisfies the equation

$$\frac{d}{dt}\left[\frac{\dot{d}(t)^2}{4} + 8e^{-\sqrt{2}d(t)}\right] = 0. \tag{120}$$

As a consequence, it can be verified that if $d(t_0) > 0$ for some $t_0 \in \mathbb{R}$, then there are real constants $v > 0$, c such that

$$d(t) = \frac{1}{\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right) \text{ for all } t \in \mathbb{R}. \tag{121}$$

In conclusion, the solution of the equations

$$\begin{aligned} \ddot{d}_1(t) &= -8\sqrt{2}e^{-\sqrt{2}d(t)}, \\ \ddot{d}_2(t) &= 8\sqrt{2}e^{-\sqrt{2}d(t)}, \\ d_2(t) - d_1(t) &= d(t) > 0, \end{aligned}$$

are given by

$$d_2(t) = a + bt + \frac{1}{2\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right), \tag{122}$$

$$d_1(t) = a + bt - \frac{1}{2\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right), \tag{123}$$

for $a,\ b$ real constants. So, we now are motivated to study how close the modulation parameters $x_1,\ x_2$ of Theorem 11 can be to functions $d_1,\ d_2$ satisfying, respectively the identities (123) and (122) for constants $v \neq 0,\ a,\ b,\ c$.

At first view, the statement of the Lemma 13 seems too complex and unnecessary for use and that a simplified version should be more useful for our objectives. However, we will show later that for a suitable choice of γ depending on the energy excess of the solution $\phi(t)$, we can get a high precision in the approximation of the modulation parameters $x_1,\ x_2$ by smooth functions $d_1,\ d_2$ satisfying (123) and (122) for a large time interval.

4. Energy Estimate Method

Before applying Lemma 13, we need to construct a function $F(t)$ to get better estimate on the value of $\|(g(t), \partial_t g(t))\|_{H^1 \times L^2}$ than that obtained in Theorem 11.

From now on, we consider $\phi(t) = H_{0,1}(x - x_2(t)) + H_{-1,0}(x - x_1(t)) + g(t,x)$, with $x_1(t),\ x_2(t)$ satisfying the orthogonality conditions of the Modulation Lemma and $x_1,\ x_2,\ (g(t), \partial_t g(t))$ and $\epsilon > 0$ satisfying all the properties of Theorem 11. Before we enunciate the main theorem of this section, we consider the following notation

$$\left\langle D^2 E_{total}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\overrightarrow{g(t)}, \overrightarrow{g(t)}\right\rangle_{L^2}$$

$$= \int_{\mathbb{R}} \partial_x g(t,x)^2 + \partial_t g(t,x)^2 + \ddot{U}\left(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)\right) g(t,x)^2 \, dx.$$

We also denote $\omega_1(t,x) = \omega\left(\frac{x - x_1(t)}{x_2(t) - x_1(t)}\right)$ for ω a smooth cut-off function with the image contained in the interval [0, 1] and satisfying the following condition

$$\omega(x) = \begin{cases} 1, & \text{if } x \le \frac{3}{4}, \\ 0, & \text{if } x \ge \frac{4}{5}. \end{cases}$$

We consider now the following function

$$\begin{aligned} F(t) = &\left\langle D^2 E_{total}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\overrightarrow{g(t)},\ \overrightarrow{g(t)}\right\rangle_{L^2 \times L^2} \\ &+ 2\int_{\mathbb{R}} \partial_t g(t) \partial_x g(t) \Big[\dot{x}_1(t)\omega_1(t,x) + \dot{x}_2(t)(1 - \omega_1(t,x))\Big] \, dx \\ &- 2\int_{\mathbb{R}} g(t)\Big(\dot{U}(H_{-1,0}^{x_1(t)}) + \dot{U}(H_{0,1}^{x_2(t)}) - \dot{U}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)})\Big) \, dx \\ &+ 2\int_{\mathbb{R}} g(t)\Big[\dot{x}_1(t)^2 \partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^2 \partial_x^2 H_{0,1}^{x_2(t)}\Big] \, dx \\ &+ \frac{1}{3}\int_{\mathbb{R}} U^{(3)}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}) g(t)^3 \, dx. \end{aligned} \tag{124}$$

Since x_1, x_2 are functions of class C^2, it is not difficult to verify that $(g(t), \partial_t g(t))$ solves the integral equation associated to the following partial differential equation

$$\begin{aligned} &\partial_t^2 g(t,x) - \partial_x^2 g(t,x) + \ddot{U}(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)) g(t,x) \\ &= -\Big[\dot{U}(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x) + g(t,x)) - \dot{U}(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)) \\ &\quad - \ddot{U}(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)) g(t,x)\Big] \\ &\quad + \dot{U}(H_{-1,0}^{x_1(t)}(x)) + \dot{U}(H_{0,1}^{x_2(t)}(x)) - \dot{U}(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)) \\ &\quad - \dot{x}_1(t)^2 \partial_x^2 H_{-1,0}^{x_1(t)}(x) - \dot{x}_2(t)^2 \partial_x^2 H_{0,1}^{x_2(t)}(x) \\ &\quad + \ddot{x}_1(t) \partial_x H_{-1,0}^{x_1(t)}(x) + \ddot{x}_2(t) \partial_x H_{0,1}^{x_2(t)}(x) \end{aligned} \tag{II}$$

in the space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$.

Theorem 14. *Assuming the hypotheses of Theorem 11 and recalling its notation, let* $\delta(t)$ *be the following quantity*

$$\begin{aligned} \delta(t) = &\left\|\overrightarrow{g(t)}\right\| \left(e^{-\sqrt{2}z(t)} \max_{j \in \{1,2\}} \left|\dot{x}_j(t)\right| + \max_{j \in \{1,2\}} \left|\dot{x}_j(t)\right|^3 e^{-\frac{\sqrt{2}z(t)}{5}}\right) \\ &+ \left\|\overrightarrow{g(t)}\right\|^2 \left(\frac{\max_{j \in \{1,2\}} \left|\dot{x}_j(t)\right|}{z(t)} + \max_{j \in \{1,2\}} \dot{x}_j(t)^2 + \max_{j \in \{1,2\}} \left|\ddot{x}_j(t)\right|\right) \\ &+ \left\|\overrightarrow{g(t)}\right\|^4 + \left\|\overrightarrow{g(t)}\right\| \max_{j \in \{1,2\}} \left|\dot{x}_j(t)\ddot{x}_j(t)\right|. \end{aligned}$$

Then, there exist positive constants $A_1,\ A_2,\ A_3$ such that the function $F(t)$ satisfies the inequalities

$$F(t) + A_1\epsilon^2 \geq A_2 \left\Vert \overrightarrow{g(t)}\right\Vert^2,\ \left|\dot{F}(t)\right| \leq A_3\delta(t).$$

Remark 10. Theorems 11 and 14 imply

$$\left|\dot{F}(t)\right| \lesssim \frac{\epsilon^{\frac{1}{2}}}{\ln\left(\frac{1}{\epsilon}\right)} \left\Vert \overrightarrow{g(t)}\right\Vert^2 + \left\Vert \overrightarrow{g(t)}\right\Vert \epsilon^{\frac{3}{2}}.$$

Proof. Since the formula defining function $F(t)$ is very large, we decompose the function in a sum of five terms $F_1,\ F_2,\ F_3,\ F_4$ and F_5. More specifically:

$$\begin{aligned}
F_1(t) =& \int_{\mathbb{R}} \partial_t g(t)^2 + \partial_x g(t)^2 + \ddot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) g(t,x)^2\,dx,\\
F_2(t) =& -2\int_{\mathbb{R}} g(t)\left[\dot{U}\left(H_{-1,0}^{x_1(t)}\right) + \dot{U}\left(H_{0,1}^{x_2(t)}\right) - \dot{U}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\right]dx,\\
F_3(t) =& 2\int_{\mathbb{R}} g(t)\left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]dx,\\
F_4(t) =& 2\int_{\mathbb{R}} \partial_t g(t)\partial_x g(t)(\dot{x}_1(t)\omega_1(t) + \dot{x}_2(t)(1-\omega_1(t)))\,dx,\\
F_5(t) =& \frac{1}{3}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)^3\,dx.
\end{aligned}$$

First, we prove that $\left|\dot{F}(t)\right| \lesssim \delta(t)$. The main idea of the proof of this item is to estimate each derivative $\frac{dF_j(t)}{dt}$, for $1 \leq j \leq 5$, with an error of size $O(\delta(t))$, then we will check that the sum of these estimates are going to be a value of order $O(\delta(t))$, which means that the main terms of the estimates of these derivatives cancel.

Step 1. (The derivative of $F_1(t)$.) By definition of $F_1(t)$, we have that

$$\begin{aligned}
&\frac{dF_1(t)}{dt}\\
&= 2\int_{\mathbb{R}}\left(\partial_t^2 g(t,x) - \partial_x^2 g(t,x) + \ddot{U}(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x))g(t,x)\right)\partial_t g(t,x)\,dx\\
&\quad - \int_{\mathbb{R}} \dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}(x)U^{(3)}\left(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)\right) g(t,x)^2\,dx\\
&\quad - \int_{\mathbb{R}} \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}(x)U^{(3)}\left(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)\right) g(t,x)^2\,dx.
\end{aligned}$$

Moreover, from the identity (II) satisfied by $g(t,x)$, we can rewrite the value of $\frac{dF_1(t)}{dt}$ as

$$\begin{aligned}
\frac{dF_1(t)}{dt} =& 2\int_{\mathbb{R}}\left[\dot{U}\left(H_{-1,0}^{x_1(t)}\right) + \dot{U}\left(H_{0,1}^{x_2(t)}\right) - \dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&-2\int_{\mathbb{R}}\left[\dot{U}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)} + g(t)\right) - \dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&+2\int_{\mathbb{R}} \ddot{U}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)\partial_t g(t)\,dx
\end{aligned}$$

$$-2\int_{\mathbb{R}}\left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]\partial_t g(t)\,dx$$
$$+2\int_{\mathbb{R}}\left[\ddot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\ddot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]\partial_t g(t)\,dx$$
$$-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2\,dx,$$

and, from the orthogonality conditions of the Modulation Lemma, we obtain

$$\begin{aligned}\frac{dF_1(t)}{dt}
&=2\int_{\mathbb{R}}\ddot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)\partial_t g(t)\,dx\\
&-2\int_{\mathbb{R}}\left[\dot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g(t)\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&+2\int_{\mathbb{R}}\left[\dot{U}\left(H_{-1,0}^{x_1(t)}\right)+\dot{U}\left(H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&-2\int_{\mathbb{R}}\left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]\partial_t g(t)\,dx\\
&+2\int_{\mathbb{R}}\left[\ddot{x}_1(t)\dot{x}_1(t)\partial_x^2 H_{-1,0}^{x_1(t)}+\ddot{x}_2(t)\dot{x}_2(t)\partial_x^2 H_{0,1}^{x_2(t)}\right]g(t)\,dx\\
&-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2\,dx,\end{aligned}$$

which implies

$$\begin{aligned}\frac{dF_1(t)}{dt}
&=2\int_{\mathbb{R}}\ddot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)\partial_t g(t,x)\,dx\\
&-2\int_{\mathbb{R}}\left[\dot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g(t)\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&+2\int_{\mathbb{R}}\left[\dot{U}\left(H_{-1,0}^{x_1(t)}\right)+\dot{U}\left(H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\right]\partial_t g(t)\,dx\\
&-2\int_{\mathbb{R}}\left[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\right]\partial_t g(t)\,dx\\
&-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2\,dx\\
&+O\left(\delta(t)\right).\end{aligned}\tag{125}$$

Step 2. (The derivative of $F_2(t)$.) It is not difficult to verify that

$$\begin{aligned}\frac{dF_2(t)}{dt}=&2\int_{\mathbb{R}}g(t)\ddot{U}\left(H_{-1,0}^{x_1(t)}\right)\partial_x H_{-1,0}^{x_1(t)}\dot{x}_1(t)\,dx\\
&+2\int_{\mathbb{R}}g(t)\ddot{U}\left(H_{0,1}^{x_2(t)}\right)\partial_x H_{0,1}^{x_2(t)}\dot{x}_2(t)\,dx\end{aligned}$$

$$-2\int_{\mathbb{R}}\partial_t g(t)\left[\dot{U}\left(H_{-1,0}^{x_1(t)}\right)+\dot{U}\left(H_{0,1}^{x_2(t)}\right)-\dot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\right]dx$$
$$-2\int_{\mathbb{R}}\ddot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\left[\partial_x H_{-1,0}^{x_1(t)}\dot{x}_1(t)+\partial_x H_{0,1}^{x_2(t)}\dot{x}_2(t)\right]g(t)\,dx.$$

From the definition of the function U, we can deduce that

$$\begin{aligned}&\ddot{U}\left(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x)\right)-\ddot{U}\left(H_{-1,0}^{x_1(t)}(x)\right)\\&\quad=O\Big(\left|H_{-1,0}^{x_1(t)}(x)H_{0,1}^{x_2(t)}(x)\right|+\left|H_{0,1}^{x_2(t)}(x)\right|^2\Big),\\&\ddot{U}\left(H_{0,1}^{x_2(t)}(x)+H_{-1,0}^{x_1(t)}(x)\right)-\ddot{U}\left(H_{0,1}^{x_2(t)}(x)\right)\\&\quad=O\Big(\left|H_{-1,0}^{x_1(t)}(x)H_{0,1}^{x_2(t)}(x)\right|+\left|H_{-1,0}^{x_1(t)}(x)\right|^2\Big),\end{aligned}$$

therefore, we obtain from Lemma 6 and Cauchy–Schwarz inequality that

$$\begin{aligned}&\left|\int_{\mathbb{R}}\left[\ddot{U}\left(H_{0,1}^{x_2(t)}\right)-\ddot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\right]\partial_x H_{0,1}^{x_2(t)}g(t)\,dx\right|\\&\quad\lesssim\left\|\overrightarrow{g(t)}\right\|e^{-\sqrt{2}z(t)},\\&\left|\int_{\mathbb{R}}\left[\ddot{U}\left(H_{-1,0}^{x_1(t)}\right)-\ddot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\right]\partial_x H_{-1,0}^{x_1(t)}g(t)\,dx\right|\\&\quad\lesssim\left\|\overrightarrow{g(t)}\right\|e^{-\sqrt{2}z(t)}.\end{aligned}$$

In conclusion, we obtain from the identity satisfied by $\frac{dF_2(t)}{dt}$ that

$$\begin{aligned}\frac{dF_2(t)}{dt}&=-2\int_{\mathbb{R}}\partial_t g(t)\left[\dot{U}\left(H_{-1,0}^{x_1(t)}\right)+\dot{U}\left(H_{0,1}^{x_2(t)}\right)\right]dx\\&\quad+2\int_{\mathbb{R}}\partial_t g(t,x)\dot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)dx+O(\delta(t)).\end{aligned}\tag{126}$$

Step 3. (The derivative of $F_3(t)$.) From the definition of $F_3(t)$, we obtain that

$$\begin{aligned}\frac{dF_3(t)}{dt}&=2\int_{\mathbb{R}}\partial_t g(t)\big[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\big]\,dx\\&\quad-2\int_{\mathbb{R}}g(t)\big[\dot{x}_1(t)^3\partial_x^3 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^3\partial_x^3 H_{0,1}^{x_2(t)}\big]\,dx\\&\quad+4\int_{\mathbb{R}}g(t)\big[\dot{x}_1(t)\ddot{x}_1(t)\partial_x^2 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\ddot{x}_2(t)\partial_x^2 H_{0,1}^{x_2(t)}\big]\,dx,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\frac{dF_3(t)}{dt}&=2\int_{\mathbb{R}}\partial_t g(t)\big[\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}\big]\,dx\\&\quad-2\int_{\mathbb{R}}g(t)\big[\dot{x}_1(t)^3\partial_x^3 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^3\partial_x^3 H_{0,1}^{x_2(t)}\big]\,dx+O(\delta(t)).\end{aligned}\tag{127}$$

Step 4. (Sum of $\frac{dF_1}{dt}$, $\frac{dF_2}{dt}$, $\frac{dF_3}{dt}$.) If we sum the estimates (125), (126) and (127), we obtain that

$$\begin{aligned}\sum_{i=1}^{3}\frac{dF_i(t)}{dt} =& 2\int_{\mathbb{R}}\ddot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)\partial_t g(t)\,dx\\ &-2\int_{\mathbb{R}}\left[\dot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g(t)\right)-\dot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\right]\partial_t g(t)\,dx\\ &-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2dx\\ &-2\int_{\mathbb{R}}g(t)\left[\dot{x}_1(t)^3\partial_x^3 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^3\partial_x^3 H_{0,1}^{x_2(t)}\right]dx+O(\delta(t)).\end{aligned}$$

More precisely, from Taylor's Expansion Theorem and since $\left\|\overrightarrow{g(t)}\right\|^4\leq\delta(t)$,

$$\begin{aligned}\sum_{i=1}^{3}\frac{dF_i(t)}{dt} =& -\int_{\mathbb{R}}\left[U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2\right]\partial_t g(t)\,dx\\ &-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2dx\\ &-2\int_{\mathbb{R}}g(t)\left[\dot{x}_1(t)^3\partial_x^3 H_{-1,0}^{x_1(t)}+\dot{x}_2(t)^3\partial_x^3 H_{0,1}^{x_2(t)}\right]dx+O(\delta(t)).\end{aligned}\tag{128}$$

Step 5. (The derivative of $F_4(t)$.) The computation of the derivative of $F_4(t)$ will be more careful since the motivation for the addition of this term is to cancel with the expression

$$-\int_{\mathbb{R}}\left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}+\dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right]U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2dx$$

of (128). The construction of functional $F_4(t)$ is based on the *momentum correction term* of Lemma 4.2 of [13]. To estimate $\frac{dF_4(t)}{dt}$ with precision of $O(\delta(t))$, it is just necessary to study the time derivative of

$$2\int_{\mathbb{R}}\partial_t g(t)\partial_x g(t)\dot{x}_1(t)\omega_1(t)\,dx,\tag{129}$$

since the estimate of the other term in $F_4(t)$ is completely analogous. First, we have the identity

$$\begin{aligned}\frac{d}{dt}\left[2\int_{\mathbb{R}}\partial_t g(t)\partial_x g(t)\dot{x}_1(t)\omega_1(t)\,dx\right] =& 2\ddot{x}_1(t)\int_{\mathbb{R}}\omega_1(t,x)\partial_t g(t)\partial_x g(t)\,dx\\ &+2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t,x)\partial_t^2 g(t)\partial_x g(t)\,dx\\ &+2\dot{x}_1(t)\int_{\mathbb{R}}\partial_t\omega_1(t)\partial_t g(t)\partial_x g(t)\,dx\\ &+2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t,x)\partial_{t,x}^2 g(t,x)\partial_t g(t)\,dx.\end{aligned}$$

From the definition of $\omega_1(t,x) = \omega\big(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\big)$, we have

$$\partial_t\omega_1(t,x) = \dot{\omega}\Big(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\Big)\left(\frac{-\dot{x}_1(t)z(t)-\dot{z}(t)(x-x_1(t))}{z(t)^2}\right). \tag{130}$$

Since in the support of $\dot{\omega}(x)$ is contained in the set $\frac{3}{4} \leq x \leq \frac{4}{5}$, we obtain the following estimate:

$$2\dot{x}_1(t)\int_{\mathbb{R}}\partial_t\omega_1(t)\partial_t g(t)\partial_x g(t)\,dx = O\left(\max_{j\in\{1,2\}}\frac{\left|\dot{x}_j(t)\right|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^2\right) = O(\delta(t)). \tag{131}$$

Clearly, from integration by parts, we deduce that

$$2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\partial^2_{t,x}g(t)\partial_t g(t)\,dx = O\left(\max_{j\in\{1,2\}}\frac{\left|\dot{x}_j(t)\right|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^2\right) = O(\delta(t)). \tag{132}$$

Also, we have

$$2\ddot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\partial_t g(t)\partial_x g(t)\,dx = O\Big(\max_{j\in\{1,2\}}\left|\ddot{x}_j(t)\right|\left\|\overrightarrow{g(t)}\right\|^2\Big) = O(\delta(t)). \tag{133}$$

So, to estimate the time derivative of (129) with precision $O(\delta(t))$, it is enough to estimate

$$2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t,x)\partial_t^2 g(t,x)\partial_x g(t,x)\,dx.$$

We have that

$$\begin{aligned}2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\partial_t^2 g(t)\partial_x g(t)\,dx =& 2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\partial_x^2 g(t)\partial_x g(t)\,dx\\ &-2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\ddot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)g(t)\partial_x g(t)\,dx\\ &+2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\left[\partial_t^2 g(t)-\partial_x^2 g(t)\right]\partial_x g(t)\,dx\\ &+2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\ddot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)g(t)\partial_x g(t)\,dx.\end{aligned} \tag{134}$$

From integration by parts, the first term of right-hand side of Eq. (134) satisfies

$$2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\partial_x^2 g(t)\partial_x g(t)\,dx = O\left(\max_{j\in\{1,2\}}\frac{\left|\dot{x}_j(t)\right|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^2\right) = O(\delta(t)). \tag{135}$$

From Taylor's Expansion Theorem, we have that

$$\left\|\dot{U}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}+g(t)\right)-\sum_{j=1}^{3}U^{(j)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)\frac{g(t)^{j-1}}{(j-1)!}\right\|_{L^2}$$

$$= O\left(\left\|\overrightarrow{g(t)}\right\|^3\right). \tag{136}$$

Also, we have verified the identity

$$\dot{U}(\phi) + \dot{U}(\theta) - \dot{U}(\phi+\theta) = 24\phi\theta(\phi+\theta) - 6\Big(\sum_{j=1}^{4}\binom{5}{j}\phi^j\theta^{5-j}\Big),$$

which clearly implies with the inequalities $(D1)$, $(D2)$ and Lemma 6 the estimate

$$\left\|\dot{U}\left(H_{0,1}^{x_2(t)}\right) + \dot{U}\left(H_{-1,0}^{x_1(t)}\right) - \dot{U}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\right\|_{L^2(\mathbb{R})} = O\left(e^{-\sqrt{2}z(t)}\right). \tag{137}$$

Finally, it is not difficult to verify that

$$\begin{aligned}&\left\|-\dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)} - \dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)} + \ddot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \ddot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right\|_{L^2(\mathbb{R})}\\ &= O\left(\max_{j\in\{1,2\}}\left|\dot{x}_j(t)\right|^2 + \left|\ddot{x}_j(t)\right|\right).\end{aligned} \tag{138}$$

Then, from estimates (136), (137) and (138) and the partial differential equation (II) satisfied by $g(t,x)$, we can obtain the estimate

$$\begin{aligned}&2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\left[\partial_t^2 g(t) - \partial_x^2 g(t) + \ddot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)g(t)\right]\partial_x g(t)\,dx\\ &= -\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)U^{(3)}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)g(t)^2\partial_x g(t)\,dx\\ &\quad - 2\dot{x}_1(t)^3\int_{\mathbb{R}}\partial_x^2 H_{-1,0}^{x_1(t)}\partial_x g(t)\,dx - 2\dot{x}_1(t)\dot{x}_2(t)^2\int_{\mathbb{R}}\omega_1(t)\partial_x^2 H_{0,1}^{x_2(t)}\partial_x g(t)\,dx\\ &\quad - 2\dot{x}_1(t)^3\int_{\mathbb{R}}(\omega_1(t)-1)\partial_x^2 H_{-1,0}^{x_1(t)}\partial_x g(t)\,dx + O\left(\left\|\overrightarrow{g(t)}\right\|^4\max_{j\in\{1,2\}}\left|\dot{x}_j(t)\right|\right)\\ &\quad + O\left(\max_{j\in\{1,2\}}\left|\ddot{x}_j(t)\dot{x}_j(t)\right|\left\|\overrightarrow{g(t)}\right\| + e^{-\sqrt{2}z(t)}\max_{j\in\{1,2\}}\left|\dot{x}_j(t)\right|\left\|\overrightarrow{g(t)}\right\|\right),\end{aligned}$$

which, by integration by parts and by Cauchy–Schwarz inequality using the estimate (96) for ω_1, we obtain that

$$\begin{aligned}&2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\left[\partial_t^2 g(t) - \partial_x^2 g(t) + \ddot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)g(t)\right]\partial_x g(t)\,dx\\ &= \frac{\dot{x}_1(t)}{3}\int_{\mathbb{R}}\omega_1(t)U^{(4)}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\left[\partial_x H_{-1,0}^{x_1(t)} + \partial_x H_{0,1}^{x_2(t)}\right]g(t)^3\,dx\\ &\quad - 2\dot{x}_1(t)^3\int_{\mathbb{R}}\partial_x^2 H_{-1,0}^{x_1(t)}\partial_x g(t)\,dx + O\left(\max_{j\in\{1,2\}}\frac{\left|\dot{x}_j(t)\right|}{z(t)}\left\|\overrightarrow{g(t)}\right\|^3\right)\\ &\quad + O\left(\max_{j\in\{1,2\}}\left|\dot{x}_j(t)\right|^3 e^{-\frac{\sqrt{2}z(t)}{5}}\left\|\overrightarrow{g(t)}\right\|\right) + O(\delta(t)).\end{aligned} \tag{139}$$

Now, to finish the estimate of $2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t,x)\partial_t^2 g(t,x)\partial_x g(t,x)\,dx$, it remains to study the integral given by

$$-2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)\ddot{U}\left(H_{-1,0}^{x_1(t)}(x)+H_{0,1}^{x_2(t)}(x)\right)g(t)\partial_x g(t)\,dx, \tag{140}$$

which by integration by parts is equal to

$$\begin{aligned}&\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)U^{(3)}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\partial_x H_{-1,0}^{x_1(t)}g(t)^2\,dx\\&+\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t)U^{(3)}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\partial_x H_{0,1}^{x_2(t)}g(t)^2\,dx+O(\delta(t)).\end{aligned} \tag{141}$$

Since the support of $\omega_1(t,x)$ is included in $\{x|\,(x-x_2(t))\le-\frac{z(t)}{5}\}$ and the support of $1-\omega_1(t,x)$ is included in $\{x|\,(x-x_1(t))\ge\frac{3z(t)}{4}\}$, from the exponential decay properties of the kink solutions in (D1), (D2), (D3), (D4) we obtain the estimates

$$\left|\dot{x}_1(t)\int_{\mathbb{R}}(\omega_1(t)-1)U^{(3)}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\partial_x H_{-1,0}^{x_1(t)}g(t)^2\,dx\right|=O(\delta(t)), \tag{142}$$

$$\left|\dot{x}_2(t)\int_{\mathbb{R}}\omega_1(t)U^{(3)}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\partial_x H_{0,1}^{x_2(t)}g(t)^2\,dx\right|=O(\delta(t)), \tag{143}$$

$$\left|\frac{1}{3}\dot{x}_1(t)\int_{\mathbb{R}}(1-\omega_1(t))U^{(4)}(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)})\partial_x H_{-1,0}^{x_1(t)}g(t)^3\,dt\right|=O(\delta(t)), \tag{144}$$

$$\left|\frac{1}{3}\dot{x}_2(t)\int_{\mathbb{R}}(\omega_1(t))U^{(4)}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\partial_x H_{0,1}^{x_2(t)}g(t)^3\,dt\right|=O(\delta(t)). \tag{145}$$

In conclusion, we obtain that the estimates (142), (143) imply the following estimate

$$\begin{aligned}&-2\dot{x}_1(t)\int_{\mathbb{R}}\omega_1(t,x)\ddot{U}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)g(t)\partial_x g(t)\,dx\\&=\int_{\mathbb{R}}\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}U^{(3)}\left(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)}\right)g(t)^2\,dx+O(\delta(t)).\end{aligned} \tag{146}$$

Then, the estimates (134), (139), (144), (145) and (146) imply that

$$\begin{aligned}&2\frac{d}{dt}\left(\int_{\mathbb{R}}\partial_t g(t)\partial_x g(t)\dot{x}_1(t)\omega_1(t)\,dx\right)\\&=-2\dot{x}_1(t)^3\int_{\mathbb{R}}\partial_x^2 H_{-1,0}^{x_1(t)}\partial_x g(t)\,dx\\&+\frac{1}{3}\int_{\mathbb{R}}U^{(4)}\left(H_{-1,0}^{x_1(t)}+H_{0,1}^{x_2(t)}\right)\left(\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}\right)g(t)^3\,dx\\&+\int_{\mathbb{R}}\left(\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}\right)U^{(3)}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)})g(t)^2\,dx+O(\delta(t)).\end{aligned}$$

By an analogous argument, we deduce that

$$2\frac{d}{dt}\left(\int_{\mathbb{R}}\partial_t g(t)\partial_x g(t)\dot{x}_2(t)(1-\omega_1(t))\,dx\right)$$

$$
\begin{aligned}
&= -2\dot{x}_2(t)^3 \int_{\mathbb{R}} \partial_x^2 H_{0,1}^{x_2(t)} \partial_x g(t)\,dx \\
&\quad + \frac{\dot{x}_2(t)}{3} \int_{\mathbb{R}} U^{(4)} \left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) \partial_x H_{0,1}^{x_2(t)} g(t)^3\,dx \\
&\quad + \int_{\mathbb{R}} \dot{x}_2(t) \partial_x H_{0,1}^{x_2(t)} U^{(3)} \left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)^2\,dx \\
&\quad + O(\delta(t)).
\end{aligned}
$$

In conclusion, we have that

$$
\begin{aligned}
\frac{dF_4(t)}{dt} &= \int_{\mathbb{R}} \left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right] U^{(3)} \left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)^2\,dx \\
&\quad - 2\dot{x}_2(t)^3 \int_{\mathbb{R}} \partial_x^2 H_{0,1}^{x_2(t)} \partial_x g(t)\,dx - 2\dot{x}_1(t)^3 \int_{\mathbb{R}} \partial_x^2 H_{-1,0}^{x_1(t)} \partial_x g(t)\,dx \\
&\quad + \int_{\mathbb{R}} \frac{1}{3} U^{(4)} \left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) \left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right] g(t)^3\,dx \\
&\quad + O(\delta(t)). \qquad (147)
\end{aligned}
$$

Step 6. (The derivative of $F_5(t)$.) We have that

$$
\begin{aligned}
\frac{dF_5(t)}{dt} &= \int_{\mathbb{R}} U^{(3)} \left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) g(t)^2 \partial_t g(t)\,dx \\
&\quad - \frac{1}{3} \int_{\mathbb{R}} U^{(4)} \left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) \left[\dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} + \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}\right] g(t)^3\,dx. \qquad (148)
\end{aligned}
$$

Step 7. (Conclusion of estimate of $|\dot{F}(t)|$) From the identities (147) and (148), we obtain that

$$
\begin{aligned}
\frac{dF_4(t)}{dt} + \frac{dF_5(t)}{dt} &= \int_{\mathbb{R}} \dot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)} U^{(3)} \left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)^2\,dx \\
&\quad + \int_{\mathbb{R}} \dot{x}_2(t)\partial_x H_{0,1}^{x_2(t)} U^{(3)} \left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)^2\,dx \\
&\quad - 2\dot{x}_1(t)^3 \int_{\mathbb{R}} \partial_x^2 H_{-1,0}^{x_1(t)} \partial_x g(t)\,dx - 2\dot{x}_2(t)^3 \int_{\mathbb{R}} \partial_x^2 H_{0,1}^{x_2(t)} \partial_x g(t)\,dx \\
&\quad + \int_{\mathbb{R}} U^{(3)} \left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) g(t)^2 \partial_t g(t)\,dx + O(\delta(t)). \qquad (149)
\end{aligned}
$$

Then, the sum of identities (128) and (149) implies $\sum_{i=1}^{5} \frac{dF_i(t)}{dt} = O(\delta(t))$, this finishes the proof of inequality $\left|\dot{F}(t)\right| = O(\delta(t))$.

Proof of $F(t) + A_1\epsilon^2 \geq A_2\epsilon^2$. The Coercivity Lemma implies that $\exists\, c > 0$, such that $F_1(t) \geq c \left\|\overrightarrow{g(t)}\right\|^2$. Also, from Theorem 11, we have the global estimate

$$
\max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|^2 + \left|\ddot{x}_j(t)\right| + e^{-\sqrt{2}z(t)} + \left\|\overrightarrow{g(t)}\right\|^2 = O(\epsilon), \qquad (150)
$$

which implies that $|F_3(t)| = O\Big(\left\|\overrightarrow{g(t)}\right\|\epsilon\Big)$, $|F_4(t)| = O\Big(\left\|\overrightarrow{g(t)}\right\|^2\epsilon^{\frac{1}{2}}\Big)$, $|F_5(t)| = O\Big(\left\|\overrightarrow{g(t)}\right\|^2\epsilon^{\frac{1}{2}}\Big)$. Also, since

$$\begin{aligned}&\left|U\left(H^{x_1(t)}_{-1,0}(x)\right)+\dot{U}\left(H^{x_2(t)}_{0,1}(x)\right)-\dot{U}\left(H^{x_2(t)}_{0,1}(x)+H^{x_1(t)}_{-1,0}(x)\right)\right|\\&\quad=O\left(\left|H^{x_1(t)}_{-1,0}(x)H^{x_2(t)}_{0,1}(x)\left[H^{x_2(t)}_{0,1}(x)+H^{x_1(t)}_{-1,0}(x)\right]\right|\right),\end{aligned}$$

Lemma 6 and Cauchy–Schwarz inequality imply that

$$|F_2(t)| = O\Big(\left\|\overrightarrow{g(t)}\right\| e^{-\sqrt{2}z(t)}\Big).$$

Then, the conclusion of $F(t)+A_1\epsilon^2 \geq A_2\left\|\overrightarrow{g(t)}\right\|^2$ follows from Young inequality for ϵ small enough. □

Remark 11. In the proof of Theorem 14, from Theorem 11 we have $|F_2(t)|+|F_3(t)| = O\left(\left\|\overrightarrow{g(t)}\right\|\epsilon\right)$. Since $|F_4(t)|+|F_5(t)| = O\left(\left\|\overrightarrow{g(t)}\right\|^2\epsilon^{\frac{1}{2}}\right)$ and $|F_1(t)| \lesssim \left\|\overrightarrow{g(t)}\right\|^2$, then Young inequality implies that

$$|F(t)| \lesssim \left\|\overrightarrow{g(t)}\right\|^2+\epsilon^2.$$

Remark 12. (*General Energy Estimate*) For any $0<\theta,\gamma<1$, we can create a smooth cut-off function $0\leq\chi(x)\leq 1$ such that

$$\chi(x) = \begin{cases}0, & \text{if } x\leq\theta(1-\gamma),\\ 1, & \text{if } x\geq\theta.\end{cases}$$

We define

$$\chi_0(t,x) = \chi\left(\frac{x-x_1(t)}{x_2(t)-x_1(t)}\right).$$

If we consider the following function

$$\begin{aligned}L(t) &= \left\langle D^2E_{total}(H^{x_2(t)}_{0,1}+H^{x_1(t)}_{-1,0})\overrightarrow{g(t)},\ \overrightarrow{g(t)}\right\rangle_{L^2\times L^2}\\&\quad+2\int_{\mathbb{R}}\partial_t g(t)\partial_x g(t)\Big[\dot{x}_1(t)\chi_0(t)+\dot{x}_2(t)\left(1-\chi_0(t)\right)\Big]dx\\&\quad-2\int_{\mathbb{R}}g(t)\left(\dot{U}\left(H^{x_1(t)}_{-1,0}\right)+\dot{U}\left(H^{x_2(t)}_{0,1}\right)-\dot{U}\left(H^{x_2(t)}_{0,1}+H^{x_1(t)}_{-1,0}\right)\right)dx\\&\quad+2\int_{\mathbb{R}}g(t)\Big[\dot{x}_1(t)^2\partial_x^2H^{x_1(t)}_{-1,0}+\dot{x}_2(t)^2\partial_x^2H^{x_2(t)}_{0,1}\Big]dx\\&\quad+\frac{1}{3}\int_{\mathbb{R}}U^{(3)}\left(H^{x_2(t)}_{0,1}+H^{x_1(t)}_{-1,0}\right)g(t)^3\,dx,\end{aligned}$$

then, by a similar proof to the Theorem 14, we obtain that if $0<\epsilon\ll 1$ and

$$\delta_1(t) = \delta(t)+\max_{j\in\{1,2\}}\left|\dot{x}_j(t)\right|^3\max(e^{-\sqrt{2}z(t)(1-\theta)},e^{-\sqrt{2}z(t)\theta(1-\gamma)})\left\|\overrightarrow{g(t)}\right\|$$

$$- \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|^3 e^{-\frac{\sqrt{2}}{5}z(t)} \left\|\overrightarrow{g(t)}\right\|, \tag{151}$$

then there are positive constants $A_1,\ A_2 > 0$ such that

$$\left|\dot{L}(t)\right| = O(\delta_1(t)),\ L(t) + A_1\epsilon^2 \geq A_2\epsilon^2.$$

Our first application of Theorem 14 is to estimate the size of the remainder $\left\|\overrightarrow{g(t)}\right\|$ during a long time interval. More precisely, this corresponds to the following theorem, which is a weaker version of Theorem 2.

Theorem 15. *There is* $\delta > 0$, *such that if* $0 < \epsilon < \delta$, $(\phi(0), \partial_t\phi(0)) \in S \times L^2(\mathbb{R})$ *and* $E_{total}(\phi(0), \partial_t\phi(0)) = 2E_{pot}(H_{0,1}) + \epsilon$, *then there exist* $x_1, x_2 \in C^2(\mathbb{R})$ *such that the unique solution of* (1) *is given, for any* $t \in \mathbb{R}$, *by*

$$\phi(t) = H_{0,1}(x - x_2(t)) + H_{-1,0}(x - x_1(t)) + g(t), \tag{152}$$

with $g(t)$ *satisfying orthogonality conditions of the Modulation Lemma and*

$$\left\|\overrightarrow{g(t)}\right\|^2_{H^1\times L^2} \leq C\left[\left\|\overrightarrow{g(0)}\right\|^2_{H^1\times L^2} + \left(\epsilon \ln\frac{1}{\epsilon}\right)^2\right]\exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln\frac{1}{\epsilon}}\right), \tag{153}$$

for all $t \in \mathbb{R}$.

Proof of Theorem 15. In notation of Theorem 14, from Theorem 14 and Remark 11, there are uniform positive constants $A_2,\ A_1$ such that for all $t \geq 0$

$$A_2\left\|\overrightarrow{g(t)}\right\|^2 \leq F(t) + A_1\epsilon^2 \leq C\left(\left\|\overrightarrow{g(t)}\right\|^2 + \epsilon^2\right). \tag{154}$$

From now on, we denote $G(t) := F(t) + A_1\left(\epsilon \ln\frac{1}{\epsilon}\right)^2$. From the inequality (154) and Remark 10, there is a constant $C > 0$ such that, for all $t \geq 0$, $G(t)$ satisfies

$$G(t) \leq G(0) + C\left(\int_0^t G(s)\frac{\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}ds\right).$$

In conclusion, from Gronwall Lemma, we obtain that $G(t) \leq G(0)\exp\left(\frac{C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)$ for all $t \geq 0$. Then, from the definition of G and inequality (154), we verify the inequality (153) for any $t \geq 0$. The proof of inequality (153) for the case $t < 0$ is completely analogous. □

5. Global Dynamics of Modulation Parameters

Lemma 16. *In notation of Theorem 2,* $\exists C > 0$, *such that if the hypotheses of Theorem 2 are true, then for* $\overrightarrow{g(0)} = (g(0,x), \partial_t g(0,x))$ *we have that there are functions* $p_1(t),\ p_2(t) \in C^1(\mathbb{R}_{\geq 0})$, *such that for* $j \in \{1,\ 2\}$ *and any* $t \geq 0$, *we have:*

$$\left|\dot{x}_j(t) - p_j(t)\right| \lesssim \left(\left\|\overrightarrow{g(0)}\right\|_{H^1\times L^2} + \epsilon\ln\frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right), \tag{155}$$

$$\left|\dot{p}_j(t) - (-1)^j 8\sqrt{2}e^{-\sqrt{2}z(t)}\right| \lesssim \frac{\left(\left\|\overrightarrow{g(0)}\right\|_{H^1\times L^2} + \epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right). \tag{156}$$

Proof. In the notation of Lemma 13, we consider the functions $p_j(t)$ for $j \in \{1,\ 2\}$ and we consider $\theta = \frac{1-\gamma}{2-\gamma}$, the value of γ will be chosen later. From Lemma 13, we have that

$$\left|\dot{x}_j(t) - p_j(t)\right| \lesssim \left[1 + \frac{1}{\gamma z(t)}\right]\Big(\max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| \left\|\overrightarrow{g(t)}\right\| + \left\|\overrightarrow{g(t)}\right\|^2\Big) + \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| z(t)e^{-\sqrt{2}z(t)}.$$

We recall from Theorem 11 the estimates $\max_{j\in\{1,\,2\}} \left|\dot{x}_j(t)\right| = O(\epsilon^{\frac{1}{2}})$, $e^{-\sqrt{2}z(t)} = O(\epsilon)$. From Theorem 15, we have that

$$\left\|\overrightarrow{g(t)}\right\| \lesssim \left(\left\|\overrightarrow{g(0)}\right\| + \epsilon \ln \frac{1}{\epsilon}\right) \exp\left(\frac{C\epsilon^{\frac{1}{2}}|t|}{\ln \frac{1}{\epsilon}}\right).$$

To simplify our computations, we denote $c_0 = \frac{\left\|\overrightarrow{g(0)}\right\| + \epsilon \ln \frac{1}{\epsilon}}{\epsilon \ln \frac{1}{\epsilon}}$. Then, we obtain for any $j \in \{1,\ 2\}$ and all $t \geq 0$ that

$$\left|\dot{x}_j(t) - p_j(t)\right| \lesssim \left[1 + \frac{1}{\gamma \ln \frac{1}{\epsilon}}\right] c_0 \epsilon^{\frac{3}{2}} \ln \frac{1}{\epsilon} \exp\left(\frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) + \left[1 + \frac{1}{\gamma \ln \frac{1}{\epsilon}}\right]\left(c_0 \epsilon \ln \frac{1}{\epsilon}\right)^2 \exp\left(\frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right). \tag{157}$$

Since $e^{-\sqrt{2}z(t)} \lesssim \epsilon$, we deduce for $\epsilon \ll 1$ that $z(t)e^{-\sqrt{2}z(t)} \lesssim \epsilon \ln \frac{1}{\epsilon} < \epsilon^{1-\frac{\gamma}{(2-\gamma)2}} \ln \frac{1}{\epsilon}$. Then, for any $t \geq 0$, we obtain from the same estimates and the definition (75) of $\alpha(t)$ that

$$\alpha(t) \lesssim c_0^2 \Big(\epsilon \ln \frac{1}{\epsilon}\Big)^2 \left[\max_{k\in\{1,\,2\}} \Big(\frac{1}{\gamma z(t)}\Big)^k + \epsilon^{\frac{1-\gamma}{2-\gamma}}\right] \exp\left(2\frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) + c_0 \left[\epsilon^{2-\frac{\gamma}{(2-\gamma)2}} \ln \frac{1}{\epsilon}\right] \exp\left(\frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) \left[1 + \frac{1}{\gamma z(t)} + \frac{\epsilon^{\frac{1}{2}}}{(\gamma z(t))^2}\right] + \frac{\epsilon^{1+\frac{2(1-\gamma)}{2-\gamma}}}{z(t)\gamma}. \tag{158}$$

However, if $\gamma \ln \frac{1}{\epsilon} \leq 1$ and $z(0) \cong \ln \frac{1}{\epsilon}$, which is possible, then the right-hand side of inequality (158) is greater than or equivalent to $\left(\epsilon \ln \frac{1}{\epsilon}\right)^2$ while $0 \leq t \lesssim \frac{\ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$. But, it is not difficult to verify for $\gamma = \frac{\ln \ln \frac{1}{\epsilon}}{\ln \frac{1}{\epsilon}}$ that the right-hand side of inequality (158) is smaller than $\left(\epsilon \ln \frac{1}{\epsilon}\right)^2$.

Therefore, from now on, we are going to study the right-hand side of (158) for $\frac{1}{\ln(\frac{1}{\epsilon})} < \gamma < 1$. Since we know that $\ln\left(\frac{1}{\epsilon}\right) \lesssim z(t)$ from Theorem 11, the inequality (158) implies for $\frac{1}{\ln(\frac{1}{\epsilon})} < \gamma < 1$ and $t \geq 0$ that

$$\alpha(t) \lesssim \beta(t) := \Big(c_0 \epsilon \ln \frac{1}{\epsilon}\Big)^2 \left[\frac{1}{\gamma \ln \frac{1}{\epsilon}} + \epsilon^{\frac{1-\gamma}{2-\gamma}}\right] \exp\left(2\frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right)$$

$$
\begin{aligned}
&+c_0\epsilon^{2-\frac{\gamma}{2(2-\gamma)}}\ln\frac{1}{\epsilon}\exp\left(\frac{C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)+\frac{\epsilon^{1+\frac{2(1-\gamma)}{2-\gamma}}}{\gamma\ln\frac{1}{\epsilon}}\\
&=\beta_1(t)+\beta_2(t)+\beta_3(t),\ \text{respectively.} \qquad (159)
\end{aligned}
$$

For $\epsilon>0$ small enough, it is not difficult to verify that if $\beta_3(t)\geq\beta_1(t)$, then $\gamma\geq\frac{\ln\ln\frac{1}{\epsilon}}{\ln\frac{1}{\epsilon}}$. Moreover, if we have that $1>\gamma>8\frac{\ln\ln\frac{1}{\epsilon}}{\ln\frac{1}{\epsilon}}$, we obtain from the following estimate

$$
\beta_3(t)=\frac{\epsilon^2\epsilon^{\frac{-\gamma}{2-\gamma}}}{\gamma\ln\frac{1}{\epsilon}}>\frac{\epsilon^2}{\ln\frac{1}{\epsilon}}\exp\left(\frac{8\ln\ln\frac{1}{\epsilon}}{2-\gamma}\right)=\frac{\epsilon^2}{\ln\frac{1}{\epsilon}}\left(\ln\frac{1}{\epsilon}\right)^{\frac{8}{2-\gamma}},
$$

that $\beta_3(t)>\frac{\left(\epsilon\ln\left(\frac{1}{\epsilon}\right)\right)^2}{\ln\ln\frac{1}{\epsilon}}$. If $\gamma\leq\frac{\ln\ln\left(\frac{1}{\epsilon}\right)}{\ln\frac{1}{\epsilon}}$, then $\frac{\left(\epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}}\lesssim\beta_1(t)$ for any $t\geq 0$.

In conclusion, for any case we have that $\frac{\left(\epsilon^2\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}}\lesssim\beta(t)$ when $t\geq 0$, so we choose $\gamma=\frac{\ln\ln\frac{1}{\epsilon}}{\ln\frac{1}{\epsilon}}$. As a consequence, there exists a constant $C_1>0$ such that, for any $t\in\mathbb{R}_{\geq 0}$,

$$
\alpha(t)\leq C_1c_0^2\frac{\left(\epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right). \qquad (160)
$$

So, the estimates (157), (160), Remark 9 and our choice of γ imply the inequalities (155) and (156). □

Remark 13. If $\frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^m}\lesssim\left\Vert\overrightarrow{g(0)}\right\Vert$ for a constant $m>0$, then, for $\gamma=\frac{1}{8}$, we have from Lemma 13 that there is $p(t)\in C^2(\mathbb{R})$ satisfying for all $t\geq 0$

$$
|\dot{z}(t)-p(t)|\lesssim\epsilon^{\frac{1}{2}}\left\Vert\overrightarrow{g(0)}\right\Vert, \qquad (161)
$$

$$
\left|\dot{p}(t)-16\sqrt{2}e^{-\sqrt{2}z(t)}\right|\lesssim\frac{\left\Vert\overrightarrow{g(0)}\right\Vert^2}{z(t)}. \qquad (162)
$$

Then, for the smooth real function $d(t)$ satisfying

$$
\ddot{d}(t)=16\sqrt{2}e^{-\sqrt{2}d(t)},\ (d(0),\dot{d}(0))=(z(0),\dot{z}(0)),
$$

and since $e^{-\sqrt{2}z(t)}\lesssim\epsilon,\ \ln\frac{1}{\epsilon}\lesssim z(t)$, we can deduce for any $t\geq 0$ that $Y(t)=(z(t)-d(t))$ satisfies the following integral inequality for a constant $K>0$

$$
|Y(t)|\leq K\left(\epsilon^{\frac{1}{2}}\left\Vert\overrightarrow{g(0)}\right\Vert t+\frac{\left\Vert\overrightarrow{g(0)}\right\Vert^2}{\ln\frac{1}{\epsilon}}t^2+\int_0^t\int_0^s\epsilon\,|Y(s_1)|\,ds_1\,ds\right)=\Lambda\left(|Y|\right)(t),
$$

$$
Y(0)=0,\ \dot{Y}(0)=0.
$$

Indeed, for any $k \in \mathbb{N}$ and all $t \geq 0$, $|Y(t)| \leq \Lambda^{(k)}(|Y|)(t)$. We also can verify for any $T > 0$ that $\Lambda^{(k)}(|Y|)(t)$ is a Cauchy sequence in the Banach space $L^{\infty}[0, T]$. In conclusion, we can deduce for any $t \geq 0$ that $|Y(t)| \lesssim Q(tK^{\frac{1}{2}})$, where $Q(t)$ is the solution of the following integral equation

$$Q(t) = \epsilon^{\frac{1}{2}} \left\|\overrightarrow{g(0)}\right\| t + \frac{\left\|\overrightarrow{g(0)}\right\|^2}{\ln \frac{1}{\epsilon}} t^2 + \int_0^t \int_0^s \epsilon Q(s_1)\, ds_1\, ds.$$

By standard ordinary differential equation techniques, we deduce for any $t \geq 0$ that

$$\begin{aligned} |z(t) - d(t)| \lesssim Q(tK^{\frac{1}{2}}) = &\left(\frac{\left\|\overrightarrow{g(0)}\right\|}{2} + \frac{\left\|\overrightarrow{g(0)}\right\|^2}{\epsilon \ln \frac{1}{\epsilon}}\right) e^{\epsilon^{\frac{1}{2}} t K^{\frac{1}{2}}} \\ &+ \left(\frac{-\left\|\overrightarrow{g(0)}\right\|}{2} + \frac{\left\|\overrightarrow{g(0)}\right\|^2}{\epsilon \ln \frac{1}{\epsilon}}\right) e^{-\epsilon^{\frac{1}{2}} t K^{\frac{1}{2}}} - 2\frac{\left\|\overrightarrow{g(0)}\right\|^2}{\epsilon \ln \frac{1}{\epsilon}}, \end{aligned} \tag{163}$$

and from $\dot{z}(0) = \dot{d}(0)$ and the estimates (161) and (162), we obtain that

$$\left|\dot{z}(t) - \dot{d}(t)\right| \lesssim |p(0) - \dot{z}(0)| + \int_0^t \epsilon\, |z(s) - d(s)|\, ds, \tag{164}$$

from which with (163) we obtain for all $t \geq 0$ that

$$\left|\dot{z}(t) - \dot{d}(t)\right| \lesssim e^{\epsilon^{\frac{1}{2}} t K^{\frac{1}{2}}} \epsilon^{\frac{1}{2}} \left(\left\|\overrightarrow{g(0)}\right\| + \frac{\left\|\overrightarrow{g(0)}\right\|^2}{\epsilon \ln \frac{1}{\epsilon}}\right). \tag{165}$$

However, the precision of the estimates (163) and (165) is very bad when $\epsilon^{-\frac{1}{2}} \ll t$, which motivate us to apply Lemma 13 to estimate the modulation parameters $x_1(t),\ x_2(t)$ for $|t| \lesssim \frac{\ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

Remark 14. We recall from Theorem 4 the definitions of the functions $d_1(t),\ d_2(t)$. If $\left\|\overrightarrow{g(0)}\right\| \geq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^5}$, then, using estimates

$$\max_{j \in \{1,2\}} \left|d_j(t) - x_j(t)\right| = O(\min(\epsilon|t|, \epsilon^{\frac{1}{2}}|t|)), \quad \max_{j \in \{1,2\}} \left|\dot{d}_j(t) - \dot{x}_j(t)\right| = O(\epsilon|t|),$$

we deduce for a positive constant C large enough the inequalities (10) and (11) of Theorem 4.

Remark 15. If

$$\left\|\overrightarrow{g(0)}\right\| \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^5},$$

the estimates of $\max_{j \in \{1,2\}} \left|x_j(t) - d_j(t)\right|$, $\max_{j \in \{1,2\}} \left|\dot{x}_j(t) - \dot{d}_j(t)\right|$ can be done by studying separated cases depending on the initial data $z(0),\ \dot{z}(0)$.

Lemma 17. *$\exists K > 0$ such that if $\left\|\overrightarrow{g(0)}\right\| \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^5}$, where $\overrightarrow{g(0)} = (g(0,x), \partial_t g(0,x))$, and all the hypotheses of Theorem 4 are true and $\frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8} \lesssim e^{-\sqrt{2}z(0)} \lesssim \epsilon$, then we have for $t \geq 0$ that*

$$\max_{j\in\{1,2\}} \left|x_j(t) - d_j(t)\right| = O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2 \left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{K\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right), \tag{166}$$

$$\max_{j\in\{1,2\}} \left|\dot{x}_j(t) - \dot{d}_j(t)\right| = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2 \frac{\left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{K\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right). \tag{167}$$

Proof of Lemma 17. First, in notation of Lemma 16, we consider

$$p(t) := p_2(t) - p_1(t),\ z(t) := x_2(t) - x_1(t),\ \dot{z}(t) := \dot{x}_2(t) - \dot{x}_1(t).$$

Also, motivated by Remark 9, we consider the smooth function $d(t)$ solution of the following ordinary differential equation

$$\begin{cases} \ddot{d}(t) = 16\sqrt{2}e^{-\sqrt{2}d(t)}, \\ (d(0), \dot{d}(0)) = (z(0), \dot{z}(0)). \end{cases}$$

Step 1. (Estimate of $z(t)$, $\dot{z}(t)$) From now on, we denote the functions $W(t) = z(t) - d(t)$, $V(t) = p(t) - \dot{d}(t)$. Then, Lemma 16 implies that W, V satisfy for any $t \in \mathbb{R}_{\geq 0}$ the following differential estimates

$$\left|\dot{W}(t) - V(t)\right| = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right),$$

$$\left|\dot{V}(t) + 16\sqrt{2}e^{-\sqrt{2}d(t)} - 16\sqrt{2}e^{-\sqrt{2}z(t)}\right| = O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\left(\frac{1}{\epsilon}\right)} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right).$$

From the above estimates and Taylor's Expansion Theorem, we deduce for $t \geq 0$ the following system of differential equations, while $|W(t)| < 1$:

$$\begin{cases} \dot{W}(t) = V(t) + O\Big(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\Big), \\ \dot{V}(t) = -32e^{-\sqrt{2}d(t)}W(t) + O\left(e^{-\sqrt{2}d(t)}W(t)^2\right) \\ \qquad + O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right). \end{cases} \tag{168}$$

Recalling Remark 9, we have that

$$d(t) = \frac{1}{\sqrt{2}} \ln \Big(\frac{8}{v^2} \cosh{(\sqrt{2}vt + c)}^2 \Big), \tag{169}$$

where $v > 0$ and $c \in \mathbb{R}$ are chosen such that $(d(0), \dot{d}(0)) = (z(0), \dot{z}(0))$. Moreover, it is not difficult to verify that

$$v = \Big(\frac{\dot{z}(0)^2}{4} + 8e^{-\sqrt{2}z(0)} \Big)^{\frac{1}{2}}, \; c = \operatorname{arctanh}\left(\frac{\dot{z}(0)}{\big[32e^{-\sqrt{2}z(0)} + \dot{z}(0)^2\big]^{\frac{1}{2}}} \right).$$

Moreover, since $8e^{-\sqrt{2}z(0)} = v^2 \operatorname{sech}{(c)}^2 \leq 4v^2 e^{-2|c|}$, we obtain from the hypothesis for $e^{-\sqrt{2}z(0)}$ that $\frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^4} \lesssim v \lesssim \epsilon^{\frac{1}{2}}$ and as a consequence the estimate $|c| \lesssim \ln{(\ln{(\frac{1}{\epsilon})})}$.

Also, it is not difficult to verify that the functions

$$n(t) = (\sqrt{2}vt + c) \tanh{(\sqrt{2}vt + c)} - 1, \; m(t) = \tanh{(\sqrt{2}vt + c)}$$

generate all solutions of the following ordinary differential equation

$$\ddot{y}(t) = -32e^{-\sqrt{2}d(t)} y(t), \tag{170}$$

which is obtained from the linear part of the system (168).

To simplify our computations, we use the following notation

$$error_1(t) = \max\Big(\Big\| \overrightarrow{(g(0))} \Big\|, \epsilon \ln \frac{1}{\epsilon} \Big) \epsilon^{\frac{1}{2}} \exp\Big(\frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}} \Big),$$

$$error_2(t) = e^{-\sqrt{2}d(t)} (z(t) - d(t))^2 + \frac{\max\Big(\Big\| \overrightarrow{g(0)} \Big\|, \epsilon \ln \frac{1}{\epsilon} \Big)^2}{\ln \ln \frac{1}{\epsilon}} \exp\Big(\frac{2C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}} \Big).$$

From the variation of parameters technique for ordinary differential equations, we can write that

$$\begin{bmatrix} W(t) \\ V(t) \end{bmatrix} = c_1(t) \begin{bmatrix} m(t) \\ \dot{m}(t) \end{bmatrix} + c_2(t) \begin{bmatrix} n(t) \\ \dot{n}(t) \end{bmatrix}, \tag{171}$$

such that for any $t \geq 0$

$$\begin{cases} \begin{bmatrix} m(t) & n(t) \\ \dot{m}(t) & \dot{n}(t) \end{bmatrix} \begin{bmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \end{bmatrix} = \begin{bmatrix} O(error_1(t)) \\ O(error_2(t)) \end{bmatrix}, \\ \\ \begin{bmatrix} m(0) & n(0) \\ \dot{m}(0) & \dot{n}(0) \end{bmatrix} \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ O\Big(\Big[\Big\| \overrightarrow{g(0)} \Big\| + \epsilon \ln \frac{1}{\epsilon} \Big] \epsilon^{\frac{1}{2}} \Big) \end{bmatrix}. \end{cases}$$

The presence of an error in the condition of the initial data $c_1(0)$, $c_2(0)$ comes from estimate (155) of Lemma 16. Since for all $t \in \mathbb{R}$ $m(t)\dot{n}(t) - \dot{m}(t)n(t) = \sqrt{2}v$, we can verify by Cramer's rule and from the fact that $\frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^4} \lesssim v$ that

$$c_1(0) = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)|c\tanh(c) - 1|\left(\ln\frac{1}{\epsilon}\right)^4\right), \tag{172}$$

$$c_2(0) = O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\left(\frac{1}{\epsilon}\right)\right)|\tanh(c)|\left(\ln\frac{1}{\epsilon}\right)^4\right), \tag{173}$$

and, for all $t \geq 0$, the estimates

$$\begin{aligned} |\dot{c}_1(t)| = & O\left(\frac{\epsilon^{\frac{1}{2}}}{v}|\dot{n}(t)|\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right) \\ & + O\left(|n(t)|v\,\mathrm{sech}(\sqrt{2}vt + c)^2|W(t)|^2\right) \\ & + O\left(|n(t)|\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2}{v\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right), \end{aligned} \tag{174}$$

$$\begin{aligned} |\dot{c}_2(t)| = & O\left(|m(t)|\,v\,\mathrm{sech}(\sqrt{2}vt + c)^2|W(t)|^2\right) \\ & + O\left(|m(t)|\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2}{v\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right) \\ & + O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\epsilon^{\frac{1}{2}}\,\mathrm{sech}(\sqrt{2}vt + c)^2\right). \end{aligned} \tag{175}$$

Since we have for all $x \geq 0$ that

$$\begin{aligned} \frac{d}{dx}\left(-\frac{\mathrm{sech}(x)^2 x}{2} + \frac{3\tanh(x)}{2}\right) &= \frac{\mathrm{sech}(x)^2}{2} + x\tanh(x)\,\mathrm{sech}(x)^2 \\ &\geq \frac{|x\tanh(x) - 1|\,\mathrm{sech}(x)^2}{2} = \frac{|n(x)|\,\mathrm{sech}(x)^2}{2}, \end{aligned}$$

we deduce from the Fundamental Theorem of Calculus, the identity $n(t) = (\sqrt{2}vt + c)\tanh(\sqrt{2}vt + c) - 1$, estimate $\frac{\epsilon^{\frac{1}{2}}}{\ln\left(\frac{1}{\epsilon}\right)^4} \lesssim v \lesssim \epsilon^{\frac{1}{2}}$ and the estimates (174), (175) that

$$\begin{aligned} |c_1(t) - c_1(0)| = & O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)\left(\ln\frac{1}{\epsilon}\right)\exp\left(\frac{2Ct\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right) \\ & + O\left(\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\|n(s)\|_{L^\infty_s[0,t]}\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2\frac{\left(\ln\frac{1}{\epsilon}\right)^5}{\epsilon\ln\ln\frac{1}{\epsilon}}\right) \end{aligned}$$

$$+O\left(\left|-\frac{\operatorname{sech}(x)^2 x}{2}+\frac{3\tanh(x)}{2}\Big|_c^{\sqrt{2}vt+c}\right|\|W(s)\|^2_{L^\infty_s[0,t]}\right), \tag{176}$$

for any $t \geq 0$. From a similar argument, we deduce that

$$\begin{aligned}|c_2(t)-c_2(0)| &= O\left(\|W(s)\|^2_{L^\infty_s[0,t]}\left[\tanh(\sqrt{2}vt+c)-\tanh(c)\right]\right)\\ &+O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2\left[\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)-1\right]\frac{(\ln\frac{1}{\epsilon})^5}{\epsilon\ln\ln\frac{1}{\epsilon}}\right)\\ &+O\left(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)\left(\ln\frac{1}{\epsilon}\right)\exp\left(2Ct\frac{\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right),\end{aligned} \tag{177}$$

for any $t \geq 0$.

From the estimates $v \lesssim \epsilon^{\frac{1}{2}}$, $|c| \lesssim \ln\ln\frac{1}{\epsilon}$, we obtain for $\epsilon \ll 1$ while $t \geq 0$ and

$$\|W(s)\|_{L^\infty_s[0,t]}\left[\epsilon^{\frac{1}{2}}t+\ln\ln\frac{1}{\epsilon}\right]\ln\ln\frac{1}{\epsilon} \leq 1, \tag{178}$$

that

$$\|W(s)\|^2_{L^\infty_s[0,t]}(1+|n(t)|) \lesssim \|W(s)\|_{L^\infty_s[0,t]}\frac{1}{\ln\ln\frac{1}{\epsilon}}. \tag{179}$$

Also, from $|n(t)| \leq (\sqrt{2}v|t|+|c|)$, we deduce for any $t \geq 0$ that

$$|n(t)| \lesssim \epsilon^{\frac{1}{2}}t+\ln\ln\frac{1}{\epsilon} \lesssim \left(\ln\frac{1}{\epsilon}\right)\exp\left(\frac{\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right) \tag{180}$$

In conclusion, the estimates (176), (177), (179), (180) and the definition of $W(t) = z(t)-d(t)$ imply that while $t \geq 0$ and the condition (178) is true, then

$$|W(t)| \lesssim f(t) = \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2(\ln\frac{1}{\epsilon})^6}{\epsilon\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{(2C+1)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right). \tag{181}$$

Then, from the expression for $V(t)$ in the equation (171) and the estimates (176), (177), (180), we obtain that if inequality (181) is true and $t \geq 0$, then

$$\begin{aligned}|V(t)| &\lesssim \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\left(\frac{1}{\epsilon}\right)\right)^2\frac{\ln(\frac{1}{\epsilon})^6}{\epsilon^{\frac{1}{2}}\ln\ln(\frac{1}{\epsilon})}\exp\left(\frac{(4C+3)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\\ &+\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^4\frac{(\ln\frac{1}{\epsilon})^{12}}{\epsilon^{\frac{3}{2}}\left[\ln\ln\frac{1}{\epsilon}\right]^2}\exp\left(\frac{(4C+3)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right),\end{aligned} \tag{182}$$

which implies the following estimate

$$\left|\dot{W}(t)\right| \lesssim \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2\frac{(\ln\frac{1}{\epsilon})^6}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{(4C+3)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right). \tag{183}$$

Indeed, from the bound $\left\|\overrightarrow{g(0)}\right\| \lesssim \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^4}$, we deduce that (178) is true if $0 \leq t \leq \frac{\left[\ln\ln\frac{1}{\epsilon}\right]\ln\frac{1}{\epsilon}}{(4C+2)\epsilon^{\frac{1}{2}}}$. As a consequence, the estimates (181) and (183) are true if $0 \leq t \leq \frac{\left[\ln\ln\frac{1}{\epsilon}\right]\ln\frac{1}{\epsilon}}{(4C+2)\epsilon^{\frac{1}{2}}}$.

But, for $t \geq 0$, we have that

$$|W(t)| \lesssim \epsilon^{\frac{1}{2}}t \lesssim 3\left(\ln\frac{1}{\epsilon}\right)\exp\left(\frac{\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right),\ \left|\dot{W}(t)\right| \lesssim \epsilon t \lesssim 3\epsilon^{\frac{1}{2}}\left(\ln\frac{1}{\epsilon}\right)\exp\left(\frac{\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right). \tag{184}$$

Since $f(t)$ defined in inequality (181) is strictly increasing and $f(0) \lesssim \frac{1}{\left(\ln\frac{1}{\epsilon}\right)^2\ln\ln\frac{1}{\epsilon}}$, there is an instant $T_M > 0$ such that

$$\exp\left(\frac{\epsilon^{\frac{1}{2}}T_M}{\ln\frac{1}{\epsilon}}\right)f(T_M) = \frac{1}{\ln\frac{1}{\epsilon}\left(\ln\ln\frac{1}{\epsilon}\right)^2}, \tag{185}$$

from which with estimate (181) and condition (178) we deduce that (181) is true for $0 \leq t \leq T_M$. Also, from the identity (185) and the fact that $\left\|\overrightarrow{g(0)}\right\| \lesssim \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^4}$ we deduce

$$\frac{1}{\ln\frac{1}{\epsilon}\left(\ln\ln\frac{1}{\epsilon}\right)^2} \lesssim \frac{1}{\left(\ln\frac{1}{\epsilon}\right)^2\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{(2C+2)\epsilon^{\frac{1}{2}}T_M}{\ln\frac{1}{\epsilon}}\right),$$

from which we obtain that $T_M \geq \frac{3}{8(C+1)}\frac{\ln\ln\frac{1}{\epsilon}\left(\ln\frac{1}{\epsilon}\right)}{\epsilon^{\frac{1}{2}}}$ for $\epsilon \ll 1$. In conclusion, since $f(t)$ is an increasing function, we have for $t \geq T_M$ and $\epsilon \ll 1$ that

$$\begin{aligned} f(t)\exp\left(\frac{[17(C+1)+4]\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right) &\geq \frac{1}{\ln\frac{1}{\epsilon}\left(\ln\ln\frac{1}{\epsilon}\right)^2}\exp\left(\frac{[17(C+1)+1]\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right) \\ &\geq \frac{\left(\ln\frac{1}{\epsilon}\right)^{1+\frac{1}{8}}}{\left(\ln\ln\frac{1}{\epsilon}\right)^2}\exp\left(\frac{\epsilon^{\frac{1}{2}}t}{3\ln\frac{1}{\epsilon}}\right), \end{aligned}$$

from which with the estimates (184) and (181) we deduce for all $t \geq 0$ that

$$|W(t)| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2\left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{(8C+9)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right). \tag{186}$$

As consequence, we obtain from the estimates (172), (173), (176), (177) and (186) that

$$\left|\dot{W}(t)\right| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2\left(\ln\frac{1}{\epsilon}\right)^6}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{(16C+18)\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right), \tag{187}$$

for all $t \geq 0$.
Step 2. (Estimate of $|x_1(t) + x_2(t)|$, $|\dot{x}_1(t) + \dot{x}_2(t)|$.) First, we define

$$M(t) := (x_1(t) + x_2(t)) - (d_1(t) + d_2(t)), \; N(t) := (p_1(t) + p_2(t)) - (\dot{d}_1(t) + \dot{d}_2(t)). \tag{188}$$

From the inequalities (155), (156) of Lemma 16, we obtain for all $t \geq 0$, respectively:

$$\left|\dot{M}(t) - N(t)\right| \lesssim \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}} \exp\left(\frac{C\epsilon^{\frac{1}{2}}t}{\ln \frac{1}{\epsilon}}\right),$$

$$\left|\dot{N}(t)\right| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2}{\ln \ln \frac{1}{\epsilon}} \exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln \frac{1}{\epsilon}}\right).$$

Also, from inequality (155) and the fact that for $j \in \{1, 2\}$ $d_j(0) = x_j(0)$, $\dot{d}_j(0) = \dot{x}_j(0)$, we deduce that $M(0) = 0$ and $|N(0)| \lesssim \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}}$. Then, from the Fundamental Theorem of Calculus, we obtain for all $t \geq 0$ that

$$N(t) = O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2 \ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}} \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{4C\epsilon^{\frac{1}{2}}t}{\ln \frac{1}{\epsilon}}\right)\right), \tag{189}$$

$$M(t) = O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2 \left(\ln \frac{1}{\epsilon}\right)^2}{\epsilon \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{4C\epsilon^{\frac{1}{2}}t}{\ln \frac{1}{\epsilon}}\right)\right). \tag{190}$$

In conclusion, for $K = 16C + 18$, we verify from triangle inequality that the estimates (186) and (190) imply (166) and the estimates (187) and (189) imply (167). □

Remark 16. The estimates (190) and (189) are true for any initial data $\overrightarrow{g(0)} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ such that the hypotheses of Theorem 4 are true.

Remark 17. (*Similar Case*) If we add the following conditions

$$e^{-\sqrt{2}z(0)} \ll \frac{\epsilon}{\left(\ln \frac{1}{\epsilon}\right)^8}, \; \frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^4} \lesssim v \lesssim \epsilon^{\frac{1}{2}}, \; -\left(\ln \frac{1}{\epsilon}\right)^2 < c < 0,$$

to the hypotheses of Theorem 4, then, by repeating the above proof of Lemma 17, we would still obtain for any $t \geq 0$ the estimates (174), (175), (176) and (177).

However, since now $|c| \leq \left(\ln \frac{1}{\epsilon}\right)^2$, if $\epsilon \ll 1$ enough, we can verify while $t \geq 0$ and

$$\|W(s)\|_{L^\infty_s[0,t]} \left(\epsilon^{\frac{1}{2}}t + \left(\ln \frac{1}{\epsilon}\right)^2\right) \ln \ln \frac{1}{\epsilon} \leq 1, \tag{191}$$

that

$$\|W(s)\|^2_{L^\infty_s[0,t]} (1 + |n(t)|) \lesssim \|W(s)\|_{L^\infty_s[0,t]} \frac{1}{\ln \ln \frac{1}{\epsilon}},$$

which implies by a similar reasoning to the proof of Lemma 17 for a uniform constant $C > 1$ and any $t \in \mathbb{R}_{\geq 0}$ the following estimates

$$|W(t)| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2 \left(\ln \frac{1}{\epsilon}\right)^7}{\epsilon \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) = f_1(t, C), \tag{192}$$

$$\left|\dot{W}(t)\right| \lesssim \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2 \frac{\left(\ln \frac{1}{\epsilon}\right)^7}{\epsilon^{\frac{1}{2}} \ln \ln \frac{1}{\epsilon}} \exp\left(\frac{C\epsilon^{\frac{1}{2}} t}{\ln \frac{1}{\epsilon}}\right) = f_2(t, C). \tag{193}$$

From the estimates (192), (193) and $\left\|\overrightarrow{g(0)}\right\| \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^5}$, we deduce that the condition (191) holds while $0 \leq t \leq \frac{\ln \ln \frac{1}{\epsilon}\left(\ln \frac{1}{\epsilon}\right)}{4(C+1)\epsilon^{\frac{1}{2}}}$. Indeed, since $\left\|\overrightarrow{g(0)}\right\|^2 \leq \frac{\epsilon}{\left(\ln \frac{1}{\epsilon}\right)^{10}}$, we can verify that there is an instant $\frac{\ln \ln \frac{1}{\epsilon}\left(\ln \frac{1}{\epsilon}\right)}{4(C+1)\epsilon^{\frac{1}{2}}} \leq T_M$ such that (191) and (192) are true for $0 \leq t \leq T_M$ and

$$f_1(T_M, C) \exp\left(\frac{\epsilon^{\frac{1}{2}} T_M}{\ln \frac{1}{\epsilon}}\right) = \frac{1}{\left(\ln \frac{1}{\epsilon}\right)^{2+\frac{1}{2}} \ln \ln \frac{1}{\epsilon}}.$$

In conclusion, we can repeat the argument in the proof of step 1 of Lemma 17 and deduce that there is $1 < K \lesssim C + 1$ such that for all $t \geq 0$

$$|W(t)| \lesssim f_1(t, K), \ \left|\dot{W}(t)\right| \lesssim f_2(t, K). \tag{194}$$

Lemma 18. *In notation of Theorem 4, $\exists K > 1,\ \delta > 0$ such that if $0 < \epsilon < \delta,\ 0 < v \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^4}$, $\overrightarrow{g(0)} = (g(0, x), \partial_t g(0, x))$ and $\left\|\overrightarrow{g(0)}\right\| \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln \frac{1}{\epsilon}\right)^5}$, then we have for all $t \geq 0$ that*

$$\max_{j \in \{1, 2\}} \left|d_j(t) - x_j(t)\right| = O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2}{\epsilon \ln \ln \frac{1}{\epsilon}} \left(\ln \frac{1}{\epsilon}\right)^2 \exp\left(\frac{Kt\epsilon^{\frac{1}{2}}}{\ln \frac{1}{\epsilon}}\right)\right), \tag{195}$$

$$\max_{j \in \{1, 2\}} \left|\dot{d}_j(t) - \dot{x}_j(t)\right| = O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}} \ln \ln \frac{1}{\epsilon}} \left(\ln \frac{1}{\epsilon}\right) \exp\left(\frac{Kt\epsilon^{\frac{1}{2}}}{\ln \frac{1}{\epsilon}}\right)\right). \tag{196}$$

Proof of Lemma 18. First, we recall that

$$d(t) = \frac{1}{\sqrt{2}} \ln\left(\frac{8}{v^2} \cosh\left(\sqrt{2}vt + c\right)\right),$$

which implies that

$$e^{-\sqrt{2}d(t)} = \frac{v^2}{8} \operatorname{sech}\left(\sqrt{2}vt + c\right)^2. \tag{197}$$

We recall the notation $W(t) = z(t) - d(t),\ V(t) = p(t) - \dot{d}(t)$. From the first inequality of Lemma 16, we have that

$$|V(0)| \lesssim \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln\frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}}. \tag{198}$$

We already verified that $W,\ V$ satisfy the following ordinary differential system

$$\begin{cases} \dot{W}(t) = V(t) + O\Big(\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln\frac{1}{\epsilon}\right)\epsilon^{\frac{1}{2}}\exp\left(\frac{C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\Big), \\ \dot{V}(t) = -32e^{-\sqrt{2}d(t)}W(t) + O\left(e^{-\sqrt{2}z(t)}W(t)^2\right) \\ \qquad + O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right). \end{cases} \tag{199}$$

However, since $v^2 \leq \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8}$, we deduce from (197) that $e^{-\sqrt{2}d(t)} \lesssim \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8}$ for all $t \geq 0$. So, while $\|W(s)\|_{L^\infty[0,t]} < 1$, we have from the system of ordinary differential equations above for some constant $C > 0$ independent of ϵ that

$$\left|\dot{V}(t)\right| \lesssim \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8}\|W(s)\|_{L^\infty[0,t]} + \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2}{\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right), \text{ for all } t \geq 0,$$

from which we deduce the following estimate for any $t \geq 0$

$$\begin{aligned} |V(t) - V(0)| = {} & O\left(\frac{\epsilon t}{\left(\ln\frac{1}{\epsilon}\right)^8}\|W(s)\|_{L^\infty[0,t]}\right) \\ & + O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\frac{1}{\epsilon}\right)^2\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right). \end{aligned}$$

In conclusion, while $\|W(s)\|_{L^\infty[0,t]} < 1$, we have that

$$\begin{aligned} \left|\dot{W}(t)\right| \leq {} & |V(0)| + O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\left(\frac{1}{\epsilon}\right)\right)^2\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}\ln\ln\frac{1}{\epsilon}}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)\right) \\ & + O\left(\frac{\epsilon t}{\left(\ln\frac{1}{\epsilon}\right)^8}\|W(s)\|_{L^\infty[0,t]}\right). \end{aligned} \tag{200}$$

Finally, since $W(0) = 0$, the Fundamental Theorem of Calculus and (200) imply the following estimate for all $t \geq 0$

$$\|W(s)\|_{L^\infty[0,t]} \leq |V(0)|t + O\left(\frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\ln\left(\frac{1}{\epsilon}\right)\right)^2\ln\left(\frac{1}{\epsilon}\right)^2}{\epsilon\ln\ln\left(\frac{1}{\epsilon}\right)}\exp\left(\frac{2C\epsilon^{\frac{1}{2}}t}{\ln\left(\frac{1}{\epsilon}\right)}\right)\right)$$

$$+ O\left(\frac{\epsilon t^2}{\ln\left(\frac{1}{\epsilon}\right)^8}\|W(s)\|_{L^\infty[0,t]}\right). \tag{201}$$

Then, the estimates (198) and (201) imply if $\epsilon \ll 1$ that

$$|W(t)| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln\frac{1}{\epsilon}\right)^2 \left(\ln\frac{1}{\epsilon}\right)^2}{\epsilon \ln\ln\frac{1}{\epsilon}} \exp\left(\frac{(2C+1)\epsilon^{\frac{1}{2}} t}{\ln\frac{1}{\epsilon}}\right), \tag{202}$$

for $0 \leq t \leq \frac{\left(\ln\frac{1}{\epsilon}\right)\ln\ln\frac{1}{\epsilon}}{(8C+4)\epsilon^{\frac{1}{2}}}$. From (202) and (200), we deduce for $0 \leq t \leq \frac{\left(\ln\frac{1}{\epsilon}\right)\ln\ln\frac{1}{\epsilon}}{(8C+4)\epsilon^{\frac{1}{2}}}$ that

$$\left|\dot{W}(t)\right| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln\frac{1}{\epsilon}\right)^2 \left(\ln\frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}} \ln\ln\frac{1}{\epsilon}} \exp\left(\frac{(2C+1)\epsilon^{\frac{1}{2}} t}{\ln\frac{1}{\epsilon}}\right). \tag{203}$$

Since $|W(t)| \lesssim \epsilon^{\frac{1}{2}} t$, $\left|\dot{W}(t)\right| \lesssim \epsilon t$ for all $t \geq 0$, we can verify by a similar argument to the proof of Step 1 of Lemma 17 that for all $t \geq 0$ there is a constant $1 < K \lesssim (C+1)$ such that

$$|W(t)| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln\frac{1}{\epsilon}\right)^2 \left(\ln\frac{1}{\epsilon}\right)^2}{\epsilon \ln\ln\frac{1}{\epsilon}} \exp\left(\frac{K\epsilon^{\frac{1}{2}} t}{\ln\frac{1}{\epsilon}}\right), \tag{204}$$

$$\left|\dot{W}(t)\right| \lesssim \frac{\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln\frac{1}{\epsilon}\right)^2 \left(\ln\frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}} \ln\ln\frac{1}{\epsilon}} \exp\left(\frac{K\epsilon^{\frac{1}{2}} t}{\ln\frac{1}{\epsilon}}\right). \tag{205}$$

In conclusion, estimates (195) and (196) follow from Remark 16, inequalities (204), (205) and triangle inequality. □

Remark 18. We recall the definition (169) of $d(t)$. It is not difficult to verify that if $\left\|\overrightarrow{g(0)}\right\| \leq \frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^5}$, $\frac{\epsilon^{\frac{1}{2}}}{\left(\ln\frac{1}{\epsilon}\right)^4} \lesssim v$ and one of the following statements

1. $e^{-\sqrt{2}z(0)} \ll \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8}$ and $c > 0$,
2. $e^{-\sqrt{2}z(0)} \ll \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8}$ and $c \leq -\left(\ln\frac{1}{\epsilon}\right)^2$

were true, then we would have that $e^{-\sqrt{2}d(t)} \ll \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8}$ for $0 \leq t \lesssim \frac{\left(\ln\frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}}}$. Moreover, assuming $e^{-\sqrt{2}z(0)}\left(\ln\frac{1}{\epsilon}\right)^8 \ll \epsilon$, if $c > 0$, then we have for all $t \geq 0$ that

$$e^{-\sqrt{2}d(t)} = \frac{v^2}{8}\operatorname{sech}\left(\sqrt{2}vt + c\right)^2 \leq \frac{v^2}{8}\operatorname{sech}(c)^2 = e^{-\sqrt{2}z(0)} \ll \frac{\epsilon}{\left(\ln\frac{1}{\epsilon}\right)^8},$$

otherwise if $c \leq -\left(\ln \frac{1}{\epsilon}\right)^2$, since $0 < v \lesssim \epsilon^{\frac{1}{2}}$, then there is $1 \lesssim K$ such that for $0 \leq t \leq \frac{K\left(\ln \frac{1}{\epsilon}\right)^2}{\epsilon^{\frac{1}{2}}}$, then $2\left|\sqrt{2}vt + c\right| > |c|$, and so

$$e^{-\sqrt{2}d(t)} \leq v^2 \operatorname{sech}\left(-\frac{c}{2}\right)^2 \ll \frac{\epsilon}{(\ln \frac{1}{\epsilon})^8}.$$

In conclusion, the result of Lemma 18 would be true for these two cases.

From the following inequality

$$\max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon \ln \frac{1}{\epsilon}\right) \leq \left(\ln \frac{1}{\epsilon}\right) \max\left(\left\|\overrightarrow{g(0)}\right\|, \epsilon\right),$$

we deduce from Lemmas 17, 18 and Remarks 16, 17 and 18 the statement of Theorem 4.

6. Proof of Theorem 2

If $\left\|\overrightarrow{g(0)}\right\| \geq \epsilon \ln \frac{1}{\epsilon}$, the result of Theorem 2 is a direct consequence of Theorem 15. So, from now on, we assume that $\left\|\overrightarrow{g(0)}\right\| < \epsilon \ln \frac{1}{\epsilon}$.

We recall from Theorem 4 the notations v, c, $d_1(t)$, $d_2(t)$ and we denote $d(t) = d_2(t) - d_1(t)$ that satisfies

$$d(t) = \frac{1}{\sqrt{2}} \ln\left(\frac{8}{v^2}\cosh\left(\sqrt{2}vt + c\right)^2\right),\ e^{-\sqrt{2}d(t)} = \frac{v^2}{8}\operatorname{sech}\left(\sqrt{2}vt + c\right)^2.$$

From the definition of $d_1(t),\ d_2(t),\ d(t)$, we know that $\max_{j\in\{1,2\}}\left|\ddot{d}_j(t)\right| + e^{-\sqrt{2}d(t)} = O\left(v^2 \operatorname{sech}\left(\sqrt{2}vt + c\right)^2\right)$ and since $z(0) = d(0),\ \dot{z}(0) = \dot{d}(0)$, we have that $v,\ c$ satisfy the following identities

$$v = \left(e^{-\sqrt{2}z(0)} + \left(\frac{\dot{x}_2(0) - \dot{x}_1(0)}{2}\right)^2\right)^{\frac{1}{2}},\ c = \operatorname{arctanh}\left(\frac{\dot{x}_2(0) - \dot{x}_1(0)}{2v}\right),$$

so Theorem 11 implies that $v \lesssim \epsilon^{\frac{1}{2}}$.

From the Corollary 5 and the Theorem 4, we deduce that $\exists C > 0$ such that if $\epsilon \ll 1$ and $0 \leq t \leq \frac{\left(\ln\ln \frac{1}{\epsilon}\right)\ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$, then we have that

$$\max_{j\in\{1,2\}}\left|\ddot{x}_j(t)\right| = O\Big(\max_{j\in\{1,2\}}\left|\ddot{d}_j(t)\right|\Big) + O\left(\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^9 \exp\left(\frac{Ct\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right), \tag{206}$$

$$\begin{aligned} e^{-\sqrt{2}z(t)} &= e^{-\sqrt{2}d(t)} + O\left(\max\left(e^{-\sqrt{2}d(t)}, e^{-\sqrt{2}z(t)}\right)|z(t) - d(t)|\right) \\ &= e^{-\sqrt{2}d(t)} + O\left(\epsilon^2\left(\ln\frac{1}{\epsilon}\right)^9 \exp\left(\frac{Ct\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)\right). \end{aligned} \tag{207}$$

Next, we consider a smooth function $0 \leq \chi_2(x) \leq 1$ that satisfies

$$\chi_2(x) = \begin{cases} 1, & \text{if } x \leq \frac{9}{20}, \\ 0, & \text{if } x \geq \frac{1}{2}. \end{cases}$$

We denote

$$\chi_2(t, x) = \chi_2\Big(\frac{x - x_1(t)}{x - x_2(t)}\Big).$$

From Theorem 14 and Remark 12, the estimates (206) and (207) of the modulation parameters imply that for the following function

$$\begin{aligned} L_1(t) = {} & \Big\langle D^2 E_{total}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\overrightarrow{g(t)},\ \overrightarrow{g(t)}\Big\rangle_{L^2 \times L^2} \\ & + 2\int_{\mathbb{R}} \partial_t g(t) \partial_x g(t)\Big[\dot{x}_1(t)\chi_2(t,x) + \dot{x}_2(t)\left(1 - \chi_2(t)\right)\Big]\, dx \\ & - 2\int_{\mathbb{R}} g(t,x)\Big(\dot{U}\left(H_{-1,0}^{x_1(t)}\right) + \dot{U}\left(H_{0,1}^{x_2(t)}\right) - \dot{U}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right)\Big)\, dx \\ & + 2\int_{\mathbb{R}} g(t,x)\Big[\dot{x}_1(t)^2 \partial_x^2 H_{-1,0}^{x_1(t)} + \dot{x}_2(t)^2 \partial_x^2 H_{0,1}^{x_2(t)}(x)\Big]\, dx \\ & + \frac{1}{3}\int_{\mathbb{R}} U^{(3)}\left(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}\right) g(t)^3\, dx, \end{aligned}$$

and the following quantity $\delta_1(t)$ denoted by

$$\begin{aligned} \delta_1(t) = {} & \left\|\overrightarrow{g(t)}\right\| \left(e^{-\sqrt{2}z(t)} \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| + \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|^3 e^{-\frac{9\sqrt{2}z(t)}{20}}\right) \\ & + \left\|\overrightarrow{g(t)}\right\| \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right| \left|\ddot{x}_j(t)\right| + \left\|\overrightarrow{g(t)}\right\|^2 \frac{\max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|}{z(t)} \\ & + \left\|\overrightarrow{g(t)}\right\|^2 \left(\max_{j\in\{1,2\}} \dot{x}_j(t)^2 + \max_{j\in\{1,2\}} \left|\ddot{x}_j(t)\right|\right) + \left\|\overrightarrow{g(t)}\right\|^4, \end{aligned}$$

we have $\left|\dot{L}_1(t)\right| = O(\delta_1(t))$ for $t \geq 0$. Moreover, estimates (206), (207) and the bound $\dot{L}_1(t) = O(\delta_1(t))$ imply that for

$$\begin{aligned} \delta_2(t) = {} & \left\|\overrightarrow{g(t)}\right\| v^2 \epsilon^{\frac{1}{2}} \operatorname{sech}\left(\sqrt{2}vt + c\right)^2 + \left\|\overrightarrow{g(t)}\right\| \epsilon^2 \left(\ln\frac{1}{\epsilon}\right)^9 \exp\left(\frac{Ct\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right) \\ & + \epsilon^{\frac{3}{2}} e^{-\frac{9\sqrt{2}z(t)}{20}} \left\|\overrightarrow{g(t)}\right\| + \max_{j\in\{1,2\}} \frac{\left|\dot{x}_j(t)\right|}{z(t)} \left\|\overrightarrow{g(t)}\right\|^2 + \left\|\overrightarrow{g(t)}\right\|^4, \end{aligned}$$

$\left|\dot{L}_1(t)\right| = O(\delta_2(t))$ if $0 \leq t \leq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

Now, similarly to the proof of Theorem 15, we denote $G(s) = \max\left(\left\|\overrightarrow{g(s)}\right\|, \epsilon\right)$. From Theorem 14 and Remark 12, we have that there are positive constants $K, k > 0$ independent of ϵ such that

$$k\left\|\overrightarrow{g(t)}\right\|^2 \leq L_1(t) + K\epsilon^2.$$

We recall that Theorem 11 implies that

$$\ln\left(\frac{1}{\epsilon}\right) \lesssim z(t),\ e^{-\sqrt{2}z(t)} + \max_{j\in\{1,2\}} \left|\dot{x}_j(t)\right|^2 + \max_{j\in\{1,\,2\}} \left|\ddot{x}_j(t)\right| = O(\epsilon),$$

from which with the definition of $G(s)$ and estimates (206) and (207) we deduce that

$$\delta_2(t) \lesssim G(t)v^2 \operatorname{sech}\left(\sqrt{2}vt + c\right)^2 \epsilon^{\frac{1}{2}} + G(t)\epsilon^{\frac{39}{20}} + G(t)^2 \frac{\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}},$$

while $0 \leq t \leq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

In conclusion, the Fundamental Theorem of Calculus implies that $\exists K > 0$ independent of ϵ such that

$$G(t)^2 \leq K\left(G(0)^2 + \int_0^t G(s)v^2 \operatorname{sech}\left(\sqrt{2}vs + c\right)^2 \epsilon^{\frac{1}{2}} + G(s)\epsilon^{\frac{39}{20}} + G(s)^2 \frac{\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\, ds\right), \tag{208}$$

while $0 \leq t \leq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$.

Since $\frac{d}{dt}[\tanh\left(\sqrt{2}vt + c\right)] = \sqrt{2}v \operatorname{sech}\left(\sqrt{2}vt + c\right)^2$, we verify that while the term $G(s)v^2 \operatorname{sech}\left(\sqrt{2}vt + c\right)^2 \epsilon^{\frac{1}{2}}$ is dominant in the integral of the estimate (208), then $G(t) \lesssim G(0)$. The remaining case corresponds when $G(s)^2\frac{\epsilon^{\frac{1}{2}}}{\ln\left(\frac{1}{\epsilon}\right)}$ is the dominant term in the integral of (208) from an instant $0 \leq t_0 \leq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$. Similarly to the proof of 15, we have for $t_0 \leq t \leq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ that $G(t) \lesssim G(t_0)\exp\left(C\frac{(t-t_0)\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right)$.

In conclusion, in any case we have for $0 \leq t \leq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ that

$$G(t) \lesssim G(0)\exp\left(C\frac{t\epsilon^{\frac{1}{2}}}{\ln\frac{1}{\epsilon}}\right). \tag{209}$$

But, for $T \geq \frac{\left(\ln\ln\frac{1}{\epsilon}\right)\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ and $K > 2$ we have that

$$\epsilon\left(\ln\frac{1}{\epsilon}\right)\exp\left(K\frac{\epsilon^{\frac{1}{2}}T}{\ln\frac{1}{\epsilon}}\right) \leq \epsilon\exp\left(\frac{2K\epsilon^{\frac{1}{2}}T}{\ln\frac{1}{\epsilon}}\right).$$

In conclusion, from the result of Theorem 15, we can exchange the constant $C > 0$ by a larger constant such that estimate (209) is true for all $t \geq 0$.

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Appendix A Auxiliary Results

We start the Appendix Section by presenting the following lemma:

Lemma 19. *With the same hypothesis as in Theorem 4 and using its notation, we have while* $\max_{j\in\{1,2\}}\left|d_j(t)-x_j(t)\right|<1$ *that* $\max_{j\in\{1,2\}}\left|\ddot{d}_j(t)-\ddot{x}_j(t)\right| = O\Big(\max_{j\in\{1,2\}}\left|d_j(t)-x_j(t)\right|\epsilon+\epsilon z(t)e^{-\sqrt{2}z(t)}+\left\|\overrightarrow{g(t)}\right\|\epsilon^{\frac{1}{2}}\Big).$

Lemma 20. *For* $U(\phi)=\phi^2(1-\phi^2)^2$, *we have that*

$$\begin{aligned}
&\dot{U}\left(H^{x_1(t)}_{-1,0}(x)+H^{x_2(t)}_{0,1}(x)\right)-\dot{U}\left(H^{x_1(t)}_{-1,0}(x)\right)-\dot{U}\left(H^{x_2(t)}_{0,1}(x)\right)\\
&=24e^{-\sqrt{2}z(t)}\left(\frac{H^{x_1(t)}_{-1,0}(x)}{(1+e^{-2\sqrt{2}(x-x_1(t))})^{\frac{1}{2}}}+\frac{H^{x_2(t)}_{0,1}(x)}{(1+e^{2\sqrt{2}(x-x_2(t))})^{\frac{1}{2}}}\right)\\
&\quad-30e^{-\sqrt{2}z(t)}\left(\frac{H^{x_1(t)}_{-1,0}(x)^3}{(1+e^{-2\sqrt{2}(x-x_1(t))})^{\frac{1}{2}}}+\frac{H^{x_2(t)}_{0,1}(x)^3}{(1+e^{2\sqrt{2}(x-x_2(t))})^{\frac{1}{2}}}\right)+r(t,x),
\end{aligned}$$

such that $\|r(t)\|_{L^2_x(\mathbb{R})}=O(e^{-2\sqrt{2}z(t)})$.

Proof. By direct computations, we verify that

$$\begin{aligned}
&\dot{U}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)-\dot{U}\left(H^{x_1(t)}_{-1,0}\right)-\dot{U}\left(H^{x_2(t)}_{0,1}\right)\\
&=-24H^{x_1(t)}_{-1,0}H^{x_2(t)}_{0,1}\left(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right)\\
&\quad+30H^{x_1(t)}_{-1,0}H^{x_2(t)}_{0,1}\left[\left(H^{x_1(t)}_{-1,0}\right)^3+\left(H^{x_2(t)}_{0,1}\right)^3\right]\\
&\quad+60\left(H^{x_1(t)}_{-1,0}H^{x_2(t)}_{0,1}\right)^2\left[H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right].
\end{aligned}$$

First, from the definition of $H_{0,1}(x)$, we verify that

$$60\left(H^{x_1(t)}_{-1,0}H^{x_2(t)}_{0,1}\right)^2\left[H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1}\right]=\frac{60e^{-2\sqrt{2}z(t)}H^{x_2(t)}_{0,1}}{(1+e^{2\sqrt{2}(x-x_2(t))})(1+e^{-2\sqrt{2}(x-x_1(t))})}+\frac{60e^{-2\sqrt{2}z(t)}H^{x_1(t)}_{-1,0}}{(1+e^{-2\sqrt{2}(x-x_1(t))})(1+e^{2\sqrt{2}(x-x_2(t))})}.$$

Using (4), we can verify using by induction for any $k\in\mathbb{N}$ that

$$\left|\frac{d^k}{dx^k}\left[\frac{1}{(1+e^{2\sqrt{2}x})}\right]\right|=\left|\frac{d^k}{dx^k}\left[1-\frac{e^{2\sqrt{2}x}}{(1+e^{2\sqrt{2}x})}\right]\right|=\left|\frac{d^k}{dx^k}\left[H_{0,1}(x)^2\right]\right|=O(1), \tag{A1}$$

and since $\frac{H_{0,1}(x)}{(1+e^{2\sqrt{2}x})}=\frac{e^{\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{3}{2}}}$ is a Schwartz function, we deduce using Lemma 6 that $60(H^{x_1(t)}_{-1,0}H^{x_2(t)}_{0,1})^2(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1})$ is in $H^k_x(\mathbb{R})$ and it satisfies for all $k>0$ the following estimate

$$\left\|\frac{\partial^k}{\partial x^k}\left[(H^{x_1(t)}_{-1,0}H^{x_2(t)}_{0,1})^2(H^{x_1(t)}_{-1,0}+H^{x_2(t)}_{0,1})\right]\right\|_{L^2}=O\left(e^{-2\sqrt{2}z(t)}\right). \tag{A2}$$

Next, using the identity

$$H^{x_1(t)}_{-1,0}(x)H^{x_2(t)}_{0,1}(x)=-\frac{e^{-\sqrt{2}z(t)}}{(1+e^{2\sqrt{2}(x-x_2(t))})^{\frac{1}{2}}(1+e^{-2\sqrt{2}(x-x_1(t))})^{\frac{1}{2}}}, \tag{A3}$$

the identity

$$1-\frac{1}{(1+e^{2\sqrt{2}x})^{\frac{1}{2}}}=\frac{e^{2\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{1}{2}}+(1+e^{2\sqrt{2}x})},$$

and Lemma 6, we deduce that

$$\left\|24(H^{x_1(t)}_{-1,0})^2H^{x_2(t)}_{0,1}+24e^{-\sqrt{2}z(t)}\frac{H^{x_1(t)}_{-1,0}(x)}{(1+e^{-2\sqrt{2}(x-x_1(t))})^{\frac{1}{2}}}\right\|_{L^2}=O\left(e^{-2\sqrt{2}z(t)}\right), \tag{A4}$$

$$\left\|30(H^{x_1(t)}_{-1,0})^4H^{x_2(t)}_{0,1}+30e^{-\sqrt{2}z(t)}\left(\frac{(H^{x_1(t)}_{-1,0}(x))^3}{(1+e^{-2\sqrt{2}(x-x_1(t))})^{\frac{1}{2}}}\right)\right\|_{L^2}=O\left(e^{-3\sqrt{2}z(t)}\right). \tag{A5}$$

The estimate of the remaining terms $-24H^{x_1(t)}_{-1,0}\left(H^{x_2(t)}_{0,1}\right)^2$, $30H^{x_1(t)}_{-1,0}\left(H^{x_2(t)}_{0,1}\right)^4$ is completely analogous to (A4) and (A5) respectively. In conclusion, all of the estimates above imply the estimate stated in the Lemma 20. □

Proof of Lemma 19. First, we recall the global estimate $e^{-\sqrt{2}z(t)} \lesssim \epsilon$. We also recall the identity (33)

$$\int_{\mathbb{R}} \left(8(H_{0,1}(x))^3 - 6(H_{0,1}(x))^5\right)e^{-\sqrt{2}x}\,dx = 2\sqrt{2},$$

which, by integration by parts, implies that

$$\int_{\mathbb{R}} 24\frac{H_{0,1}(x)\partial_x H_{0,1}(x)}{(1+e^{2\sqrt{2}(x)})^{\frac{1}{2}}} - 30\frac{(H_{0,1}(x))^3\partial_x H_{0,1}(x)}{(1+e^{2\sqrt{2}(x)})^{\frac{1}{2}}}\,dx = 4. \tag{A6}$$

We recall $d_1(t)$, $d_2(t)$ defined in (8) and (9) respectively and $d(t) = d_2(t) - d_1(t)$. Since $\ddot{d}_j(t) = (-1)^j 8\sqrt{2}e^{-\sqrt{2}d(t)}$ for $j \in \{1, 2\}$, we have $\ddot{d}(t) = 16\sqrt{2}e^{-\sqrt{2}d(t)}$, which implies clearly with the identities

$$\left\|\partial_x H_{0,1}\right\|_{L^2}^2 = \left\|\partial_x^2 H_{0,1}\right\|_{L^2}^2 = \frac{1}{2\sqrt{2}}$$

that $\ddot{d}_j(t)\left\|\partial_x H_{0,1}\right\|_{L^2}^2 = (-1)^j 4e^{-\sqrt{2}d(t)}$. We also recall the partial differential equation satisfied by the remainder $g(t,x)$ (II), which can be rewritten as

$$\begin{aligned}
&\dot{U}\left(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)\right) - \dot{U}\left(H_{-1,0}^{x_1(t)}(x)\right) - \dot{U}\left(H_{0,1}^{x_2(t)}(x)\right) - \ddot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}(x)\\
&\quad= -\left(\partial_t^2 g(t,x) - \partial_x^2 g(t,x) + \ddot{U}\left(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)\right)g(t,x)\right)\\
&\qquad+ \sum_{k=3}^{6} U^{(k)}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right)\frac{g(t)^{k-1}}{(k-1)!} - \dot{x}_1(t)^2\partial_x^2 H_{-1,0}^{x_1(t)}(x)\\
&\qquad- \dot{x}_2(t)^2\partial_x^2 H_{0,1}^{x_2(t)}(x) + \ddot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}(x).
\end{aligned} \tag{A7}$$

Furthermore, from the estimate (A6), Lemma 20 and Lemma 6, we obtain that

$$\begin{aligned}
&\left\langle \dot{U}\left(H_{-1,0}^{x_1(t)} + H_{0,1}^{x_2(t)}\right) - \dot{U}\left(H_{-1,0}^{x_1(t)}\right) - \dot{U}\left(H_{0,1}^{x_2(t)}\right),\ \partial_x H_{0,1}^{x_2(t)}\right\rangle_{L^2}\\
&\quad= \ddot{x}_2(t)\left\|\partial_x H_{0,1}\right\|_{L^2}^2 - (\ddot{x}_2(t) - \ddot{d}_2(t))\left\|\partial_x H_{0,1}\right\|_{L^2}^2\\
&\qquad+ O\left(|\ddot{x}_1(t)|\,z(t)e^{-\sqrt{2}z(t)}\right)\\
&\qquad+ O\left(e^{-\sqrt{2}z(t)}\max_{j\in\{1,2\}}\left|x_j(t) - d_j(t)\right| + e^{-2\sqrt{2}z(t)}z(t)\right).
\end{aligned} \tag{A8}$$

We recall from the proof of Theorem 14 the following estimate

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\left[\ddot{U}\left(H_{0,1}^{x_2(t)}(x)\right) - \ddot{U}\left(H_{0,1}^{x_2(t)}(x) + H_{-1,0}^{x_1(t)}(x)\right)\right]\partial_x H_{0,1}^{x_2(t)}(x)g(t,x)\,dx\right|\\
&\quad= O\left(\left\|\overrightarrow{g(t)}\right\|e^{-\sqrt{2}z(t)}\right).
\end{aligned}$$

Also, from the Modulation Lemma, we have that

$$\langle\partial_t^2 g(t), \partial_x H_{0,1}^{x_2(t)}\rangle_{L^2} = \frac{d}{dt}\left[\langle\partial_t g(t), \partial_x H_{0,1}^{x_2(t)}\rangle_{L^2}\right] + \dot{x}_2(t)\langle\partial_t g(t), \partial_x H_{0,1}^{x_2(t)}\rangle_{L^2}$$

$$= \frac{d}{dt}\Big[\dot{x}_2(t)\langle g(t), \partial_x^2 H_{0,1}^{x_2(t)}\rangle_{L^2}\Big] + \dot{x}_2(t)\langle \partial_t g(t), \partial_x H_{0,1}^{x_2(t)}\rangle_{L^2}$$
$$= \ddot{x}_2(t)\langle g(t), \partial_x^2 H_{0,1}^{x_2(t)}\rangle_{L^2} + 2\dot{x}_2(t)\langle \partial_t g(t), \partial_x H_{0,1}^{x_2(t)}\rangle_{L^2}$$
$$- \dot{x}_2(t)^2\langle g(t), \partial_x^3 H_{0,1}^{x_2(t)}\rangle_{L^2}.$$

In conclusion, since $\partial_x H_{0,1}^{x_2(t)} \in \ker D^2 E_{pot}\left(H_{0,1}^{x_2(t)}\right)$ and $e^{-\sqrt{2}z(t)} = O\left(\epsilon^{\frac{1}{2}}\right)$, we obtain from (A8) and (A7) that

$$\left|\ddot{x}_2(t) - \ddot{d}_2(t)\right| = O\Big(\max_{j\in\{1,2\}}\left|d_j(t) - x_j(t)\right|\epsilon + \epsilon z(t)e^{-\sqrt{2}z(t)} + \left\|\overrightarrow{g(t)}\right\|\epsilon^{\frac{1}{2}}\Big),$$

the estimate of $\left|\ddot{x}_1(t) - \ddot{d}_1(t)\right|$ is completely analogous, which finishes the proof of Lemma 19. □

Lemma 21. *For any $\delta > 0$ there is a $\epsilon(\delta) > 0$ such that if*

$$\left\|\phi(x) - H_{0,1}(x)\right\|_{H^1} < +\infty,\ 0 < E_{pot}(\phi(x)) - E_{pot}(H_{0,1}) < \epsilon(\delta), \tag{A9}$$

then there is a real number y such that

$$\left\|\phi(x) - H_{0,1}(x-y)\right\|_{H^1} \leq \delta.$$

Proof of Lemma 21. The proof of Lemma 21 will follow by a contradiction argument. We assume the existence of a sequence of real functions $(\phi_n(x))_n$ satisfying

$$\lim_{n\to+\infty} E_{pot}(\phi_n) = E_{pot}(H_{0,1}), \tag{A10}$$
$$\left\|\phi_n(x) - H_{0,1}(x)\right\|_{H^1} <+\infty, \tag{A11}$$

such that

$$\lim_{n\to+\infty}\inf_{y\in\mathbb{R}}\left\|\phi_n(x) - H_{0,1}(x+y)\right\|_{H^1} > 0. \tag{A12}$$

First, the condition (A10) and the fact that $\lim_{\phi\to+\infty} U(\phi) = +\infty$ imply the existence of a positive constant c, which satisfies $\|\phi_n\|_{L^\infty} < c$ if $n \gg 1$.

Next, since $U(\phi) = \phi^2(1-\phi^2)^2$ and $\left|E_{pot}(\phi_n) - E_{pot}(H_{0,1})\right| \ll 1$ for $1 \ll n$, it is not difficult to verify from the definition of the potential energy functional E_{pot} that if $1 \ll n$, then

$$\|\phi_n(x) - 1\|^2_{L^2(\{x|\phi_n(x)>1\})} + \left\|\frac{d\phi_n(x)}{dx}\right\|^2_{L^2(\{x|\phi_n(x)>1\})} \lesssim \left|E_{pot}(\phi_n) - E_{pot}(H_{0,1})\right|.$$

By an analogous argument, we can verify that

$$\|\phi_n(x)\|^2_{L^2(\{x|-\frac{1}{2}<\phi_n(x)<0\})} + \left\|\frac{d\phi_n(x)}{dx}\right\|^2_{L^2(\{x|-\frac{1}{2}<\phi_n(x)<0\})}$$
$$\lesssim \left|E_{pot}(\phi_n) - E_{pot}(H_{0,1})\right|,$$

and if there is $x_0 \in \mathbb{R}$ such that $\phi_n(x_0) \leq -\frac{1}{2}$, we would obtain that

$$\int_{x_0}^{+\infty} \frac{1}{2}\frac{d\phi_n(x)}{dx}^2 + U(\phi_n(x))\,dx$$

$$= \int_{x_0}^{+\infty} \sqrt{2U(\phi_n(x))} \left|\frac{d\phi_n(x)}{dx}\right| dx + \frac{1}{2}\int_{x_0}^{+\infty} \left(\left|\frac{d\phi_n(x)}{dx}\right| - \sqrt{2U(\phi_n(x))}\right)^2 dx$$
$$\geq \int_{-\frac{1}{2}}^{1} \sqrt{2U(\phi)}\, d\phi = E_{pot}(H_{0,1}) + \int_{-\frac{1}{2}}^{0} \sqrt{2U(\phi)}\, d\phi > E_{pot}(H_{0,1}),$$

which contradicts (A10) if $n \gg 1$. Thus, if we consider the following function

$$\varphi_n(x) = \min\left(\max\left(\phi_n(x), 0\right), 1\right),$$

which satisfies $E_{pot}\left(\varphi_n\right) \geq E_{pot}\left(H_{0,1}\right)$ and

$$\frac{d\varphi_n(x)}{dx} = \begin{cases} \frac{d\phi_n(x)}{dx}, \text{ if } 0 < \phi_n(x) < 1, \\ 0, \text{ for almost every } x \in \mathbb{R} \text{ satisfying either } \phi_n(x) \leq 0 \text{ or } \phi_n(x) \geq 1, \end{cases}$$

we can deduce with the estimates above and inequality $\limsup_{n\to+\infty} \|\phi_n\|_{L^\infty} < c$ that if $n \gg 1$, then

$$\|\phi_n(x) - \varphi_n(x)\|_{L^2}^2 + \left\|\frac{d\phi_n(x)}{dx} - \frac{d\varphi_n(x)}{dx}\right\|_{L^2}^2 \lesssim \left|E_{pot}\left(\phi_n\right) - E_{pot}\left(H_{0,1}\right)\right|,$$
$$\left|E_{pot}\left(\phi_n\right) - E_{pot}\left(\varphi_n\right)\right| \lesssim \left|E_{pot}\left(\phi_n\right) - E_{pot}\left(H_{0,1}\right)\right|.$$

Consequently, using triangle inequality and conditions (A10), (A12), we would obtain that

$$\lim_{n\to+\infty} \inf_{y\in\mathbb{R}} \left\|\varphi_n(x) - H_{0,1}(x+y)\right\|_{H^1} > 0.$$

In conclusion, we can restrict the proof to the case where $0 \leq \phi_n(x) \leq 1$ and $n \gg 1$.

Now, from the density of $H^2(\mathbb{R})$ in $H^1(\mathbb{R})$, we can also restrict the contradiction hypotheses to the situation where $\frac{d\phi_n}{dx}(x)$ is a continuous function for all $n \in \mathbb{N}$. Also, we have that if $\left\|\phi(x) - H_{0,1}(x)\right\|_{H^1} < +\infty$, then $E_{pot}(\phi(x)) \geq E_{pot}(H_{0,1}(x))$. In conclusion, there is a sequence of positive numbers $(\epsilon_n)_n$ such that

$$E_{pot}(\phi_n) = E_{pot}(H_{0,1}) + \epsilon_n, \ \lim_{n\to+\infty} \epsilon_n = 0.$$

Also, $\tau_y\phi(x) = \phi(x-y)$ satisfies $E_{pot}(\phi(x)) = E_{pot}(\tau_y\phi(x))$ for any $y \in \mathbb{R}$. In conclusion, since for all $n \in \mathbb{N}$, $\lim_{x\to+\infty} \phi_n(x) = 1$ and $\lim_{x\to-\infty} \phi_n(x) = 0$, we can restrict to the case where

$$\phi_n(0) = \frac{1}{\sqrt{2}},$$

for all $n \in \mathbb{N}$.

Next, we consider the notations $(v)_+ = \max(v, 0)$ and $(v)_- = -\left(v - (v)_+\right)$. Since $\frac{d\phi_n(x)}{dx}$ is a continuous function on x, we deduce that $\left(\frac{d\phi_n(x)}{dx}\right)_+$ and $\left(\frac{d\phi_n(x)}{dx}\right)_-$ are also continuous functions on x for all $n \in \mathbb{N}$. In conclusion, for any $n \in \mathbb{N}$, we have that the set

$$U = \left\{x \in \mathbb{R} \middle| \frac{d\phi_n(x)}{dx} < 0\right\} \tag{A13}$$

is an enumerable union of disjoint open intervals $(a_{k,n}, b_{k,n})_{k\in\mathbb{N}}$, which are bounded, since $\lim_{x\to+\infty}\phi_n(x) = 1$, $\lim_{x\to-\infty}\phi_n(x) = 0$ and $0 \leq \phi_n(x) \leq 1$.

Now, let E be a set of disjoint open bounded intervals $(h_{i,n}, l_{i,n}) \subset \mathbb{R}$ satisfying the conditions

$$\phi_n(h_{i,n}) = \phi_n(l_{i,n}), \tag{A14}$$

and $\{i|\,(h_{i,n}, l_{i,n}) \in E\} = I \subset \mathbb{Z}$. For any $i \in I$, the following function

$$f_{i,n}(x) = \begin{cases} \phi_n(x) \text{ if } x \leq h_{i,n}, \\ \phi_n(x + l_{i,n} - h_{i,n}) \text{ if } x > h_{i,n}, \end{cases}$$

satisfies $E_{pot}(H_{0,1}) \leq E_{pot}(f_{i,n}) \leq E_{pot}(\phi_n) = E_{pot}(H_{0,1}) + \epsilon_n$, which implies that

$$\int_{h_{i,n}}^{l_{i,n}} \frac{1}{2}\frac{d\phi_n(x)}{dx}^2 + U(\phi_n(x)) \leq \epsilon_n.$$

Furthermore, we can deduce from Lebesgue's dominated convergence theorem that

$$\sum_{i\in I}\int_{h_{i,n}}^{l_{i,n}} \frac{1}{2}\frac{d\phi_n(x)}{dx}^2 + U(\phi_n(x)) \leq \epsilon_n, \tag{A15}$$

for every finite or enumerable collection E of disjoint open bounded intervals $(h_{i,n}, l_{i,n}) \subset \mathbb{R}$, $i \in I \subset \mathbb{Z}$ such that $\phi_n(h_{i,n}) = \phi_n(l_{i,n})$. In conclusion, we can deduce from (A15) that

$$\int_{\mathbb{R}} \left(\frac{d\phi_n(x)}{dx}\right)_-^2 dx \leq 2\epsilon_n, \tag{A16}$$

and so for $1 \ll n$ we have that

$$\left\|\frac{d\phi_n(x)}{dx} - \left|\frac{d\phi_n(x)}{dx}\right|\right\|_{L^2}^2 \leq 8\epsilon_n,\ \phi_n(0) = \frac{1}{\sqrt{2}}. \tag{A17}$$

Moreover, we can verify that

$$E_{pot}(\phi_n) = \frac{1}{2}\left[\int_{\mathbb{R}}\left(\left|\frac{d\phi_n(x)}{dx}\right| - \sqrt{2U(\phi_n(x))}\right)^2 dx\right] + \int_{\mathbb{R}}\sqrt{2U(\phi_n(x))}\left|\frac{d\phi_n(x)}{dx}\right| dx,$$

from which we deduce with $\lim_{x\to-\infty}\phi_n(x) = 0$ and $\lim_{x\to+\infty}\phi_n(x) = 1$ that

$$\begin{aligned} E_{pot}(H_{0,1}) + \epsilon_n \geq & \frac{1}{2}\left[\int_{\mathbb{R}}\left(\left|\frac{d\phi_n(x)}{dx}\right| - \sqrt{2U(\phi_n(x))}\right)^2 dx\right] + \int_0^1 \sqrt{2U(\phi)}\,d\phi \\ = & \frac{1}{2}\left[\int_{\mathbb{R}}\left(\left|\frac{d\phi_n(x)}{dx}\right| - \sqrt{2U(\phi_n(x))}\right)^2 dx\right] + E_{pot}(H_{0,1}). \end{aligned}$$

Then, from estimate (A17), we have that

$$\frac{d\phi_n(x)}{dx} = \sqrt{2U(\phi_n(x))} + r_n(x),\ \phi_n(0) = \frac{1}{\sqrt{2}}, \tag{A18}$$

with $\|r_n\|_{L^2}^2 \lesssim \epsilon_n$ for all $1 \ll n$.

We recall that $U(\phi) = \phi^2(1-\phi^2)^2$ is a Lipschitz function in the set $\{\phi | 0 \leq \phi \leq 1\}$. Then, because $H_{0,1}(x)$ is the unique solution of the following ordinary differential equation

$$\begin{cases} \frac{d\phi(x)}{dx} = \sqrt{2U(\phi(x))}, \\ \phi(0) = \frac{1}{\sqrt{2}}, \end{cases}$$

we deduce from Gronwall Lemma that for any $K > 0$ we have

$$\lim_{n\to+\infty} \left\|\phi_n(x) - H_{0,1}(x)\right\|_{L^\infty[-K,K]} = 0, \lim_{n\to+\infty} \left\|\frac{d\phi_n(x)}{dx} - \dot{H}_{0,1}(x)\right\|_{L^2[-K,K]} = 0. \tag{A19}$$

Also, if $1 \ll n$, then $\left\|\frac{d\phi_n(x)}{dx}\right\|_{L^2}^2 < 2E_{pot}(H_{0,1}) + 1$, and so we obtain from Cauchy–Schwarz inequality that

$$|\phi_n(x) - \phi_n(y)| \leq |x-y|^{\frac{1}{2}} \left\|\frac{d\phi_n}{dx}\right\|_{L^2}^2 < M\,|x-y|^{\frac{1}{2}}, \tag{A20}$$

for a constant $M > 0$. The inequality (A20) implies that for any $1 > \omega > 0$ there is a number $h(\omega) \in \mathbb{N}$ such that if $n \geq h(\omega)$ then

$$\left\|\phi_n(x) - H_{0,1}(x)\right\|_{L^\infty\{x|\,\frac{1}{\omega}<|x|\}} < \omega, \tag{A21}$$

otherwise we would obtain that there are $0 < \theta < \frac{1}{4}$, a subsequence $(m_n)_{n\in\mathbb{N}}$ and a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ with $\lim_{n\to+\infty} m_n = +\infty$, $|x_n| > n+1$ such that

$$\left|\phi_{m_n}(x_n) - 1\right| > \theta \text{ if } x_n > 0, \tag{A22}$$
$$\left|\phi_{m_n}(x_n)\right| > \theta \text{ if } x_n < 0. \tag{A23}$$

However, since we are considering $\phi_n(x) \in C^1(\mathbb{R})$ and $0 \leq \phi_n \leq 1$, we would obtain from the intermediate value theorem that there would exist a sequence $(y_n)_n$ with $y_n > x_n > n+1$ or $y_n < x_n < -n-1$ such that

$$1-\theta \leq \phi_{m_n}(y_n) \leq 1+\theta, \text{ if } y_n > 0, \tag{A24}$$
$$\phi_{m_n}(y_n) = \theta \text{ otherwise}. \tag{A25}$$

But, estimates (A20), (A24), (A25) and identity $U(\phi) = \phi^2(1-\phi^2)^2$ would imply that

$$1 \lesssim \int_{|x|\geq n-2} U(\phi_{m_n}(x))\,dx \text{ for all } n \gg 1, \tag{A26}$$

and because of estimate (A19) and the following identity

$$\lim_{K\to+\infty} \int_{-K}^{K} \frac{1}{2}\dot{H}_{0,1}(x)^2 + U(H_{0,1}(x)) = E_{pot}(H_{0,1}(x)), \tag{A27}$$

estimate (A26) would imply that $\lim_{n\to+\infty} E_{pot}(\phi_{m_n}) > E_{pot}(H_{0,1})$ which contradicts our hypotheses.

In conclusion, for any $1 > \omega > 0$ there is a number $h(\omega)$ such that if $n \geq h(\omega)$ then (A21) holds. So we deduce for any $0 < \omega < 1$ that there is a number $h_1(\omega)$ such that

$$\text{if } n \geq h_1(\omega), \text{ then } \left|\phi_n(x) - H_{0,1}(x)\right| \leq \omega \text{ for all } x \in \mathbb{R}. \tag{A28}$$

Then, if $\omega \leq \frac{1}{100}$, $n \geq h(\omega)$ and $K \geq 200$, estimates (A28) and (A19) imply that

$$\int_K^{+\infty} U(\phi_n(x)) + \frac{1}{2}\frac{d\phi_n(x)}{dx}^2 dx \geq \frac{1}{2}\int_K^{+\infty} (1-\phi_n(x))^2 + \frac{d\phi_n(x)}{dx}^2 dx, \tag{A29}$$

$$\int_{-\infty}^{-K} U(\phi_n(x)) + \frac{1}{2}\frac{d\phi_n(x)}{dx}^2 dx \geq \frac{1}{2}\int_{-\infty}^{-K} \phi_n(x)^2 + \frac{d\phi_n(x)}{dx}^2 dx. \tag{A30}$$

In conclusion, from estimates (A28), (A29), (A30) and

$$\lim_{K\to+\infty}\int_{|x|\geq K} \frac{1}{2}\dot{H}_{0,1}(x)^2 + U(H_{0,1}(x))\,dx = 0,$$

we obtain that $\lim_{n\to+\infty}\left\|\phi_n(x) - H_{0,1}(x)\right\|_{L^2} = 0$ and, from the initial value problem (A18) satisfied for each ϕ_n, we conclude that $\lim_{n\to+\infty}\left\|\frac{d\phi_n}{dx}(x) - \dot{H}_{0,1}(x)\right\|_{L^2} = 0$. In conclusion, inequality (A12) is false. □

From Lemma 21, we obtain the following corollary:

Corollary 22. *For any $\delta > 0$ there exists $\epsilon_0 > 0$ such that if $0 < \epsilon \leq \epsilon_0$, $\left\|\phi(x) - H_{0,1}(x) - H_{-1,0}(x)\right\|_{H^1} < +\infty$ and $E_{pot}(\phi) = 2E_{pot}(H_{0,1}) + \epsilon$, then there exist $x_2, x_1 \in \mathbb{R}$ such that*

$$x_2 - x_1 \geq \frac{1}{\delta},\ \left\|\phi(x) - H_{0,1}(x-x_2) + H_{-1,0}(x-x_1)\right\|_{H^1} \leq \delta. \tag{A31}$$

Proof of Corollary 22. First, from a similar reasoning to the proof of Lemma 21 we can assume by density that $\frac{d\phi(x)}{dx} \in H^1_x(\mathbb{R})$. Next, from hypothesis $\left\|\phi(x) - H_{0,1}(x) - H_{-1,0}(x)\right\|_{H^1(\mathbb{R})} < +\infty$, we deduce using the intermediate value theorem that there is a $y \in \mathbb{R}$ such that $\phi(y) = 0$. Now, we consider the functions

$$\phi_-(x) = \begin{cases} \phi(x) & \text{if } x \leq y, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi_+(x) = \begin{cases} 0 & \text{if } x \leq y, \\ \phi(x) & \text{otherwise.} \end{cases}$$

Clearly, $\phi(x) = \phi_-(x)$ for $x < y$ and $\phi(x) = \phi_+(x)$ for $x > y$. From identity $U(0) = 0$, we deduce that

$$E_{pot}(\phi) = E_{pot}(\phi_-) + E_{pot}(\phi_+),$$

also, we have that

$$E_{pot}(H_{-1,0}) < E_{pot}(\phi_-),\ E_{pot}(H_{0,1}) < E_{pot}(\phi_+).$$

In conclusion, since $E_{pot}(\phi) = 2E_{pot}(H_{0,1}) + \epsilon$, Lemma 21 implies that if $\epsilon < \epsilon_0 \ll 1$, then there exist $x_2,\ x_1 \in \mathbb{R}$ such that

$$\begin{aligned}&\left\|\phi(x) - H_{0,1}(x-x_2) - H_{-1,0}(x-x_1)\right\|_{H^1}\\ &\quad\leq \left\|\phi_+ - H_{0,1}(x-x_2)\right\|_{H^1} + \left\|\phi_- - H_{-1,0}(x-x_1)\right\|_{H^1} \leq e^{-\frac{4}{\delta}} \ll \delta. \quad \text{(A32)}\end{aligned}$$

So, to finish the proof of Corollary 22, we need only to verify that we have $x_2 - x_1 \geq \frac{1}{\delta}$ if $0 < \epsilon_0 \ll 1$. But, we recall that $H_{0,1}(0) = \frac{1}{\sqrt{2}}$, from which with estimate (A32) we deduce that

$$\left|\phi_+(x_2) - \frac{1}{\sqrt{2}}\right| \lesssim \delta,\ \left|\phi_-(x_1) + \frac{1}{\sqrt{2}}\right| \lesssim \delta, \tag{A33}$$

so if $\epsilon_0 \ll 1$, then $x_1 < y < x_2$. Using the fact that U is a smooth function, Lemma 10 and identity (35), we can verify the existence of a constant $C > 0$ satisfying the following inequality

$$\left|DE_{pot}\left(H_{0,1}(x-x_2) + H_{-1,0}(x-x_1) + u\right)(v)\right| \leq C\,\|v\|_{H^1}.$$

for any $u,\ v \in H^1(\mathbb{R})$ such that $\|u\|_{H^1} \leq 1$. Therefore, using estimate (A32) and the Fundamental Theorem of Calculus, we deduce that if $0 < \epsilon_0 \ll 1$, then

$$\left|E_{pot}(\phi) - E_{pot}\left(H_{0,1}(x-x_2) + H_{-1,0}(x-x_1)\right)\right| < e^{-2\sqrt{2}\frac{1}{\delta}}. \tag{A34}$$

Furthermore, since the function $A(z) = E_{pot}\left(H^z_{0,1}(x) + H_{-1,0}(x)\right)$ is a continuous function on $\mathbb{R}_{\geq 0}$ and $A(z) > 2E_{pot}\left(H_{0,1}\right)$ for any $z \geq 0$, we have for any $k > 0$ that there exists $\delta_k > 0$ satisfying

$$\sup_{\{z\in[0,k]\}} A(z) > 2E_{pot}\left(H_{0,1}\right) + \delta_k.$$

In conclusion, we obtain from Lemma 7 and the estimate (A34) that $x_2 - x_1 \geq \frac{1}{\delta}$ if $0 < \epsilon_0 \ll 1$ and $\epsilon < \epsilon_0$. □

Now, we complement our manuscript by presenting the proof of identity (33).

Proof of Identity (33). From the definition of the function $H_{0,1}(x)$, we have

$$\int_{\mathbb{R}} \left(8(H_{0,1}(x))^3 - 6(H_{0,1}(x))^5\right)e^{-\sqrt{2}x}\,dx = \int_{\mathbb{R}} \frac{8e^{2\sqrt{2}x} + 2e^{4\sqrt{2}x}}{(1+e^{2\sqrt{2}x})^{\frac{5}{2}}}\,dx,$$

by the change of variable $y(x) = (1 + e^{2\sqrt{2}x})$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}} \left(8(H_{0,1}(x))^3 - 6(H_{0,1}(x))^5\right)e^{-\sqrt{2}x}\,dx\\ &\quad= \frac{1}{2\sqrt{2}}\int_1^{\infty} \frac{8}{y^{\frac{5}{2}}} + \frac{2(y-1)}{y^{\frac{5}{2}}}\,dy\\ &\quad= \frac{1}{2\sqrt{2}}\int_1^{\infty} \frac{6}{y^{\frac{5}{2}}} + \frac{2}{y^{\frac{3}{2}}}\,dy, = \frac{1}{2\sqrt{2}}(-4y^{-\frac{3}{2}} - 4y^{-\frac{1}{2}})\Big|_1^{\infty} = 2\sqrt{2}.\end{aligned}$$

□

Appendix B Proof of Theorem 3

Proof of Theorem 3. We use the notations of Theorems 2 and 4. Clearly, if the result of Theorem 3 is false, then by contradiction for any $N \gg 1$ the inequality

$$\left\|\overrightarrow{g(t)}\right\| \leq \frac{\epsilon}{N} \tag{B35}$$

could be possible for all $0 \leq t \leq N\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}} = T$ if $\epsilon \ll 1$ enough.

From Modulation Lemma, we can denote the solution $\phi(t,x)$ as

$$\phi(t,x) = H^{x_1(t)}_{-1,0}(x) + H^{x_2(t)}_{0,1}(x) + g(t,x),$$

such that

$$\langle g(t,x),\ \partial_x H^{x_1(t)}_{-1,0}(x)\rangle_{L^2} = 0,\ \langle g(t,x),\ \partial_x H^{x_2(t)}_{0,1}(x)\rangle_{L^2} = 0.$$

Also, for all $t \geq 0$, we have that $g(t,x)$ has a unique representation as

$$g(t,x) = P_1(t)\partial_x^2 H^{x_1(t)}_{-1,0}(x) + P_2(t)\partial_x^2 H^{x_2(t)}_{0,1}(x) + r(t,x), \tag{B36}$$

such that $r(t)$ satisfies the following new orthogonality conditions

$$\left\langle r(t),\ \partial_x^2 H^{x_1(t)}_{-1,0}\right\rangle_{L^2} = 0,\ \left\langle r(t),\ \partial_x^2 H^{x_2(t)}_{0,1}\right\rangle_{L^2} = 0. \tag{B37}$$

In conclusion, we deduce that

$$\|g(t)\|^2_{L^2} = \left\|\partial_x^2 H_{0,1}\right\|^2_{L^2}(P_1^2 + P_2^2) + \|r(t)\|^2_{L^2} + 2P_1P_2\left\langle \partial_x^2 H^{z(t)}_{0,1},\ \partial_x^2 H_{-1,0}\right\rangle_{L^2}. \tag{B38}$$

We recall from Theorem 11 that $\frac{1}{\sqrt{2}}\ln\frac{1}{\epsilon} < z(t)$ for all $t \geq 0$. Since, from Lemma 6, we have that $\left\langle \partial_x^2 H^{x_1(t)}_{-1,0},\ \partial_x^2 H^{x_2(t)}_{0,1}\right\rangle_{L^2} \lesssim z(t)e^{-\sqrt{2}z(t)}$ and $z(t)e^{-\sqrt{2}z(t)} \lesssim \epsilon\ln\frac{1}{\epsilon}$ if $0 < \epsilon \ll 1$, we deduce from the Eq. (B38) that there is a uniform constant $K > 1$ such that for all $t \geq 0$ we have the following estimate

$$\frac{\|g(t)\|_{L^2}}{K} \leq |P_1(t)| + |P_2(t)| + \|r(t)\|_{L^2} \leq K\left\|\overrightarrow{g(t)}\right\|. \tag{B39}$$

From Theorem 11 and the orthogonality conditions (B37), we deduce that

$$\begin{aligned}
\left\langle \partial_t r(t),\ \partial_x^2 H^{x_2(t)}_{0,1}\right\rangle_{L^2} &= \dot{x}_2(t)\left\langle r(t),\ \partial_x^3 H^{x_2(t)}_{0,1}\right\rangle_{L^2} = O\left(\|r(t)\|_{L^2}\epsilon^{\frac{1}{2}}\right),\\
\left\langle \partial_t r(t),\ \partial_x^2 H^{x_1(t)}_{-1,0}\right\rangle_{L^2} &= \dot{x}_2(t)\left\langle r(t),\ \partial_x^3 H^{x_1(t)}_{-1,0}\right\rangle_{L^2} = O\left(\|r(t)\|_{L^2}\epsilon^{\frac{1}{2}}\right).
\end{aligned}$$

In conclusion, estimate (B39) and Lemma 6 imply that there is a $K > 1$ such that

$$\left|\dot{P}_1(t)\right| + \left|\dot{P}_2(t)\right| + \|\partial_t r(t)\|_{L^2} \leq K\left\|\overrightarrow{g(t)}\right\| \tag{B40}$$

for all $t \geq 0$. Finally, Minkowski inequality and estimate (B39) imply that there is a uniform constant $K > 1$ such that

$$\|\partial_x r(t,x)\|_{L^2} \leq K\left\|\overrightarrow{g(t)}\right\|. \tag{B41}$$

We recall from Theorem 12 the following estimate

$$\frac{\epsilon}{K} \leq \left\|\overrightarrow{g(t)}\right\|^2 + \dot{x}_1(t)^2 + \dot{x}_2(t)^2 + e^{-\sqrt{2}z(t)} \leq K\epsilon \tag{B42}$$

for some uniform constant $K > 1$. Now, from hypothesis (B35), we obtain from Theorem 4 and Corollary 5 that there are constants $M \in \mathbb{N}$ and $C > 0$ such that for all $t \geq 0$ the following inequalities are true

$$\max_{j\in\{1,2\}} \left|x_j(t) - d_j(t)\right| \leq \epsilon \left(\ln\frac{1}{\epsilon}\right)^{M+1} \exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right), \tag{B43}$$

$$\max_{j\in\{1,2\}} \left|\dot{x}_j(t) - \dot{d}_j(t)\right| \leq \epsilon^{\frac{3}{2}} \left(\ln\frac{1}{\epsilon}\right)^{M} \exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right), \tag{B44}$$

$$\max_{j\in\{1,2\}} \left|\ddot{x}_j(t) - \ddot{d}_j(t)\right| \leq \epsilon^{\frac{3}{2}} \left(\ln\frac{1}{\epsilon}\right) \exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\left(\frac{1}{\epsilon}\right)}\right), \tag{B45}$$

for a uniform constant $C > 0$.

From the partial differential equation (1) satisfied by $\phi(t,x)$ and the representation (B36) of $g(t,x)$, we deduce in the distributional sense that for any $h(x) \in H^1(\mathbb{R})$ that

$$\begin{aligned}
&\left\langle h(x),\ (\ddot{P}_1(t) + \dot{x}_1(t)^2)\partial_x^2 H_{-1,0}^{x_1(t)} + (\ddot{P}_2(t) + \dot{x}_2(t)^2)\partial_x^2 H_{0,1}^{x_2(t)}\right\rangle_{L^2} \\
&= -\left\langle h(x),\, P_1(t)\left[\left(-\partial_x^2 + \ddot{U}(H_{-1,0}^{x_1(t)})\right)\partial_x^2 H_{-1,0}^{x_1(t)}\right]\right\rangle_{L^2} \\
&\quad - \left\langle h(x),\, P_2(t)\left[\left(-\partial_x^2 + \ddot{U}(H_{0,1}^{x_2(t)})\right)\partial_x^2 H_{0,1}^{x_2(t)}\right]\right\rangle_{L^2} \\
&\quad - \left\langle h(x),\, \left[\partial_t^2 r(t) - \partial_x^2 r(t) + \ddot{U}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)})r(t)\right]\right\rangle_{L^2} \\
&\quad - \left\langle h(x),\, \left[\dot{U}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}) + \dot{U}(H_{0,1}^{x_2(t)}) - \dot{U}(H_{-1,0}^{x_1(t)})\right]\right\rangle_{L^2} \\
&\quad + \left\langle h(x),\ \ddot{x}_1(t)\partial_x H_{-1,0}^{x_1(t)}(x) + \ddot{x}_2(t)\partial_x H_{0,1}^{x_2(t)}(x)\right\rangle_{L^2} \\
&\quad - \left\langle h(x),\ P_1(t)\left[\left(\ddot{U}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}) - \ddot{U}(H_{-1,0}^{x_1(t)})\right)\partial_x^2 H_{-1,0}^{x_1(t)}\right]\right\rangle_{L^2} \\
&\quad - \left\langle h(x),\, P_2(t)\left[\left(\ddot{U}(H_{0,1}^{x_2(t)} + H_{-1,0}^{x_1(t)}) - \ddot{U}(H_{0,1}^{x_2(t)})\right)\partial_x^2 H_{0,1}^{x_2(t)}\right]\right\rangle_{L^2} \\
&\quad + O\left(\|h\|_{L^2}\left[\|g(t)\|_{H^1}^2 + \max_{j\in\{1,2\}}\left|\ddot{x}_j(t)\right|\right]\right) \\
&\quad + O\left(\|h\|_{L^2}\left[\max_{j\in\{1,2\}}\left|\dot{P}_j(t)\dot{x}_j(t)\right| + \max_{j\in\{1,2\}}\left|P_j(t)\right| e^{-\sqrt{2}z(t)}\right]\right) \\
&\quad + O\left(\left|P_j(t)\ddot{x}_j(t)\right| + \left|P_j(t)\dot{x}_j(t)^2\right|\right).
\end{aligned} \tag{B46}$$

From Lemma 20 and estimates (B43) and (B45), we obtain from (B46) that

$$\begin{aligned}
&\left\langle h(x),\ (\ddot{P}_1(t) + \dot{x}_1(t)^2)\partial_x^2 H_{-1,0}^{x_1(t)} + (\ddot{P}_2(t) + \dot{x}_2(t)^2)\partial_x^2 H_{0,1}^{x_2(t)}\right\rangle_{L^2} \\
&= -\left\langle h(x),\, P_1(t)\left[\left(-\partial_x^2 + \ddot{U}(H_{-1,0}^{x_1(t)})\right)\partial_x^2 H_{-1,0}^{x_1(t)}\right]\right\rangle_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&-\left\langle h(x),\, P_2(t)\left[\left(-\partial_x^2+\ddot{U}(H_{0,1}^{x_2(t)})\right)\partial_x^2 H_{0,1}^{x_2(t)}\right]\right\rangle_{L^2}\\
&-\left\langle h(x),\left[\partial_t^2 r(t)-\partial_x^2 r(t)+\ddot{U}(H_{0,1}^{x_2(t)}+H_{-1,0}^{x_1(t)})r(t)\right]\right\rangle_{L^2}\\
&+O\left(\|h\|_{L^2}\left[\max_{j\in\{1,\,2\}}\left|\ddot{x}_j(t)-\ddot{d}_j(t)\right|+e^{-\sqrt{2}d(t)}\right]\right)\\
&+O\left(\|h\|_{L^2}\left[|z(t)-d(t)|\,e^{-\sqrt{2}z(t)}+e^{-2\sqrt{2}z(t)}\right]\right)\\
&+O\left(\|h\|_{L^2}\left[\|g(t)\|_{H^1}^2+\max_{j\in\{1,\,2\}}\left|\ddot{x}_j(t)\right|\right]\right)\\
&+O\left(\|h\|_{L^2}\left[\max_{j\in\{1,\,2\}}\left|\dot{P}_j(t)\dot{x}_j(t)\right|+\max_{j\in\{1,\,2\}}\left|P_j(t)\right|e^{-\sqrt{2}z(t)}+\left|P_j(t)\ddot{x}_j(t)\right|\right]\right)\\
&+O\left(\|h\|_{L^2}\left|P_j(t)\dot{x}_j(t)^2\right|\right).
\end{aligned}
\tag{B47}
$$

From the condition (B37), we deduce that

$$
\left\langle\partial_t^2 r(t),\ \partial_x^2 H_{0,1}^{x_2(t)}\right\rangle_{L^2}=\frac{d}{dt}\left[\dot{x}_2(t)\left\langle r(t),\ \partial_x^3 H_{0,1}^{x_2(t)}\right\rangle\right]+\dot{x}_2(t)\left\langle\partial_t r(t),\ \partial_x^3 H_{0,1}^{x_2(t)}\right\rangle_{L^2},
$$

$$
\left\langle\partial_t^2 r(t),\ \partial_x^2 H_{-1,0}^{x_1(t)}\right\rangle_{L^2}=\frac{d}{dt}\left[\dot{x}_1(t)\left\langle r(t),\ \partial_x^3 H_{-1,0}^{x_1(t)}\right\rangle_{L^2}\right]+\dot{x}_1(t)\left\langle\partial_t r(t),\ \partial_x^3 H_{-1,0}^{x_1(t)}\right\rangle_{L^2},
$$

which imply with Theorem 11 the existence of a uniform constant $C>0$ such that

$$
\left|\left\langle\partial_t^2 r(t),\ \partial_x^2 H_{0,1}^{x_2(t)}\right\rangle_{L^2}\right|\leq C\epsilon^{\frac{1}{2}}\left\|\overrightarrow{r(t)}\right\|,\ \left|\left\langle\partial_t^2 r(t),\ \partial_x^2 H_{-1,0}^{x_1(t)}\right\rangle_{L^2}\right|\leq C\epsilon^{\frac{1}{2}}\left\|\overrightarrow{r(t)}\right\|.
\tag{B48}
$$

From (B39), (B40) and (B41), we obtain that $\left\|\overrightarrow{r(t)}\right\|\lesssim\left\|\overrightarrow{g(t)}\right\|.$

In conclusion, after we apply the partial differential equation (B47) in the distributional sense to $\partial_x^2 H_{0,1}^{x_2(t)}$, $\partial_x^2 H_{-1,0}^{x_1(t)}$, the estimates (B39), (B40), (B41), (B43), (B45) and (B48) imply that there is a uniform constant $K_1>0$ such that if $\epsilon\ll 1$ enough, then for $j\in\{1,\ 2\}$ we have that for $0\leq t\leq\frac{N\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$

$$
\left|\ddot{P}_j(t)+\dot{x}_j(t)^2\right|\leq K_1\left(e^{-\sqrt{2}d(t)}+\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)+\frac{\epsilon}{N}\right),
$$

from which we deduce for all $0\leq t\leq N\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ that

$$
\left|\sum_{j=1}^{2}\ddot{P}_j(t)+\dot{x}_j(t)^2\right|\leq 2K_1\left(e^{-\sqrt{2}d(t)}+\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\left(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\right)+\frac{\epsilon}{N}\right).
\tag{B49}
$$

Since $\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right|\geq-\left|\sum_{j=1}^{2}\ddot{P}_j(t)+\dot{x}_j(t)^2\right|+\sum_{j=1}^{2}\dot{x}_j(t)^2$, we deduce from the estimates (B49) and (B42) that

$$
\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right|\geq\frac{\epsilon}{K}-\left[e^{-\sqrt{2}z(t)}+\left\|\overrightarrow{g(t)}\right\|^2\right]
$$

$$- 2K_1\Big[e^{-\sqrt{2}d(t)} + \epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\Big(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\Big)\Big] - \frac{2K_1\epsilon}{N}. \tag{B50}$$

We recall that from the statement of Theorem 4 that $e^{-\sqrt{2}d(t)} = \frac{v^2}{8}\operatorname{sech}(\sqrt{2}vt + c)^2$, with $v = \left(\frac{\dot{z}(0)^2}{4} + 8e^{-\sqrt{2}z(0)}\right)^{\frac{1}{2}}$, which implies that $v \lesssim \epsilon^{\frac{1}{2}}$. Since we have verified in Theorem 11 that $e^{-\sqrt{2}z(t)} \lesssim \epsilon$, the mean value theorem implies that $\left|e^{-\sqrt{2}z(t)} - e^{-\sqrt{2}d(t)}\right| = O(\epsilon\,|z(t) - d(t)|)$, from which we deduce from (B43) that

$$\left|e^{-\sqrt{2}z(t)} - e^{-\sqrt{2}d(t)}\right| = O\left(\epsilon^2\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\Big(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\Big)\right).$$

In conclusion, if $\epsilon \ll 1$ enough, we obtain for $0 \leq t \leq \frac{N\ln(\frac{1}{\epsilon})}{\epsilon^{\frac{1}{2}}}$ from (B50) that

$$\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right| \geq \frac{\epsilon}{K} - \Big[e^{-\sqrt{2}d(t)} + \left\|\overrightarrow{g(t)}\right\|^2\Big]$$
$$- 4K_1\Big[e^{-\sqrt{2}d(t)} + \epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\Big(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\Big)\Big] - \frac{2K_1\epsilon}{N}. \tag{B51}$$

The conclusion of the demonstration will follow from studying separate cases in the choice of $v > 0,\ c$. We also observe that $K,\ K_1$ are uniform constants and the value of $N \in \mathbb{N}_{>0}$ can be chosen at the beginning of the proof to be as much large as we need.

Case 1. ($v^2 \leq \frac{8\epsilon}{(1+4K_1)2K}$.) From inequality (B51), we deduce that

$$\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right| \geq \frac{\epsilon}{2K} - \left\|\overrightarrow{g(t)}\right\|^2 - 4K_1\Big(\epsilon^{\frac{3}{2}}\left(\ln\frac{1}{\epsilon}\right)^{M+1}\exp\Big(\frac{10C\epsilon^{\frac{1}{2}}t}{\ln\frac{1}{\epsilon}}\Big)\Big) - \frac{2K_1\epsilon}{N},$$

then, from (B35) we deduce for $0 \leq t \leq \frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ that if ϵ is small enough and $N > 10KK_1$, then $\left|\sum_{j=1}^{2}\ddot{P}_j(t)\right| \geq \frac{\epsilon}{4K}$, and so,

$$\left|\sum_{j=1}^{2}\dot{P}_j(t)\right| \geq \frac{\epsilon t}{4K} - \left|\sum_{j=1}^{2}\dot{P}_j(0)\right|,$$

which contradicts the fact that (B40) and (B35) should be true for $\epsilon \ll 1$.

Case 2. ($v^2 \geq \frac{8\epsilon}{(1+4K_1)2K},\ |c| > 2\ln(\frac{1}{\epsilon})$.) It is not difficult to verify that for $0 \leq t \leq \min(\frac{|c|}{2\sqrt{2}v}, N\frac{\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}})$, we have that $e^{-\sqrt{2}d(t)} \leq \frac{v^2}{8}\operatorname{sech}(\frac{c}{2})^2 \lesssim \epsilon^3$. Therefore, if $N >$

$10KK_1$ and $\epsilon > 0$ is small enough, estimate (B51) would imply that $\left|\sum_{j=1}^{2} \ddot{P}_j(t)\right| \geq \frac{\epsilon}{4K}$ is true in this time interval. Also, since now $v \cong \epsilon^{\frac{1}{2}}$, we have that

$$\frac{\ln \frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}} \lesssim \frac{|c|}{2\sqrt{2}v},$$

so we obtain a contradiction by a similar argument to the Case 1.

Case 3. ($v^2 \geq \frac{8\epsilon}{(1+4K_1)2K}$ and $|c| \leq 2\ln\frac{1}{\epsilon}$.) For $N \gg 1$ and $t_0 = \frac{(1+4K_1)^{\frac{1}{2}}K^{\frac{1}{2}}\sqrt{2}\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$, we have during the time interval $\left\{t_0 \leq t \leq 2\frac{(1+4K_1)^{\frac{1}{2}}K^{\frac{1}{2}}\sqrt{2}\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}\right\}$ that $e^{-\sqrt{2}d(t)} \leq \frac{v^2}{8}$ sech $\left(2\ln\frac{1}{\epsilon}\right)^2 \lesssim \epsilon^5$ and $\frac{\epsilon}{N} < \frac{\epsilon}{20K}$. In conclusion, estimate (B50) implies that $\left|\sum_{j=1}^{2} \ddot{P}_j(t)\right| \geq \frac{\epsilon}{4K}$ is true in this time interval. From the Fundamental Calculus Theorem, we have that

$$\left|\sum_{j=1}^{2} \dot{P}_j(t)\right| \geq \frac{\epsilon(t-t_0)}{4K} - \left|\sum_{j=1}^{2} \dot{P}_j(t_0)\right|.$$

In conclusion, hypothesis (B35) and estimate (B40) imply for $T = 2\frac{(1+2K_1)^{\frac{1}{2}}K^{\frac{1}{2}}\sqrt{2}\ln\frac{1}{\epsilon}}{\epsilon^{\frac{1}{2}}}$ and $N \gg 1$ that

$$\left|\sum_{j=1}^{2} \dot{P}_j(T)\right| \geq \frac{\epsilon^{\frac{1}{2}}(1+2K_1)^{\frac{1}{2}}\sqrt{2}\ln\frac{1}{\epsilon}}{8K^{\frac{1}{2}}},$$

which contradicts the fact that (B35) and (B40) should be true, which finishes our proof. □

References

1. Bethuel, F., Orlandi, G., Smets, D.: Dynamics of multiple degree Ginzburg–Landau vortices. C. R. Math. **342**, 837–842 (2006)
2. Bishop, A.R., Schneider, T., Matter, S.N.S.D.C.: National Science Foundation (U.S.). In: Solitons and Condensed Matter Physics: Proceedings of the Symposium on Nonlinear (Soliton) Structure and Dynamics in Condensed Matter, Oxford, England, June 27–29, 1978. Lecture Notes in Economic and Mathematical Systems. Springer (1978)
3. Coddington, A., Levinson, N.: Theory of Ordinary Differential Equations. International Series in Pure and Applied Mathematics. McGraw-Hill Companies, New York (1955)
4. Colliander, J.E., Jerrard, R.L.: Vortex dynamics for the Ginzburg–Landau–Schrödinger equation. Int. Math. Res. Not. **333–358**, 1998 (1998)
5. Delort, J.-M., Masmoudi, N.: Long-Time Dispersive Estimates for Perturbations of a Kink Solution of One-Dimensional Cubic Wave Equations Space Time Dimensions, volume 1 of Memoirs of European Mathematical Society. EMS Press, Zurich (2022)
6. Dorey, P., Mersh, K., Romanczukiewicz, T., Shnir, Y.: Kink-antikink collisions in the ϕ^6 model. Phys. Rev. Lett. **107**, 091602 (2011)
7. Dunajski, M., Manton, N.S.: Reduced dynamics of Ward solitons. Nonlinearity **18**(4), 1677–1689 (2005)
8. Gani, V.A., Kudryavtsev, A.E., Lizunova, M.A.: Kink interactions in the (1+1)-dimensional ϕ^6 model. Phys. Rev. D Part. Fields **89**, 125009 (2014)
9. Giblin, J.T., Hui, L., Lim, E.A., Yang, I.-S.: How to run through walls: dynamics of bubble and soliton collisions. Phys. Rev. D Part. Fields **82**, 045019 (2010)

10. Gorshkov, K.A., Ostrovsky, L.A.: Interactions of solitons in nonintegrable systems: direct perturbation method and applications. Phys. D **3**, 428–438 (1981)
11. Gustafson, S., Sigal, I.M.: Effective dynamics of magnetic vortices. Adv. Math. **199**, 448–498 (2006)
12. Hawking, S.W., Moss, I.G., Stewart, J.M.: Bubble collisions in the very early universe. Phys. Rev. D **26**, 2681–2713 (1982)
13. Jendrej, J., Kowalczyk, M., Lawrie, A.: Dynamics of strongly interacting kink-antikink pairs for scalar fields on a line. Duke Math. J. **171**(18), 3643–3705 (2022)
14. Jerrard, R.L., Soner, H.M.: Dynamics of Ginzburg–Landau vortices. Arch. Ration. Mech. Anal. **142**, 99–125 (1998)
15. Jerrard, R.L., Spirn, D.: Refined Jacobian estimates and Gross–Pitaevsky vortex dynamics. Arch. Ration. Mech. Anal. **190**, 425–475 (2008)
16. Kowalczyk, M., Martel, Y., Muñoz, C.: Kink dynamics in the ϕ^4 model: asymptotic stability for odd perturbations in the energy space. J. Am. Math. Soc. **30**, 769–798 (2017)
17. Kowalczyk, M., Martel, Y., Muñoz, C., Van Den Bosch, H.: A sufficient condition for asymptotic stability of kinks in general (1+1)-scalar field models. Ann. PDE **7**, 1–98 (2021)
18. Lohe, M.A.: Soliton structures in $p(\phi)_2$. Phys. Rev. D **20**(12), 3120 (1979)
19. Manton, N., Sutcliffe, P.: Topological Solitons. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (2004)
20. Manton, N.S., Speight, J.M.: Asymptotic interactions of critically coupled vortices. Commun. Math. Phys. **236**, 535–555 (2003)
21. Manton, N.: Forces between kinks and antikinks with long-range tails. J. Phys. A Math. Theor. **52**(6), 065401 (2019)
22. Martel, Y., Merle, F., Tsai, T.-P.: Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations. Commun. Math. Phys. **231**, 347–373 (2002)
23. Muñoz, C.: The Gardner equation and the stability of multi-kink solutions of the mKdV equation. Discrete Contin. Dyn. Syst. **16**(7), 3811–3843 (2016)
24. Ovchinnikov, Y.N., Sigal, I.M.: The Ginzburg–Landau equation III. Vortices dynamics. Nonlinearity **11**(5), 1277–1294 (1998)
25. Raphaël, P., Szeftel, J.: Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical nls. J. Am. Math. Soc. **24**(2), 471–546 (2010)
26. Sandier, E., Serfaty, S.: Gamma-convergence of gradient flows and applications to Ginzburg–Landau vortex dynamics. Commun. Pure Appl. Math. **57**, 1627–1672 (2004)
27. Shatah, J., Struwe, M.: Geometric Wave Equation, volume 2 of Courant Lecture Notes. AMS, Boston (2000)
28. Stuart, D.: Dynamics of Abelian Higgs vortices in the near Bogolmony regime regime. Commun. Math. Phys. **159**, 51–91 (1994)
29. Stuart, D.: The geodesic approximation for Yang–Mills–Highs equations. Commun. Math. Phys. **166**, 149–190 (1994)
30. Stuart, D.: Analysis of the adiabatic limit for solitons in classical field theory. Proc. R. Soc. A **463**, 2753–2781 (2007)
31. Vilenkin, A., Shellard, E.P.S.: Cosmic Strings and Other Topological Defects. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge (1994)

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