Commun. Math. Phys. 401, 1237–1290 (2023)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-023-04664-2

Communications in
Mathematical Physics

A Proof of the Fusion Rules Theorem

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Received: 12 April 2022 / Accepted: 24 January 2023
Published online: 11 February 2023 –

Abstract: We prove that the space of intertwining operators associated with certain admissible modules over vertex operator algebras is isomorphic to a quotient of the vector space of conformal blocks on a three-pointed rational curve defined by the same data. This provides a new proof and alternative version of Frenkel and Zhu's fusion rules theorem, in terms of the dimension of certain bimodules over Zhu's algebra, without the assumption of rationality.

1. Introduction

The space of intertwining operators of vertex operator algebras (see [1,4,5]) and its dimension, the so-called fusion rule in the physics literature [9–11], plays an essential role in studying the tensor product of modules over vertex operator algebras. In the semi-simple case, the fusion rule is the multiplicity of an irreducible module in a tensor product. For the affine Lie algebras or the associated affine vertex operator algebras [6], the fusion rules in case $\widehat{sl_2(\mathbb{C})}$ were computed in [10], and a general version was stated in [11] without proof. In [6], Frenkel and Zhu proposed a formula (Theorem 1.5.2 in [6]) to compute the fusion rules for arbitrary vertex operator algebras by using Zhu's algebra $A(V)$ defined in [13] and some of its (bi)modules. Given irreducible modules M^1, M^2 and M^3 over a vertex operator algebra V, Frenkel and Zhu's fusion rules theorem claimed that the space of intertwining operators $I\binom{M^3}{M^1\ M^2}$ can be identified with the vector space $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$, where $A(M^1)$ is a bimodule over the Zhu's algebra $A(V)$, and $M^2(0)$ and $M^3(0)$ are the bottom levels of the V-modules M^2 and M^3, which are modules over $A(V)$, see Section 1 in [6] for more details.

However, it was later realized by Li (see [8]) that some additional conditions are needed in Frenkel and Zhu's fusion rules theorem. Li gave a counter-example in [8] in the case of the universal Virasoro vertex operator algebra that shows that $I\binom{M^3}{M^1\ M^2}$ is not isomorphic to $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$ in general. Li also proposed in

[8] that the fusion rules theorem is true when M^2 and M^3 are the so-called generalized Verma modules constructed in [2]. In particular, it is true for the rational vertex operator algebras (see Section 2 in [8] for more detailed discussions and the counter-example).

In this paper, we give an alternative version of the fusion rules theorem for general vertex operator algebras. It can be stated as follows:

Theorem 1.1. *Let V be a CFT-type vertex operator algebra, and let M^1, M^2, and M^3 be V-modules with conformal weights h_1, h_2, and h_3, respectively. Assume $M^2(0)$ and $M^3(0)$ are irreducible $A(V)$-modules, then we have the following isomorphism of vector spaces:*

$$I\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))} \cong I\binom{\bar{M}^3}{M^1\ \bar{M}^2} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*,$$

where $h = h_1 + h_2 - h_3$, and $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ and $\bar{M}^{3'} = \tilde{M}/\mathrm{Rad}\tilde{M}$ are quotient modules of the generalized Verma modules $\bar{M}(M^2(0))$ and $\bar{M}(M^3(0)^*)$, respectively.

In our version of the fusion rules theorem, we replaced the $A(V)$-bimodule $A(M^1)$ by a newly defined $A(V)$-bimodule $B_h(M^1)$, which is given by $B_h(M^1) = M^1/span\{a \circ u,\ L(-1)v + (L(0) + h_2 - h_3)v : a \in V,\ u, v \in M^1\}$. We will show that $B_h(M^1)$ is a quotient module of $A(M^1)$, and we will give examples to show that the vector spaces $(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$ and $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$ are not isomorphic in general. We need to mod out the additional terms $L(-1)v + (L(0)v + h_2 - h_3)v$ in $A(M^1)$ because otherwise, the $L(-1)$-derivation property of the intertwining operators cannot be correctly reflected. We will also give sufficient conditions for modules $\bar{M}^2$ and $\bar{M}^3$ to be irreducible. In particular, for a CFT-type rational vertex operator algebra V, the modules $\bar{M}^2$ and $\bar{M}^3$ are automatically irreducible, then the fusion rule $\dim I\binom{M^3}{M^1\ M^2}$ for three irreducible V-modules is equal to the dimension of $(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$.

Our proof of Theorem 1.1 is different than Li's proof of Theorem 2.11 in [8]. We prove Theorem 1.1 based on a combination of ideas the ideas from [11] and extensions made in [12], wherein a system of correlation functions is associated with every vector in the space of conformal blocks (see Theorem 6.2 in [12]). Based on the properties of the following prototype system of $(n + 3)$-point correlation functions on $\mathbb{P}^1_{\mathbb{C}}$:

$$(v_3', Y_{M^3}(a_1, z_1) \dots Y_{M^3}(a_k, z_k) I(v, w) Y_{M^2}(a_{k+1}, z_{k+1}) \dots Y_{M^2}(a_n, z_n) v_2), \quad (1.1)$$

where $v_3' \in M^3(0)^*$, $v \in M^1$, $v_2 \in M^2$, $a_1, \dots, a_n \in V$, and I is an intertwining operator of type $\binom{M^3}{M^1\ M^2}$, we introduce the notion of space of correlation functions associated with V-modules M^1, M^2, and M^3, denoted by $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$. It is essentially a quotient of the vector space of three-point genus zero conformal blocks, the dual space to a certain quotient of the tensor product of 3 admissible V-modules (see [11,12]). Then we prove that $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ is isomorphic to $I\binom{M^3}{M^1\ M^2}$.

In order to relate $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ with the modules over $A(V)$, we introduce an auxiliary notion of the space of correlation functions associated with M^1, $M^2(0)$, and $M^3(0)$, denoted by $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. This space can be viewed as the space $A(V)$-conformal blocks on the 3-pointed rational curve $\mathbb{P}^1_{\mathbb{C}}$ defined from the representations of Zhu's algebra

$A(V)$. The axioms we imposed on this space are based on the restriction of (1.1) onto the bottom levels $M^2(0)$ and $M^3(0)^*$. Then we use certain generating formulas satisfied by the correlation function (1.1) and prove that $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ is isomorphic to both $\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}$ and $\mathrm{Cor}\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))}$ when $M^2(0)$ and $M^3(0)$ are irreducible modules over $A(V)$. However, unlike building V-modules from $A(V)$-modules (see Theorem 2.2.1 in [13]) based on the ordinary correlation functions $(v', Y(a_1, z_1)\dots Y(a_n, z_n)v)$, in our case, due to the appearance of intertwining operator $I(v, w)$ in (1.1), the modules $\bar{M}^2$ and $\bar{M}^3$ constructed by (1.1) are not necessarily irreducible. This issue was first observed by Li in [8]. The V-modules $\bar{M}^2$ and $\bar{M}^3$ are quotient modules of certain generalized Verma modules. They can be proved to be irreducible if a technical condition depends only on the (bi)modules over $A(V)$ is satisfied.

We then prove that $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ is isomorphic to $(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$. Given a linear function f on $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)$, we shall use the recursive formulas satisfied by (1.1) and reconstruct a system of correlation functions in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. There is one recursive formula ((2.2.1) in [13]) of the correlation functions $S(v', (a_1, z_1)\dots(a_n, z_n)v) = (v', Y(a_1, z_1)\dots Y(a_n, z_n)v)$, where $v \in M(0)$ and $v' \in M(0)^*$, obtained by expanding the left-most term $Y(a_1, z_1)$. However, in our case, this formula alone is not enough to rebuild the correlation functions from f. The reason is again because of the appearance of $I(v, w)$ in the correlation functions, which makes expanding the left-most term (v, w) in $S(v_3', (v, w)(a_1, z_1)\dots(a_n, z_n)v_2)$ unreasonable, as the action $v(n)a_i = \mathrm{Res}_z w^{n+h} I(v, w)a_i$ is not yet defined. We remedy this situation by introducing an additional recursive formula for the correlation functions (1.1) obtained by expanding the right-most term $Y(a_n, z_n)$ in $(v_3', I(v, w)Y(a_1, z_1)\dots Y(a_n, z_n)v_2)$, where $v_3' \in M^3(0)^*$ and $v_2 \in M^2(0)$, and we use both the recursive formulas to reconstruct the correlation functions from f. Then Theorem 1.1 follows from the isomorphisms $I\binom{\bar{M}^3}{M^1\ \bar{M}^2} \cong \mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2} \cong \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$.

This paper is organized as follows: In Section 2, we define $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ and prove that it is isomorphic to $I\binom{M^3}{M^1\ M^2}$. In Section 3, we define $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ for irreducible $A(V)$-modules $M^2(0)$ and $M^3(0)$ and prove that $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ is isomorphic to both $\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}$ and $\mathrm{Cor}\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))}$. In section 4, we define the $A(V)$-bimodule $B_h(M^1)$ and prove that $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ is isomorphic to $(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$, which finishes the proof of Theorem 1.1. Then we verify this theorem on some particular examples, one of which shows that the counter-example given by Li in [8] does not contradict Theorem 1.1.

We expect the readers are familiar with the concept of vertex operator algebras, modules over vertex operator algebras, and the $A(V)$-theory, see [1,4,13].

2. The Space of Correlation Functions Associated with M^1, M^2, and M^3

We fix some notations that will be in force throughout this paper. We denote by $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ the set of complex numbers, the set of integers, and the set of natural numbers, including 0. All vector spaces are defined over $\mathbb{C}$.

Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (VOA) which is of the CFT-type: $V = \bigoplus_{n=0}^{\infty} V_n$, with $V_0 = \mathbb{C}\mathbf{1}$. A module M over V is an ordinary V-module: $M = \bigoplus_{n=0}^{\infty} M_{\lambda+n}$, where each $M_{\lambda+n}$ is an eigenspace of $L(0)$ with eigenvalue $\lambda+n$. Any V-module M is $\mathbb{N}$-gradable (or admissible): $M = \bigoplus_{n=0}^{\infty} M(n)$, with $M(n) = M_{\lambda+n}$ for each n. We write $Y_M(a,z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$, for all $a \in V$, and we write $Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$. One can find more details about the definitions in [2,4,5,13].

When we use the integral sign $\int_C f(z)dz$, where C is a simple closed contour of z, it means $\frac{1}{2\pi i}\int_C f(z)dz$.

2.1. The $(n+3)$*-Point Correlation Functions.* Let M^1, M^2, and M^3 be V-modules with conformal weights h_1, h_2, and h_3, respectively, and let $I \in I\binom{M^3}{M^1\ M^2}$ be an intertwining operator. Recall that $I(v,w) = \sum_{n\in\mathbb{Z}} v(n)w^{-n-1}\cdot w^{-h}$, where $h = h_1 + h_2 - h_3$, and $v(n) = \mathrm{Res}_w I(v,w)w^{n+h}$. Moreover, $v(n)M^2(m) \subseteq M^3(\deg v - n - 1 + m)$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, see [6] for more details. Consider the power series

$$\langle v_3', Y(a_1,z_1)\dots I(v,w)\dots Y(a_n,z_n)v_2\rangle w^h \tag{2.1}$$

in $n+1$ complex variables $z_1,\dots,z_n,w$ with integer powers, where $a_1,\dots,a_n \in V$, $v \in M^1$, $v_2 \in M^2$, and $v_3' \in M^{3'}$ which is the contragredient module of M^3 (cf. [4]). We multiply the term w^h to avoid the appearance of the logarithm when computing the integrations.

Recall that the power series (2.1) converges in the domain

$$\mathbb{D} = \{(z_1,\dots,z_n,w) \in \mathbb{C}^{n+1} | |z_1| > |z_2| > \cdots > |w| > \cdots > |z_n| > 0\}$$

to a rational function in $z_1, ..., z_n, w,\ z_i - z_j$ and $z_k - w$, where $1 \le i \ne j \le n$ and $1 \le k \le n$. We denote this rational function by:

$$(v_3', Y(a_1,z_1)\dots I(v,w)\dots Y(a_n,z_n)v_2), \tag{2.2}$$

also recall that the only possible poles of (2.2) are at $z_i = 0,\ w = 0,\ z_i = z_j$ and $z_k = w$, see [4] for more details.

Moreover, it is also essentially proved in [4] that the rational function (2.2) is invariant under the permutation of the terms $Y(a_1,z_1),\dots,Y(a_n,z_n)$, and $I(v,w)$. In other words, the power series (2.1) and the power series $\langle v_3', Y(a_{i_1},z_{i_1})\dots I(v,w)\dots Y(a_{i_n},z_{i_n})v_2\rangle w^h$ have the same limit function (2.2) on their corresponding domain of convergence.

We use the symbol S_I as in [13] to denote the limit function (2.2):

$$\begin{aligned} &S_I(v_3', (a_1,z_1)\dots(v,w)\dots(a_n,z_n)v_2) :\\ &= (v_3', Y(a_1,z_1)\dots I(v,w)\dots Y(a_n,z_n)v_2). \end{aligned} \tag{2.3}$$

Then we have a system of linear maps $S_I = \{(S_I)^n_{V\dots M^1\dots V}\}_{n=0}^{\infty}$:

$$\begin{aligned} &(S_I)^n_{V\dots M^1\dots V} : M^{3'} \times V \times \dots \times M^1 \times \dots V \times M^2 \to \mathcal{F}(z_1,\dots,z_n,w),\\ &(v_3', a_1,\dots,v,\dots,a_n,v_2) \mapsto S_I(v_3', (a_1,z_1)\dots(v,w)\dots(a_n,z_n)v_2), \end{aligned} \tag{2.4}$$

where $\mathcal{F}(z_1,\dots,z_n,w)$ is the space of rational functions in $n+1$ variables $z_1,z_2,\dots,z_n$, w, with only possible poles at $z_i = 0,\ w = 0,\ z_i = z_j,\ z_k = w$. For a fixed $n \in \mathbb{N}$, we have $(S_I)^n_{M^1V\dots V} = (S_I)^n_{VM^1\dots V} = \cdots = (S_I)^n_{V\dots VM^1}$, since the terms $(a_1,z_1),\dots,(a_n,z_n)$, and (v,w) can be permuted within S_I in (2.3).

We introduce the following notion that generalizes Definition 4.1.1 in [13]:

Definition 2.1. A system of linear maps $S = \{S^n_{V\ldots M^1\ldots V}\}_{n=0}^{\infty}$,

$$S^n_{V\ldots M^1\ldots V} : M^{3'} \times V \times \ldots \times M^1 \times \ldots V \times M^2 \to \mathcal{F}(z_1, \ldots, z_n, w),$$
$$(v_3', a_1, \ldots, v, \ldots, a_n, v_2) \mapsto S(v_3', (a_1, z_1) \ldots (v, w) \ldots (a_n, z_n)v_2),$$

is said to satisfy the *genus-zero property associated with* M^1, M^2, *and* M^3 if it satisfies

(1) (The truncation property) For fixed $v \in M^1$ and $v_2 \in M^2$, the Laurent series expansion of $S(v_3', (v, w)v_2)$ around $w = 0$ has a uniform lower bound for w independent of $v_3' \in M^{3'}$. i.e., $S(v_3', (v, w)v_2) = \sum_{n \le N} a_n w^{-n-1}$ for all $v_3' \in M^{3'}$.
(2) (The locality) The terms $(a_1, z_1), \ldots, (a_n, z_n)$, and (v, w) can be permuted arbitrarily within S. i.e., $S^n_{M^1 V\ldots V} = S^n_{V M^1\ldots V} = \cdots = S^n_{V\ldots V M^1}$ for any fixed $n \in \mathbb{N}$.
(3) (The vacuum property)

$$\begin{aligned} &S(v_3', (\mathbf{1}, z)(a_1, z_1) \ldots (v, w) \ldots (a_n, z_n)v_2) \\ &\quad = S(v_3', (a_1, z_1) \ldots (v, w) \ldots (a_n, z_n)v_2). \end{aligned} \tag{2.5}$$

(4) (The $L(-1)$-derivation property)

$$\begin{aligned} &S(v_3', (L(-1)a_1, z_1) \ldots (a_n, z_n)(v, w)v_2) \\ &\quad = \frac{d}{dz_1} S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)v_2), \\ &S(v_3', (L(-1)v, w)(a_1, z_1) \ldots v_2)w^{-h} \\ &\quad = \frac{d}{dw}\left(S(v_3', (v, w)(a_1, z_1) \ldots v_2)w^{-h}\right). \end{aligned} \tag{2.6}$$

(5) (The associativity)

$$\begin{aligned} &\int_C S(v_3', (a_1, z_1)(v, w) \ldots (a_n, z_n)v_2)(z_1 - w)^k dz_1 \\ &= S(v_3', (a_1(k)v, w) \ldots (a_n, z_n)v_2), \\ &\int_C S(v_3', (a_1, z_1)(a_2, z_2) \ldots (v, w)v_2)(z_1 - z_2)^k dz_1 \\ &= S(v_3', (a_1(k)a_2, z_2) \ldots (v, w)v_2), \end{aligned} \tag{2.7}$$

where in the first equation of (2.7), C is a contour of z_1 surrounding w, with z_2, ..., z_n outside of C; while in the second equation of (2.7), C is a contour of z_1 surrounding z_2, with z_3, ..., z_n, w outside of C.
(6) (The Virasoro relation) Let $\omega \in V$ be the Virasoro element, and let $x, x_1, \ldots, x_m$ be complex variables, denote the rational function

$$S(v_3', (\omega, x_1) \ldots (\omega, x_m)(a_1, z_1) \ldots (v, w) \ldots (a_n, z_n)v_2)$$

by S for simplicity. Assume that $v_3', v, v_2, a_1, \ldots, a_n$ are highest weight vectors for the Virasoro algebra, then we have:

$$\begin{aligned} &S(v_3', (\omega, x)(\omega, x_1) \ldots (\omega, x_m)(a_1, z_1) \ldots (v, w) \ldots (a_n, z_n)v_2) \\ &= \sum_{k=1}^{n} \frac{x^{-1}z_k}{x - z_k} \frac{d}{dz_k} S + \sum_{k=1}^{n} \frac{\mathrm{wt} a_k}{(x - z_k)^2} S + \frac{x^{-1}w}{x - w} w^h \frac{d}{dw}(S \cdot w^{-h}) \end{aligned}$$

$$+\frac{\mathrm{wt}v}{(x-w)^2}S+\frac{\mathrm{wt}v_2}{x^2}S+\sum_{k=1}^{m}\frac{x^{-1}w_k}{x-x_k}\frac{d}{dx_k}S+\sum_{k=1}^{m}\frac{2}{(x-x_k)^2}S$$
$$+\frac{c}{2}\sum_{k=1}^{m}\frac{1}{(x-x_k)^4}S(v_3',(\omega,x_1)\ldots$$
$$\widehat{(\omega,x_k)}\ldots(\omega,x_m)(a_1,z_1)\ldots(v,w)\ldots(a_n,z_n)v_2) \tag{2.8}$$

(7) (The generating property for M^2) For any $a\in V$ and $m\in\mathbb{Z}$, we have:

$$\begin{aligned}&S(v_3',(a_1,z_1)\ldots(v,w)\ldots(a_n,z_n)a(m)v_2)\\&\quad=\int_C S(v_3',(a_1,z_1)\ldots(v,w)\ldots(a_n,z_n)(a,z)v_2)z^m dz,\end{aligned} \tag{2.9}$$

where $C=C_R(0)$ is a contour of z surrounding 0 with $z_1,\ldots,z_n,w$ lying outside.

(8) (The generating property for $M^{3'}$) Denote $(e^{z^{-1}L(1)}(-z^2)^{L(0)}a,z)$ by $(a,z)'$, then

$$\begin{aligned}&S(a(m)v_3',(a_1,z_1)\ldots(v,w)\ldots(a_n,z_n)v_2)\\&\quad=\int_{C'} S(v_3',(a,z)'(a_1,z_1)\ldots(v,w)\ldots(a_n,z_n)v_2)z^{-m-2}dz,\end{aligned} \tag{2.10}$$

where $C'=C_r(0)$ is a contour of z surrounding 0 with $z_1,\ldots,z_n,w$ lying inside.

Definition 2.2. The vector space of the system of linear maps $S=\{S^n_{V\ldots M^1\ldots V}\}_{n=0}^{\infty}$ satisfying the genus-zero property associated with M^1, M^2, and M^3 is called the *space of correlation functions associated with* M^1, M^2, and M^3. We denote it by $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$.

Proposition 2.3. *The system of functions S_I given by* (2.3) *and* (2.4) *satisfies the genus-zero property associated with M^1, M^2, and M^3 in Definition 2.1. Thus $S_I\in\mathrm{Cor}\binom{M^3}{M^1\ M^2}$.*

Proof. The properties (1) - (6) for S_I follow immediately from the axioms satisfied by the intertwining operator I and the vertex operator Y; see Section 5.6 in [4] for more details.

To prove (2.9), we note that the Laurent series expansion of the rational function (2.3) on the domain $|z|<|z_i|,|w|$ for all i is $\sum_{m\in\mathbb{Z}}(v_3',Y(a_1,z_1)\ldots I(v,w)\ldots a(m)v_2)z^{-m-1}$. The coefficient of z^{-m-1} in the Laurent series is also

$$\int_C(v_3',Y(a_1,z_1)\ldots I(v,w)\ldots Y(a_n,z_n)Y(a,z)v_2)z^m dz,$$

where $C=C_R(0)$ is a contour of z surrounding 0 with $z_1,\ldots,z_n$ and w lying outside. This proves (2.9). To prove (2.10), we denote the term $\sum_{j\geq 0}\frac{1}{j!}(-1)^{\mathrm{wt}a}(L(1)a^j)(2\mathrm{wt}a-m-j-2)$ by $a'(m)$, then by the definition of contragredient module (see (5.2.4) in [4]), the series

$$\begin{aligned}&\sum_{m\in\mathbb{Z}}(a(m)v_3',Y(a_1,z_1)\ldots I(v,w)\ldots Y(a_n,z_n)Y(a,z)v_2)z^{-m-1}\\&\quad=\sum_{m\in\mathbb{Z}}(v_3',a'(m)Y(a_1,z_1)\ldots I(v,w)\ldots Y(a_n,z_n)v_2)z^{-m-1}\end{aligned}$$

is the expansion of $(v_3', Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)v_2)$ on the domain $|z^{-1}| > |z_i|, |w|$, or equivalently, $|z| < 1/|z_i|, 1/|w|$, for $i = 1, \dots, n$. By comparing the Laurent coefficient of z^{-m-1}, we have:

$$\begin{aligned}&(a(m)v_3', Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)Y(a, z)v_2)\\&\quad = \int_{C_R(0)} (v_3', Y(e^{zL(1)}(-z^{-2})^{L(0)}a, z^{-1})Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)v_2)z^m dz,\end{aligned} \tag{2.11}$$

where R is small enough such that $R < 1/|z_i|, 1/|w|$, for $i = 1, \dots, n$. Change the variable $z \to 1/z$ in the integral (2.11). Note that the parametrization of $1/z$ is $(1/R)e^{-i\theta}$, which gives us a clockwise orientation, and $d(1/z) = -(1/z^2)dz$. Let $C' = C_r(0)$, with radius $r = 1/R > |z_i|, |w|$ for $i = 1, \dots, n$, equipped with the counterclockwise orientation. Then $z_1, \dots, z_n, w$ are inside of C', and

$$\begin{aligned}(2.11) &= -\int_{C'} (v_3', Y(e^{z^{-1}L(1)}(-z^2)^{L(0)}a, z)Y(a_1, z_1)\dots I(v, w)\\&\quad \dots Y(a_n, z_n)v_2)z^{-m}(-z^{-2})dz\\&= \int_{C'} (v_3', Y(e^{z^{-1}L(1)}(-z^2)^{L(0)}a, z)Y(a_1, z_1)\dots I(v, w)\dots Y(a_n, z_n)v_2)z^{-m-2}dz\\&= \int_{C'} S_I(v_3', (a, z)'(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2)z^{-m-2}dz.\end{aligned}$$

This proves (2.10). □

Remark 2.4. Let $S \in \mathrm{Cor}\binom{M^3}{M^1\ M^2}$. With the notations of Proposition 2.3, we have:

$$\begin{aligned}&S(a'(m)v_3', (a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2)\\&\quad = \sum_{j\geq 0}\frac{1}{j!}(-1)^{\mathrm{wt}a}\int_{C'} S(v_3', (e^{z^{-1}L(1)}(-z^2)^{L(0)}(L(1)^j a), z)(a_1, z_1)\dots v_2)z^{-2\mathrm{wt}a+m+j}dz\\&\quad = \int_{C'} S(v_3', (e^{z^{-1}L(1)}(-z^2)^{L(0)}e^{zL(1)}(-z^{-2})^{L(0)}a, z)(a_1, z_1)\dots v_2)z^m dz\\&\quad = \int_{C'} S(v_3', (e^{z^{-1}L(1)}e^{-z^{-1}L(1)}a, z)(a_1, z_1)\dots v_2)z^m dz\\&\quad = \int_{C'} S(v_3', (a, z)(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2)z^m dz.\end{aligned}$$

Hence the generating property for $M^{3'}$ (8) in Definition 2.1 is equivalent to:

$$\begin{aligned}&S(a'(m)v_3', (a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2)\\&\quad = \int_{C'} S(v_3', (a, z)(a_1, z_1)\dots(v, w)\dots(a_n, z_n)v_2)z^m dz,\end{aligned} \tag{2.12}$$

where $a'(m) = \sum_{j\geq 0}\frac{1}{j!}(-1)^{\mathrm{wt}a}(L(1)^j a)(2\mathrm{wt}a - m - j - 2)$ and $C' = C_r(0)$ as in (8).

As a consequence of Proposition 2.3, we have a well-defined linear map:

$$\alpha : I\begin{pmatrix} M^3 \\ M^1\ M^2\end{pmatrix} \to \mathrm{Cor}\begin{pmatrix} M^3 \\ M^1\ M^2\end{pmatrix}, \qquad I \mapsto S_I, \tag{2.13}$$

where S_I is given by (2.3) and (2.4).

2.2. The Space of Correlation Functions and the Space of Intertwining Operators. Although the genus-zero property associated with three V-modules in Definition 2.1 seems long and intrinsic, it is good enough to characterize an intertwining operator. In other words, we can construct an inverse of the map α in (2.13).

Fix a system of correlation functions S in $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$, we construct an intertwining operator $I_S \in I\binom{M^3}{M^1\ M^2}$ in the following way:

Let $v \in M^1$, define a linear map $v(n) : M^2 \to M^3$ by the formula:

$$\langle v_3', v(n)v_2\rangle := \int_C S(v_3', (v, w)v_2)w^n dw, \tag{2.14}$$

where C is a contour of w surrounding 0. Note that an element $u \in M^3$ is uniquely determined by the value $\langle v_3', u\rangle$ for $v_3' \in M^{3'}$, so we have a well-defined element $v(n)v_2$ in M^3. Define $I(v, w)$ by

$$I_S(v, w) := \sum_{n\in\mathbb{Z}} v(n)w^{-n-1} \cdot w^{-h}, \tag{2.15}$$

where $h = h_1 + h_2 - h_3$. Then $I(v, w) \in \mathrm{Hom}(M^2, M^3)\{z\}$.

Theorem 2.5. *The series $I_S(v, w)$ defined by (2.14) and (2.15) is an intertwining operator of type $\binom{M^3}{M^1\ M^2}$.*

Proof. By Definition 2.1, $S(v_3', (v, w)v_2)$ is a rational function in w with the only possible pole at $w = 0$, and the term (2.14) is the Laurent coefficient of $S(v_3', (v, w)v_2)$. Thus the series $\langle x_3', I_S(v, w)x_2\rangle w^h$ is the Laurent series expansion of $S(x_3', (v, w)x_2)$ around $w = 0$ by (2.15). In particular, if we denote the limit of the Laurent series $\langle v_3', I(v, w)v_2\rangle w^h$ by $(v_3', I(v, w)v_2)$, then we have the following equality of rational functions:

$$(v_3', I_S(v, w)v_2) = S(v_3', (v, w)v_2) \tag{2.16}$$

Since S satisfies the property (1) in Definition 2.1, for $v \in M^1$ and $v_2 \in M^2$, there exists $N \in \mathbb{Z}$ such that $\langle v_3', I_S(v, w)v_2\rangle w^h = \sum_{n\le N} \left(\int_C S(v_3', (v, w)v_2)w^n dw\right) w^{-n-1}$, for all $v_3' \in M^{3'}$. Hence we have $v(n)v_2 = 0$ for $n \gg 0$. By the locality of S, together with (2.15), we have:

$$\langle v_3', I_S(L(-1)v, w)v_2\rangle = \frac{d}{dw}(S(v_3', (v, w)v_2)w^{-h}) = \frac{d}{dw}\langle v_3', I_S(v, w)v_2\rangle.$$

Hence $I_S(L(-1)v, w) = \frac{d}{dw} I_S(v, w)$. Moreover, we claim that the following equation holds:

$$\begin{aligned}\sum_{i=0}^{\infty} &\binom{m}{i}(a(l+i)v)(m+n-i)v_2\\ &= \sum_{i=0}^{\infty}(-1)^i\binom{l}{i}a(m+l-i)v(n+i)v_2 - \sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}v(n+l-i)a(m+i)v_2,\end{aligned} \tag{2.17}$$

for all $m, n, l \in \mathbb{Z}$, $a \in V$, $v \in M^1$, and $v_2 \in M^2$. Note that (2.17) is the component form of the Jacobi identity for the intertwining operator I_S (see (1.2.9) in [13]).

Indeed, by (2.14) and the generating property (2.12) of S, we have:

$$\begin{aligned}
&\langle v_3', \sum_{i=0}^{\infty}(-1)^i\binom{l}{i}a(m+l-i)v(n+i)v_2\rangle\\
&=\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}\int_{C_1'} S(a'(m+l-i)v_3', (v,w)v_2)w^{n+i}dw\\
&=\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}\int_{C_1'}\int_{C_2'} S(v_3', (a,z)(v,w)v_2)z^{m+l-i}w^{n+i}dw,
\end{aligned}\tag{2.18}$$

where C_1' is a contour of w, and C_2' is a contour of z which contains C_1'. On the other hand, by (2.14) and the generating property (2.9) of S, we have:

$$\begin{aligned}
&\langle v_3', \sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}v(n+l-i)a(m+i)v_2\rangle\\
&=\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}\int_{C_1} S(v_3', (v,w)a(m+i)v_2)w^{n+l-i}dw\\
&=\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}\int_{C_1}\int_{C_2} S(v_3', (v,w)(a,z)v_2)z^{m+i}w^{n+l-i}dzdw,
\end{aligned}\tag{2.19}$$

where C_1 and C_2 are contours in w and z, respectively, and C_2 is contained in C_1.

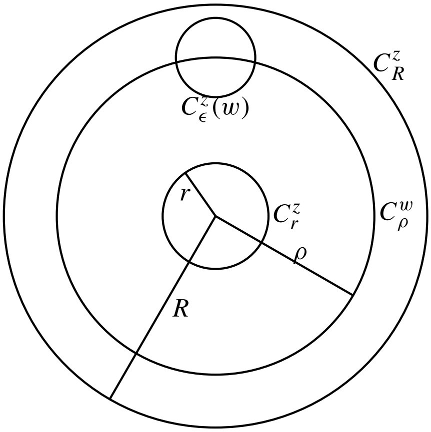
We adopt the notations in Proposition A.2.8 in [5]. Choose the contours C_1, C_2, C_1', and C_2' in the following way: Let C_α^z be a circle in the variable z centered at 0, with radius α, and $C_\epsilon^1(w_2)$ be the circle of w_1 centered at w_2 with radius ϵ. We may choose ϵ small enough so that $|w_1 - w_2| < |w_2|$ for any w_1 lying on $C_\epsilon^1(w_2)$. Choose $R, r, \rho > 0$ so that $1 > R > \rho > r$. Let $C_1' = C_\rho^w$, $C_2' = C_R^z$, $C_1 = C_\rho^w$, and $C_2 = C_r^z$. Then by (2.14), (2.18), and (2.19), together with (2) and (5) in Definition 2.1, we have:

$$\begin{aligned}
&\langle v_3', \sum_{i=0}^{\infty}(-1)^i\binom{l}{i}a(m+l-i)v(n+i)v_2 - \sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}v(n+l-i)a(m+i)v_2\rangle\\
&=\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}\int_{C_\rho^w}\int_{C_R^z} S(v_3', (a,z)(v,w)v_2)z^{m+l-i}w^{n+i}dwdz\\
&-\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}\int_{C_\rho^w}\int_{C_r^z} S(v_3', (v,w)(a,z)v_2)z^{m+i}w^{n+l-i}dzdw\\
&=\int_{C_\rho^w}\int_{C_R^z} S(v_3', (a,z)(v,w)v_2)\iota_{z,w}(z-w)^l z^m w^n dwdz\\
&-\int_{C_\rho^w}\int_{C_r^z} S(v_3', (v,w)(a,z)v_2)\iota_{w,z}(z-w)^l z^m w^n dzdw\\
&=\int_{C_\rho^w}\int_{C_\epsilon^z(w)} S(v_3', (a,z)(v,w)v_2)(z-w)^l z^m w^n dzdw
\end{aligned}$$

$$
\begin{aligned}
&= \int_{C_\rho^w} \int_{C_\epsilon^z(w)} S(v_3', (a,z)(v,w)v_2)(z-w)^l \iota_{w,z-w}(w+(z-w))^m w^n dz dw \\
&= \sum_{i\geq 0} \binom{m}{i} \int_{C_\rho^w} \int_{C_\epsilon^z(w)} S(v_3', (a,z)(v,w)v_2)(z-w)^{l+i} w^{n+m-i} dz dw \\
&= \sum_{i\geq 0} \binom{m}{i} \int_{C_\rho^w} S(v_3', (a(l+i)v, w)v_2) w^{m+n-i} \\
&= \sum_{i\geq 0} \binom{m}{i} \langle v_3', (a(l+i)v)(m+n-i)v_2 \rangle. \qquad (2.20)
\end{aligned}
$$

□

The graph of the contours appear in (2.20) can be sketched as follows:



Since v_3' in (2.20) can be choosen arbitraily, the Jacobi identity (2.17) follows, and so I_S given by (2.15) is an intertwining operator of type $\binom{M^3}{M^1\ M^2}$. □

Corollary 2.6. *The vector space of intertwining operators* $I\binom{M^3}{M^1\ M^2}$ *is isomorphic to the vector space* $\mathrm{Cor}\binom{M^3}{M^1\ M^2}$ *in Definition 2.2.*

Proof. Theorem 2.5 indicates that there exists a well-defined linear map:

$$
\beta : \mathrm{Cor}\begin{pmatrix} M^3 \\ M^1\ M^2 \end{pmatrix} \to I\begin{pmatrix} M^3 \\ M^1\ M^2 \end{pmatrix}, \quad S \mapsto I_S. \qquad (2.21)
$$

By (2.3) and (2.16), it is clear that β is an inverse of the linear map α in (2.13). Hence $I\binom{M^3}{M^1\ M^2} \cong \mathrm{Cor}\binom{M^3}{M^1\ M^2}$ as vector spaces. □

3. Extension of Correlation Functions from the Bottom Levels

Let M^2 and M^3 be any V-modules with bottom levels $M^2(0)$ and $M^3(0)$, respectively.

Recall the bottom level $M(0)$ of any $\mathbb{N}$-gradable V-module $M = \bigoplus_{n=0}^\infty M(n)$ is a module over the Zhu's algebra $A(V)$ defined in [13] or the generalized Zhu's algebra $A_n(V)$ defined in [3] under the module action:

$$
[a].v = o(a)v = a(\mathrm{wt} a - 1)v,
$$

for all $[a] \in A(V)$ or $A_n(V)$, and $v \in M(0)$ (see Theorem 2.1.2 in [13]).

In this section, we assume that the $A(V)$-modules $M^2(0)$ and $M^3(0)$ are irreducible.

3.1. The Space of Correlation Functions Associated with M^1, $M^2(0)$, and $M^3(0)$. Let $S \in \mathrm{Cor}\binom{M^3}{M^1\, M^2}$, and let $I \in I\binom{M^3}{M^1\, M^2}$ be its corresponding intertwining operator under the isomorphism β in (2.21). For each $n \in \mathbb{N}$, consider the restriction of S onto the bottom levels $M^2(0)$ and $M^3(0)^*$:

$$S|_{M^3(0)^*\times \ldots M^1 \ldots \times M^2(0)} : M^3(0)^* \times V \times \cdots \times M^1 \cdots \times V \times M^2(0) \to \mathcal{F}(z_1, \ldots, z_n, w). \tag{3.1}$$

To simplify our notation, we use the same symbol S to denote the restricted function (3.1). Clearly, S in (3.1) satisfies properties (1)-(6) in Definition 2.1, with the elements v_3' and v_2 in these properties belong to $M^3(0)^*$ and $M^2(0)$, respectively. Moreover, since $(v_3', I(v, w)v_2) = S(v_3', (v, w)v_2)$ by (2.16), and $v(n)M^2(m) \subseteq M^3(m+\deg v - n - 1)$ for all $v \in M^1$ homogeneous, $n \in \mathbb{Z}$, and $m \in \mathbb{N}$ (see (1.5.4) in [6]), then we have:

$$S(v_3', (v, w)v_2) = \langle v_3', v(\deg v - 1)(v_2)\rangle w^{-\deg v}. \tag{3.2}$$

We introduce the following intermediate notion based on the properties satisfied by the system of restricted correlation functions (3.1).

Definition 3.1. Let $M^2(0)$ and $M^3(0)$ be irreducible $A(V)$-modules. A system of linear maps $S = \{S^n_{V\ldots M^1\ldots V}\}_{n=0}^{\infty}$,

$$\begin{aligned} S^n_{V\ldots M^1\ldots V} &: M^3(0)^* \times V \times \ldots \times M^1 \times \ldots V \times M^2(0) \to \mathcal{F}(z_1, \ldots, z_n, w),\\ &(v_3', a_1, \ldots, v, \ldots, a_n, v_2) \mapsto S(v_3', (a_1, z_1) \ldots (v, w) \ldots (a_n, z_n)v_2), \end{aligned}$$

is said to satisfy the genus-zero property associated with M^1, $M^2(0)$, and $M^3(0)$ if it satisfies the following:

(1) Properties $(2) - (6)$ in Definition 2.1, with the elements v_3' and v_2 in these properties belong to $M^3(0)^*$ and $M^2(0)$, respectively.
(2) There exists a linear functional $f : M^1 \to \mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0))$, $v \mapsto f_v$, such that
$$S(v_3', (v, w)v_2) = \langle v_3', f_v(v_2)\rangle w^{-\deg v}, \tag{3.3}$$
for all $v_2 \in M^2(0)$ and $v_3' \in M^3(0)^*$.
(3) (The recursive formula for $M^3(0)^*$) For any $v_3' \in M^3(0)^*$, $v \in M^1$, $v_2 \in M^2(0)$, and $a_1, \ldots, a_n \in V$,

$$\begin{aligned} S(v_3', (a, z)(a_1, z_1) \ldots (a_n, z_n)(v, w)v_2) &= S(v_3' o(a), (a_1, z_1) \ldots (a_n, z_n)(v, w)v_2) z^{-\mathrm{wt} a}\\ &+ \sum_{k=1}^{n} \sum_{i \geq 0} F_{\mathrm{wt} a, i}(z, z_k) S(v_3', (a_1, z_1) \ldots (a(i)a_k, z_k) \ldots (a_n, z_n)(v, w)v_2)\\ &+ \sum_{i \geq 0} F_{\mathrm{wt} a, i}(z, w) S(v_3', (a_1, z_1) \ldots (a_n, z_n)(a(i)v, w)v_2), \end{aligned} \tag{3.4}$$

where $F_{\mathrm{wt}a,i}(z,w)$ is a rational function in z, w given by:

$$\iota_{z,w}(F_{\mathrm{wt}a,i}(z,w)) = \sum_{j\geq 0}\binom{\mathrm{wt}a+j}{i} z^{-\mathrm{wt}a-j-1}w^{\mathrm{wt}a+j-i},$$
$$F_{m,i}(z,w) = \frac{z^{-m}}{i!}\left(\frac{d}{dw}\right)^{i}\frac{w^m}{z-w}, \tag{3.5}$$

for any $m \in \mathbb{N}$, and $v_3'o(a)$ is given by the natural right module action on $M^3(0)^*$.

(4) (The recursive formula for $M^2(0)$) For any $v_3' \in M^3(0)^*$, $v \in M^1$, $v_2 \in M^2(0)$, and $a_1, \ldots, a_n \in V$, we have:

$$S(v_3', (a_1, z_1)\ldots(a_n, z_n)(v, w)(a, z)v_2) = S(v_3', (a_1, z_1)\ldots(a_n, z_n)(v, w)o(a)v_2)z^{-\mathrm{wt}a}$$
$$+\sum_{k=1}^{n}\sum_{i\geq 0} G_{\mathrm{wt}a,i}(z,w)S(v_3', (a_1, z_1)\ldots(a(i)a_k, z_k)...(a_n, z_n)(v, w)v_2)$$
$$+\sum_{i\geq 0} G_{\mathrm{wt}a,i}(z,w)S(v_3', (a_1, z_1)\ldots(a_n, z_n)(a(i)v, w)(a, z)v_2), \tag{3.6}$$

where $G_{\mathrm{wt}a,i}(z,w)$ is a rational function defined by

$$\iota_{w,z}(G_{\mathrm{wt}a,i}(z,w)) = -\sum_{j\geq 0}\binom{\mathrm{wt}a-2-j}{i} w^{\mathrm{wt}a-j-2-i}z^{-\mathrm{wt}a+1+j},$$
$$G_{m,i}(z,w) = \frac{z^{-m+1}}{i!}\left(\frac{d}{dw}\right)^{i}\left(\frac{w^{m-1}}{z-w}\right), \tag{3.7}$$

for any $m \in \mathbb{N}$.

The vector space of the system of functions satisfying the genus-zero property associated with M^1, $M^2(0)$, and $M^3(0)$ is denoted by $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

We observe that the rational functions F and G given by (3.5) and (3.7) satisfy the following relation:

$$F_{m,i}(z,w) - G_{m,i}(z,w) = \frac{z^{-m}}{i!}\left(\frac{d}{dw}\right)^{i}\left(\frac{w^m}{z-w} - \frac{zw^{m-1}}{z-w}\right)$$
$$= -\binom{m-1}{i}z^{-m}w^{m-1-i},$$

for all $m \in \mathbb{N}$. In particular, we have

$$F_{\mathrm{wt}a,i}(z,w) - G_{\mathrm{wt}a,i}(z,w) = -\binom{\mathrm{wt}a-1}{i}z^{-\mathrm{wt}a}w^{\mathrm{wt}a-1-i}. \tag{3.8}$$

The equation (3.8) will be used multiple times in Section 4 when we build a system of correlation functions S from a linear map on a tensor product of $A(V)$-modules.

Proposition 3.2. *Let* $S \in \mathrm{Cor}\binom{M^3}{M^1\ M^2}$*. Then the system of restricted functions* S *in* (3.1) *satisfies the genus-zero property associated with* M^1*,* $M^2(0)$*, and* $M^3(0)$*.*

Proof. By our discussion in the begining of this subsection, S in (3.1) satisfies (1) and (2) in Definition 3.1, where the f_v in (3.3) is given by $f_v = v(\deg v - 1)$, for all $v \in M^1$. The proof of (3.4) is similar to the proof of Lemma 2.2.1 in [13]. We omit the details. To prove (3.6), we only consider the case when $n = 0$ (the general case follows from a similar argument.) Note that $a(n)v_2 = 0$ if $\mathrm{wt}a - n - 1 < 0$, it follows that $\langle v_3', I(v, w)Y(a, z)v_2\rangle = \langle v_3', I(v, w)o(a)v_2\rangle z^{-\mathrm{wt}a} + \sum_{\mathrm{wt}a-n-1>0}\langle v_3', I(v, w)a(n)v_2\rangle z^{-n-1}$. By the definition of contragredient modules, we have $\langle v_3', a(n)u\rangle = \sum_{i\geq 0}\frac{1}{i!}(-1)^i\langle (L(i)a)(2\mathrm{wt}a - n - i - 2)v_3', u\rangle$, for any $n \in \mathbb{Z}$. But $(L(i)a)(2\mathrm{wt}a - n - i - 2)v_3' \in M^{3'}(-\mathrm{wt}a + n + 1) = 0$ when $\mathrm{wt}a - n - 1 > 0$. Thus

$$
\begin{aligned}
&\sum_{wta-n-1>0} \langle v_3', I(v, w)a(n)v_2\rangle z^{-n-1}\\
&= - \sum_{\mathrm{wt}a-n-1>0} \langle v_3', [a(n), I(v, w)]v_2\rangle z^{-n-1}\\
&= - \sum_{\mathrm{wt}a-n-1>0}\ \sum_{i\geq 0} \binom{n}{i}\langle v_3', I(a(i)v, w)v_2\rangle z^{-n-1}w^{n-i}\\
&= - \sum_{j\geq 0}\sum_{i\geq 0}\binom{\mathrm{wt}a - j - 2}{i} z^{-\mathrm{wt}a+j+2-1} w^{\mathrm{wt}a-j-2-i}\langle v_3', I(a(i)v, w)v_2\rangle\\
&= \sum_{i\geq 0} \iota_{w,z}(G_{\mathrm{wt}a,i}(z, w))\langle v_3', I(a(i)v, w)v_2\rangle,
\end{aligned}
$$

where the last equality follows from (3.7). Hence we have:

$$
\begin{aligned}
\langle v_3', I(v, w)Y(a, z)\rangle &= \langle v_3', I(v, w)o(a)v_2\rangle z^{-\mathrm{wt}a}\\
&\quad + \sum_{i\geq 0} \iota_{w,z}(G_{\mathrm{wt}a,i}(z, w))\langle v_3', I(a(i)v, w)v_2\rangle
\end{aligned}
$$

as power series. By taking the limit of this series, we obtain (3.6) for $n = 0$. □

As a consequence of Proposition 3.2, we have a well-defined restriction map:

$$
\varphi : \mathrm{Cor}\begin{pmatrix} M^3 \\ M^1\ M^2\end{pmatrix} \to \mathrm{Cor}\begin{pmatrix} M^3(0) \\ M^1\ M^2(0)\end{pmatrix}, \quad S \mapsto S|_{M^3(0)^*\times\ldots M^1\ldots\times M^2(0)}, \tag{3.9}
$$

where M^2 and M^3 are any V-modules with bottom levels $M^2(0)$ and $M^3(0)$,

The following Lemma will be used in the next subsection:

Lemma 3.3. *Let* $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$*, and let* $f : M^1 \to \mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0))$*,* $v \mapsto f_v$ *be the linear functional in Definition 3.1. Suppose that* $f_v = 0$ *for all* $v \in M^1$*. Then* $S = 0$*.*

Proof. We use induction on n to show that $S(v_3', (a_1, z_1)\ldots(a_n, z_n)(v, w)v_2) = 0$ for all $v_3' \in M^3(0)^*$, $v \in M^1$, $v_2 \in M^2(0)$, and $a_1, \ldots, a_n \in V$. When $n = 0$, by the assumption and (3.3), we have: $S(v_3', (v, w)v_2) = \langle v_3', f_v(v_2)\rangle w^{-\deg v} = \langle v_3', 0\rangle w^{-\deg w} = 0$, for all $v_3' \in M^3(0)^*$, $v \in M^1$, and $v_2 \in M^2(0)$. For $n > 0$, by the recursive formula (3.4), we have

$$
\begin{aligned}
&S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)v_2) = S(v_3' o(a_1), (a_2, z_2) \dots (a_n, z_n)(v, w))z^{-\mathrm{wt} a_1} \\
&\quad + \sum_{k=2}^{n} \sum_{i \geq 0} F_{\mathrm{wt} a_1, i}(z_1, z_k) S(v_3', (a_2, z_2) \dots (a_1(i)a_k, z_k) \dots (a_n, z_n)(v, w)v_2) \\
&\quad + \sum_{i \geq 0} F_{\mathrm{wt} a_1, i}(z_1, w) S(v_3', (a_2, z_2) \dots (a_n, z_n)(a_1(i)v, w)v_2).
\end{aligned}
$$

Since each term on the right-hand side has a smaller length, the right-hand side is equal to 0 by the induction hypothesis, so we have $S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)v_2) = 0$. □

3.2. Extension from the Bottom Levels. In this subsection, we will show that the restriction map φ in (3.9) has an inverse for certain V-modules M^2 and M^3, with (irreducible) bottom levels $M^2(0)$ and $M^3(0)$, respectively.

Recall that for any irreducible $A(V)$-module U, Dong, Li, and Mason constructed a generalized Verma module $\bar{M}(U)$ in [2]. By construction, $\bar{M}(U) = (U(\mathcal{L}(V)) \otimes_{U(\mathcal{L}(V)_{\geq 0})} U)/U(\mathcal{L}(V))W$, where

$$
\mathcal{L}(V) = V \otimes \mathbb{C}[t, t^{-1}]/(L(-1) \otimes 1 + 1 \otimes \frac{d}{dt})(V \otimes \mathbb{C}[t, t^{-1}]) \tag{3.10}
$$

is the Lie algebra associated with the VOA V (cf.[1,2]), and W is the subspace of $U(\mathcal{L}(V)) \otimes_{U(\mathcal{L}(V)_{\geq 0})} U$ spanned by the coefficients of the weak associativity equality, see Section 5 in [2] for more details.

$\bar{M}(U)$ is $\mathbb{N}$-gradable: $\bar{M}(U) = \bigoplus_{n=0}^{\infty} \bar{M}(n)$, with the bottom level $\bar{M}(U)(0) = U$. It satisfies a universal property in the sense that any $\mathbb{N}$-gradable V-module with bottom level U is a quotient module of $\bar{M}(U)$ (Theorem 6.2 in [2]). Moreover, $\bar{M}(U)$ admits a unique maximal graded $\mathcal{L}(V)$-submodule J subject to $J \cap U = 0$, and $L(U) = \bar{M}(U)/J$ is an irreducible V-module (Theorem 6.3 in [3]).

In Section 2 of [8], Li gave an alternative definition of the generalized Verma module $\bar{F}(U)$ associated with U, namely, $\bar{F}(U) = (U(\mathcal{L}(V)) \otimes_{U(\mathcal{L}(V)_{\geq 0})} U)/J(U)$, where $J(U)$ is the intersection of $\ker \alpha$, where α runs over all $\mathcal{L}(V)$-homomorphisms from $\bar{F}(U)$ to weak V-modules. Clearly, $\bar{M}(U) = \bar{F}(U)$ since they satisfy the same universal property.

Choose an element

$$
S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times M^2(0) \to \mathcal{F}(z_1, ..., z_n, w) \tag{3.11}
$$

in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. We will extend the first and the last input vector spaces from $M^3(0)^*$ and $M^2(0)$ to some V-modules $\tilde{M}/\mathrm{Rad}\tilde{M}$ and $\bar{M}/\mathrm{Rad}\bar{M}$, which are certain quotient modules of the generalized Verma modules $\bar{M}(M^3(0)^*)$ and $\bar{M}(M^2(0))$, respectively.

We first extend $M^2(0)$, and we will proceed like the proof of Theorem 2.2.1 in [13]. In our case, however, the extended V-module is *not* necessarily irreducible like the extended module in Theorem 2.2.1 [13] .

Let $\bar{M} := T(\mathcal{L}(V)) \otimes_{\mathbb{C}} M^2(0)$, where $T(\mathcal{L}(V))$ is the tensor algebra of $\mathcal{L}(V)$. To simplify our notation, we omit the tensor symbol in an element of $\bar{M}$ and denote an element $\overline{b \otimes t^n}$ in $\mathcal{L}(V)$ by (b, n), then an element in $\bar{M}$ can be written as:

$$
(b_1, i_1)(b_1, i_2) \dots (b_m, i_m)v_2 \tag{3.12}
$$

where $b_i \in V$, $i_k \in \mathbb{Z}$, $v_2 \in M^2(0)$, and (b, i) linear in b. Denote the vector in (3.12) by x. Extend $M^2(0)$ to $\bar{M}$ by repeatedly using the generating formula (2.9). i.e., we let:

$$\begin{aligned}
&S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M} \to \mathcal{F}(z_1, ..., z_n, w),\\
&S(v_3', (a_1, z_1) \dots (a_n, v_n)(v, w)x)\\
&:= \int_{C_1} \dots \int_{C_m} S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)(b_1, w_1)\\
&\dots (b_m, w_m)v_2) w_1^{i_1} \dots w_m^{i_m} dw_1 \dots dw_m,
\end{aligned} \tag{3.13}$$

where C_k is a contour of w_k, C_k contains C_{k+1} for each k, C_m contains 0, and $z_1, \dots, z_n, w$ are lying outside of C_1. For the well-definedness of S in (3.13), by (3.10), we just need to show that S in (3.13) agrees on the elements:

$$\begin{gathered}
(b_1, i_1) \dots (L(-1)b_k, i_k) \dots (b_m, i_m)v_2, \quad \text{and}\\
-i_k(b_1, i_1) \dots (b_k, i_k - 1) \dots (b_m, i_m)v_2.
\end{gathered}$$

Indeed, by the Definition 3.1, S in (3.11) satisfies (2.6). Thus,

$$\begin{aligned}
&S(v_3', (a_1, z_1)...(a_n, v_n)(v, w)(b_1, i_1)...(L(-1)b_k, i_k)...(b_m, i_m)v_2)\\
&= \int_{C_1} ... \int_{C_m} \frac{d}{dw_k} S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)\\
&...(b_k, w_k)...v_2) w_1^{i_1} ... w_k^{i_k} ... w_m^{i_m} dw_1 ... dw_m\\
&= -\int_{C_1} ... \int_{C_m} S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)...(b_k, w_k)\\
&...v_2) w_1^{i_1} ... (i_k) w_k^{i_k - 1} ... w_m^{i_m} dw_1 ... dw_m\\
&= S(v_3', (a_1, z_1)...(a_n, v_n)(v, w)(-i_k)(b_1, i_1)...(b_k, i_k - 1)...(b_m, i_m)v_2).
\end{aligned}$$

Introduce a gradation on $\bar{M}$ by letting

$$\deg((b_1, i_1)(b_1, i_2) \dots (b_m, i_m)v_2) := \sum_{k=1}^{m} (\mathrm{wt} b_k - i_k - 1), \tag{3.14}$$

and denote the degree n subspace by $\bar{M}(n)$. Then $\bar{M} = \bigoplus_{n \in \mathbb{Z}} \bar{M}(n)$, with $M^2(0) \subseteq \bar{M}(0)$.

Similar to (2.2.30) in [13], we define the radical of S on $\bar{M}$ by

$$\begin{aligned}
\mathrm{Rad}(S) := \{&x \in \bar{M} | S(v_3', (a_1, z_1) \dots (a_n, z_n)(v, w)x) = 0,\\
&\forall n \geq 0,\ a_1, \dots a_n \in V,\ v \in M^1,\ v_3 \in M^3(0)^*\},
\end{aligned} \tag{3.15}$$

then let $\mathrm{Rad}(\bar{M}) := \bigcap_S \mathrm{Rad}(S)$, where the intersection is taken over all $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. In fact, we can take the intersection over all nonzero S, since $\mathrm{Rad}(S) = \bar{M}$ if $S = 0$.

It is clear that the extended S in (3.13) factors through $\bar{M}/\mathrm{Rad}(\bar{M})$. Next, we show that $\bar{M}/\mathrm{Rad}(\bar{M})$ carries a structure of $\mathbb{N}$-gradable V-module whose bottom level is $M^2(0)$.

Lemma 3.4. *Let W be the subspace of $\bar{M}$ spanned by the following elements:*

$$\sum_{i=0}^{\infty}\binom{m}{i}(a(l+i)b, m+n-i)x - \Bigg(\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}(a, m+l-i)(b, n+i)x \\ - \sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}(b, n+l-i)(a, m+i)x\Bigg), \tag{3.16}$$

where $a, b \in V$, $m, n, l \in \mathbb{Z}$, and $x \in \bar{M}$. Then we have $W \subset \mathrm{Rad}(\bar{M})$.

Proof. By the formula (3.13), it is easy to see that for the following element in $\bar{M}$:

$$x' = (b_1, i_1)\ldots(b_m, i_m)x,$$

where $x = (c_1, j_1)\ldots(c_n, j_n)v_2$ for some $b_i, c_j \in V$ and $i_k, j_l \in \mathbb{Z}$, we have:

$$\begin{aligned} &S(v_3', (a_1, z_1)...(a_n, v_n)(v, w)x') \\ &= \int_{C_1}\ldots\int_{C_m} S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)(b_1, w_1) \\ &\ldots(b_m, w_m)x)w_1^{i_1}\ldots w_m^{i_m}dw_1\ldots dw_m, \end{aligned} \tag{3.17}$$

where C_k is a contour of w_k, C_{k+1} is inside of C_k for each k, C_m contains 0, and $z_1, \ldots, z_n, w$ are lying outside of C_1. Now we fix a nonzero element $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

Denote the element (3.16) by y. We adopt the notations in Proposition A.2.8 in [5] again. Let C_R^i be the circle of w_i, $i = 1, 2$, centered at 0 with radius R, and let $C_\epsilon^1(w_2)$ be the circle of w_1 centered at w_2 with radius ϵ. We may choose ϵ small enough so that $|w_1 - w_2| < |w_2|$ for any w_1 lying on $C_\epsilon^1(w_2)$. Choose $R, r, \rho > 0$ so that $R > \rho > r$. By (3.17) and the locality (2) in Definition 2.1 of S, we have:

$$\begin{aligned} &S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)y) \\ &= \int_{C_\rho^2}\sum_{i=0}^{\infty}\binom{m}{i}S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)(a(l+i)b, w_2)x)w_2^{m+n-i}dw_2 \\ &- \int_{C_R^1}\int_{C_\rho^2}\sum_{i=0}^{\infty}(-1)^i\binom{l}{i}S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)(a, w_1) \\ &(b, w_2)x)w_1^{m+l-i}w_2^{n+i}dw_1dw_2 \\ &+ \int_{C_\rho^2}\int_{C_r^1}\sum_{i=0}^{\infty}(-1)^{l+i}\binom{l}{i}S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)(b, w_2) \\ &(a, w_1)x)w_1^{m+i}w_2^{n+l-i}dw_1dw_2 \\ &= \int_{C_\rho^2}\sum_{i=0}^{\infty}\binom{m}{i}S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)(a(l+i)b, w_2)x)w_2^{m+n-i}dw_2 \\ &- \int_{C_R^1}\int_{C_\rho^2} S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)(a, w_1)(b, w_2)x) \\ &\cdot \iota_{w_1,w_2}((w_1 - w_2)^l)w_1^m w_2^n dw_1dw_2 \end{aligned}$$

$$
\begin{aligned}
&+\int_{C_\rho^2}\int_{C_r^1} S(v_3', (a_1,z_1)...(a_n,z_n)(v,w)(b,w_2)(a,w_1)x)\\
&\cdot \iota_{w_2,w_1}((-w_2+w_1)^l)w_1^m w_2^n dw_1dw_2\\
&=\int_{C_\rho^2}\sum_{i=0}^{\infty}\binom{m}{i}S(v_3', (a_1,z_1)...(a_n,z_n)(v,w)(a(l+i)b,w_2)x)w_2^{m+n-i}dw_2\\
&-\int_{C_\rho^2}\int_{C_\epsilon^1(w_2)} S(v_3', (a_1,z_1)...(a_n,z_n)(v,w)(a,w_1)\\
&(b,w_2)v_2)(w_1-w_2)^l w_1^m w_2^n dw_1dw_2.\\
&=\int_{C_\rho^2}\sum_{i=0}^{\infty}\binom{m}{i}S(v_3', (a_1,z_1)...(a_n,z_n)(v,w)(a(l+i)b,w_2)x)w_2^{m+n-i}dw_2\\
&-\int_{C_\rho^2}\int_{C_\epsilon^1(w_2)}\sum_{i=1}^{\infty}\binom{m}{i}S(v_3', (a_1,z_1)...(v,w)(a,w_1)(b,w_2)v_2)\\
&(w_1-w_2)^{l+i}w_2^{m+n-i}dw_1dw_2=0,
\end{aligned}
$$

for all $v_3' \in M^3(0)^*$, $a_1,\dots a_n \in V$, and $v \in M^1$, where the last equality follows from the associativity (5) in Definition 2.1. This shows $y \in \mathrm{Rad}(S)$. But S is chosen arbitrarily. Hence we have $y \in \mathrm{Rad}(\bar{M})$. □

The following facts are satisfied by $\mathrm{Rad}(\bar{M})$:

Lemma 3.5 *(a) If $x \in \mathrm{Rad}(\bar{M})$, then $(b,i)x \in \mathrm{Rad}(\bar{M})$, for any $b \in V$ and $i \in \mathbb{Z}$.*
(b) $M^2(0) \cap \mathrm{Rad}(\bar{M}) = 0$.
(c) $\bar{M}(n) \subset \mathrm{Rad}(\bar{M})$ for all $n < 0$.

Proof. Since $\mathrm{Rad}(\bar{M}) = \bigcap_S \mathrm{Rad}(S)$, we just need to show that (a), (b), and (c) hold for $\mathrm{Rad}(S)$, where $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ is nonzero.

(a) Let $x \in \mathrm{Rad}(S)$, by (3.13) and the definition (3.15) of $\mathrm{Rad}(S)$, we have

$$
\begin{aligned}
S(v_3', (a_1,z_1)...(v,w)(b,i)x) &= \int_C S(v_3', (a_1,z_1)...(v,w)(b,w_1)x)w_1^i dw_1\\
&=\int_C 0\cdot w_1^i dw_1 = 0,
\end{aligned}
$$

where C is a contour of w_1, with $z_1,\dots,z_n,w$ lying outside. Thus $(b,i)x \in \mathrm{Rad}(S)$.

(b) Suppose there exists some $v_2 \neq 0$ in $M^2(0) \cap \mathrm{Rad}(S)$, then by (3.3) and the recursive formula (3.6), we have

$$
\begin{aligned}
0 &= \iota_{w,z}(S(v_3', (a,z)(v,w)v_2))\\
&= S(v_3', (v,w)o(a)v_2)z^{-\mathrm{wt} a} + \sum_{i\ge 0}\iota_{w,z}(G_{\mathrm{wt} a,i}(z,w))S(v_3', (a(i)v,w)v_2)\\
&= \langle v_3', f_v(o(a)v_2)\rangle z^{-\mathrm{wt} a}w^{-\deg w} - \sum_{i,j\ge 0}\binom{\mathrm{wt} a-2-j}{i}w^{\deg v-j-1}z^{-\mathrm{wt} a+1+j}\langle v_3', f_{a(i)v}(v_2)\rangle,
\end{aligned}
\tag{3.18}
$$

for any $a \in V$, $v_3' \in M^3(0)^*$, and $v \in M^1$. By comparing the coefficients of $z^{-\mathrm{wt}a} w^{-\deg w}$ on both sides of (3.18), we have $\langle v_3', f_v(o(a)v_2)\rangle = 0$ for all $v_3 \in M^3(0)^*$, $a \in V$, and $v \in M^1$. Then $f_v(M^2(0)) = 0$, since $M^2(0)$ is an irreducible $A(V)$-module, and $M^2(0) = A(V).v_2 = span\{o(a)v_2 | a \in V\}$. It follows that $f_v = 0$ for all $v \in M^1$. By Lemma 3.3, we have $S = 0$, which is a contradiction.

(*c*) Let $x = (b_m, i_m) \dots (b_1, i_1)v_2$, with $\sum_{k=1}^{m} (\mathrm{wt}b_k - i_k - 1) < 0$. We use induction on the length m of x to show that $x \in \mathrm{Rad}(S)$. For the base case, let $x = (b, t)v_2$ with $\mathrm{wt}b - t - 1 < 0$, then by (3.13) and (3.6), we have

$$\begin{aligned}
S(v_3', (a_1, z_1)...(v, w)x) &= \int_C S(v_3', (a_1, z_1)...(v, w)(b, z)v_2)z^t dz \\
&= \int_C S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)o(b)v_2)z^{t-\mathrm{wt}b} dz \\
&+ \int_C \sum_{k=1}^{n} \sum_{i \geq 0} G_{\mathrm{wt}b,i}(z, z_k) S(v_3', (a_1, z_1)...(b(i)a_k, z_k)...(v, w)v_2)z^t dz \\
&+ \int_C \sum_{i \geq 0} G_{\mathrm{wt}b,i}(z, w) S(v_3', (a_1, z_1)...(b(i)v, w)v_2)z^t dz,
\end{aligned} \tag{3.19}$$

where C is a contour of z surrounding 0, with all other variables lying outside C. In particular, we have $|z| < |z_k|$ for all k, and $|z| < |w|$. Then by (3.7),

$$\int_C G_{\mathrm{wt}b,i}(z, z_k) z^t dz = \int_C \frac{z^{-\mathrm{wt}b+1+t}}{i!} \left(\frac{d}{dz_k}\right)^i \left(\frac{z_k^{\mathrm{wt}b-1}}{z - z_k}\right) dz = 0, \tag{3.20}$$

since $-\mathrm{wt}b + 1 + t > 0$, and $1/(z - z_k)$ is a sum of nonnegative powers in z for all z lying on the contour C. We also have $\int_C z^{t-\mathrm{wt}b} dz = 0$, since $t - \mathrm{wt}b > -1$. It follows that all the integrals on the right-hand side of (3.19) are equal to 0. This finishes the base case.

Now let $m > 0$, and consider $x = (b_m, i_m)...(b_1, i_1)v_2 \in \bar{M}$. We have:

$$\begin{aligned}
&S(v_3', (a_1, z_1)...(v, w)x) \\
&= \int_{C_m} ... \int_{C_1} S(v_3', (a_1, z_1)...(v, w)(b_m, w_m)...(b_1, w_1)v_2) w_m^{i_m} ... w_1^{i_1} dw_1 ... dw_m \\
&= \underbrace{\int_{C_m} ... \int_{C_1} S(v_3', (a_1, z_1)...(v, w)(b_m, w_m)...o(b_1)v_2) w_m^{i_m} ... w_1^{-\mathrm{wt}b_1+i_1} dw_1 ... dw_m}_{(1)} \\
&+ \underbrace{\int_{C_m} ... \int_{C_1} \sum_{k=1}^{n} \sum_{i \geq 0} G_{\mathrm{wt}b_1,i}(w_1, z_k) S(v_3', ...(b_1(i)a_k, z_k)...(v, w)...v_2) w_m^{i_m} ... w_1^{i_1} dw_1 ... dw_m}_{(2)} \\
&+ \underbrace{\int_{C_m} ... \int_{C_1} \sum_{i \geq 0} G_{\mathrm{wt}b_1,i}(w_1, w) S(v_3', ...(b_1(i)v, w)(b_m, w_m)...v_2) w_m^{i_m} ... w_1^{i_1} dw_1 ... dw_m}_{(3)} \\
&+ \underbrace{\int_{C_m} ... \int_{C_1} \sum_{l=2}^{m} \sum_{i \geq 0} G_{\mathrm{wt}b_1,i}(w_1, w_l) S(v_3', ...(v, w)...(b_1(i)b_l, w_l)...v_2) w_m^{i_m} ... w_1^{i_1} dw_1 ... dw_m}_{(4)} \\
&= (1) + (2) + (3) + (4),
\end{aligned}$$

where C_1 is a contour of w_1 surrounding 0, with all other variables lying outside. We need to show that the sum of these integrals equals 0. i.e., (1) + (2) + (3) + (4) = 0.

Case 1. $\mathrm{wt} b_1 - i_1 - 1 < 0$.

Similar to (3.20), we have $\int_{C_1} G_{\mathrm{wt} b_1, i}(w_1, z) w_1^{i_1} dw_1 = 0$, for $z = z_k$, w or w_l. Thus we have (2) = (3) = (4) = 0. We also have (1) = 0 because $-\mathrm{wt} b_1 + i_1 > -1$.

Case 2. $\mathrm{wt} b_1 - i_1 - 1 > 0$.

Then $-\mathrm{wt} b_1 + i_1 < -1$, which implies (1) = 0. Moreover, by (3.7) we have:

$$\begin{aligned}
&\int_{C_1} G_{\mathrm{wt} b_1, i}(w_1, z) w_1^{i_1} dw_1 \\
&\quad = \mathrm{Res}_{w_1=0}\left(-\sum_{j \geq 0} \binom{\mathrm{wt} b_1 - 2 - j}{i} z^{\mathrm{wt} b_1 - j - 2 - i} w_1^{-\mathrm{wt} b_1 + 1 + j + i_1} \right) \\
&\quad = -\binom{i_1}{i} z^{i_1 - i}.
\end{aligned} \tag{3.21}$$

for $z = z_k$, w or w_l. Apply (3.21) to (2), (3), and (4), and we have:

$$\begin{aligned}
(2) &= -\int_{C_m} \ldots \int_{C_2} \sum_{k=1}^{n} \sum_{i \geq 0} \binom{i_1}{i} z_k^{i_1 - i} S(v_3', \ldots (b_1(i) a_k, z_k) \\
&\quad \ldots (v, w)(b_m, w_m) \ldots (b_2, w_2) v_2) \\
&= -\sum_{k=1}^{n} \sum_{i \geq 0} \binom{i_1}{i} z_k^{i_1 - i} S(v_3', (a_1, z_1) \ldots (b_1(i) a_k, z_k) \ldots (a_n, z_n)(v, w) y),
\end{aligned}$$

where $y = (b_m, i_m) \ldots (b_2, i_2) v_2$. Note that $\deg y = \deg x - (\mathrm{wt} b_1 - i_1 - 1) < 0$, and the length of y is $m - 1$, then by the induction hypothesis we have (2) = 0. Similarly, (3) = 0.

$$\begin{aligned}
(4) &= \int_{C_m} \ldots \int_{C_1} \sum_{l=2}^{m} \sum_{i \geq 0} \binom{i_1}{i} w_l^{i_1 - i} S(v_3', \ldots (v, w) \ldots (b_1(i) b_l, w_l) \\
&\quad \ldots v_2) w_m^{i_m} \ldots w_1^{i_1} dw_1 \ldots dw_m \\
&= \sum_{l=2}^{m} \sum_{i \geq 0} \binom{i_1}{i} S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w) y_l),
\end{aligned}$$

where $y_l = (b_m, i_m) \ldots (b_1(i) b_l, i_1 + i_l - i) \ldots (b_2, i_2) v_2$. Note that

$$\deg(b_1(i) b_l, i_1 + i_l - i) = \mathrm{wt} b_1 + \mathrm{wt} b_l - i - 1 - i_1 - i_l + i - 1 = \deg(b_1, i_1) + \deg(b_l, i_l).$$

Thus, $\deg y_l = \sum_{k=1}^{m} \mathrm{wt}(b_k, i_k) = \deg x < 0$, and the length of y_l is $m - 1$ for each l. Hence (4) = 0 by the induction hypothesis.

Case 3. $\mathrm{wt} b_1 - i_1 - 1 = 0$.

In this case, we have: $\int_{C_1} G_{\mathrm{wt} b_1, i}(w_1, z) w_1^{i_1} dw_1 = 0$ in view of (3.20). Hence (2) = (3) = (4) = 0. Moreover, since $-\mathrm{wt} b_1 + i_1 = -1$, we have:

$$\begin{aligned}
(1) &= \int_{C_m} \ldots \int_{C_2} S(v_3', (a_1, z_1) \ldots (v, w)(b_m, w_m) \ldots o(b_1) v_2) w_m^{i_m} \\
&\quad \ldots w_2^{i_2} dw_2 \ldots dw_m = S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w) y),
\end{aligned}$$

where $y = (b_m, i_m) \ldots (b_2, i_2) v_2$. Since $\deg y = \deg x < 0$, and the length of y is $m - 1$, we have (1) = 0 by the induction hypothesis. Now the proof of (c) is complete. □

We define a vertex operator $Y_{\bar{M}^2}$ on the quotient space $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ as follows:

$$Y_{\bar{M}^2}(a,z)(b_1,i_1)\dots(b_m,i_m)v_2 := \sum_{n\in\mathbb{Z}}(a,n)(b_1,i_1)\dots(b_m,i_m)v_2 z^{-n-1}, \tag{3.22}$$

where $a \in V$, $(b_1,i_1)\dots(b_m,i_m)v_2 \in \bar{M}^2$, and we use the same notation $(b_1,i_1)\dots(b_m,i_m)v_2$ for its image in the quotient space $\bar{M}^2$. We can express (3.22) in the component form:

$$a(n)(b_1,i_1)\dots(b_m,i_m)v_2 = (a,n)(b_1,i_1)\dots(b_m,i_m)v_2, \tag{3.23}$$

for all $a \in V,\ n \in \mathbb{Z}$, and $(b_1,i_1)\dots(b_m,i_m)v_2 \in \bar{M}$.

Proposition 3.6. *$\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$, together with $Y_{\bar{M}^2} : V \to \mathrm{End}(\bar{M}^2)[[z,z^{-1}]]$ given by (3.22) and (3.23), is a weak V-module.*

Proof. By (a) of Lemma 3.5, we have $a(n)\mathrm{Rad}(\bar{M}) \subseteq \mathrm{Rad}(\bar{M})$. Hence $Y_{\bar{M}^2}$ is well-defined. Let $x = (b_1,i_1)\dots(b_m,i_m)v_2 \in \bar{M}^2$, we claim that $\mathbf{1}(-1)x = x$ and $\mathbf{1}(n)x = 0$ for any $n \neq -1$. Indeed, for any $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, by the definition formula (3.13), the recursive formula (3.6), together with the fact that $\mathbf{1}(j)a = 0$ for all $j \geq 0$, $a \in V$, and $\mathbf{1}(j)v = 0$ for all $j \geq 0$, $v \in M^1$, we have:

$$\begin{aligned}
&S(v_3',(a_1,z_1)...(v,w)\mathbf{1}(n)x)\\
&= \int_{C_0}\int_{C_m}...\int_{C_1} S(v_3',(\mathbf{1},w_0)(a_1,z_1)...(v,w)(b_1,w_1)...v_2)w_0^n w_1^{i_1}...w_m^{i_m}dw_1...dw_m dw_0\\
&= \int_{C_0}\int_{C_m}...\int_{C_1} S(v_3' o(\mathbf{1}),(a_1,z_1)...(v,w)(b_1,w_1)...v_2)w_0^n w_1^{i_1}...w_m^{i_m}dw_1...dw_m dw_0\\
&= \delta_{n+1,0}\cdot S(v_3',(a_1,z_1)...(v,w)x),
\end{aligned}$$

where the last equality follows from the fact that $\int_{C_0} w_0^n dw_0 = \delta_{n+1,0}$. Thus, $(\mathbf{1}(n)x - \delta_{n+1,0}x) \in \mathrm{Rad}(\bar{M})$, and so $\mathbf{1}(n)x = \delta_{n+1,0}x$ in $\bar{M}^2$. Moreover, given homogeneous elements $x \in \bar{M}$ and $a \in V$, by (3.14) and (3.23), $\deg(a(n).x) = \mathrm{wt}a - n - 1 + \deg x < 0$ when $n >> 0$. Then by part (c) of Lemma 3.5, we have $a(n)x = 0$ in $\bar{M}^2$ when n is large enough. Finally, by Lemma 3.4 and (3.23), $(\bar{M}^2, Y_{\bar{M}^2})$ satisfies the Jacobi identity. Hence it is a weak V-module. □

Proposition 3.7. *$\bar{M}^2$ has a gradation $\bar{M}^2 = \bigoplus_{n=0}^{\infty} \bar{M}^2(n)$, where $\bar{M}^2(n)$ is an eigenspace of $L(0)$ of eigenvalue $\lambda + n$ for each $n \in \mathbb{N}$, and $\bar{M}^2(0) = M^2(0)$. In particular, $\bar{M}^2$ is an ordinary V-module, and if $M^2(0)$ is the bottom level of some ordinary V-module M^2, with conformal weight h_2, then $\lambda = h_2$.*

Proof. Let $\bar{M}^2(n)$ be the image of $\bar{M}(n)$ under the quotient map $\bar{M} \to \bar{M}^2$. By Lemma 3.5, we have $\bar{M}^2 = \sum_{n\geq 0} \bar{M}^2(n)$ and $M^2(0) \subseteq \bar{M}^2(0)$. We claim that

$$a(\mathrm{wt}a - 1)v_2 = o(a)v_2, \tag{3.24}$$

for all $v_2 \in M^2(0)$ and homogeneous $a \in V$. Indeed, we only need to show that $(a, \mathrm{wt}a-1)v_2 - o(a)v_2 \in \mathrm{Rad}(S)$, for all $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. By (3.13) and (3.6),

$$\begin{aligned}
&S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)(a, \mathrm{wt}a-1)v_2)\\
&\quad= \int_C S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)(a, w_1)v_2)w_1^{\mathrm{wt}a-1}dw_1\\
&\quad= \int_C S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)o(a)v_2)w_1^{-\mathrm{wt}a}w_1^{\mathrm{wt}a-1}dw_1\\
&\quad+ \sum_{k=1}^{n}\sum_{i\geq 0}\int_C G_{\mathrm{wt}a,i}(w_1, z_k)S(v_3', (a_1, z_1)...(a(i)a_k, z_k)...(a_n, z_n)(v, w)v_2)w_1^{\mathrm{wt}a-1}dw_1\\
&\quad+ \sum_{i\geq 0}\int_C G_{\mathrm{wt}a,i}(w_1, w)S(v_3', (a_1, z_1)...(a_n, z_n)(a(i)v, w)v_2)w_1^{\mathrm{wt}a-1}dw_1,
\end{aligned}$$

where C is a contour of w_1 surrounding 0, with all other variables lying outside of C. Since $|z_k|, |w| > |w_1|$ for all k, where w_1 is lying on C, then we have

$$\int_C G_{\mathrm{wt}a,i}(w_1, z)w_1^{\mathrm{wt}a-1}dw_1 = \int_C w_1^{\mathrm{wt}a-1}\frac{w_1^{-\mathrm{wt}a+1}}{i!}\left(\frac{d}{dz}\right)^i\left(\frac{z^{\mathrm{wt}a-1}}{w_1-z}\right)dw_1 = 0,$$

for $z = z_k$ or w. Hence $(a, \mathrm{wt}a-1)v_2 - o(a)v_2 \in \mathrm{Rad}(S)$. This shows (3.24).

Since $L(0) = \omega(\mathrm{wt}\omega - 1)$ on $\bar{M}^2$, it follows from (3.24) that $L(0)$ preserves $M^2(0)$. On the other hand, we have $[L(0), a(n)] = (\mathrm{wt}a - n - 1)a(n)$ (see (4.2.2) in [4]). Then by (3.24) again, we have $[L(0), o(a)]v_2 = [L(0), a(\mathrm{wt}a-1)]v_2 = 0$. Since $M^2(0)$ is an irreducible $A(V)$-module which is of countable dimension, then by the Schur's Lemma (Lemma 1.2.1 in [13]), there exists $\lambda \in \mathbb{C}$ such that $L(0) = \lambda\cdot \mathrm{Id}$ on $M^2(0)$. If $M^2(0)$ is the bottom level of M^2, with conformal weight h_2, then $L(0) = h_2 \cdot \mathrm{Id}$ on $M^2(0)$, and so $h_2 = \lambda$.

Now for any spanning element $x = (b_1, i_1)\dots(b_m, i_m)v_2 = b_1(i_1)\dots b_m(i_m)v_2$ of $\bar{M}^2(n)$, we have $L(0)x = (\sum_{k=1}^m(\mathrm{wt}b_k - i_k - 1) + \lambda)x = (n+\lambda)x$. Therefore, $\bar{M}^2(n)$ is an eigenspace of $L(0)$ of eigenvalue $n+\lambda$ for every $n \in \mathbb{N}$, and $\bar{M}^2 = \bigoplus_{n=0}^\infty \bar{M}^2(n)$.

Finally, for any spanning element $x = b_1(i_1)\dots b_m(i_m)v_2$ of $\bar{M}^2(0)$, it follows from (3.24) and an easy induction that $x \in M^2(0)$, therefore $\bar{M}^2(0) = M^2(0)$. □

Remark 3.8. Unlike the construction of V-modules from the correlation functions in Theorem 2.2.1 in [13], in our case, it is unclear whether $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ is an irreducible V-module. The reason is the following:

Assume $N \leq \bar{M}^2$ is a submodule, by Proposition 3.7 we have $N = \bigoplus_{n=0}^\infty N(n)$, with $N(n) = N \cap \bar{M}^2(n)$ for each n. If $N(0) \neq 0$, then clearly $N = \bar{M}^2$. So to show $\bar{M}^2$ is irreducible, we need to show that $N = 0$ when $N(0) = 0$.

This is true for the module $\bar{M}/\mathrm{Rad}(\bar{M})$ constructed in Theorem 2.2.1 in [13], wherein the correlation function $S(v', (a_1, z_1)\dots(a_n, z_n)N)$, with $v' \in M^2(0)$, is essentially the limit function of $\langle v', Y(a_1, z_1)\dots Y(a_n, z_n)N\rangle$. It is zero because $Y(a, z)N \subset N((z))$, and the bottom level of N is 0. Thus, $N \subseteq \mathrm{Rad}(S)$, and so $N = 0$ in $\bar{M}/\mathrm{Rad}(\bar{M})$. However, in our case, $S(v_3', (a_1, z_1)\dots(a_n, z_n)(v, w)N)$ with $v_3' \in M^3(0)^*$ is essentially the limit function of $\langle v_3', I(v, w)Y(a_1, z_1)\dots Y(a_n, z_n)N\rangle w^{-h}$. Although the components of $Y(a, z)$ still leave N invariant, the intertwining operator $I(v, w)$ could send some element in N to a nonzero element of $M^3(0)$. Hence we cannot conclude that $N \subseteq \mathrm{Rad}(\bar{M})$ in general.

We give a sufficient condition under which $\bar{M}^2$ is irreducible.

Lemma 3.9. *Suppose* $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ *satisfies:*

$$\sum_{i\geq 0}\binom{n}{i}\langle v_3', f_{b(i)v}(v_2)\rangle = 0, \tag{3.25}$$

for all $b \in V$, $n \in \mathbb{Z}$ *such that* $\mathrm{wt}b - n - 1 > 0$, $v \in M^1$, $v_3' \in M^3(0)^*$, *and* $v_2 \in M^2(0)$. *Then* $S(v_3', (v, w)y) = 0$ *for any* $y \in M(m)$ *with* $m \geq 1$, $v_3' \in M^3(0)^*$, *and* $v \in M^1$.

Proof. It follows from an easy induction that y can be written as a sum of the terms $(b_m, n_m)\ldots(b_1, n_1)v_2$ for some $m \geq 1$ and $v_2 \in M^2(0)$, with $\mathrm{wt}b_j - n_j - 1 > 0$ for all j.

Let $y = (b_m, n_m)\ldots(b_1, n_1)v_2$. We use induction on m to show that $S(v_3', (v, w)y) = 0$. For the base case $m = 1$ and $y = (b, n)v_2$, with $\mathrm{wt}b - n - 1 > 0$, by (3.13), (3.3), (3.6), (3.7), and the assumption (3.25), we have:

$$\begin{aligned}
S(v_3', (v, w)y) &= \int_C S(v_3', (v, w)(b, z)v_2)z^n dz\\
&= \int_C S(v_3', (v, w)o(b)v_2)z^{-\mathrm{wt}b+n}dz\\
&+ \int_C \sum_{i\geq 0} G_{\mathrm{wt}b,i}(z, w)S(v_3', (b(i)v, w)v_2)z^n dz\\
&= 0 + \sum_{i\geq 0}\int_C -\sum_{j\geq 0}\binom{\mathrm{wt}b-2-j}{i}w^{\mathrm{wt}b-j-2-i}z^{n-\mathrm{wt}b+1+j}\\
&S(v_3', (b(i)v, w)v_2)dz\\
&= -\sum_{i\geq 0}\binom{n}{i}\langle v_3', f_{b(i)v}v_2\rangle w^{-\mathrm{wt}b-\deg v+1+n} = 0.
\end{aligned}$$

Now let $m > 1$. Then by (3.13) and (3.6), we have

$$\begin{aligned}
S(v_3', (v, w)y) &= \int_{C_m}\ldots\int_{C_1} S(v_3', (v, w)(b_m, z_m)\\
&\ldots(b_1, z_1)v_2)z_1^{n_1}\ldots z_m^{n_m}dz_1\ldots dz_m\\
&= \int_{C_m}\ldots\int_{C_1} S(v_3', (v, w)(b_m, z_m)\\
&\ldots(b_2, z_2)o(b_1)v_2)z_1^{-\mathrm{wt}b_1+n_1}\ldots z_m^{n_m}dz_1\ldots dz_m\\
&+ \int_{C_m}\ldots\int_{C_1}\sum_{i\geq 0} G_{\mathrm{wt}b_1,i}(z_1, w)S(v_3', (b_1(i)v, w)(b_m, z_m)\\
&\ldots(b_2, z_2)v_2)z_1^{n_1}\ldots z_m^{n_m}dz_1\ldots dz_m\\
&+ \int_{C_m}\ldots\int_{C_1}\sum_{k=2}^{m}\sum_{i\geq 0} G_{\mathrm{wt}b_1,i}(z_1, z_k)S(v_3', (v, w)\ldots(b_1(i)b_k, z_k)\\
&\ldots(b_2, z_2)v_2)z_1^{n_1}\ldots z_m^{n_m}dz_1\ldots dz_m
\end{aligned}$$

$$
\begin{aligned}
&= 0 + \int_{C_m} \cdots \int_{C_2} \sum_{i \geq 0} \int_{C_1} - \sum_{j \geq 0} \binom{\mathrm{wt} b_1 - 2 - j}{i} w^{\mathrm{wt} b_1 - j - 2 - i} z_1^{n_1 - \mathrm{wt} b_1 + 1 + j} \\
&\cdot S(v_3', (b_1(i)v, w)(b_m, z_m) \ldots (b_2, z_2)v_2) z_2^{n_2} \ldots z_m^{n_m} dz_2 \ldots dz_m \\
&+ \int_{C_m} \cdots \int_{C_2} \sum_{k=2}^{m} \sum_{i \geq 0} \int_{C_1} - \sum_{j \geq 0} \binom{\mathrm{wt} b_1 - 2 - j}{i} z_k^{n_k + \mathrm{wt} b_1 - j - 2 - i} z_1^{n_1 - \mathrm{wt} b_1 + 1 + j} \\
&\cdot S(v_3', (v, w)(b_m, z_m) \ldots (b_1(i) b_k, z_k) \ldots (b_2, z_2) v_2) z_2^{n_2} \ldots \widehat{z_k^{n_k}} \ldots z_m^{n_m} dz_2 \ldots dz_m \\
&= - \int_{C_m} \cdots \int_{C_2} \sum_{i \geq 0} \binom{n_1}{i} w^{n_1 - i} S(v_3', (b_1(i)v, w)(b_m, z_m) \\
&\ldots (b_2, z_2) v_2) z_2^{n_2} \ldots z_k^{n_k} dz_2 \ldots dz_m \\
&- \int_{C_m} \cdots \int_{C_2} \sum_{k=2}^{m} \sum_{i \geq 0} \binom{n_1}{i} S(v_3', (v, w) \\
&\ldots (b_1(i) b_k, z_k) \ldots v_2) z_2^{n_2} \ldots z_k^{n_1 + n_k - i} \ldots z_m^{n_m} dz_2 \ldots dz_m \\
&= - \sum_{i \geq 0} \binom{n_1}{i} w^{n_1 - i} S(v_3', (b_1(i)v, w)(b_m, n_m) \ldots (b_2, n_2) v_2) \\
&- \sum_{k=2}^{m} \sum_{i \geq 0} \binom{n_1}{i} S(v_3', (v, w)(b_m, n_m) \ldots (b_1(i) b_k, n_1 + n_k - i) \ldots (b_2, n_2) v_2) = 0,
\end{aligned}
$$

where the last equality follows from the induction hypothesis, together with the fact that $\deg(b_1(i)b_k, n_1 + n_k - i) = \mathrm{wt} b_1 - n_1 - 1 + \mathrm{wt} b_k - n_k - 1 > 0$, for any $i \geq 0$. □

Corollary 3.10. *For any fixed $v \in M^1$ and $y \in \bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$, let $n \in \mathbb{Z}$ be an integer such that $n > \deg v + \deg y - 1$. Then we have*

$$
\int_C S(v_3', (v, w)y) w^n dw = 0, \tag{3.26}
$$

for all $v_3' \in M^3(0)$, where C is a contour of w surrounding 0. In particular, for fixed $v \in M^1$ and $y \in \bar{M}^2$, the power series expansion of $S(v_3', (v, w)y)$ has a uniform lower bound for w independent of $v_3' \in M^3(0)^*$.

Proof. It suffices to show (3.26) for $y = (b_m, n_m) \ldots (b_1, n_1) v_2$, where $v_2 \in M^2(0)$, $m \geq 0$, and $\mathrm{wt} b_j - n_j - 1 > 0$ for all j. Again, we use induction on m. When $m = 0$, we have $y = v_2$ and $\deg y = 0$. Then by (3.3) and $-\deg v + n > -1$, we have: $\int_C S(v_3', (v, w)v_2) w^n dw = \int_C \langle v_3', f_v(v_2) \rangle w^{-\deg v + n} dw = 0$. Now let $m > 0$, and let $n \in \mathbb{Z}$ be such that $n > \deg v + \deg y - 1$. Since $-\mathrm{wt} b_1 + n_1 < -1$, by the calculations in Lemma 3.9, we have:

$$
\begin{aligned}
&\int_C S(v_3', (v, w)y) w^n dw \\
&= - \sum_{i \geq 0} \int_C \binom{n_1}{i} w^{n + n_1 - i} \underset{(1)}{S(v_3', (b_1(i)v, w)(b_m, n_m) \ldots (b_2, n_2) v_2)} dw
\end{aligned}
$$

$$-\sum_{k=2}^{m}\sum_{i\geq 0}\int_C \binom{n_1}{i} w^n S(v_3', (v, w)(b_m, n_m)...\underset{(2)}{(b_1(i)b_k, n_1 + n_k - i)}...(b_2, n_2)v_2)dw$$
$$= (1) + (2).$$

Since $n > \deg v + \deg y - 1$, we have $n + n_1 - i > \deg(b_1(i)v) + \sum_{j=2}^{m}(\mathrm{wt} b_j - n_j - 1) - 1$ for all $i \geq 0$. Then by the induction hypothesis, $(1) = 0$ for all $v_3' \in M^3(0)^*$. On the other hand, since $\deg(b_1(i)b_k, n_1 + n_k - i) = \mathrm{wt} b_1 - n_1 - 1 + \mathrm{wt} b_k - n_k - 1$ for all $i \geq 0$, we have $(2) = 0$ for all $v_3' \in M^3(0)^*$. Thus $\int_C S(v_3', (v, w)y)w^n dw = 0$. □

Proposition 3.11. *Suppose every $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ satisfies the condition (3.25), then $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ is an irreducible V-module with bottom level $M^2(0)$. In particular, $\bar{M}^2$ is isomorphic to $L(M^2(0))$, the unique irreducible V-module with bottom level is $M^2(0)$.*

Proof. Note that for any $x \in M$, $S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)x)$ is also a rational function in $z_1, \ldots, z_n, w$ by (3.13) and (3.23), and it has Laurent series expansion:

$$S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)x) = S(v_3', (v, w)(a_1, z_1)...(a_n, z_n)x)$$
$$= \sum_{i_1,...,i_n \in \mathbb{Z}} \left(\int_{C_n} ... \int_{C_1} S(v_3', (v, w)(a_n, z_n)...(a_1, z_1)x)z_1^{i_1}...z_n^{i_n} dz_1...dz_n\right) z_1^{-i_1-1}...z_n^{-i_n-1}$$
$$= \sum_{i_1,...,i_n \in \mathbb{Z}} S(v_3', (v, w)a_n(i_n)...a_1(i_1)x)z_1^{-i_1-1}...z_n^{-i_n-1} \tag{3.27}$$

on the domain $\mathbb{D} = \{(z_1, \ldots, z_n, w) | |w| > |z_n| > \cdots > |z_1| > 0\}$. Let N be a submodule of $\bar{M}^2$ such that $N(0) = 0$, we need to show that $N = 0$. Let $x \in N$, we have $y = a_n(i_n) \ldots a_1(i_1)x \in N$, and if $y \neq 0$ then $\deg(y) > 0$. By Lemma 3.9, we have $S(v_3', (v, w)y) = 0$. Thus, the rational function $S(v_3', (a_1, z_1) \ldots (a_n, z_n)(v, w)x)$ is equal to 0 by (3.27). i.e., $x \in \mathrm{Rad}(S)$ for all $S \in \mathrm{Cor}\binom{M^3}{M^1\ M^2}$. Thus $N = 0$. □

In conclusion, given a $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, the extended S in (3.13) factors though an $\mathbb{N}$-gradable V-module $\bar{M}^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ whose bottom level is $M^2(0)$. It is irreducible if the condition (3.25) is satisfied. Therefore, by (3.13) and (3.23), we have a well-defined system of $(n + 3)$-point correlation functions:

$$S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M}^2 \to \mathcal{F}(z_1, \ldots, z_n, w),$$
$$S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)b_1(i_1)...b_m(i_m)v_2)$$
$$= \int_{C_1} ... \int_{C_m} S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)(b_1, w_1)...(b_m, w_m)v_2)w_1^{i_1}...w_m^{i_m} dw_1...dw_m, \tag{3.28}$$

for all $b_1(i_1) \ldots b_m(i_m)v_2 \in \bar{M}^2$, where C_k is a contour of w_k, C_k contains C_{k+1} for all k, C_m contains 0, and $z_1, \ldots, z_n, w$ are outside of C_1.

In particular, S in (3.28) satisfies the generating formula (2.9) with $M^2 = \bar{M}^2$, since the extended S is defined by this formula. Moreover, by Corollary 3.10 and the fact that the orginal S in (3.11) belongs to $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, it is easy to see that the S in (3.28) also satisfies the properties $(1)-(6)$ in Definition 2.1, with $v_3' \in M^3(0)^*$ and $v_2 \in \bar{M}^2$.

We adopt a similar method to extend the first input component of S in (3.28) from $M^3(0)^*$ to a V-module by using the other generating formula (2.10). First, we let

$$\tilde{M} := T(\mathcal{L}(V)) \otimes_{\mathbb{C}} M^3(0)^*.$$

Then $\tilde{M}$ is spanned by elements of the form: $y = (b_1, i_1) \dots (b_m, i_m)v_3'$, where $b_j \in V$, $i_j \in \mathbb{Z}$ for $j = 1, \dots, m$, and $v_3' \in M^3(0)^*$. Next, we extend S in (3.28) by iterating the generating formula (2.10). i.e., we define:

$$\begin{aligned} &S : \tilde{M} \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M}^2 \to \mathcal{F}(z_1, \dots, z_n, w) \\ &\quad S((b_1, i_1)...(b_m, i_m)v_3', (a_1, z_1)...(a_n, z_n)(v, w)x_2) \\ &\quad := \int_{C_1} ... \int_{C_m} S(v_3', (b_m, w_m)'...(b_1, w_1)'(a_1, z_1)...(v, w)x_2) w_1^{-i_1-2} ... w_m^{-i_m-2} dw_m ... dw_1, \end{aligned} \tag{3.29}$$

where $(b, w)' = (e^{w^{-1}L(1)}(-w^2)^{L(0)}b, w)$, C_k is a contour of w_k s.t. C_k contains C_{k-1} for each k, and C_1 contains all the variables $z_1, \dots, z_n, w$. For S in (3.29), we similarly define

$$\mathrm{Rad}(S) := \{y \in \tilde{M} : S(y, (a_1, z_1) \dots (a_n, z_n)(v, w)x) = 0, \forall a_i \in V, v \in M^1, x \in \bar{M}^2\},$$

and let $\mathrm{Rad}(\tilde{M}) := \bigcap \mathrm{Rad}(S)$, where the intersection is taken over all $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, with the extension given by (3.29). Clearly, S factors though $\tilde{M}/\mathrm{Rad}(\tilde{M})$.

Similar to our previous argument, one can show that $\bar{M}^{3'} = \tilde{M}/\mathrm{Rad}(\tilde{M})$ has a natural $\mathbb{N}$-gradable V-module structure $\bar{M}^{3'} = \bigoplus_{n=0}^{\infty} \bar{M}^{3'}(n)$, with $\bar{M}^{3'}(0) = M^3(0)^*$. Moreover, $\bar{M}^{3'} = \tilde{M}/\mathrm{Rad}(\tilde{M})$ is irreducible if the condition 3.25 is satisfied. Thus we have a well-defined system of correlation functions S:

$$\begin{aligned} &S : \bar{M}^{3'} \times V \times \cdots \times M^1 \times \cdots \times V \times \bar{M}^2 \to \mathcal{F}(z_1, \dots, z_n, w), \\ &\quad S(b_1(i_1)...b_m(i_m)v_3', (a_1, z_1)...(a_n, z_n)(v, w)x_2) \\ &\quad = \int_{C_1} ... \int_{C_m} S(v_3', (b_m, w_m)'...(b_1, w_1)'(a_1, z_1)...(v, w)x_2) w_1^{-i_1-2} ... w_m^{-i_m-2} dw_m ... dw_1, \end{aligned} \tag{3.30}$$

for all $b_1(i_1) \dots b_m(i_m)v_3' \in \bar{M}^{3'}$ and $x_2 \in \bar{M}^2$. Moreover, by Remark 2.4, we also have:

$$\begin{aligned} &S(b_1'(i_1)...b_m'(i_m)v_3', (a_1, z_1)...(a_n, z_n)(v, w)x_2) \\ &\quad = \int_{C_1} ... \int_{C_m} S(v_3', (b_m, w_m)...(b_1, w_1)(a_1, z_1) \\ &\quad ...(a_n, z_n)(v, w)x_2) w_1^{i_1} ... w_m^{i_m} dw_m ... dw_1, \end{aligned} \tag{3.31}$$

where $b'(i) = \sum_{j\geq 0}\frac{1}{j!}(-1)^{\mathrm{wt}b}(L(1)^j b)(2\mathrm{wt}b - i - j - 2)$, C_k is a contour of w_k such that C_k contains C_{k-1} for each k, and $z_1, \dots, z_n, w$ are inside of C_1. Since (3.30) and (3.31) are given by iterating the generating formula (2.10), it is clear that S in (3.30) also satisfies (2.10) with $M^2 = \bar{M}^2$ and $M^{3'} = \bar{M}^{3'}$. Denote the contragredient module of $\bar{M}^{3'}$ by $\bar{M}^3$.

Theorem 3.12. *The system of extended correlation functions S in (3.30) lies in* $\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}$. *Hence we have an isomorphism of vector spaces* $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong \mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2} \cong I\binom{\bar{M}^3}{M^1\ \bar{M}^2}$.

Proof. We have already proven that S satisfies (7) and (8) in Definition 2.1, with $M^2 = \bar{M}^2$ and $M^{3'} = \bar{M}^{3'}$. It remains to show that S in (3.30) satisfies the properties $(1)-(6)$ in Definition 2.1, with $M^2 = \bar{M}^2$ and $M^3 = \bar{M}^3$. In fact, by the definition formulas (3.28) and (3.31), together with the fact that the orginal S in (3.11) lies in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, the properties $(2)-(6)$ are straightforward.

To prove (1), we need an intermediate result first. We introduce the following notation:

$$S(v_3', b_1(n_1)...b_m(n_m)(v, w)x_2) := \int_{C_m}...\int_{C_1} S(v_3', (b_1, z_1)...(b_m, z_m)(v, w)x_2) \cdot z_1^{n_1}...z_m^{n_m} dz_1...dz_m, \tag{3.32}$$

where $m \geq 0$, $x_2 \in \bar{M}^2$, $b_k \in V$, $n_k \in \mathbb{Z}$, C_k is a contour of z_k s.t. C_k contains C_{k+1} for all k, and w is inside of C_m. Assume $\mathrm{wt}b_1 - n_1 - 1 < 0$. We claim that:

$$\begin{aligned} &S(v_3', b_1(n_1)...b_m(n_m)(v, w)x_2)\\ &= \sum_{l=2}^{m}\sum_{i\geq 0}\binom{n_1}{i} S(v_3', b_2(n_2)...(b_1(i)b_l)(n_1 + n_l - i)...b_m(n_m)(v, w)x_2)\\ &+ \sum_{i\geq 0}\binom{n_1}{i} S(v_3', b_2(n_2)...b_m(n_m)(b_1(i)v, w)x_2)w^{n_1-i}\\ &+ S(v_3', b_2(n_2)...b_m(n_m)(v, w)(b_1(n_1)x_2)). \end{aligned}\tag{3.33}$$

Let $x_2 = c_1(i_1)\dots c_r(i_r)v_2$, for some $c_j \in V$, $i_j \in \mathbb{Z}$ for all j, and $v_2 \in M^2(0)$. Note that $b_1(n_1)v_2 = 0$ as $\mathrm{wt}b_1 - n_1 - 1 < 0$. For $|z_1| > |w|$, by (3.5) we have:

$$\begin{aligned}\int_{C_1} F_{\mathrm{wt}b_1,i}(z_1, w) z_1^{n_1} dz_1 &= \sum_{j\geq 0}\int_{C_1}\binom{\mathrm{wt}b_1 + j}{i} z_1^{n_1-\mathrm{wt}b_1-j-1} w^{\mathrm{wt}b_1+j-i+i_t} dz_1\\ &= \binom{n_1}{i} w^{n_1-i},\end{aligned}$$

where C_1 is a contour of z_1, with w lying inside. Then by (3.32), (3.28), the recursive formula (3.4), together with the fact that $-\mathrm{wt}b_1 + n_1 > -1$, we have:

$$
\begin{aligned}
& S(v_3', b_1(n_1)...b_m(n_m)(v, w)x_2) \\
& \quad = \int_{C_m} \cdots \int_{C_1} \sum_{l=2}^{m} \sum_{i \geq 0} F_{\mathrm{wt} b_1, i}(z_1, z_l) S(v_3', (b_2, z_2) \\
& \quad ...(b_1(i)b_l, z_l)...(v, w)x_2) z_1^{n_1}...z_m^{n_m} dz_1...dz_m \\
& \quad + \int_{C_m} \cdots \int_{C_1} \sum_{i \geq 0} F_{\mathrm{wt} b_1, i}(z_1, w) S(v_3', (b_2, z_2) \\
& \quad ...(b_m, z_m)(b_1(i)v, w)x_2) z_1^{n_1}...z_m^{n_m} dz_1...dz_m \\
& \quad + \int_{C_m} \cdots \int_{C_1} \left(\int_{C_1'} \cdots \int_{C_r'} \sum_{i \geq 0} F_{\mathrm{wt} b_1, i}(z_1, w_t) S(v_3', (b_2, z_2) \right. \\
& \quad ...(b_m, z_m)(v, w)(c_1, w_1)... \\
& \quad (b_1(i)c_t, w_t)...(c_r, w_r)v_2) \cdot w_1^{i_1}...w_r^{i_r} dw_r...dw_1) z_1^{n_1}...z_m^{n_m} dz_1...dz_m \\
& \quad = \int_{C_m} \cdots \int_{C_2} \sum_{l=2}^{m} \sum_{i \geq 0} \binom{n_1}{i} S(v_3', (b_2, z_2)...(b_1(i)b_l, z_l)...(b_m, z_m)(v, w)x_2) \\
& \quad \cdot z_2^{n_2}...z_l^{n_1 - i + n_l}...z_m^{n_m} dz_2...dz_m + \int_{C_m} \cdots \int_{C_2} \sum_{i \geq 0} \binom{n_1}{i} S(v_3', \\
& \quad (b_2, z_2)...(b_m, z_m)(b_1(i)v, w)x_2) w^{n_1 - i} z_2^{n_2}...z_m^{n_m} dz_2...dz_m \\
& \quad + \int_{C_m} \cdots \int_{C_2} \sum_{i \geq 0} \binom{n_1}{i} S(v_3', (b_2, z_2) \\
& \quad ...(v, w)\, (c_1(i_1)...(b_1(i)c_t)(n_1 - i + i_t)...c_r(i_r)v_2)) \\
& \quad \cdot z_2^{n_2}...z_m^{n_m} dz_2...dz_m \\
& \quad = \sum_{l=2}^{m} \sum_{i \geq 0} \binom{n_1}{i} S(v_3', b_2(n_2)...(b_1(i)b_l)(n_1 + n_l - i)...b_m(n_m)(v, w)x_2) \\
& \quad + \sum_{i \geq 0} \binom{n_1}{i} S(v_3', b_2(n_2)...b_m(n_m)(b_1(i)v, w)x_2) w^{n_1 - i} \\
& \quad + S(v_3', b_2(n_2)...b_m(n_m)(v, w)(b_1(n_1)x_2)).
\end{aligned}
$$

This proves (3.33). Now let $x_3' = b_m(n_m) \dots b_1(n_1)v_3' \in \bar{M}^{3'}$, with $\mathrm{wt} b_i - n_i - 1 > 0$ for all i. We use induction on m to show that

$$
\int_C S(b_m(n_m)...b_1(n_1)v_3', (v, w)x_2) w^n dw = 0, \tag{3.34}
$$

for any fixed $v \in M^1$, $x_2 \in \bar{M}^2$, and $n \in \mathbb{Z}$ such that $n > \deg v + \deg x_2 - 1$. The base case $m = 0$ follows from the Corollary 3.10. Let $m > 0$, then by (3.30) and (3.32), we have:

$$
\int_C S(b_m(n_m)...b_1(n_1)v_3', (v, w)x_2) w^n dw
$$

$$= \int_C \int_{C_m} \dots \int_{C_1} S(v_3', (b_1, z_1)' \dots (b_m, z_m)'(v, w)x_2) z_1^{-n_1-2}$$
$$\dots z_m^{-n_m-2} w^n dz_1 \dots dz_m dw$$
$$= \sum_{j_1 \geq 0, \dots, j_m \geq 0} \frac{(-1)^{\mathrm{wt} b_1 + \dots + \mathrm{wt} b_m}}{j_1! \dots j_m!} \int_C \int_{C_m}$$
$$\dots \int_{C_1} S(v_3', (L(1)^{j_1} b_1, z_1) \dots (L(1)^{j_m} b_m, z_m)(v, w)x_2)$$
$$\cdot z_1^{2\mathrm{wt} b_1 - n_1 - 2 - j_1} \dots z_m^{2\mathrm{wt} b_m - n_m - 2 - j_m} w^n dz_1 \dots dz_m dw.$$
$$= \sum_{j_1 \geq 0, \dots, j_m \geq 0} \frac{(-1)^{\mathrm{wt} b_1 + \dots + \mathrm{wt} b_m}}{j_1! \dots j_m!} \int_C S(v_3', (L(1)^{j_1} b_1)(2\mathrm{wt} b_1 - n_1 - 2 - j_1) \dots$$
$$\dots (L(1)^{j_m} b_m)(2\mathrm{wt} b_m - n_m - 2 - j_m)(v, w)x_2). \tag{3.35}$$

It suffices to show that each summand in (3.35) is 0. For simplicity, we denote the term $(L(1)^{j_i} b_i)(2\mathrm{wt} b_i - n_i - 2 - j_i)$ by $c_i(r_i)$ for each i, note that

$$\mathrm{wt} c_1(r_1) = \mathrm{wt}(L(1)^{j_1} b_1)(2\mathrm{wt} b_1 - n_1 - 2 - j_1) = -\mathrm{wt} b_1 + n_1 + 1 < 0.$$

Then by (3.33), together with the definition formulas (3.32) and (3.31), we have:

$$\int_C S(v_3', c_1(r_1) \dots c_m(r_m)(v, w)x_2) w^n dw$$
$$= \sum_{l=2}^m \sum_{i \geq 0} \binom{r_1}{i} \int_C S(v_3', c_2(r_2) \dots (c_1(i) c_l)(r_1 + r_l - i) \dots c_m(r_m)(v, w)x_2) w^n dw$$
$$+ \sum_{i \geq 0} \binom{r_1}{i} \int_C S(v_3', c_2(r_2) \dots c_m(r_m)(c_1(i)v, w)x_2) w^{n + r_1 - i} dw$$
$$+ \int_C S(v_3', c_2(r_2) \dots c_m(r_m)(v, w)(c_1(r_1)x_2)) w^n dw$$
$$= \sum_{l=2}^m \sum_{i \geq 0} \binom{r_1}{i} \underbrace{\int_C S(c_m'(r_m) \dots (c_1(i) c_l)'(r_1 + r_l - i) \dots c_2'(r_2) v_3', (v, w)x_2) w^n dw}_{(1)}$$
$$+ \sum_{i \geq 0} \binom{r_1}{i} \underbrace{\int_C S(c_m'(r_m) \dots c_2'(r_2) v_3', (c_1(i)v, w)x_2) w^{n + r_1 - i} dw}_{(2)}$$
$$+ \underbrace{\int_C S(c_m'(r_m) \dots c_2'(r_2) v_3', (v, w)(c_1(r_1)x_2)) w^n dw}_{(3)}$$
$$= (1) + (2) + (3).$$

Since $\mathrm{wt} c_1 - r_1 - 1 < 0$ and $n > \deg v + \deg x_2 - 1$, we have

$$\deg(c_1(i)v) + \deg x_2 - 1 = \deg v + \deg x_2 - 1 + \mathrm{wt} c_1 - i - 1 < n + r_1 - i,$$
$$\deg v + \deg(c_1(r_1)x_2) - 1 = \deg v + \deg x_2 + \mathrm{wt} c_1 - r_1 - 1 - 1 < n,$$

for all $i \geq 0$. Then by the induction hypothesis, we have $(1) = (2) = (3) = 0$. This finishes the proof of (3.34). Hence S in (3.30) belongs to $\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}$. $\square$

So far in this subsection, by abuse of notations, we used the same symbol S (3.30) for the extension of a system of correlation functions S in (3.11). We denote the extended S in (3.30) by $\psi(S)$ for the rest of this subsection. Then by the Theorem 3.12, we have a linear map:

$$\psi : \mathrm{Cor}\begin{pmatrix} M^3(0) \\ M^1 \ M^2(0) \end{pmatrix} \to \mathrm{Cor}\begin{pmatrix} \bar{M}^2 \\ M^1 \ \bar{M}^2 \end{pmatrix}, \quad S \mapsto \psi(S), \tag{3.36}$$

which is an inverse of the restriction map φ in (3.9), with $M^2 = \bar{M}^2$ and $M^3 = \bar{M}^3$.

Corollary 3.13. *Let* $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. *Then the linear functional* f *in Definition 3.1 is given by* $f_v = o(v) = v(\deg v - 1) = \mathrm{Res}_z I(z, w) w^{\deg v - 1 + h}$, *where* $I \in I\binom{\bar{M}^2}{M^1\ \bar{M}^2}$ *is the intertwining operator corresponds to* $\psi(S)$ *in* $\mathrm{Cor}\binom{\bar{M}^2}{M^1\ \bar{M}^2}$.

Proof. By (3.3), we have $S(v_3', (v, w)v_2) = \langle v_3', f_v(v_2)\rangle w^{-\deg v}$, for all $v_3' \in M^3(0)^*$, $v_2 \in M^2(0)$, and $v \in M^1$. On the other hand, by (2.16),

$$S(v_3', (v, w)v_2) = \psi(S)(v_3', (v, w)v_2) = (v_3', I(v, w)v_2) = \langle v_3', v(\deg v - 1)v_2\rangle w^{-\deg v},$$

since $v(m)M^2(0) \subseteq M^3(\deg v - m - 1)$ for any $m \in \mathbb{Z}$. Thus, $f_v = v(\deg v - 1)$. $\square$

We finish this subsection by showing another property of the space of correlation functions associated with three modules. By (3.28) and (3.30), the $\psi(S)$ in (3.36) satisfies:

$$\begin{aligned} &\psi(S)(c_1(j_1)...c_m(j_m)v_3', (a_1, z_1)...(a_p, z_p)(v, w)b_1(i_1)...b_n(i_n)v_2) \\ &= \int_{C_1'} \cdots \int_{C_m'} \int_{C_n} \cdots \int_{C_1} S(v_3', (c_m, w_m)'...(c_1, w_1)'(a_1, z_1)...(v, w)(b_1, x_1)...(b_n, x_n)v_2) \\ &\cdot x_1^{i_1}...x_n^{i_n} w_1^{-j_1-2}...w_m^{-j_m-2} dx_1...dx_n dw_m...dw_1, \end{aligned} \tag{3.37}$$

where $v_3' \in M^3(0)^*$, $v_2 \in M^2(0)$, $v \in M^1$, $a_r, b_s, c_t \in V$ for all r, s, t, C_k' is a contour of w_k, C_l is a contour of x_l for all k, l, such that $C_1 \subset \cdots \subset C_n \subset C_1' \subset \cdots \subset C_m'$ (we use the subset symbol to indicate one contour is inside of the other), and $z_1, \ldots, z_n, w$ are outside of C_1' but inside of C_n.

By Proposition 3.7 and Theorem 6.2 in [2], we have an epimorphism of V-modules $\pi : \bar{M}(M^2(0)) \to \bar{M}^2$, where $\bar{M}(M^2(0))$ is the generalized Verma module with bottom level $M^2(0)$. Similarly, there is an epimorphism $\pi : \bar{M}(M^3(0)^*) \to \bar{M}^{3'}$. More generally, let N^2 and N^3 be any V-modules that are generated by their corresponding bottom levels, and assume that $N^2(0) = M^2(0)$ and $N^3(0) = M^3(0)$. Suppose there exist epimorphisms $\pi : N^2 \to \bar{M}^2$ and $\pi : N^{3'} \to \bar{M}^{3'}$.

If we write $\mathrm{Res}_z Y_N(b, z)z^j = b_j$ and $\mathrm{Res}_z Y_{\bar{M}}(b, z)z^j = b(j)$, then we have

$$\pi(c^1_{j_1}...c^m_{j_m} v_3') = c^1(j_1)...c^m(j_m)v_3', \quad \text{and} \quad \pi(b^1_{i_1}...b^n_{i_n} v_2) = b^1(i_1)...b^n(i_n)v_2,$$

where $c^k, b^l \in V$, $j_k, i_l \in \mathbb{Z}$ for all k, l, $v_3' \in M^3(0)^*$, and $v_2 \in M^2(0)$. Thus we have a linear map: $\pi^* : \mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2} \to \mathrm{Cor}\binom{N^3}{M^1\ N^2}$ that is given by:

$$\pi^*(S)(c^1_{j_1}...c^m_{j_m} v_3', (a_1, z_1)...(a_n, z_n)(v, w)b^1_{i_1}...b^n_{i_n} v_2)$$

$$= S(c^1(j_1)...c^m(j_m)v_3', (a_1, z_1)...(a_n, z_n)(v, w)b^1(i_1)...b^n(i_n)v_2). \tag{3.38}$$

Compose ψ and π^*, we have a linear map $\pi^*\psi : \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \to \mathrm{Cor}\binom{N^3}{M^1\ N^2}$. We claim that $\pi^*\psi$ is the inverse of the restriction map $\varphi : \mathrm{Cor}\binom{N^3}{M^1\ N^2} \to \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

Indeed, for $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, by (3.37) and (3.38), we have:

$$\begin{aligned}
&\varphi(\pi^*\psi)(S)(v_3', (a_1, z_1)...(a_n, z_n)(v, w)v_2) \\
&\quad = \psi(S)(\pi(v_3'), (a_1, z_1)...(a_n, z_n)(v, w)\pi(v_2)) \\
&\quad = S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)v_2),
\end{aligned}$$

where $v_2 \in M^2(0)$ and $v_3' \in M^3(0)^*$. Hence $\varphi(\pi^*\psi) = 1$. On the other hand, for $S \in \mathrm{Cor}\binom{N^3}{M^1\ N^2}$, again by (3.37) and (3.38), together with the fact that S satisfies (2.9) and (2.10), we have for any $c^1_{j_1}...c^m_{j_m}v_3' \in N^{3'}$, $b^1_{i_1}...b^n_{i_n}v_2 \in N^2$, $a_1, ..., a_n \in V$, and $v \in M^1$,

$$\begin{aligned}
&(\pi^*\psi)\varphi(S)(c^1_{j_1}...c^m_{j_m}v_3', (a_1, z_1)...(a_n, z_n)(v, w)b^1_{i_1}...b^n_{i_n}v_2) \\
&\quad = \psi(\varphi(S))(c^1(j_1)...c^m(j_m)v_3', (a_1, z_1)...(a_n, z_n)(v, w)b^1(i_1)...b^n(i_n)v_2) \\
&\quad = \int_{C_1'} ... \int_{C_m'} \int_{C_n} ... \int_{C_1} \varphi(S)(v_3', (c^m, w_m)' \\
&\quad ...(c^1, w_1)'(a_1, z_1)...(v, w)(b^1, x_1)...(b^n, x_n)v_2) \\
&\quad \cdot x_1^{i_1}...x_n^{i_n}w_1^{-j_1-2}...w_m^{-j_m-2}dx_1...dx_n dw_m...dw_1, \\
&= \int_{C_1'} ... \int_{C_m'} \int_{C_n} ... \int_{C_1} S(v_3', (c^m, w_m)'...(c^1, w_1)'(a_1, z_1) \\
&\quad ...(v, w)(b^1, x_1)...(b^n, x_n)v_2) \\
&\quad \cdot x_1^{i_1}...x_n^{i_n}w_1^{-j_1-2}...w_m^{-j_m-2}dx_1...dx_n dw_m...dw_1, \\
&\quad = S(c^1_{j_1}...c^m_{j_m}v_3', (a_1, z_1)...(a_n, z_n)(v, w)b^1_{i_1}...b^n_{i_n}v_2).
\end{aligned}$$

This shows $(\pi^*\psi)\varphi = 1$, and so we have $\mathrm{Cor}\binom{N^3}{M^1\ N^2} \cong \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. In particular, choose $N^2 = \bar{M}(M^2(0))$ and $N^3 = \bar{M}(M^3(0)^*)'$, then we have:

$$\mathrm{Cor}\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))} \cong \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong \mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2} \tag{3.39}$$

Now by (3.39), Corollary 2.6 and Theorem 3.12, we have the following theorem:

Theorem 3.14. *Let M^1 be a V-module, and let $M^2(0)$ and $M^3(0)$ be irreducible $A(V)$-modules, then we have the following isomorphism of vector spaces:*

$$I\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))} \cong \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong I\binom{\bar{M}^3}{M^1\ \bar{M}^2}. \tag{3.40}$$

If the VOA V is rational, then the generalized Verma module $\bar{M}(U)$ is an irreducible V-module for any irreducible $A(V)$-module U. Thus, $\bar{M}(M^2(0)) = \bar{M}^2 = L(M^2(0))$, and $\bar{M}(M^3(0)^*)' = \bar{M}^3 = L(M^3(0))$. On the other hand, by Theorem 2.2.2 in [13], if M^2 and M^3 are irreducible V-module, then $M^2(0)$ and $M^3(0)$ are irreducible $A(V)$-module.

Corollary 3.15. *Let V be an rational VOA, and let M^1, M^2, and M^3 be V-modules. Suppose M^2 and M^3 are irreducible, then we have* $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong I\binom{M^3}{M^1\ M^2}$.

Remark 3.16. Let W^2 and W^3 be any $\mathbb{N}$-gradable V-module that are generated by their corresponding bottom levels, and assume that $W^2(0) = M^2(0)$ and $W^3(0) = M^3(0)$. Then there exist epimorphisms: $\pi : \bar{M}(M^2(0)) \to W^2$, and $\pi : \bar{M}(M^3(0)^*) \to W^{3'}$. Similar to (3.38), π induces a linear map: $\pi^* : \mathrm{Cor}\binom{W^3}{M^1\ W^2} \hookrightarrow \mathrm{Cor}\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))}$, which is injective since π are surjective. Then by Corollary 2.6, (3.39), and (3.40), we have the following estimate for the fusion rule:

$$\dim I\begin{pmatrix} W^3 \\ M^1\ W^2 \end{pmatrix} \le \dim \mathrm{Cor}\begin{pmatrix} M^3(0) \\ M^1\ M^2(0) \end{pmatrix}. \tag{3.41}$$

4. $A(V)$-Bimodules and the Correlation Function S

In this section, we again assume that $M^2(0)$ and $M^3(0)$ are irreducible $A(V)$-modules. By Proposition 3.7, $L(0) = o(\omega) = h_2 \cdot \mathrm{Id}$ on $M^2(0)$, and $L(0) = h_3 \cdot \mathrm{Id}$ on $M^3(0)$, for some $h_2, h_3 \in \mathbb{C}$. Moreover, h_2 and h_3 are the conformal weights of $\bar{M}^2$ and $\bar{M}^3$, respectively.

We will show that $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$ can be identified with the vector space $(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$, where $B_h(M^1)$ is an $A(V)$-bimodule that is similar to the $A(V)$-bimodule $A_0(M^1)$ constructed in [7].

However, there are counter-examples showing that this identification is false if one replaces $B_h(M^1)$ by the $A(V)$-bimodule $A(M^1)$ constructed in Theorem 1.5.1 in [6] or $A_0(M^1)$ constructed in Section 4 of [7]. The reason is that the correct $L(-1)$-derivation property of the intertwining operators cannot be captured by $A(M^1)$ nor $A_0(M^1)$. We will see this by the end of this section.

4.1. The $A(V)$-Bimodule $B_\lambda(W)$.

Let W be a V-module with conformal weight h'. A sequence of $A_N(V)$-bimodules $A_N(W)$ are constructed by Huang and Yang in Section 4 of [7]. In particular, the $A_0(V) = A(V)$-bimodule $A_0(W)$ is defined as follows:

$A_0(W) = W/O_0(W)$, where $O_0(W) = span\{a \circ u, L(-1)u + (L(0) - h')u : \forall a \in V, u \in W\}$. It is proved (see Theorem 4.7 in [7]) that $A_0(W)$ is an $A(V)$-bimodule under the left and right actions: $a *_0 u = \mathrm{Res}_z Y_W(a, z)u \frac{(1+z)^{\mathrm{wt} a}}{z}$ and $v *_0 a = \mathrm{Res}_z Y_{WV}^W(u, z)a \frac{(1+z)^{\deg u}}{z}$, where Y_{WV}^W is defined by the skew-symmetry formula (5.1.5) in [4]:

$$Y_{WV}^W(u, z)a = e^{zL(-1)} Y_W(a, -z)u. \tag{4.1}$$

Now let $\lambda \in \mathbb{C}$ be a fixed complex number, we construct another $A(V)$-bimodule $B_\lambda(W)$ that is similar to $A_0(W)$ in the following way:

Definition 4.1. For homogeneous elements $u \in W$ and $a \in V$, define:

$$u \circ_{\substack{W\\WV}} a := \mathrm{Res}_z \left(Y_{WV}^W(u,z)a \frac{(1+z)^{\deg u+\lambda}}{z^2} \right), \tag{4.2}$$

then extend $\circ$ bilinearly to $\circ : W \times V \to W$. Let $O_{WV}^W(W)$ be the vector space spanned by elements (4.2) for all $a \in V$ and $u \in W$, and let $B_\lambda(W) := W/(O(W) + O_{WV}^W(W))$, where $O(W) = span\{a \circ u = \mathrm{Res}_z \left(Y_W(a,z)u \frac{(1+z)^{\mathrm{wt} a}}{z^2} \right) : \forall a \in V, u \in W\}$.

Lemma 4.2. *Let $u \in W$ and $a \in V$ by homogeneous elements, and $m \geq n \geq 0$. Then*

$$\mathrm{Res}_z Y_{WV}^W(u,z)a \frac{(1+z)^{\deg u+\lambda+n}}{z^{2+m}} \in O_{WV}^W(W). \tag{4.3}$$

Proof. Since $Y_{WV}^W(L(-1)u,z) = \frac{d}{dz} Y_{WV}^W(u,z)$, the proof of (4.3) is almost the same as the proof of Lemma 2.1.2 in [13], we omit the details. □

Recall that the module actions of $A(V)$ on its bimodule $A(W)$ are given by:

$$b * v = \mathrm{Res}_z \left(Y_W(b,z)v \frac{(1+z)^{\mathrm{wt} b}}{z} \right), \quad \text{and} \quad v * b = \mathrm{Res}_z \left(Y_W(b,z)v \frac{(1+z)^{\mathrm{wt} b-1}}{z} \right),$$

where $b \in V$ is homogeneous and $v \in W$ (see Definition 1.5.2 in [6]).

Lemma 4.3. $b * O_{WV}^W(W) \subseteq O_{WV}^W(W)$ *and* $O_{WV}^W(W) * b \subseteq O_{WV}^W(W)$, *for all* $b \in V$.

Proof. Let $u \in W$ and $b \in V$ be homogeneous, and let $a \in V$. By Definition 4.1, Lemma 4.2, and the Jacobi identity of the intertwining operator Y_{WV}^W, we have:

$$\begin{aligned}
b * (u \circ_{\substack{W\\WV}} a) &\equiv \mathrm{Res}_{z_1} Y_W(b,z_1) \frac{(1+z_1)^{\mathrm{wt} b}}{z_1} \mathrm{Res}_{z_2} Y_{WV}^W(u,z_2)a \frac{(1+z_2)^{\deg u+\lambda}}{z_2^2} \\
&\quad - \mathrm{Res}_{z_2} Y_{WV}^W(u,z_2) \frac{(1+z_2)^{\deg u+\lambda}}{z_2^2} \mathrm{Res}_{z_1} Y_V(u,z_1)a \frac{(1+z_1)^{\mathrm{wt} b}}{z_1} \pmod{O_{WV}^W(W)} \\
&= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} Y_{WV}^W(Y_W(b,z_0),z_2)a \frac{(1+z_2+z_0)^{\mathrm{wt} b}}{z_2+z_0} \cdot \frac{(1+z_2)^{\deg u+\lambda}}{z_2^2} \\
&= \mathrm{Res}_{z_0} \mathrm{Res}_{z_2} \sum_{i=0}^{\mathrm{wt} b} \sum_{j \geq 0} Y_{WV}^W(Y_W(b,z_0)u,z_2)a \binom{\mathrm{wt} b}{i} (-1)^j z_0^{i+j} \frac{(1+z_2)^{\deg u+\lambda+\mathrm{wt} b-i}}{z_2^{2+j+1}} \\
&= \sum_{i=0}^{\mathrm{wt} b} \sum_{j \geq 0} \binom{\mathrm{wt} b}{i} \mathrm{Res}_{z_2} Y_{WV}^W(b_{i+j}u,z_2)a \frac{(1+z_2)^{\deg(b_{i+j}u)+\lambda+(j+1)}}{z_2^{2+(j+1)}} \\
&\equiv 0 \pmod{O_{WV}^W(W)},
\end{aligned}$$

where the last congruence follows from Lemma 4.2. By a similar computation, we have:

$$\begin{aligned}
(u \circ_{\substack{W\\WV}} a) * b &\equiv \sum_{i=0}^{\mathrm{wt} b-1} \sum_{j \geq 0} \binom{\mathrm{wt} b-1}{i} \mathrm{Res}_{z_2} Y_{WV}^W(b_{i+j}u,z_2)a \frac{(1+z_2)^{\deg(b_{i+j}u)+\lambda+j}}{z_2^{2+(j+1)}} \\
&\equiv 0 \pmod{O_{WV}^W(W)}.
\end{aligned}$$

Hence we have $b * O_{WV}^W(W) \subseteq O_{WV}^W(W)$, and $O_{WV}^W(W) * b \subseteq O_{WV}^W(W)$. □

By Lemma 4.3 and Theorem 1.5.1 in [6], $B_\lambda(W) = W/(O(W) + O_{WV}^{W}(W))$ has an $A(V)$-bimodule structure with respect to $b * v$ and $v * b$. Moreover, $B_\lambda(W)$ is a quotient module of $A(W)$. In particular, we have the following formula holds in $B_\lambda(W)$:

$$a * u - u * a \equiv \sum_{j \geq 0} \binom{\mathrm{wt} a - 1}{j} a(j)u \pmod{O_{WV}^{W}(W) + O(W)}, \tag{4.4}$$

where $a \in V$ homogeneous, and $u \in W$. Let

$$O_\lambda(W) := span\{a \circ u,\ L(-1)u + (L(0) - h' + \lambda)u : \forall a \in V, u \in W\} \subset W. \tag{4.5}$$

Lemma 4.4. *For any $u \in W$, we have $L(-1)u + (L(0) - h' + \lambda)u \in O_{WV}^{W}(W)$.*

Proof. Let $u \in W$ be homogeneous. Since $\deg u = (L(0) - h')u$, we have:

$$\begin{aligned} u \circ_{\substack{W \\ WV}} \mathbf{1} &= \mathrm{Res}_z e^{zL(-1)} Y_W(\mathbf{1}, -z)u \frac{(1+z)^{\deg u + \lambda}}{z^2} \\ &= \mathrm{Res}_z \sum_{j \geq 0} \frac{z^j}{j!} L(-1)^j \sum_{i=0}^{\deg u + \lambda} \binom{\deg u + \lambda}{i} z^{i-2} \\ &= \binom{\deg u + \lambda}{0} L(-1)u + \binom{\deg u + \lambda}{1} L(-1)^0 u \\ &= (L(-1) + L(0) - h' + \lambda)u. \end{aligned}$$

Hence $(L(-1) + (L(0) - h' + \lambda))u \in O_{WV}^{W}(W)$. □

Lemma 4.5. *We have $O(W) + O_{WV}^{W}(W) = O_\lambda(W)$. In particular, $B_\lambda(W) = W/O_\lambda(W)$.*

Proof. By Lemma 4.4, we only need to show that $O_{WV}^{W}(W) \subseteq O_\lambda(W)$. Similar to the proof of Lemma 2.1.3 in [13], for any homogeneous $u \in W$ and $a \in V$, we have: $Y_{WV}^{W}(u, z)a \equiv (1+z)^{-\deg u - \lambda - \mathrm{wt} a} Y_W\left(a, \frac{-z}{1+z}\right) u \pmod{O_\lambda(W)}$. It follows that

$$\begin{aligned} u \circ_{\substack{W \\ WV}} a &= \mathrm{Res}_z Y_{WV}^{W}(u, z)a \frac{(1+z)^{\deg u + \lambda}}{z^2} \\ &\equiv \mathrm{Res}_z Y_W\left(a, \frac{-z}{1+z}\right) u \frac{(1+z)^{-\mathrm{wt} a}}{z^2} \pmod{O_\lambda(W)} \\ &\equiv -\mathrm{Res}_w Y_W(a, w)u \frac{(1+w)^{\mathrm{wt} a}}{w^2} \pmod{O_\lambda(W)}. \end{aligned}$$

Hence $u \circ_{\substack{W \\ WV}} a \equiv -a \circ u \pmod{O_\lambda(W)}$, and so $O_{WV}^{W}(W) + O(W) = O_\lambda(W)$. □

Now let $W = M^1$, and $\lambda = h = h_1 + h_2 - h_3$. Then by (4.5) and Lemma 4.5, $B_h(M^1) = M^1/O_h(M^1)$, where $O_h(M^1) = span\{a \circ u,\ L(-1)u + (L(0) + h_2 - h_3)u : \forall a \in V, u \in M^1\}$.

Lemma 4.6. *Let $I \in I\binom{\bar{M}^3}{M^1\ \bar{M}^2}$, then the linear map*

$$o : M^1 \to \mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0)),\ o(v) = v(\deg v - 1) = \mathrm{Res}_z I(v, z) z^{\deg v - 1 + h}$$

factors through $B_h(M^1) = M^1/O_h(M^1)$.

Proof. By Lemma 4.5, we need to show that $o(O_h(M^1)) = 0$. By Lemma 1.5.2 in [6], we already have $o(a \circ u) = 0$ for all $a \in V$ and $u \in M^1$. Furthermore, by the $L(-1)$-derivation property of I, we have:

$$\begin{aligned} o(L(-1)v) &= \mathrm{Res}_z I(L(-1)v, z) z^{\deg v + 1 - 1 + h} \\ &= \mathrm{Res}_z \left(\frac{d}{dz} I(v, z) \right) z^{\deg v + h} \\ &= \mathrm{Res}_z I(v, z)(-\deg v - h) z^{\deg v + h - 1} \\ &= -((L(0) - h_1 + h)v)(\deg v - 1) \\ &= -o((L(0) + h_2 - h_3)v). \end{aligned}$$

Hence $o(O_h(M^1)) = 0$, and so o factors through $B_h(M^1)$. ◻

Proposition 4.7. *There exists an injective linear map:*

$$\begin{aligned} \nu : \mathrm{Cor}\begin{pmatrix} M^3(0) \\ M^1 \; M^2(0) \end{pmatrix} &\to (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^* \\ S \mapsto f_S, \qquad & f_S(v_3' \otimes v \otimes v_2) := \langle v_3', f_v(v_2) \rangle, \end{aligned} \tag{4.6}$$

where we use the same symbol v for its image in $B_h(M^1)$.

Proof. First, we have $f_v = o(v)$ by Corollary 3.13, where $o(v) = \mathrm{Res}_w I(v, w) w^{\deg v - 1 + h}$, and $I \in I\binom{\bar{M}^3}{M^1\ \bar{M}^2}$ is the intertwining operator corresponds to $\psi(S)$ in $\mathrm{Cor}\binom{\bar{M}^3}{M^1\ \bar{M}^2}$, see Theorem 3.14. Moreover, it follows from Lemma 4.6 that $o(O_h(M^1)) = 0$. Hence ν is well-defined. The injectivity of ν follows from Lemma 3.3. ◻

Remark 4.8. Although our definition for $B_h(M^1)$ is similar to the $A(V)$-bimodule $A_0(M^1)$ constructed by Huang and Yang in [7], they are not isomorphic as $A(V)$-bimodules. We will give a counter-example in the next subsection.

Our goal next is to construct an inverse map of ν in (4.6). Given a $f \in (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$, we need to construct a corresponding system of correlation functions S in $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. Our strategy is to use the recursive formulas (3.4) and (3.6) and construct the system of functions S inductively. The key is to show the locality ((2) in Definition 2.1) in each step, which can be achieved by the properties of the $A(V)$-bimodule $B_h(M^1)$, together with the formula (3.8).

4.2. The Construction of 4*-Point and* 5*-Point Functions.* From now on, we fix a linear function f on the vector space $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)$.

Definition 4.9. Define $S_M : M^3(0)^* \times M^1 \times M^2(0) \to \mathcal{F}(w)$ by

$$S_M(v_3', (v, w)v_2) := f(v_3' \otimes v \otimes v_2) w^{-\deg v}, \tag{4.7}$$

where on the right-hand side we use the same symbol v for its image $v + O(M^1)$ in $B_h(M^1)$.

Define $S^L_{VM}: M^3(0)^* \times V \times M^1 \times M^2(0) \to \mathcal{F}(z, w)$ by

$$\begin{aligned} S^L_{VM}(v_3', (a, z)(v, w)v_2) &:= S_M(v_3' o(a), (v, w)v_2) z^{-\mathrm{wt} a} \\ &+ \sum_{i\geq 0} F_{\mathrm{wt} a, i}(z, w) S_M(v_3', (a(i)v, w)v_2). \end{aligned} \tag{4.8}$$

Finally, define $S^R_{MV}: M^3(0)^* \times M^1 \times V \times M^2(0) \to \mathcal{F}(z, w)$ by

$$\begin{aligned} S^R_{MV}(v_3', (v, w)(a, z)v_2) &:= S_M(v_3', (v, w) o(a) v_2) z^{-\mathrm{wt} a} \\ &+ \sum_{i\geq 0} G_{\mathrm{wt} a, i}(z, w) S_M(v_3', (a(i)v, w)v_2). \end{aligned} \tag{4.9}$$

The upper index L (resp. R) in the 4-point functions S indicates that we use the expansion formula for the left (resp. right) most term, namely, (3.4) (resp.(3.6)) to construct the new S from the 3-point function. We will denote the 3-point function S_M by S.

Proposition 4.10. *As rational functions in $\mathcal{F}(z, w)$, we have:*

$$S^L_{VM}(v_3', (a, z)(v, w)v_2) = S^R_{MV}(v_3', (v, w)(a, z)v_2).$$

Proof. By Definition 4.9, (3.8), and the property of $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)$,

$$\begin{aligned} &S^L_{VM}(v_3', (a, z)(v, w)v_2) - S^R_{MV}(v_3', (v, w)(a, z)v_2) \\ &= f(v_3' o(a) \otimes v \otimes v_2) w^{-\deg v} z^{-\mathrm{wt} a} - f(v_3' \otimes v \otimes o(a) v_2) w^{-\deg v} z^{-\mathrm{wt} a} \\ &+ \sum_{i\geq 0} (F_{\mathrm{wt} a, i}(z, w) - G_{\mathrm{wt} a, i}(z, w)) S_M(v_3', (a(i)v, w)v_2) \\ &= f(v_3' \otimes a * v \otimes v_2) w^{-\deg v} z^{-\mathrm{wt} a} - f(v_3' \otimes v * a \otimes v_2) w^{-\deg v} z^{-\mathrm{wt} a} \\ &- \sum_{i\geq 0} \binom{\mathrm{wt} a - 1}{i} f(v_3' \otimes a(i)v \otimes v_2) w^{-\deg v - \mathrm{wt} a + i + 1} z^{-\mathrm{wt} a} w^{\mathrm{wt} a - 1 - i} \\ &= f(v_3' \otimes (a * v - v * a) \otimes v_2) w^{-\deg v} z^{-\mathrm{wt} a} \\ &- \sum_{i\geq 0} \binom{\mathrm{wt} a - 1}{i} f(v_3' \otimes a(i)v \otimes v_2) z^{-\mathrm{wt} a} w^{-\deg v}. \end{aligned}$$

By (4.4), we also have $a * v - v * a = \sum_{i\geq 0} \binom{\mathrm{wt} a - 1}{i} a(i)v$ holds in the $A(V)$-bimodule $B_h(M^1)$. Hence $S^L_{VM}(v_3', (a, z)(v, w)v_2) - S^R_{MV}(v_3', (v, w)(a, z)v_2) = 0$. □

By Proposition 4.10, the 4-point functions S^L_{VM} and S^R_{MV} in definition 4.9 give rise to one single 4-point function S that satisfies

$$S(v_3', (a, z)(v, w)v_2) = S(v_3', (v, w)(a, z)v_2), \tag{4.10}$$

and this function can be defined either by (4.8) or (4.9).

We adopt a similar method to construct 5-point functions. As long as the term (v, w) does not appear at the left-most place, we use the formula (3.4) to construct S from the 4-point function; if (v, w) appears at the left-most place, we use (3.6) to construct S.

Definition 4.11. Define the 5-point functions with the upper index L,

$$S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2), \quad \text{and} \quad S^L_{VVM}(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2),$$

by expanding (a_1, z_1) from the left, which is given by the common formula:

$$\begin{aligned} &S(v_3' o(a_1), (v, w)(a_2, z_2)v_2)z_1^{-\mathrm{wt} a_1} + \sum_{j\geq 0} F_{\mathrm{wt} a_1, j}(z_1, w) S(v_3', (a_1(j)v, w)(a_2, z_2)v_2) \\ &\quad + \sum_{j\geq 0} F_{\mathrm{wt} a_1, j}(z_1, z_2) S(v_3', (v, w)(a_1(j)a_2, z_2)v_2). \end{aligned} \tag{4.11}$$

Define the 5-point functions with upper index R,

$$S^R_{VMV}(v_3', (a_2, z_2)(v, w)(a_1, z_1)v_2), \quad \text{and} \quad S^R_{MVV}(v_3', (v, w)(a_2, z_2)(a_1, z_1)v_2),$$

by expanding (a_1, z_1) from the right, which is given by the common formula:

$$\begin{aligned} &S(v_3', (a_2, z_2)(v, w)o(a_1)v_2)z_1^{-\mathrm{wt} a_1} \\ &\quad + \sum_{j\geq 0} G_{\mathrm{wt} a_1, j}(z_1, w) S(v_3', (a_2, z_2)(a_1(j)v, w)v_2) \\ &\quad + \sum_{j\geq 0} G_{\mathrm{wt} a_1, j}(z_1, z_2) S(v_3', (a_1(j)a_2, z_2)(v, w)v_2). \end{aligned} \tag{4.12}$$

The function S in (4.11) and (4.12) is the (common) 4-point function in Definition 4.9. By (4.10), it makes sense to define S^L_{VMV} and S^L_{VVM} by the same formula, same for S^R_{VMV} and S^R_{MVV}. We will show that all the 5-point functions in Definition 4.11 are the same. First, we observe that the term $S_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2)$ has the following two expressions: $S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2)$ and $S^R_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2)$.

Proposition 4.12. *If* (4.11)=(4.12)*, then we have:*

$$S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) = S^R_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2).$$

Proof. Note that (4.11) is a generalization of the function (2.2.6) in [13]. By a similar calculation, it is easy to see that the formula (2.2.11) in [13] also holds for our case. i.e., we can swap the terms (a_1, z_1) and (a_2, z_2) in S^L_{VVM}:

$$S^L_{VVM}(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2) = S^L_{VVM}(v_3', (a_2, z_2)(a_1, z_1)(v, w)v_2). \tag{4.13}$$

By the assumption that (4.11)=(4.12), Definition 4.11, and (4.13), we have:

$$\begin{aligned} &S^L_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) \\ &\quad = S^L_{VVM}(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2) \\ &\quad = S^L_{VVM}(v_3', (a_2, z_2)(a_1, z_1)(v, w)v_2) \\ &\quad = S^L_{VMV}(v_3', (a_2, z_2)(v, w)(a_1, z_1)v_2) \\ &\quad = S^R_{VMV}(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2), \end{aligned}$$

where the last equality follows from the assumption that (4.11)=(4.12). $\square$

Next, we show that (4.11)=(4.12). We use symbols (1), (2), and (3) to denote the difference of the three summands in the term (4.11)-(4.12):

$$S(v_3'o(a_1),(v,w)(a_2,z_2)v_2)z_1^{-\mathrm{wt}a_1}-S(v_3',(a_2,z_2)(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}. \tag{1}$$

$$\sum_{j\geq 0}(F_{\mathrm{wt}a_1,j}(z_1,w)-G_{\mathrm{wt}a_1,j}(z_1,w))S(v_3',(a_1(j)v,w)(a_2,z_2)v_2). \tag{2}$$

$$\sum_{j\geq 0}(F_{\mathrm{wt}a_1,j}(z_1,z_2)-G_{\mathrm{wt}a_1,j}(z_1,z_2))S(v_3',(v,w)(a_1(j)a_2,z_2)v_2). \tag{3}$$

So we need to show that (1)+(2)+(3)=0.

By (4.10), we may use the formula (4.8) and expand both terms in (1) with respect to (a_2,z_2) from the left. Then (1) can be expressed as:

$$\begin{aligned}
&S(v_3'o(a_1),(v,w)(a_2,z_2)v_2)z_1^{-\mathrm{wt}a_1}-S(v_3',(a_2,z_2)(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}\\
&\quad=S(v_3'o(a_1)o(a_2),(v,w)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}\\
&\quad+\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2,w)S(v_3'o(a_1),(a_2(i)v,w)v_2)z_1^{-\mathrm{wt}a_1}\\
&\quad-S(v_3'o(a_2),(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}\\
&\quad+\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2,w)S(v_3',(a_2(i)v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}\\
&\quad=\underbrace{f(v_3'\otimes a_1*a_2*v\otimes v_2)w^{-\deg v}z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}}_{(11)}\\
&\quad-\underbrace{f(v_3'\otimes a_2*v*a_1\otimes v_2)w^{-\deg v}z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}}_{(12)}\\
&\quad+\sum_{i\geq 0}\underbrace{F_{\mathrm{wt}a_2,i}}_{(13)}(z_2,w)f(v_3'\otimes(a_1*(a_2(i)v))\\
&-(a_2(i)v)*a_1)\otimes v_2)w^{-\mathrm{wt}a_2-\deg v+i+1}z_1^{-\mathrm{wt}a_1}\\
&=(11)+(12)+(13).
\end{aligned}$$

For the term (2), we use the formula (4.8) agian and expand each summand in (2) with respect to (a_2,z_2) from the left. Then by (3.8), (2) can be expressed as:

$$\begin{aligned}
(2)&=\sum_{j\geq 0}F_{\mathrm{wt}a_1,j}(z_1,w)S(v_3'o(a_2),(a_1(j)v,w)v_2)z_2^{-\mathrm{wt}a_2}\\
&+\sum_{j\geq 0}\sum_{i\geq 0}F_{\mathrm{wt}a_1,j}(z_1,w)F_{\mathrm{wt}a_2,i}(z_2,w)S(v_3',(a_2(i)a_1(j)v,w)v_2)\\
&-\sum_{j\geq 0}G_{\mathrm{wt}a_1,j}(z_1,w)S(v_3'o(a_2),(a_1(j)v,w)v_2)z_2^{-\mathrm{wt}a_2}\\
&-\sum_{j\geq 0}\sum_{i\geq 0}G_{\mathrm{wt}a_1,j}(z_1,w)F_{\mathrm{wt}a_2,i}(z_2,w)S(v_3',(a_2(i)a_1(j)v,w)v_2)\\
&=\sum_{j\geq 0}-\binom{\mathrm{wt}a_1-1}{j}\underbrace{S(v_3'o(a_2),(a_1(j)v,w)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}w^{\mathrm{wt}a_1-1-j}}_{(21)}
\end{aligned}$$

$$+\sum_{j\geq 0}\sum_{i\geq 0}-\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}w^{\mathrm{wt}a_1-1-j}\underbrace{F_{\mathrm{wt}a_2,w}(z_2,w)S(v_3',(a_2(i)a_1(j)v,w)v_2)}_{(22)}$$
$$=(21)+(22).$$

Finally, for the term (3), we expand each of its summand with respect to $(a_1(j)a_2, z_2)$ from the left, so (3) can be expressed as:

$$\begin{aligned}
(3) &= \sum_{j\geq 0}F_{\mathrm{wt}a_1,j}(z_1,z_2)S(v_3'o(a_1(j)a_2),(v,w)v_2)z_2^{-\mathrm{wt}a_1-\mathrm{wt}a_2+j+1}\\
&+\sum_{j\geq 0}\sum_{i\geq 0}F_{\mathrm{wt}a_1,j}(z_1,z_2)F_{\mathrm{wt}a_1+\mathrm{wt}a_2-j-1,i}(z_2,w)S(v_3',((a_1(j)a_2)(i)v,w)v_2)\\
&-\sum_{j\geq 0}G_{\mathrm{wt}a_1,j}(z_1,z_2)S(v_3'o(a_1(j)a_2),(v,w)v_2)z_2^{-\mathrm{wt}a_1-\mathrm{wt}a_2+j+1}\\
&+\sum_{j\geq 0}\sum_{i\geq 0}G_{\mathrm{wt}a_1,j}(z_1,z_2)F_{\mathrm{wt}a_1+\mathrm{wt}a_2-j-1,i}(z_2,w)S(v_3',((a_1(j)a_2)(i)v,w)v_2)\\
&=\sum_{j\geq 0}-\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}\underbrace{z_2^{\mathrm{wt}a_1-1-j}S(v_3'o(a_1(j)a_2),(v,w)v_2)z_2^{-\mathrm{wt}a_1-\mathrm{wt}a_2+j+1}}_{(31)}\\
&+\sum_{j\geq 0}\sum_{i\geq 0}-\binom{\mathrm{wt}a_1-1}{j}\\
&\quad z_1^{-\mathrm{wt}a_1}z_2^{\mathrm{wt}a_1-1-j}\underbrace{F_{\mathrm{wt}a_1+\mathrm{wt}a_2-j-1,i}(z_2,w)S(v_3',(a_1(j)a_2)(i)(v,w)v_2)}_{(32)}\\
&=(31)+(32).
\end{aligned}$$

We need to show that $(11)+(12)+(13)+(21)+(22)+(31)+(32)=0$. In fact, since $a*v-v*a=\mathrm{Res}_zY(a,z)v(1+z)^{\mathrm{wt}a-1}=\sum_{j\geq 0}\binom{\mathrm{wt}a-1}{j}a(j)v$ in $B_h(M^1)$, see (4.4), and $a_1*a_2-a_2*a_1=\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}a_1(j)a_2$ in $A(V)$, we can rewrite (21) and (31) as:

$$\begin{aligned}
(21) &= -\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}w^{-\mathrm{wt}a_1-\deg v+j+1}z_1^{\mathrm{wt}a_1}z_2^{\mathrm{wt}a_2}w^{\mathrm{wt}a_1-j-1}\\
&\quad f(v_3'o(a_2)\otimes a_1(j)v\otimes v_2)\\
&\quad =-w^{-\deg v}z_1^{-\mathrm{wt}a_1}z_2^{\mathrm{wt}a_2}f(v_3'\otimes(a_2*a_1*v-a_2*v*a_1)\otimes v_2);\\
(31) &= -\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}w^{-\deg v}f(v_3'o(a_1(j)a_2)\otimes v\otimes v_2)\\
&\quad =-z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}w^{-\deg v}f(v_3'\otimes(a_1*a_2*v-a_2*a_1*v)\otimes v_2).
\end{aligned}$$

Then by the bimodule property of $B_h(M^1)$, we have:

$$\begin{aligned}
&(11)+(12)+(21)+(31)\\
&\quad = f(v_3'\otimes a_1*a_2*v\otimes v_2)w^{-\deg v}z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}\\
&\quad - f(v_3'\otimes a_2*v*a_1\otimes v_2)w^{-\deg v}z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}
\end{aligned}$$

$$- w^{-\deg v} z_1^{-\mathrm{wt} a_1} z_2^{\mathrm{wt} a_2} f(v_3' \otimes (a_2 * a_1 * v - a_2 * v * a_1) \otimes v_2)$$
$$- z_1^{-\mathrm{wt} a_1} z_2^{-\mathrm{wt} a_2} w^{-\deg v} f(v_3' \otimes (a_1 * a_2 * v - a_2 * a_1 * v) \otimes v_2) = 0.$$

It remains to show that (13) + (22) + (32) = 0.

Lemma 4.13. *Let M be a V module, and let $a_1, a_2 \in V$, $v \in M$, and $n \in \mathbb{N}$. We have:*

$$\begin{aligned}&\sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j}\binom{\mathrm{wt} a_2 + n}{i}(a_1(j)a_2(i)v - a_2(i)a_1(j)v)\\ &\quad = \sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j}\binom{\mathrm{wt} a_1 + \mathrm{wt} a_2 - j - 1 + n}{i}(a_1(j)a_2)(i)v\end{aligned} \tag{4.14}$$

Proof: Choose complex variables z_1, z_2 in the domain $|z_1| < 1,\ |z_2| < 1,\ |z_1 - z_2| < |1 + z_2|$.
By the Jacobi identity in the residue form, the left-hand side of (4.14) can be written as:

$$\mathrm{Res}_{z_1,z_2} \sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j}\binom{\mathrm{wt} a_2 + n}{i} z_1^j z_2^i (Y(a_1, z_1)Y(a_2, z_2)v - Y(a_2, z_2)Y(a_1, z_1)v)$$
$$= \mathrm{Res}_{z_1,z_2}(1 + z_1)^{\mathrm{wt} a_1 - 1}(1 + z_2)^{\mathrm{wt} a_2 + n}(Y(a_1, z_1)Y(a_2, z_2)v - Y(a_2, z_2)Y(a_1, z_1)v)$$
$$= \mathrm{Res}_{z_2}\mathrm{Res}_{z_1 - z_2}(1 + z_2 + (z_1 - z_2))^{\mathrm{wt} a_1 - 1}(1 + z_2)^{\mathrm{wt} a_2 + n} Y(Y(a_1, z_1 - z_2)a_2, z_2)v$$
$$= \mathrm{Res}_{z_2}\mathrm{Res}_{z_1 - z_2} \sum_{j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j}(1 + z_2)^{\mathrm{wt} a_1 - 1 - j + \mathrm{wt} a_2 + n}$$
$$(z_1 - z_2)^j Y(Y(a_1, z_1 - z_2)a_2, z_2)v$$
$$= \sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j}\binom{\mathrm{wt} a_1 + \mathrm{wt} a_2 - j - 1 + n}{i}(a_1(j)a_2)(i)v,$$

which is the right-hand side of (4.14). □

We use the formula (4.4) again and rewrite (13) as:

$$(13) = \sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} z_1^{-\mathrm{wt} a_1} w^{-\mathrm{wt} a_2 - \deg v + i + 1} F_{\mathrm{wt} a_2, i}(z_2, w) f(v_3' \otimes a_1(j)a_2(i)v \otimes v_2).$$

Since the map $\iota_{z_2,w}$ is injective (see Section 3 in [4]), we only need to show that $\iota_{z_2,w}((13) + (22) + (32)) = 0$. By (3.5), $\iota_{z_2,w}(F_{\mathrm{wt} a_2,i}(z_2, w))$ can be written as:

$$\iota_{z_2,w}(F_{\mathrm{wt} a_2,i}(z_2, w)) = \sum_{n\geq 0} \binom{\mathrm{wt} a_2 + n}{i} w^{\mathrm{wt} a_2 + n - i} z_2^{-\mathrm{wt} a_2 - n - 1}$$

To simplify our notation, we denote $z_1^{\mathrm{wt} a_1} w^{-\deg v + n + 1} z_2^{-\mathrm{wt} a_2 - n - 1}$ by γ. By Lemma 4.13,

$$\iota_{z_2,w}(13) + \iota_{z_2,w}(22)$$
$$= \sum_{i,j\geq 0} \binom{\mathrm{wt} a_1 - 1}{j} z_1^{\mathrm{wt} a_1} w^{-\mathrm{wt} a_2 - \deg v + i + 1} \left(\sum_{n\geq 0} \binom{\mathrm{wt} a_2 + n}{i} w^{\mathrm{wt} a_2 + n - i} z_2^{-\mathrm{wt} a_2 - n - 1}\right)$$
$$\cdot (f(v_3' \otimes a_1(j)a_2(i)v \otimes v_2) - f(v_3' \otimes a_2(i)a_1(j)v \otimes v_2))$$

$$= \sum_{i,j,n\geq 0} \binom{\mathrm{wt}a_1 - 1}{j}\binom{\mathrm{wt}a_2 + n}{i}\gamma \cdot f(v_3' \otimes (a_1(j)a_2(i)v - a_2(i)a_1(j)v) \otimes v_2)$$
$$= \sum_{i,j,n\geq 0} \binom{\mathrm{wt}a_1 - 1}{j}\binom{\mathrm{wt}a_1 + \mathrm{wt}a_2 + n - j - 1}{i}\gamma \cdot f(v_3' \otimes (a_1(j)a_2)(i)v \otimes v_2)$$
$$= -\iota_{z_2,w}(32).$$

Now the proof of (4.11)=(4.12) is complete.

Therefore, the 5 point functions in Definition 4.11 give rise to one single 5-point function S that satisfies:

$$\begin{aligned} &S(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2) = S(v_3', (a_2, z_2)(a_1, z_1)(v, w)v_2) \\ &= S(v_3', (a_1, z_1)(v, w)(a_2, z_2)v_2) = S(v_3', (a_2, z_2)(v, w)(a_1, z_1)v_2) \\ &= S(v_3', (v, w)(a_1, z_1)(a_2, z_2)v_2) = S(v_3', (v, w)(a_2, z_2)(a_1, z_1)v_2). \end{aligned} \tag{4.15}$$

In particular, the 5-point function S satisfies the locality in Definition 2.1, with $v_3' \in M^3(0)^*$ and $v_2 \in M^2(0)$. Moreover, $S(v_3', (a_1, z_1)(a_2, z_2)(v, w)v_2)$ also satisfies both of the recursive formula (3.4) and (3.6) by its definition.

4.3. Construction of $(n+3)$*-Point Functions.* We construct the general $(n+3)$-point function S using induction on n. We have finished the base cases $n = 1, 2$ in the previous subsection. Now assume the $(n+2)$-point function:

$$S : M^3(0)^* \times V \times \cdots \times M^1 \times \cdots \times V \times M^2(0) \to \mathcal{F}(z_1, \ldots, z_{n-1}, w)$$

exist and satisfy the following two properties: Let $\{(b_1, w_1), (b_2, w_2), \ldots, (b_n, w_n)\}$ be the same set as $\{(a_1, z_1), \ldots, (a_{n-1}, z_{n-1}), (v, w)\}$. The first property is the locality:

$$S(v_3', (a_1, z_1)(a_2, z_2)...(a_{n-1}, z_{n-1})(v, w)v_2) = S(v_3', (b_1, w_1)(b_2, w_2)...(b_n, w_n)v_2), \tag{I}$$

that is, the terms (a_1, z_1),(a_2, z_2), $\ldots$, (a_{n-1}, z_{n-1}), and (v, w) can be permutated arbitrarily within S. Denote by S^L (resp. S^R) the expansion of the $(n+1)$-point function S with respect to the left (resp. right)-most term using (3.4) (resp. (3.6)). The second property is that

$$\begin{aligned} S(v_3', (b_1, w_1)(b_2, w_2)...(b_n, w_n)v_2) &= S^L(v_3', (b_1, w_1)(b_2, w_2)...(b_n, w_n)v_2) \\ &= S^R(v_3', (b_1, w_1)(b_2, w_2)...(b_n, w_n)v_2), \end{aligned} \tag{II}$$

where (b_1, w_1) in S^L is not (v, w), and (b_n, w_n) in S^R is not (v, w).

Note that properties (I) and (II) are satisfied by the 4-point and 5-point functions (see (4.10) and (4.15).) We construct $(n+3)$-point functions as follows:

Definition 4.14. Assume the number of V in the sub-indices of $S^L_{VV\ldots M^1\ldots V}$ and $S^R_{V\ldots M^1\ldots VV}$ are both equal to n, the sub-index M^1 in S^L is not at the first place, and the sub-index M^1 in S^R is not at the last place. We define $S^L_{VV\ldots M^1\ldots V}$ by

$$S^L_{VV\ldots M^1\ldots V}(v_3', (a_1, z_1)\ldots(v, w)\ldots v_2) := S(v_3' o(a_1), (a_2, z_2)\ldots(a_n, z_n)(v, w)v_2)z_1^{-\mathrm{wt}a_1}$$

$$+\sum_{k=2}^{n}\sum_{j\geq 0}F_{\mathrm{wt}a_1,j}(z_1,z_k)S(v_3',(a_2,z_2)...(a_1(j)a_k,z_k)...(a_n,z_n)(v,w)v_2)$$
$$+\sum_{j\geq 0}F_{\mathrm{wt}a_1,j}(z_1,w)S(v_3',(a_2,z_2)...(a_n,z_n)(a_1(j)v,w)v_2), \tag{4.16}$$

and define $S^R_{V...M^1...VV}$ by

$$S^R_{V...M^1...VV}(v_3',...(v,w)...(a_1,z_1)v_2) := S(v_3',(a_2,z_2)...(a_n,z_n)(v,w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}$$
$$+\sum_{k=2}^{n}\sum_{j\geq 0}G_{\mathrm{wt}a_1,j}(z_1,z_k)S(v_3',(a_2,z_2)...(a_1(j)a_k,z_k)...(a_n,z_n)(v,w)v_2)$$
$$+\sum_{j\geq 0}G_{\mathrm{wt}a_1,j}(z_1,w)S(v_3',(a_2,z_2)...(a_n,z_n)(a_1(j)v,w)v_2), \tag{4.17}$$

where the S on right-hand sides of (4.16) and (4.17) is the $(n+2)$-point function.

The definition above indicates that $S^L_{VMV...V} = S^L_{VVM...V} = \cdots = S^L_{VV...VM}$, which is reasonable because the $(n+2)$-point function S on the right-hand side of (4.16) satisfies the locality property (I). For a similar reason, we can also expect that $S^R_{MV...VV} = S^R_{VM...VV} = \cdots = S^R_{V...VMV}$. We need to show that

$$\begin{aligned}&S^L_{V...M...V}(v_3',(a_1,z_1)...(v,w)...(a_2,z_2)v_2)\\&=S^R_{V...M...V}(v_3',(a_1,z_1)...(v,w)...(a_2,z_2)v_2),\end{aligned} \tag{4.18}$$

for all $S^L_{VV...M...V}$ and $S^R_{V...M...VV}$.

Indeed, as we mentioned in Proposition 4.10, since (4.16) is the generalization of (2.2.6) in [13], by a similar argument as the proof of (2.2.11) in [13], we have:

$$\begin{aligned}&S^L_{VV...M...V}(v_3',(a_1,z_1)(a_2,z_2)...(v,w)...v_2)\\&=S^L_{VV...M...V}(v_3',(a_2,z_2)(a_1,z_1)...(v,w)...v_2).\end{aligned} \tag{4.19}$$

Proposition 4.15. *If $S^L_{VV...M...V}(v_3',(a_1,z_1)...v_2) = S^R_{V...M...VV}(v_3',...(a_1,z_1)v_2)$, i.e. if the right-hand side of (4.16) is equal to the right-hand side of (4.17), then (4.18) holds.*

Proof: The proof is similar to the proof of Proposition 4.12. By (4.19) and the assumption,

$$\begin{aligned}&S^L_{V...M...V}(v_3',(a_1,z_1)...(v,w)...(a_2,z_2)v_2)\\&=S^L_{VV...M...V}(v_3',(a_1,z_1)(a_2,z_2)...(v,w)...v_2)\\&=S^L_{VV...M...V}(v_3',(a_2,z_2)(a_1,z_1)...(v,w)...v_2)\\&=S^R_{V...M...VV}(v_3',(a_1,z_1)...(v,w)...(a_2,z_2)v_2)\end{aligned}$$

as asserted. □

Now we are left to show that:

$$S^L_{VV...M...V}(v_3',(a_1,z_1)...(v,w)...v_2) = S^R_{V...M...VV}(v_3',...(v,w)...(a_1,z_1)v_2). \tag{4.20}$$

Similar to the previous subsection, we use the symbols (1), (2), (3), and (4) to denote the following summands on the right-hand side of (4.16)-(4.17):

$$S(v_3'o(a_1), (a_2, z_2)...(v, w)v_2)z^{-\mathrm{wt}a_1} - S(v_3', (a_2, z_2)...(v, w)o(a_1)v_2)z^{-\mathrm{wt}a_1}. \tag{1}$$

$$\sum_{j\geq 0}(F_{\mathrm{wt}a_1,j}(z_1, z_2) - G_{\mathrm{wt}a_1,j}(z_1, z_2))S(v_3', (a_1(j)a_2, z_2)...(a_n, z_n)(v, w)v_2). \tag{2}$$

$$\sum_{k=3}^{n}\sum_{j\geq 0}(F_{\mathrm{wt}a_1,j}(z_1, z_k) - G_{\mathrm{wt}a_1,j}(z_1, z_k))S(v_3', (a_2, z_2)...(a_1(j)a_k, z_k)...(v, w)v_2). \tag{3}$$

$$\sum_{j\geq 0}((F_{\mathrm{wt}a_1,j}(z_1, w) - G_{\mathrm{wt}a_1,j}(z_1, w))S(v_3', (a_2, z_2)...(a_n, z_n)(a_1(j)v, w)v_2). \tag{4}$$

Then we need to show that (1)+(2)+(3)+(4)=0.

Our strategy is to apply the expansion formula (3.4) and expand each summand of (1)-(4) with respect to the left-most term. Then we add them all up and show that the sum equals 0. (Since we are using the recursive formula (3.4) twice and the 3-point function cannot be expanded, the construction of the 5-point function in the previous subsection is necessary for our induction process.)

Start with (1), note that $S(v_3'o(a_1), (a_2, z_2)...(a_n, z_n)(v, w)v_2)z^{-\mathrm{wt}a_1}$ can be written as:

$$\begin{aligned}&S(v_3'o(a_1)o(a_2), (a_3, z_3)...(a_n, z_n)(v, w)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2} \qquad (*)\\
&\quad+\sum_{t=3}^{n}\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2, z_t)S(v_3'o(a_1), (a_3, z_3)...(a_2(i)a_t, z_t)...(a_n, z_n)(v, w)v_2)z_1^{-\mathrm{wt}a_1}\\
&\quad+\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2, w)S(v_3'o(a_1), (a_3, z_3)...(a_n, z_n)(a_2(i)v, w)v_2)z_1^{-\mathrm{wt}a_1},\end{aligned}$$

and $S(v_3', (a_2, z_2)...(a_n, z_n)(v, w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}$ can be written as

$$\begin{aligned}&S(v_3'o(a_2), (a_3, z_3)...(a_n, z_n)(v, w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2} \qquad (**)\\
&\quad+\sum_{t=3}^{n}\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2, z_t)S(v_3', (a_3, z_3)\\
&\quad...(a_2(i)a_t, z_t)...(a_n, z_n)(v, w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}\\
&\quad+\sum_{i\geq 0}F_{\mathrm{wt}a_2,i}(z_2, w)S(v_3', (a_3, z_3)...(a_n, z_n)(a_2(i)v, w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}.\end{aligned}$$

We denote the first, second, and third corresponding terms in $(*)$-$(**)$ by (11), (12), and (13), respectively. In particular, (11) is

$$\begin{aligned}&S(v_3'o(a_1)o(a_2), (a_3, z_3)...(a_n, z_n)(v, w)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2} \qquad (11)\\
&\quad- S(v_3'o(a_2), (a_3, z_3)...(a_n, z_n)(v, w)o(a_1)v_2)z_1^{-\mathrm{wt}a_1}z_2^{-\mathrm{wt}a_2}.\end{aligned}$$

Lemma 4.16. *As* $(n+1)$*-point function, we have:*

$$S(v_3' o(a_1), (a_3, z_3)...(a_n, z_n)(v, w)v_2) - S(v_3', (a_3, z_3)...(a_n, z_n)(v, w)o(a_1)v_2)$$
$$= \sum_{k=3}^{n}\sum_{j\geq 0}\binom{\mathrm{wt}a_1 - 1}{j} z_k^{\mathrm{wt}a_1 - j - 1} S(v_3', (a_3, z_3)...(a_1(j)a_k, z_k)...(a_n, z_n)(v, w)v_2)$$
$$+ \sum_{j\geq 0}\sum_{j\geq 0}\binom{\mathrm{wt}a_1 - 1}{j} w^{\mathrm{wt}a_1 - j - 1} S(v_3', (a_3, z_3)...(a_n, z_n)(a_1(j)v, w)v_2) \tag{4.21}$$

Proof: By the induction hypothesis for the $(n+2)$-point functions and (3.8), we have:

$$\begin{aligned}
0 &= S(v_3', (a_1, z_1)(a_3, z_3)...(a_n, z_n)(v, w)v_2)\\
&\quad - S(v_3', (a_3, z_3)...(a_n, z_n)(a_1, z_1)(v, w)v_2)\\
&= S(v_3' o(a_1)(a_3, z_3)...(a_n, z_n)(v, w)v_2) z_1^{-wta_1}\\
&\quad - S(v_3'(a_3, z_3)...(a_n, z_n)(v, w)o(a_1)v_2) z_1^{-\mathrm{wt}a_1}\\
&\quad + \sum_{k=3}^{n}\sum_{j\geq 0}(F_{\mathrm{wt}a_1, j}(z_1, z_k) - G_{\mathrm{wt}a_1, j}(z_1, z_k)) S(v_3', (a_3, z_3)\\
&\quad ...(a_1(j)a_k, z_k)...(a_n, z_n)(v, w)v_2)\\
&\quad + \sum_{j\geq 0}(F_{\mathrm{wt}a_1, j}(z_1, w) - G_{\mathrm{wt}a_1, j}(z_1, w)) S(v_3', (a_3, z_3)...(a_n, z_n)(a_1(j)v, w)v_2)\\
&= S(v_3' o(a_1)(a_3, z_3)...(a_n, z_n)(v, w)v_2) z_1^{-\mathrm{wt}a_1}\\
&\quad - S(v_3'(a_3, z_3)...(a_n, z_n)(v, w)o(a_1)v_2) z_1^{-\mathrm{wt}a_1}\\
&\quad + \sum_{k=3}^{n}\sum_{j\geq 0} - \binom{\mathrm{wt}a_1 - 1}{j} z_k^{\mathrm{wt}a_1 - j - 1} S(v_3', (a_3, z_3)...(a_1(j)a_k, z_k)...(a_n, z_n)(v, w)v_2)\\
&\quad + \sum_{j\geq 0} - \binom{\mathrm{wt}a_1 - 1}{j} w^{\mathrm{wt}a_1 - j - 1} S(v_3', (a_3, z_3)...(a_n, z_n)(a_1(j)v, w)v_2).
\end{aligned}$$

This proves (4.21). □

It follows from the Lemma 4.16 that (12) and (13) can be written as:

$$\begin{aligned}
(12) &= \sum_{t=3}^{n}\sum_{k=3, k\neq t}^{n}\sum_{i, j\geq 0} F_{\mathrm{wt}a_2, i}(z_2, z_t)\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_k^{\mathrm{wt}a_1 - 1 - j}\\
&\quad \cdot \underbrace{S(v_3', (a_3, z_3)...(a_1(j)a_k, z_k)...(a_2(i)a_t, z_t)...(a_n, z_n)(v, w)v_2)}_{(121)}\\
&\quad + \sum_{t=3}^{n}\sum_{i, j\geq 0} F_{\mathrm{wt}a_2, i}(z_2, z_t)\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_t^{\mathrm{wt}a_1 - 1 - j}\\
&\quad \cdot \underbrace{S(v_3', (a_3, z_3)...(a_1(j)a_2(i)a_t, z_t)...(a_n, z_n)(v, w)v_2)}_{(122)}\\
&\quad + \sum_{t=3}^{n}\sum_{i, j\geq 0} F_{\mathrm{wt}a_2, i}(z_2, w)\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} w^{\mathrm{wt}a_1 - 1 - j}
\end{aligned}$$

$$\cdot \underset{(123)}{S(v_3', (a_3, z_3)...(a_2(i)a_t, z_t)...(a_n, z_n)(a_1(j)v, w)v_2)}$$

$$= (121) + (122) + (123),$$

$$(13) = \sum_{k=3}^{n} \sum_{i,j\geq 0} F_{\mathrm{wt}a_2,i}(z_2, z_k) \binom{\mathrm{wt}a_1 - 1}{j} z_1^{-wta_1} z_k^{\mathrm{wt}a_1 - 1 - j}$$

$$\cdot \underset{(131)}{S(v_3', (a_3, z_3)...(a_1(j)a_k, z_k)...(a_n, z_n)(a_2(i)v, w)v_2)}$$

$$+ \sum_{i,j\geq 0} F_{\mathrm{wt}a_2,i}(z_2, w) \binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} w^{\mathrm{wt}a_1 - 1 - j}$$

$$\cdot \underset{(132)}{S(v_3', (a_3, z_3)...(a_n, z_n)(a_1(j)a_2(i)v, w)v_2)}$$

$$= (131) + (132).$$

Then (1)=(11)+(121)+(122)+(123)+(131)+(132).

Now we expand (2), (3), and (4) with respect to their corresponding left-most terms. By (3.8), they can be expressed as follows:

$$(2) = \sum_{j\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_2^{-\mathrm{wt}a_2} \underset{(21)}{S(v_3' o(a_1(j)a_2), (a_3, z_3)...(a_n, z_n)(v, w)v_2)}$$

$$+ \sum_{k=3}^{n} \sum_{i,j\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_2^{\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_1 + \mathrm{wt}a_2 - j - 1, i}(z_2, z_k)$$

$$\cdot \underset{(22)}{S(v_3', (a_3, z_3)...((a_1(j)a_2)(i)a_k, z_k)...(a_n, z_n)(v, w)v_2)}$$

$$+ \sum_{i,j\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_2^{-\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_1 + \mathrm{wt}a_2 - j - 1, i}(z_2, w)$$

$$\cdot \underset{(23)}{S(v_3', (a_3, z_3)...(a_n, z_n)((a_1(j)a_2)(i)v, w)v_2)}$$

$$= (21) + (22) + (23). (3) = \sum_{k=3}^{n} \sum_{j\geq 0}$$

$$- \binom{wta_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_k^{\mathrm{wt}a_1 - 1 - j} \underset{(31)}{S(v_3' o(a_2), (a_3, z_3)...(a_1(j)a_k, z_k)...v_2)} z_2^{-\mathrm{wt}a_2}$$

$$+ \sum_{k=3}^{n} \sum_{j,i\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_k^{\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_2,i}(z_2, w)$$

$$\cdot \underset{(32)}{S(v_3', (a_3, z_3)...(a_1(j)a_k, z_k)...(a_n, z_n)(a_2(i)v, w)v_2)}$$

$$+ \sum_{k=3}^{n} \sum_{t=3, t\neq k}^{n} \sum_{j,i\geq 0} -\binom{\mathrm{wt}a_1 - 1}{j} z_1^{-\mathrm{wt}a_1} z_k^{\mathrm{wt}a_1 - 1 - j} F_{\mathrm{wt}a_2,i}(z_2, z_t)$$

$$\cdot \underset{(33)}{S(v_3', (a_3, z_3)...(a_2(i)a_t, z_t)...(a_1(j)a_k, z_k)...(a_n, z_n)(v, w)v_2)}$$

$$+\sum_{k=3}^{n}\sum_{j,i\geq 0}-\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}z_k^{\mathrm{wt}a_1-1-j}F_{\mathrm{wt}a_2,i}(z_2,z_k)$$

$$\cdot\underbrace{S(v_3',(a_3,z_3)...(a_2(i)a_1(j)a_k,z_k)...(a_n,z_n)(v,w)v_2)}_{(34)}$$

$$=(31)+(32)+(33)+(34).(4)=\sum_{j\geq 0}$$

$$-\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}w^{\mathrm{wt}a_1-1-j}\underbrace{S(v_3'o(a_2),(a_3,z_3)...(a_n,z_n)(a_1(j)v,w)v_2)z_2^{-\mathrm{wt}a_2}}_{(41)}$$

$$+\sum_{t=3}^{n}\sum_{j,i\geq 0}-\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}w^{\mathrm{wt}a_1-1-j}F_{\mathrm{wt}a_2,i}(z_2,z_k)$$

$$\cdot\underbrace{S(v_3',(a_3,z_3)...(a_2(i)a_k,z_k)...(a_n,z_n)(a_1(j)v,w)v_2)}_{(42)}$$

$$+\sum_{j,i\geq 0}-\binom{\mathrm{wt}a_1-1}{j}z_1^{-\mathrm{wt}a_1}w^{\mathrm{wt}a_1-1-j}F_{\mathrm{wt}a_2,i}(z_2,w)$$

$$\cdot\underbrace{S(v_3',(a_3,z_3)...(a_n,z_n)(a_2(i)a_1(j)v,w)v_2)}_{(43)}$$

$$=(41)+(42)+(43).$$

By Lemma 4.13 and the formula (3.5) of $\iota_{z_2,z_t}F_{n,i}(z_2,z_t)$, we have:

$$\sum_{i,j\geq 0}\binom{\mathrm{wt}a_1-1}{j}F_{\mathrm{wt}a_2,i}(z_2,z_t)a_1(j)a_2(i)a_t$$
$$+\sum_{i,j\geq 0}-\binom{\mathrm{wt}a_1-1}{j}F_{\mathrm{wt}a_2,i}(z_2,z_t)a_1(j)a_2(i)a_t$$
$$+\sum_{i,j\geq 0}-\binom{\mathrm{wt}a_1-1}{j}F_{\mathrm{wt}a_1+\mathrm{wt}a_2-j-1,i}(z_2,z_t)(a_1(j)a_2)(i)a_t=0, \tag{4.22}$$

and the same equation holds if we replace z_t with w and a_i with v. Using (4.22), we have the cancelations (122) + (22) + (34) = 0, and (132) + (23) + (43) = 0. Moreover, it follows directly from the expressions of the terms (123), (42), (121), (33), (131), and (32) that

$$(123)+(42)=0,\quad (121)+(33)=0,\quad \text{and}\quad (131)+(32)=0.$$

Now it remains to show 11+(21)+(31)+(41)=0, or equivalently,

$$S(v_3'o(a_1)o(a_2),(a_3,z_3)...(a_n,z_n)(v,w)v_2)$$
$$-S(v_3'o(a_2),(a_3,z_3)...(a_n,z_n)(v,w)o(a_1)v_2)$$
$$=\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}S(v_3'o(a_1(j)a_2),(a_3,z_3)...(a_n,z_n)(v,w)v_2)$$
$$+\sum_{k=3}^{n}\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}z_k^{\mathrm{wt}a_1-1-j}S(v_3'o(a_2),(a_3,z_3)...(a_1(j)a_k,z_k)...(v,w)v_2)$$

$$+\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}w^{\mathrm{wt}a_1-1-j}S(v_3'o(a_2),(a_3,z_3)\ldots(a_n,z_n)(a_1(j)v,w)v_2), \quad (4.23)$$

but this is a consequence of Lemma 4.16. In fact,

$$\begin{aligned} &L.H.S.\ of\ (4.23)\\ &\quad = S(v_3'o(a_1)o(a_2),(a_3,z_3)\ldots(a_n,z_n)(v,w)v_2)\\ &\quad - S(v_3'o(a_2)o(a_1),(a_3,z_3)\ldots(a_n,z_n)(v,w)v_2)\\ &\quad + S(v_3'o(a_2)o(a_1),(a_3,z_3)\ldots(a_n,z_n)(v,w)v_2)\\ &\quad - S(v_3'o(a_2),(a_3,z_3)\ldots(a_n,z_n)(v,w)o(a_1)v_2). \end{aligned}$$

Since S is linear in the place $M^3(0)^*$, we have

$$\begin{aligned} &S(v_3'o(a_1)o(a_2),(a_3,z_3)\ldots(a_n,z_n)(v,w)v_2)\\ &\quad - S(v_3'o(a_2)o(a_1),(a_3,z_3)\ldots(a_n,z_n)(v,w)v_2)\\ &\quad = S(v_3'[o(a_1),o(a_2)],(a_3,z_3)\ldots(a_n,z_n)(v,w)v_2)\\ &\quad = \sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}S(v_3'o(a_1(j)a_2),(a_3,z_3)\ldots(a_n,z_n)(v,w)v_2), \end{aligned}$$

which is the first term on the right-hand side of (4.23). Moreover, by Lemma 4.16,

$$\begin{aligned} &S(v_3'o(a_2)o(a_1),(a_3,z_3)\ldots(a_n,z_n)(v,w)v_2)\\ &\quad - S(v_3'o(a_2),(a_3,z_3)\ldots(a_n,z_n)(v,w)o(a_1)v_2)\\ &\quad = \sum_{k=3}^{n}\sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}z_k^{\mathrm{wt}a_1-1-j}S(v_3'o(a_2),(a_3,z_3)\ldots(a_1(j)a_k,z_k)\ldots(v,w)v_2)\\ &\quad + \sum_{j\geq 0}\binom{\mathrm{wt}a_1-1}{j}w^{\mathrm{wt}a_1-1-j}S(v_3'o(a_2),(a_3,z_3)\ldots(a_n,z_n)(a_1(j)v,w)v_2), \end{aligned}$$

which gives us the last two summands on the right-hand side of (4.23). This proves (4.23). Hence 1+2+3+4= 0, and so (4.20) holds.

Then by Proposition 4.15, all the $(n+3)$-point functions $S^L_{VV\ldots M\ldots V}$ and $S^R_{V\ldots M\ldots VV}$ defined by (4.16) and (4.17) give rise to one single $(n+3)$-point function:

$$S: M^3(0)^*\times V\times\cdots\times M^1\times\cdots\times V\times M^2(0)\to\mathcal{F}(z_1,\ldots,z_n,w), \quad (4.24)$$

where M^1 can be placed anywhere in between the first and the last place of V. Moreover, by Definition 4.14 and (4.18), S in (4.24) satisfies the locality (I) and the expansion property (II), with n replaced by $n+1$. Therefore, the induction step is complete.

Theorem 4.17. *The system of* $(n+3)$*-point functions S we constructed by Definitions 4.9, 4.11, and 4.14 in this subsection lies in* $\mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$.

Proof. Since S is constructed inductively by the recursive formulas (3.4) and (3.6) in view of Defintions 4.9, 4.11, and 4.14, it obviously satisfies (3.4) and (3.6). By (4.7), we have $S(v_3',(v,w)v_2)=f(v_3'\otimes v\otimes v_2)w^{-\deg v}$,for any $v_3'\in M^3(0)^*$, $v\in M^1$, and $v_2\in M^2(0)$. By the Hom-tensor duality, we have a well-defined element $f_v\in$

$\mathrm{Hom}_{\mathbb{C}}(M^2(0), M^3(0))$ such that $\langle v_3', f_v(v_2)\rangle = f(v_3' \otimes v \otimes v_2)$ for each $v \in M^1$. Hence S satisfies (3.3).

In view of Definition 3.1, it remains to show that S satisfies (2) – (6) in Definition 2.1 for $v_2 \in M^2(0)$ and $v_3' \in M^3(0)^*$. Indeed, the locality follows from (I), and by (4.16),

$$\begin{aligned}
&S(v_3', (\mathbf{1}, z)(a_1, z_1)...(a_n, z_n)(v, w)v_2)\\
&\quad = S(v_3' o(\mathbf{1}), (a_1, z_1)...(a_n, z_n)(v, w)v_2)z^{-\mathrm{wt}\mathbf{1}}\\
&\quad + \sum_{k=1}^{n}\sum_{j\geq 0} F_{\mathrm{wt}\mathbf{1},j}(z, z_j)S(v_3', (a_1, z_1)...(\mathbf{1}(j)a_k, z_k)...(a_n, z_n)(v, w)v_2)\\
&\quad + \sum_{j\geq 0} F_{\mathrm{wt}\mathbf{1},j}(z, w)S(v_3', (a_1, z_1)...(a_n, z_n)(\mathbf{1}(j)v, w)v_2)\\
&\quad = S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)v_2),
\end{aligned}$$

since $\mathbf{1}(j)a_k = \mathbf{1}(j)v = 0$ when $j \geq 0$, and $o(\mathbf{1}) = \mathrm{Id}$.

Again because S in (4.24) satisfies (4.16), it is easy to verify the following associativity formulas by a similar argument to the proof of (2.2.9) in [13]:

$$\begin{aligned}
&\int_C S(v_3', (a_1, z_1)(v, w)...(a_n, z_n)v_2)(z_1 - w)^n dz_1\\
&\quad = S(v_3', (a_1(k)v, w)...(a_n, z_n)v_2),\\
&\int_C S(v_3', (a_1, z_1)(a_2, z_2)...(v, w)v_2)(z_1 - z_2)^n dz_1\\
&\quad = S(v_3', (a_1(k)a_2, z_2)...(v, w)v_2),
\end{aligned} \tag{4.25}$$

where in the first equation of (4.25), C is a contour of z_1 surrounding w with $z_2, ..., z_n$ outside of C; while in the second equation of (4.25), C is a contour of z_1 surrounding z_2 with $z_3, ..., z_n, w$ outside of C. We also have:

$$\begin{aligned}
&S(v_3', (L(-1)a_1, z_1)...(a_n, z_n)(v, w)v_2)\\
&\quad = \frac{d}{dz_1}S(v_3', (a_1, z_1)...(a_n, z_n)(v, w)v_2),\\
&S(v_3', (L(-1)v, w)(a_1, z_1)...(a_n, z_n)v_2)w^{-h}\\
&\quad = \frac{d}{dw}(S(v_3', (v, w)(a_1, z_1)...v_2)w^{-h}).
\end{aligned} \tag{4.26}$$

The first equation in (4.26) is similar to (2.2.8) in [13]. We omit the details of the proof. To show the second equation in (4.26), we use induction on n. When $n = 0$, by (4.5) and Lemma 4.5, we have: $L(-1)v + (L(0) + h_2 - h_3)v \equiv 0 \mod O_h(M^1)$ for all $v \in M^1$. Then

$$\begin{aligned}
S(v_3', (L(-1)v, w)v_2)w^{-h} &= f(v_3' \otimes L(-1)v \otimes v_2)w^{-\deg v - 1 - h}\\
&= -f(v_3' \otimes (L(0) + h_2 - h_3)v \otimes v_2)w^{-\deg v - 1 - h}\\
&= f(v_3' \otimes v \otimes v_2)\frac{d}{dw}(w^{-\deg v - h})\\
&= \frac{d}{dw}(S(v_3', (v, w)v_2)w^{-h}).
\end{aligned} \tag{4.27}$$

Now assume the second equation of (4.26) holds for the $(n+2)$-point function, then by the properties (I) and (II) of S, we have:

$$\begin{aligned}
&S(v_3', (L(-1)v, w)(a_1, z_1)...(a_n, z_n)v_2)w^{-h}\\
&\quad = S^L(v_3', (a_1, z_1)...(a_n, z_n)(L(-1)v, w)v_2)w^{-h}\\
&\quad = S(v_3' o(a_1), (a_2, z_2)...(a_n, z_n)(L(-1)v, w)v_2)z_1^{-\mathrm{wt}a_1}w^{-h}\\
&\quad + \sum_{k=2}^{n}\sum_{j\geq 0} F_{\mathrm{wt}a_1, j}(z_1, z_k)S(v_3', (a_2, z_2)...(a_1(j)a_k, z_k)...(a_n, z_n)(L(-1)v, w)v_2)w^{-h}\\
&\quad + \sum_{j\geq 0} F_{\mathrm{wt}a_1, j}(z_1, w)S(v_3', (a_2, z_2)...(a_n, z_n)(a_1(j)L(-1)v, w)v_2)w^{-h}.
\end{aligned} \tag{4.28}$$

Note that we can apply the induction hypothesis to the first two terms of (4.28). Moreover, by the $L(-1)$-bracket formula (4.2.1) in [4], we have:

$$a_1(j)L(-1)v_2 = L(-1)a_1(j)v_2 - [L(-1), a_1(j)]v_2 = L(-1)a_1(j)v_2 + ja_1(j-1)v_2.$$

It follows from the induction hypothesis and (3.5) that

$$\begin{aligned}
&\sum_{j\geq 0} F_{\mathrm{wt}a_1, j}(z_1, w)S(v_3', (a_2, z_2)...(a_n, z_n)(a_1(j)L(-1)v, w)v_2)w^{-h}\\
&\quad = \sum_{j\geq 0} F_{\mathrm{wt}a_1, j}(z_1, w)\frac{d}{dw}(S(v_3', (a_2, z_2)...(a_n, z_n)(a_1(j)v, w)v_2)w^{-h})\\
&\quad + \sum_{j\geq 1}\frac{z_1^{-\mathrm{wt}a_1}}{(j-1)!}\left(\frac{d}{dw}\right)^j\left(\frac{w^{\mathrm{wt}a_1}}{z_1-w}\right)S(v_3', (a_2, z_2)...(a_n, z_n)(a_1(j-1)v, w)v_2)w^{-h}\\
&\quad = \frac{d}{dw}\sum_{j\geq 0} F_{\mathrm{wt}a_1, j}(z_1, w)S(v_3', (a_2, z_2)...(a_n, z_n)(a_1(j)v, w)v_2)w^{-h}.
\end{aligned}$$

This proves (4.26). Finally, let $v_3' \in M^3(0)^*$, $v \in M^1$, $v_2 \in M^2(0)$, and $a_1, \ldots, a_n \in V$ be highest weight vectors of the Virasoro algebra. By property (I) and (4.26) of S, we have:

$$\begin{aligned}
&S(v_3', (\omega, x)(\omega, x_1)...(\omega, x_m)(a_1, z_1)...(a_n, z_n)(v, w)v_2)\\
&\quad = S(v_{3'}, (\omega, x_1)...(a_n, z_n)(v, w)o(\omega)v_2)x^{-2}\\
&\quad + \sum_{k=1}^{m}\sum_{j\geq 0} G_{2,j}(x, x_k)S(v_3', (\omega, x_1)...(\omega_j\omega, x_k)...(a_n, z_n)(v, w)v_2)\\
&\quad + \sum_{k=1}^{n}\sum_{j\geq 0} G_{2,j}(x, z_k)S(v_3', (\omega, x_1)...(\omega_j a_k, z_k)...(a_n, z_n)(v, w)v_2)\\
&\quad + \sum_{j\geq 0} G_{2,j}(x, w)S(v_3', (\omega, x_1)...(a_n, z_n)(\omega_j v, w)v_2).
\end{aligned}$$

By the definition formula (3.7), it is easy to verify that:

$$G_{2,0}(x, z) = \frac{x^{-1}z}{x-z}, \quad G_{2,1}(x, z) = \frac{1}{(x-z)^2},$$

$$G_{2,3}(x,z) = \frac{1}{(x-z)^4}.$$

Then by using the properties of the Virasoro element ω (see Section 2.3 in [4]), we have:

$$\begin{aligned}
&S(v_3', (\omega,x)(\omega,x_1)...(\omega,x_m)(a_1,z_1)...(v,w)...(a_n,z_n)v_2)\\
&= \sum_{k=1}^{n} \frac{x^{-1}z_k}{x-z_k}\frac{d}{dz_k}S + \sum_{k=1}^{n}\frac{\mathrm{wt} a_k}{(x-z_k)^2}S + \frac{x^{-1}w}{x-w}w^h\frac{d}{dw}(S\cdot w^{-h}) + \frac{\mathrm{wt} v}{(x-w)^2}S\\
&+ \frac{h_2}{x^2}S + \sum_{k=1}^{m}\frac{x^{-1}wx_k}{x-x_k}\frac{d}{dx_k}S + \sum_{k=1}^{m}\frac{2}{(x-x_k)^2}S\\
&+ \frac{c}{2}\sum_{k=1}^{m}\frac{1}{(x-x_k)^4}S(v_3', (\omega,x_1)...\widehat{(\omega,x_k)}...(\omega,x_m)(a_1,z_1)...(v,w)...(a_n,z_n)v_2),
\end{aligned}$$

where $S = S(v_3', (\omega,x_1)...(\omega,x_m)(a_1,z_1)...(a_n,z_n)(v,w)v_2)$. This shows that the S in (3.24) also satisfies (2.8), with $v_3' \in M^3(0)^*$ and $v_2 \in M^2(0)$. Therefore, $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$. □

Remark 4.18. By equation (4.27), we see that it is necessary to have the equality $L(-1)v + (L(0)+h_2-h_3)v = 0$ hold in the bimodule $B_h(M^1)$ to show the $L(-1)$-derivation property (4.26) of S. However, in general, such equality does not hold in the bimodule $A(M^1)$ in [6] by its construction. This is the reason why $I\binom{M^3}{M^1\ M^2}$ is not isomorphic to $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$ in general.

Theorem 4.17 indicates that we have a well-defined linear map:

$$\mu : (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^* \to \mathrm{Cor}\begin{pmatrix} M^3(0) \\ M^1\ M^2(0)\end{pmatrix}, \quad f \mapsto S_f, \tag{4.29}$$

where S_f is the S we constructed in this subsection by Defintions 4.9, 4.11, and 4.14.

Since we have $S_f(v_3', (v,w)v_2) = f(v_3'\otimes v\otimes v_2)w^{-\deg v}$ by (4.7), and $f_{S_f}(v_3'\otimes v\otimes v_2)w^{-\deg v} = S_f(v_3', (v,w)v_2)$ by (4.6) and Definition 3.1, then $f_{S_f} = f$. i.e., $\nu\mu = 1$. On the other hand, for $S \in \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)}$, again by (4.7) and (4.6), we have:

$$S_{f_S}(v_3', (v,w)v_2) = f_S(v_3'\otimes v\otimes v_2)w^{-\deg v} = S(v_3', (v,w)v_2).$$

Moreover, S_{f_S} and S satisfy the same recursive formulas by (4.16), (4.17), (3.4), and (3.6), then it follows from an easy induction that $S_{f_S} = S$. i.e., $\mu\nu = 1$, and so μ is an isomorphism. Now we have our main result:

Theorem 4.19. *Let M^1, M^2, and M^3 be V-modules, with conformal weight h_1, h_2, and h_3, respectively. Assume $M^2(0)$ and $M^3(0)$ are irreducible $A(V)$-modules. Then we have the following isomorphism of vector spaces:*

$$\begin{aligned}
&I\begin{pmatrix} \bar{M}(M^3(0)^*)' \\ M^1\ \bar{M}(M^2(0))\end{pmatrix} \cong I\begin{pmatrix} \bar{M}^3 \\ M^1\ \bar{M}^2\end{pmatrix} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*,\\
&I \mapsto f_I, \quad f_I(v_3'\otimes v\otimes v_2) = \langle v_3', o(v)v_2\rangle,
\end{aligned} \tag{4.30}$$

for all $v_3' \in M^3(0)^*$, $v \in M^1$, *and* $v_2 \in M^2(0)$, *where* $h = h_1 + h_2 - h_3$, *and* $M^2 = \bar{M}/\mathrm{Rad}(\bar{M})$ *and* $M^3 = (\tilde{M}/\mathrm{Rad}\tilde{M})'$ *are quotient modules of the generalized Verma module* $\bar{M}(M^2(0))$ *and* $\bar{M}(M^3(0))$, *respectively.*

Proof. This is a direct consequence of Corollary 2.6, Theorem 3.14, and Theorem 4.17, together of which give us the isomorphism: $I\binom{\bar{M}(M^3(0)^*)'}{M^1\ \bar{M}(M^2(0))} \cong I\binom{\bar{M}^3}{M^1\ \bar{M}^2} \cong \mathrm{Cor}\binom{M^3(0)}{M^1\ M^2(0)} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$, such that $I \mapsto f_I$ as in (4.30). □

Recall that V-modules $\bar{M}^2$ and $\bar{M}^{3'}$ are irreducible if condition (3.25) is satisfied (see Proposition 3.11). By the isomorphism (4.29), condition (3.25) translates to the following:

For any $f \in (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$, one has:

$$\sum_{i\geq 0} \binom{n}{i} f(v_3' \otimes b(i)v \otimes v_2) = 0, \tag{4.31}$$

for all $b \in V$, $n \in \mathbb{Z}$ such that $\mathrm{wt}b - n - 1 > 0$, $v \in M^1$, $v_3' \in M^3(0)^*$, and $v_2 \in M^2(0)$.

Corollary 4.20. *Let* M^1, M^2, *and* M^3 *be* V*-modules, with conformal weight* h_1, h_2, *and* h_3, *respectively. Suppose* M^2 *and* M^3 *are irreducible, and condition* (4.31) *is satisfied, then we have an isomorphism:* $I\binom{M^3}{M^1\ M^2} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*$.

Suppose M^2 and M^3 are V-modules (not necessarily irreducible) that are generated by their corresponding bottom levels $M^2(0)$ and $M^3(0)$, which are irreducible $A(V)$-modules. Then by (3.41) and (4.30), we have the following estimate of the fusion rule:

$$\dim I\binom{M^3}{M^1\ M^2} \leq \dim(M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*. \tag{4.32}$$

Finally, when V is rational, by Theorem 4.19 and Corollary 3.15, we have:

Corollary 4.21. *Let* V *be a rational VOA, and let* M^1, M^2, *and* M^3 *be* V *modules, with conformal weight* h_1, h_2, *and* h_3, *respectively. Suppose* M^2 *and* M^3 *are irreducible, then*

$$I\binom{M^3}{M^1\ M^2} \cong (M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0))^*. \tag{4.33}$$

4.4. Examples. In this subsection, we will use (4.30) and the estimating formula (4.32) and compute the fusion rules for certain modules over the Virasoro VOAs and the Heisenberg VOAs.

Example 4.22. A counter-example that shows $I\binom{M^3}{M^1\ M^2}$ is not isomorphic to $(M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0))^*$ was presented in Section 2 in [8]. It was given as follows:

Recall that the (universal) Virasoro VOA $M_c = M(c, 0)/\langle L(-1)v_{c,0}\rangle$ defined in [6] has Zhu's algebra $A(M_c) \cong \mathbb{C}[t]$, with $[\omega]^n \mapsto t^n$. Let $M(c, h)$ be the Verma module of highest weight h and central charge c over the Virasoro algebra, then $M(c, h)$ is a module over M_c, and we have the following equalities held in $A(M(c, h))$:

$$[b] * [\omega]^n = [(L(-2) + L(-1))^n b], \qquad [\omega]^n * [b] = [(L(-2) + 2L(-1) + L(0))^n b],$$

for all $b \in M(c,h)$ and $n \in \mathbb{N}$. Hence there is an identification of $\mathbb{C}[t] \cong A(M_c)$-bimodules:

$$\begin{aligned}&\mathbb{C}[t_1,t_2] \cong A(M(c,h))\\ &f(t_1,t_2) \mapsto f(L(-2)+2L(-1)+L(0), L(-2)+L(-1))v_{c,h},\end{aligned} \tag{4.34}$$

where $C[t_1,t_2]$ is a bimodule over $\mathbb{C}[t]$ on which the actions are given by:

$$t^n.f(t_1,t_2) = t_1^n f(t_1,t_2), \qquad f(t_1,t_2).t^n = t_2^n f(t_1,t_2).$$

For $h_1, h_2 \in \mathbb{C}$ such that $M(c,h_1)$ and $M(c,h_2)$ are irreducible, it is proved (see (2.37) in [8]) that $I\binom{M(c,h_2)}{M(c,h_1)\ M_c} = 0$, while $\dim(M(c,h_2)(0)^* \otimes_{A(M_c)} A(M(c,h_1)) \otimes_{A(M_c)} M_c(0))^* = 1$.

Although $M^2 = M_c$ is neither a generalized Verma module nor irreducible, we can still use (4.30) and (4.32) to obtain the correct fusion rules. Indeed, since M_c and $M(c,h_2)$ are both generalized by their bottom levels, by (4.32), we have:

$$\dim I\begin{pmatrix} M(c,h_2) \\ M(c,h_1)\ M_c\end{pmatrix} \leq \dim(M(c,h_2)(0)^* \otimes_{A(M_c)} B_h(M(c,h_1)) \otimes_{A(M_c)} M_c(0))^*. \tag{4.35}$$

Moreover, since $h = h_1 + 0 - h_2$, it follows from Lemma 4.4 and Lemma 4.5 that

$$B_h(M(c,h_1)) = A(M(c,h_1))/\mathrm{span}\{(L(-1)+L(0)-h_2)[b] : b \in M(c,h_1)\}.$$

Then $[L(-1)b] = -[(\deg b + h_1 - h_2)b]$ in $B_h(M(c,h_1))$. It follows from (4.34) that

$$B_h(M(c,h_1)) \cong \mathbb{C}[t_0], \quad \text{with} \quad [(L(-2)-L(0)+h_2)^n v_{c,h_1}] \mapsto t_0^n,$$

and $\mathbb{C}[t_0]$ is a $\mathbb{C}[t](\cong A(M_c))$-bimodule on which the actions are given by:

$$f(t_0).t^n = t_0^n f(t_0), \quad \text{and} \quad t.f(t_0) = (t_0+h_2)^n f(t_0).$$

Hence we have $B_h(M(c,h_1)) \otimes_{A(M_c)} M_c(0) \cong \mathbb{C}[t_0] \otimes_{\mathbb{C}[t]} M_c(0) \cong M_c(0)$, and so

$$\begin{aligned}&(M(c,h_2)(0)^* \otimes_{A(M_c)} B_h(M(c,h_1)) \otimes_{A(M_c)} M_c(0))^*\\ &\cong \mathrm{Hom}_{A(M_c)}(M_c(0), M(c,h_2)(0)) = 0,\end{aligned}$$

since $o(\omega)v_{c,0} = 0, o(\omega)v_{c,h_2} = h_2 v_{c,h_2}$ and $h_2 \neq 0$. Thus, $I\binom{M(c,h_2)}{M(c,h_1)\ M_c} = 0$ by (4.35).

We give another example that shows that the bimodule $B_h(M^1)$ in (4.30) cannot be replaced by the $A(V)$-bimodule $A_0(M^1)$ defined in [7] either.

Example 4.23. Let $V = M_{\widehat{\mathfrak{h}}}(1,0)$ be the Heisenberg VOA of level 1 associated to a one-dimensional vector space $\mathfrak{h} = \mathbb{C}\alpha$ with $(\alpha|\alpha) = 1$. By Theorem 3.1.1 in [6], one has $A(M_{\widehat{\mathfrak{h}}}(1,0)) \cong \mathbb{C}[x]$, with $[\alpha(-i_1-1)...\alpha(-i_n-1)\mathbf{1}] \mapsto (-1)^{i_1+...+i_n}x^n$.

Let $\lambda \in \mathfrak{h}$, we have a V-module $M_{\widehat{\mathfrak{h}}}(1,\lambda) = M_{\widehat{\mathfrak{h}}}(1,0) \otimes_{\mathbb{C}} \mathbb{C}e^\lambda$, with conformal weight $h = \frac{(\lambda|\lambda)}{2}$. Note that $M_{\widehat{\mathfrak{h}}}(1,\lambda)$ is the Verma module over the Heisenberg Lie algebra $\widehat{\mathfrak{h}}$. Since $M_{\widehat{\mathfrak{h}}}(1,\lambda)$ is irreducible, it is automatically a generalized Verma module associated with its bottom level $\mathbb{C}e^\lambda$. By Theorem 3.2.1 in [6], we have:

$$A(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \cong \mathbb{C}e^{\lambda} \otimes_{\mathbb{C}} \mathbb{C}[x], \quad \text{with}$$
$$[\alpha(-i_1-1)...\alpha(-i_n-1)e^{\lambda}] \mapsto (-1)^{i_1+...+i_n} e^{\lambda} \otimes x^n,$$

where the bimodule actions are given by $x.(e^{\lambda} \otimes x^n) = e^{\lambda} \otimes x^{n+1} + (\lambda|\alpha)e^{\lambda} \otimes x^n$, and $(e^{\lambda} \otimes x^n).x = e^{\lambda} \otimes x^{n+1}$ for all $n \in \mathbb{N}$. By definition in Section 4 of [7],

$$A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda)) = A(M_{\widehat{\mathfrak{h}}}(1,\lambda))/span\{[(L(-1)+L(0)-(\lambda|\lambda)/2)b] : b \in M_{\widehat{\mathfrak{h}}}(1,\lambda)\}.$$

Choose $\lambda \in \mathfrak{h}$ such that $(\lambda|\alpha) \neq 0$. Recall that $\omega = \frac{1}{2}\alpha(-1)^2\mathbf{1}$, and so

$$L(-1)e^{\lambda} = \mathrm{Res}_z Y_W(\omega, z)e^{\lambda} = \sum_{i\geq 0} \alpha(-1-i)\alpha(i)e^{\lambda} = (\lambda|\alpha)\alpha(-1)e^{\lambda}.$$

Then we have $[(\lambda|\alpha)\alpha(-1)e^{\lambda}] = [L(-1)e^{\lambda}] = -[(L(0)-(\lambda|\lambda)/2)e^{\lambda}] = 0$ in $A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda))$, and $[\alpha(-1)e^{\lambda}] = 0$ in $A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda))$. For any spanning element $[\alpha(-i_1-1)...\alpha(-i_n-1)e^{\lambda}]$ of $A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda))$, we then have $[\alpha(-i_1-1)...\alpha(-i_n-1)e^{\lambda}] = (-1)^{i_1+...+i_n}[\alpha(-1)^n e^{\lambda}] = 0$ for $n > 0$. Thus, $A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \cong \mathbb{C}[e^{\lambda}]$, with the module actions given by:

$$x.[e^{\lambda}] = (\lambda|\alpha)[e^{\lambda}], \quad \text{and} \quad [e^{\lambda}].x = 0. \tag{4.36}$$

Now choose $\mu \in \mathfrak{h}$ such that $(\mu|\alpha) \neq 0$, it is well-known that $\dim I\binom{M_{\widehat{\mathfrak{h}}}(1,\lambda+\mu)}{M_{\widehat{\mathfrak{h}}}(1,\lambda)\ M_{\widehat{\mathfrak{h}}}(1,\mu)} = 1$. But

$$A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0) \cong \mathbb{C}[e^{\lambda}] \otimes_{\mathbb{C}[x]} \mathbb{C}e^{\mu} = 0,$$

since it follows from (4.36) that $[e^{\lambda}] \otimes e^{\mu} = \frac{1}{(\mu|\alpha)}[e^{\lambda}] \otimes o(\alpha(-1)\mathbf{1})e^{\mu} = \frac{1}{(\mu|\alpha)}[e^{\lambda}].x \otimes e^{\mu} = 0$ in the tensor product above. Then we have:

$$\dim(M_{\widehat{\mathfrak{h}}}(1,\lambda+\mu)(0)^* \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} A_0(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0))^* = 0 \neq 1.$$

This shows that the isomorphism (4.30) is not true if one replaces $B_h(M^1)$ with $A_0(M^1)$.

Now we verify (4.30) in this case. Indeed, since $\mathfrak{h} = \mathbb{C}\alpha$, then $(\lambda|\alpha) \neq 0$ and $(\mu|\alpha) \neq 0$ imply that $\lambda = m\alpha$ and $\mu = n\alpha$, with $m \neq 0$ and $n \neq 0$. Hence

$$h = \frac{(\lambda|\lambda)}{2} + \frac{(\mu|\mu)}{2} - \frac{(\lambda+\mu|\lambda+\mu)}{2} = -(\lambda|\mu) = -mn \neq 0.$$

By definition 4.1, we have the following equality holds in $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda))$:

$$[(\lambda|\alpha)\alpha(-1)e^{\lambda}] = [L(-1)e^{\lambda}] = -[(L(0) - \frac{(\lambda|\lambda)}{2} + h)e^{\lambda}] = -(\lambda|\mu)[e^{\lambda}]$$

Then for any spanning element $[\alpha(-i_1-1)...\alpha(-i_n-1)e^{\lambda}]$ of $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda))$, we have:

$$\begin{aligned}[\alpha(-i_1-1)...\alpha(-i_n-1)e^{\lambda}] &= (-1)^{i_1+...+i_n}[\alpha(-1)^n e^{\lambda}] \\ &= (-1)^{i_1+...+i_n}\left(\frac{-(\lambda|\mu)}{(\lambda|\alpha)}\right)^n [e^{\lambda}].\end{aligned}$$

Thus $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) = \mathbb{C}[e^\lambda]$, with the module actions given by

$$[e^\lambda].x = \frac{-(\lambda|\mu)}{(\lambda|\alpha)}[e^\lambda](\neq 0), \quad \text{and} \quad x.[e^\lambda] = \frac{-(\lambda|\mu)}{(\lambda|\alpha)}[e^\lambda] + (\lambda|\alpha)[e^\lambda]. \tag{4.37}$$

Then by (4.37), we have $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0) \cong \mathbb{C}[e^\lambda] \otimes_{\mathbb{C}[x]} \mathbb{C}e^\mu$ is a one-dimensional vector space, with $x.[e^\lambda] \otimes e^\mu = [e^\lambda].x \otimes e^\mu + (\lambda|\alpha)[e^\lambda] \otimes e^\mu = (\lambda+\mu|\alpha)[e^\lambda] \otimes e^\mu$. On the other hand, $x.e^{\lambda+\mu} = (\lambda+\mu|\alpha)e^{\lambda+\mu}$. Thus we have:

$$\dim \mathrm{Hom}_{A(M_{\widehat{\mathfrak{h}}}(1,0))}(B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0), M_{\widehat{\mathfrak{h}}}(1,\lambda+\mu)(0)) = 1.$$

This shows (4.30) is true for $M^1 = M_{\widehat{\mathfrak{h}}}(1,\lambda)$, $M^2 = M_{\widehat{\mathfrak{h}}}(1,\mu)$, and $M^3 = M_{\widehat{\mathfrak{h}}}(1,\lambda+\mu)$.

Furthermore, the argument above also shows that $B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0)$ is a one-dimensional vector space spanned by an eigenvector of $\mathfrak{h}$ of eigenfunction $(\lambda+\mu|\cdot)$. Hence we have:

$$\mathrm{Hom}_{A(M_{\widehat{\mathfrak{h}}}(1,0))}(B_h(M_{\widehat{\mathfrak{h}}}(1,\lambda)) \otimes_{A(M_{\widehat{\mathfrak{h}}}(1,0))} M_{\widehat{\mathfrak{h}}}(1,\mu)(0), M_{\widehat{\mathfrak{h}}}(1,\gamma)(0)) = 0,$$

if $\gamma \neq \lambda+\mu$. On the other hand, for $\gamma \neq \lambda+\mu$, it is well-known that $I\binom{M_{\widehat{\mathfrak{h}}}(1,\gamma)}{M_{\widehat{\mathfrak{h}}}(1,\lambda)\ M_{\widehat{\mathfrak{h}}}(1,\mu)} = 0$. Thus, the rank one Heisenberg VOA verifies (4.30).

Although the bimodule $B_h(M^1)$ by its construction is a quotient module of $A(M^1)$, the vectors spaces $M^3(0)^* \otimes_{A(V)} B_h(M^1) \otimes_{A(V)} M^2(0)$, and $M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)$ might be isomorphic to each other, it is easy to see that the case of the rank one Heisenberg VOA in Example 4.23 above is such an example.

Remark 4.24. Note that in $A(M^1)$ we have: $[\omega]*[u]-[u]*[\omega] = \mathrm{Res}_z Y_{M^1}(\omega,z)u(1+z)^{\mathrm{wt}\omega-1} = [L(-1)u + L(0)u]$, for all $u \in M^1$. Hence $[(L(-1)+L(0)+h_2-h_3)u] = [\omega]*[u]-[u]*[\omega]+(h_2-h_3)[u]$, and by Lemma 4.5, we have $B_h(M^1) = A(M^1)/J$, where

$$J = span\{[\omega]*[u]-[u]*[\omega]+(h_2-h_3)[u] : u \in M^1\}.$$

We have $M^3(0)^* \otimes J \otimes M^2(0) = 0$ in $M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)$. Indeed, for any $v_3' \in M^3(0)^*$ and $v_2 \in M^2(0)$,

$$\begin{aligned} &v_3' \otimes ([\omega]*[u]-[u]*[\omega]+(h_2-h_3)[u]) \otimes v_2 \\ &= v_3'(o(\omega)-h_3) \otimes [u] \otimes v_2 - v_3' \otimes [u] \otimes (o(\omega)-h_2)v_2 \\ &= v_3'(L(0)-h_3) \otimes [u] \otimes v_2 - v_3' \otimes [u] \otimes (L(0)-h_2)v_2 \\ &= 0. \end{aligned}$$

However, in general we do not have $M^3(0)^* \otimes_{A(V)} (A(M^1)/J) \otimes_{A(V)} M^2(0)$ isomorphic to $M^3(0)^* \otimes_{A(V)} A(M^1) \otimes_{A(V)} M^2(0)/(M^3(0)^* \otimes J \otimes M^2(0))$, see Example 4.22.

Acknowledgements. I want to thank Professor Haisheng Li for the conversations and for pointing out an error in an earlier version of this paper. I am also grateful to the anonymous referee for valuable comments and suggestions.

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Communicated by C. Schweigert