



Holographic Relative Entropy in Infinite-Dimensional Hilbert Spaces

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Abstract: We reformulate entanglement wedge reconstruction in the language of operator-algebra quantum error correction with infinite-dimensional physical and code Hilbert spaces. Von Neumann algebras are used to characterize observables in a boundary subregion and its entanglement wedge. Assuming that the infinite-dimensional von Neumann algebras associated with an entanglement wedge and its complement may both be reconstructed in their corresponding boundary subregions, we prove that the relative entropies measured with respect to the bulk and boundary observables are equal. We also prove the converse: when the relative entropies measured in an entanglement wedge and its complement equal the relative entropies measured in their respective boundary subregions, entanglement wedge reconstruction is possible. Along the way, we show that the bulk and boundary modular operators act on the code subspace in the same way. For holographic theories with a well-defined entanglement wedge, this result provides a well-defined notion of holographic relative entropy.

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1. Introduction

Entanglement entropy has many applications in quantum field theory, ranging from the study of renormalization group flows [1,2] to confinement [3] to topological orders [4,5]. With the discovery of the Ryu–Takayangi formula [6], entanglement entropy has been especially useful in studying holographic quantum field theories. For holographic theories, it is important to understand the emergent low-energy bulk physics in d -dimensions from the conformal field theory in $(d - 1)$ -dimensions. Since local bulk operators can be expressed as boundary operators smeared over either the entire spatial slice or compact spatial subregions [7,8], a single bulk operator can be reconstructed in different subregions [9]. Quantum error correction provides a convenient setup where bulk operators are defined only on a code subspace of the physical Hilbert space of the conformal field theory. In order to resolve apparent inconsistencies with space-like commutativity of local operators in quantum field theory, bulk reconstruction was studied in the context of quantum error correcting codes [9]. Using the Ryu–Takayangi formula, [10] showed that the relative entropy of nearby states computed in a boundary subregion is equivalent to the relative entropy computed in the dual entanglement wedge [11], up to corrections on the order of Newton’s constant G_N . These results were used in [12,13] to argue that CFT operators in a boundary subregion can be used to reconstruct bulk operators in the entanglement wedge.

Much of the literature on entanglement entropy contains assumptions that are only true for quantum mechanical systems with finite-dimensional Hilbert spaces. For instance, entanglement entropy has often been defined by assuming that the Hilbert space \mathcal{H} can be written as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$, where A refers to a subregion of space and A^c refers to the complement of A . The entanglement entropy is the von Neumann entropy of the reduced density matrix one obtains after performing a partial trace on the Hilbert space \mathcal{H}_{A^c} . However, the infinite-dimensional Hilbert space \mathcal{H} does not factorize in this way because the entanglement entropy contains a universal area-law divergence [14].

Von Neumann algebras are a mathematical structure that arise naturally in quantum field theory. Instead of assuming that the Hilbert space factorizes, we should characterize a causally complete region of spacetime¹ by an associated von Neumann algebra [15]. Formulating quantum field theory with von Neumann algebras is powerful because it allows one to make use of the mathematical machinery of Tomita–Takesaki theory to study entanglement. The modular operator is an important object in Tomita–Takesaki theory, and Araki [16] has used it to define relative entropy in quantum field theory. Theorem 3.8, a central result of Tomita–Takesaki theory, formalizes the notion of modular flow. A demonstration of how von Neumann algebras are associated with the left and right Rindler wedges of Minkowski space was provided by Bisognano and Wichmann in [17]. More recently, an explicit computation of mutual information for free fermions in 1+1 dimensions was performed in [18].

¹ The causal complement of a region R , denoted by R' , is defined to be all of the points in spacetime which are spacelike separated from every point in R . A region R is causally complete if $R'' = R$. Note that any von Neumann algebra M satisfies $M = M'$, where the \prime denotes the commutant.

Given the role that entanglement entropy plays in our understanding of holography and the role that von Neumann algebras play in our understanding of entanglement entropy, it is natural to ask whether statements in the bulk reconstruction literature can be recast in a way that dispenses with the fiction that the boundary Hilbert space can be written as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$ for an arbitrary subregion A . In the context of quantum error correction with finite dimensional Hilbert spaces [13], formulates and completes the equivalence of the Ryu–Takayangi formula, entanglement wedge reconstruction, and the equality of bulk and boundary relative entropies. With the exception of the Ryu–Takayangi formula, there are natural ways to generalize these statements to the case where the Hilbert space is infinite-dimensional. The Ryu–Takayangi formula, on the other hand, computes the entanglement entropy of an arbitrary subregion in the boundary field theory, which is infinite.

In this paper, we prove that in the context of quantum error correction with infinite-dimensional Hilbert spaces, the equivalence of bulk and boundary relative entropies is a necessary and sufficient condition for entanglement wedge reconstruction. This is presented more precisely in Theorem 1.1. We define cyclic and separating states in Definitions 3.1 and 3.2, and relative entropy in Definition 3.6.

Theorem 1.1. *Let $u : \mathcal{H}_{code} \rightarrow \mathcal{H}_{phys}$ be an isometry² between two Hilbert spaces. Let M_{code} and M_{phys} be von Neumann algebras on \mathcal{H}_{code} and \mathcal{H}_{phys} respectively. Let M'_{code} and M'_{phys} respectively be the commutants of M_{code} and M_{phys} . Suppose that the set of cyclic and separating vectors with respect to M_{code} is dense in \mathcal{H}_{code} . Also suppose that if $|\Psi\rangle \in \mathcal{H}_{code}$ is cyclic and separating with respect to M_{code} , then $u|\Psi\rangle$ is cyclic and separating with respect to M_{phys} . Then the following two statements are equivalent:*

1. Bulk reconstruction

$$\forall \mathcal{O} \in M_{code} \ \forall \mathcal{O}' \in M'_{code}, \quad \exists \tilde{\mathcal{O}} \in M_{phys} \ \exists \tilde{\mathcal{O}}' \in M'_{phys} \quad \text{such that}$$

$$\forall |\Theta\rangle \in \mathcal{H}_{code} \quad \begin{cases} u\mathcal{O}|\Theta\rangle = \tilde{\mathcal{O}}u|\Theta\rangle, & u\mathcal{O}'|\Theta\rangle = \tilde{\mathcal{O}}'u|\Theta\rangle, \\ u\mathcal{O}^\dagger|\Theta\rangle = \tilde{\mathcal{O}}^\dagger u|\Theta\rangle, & u\mathcal{O}'^\dagger|\Theta\rangle = \tilde{\mathcal{O}}'^\dagger u|\Theta\rangle. \end{cases}$$

2. Boundary relative entropy equals bulk relative entropy

For any $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_{code}$ with $|\Psi\rangle$ cyclic and separating with respect to M_{code} ,

$$\mathcal{S}_{\Psi|\Phi}(M_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M_{phys}), \text{ and } \mathcal{S}_{\Psi|\Phi}(M'_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M'_{phys}),$$

where $\mathcal{S}_{\Psi|\Phi}(M)$ is the relative entropy.

Theorem 1.1 has two separate statements regarding bulk reconstruction and relative entropy. Early attempts to express bulk operators as nonlocal operators on the boundary were made in [7, 8], and [9] made the connection between bulk reconstruction and quantum error correction. The statement that relative entropy equals bulk relative entropy is due to [10].

Given the assumptions of Theorem 1.1, M_{code} may be viewed as a von Neumann subalgebra of M_{phys} . For a specific setting when the relative entropy of two states defined with respect to M_{code} is identical to the relative entropy defined with respect to M_{phys} , M_{code} is called a weakly sufficient subalgebra with respect to the two states.

² This means that u is a norm-preserving map. The map u need not be a bijection. In general, $u^\dagger u$ is the identity on \mathcal{H}_{code} and uu^\dagger is a projection on \mathcal{H}_{phys} .

This particular case is studied in [19]. However, Theorem 1.1 is concerned with the case when the relative entropies agree for all states in the code subspace.

Theorem 1.1 contains an assumption that the set of cyclic and separating vectors with respect to M_{code} is dense in \mathcal{H}_{code} . This assumption can be relaxed to the weaker assumption that there exists in \mathcal{H}_{code} a cyclic and separating vector with respect to M_{code} . The existence of this vector is guaranteed by the Reeh–Schlieder Theorem [20]. If M_{code} and M'_{code} are of type III₁, then this conclusion also follows from a result of Connes–Størmer [21]. See Sect. 6.4 for further discussion.

The proof of Theorem 1.1 requires two parts: statement 1 implies statement 2, and statement 2 implies statement 1 as well. Unlike the other direction, our proof that statement 1 implies statement 2 does not require all of the assumptions of the theorem. We highlight this by presenting Theorem 1.2:

Theorem 1.2. *Let $u : \mathcal{H}_{code} \rightarrow \mathcal{H}_{phys}$ be an isometry between two Hilbert spaces. Let M_{code} and M_{phys} be von Neumann algebras on \mathcal{H}_{code} and \mathcal{H}_{phys} respectively. Let M'_{code} and M'_{phys} respectively be the commutants of M_{code} and M_{phys} . Suppose that*

- *There exists some state $|\Omega\rangle \in \mathcal{H}_{code}$ such that $u|\Omega\rangle \in \mathcal{H}_{phys}$ is cyclic and separating with respect to M_{phys} .*
- *$\forall \mathcal{O} \in M_{code} \forall \mathcal{O}' \in M'_{code}, \exists \tilde{\mathcal{O}} \in M_{phys} \exists \tilde{\mathcal{O}}' \in M'_{phys}$ such that*

$$\forall |\Theta\rangle \in \mathcal{H}_{code} \quad \begin{cases} u\mathcal{O}|\Theta\rangle = \tilde{\mathcal{O}}u|\Theta\rangle, & u\mathcal{O}'|\Theta\rangle = \tilde{\mathcal{O}}'u|\Theta\rangle, \\ u\mathcal{O}^\dagger|\Theta\rangle = \tilde{\mathcal{O}}^\dagger u|\Theta\rangle, & u\mathcal{O}'^\dagger|\Theta\rangle = \tilde{\mathcal{O}}'^\dagger u|\Theta\rangle. \end{cases}$$

Then, for any $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_{code}$ with $|\Psi\rangle$ cyclic and separating with respect to M_{code} ,

- *$u|\Psi\rangle$ is cyclic and separating with respect to M_{phys} and M'_{phys} ,*
- *$\mathcal{S}_{\Psi|\Phi}(M_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M_{phys}), \quad \mathcal{S}_{\Psi|\Phi}(M'_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M'_{phys}),$*

where $\mathcal{S}_{\Psi|\Phi}(M)$ is the relative entropy.

Theorem 1.1, our main result, has a natural interpretation in the context of AdS/CFT. As the notation suggests, \mathcal{H}_{code} may be interpreted as a code subspace of the physical Hilbert space \mathcal{H}_{phys} that consists of states with semi-classical bulk duals. The von Neumann algebra M_{phys} denotes an algebra of boundary operators associated with a subregion on the boundary, and M_{code} denotes an algebra of bulk operators associated with the dual entanglement wedge. The commutant algebra M'_{phys} is associated with the complementary boundary subregion, and M'_{code} is associated with the complement of the entanglement wedge of M_{code} .

Theorem 1.1 provides a necessary and sufficient criterion for a subalgebra of bulk operators and its commutant to respectively be reconstructed in a subregion in the boundary and its complement. We need the results of [10] to argue that M_{code} and M'_{code} are associated with entanglement wedges. While Theorem 1.1 may not come as a surprise to readers familiar with [12, 13], we emphasize that studying the infinite-dimensional case can potentially yield new physical insights in AdS/CFT. As an example in quantum field theory, the Reeh–Schlieder Theorem [20] cannot be anticipated by studying a finite-dimensional spin–lattice model where the Hilbert space factorizes as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$ where \mathcal{H}_i denotes the finite-dimensional Hilbert space at each site.

While proving Theorem 1.2, we show in Eq. (4.35) that the modular operators associated with the bulk and boundary subregions act the same way on \mathcal{H}_{code} . Furthermore,

while proving bulk reconstruction in Theorem 1.1, we explicitly show how to define a boundary operator that represents a given bulk operator on the code subspace. In Sect. 6, we discuss the implications of the Reeh–Schlieder Theorem for infinite- and finite-dimensional quantum error correction and make contact with the results of [13].

An outline of our proof of Theorem 1.2 is the following.

- We prove that for any $|\Psi\rangle \in \mathcal{H}_{code}$ which is cyclic and separating with respect to M_{code} , $u|\Psi\rangle$ is cyclic and separating with respect to M_{phys} .³ If this were not true, then the relative entropy between $u|\Psi\rangle$ and $u|\Phi\rangle$ could not be defined, as the relative modular operator requires that $u|\Psi\rangle$ is cyclic and separating with respect to M_{phys} .
- Using the fact that M_{phys} and M'_{phys} are commutants of each other, we show that for any $\mathcal{P} \in M_{phys}$, $u^\dagger \mathcal{P} u \in M_{code}$.
- Let $S_{\Psi|\Phi}^c$ and $S_{u\Psi|u\Phi}^p$ denote relative Tomita operators defined with respect to M_{code} and M_{phys} respectively. We relate $S_{\Psi|\Phi}^c$ and $S_{u\Psi|u\Phi}^p$ and derive $uS_{\Psi|\Phi}^c = S_{u\Psi|u\Phi}^p u$ for generically unbounded operators. In particular, we show that their domains are equal and $S_{u\Psi|u\Phi}^p$ restricted to the vector space $(\text{Im } u)^\perp$ has a range contained within $(\text{Im } u)^\perp$.
- We derive a relation for the relative modular operators associated with $S_{\Psi|\Phi}^c$ and $S_{u\Psi|u\Phi}^p$.⁴ This is related to the physical notion that bulk modular flow is dual to boundary modular flow. Likewise, we show that $\Delta_{u\Psi|u\Phi}^p$ restricted to the vector space $(\text{Im } u)^\perp$ has a range contained within $(\text{Im } u)^\perp$.
- Using the spectral theorem, we show that the spectral projections commute with the projector uu^\dagger .⁵ We derive that the spectral projections of $\Delta_{\Psi|\Phi}^c$ are given by $u^\dagger P_\Omega^p u$, where P_Ω^p denotes the spectral projections of $\Delta_{u\Psi|u\Phi}^p$.⁶
- Any function of $\Delta_{u\Psi|u\Phi}^p$ or $\Delta_{\Psi|\Phi}^c$ can be constructed once the spectral projections are known. It follows that $\langle \Psi | \log \Delta_{\Psi|\Phi}^c | \Psi \rangle = \langle u\Psi | \log \Delta_{u\Psi|u\Phi}^p | u\Psi \rangle$, and thus the relative entropies are equal.

We note that Theorem 1.2 dictates that statement 1 of Theorem 1.1 implies statement 2 of Theorem 1.1. A sketch of our proof of the converse is the following. This completes the proof of Theorem 1.1.

- For any $|\Phi\rangle \in \mathcal{H}_{code}$ that is cyclic and separating with respect to M_{code} , and for any unitary $U' \in M'_{code}$, the properties of relative entropy and the assumptions of the theorem imply that $0 = \mathcal{S}_{\Phi|U'\Phi}(M_{code}) = \mathcal{S}_{u\Phi|uU'\Phi}(M_{phys})$.
- Following the logic of [14], one may show that $\langle uU'\Phi | \mathcal{P} u U'\Phi \rangle = \langle u\Phi | \mathcal{P} | u\Phi \rangle$ for all $\mathcal{P} \in M_{phys}$. Using the assumption that cyclic and separating states with respect to M_{code} are dense in \mathcal{H}_{code} , it follows that $u^\dagger \mathcal{P} u U' = U' u^\dagger \mathcal{P} u$. The same logic also implies that for $\mathcal{P}' \in M'_{phys}$ and any unitary $U \in M_{code}$, $u^\dagger \mathcal{P}' u U = U u^\dagger \mathcal{P}' u$.

³ This is because we may act with an operator in M_{code} to send $|\Psi\rangle$ to a vector arbitrarily close to $|\Omega\rangle$, and we may act with an operator in M_{phys} to send $u|\Omega\rangle$ arbitrarily close to any vector in \mathcal{H}_{phys} .

⁴ With the relation for the Tomita operators we derived above, we obtain a relation for the relative modular operators $\Delta_{\Psi|\Phi}^c$ and $\Delta_{u\Psi|u\Phi}^p$ to be $u\Delta_{\Psi|\Phi}^c = \Delta_{u\Psi|u\Phi}^p u$.

⁵ We apply the spectral theorem separately for the restriction of $\Delta_{u\Psi|u\Phi}^p$ to $\text{Im } u$ and $(\text{Im } u)^\perp$.

⁶ We use the relation $\Delta_{\Psi|\Phi}^c = u^\dagger \Delta_{u\Psi|u\Phi}^p u$. For the projections, Ω denotes a measurable subset of \mathbb{R}

- We define a linear map $X'^{\Phi U'} : \mathcal{H}_{phys} \rightarrow \mathcal{H}_{phys}$ by $X'^{\Phi U'} \mathcal{P}u |\Phi\rangle := \mathcal{P}u U' |\Phi\rangle \forall \mathcal{P} \in M_{phys}$, and we show that $X'^{\Phi U'}$ is unitary and that $X'^{\Phi U'} \in M'_{phys}$.
- Since $u^\dagger X'^{\Phi U'} u U = U u^\dagger X'^{\Phi U'} u$ and any operator in M_{code} may be written as a linear combination of four unitary operators in M_{code} , we show that $u^\dagger X'^{\Phi U'} u = U'$. We also show that $X'^{\Phi U'}$ maps the vector space $\text{Im } u \rightarrow \text{Im } u$. Hence, $X'^{\Phi U'} u = u U'$
- Using similar methods, we then show that $(X'^{\Phi U'})^\dagger u = u (U')^\dagger$. Thus, the unitary operator $U' \in M_{code}$ may be reconstructed as $X'^{\Phi U'}$ for some choice of $|\Phi\rangle \in \mathcal{H}_{code}$ that is cyclic and separating with respect to M_{code} .
- Since any operator in M_{code} may be written as a linear combination of four unitary operators in M_{code} , we have a way to represent any operator in M_{code} as an operator in M_{phys} . The same logic applies to show that any operator in M'_{code} may be represented as an operator in M'_{phys} .

The rest of this paper is summarized as follows. In Sect. 2, we define von Neumann algebras and functions of operators, and we review the spectral theorem (for unbounded operators). In Sect. 3, we review the relative modular operator from Tomita–Takesaki theory, and define the relative entropy. In Sect. 4, we prove that when the bulk reconstruction is satisfied, the relative entropy is equivalent between the boundary and the bulk (Theorem 1.2). In Sect. 5, we prove the converse, completing the proof of Theorem 1.1. In Sect. 6, we physically interpret Theorem 1.1 and relate our work to previous work on finite-dimensional quantum error correction and holography.

2. Bounded and Unbounded Operators

In this section, we review some results in functional analysis that are used in the proofs of Theorems 1.2 and 1.1. In particular, we explain how to define a function of a bounded self-adjoint operator and we review the spectral theorem (for unbounded operators). We mostly follow reference [22].

Definition 2.1. An operator on a Hilbert space \mathcal{H} is a linear map from its domain, a linear subspace of \mathcal{H} , into \mathcal{H} .

Definition 2.2. A bounded operator is an operator \mathcal{O} that satisfies $\|\mathcal{O} |\psi\rangle\| \leq K \|\psi\| \forall |\psi\rangle \in \mathcal{H}$ for some $K \in \mathbb{R}$. We let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded operators on \mathcal{H} .

Definition 2.3. The commutant of a subset $S \subset \mathcal{B}(\mathcal{H})$ is the set S' of bounded operators that commute with all operators in S , i.e. $S' = \{\mathcal{O} \in \mathcal{B}(\mathcal{H}) : \mathcal{O} \mathcal{P} = \mathcal{P} \mathcal{O} \forall \mathcal{P} \in S\}$. The double commutant of S is the commutant of S' .

Definition 2.4. A von Neumann algebra is an algebra of bounded operators that contains the identity operator, is closed under hermitian conjugation, and is equal to its double commutant.

Theorem 2.5. Let $\mathcal{O} \in \mathcal{B}(\mathcal{H})$. Let $\{|\Psi_n\rangle\} \in \mathcal{H}$ be a sequence of vectors such that its limit vanishes, i.e. $\lim_{n \rightarrow \infty} |\Psi_n\rangle = 0$. Then, $\lim_{n \rightarrow \infty} \mathcal{O} |\Psi_n\rangle = 0$.

Theorem 2.5 implies that bounded operators define a continuous linear map on the Hilbert space. Any bounded operator that annihilates a dense subspace of the Hilbert space is identically zero.

Definition 2.6. A densely defined operator on a Hilbert space \mathcal{H} is an operator whose domain is a dense subspace of \mathcal{H} .

2.1. *Functions of bounded operators.* In this section, we will explain how to understand functions of bounded operators.

Definition 2.7. The *spectrum* of $\mathcal{O} \in \mathcal{B}(\mathcal{H})$ is defined as

$$\sigma(\mathcal{O}) := \{\lambda \in \mathbb{C} : \mathcal{O} - \lambda I \text{ is not invertible}\},$$

where I denotes the identity operator.

We will make use of the mathematical facts that $\sigma(\mathcal{O})$ is a nonempty closed bounded subset of \mathbb{C} and that when \mathcal{O} is self-adjoint, $\sigma(\mathcal{O}) \subset \mathbb{R}$ and $\|\mathcal{O}\| = \sup_{\lambda \in \sigma(\mathcal{O})} |\lambda|$ [23] [22].

Definition 2.8. Let $\mathcal{O} \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator. We denote the set of continuous \mathbb{R} -valued functions defined on $\sigma(\mathcal{O})$ by $C(\sigma(\mathcal{O}))$.

Definition 2.9. For every self-adjoint operator $\mathcal{O} \in \mathcal{B}(\mathcal{H})$, we define the L_∞ norm (denoted by $\|\cdot\|_\infty$) of $f \in C(\sigma(\mathcal{O}))$ by

$$\|f\|_\infty = \sup_{x \in \sigma(\mathcal{O})} |f(x)|.$$

Theorem 2.10 ([22], page 121). *Given a self-adjoint operator $\mathcal{O} \in \mathcal{B}(\mathcal{H})$, the set of polynomials (with \mathbb{R} -valued coefficients) is dense in $C(\sigma(\mathcal{O}))$ in the L_∞ norm.*

Definition 2.11. For any polynomial $p(x) = \sum_{n=0}^N a_n x^n$ with $a_n \in \mathbb{R}$, we define $p(\mathcal{O}) := \sum_{n=0}^N a_n \mathcal{O}^n$ for $\mathcal{O} \in \mathcal{B}(\mathcal{H})$.

Theorem 2.12 ([22], page 223). *Let $p(x) = \sum_{n=0}^N a_n x^n$ with $a_n \in \mathbb{R}$. Let $\mathcal{O} \in \mathcal{B}(\mathcal{H})$.⁷ Then*

$$\sigma(p(\mathcal{O})) = \{p(\lambda) | \lambda \in \sigma(\mathcal{O})\}.$$

Theorem 2.13 ([22], page 223). *For any self-adjoint operator $\mathcal{O} \in \mathcal{B}(\mathcal{H})$ and any polynomial $p \in C(\sigma(\mathcal{O}))$,*

$$\|p(\mathcal{O})\| = \|p\|_\infty.$$

Proof. $\|p(\mathcal{O})\| = \sup_{\lambda \in \sigma(p(\mathcal{O}))} |\lambda| = \sup_{\lambda \in \sigma(\mathcal{O})} |p(\lambda)| = \|p\|_\infty.$ □

Let $\mathcal{O} \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Let P denote the space of polynomials defined on \mathbb{R} with \mathbb{R} -valued coefficients. Define a map $\tilde{\phi}_\mathcal{O} : P \rightarrow \mathcal{B}(\mathcal{H})$ such that $\tilde{\phi}_\mathcal{O}(p) = p(\mathcal{O})$ for any polynomial $p \in P$. The map $\tilde{\phi}_\mathcal{O}$ is a bounded linear operator because $\|\tilde{\phi}_\mathcal{O}(p)\| = \|p\|_\infty$. Hence, $\tilde{\phi}_\mathcal{O}$ may be uniquely extended to a bounded linear operator $\phi_\mathcal{O} : C(\sigma(\mathcal{O})) \rightarrow \mathcal{B}(\mathcal{H})$. For $f \in C(\sigma(\mathcal{O}))$, we define $f(\mathcal{O}) := \phi_\mathcal{O}(f)$. If $\{p_n\} \in P$ denotes a sequence of polynomials such that $\lim_{n \rightarrow \infty} p_n = f$ (where the limit converges in the L_∞ norm), then we may also write

$$f(\mathcal{O}) = \lim_{n \rightarrow \infty} p_n(\mathcal{O}), \tag{2.1}$$

where the limit converges in the norm topology. If $f, g \in C(\sigma(\mathcal{O}))$, then one may show [22] that $f(\mathcal{O})g(\mathcal{O}) = (fg)(\mathcal{O})$ and that $(f^*)(\mathcal{O}) = f(\mathcal{O})^\dagger$.

⁷ Note that \mathcal{O} need not be self-adjoint.

Theorem 2.14 ([23], page 19). *Let M be a von Neumann algebra. Any operator in M is a linear combination of four unitary operators in M .*

Proof. Let $\mathcal{O} \in M$. We may write

$$\mathcal{O} = \frac{1}{2}(\mathcal{O} + \mathcal{O}^\dagger) - \frac{i}{2}(i(\mathcal{O} - \mathcal{O}^\dagger)).$$

This shows that \mathcal{O} may be written as a linear combination of two self-adjoint operators in M . Next, let $\mathcal{Q} \in M$ be a self-adjoint operator that satisfies $\|\mathcal{Q}\| < 1$. The condition $\|\mathcal{Q}\| < 1$ is important because the function $f(x) = \sqrt{1 - x^2}$ is \mathbb{R} -valued and continuous only for $|x| < 1$. Define $U := \mathcal{Q} + i\sqrt{1 - \mathcal{Q}^2}$. Then U is unitary, $U \in M$, and $\mathcal{Q} = \frac{U + U^\dagger}{2}$.

□

2.2. Unbounded operators. Unbounded operators are generically not defined on the entire Hilbert space. The domain of an operator \mathcal{O} is denoted by $D(\mathcal{O})$. The definition of \mathcal{O}^\dagger is subtle when \mathcal{O} is unbounded, as \mathcal{O}^\dagger may not be defined on the entire Hilbert space.

Definition 2.15. A densely defined operator \mathcal{O} is *closed* when $\mathcal{O}(\lim_{n \rightarrow \infty} |\psi_n\rangle) = \lim_{n \rightarrow \infty} \mathcal{O}|\psi_n\rangle$ whenever both limits exist.

Definition 2.16. Let \mathcal{O} be a densely defined operator on \mathcal{H} . The domain of the adjoint \mathcal{O}^\dagger is defined by

$$D(\mathcal{O}^\dagger) = \{|\phi\rangle : \exists |\eta\rangle \in \mathcal{H} \text{ such that } \langle \phi | \mathcal{O} | \psi \rangle = \langle \eta | \psi \rangle \quad \forall |\psi\rangle \in D(\mathcal{O})\}.$$

For $|\phi\rangle \in D(\mathcal{O}^\dagger)$ there is precisely one $|\eta\rangle$ that satisfies the above criterion. We define

$$\mathcal{O}^\dagger |\phi\rangle := |\eta\rangle.$$

Theorem 2.17 ([22], page 252). *If \mathcal{O} is a densely defined operator on \mathcal{H} , then \mathcal{O}^\dagger is closed. If \mathcal{O} is closed, $D(\mathcal{O}^\dagger)$ is dense in \mathcal{H} .*

Definition 2.18. A densely defined operator \mathcal{O} is *self-adjoint* when $\mathcal{O} = \mathcal{O}^\dagger$. In particular, $D(\mathcal{O}) = D(\mathcal{O}^\dagger)$.

Definition 2.19. A densely defined operator is *positive* when $\langle \psi | \mathcal{O} | \psi \rangle \geq 0 \quad \forall |\psi\rangle \in D(\mathcal{O})$.

Definition 2.20. Let \mathcal{O} be a closed operator on a Hilbert space \mathcal{H} . $\lambda \in \mathbb{C}$ is in the *resolvent set* of \mathcal{O} if $\lambda I - \mathcal{O}$ is a bijection of $D(\mathcal{O})$ onto \mathcal{H} . The *spectrum* of \mathcal{O} , denoted $\sigma(\mathcal{O})$, is defined to be the set of all complex numbers that are not in the resolvent set of \mathcal{O} .

Theorem 2.21. *Let \mathcal{O} be a self-adjoint positive operator. Then the spectrum of \mathcal{O} is a subset of $[0, \infty)$.*

Proof. For any $|\chi\rangle \in D(\mathcal{O})$ and any $\lambda = \lambda_1 + i\lambda_2$ for $\lambda_1, \lambda_2 \in \mathbb{R}$, note that⁸

$$\|(\mathcal{O} - \lambda I)|\chi\rangle\|^2 = \lambda_2^2\|\chi\|^2 + \|(\mathcal{O} - \lambda_1 I)|\chi\rangle\|^2 \geq \lambda_2^2\|\chi\|^2. \tag{2.2}$$

Let us consider the case when $\lambda_2 \neq 0$. Then $\ker(\mathcal{O} - \lambda I) = \{0\}$ so that $\mathcal{O} - \lambda I$ is an injection. Using the fact that $D(\mathcal{O})$ is dense in \mathcal{H} , one may show that the orthocomplement of the range of $(\mathcal{O} - \lambda I)$ is trivial, implying that the range of $(\mathcal{O} - \lambda I)$ is dense in \mathcal{H} . Then, the previous equation implies that if $\{|\chi_n\rangle\} \in D(\mathcal{O})$ is a sequence such that $\lim_{n \rightarrow \infty} (\mathcal{O} - \lambda I)|\chi_n\rangle$ exists, then $\lim_{n \rightarrow \infty} |\chi_n\rangle$ also exists. Since \mathcal{O} is a closed operator, the range of $(\mathcal{O} - \lambda I)$ is also closed. Thus, $(\mathcal{O} - \lambda I)$ is a bijection from $D(\mathcal{O})$ onto \mathcal{H} , demonstrating that λ is in the resolvent set of \mathcal{O} .

Now, consider the case when $\lambda \in \mathbb{R}, \lambda < 0$. For any $|\chi\rangle \in D(\mathcal{O})$,

$$\|(\mathcal{O} - \lambda I)|\chi\rangle\|^2 = |\lambda|^2\|\chi\|^2 - 2\langle\chi|\mathcal{O}|\chi\rangle\lambda + \|\mathcal{O}|\chi\rangle\|^2. \tag{2.3}$$

As \mathcal{O} is a positive operator,

$$\|(\mathcal{O} - \lambda I)|\chi\rangle\|^2 \geq |\lambda|^2\|\chi\|^2. \tag{2.4}$$

The same logic used in the previous case establishes that λ is in the resolvent set of \mathcal{O} . Hence, the spectrum of \mathcal{O} must be a subset of $[0, \infty)$. \square

Theorem 2.22 ([22], page 316). *Let \mathcal{O} be a closed operator. Then $D(\mathcal{O}^\dagger\mathcal{O}) = \{|\psi\rangle : |\psi\rangle \in D(\mathcal{O}), \mathcal{O}|\psi\rangle \in D(\mathcal{O}^\dagger)\}$ is dense in the Hilbert space and $\mathcal{O}^\dagger\mathcal{O}$ is self-adjoint and positive.*

2.3. The spectral theorem for unbounded operators. In this section, we closely follow [22] (pages 262–263), to which we refer the reader for more details on the spectral theorem. Note that a projection $P \in \mathcal{B}(\mathcal{H})$ is idempotent and hermitian i.e. $P = P^2 = P^\dagger$.

Definition 2.23. A projection valued measure assigns a projection P_Ω to every Borel set $\Omega \subset \mathbb{R}$ such that

- $P_\emptyset = 0, P_{(-\infty, \infty)} = I$
- $P_{\Omega_1}P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$
- If $\Omega = \cup_{n=1}^\infty \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ if $n \neq m$, then P_Ω is a strong limit of $\sum_{n=1}^N P_{\Omega_n}$.

Given any vector $|\psi\rangle \in \mathcal{H}$, $\langle\psi|P_\Omega|\psi\rangle$ defines an integration measure for Borel functions, which we will use in Definition 2.25.

Theorem 2.24 (Spectral Theorem [22], page 263). *There is a one-to-one correspondence between self-adjoint operators \mathcal{O} and projection valued measures $P_\Omega^\mathcal{O}$. The correspondence is given by*

$$\mathcal{O} = \int_{\mathbb{R}} \lambda d(P_\lambda^\mathcal{O}).$$

⁸ To be explicit, we have that

$$\langle(\mathcal{O} - \lambda I)\chi|(\mathcal{O} - \lambda I)\chi\rangle = \langle(\mathcal{O} - \lambda_1 I)\chi|(\mathcal{O} - \lambda_1 I)\chi\rangle + \langle\lambda_2\chi|\lambda_2\chi\rangle + i\lambda_2\langle\chi|(\mathcal{O} - \lambda_1 I)\chi\rangle - i\lambda_2\langle(\mathcal{O} - \lambda_1 I)\chi|\chi\rangle.$$

The last two terms cancel because \mathcal{O} is self-adjoint and λ_1 is real.

The notation means that we are integrating the function $f(\lambda) = \lambda$ on \mathbb{R} with the projection-valued measure given by $P_\Omega^\mathcal{O}$. The integral converges strongly.⁹

Intuitively, $P_\Omega^\mathcal{O}$ is the projection onto the “eigenspace” spanned by all “eigenvalues” in Ω . We will need that $P_{(-\infty, \infty)}^\mathcal{O} = P_{\sigma(\mathcal{O})}^\mathcal{O}$, where $\sigma(\mathcal{O})$ denotes the spectrum of \mathcal{O} . For the details on how the spectral projections associated with a self-adjoint operator \mathcal{O} are constructed, see Theorem VIII.4 and discussions afterwards in Section VIII.3 of [22].

Definition 2.25. Given a self-adjoint positive operator \mathcal{O} , the diagonal matrix element of $\log \mathcal{O}$ is given by

$$\langle \psi | \log \mathcal{O} | \psi \rangle := \int_0^\infty \log \lambda d(\langle \psi | P_\lambda^\mathcal{O} | \psi \rangle),$$

for all $|\psi\rangle \in \mathcal{H}$ such that the above integral converges, where $P_\Omega^\mathcal{O}$ is the unique projection valued measure associated with \mathcal{O} by the spectral theorem.

Note that $\log x$ is continuous for $x \in (0, \infty)$. Thus, $\log x$ is a Borel function. One can define a self-adjoint operator using any real-valued Borel function on \mathbb{R} . See page 264 of [22].

Theorem 2.26. Let \mathcal{O} be a self-adjoint positive operator. For all $|\psi\rangle \in D(\mathcal{O})$ such that $\langle \psi | \log \mathcal{O} | \psi \rangle$ is defined,

$$\langle \psi | \log \mathcal{O} | \psi \rangle \leq \langle \psi | \mathcal{O} | \psi \rangle - \langle \psi | \psi \rangle,$$

and the inequality is saturated if and only if $\mathcal{O} |\psi\rangle = |\psi\rangle$.

Proof. Assume $|\psi\rangle \neq 0$. For $x > 0$, note that $\log x \leq x - 1$. It follows that

$$\begin{aligned} \langle \psi | \log \mathcal{O} | \psi \rangle &= \int_0^\infty \log \lambda d(\langle \psi | P_\lambda^\mathcal{O} | \psi \rangle) \leq \int_0^\infty \lambda d(\langle \psi | P_\lambda^\mathcal{O} | \psi \rangle) \\ &\quad - \int_0^\infty 1 d(\langle \psi | P_\lambda^\mathcal{O} | \psi \rangle). \end{aligned} \tag{2.5}$$

The first integral on the right hand side converges because $|\psi\rangle \in D(\mathcal{O})$. The second integral converges to $\langle \psi | \psi \rangle$ because the spectrum of \mathcal{O} is a subset of $[0, \infty)$, which implies that $P_{[0, \infty)}^\mathcal{O} = P_{(-\infty, \infty)}^\mathcal{O} = I$. Hence,

$$\langle \psi | \log \mathcal{O} | \psi \rangle \leq \langle \psi | \mathcal{O} | \psi \rangle - \langle \psi | \psi \rangle. \tag{2.6}$$

As $\log x \leq x - 1$ is only saturated for $x = 1$, the inequality in Eq. (2.5) is only saturated when the measure $\langle \psi | P_\Omega^\mathcal{O} | \psi \rangle$ is such that $\langle \psi | P_\Omega^\mathcal{O} | \psi \rangle = 0$ when $1 \notin \Omega$. If $1 \notin \Omega$, then $\langle \psi | P_\Omega^\mathcal{O} | \psi \rangle = \langle P_\Omega^\mathcal{O} \psi | P_\Omega^\mathcal{O} \psi \rangle$ implies that $P_\Omega^\mathcal{O} |\psi\rangle = 0$. If $1 \in \Omega$, then the fact that $\int_{\mathbb{R}} 1 d(\langle \psi | P_\lambda^\mathcal{O} | \psi \rangle) = \langle \psi | \psi \rangle$ implies that $\langle P_\Omega^\mathcal{O} \psi | P_\Omega^\mathcal{O} \psi \rangle = \langle \psi | P_\Omega^\mathcal{O} | \psi \rangle = \langle \psi | \psi \rangle$. For $1 \in \Omega$, note that the Cauchy-Schwartz inequality $|\langle \psi | P_\Omega^\mathcal{O} | \psi \rangle| \leq \|\psi\| \cdot \|P_\Omega^\mathcal{O} |\psi\rangle\|$ is saturated, which implies that $P_\Omega^\mathcal{O} |\psi\rangle$ is a multiple of $|\psi\rangle$, and this multiple must be 1. Thus, for $1 \in \Omega$, $P_\Omega^\mathcal{O} |\psi\rangle = |\psi\rangle$. This implies that

$$\mathcal{O} |\psi\rangle = \int_{\mathbb{R}} \lambda d(P_\lambda^\mathcal{O} |\psi\rangle) = |\psi\rangle. \tag{2.7}$$

□

⁹ For any $|\psi\rangle \in D(\mathcal{O})$, the integral $\int_{\mathbb{R}} \lambda d(P_\lambda^\mathcal{O} |\psi\rangle)$ with vector-valued measure $P_\Omega^\mathcal{O} |\psi\rangle$ converges in the Hilbert space norm to $\mathcal{O} |\psi\rangle$. The integral does not converge for $|\psi\rangle \notin D(\mathcal{O})$.

3. Review of Tomita–Takesaki theory

Previous works on entanglement entropy and AdS/CFT [10,12,24,25] have used the definition for the relative entropy as $S(\rho, \sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma)$. Since $S(\rho, \sigma)$ does not increase upon performing a partial trace on ρ and σ , the relative entropy may be intuitively thought of as a measure of distinguishability between two states. Araki’s definition of the relative entropy [16] also has a monotonicity property, and it reduces to $S(\rho, \sigma)$ when the Hilbert space is finite-dimensional [14]. Hence, we might expect that statements about relative entropy in AdS/CFT can be reformulated for infinite-dimensional Hilbert spaces.

We want to understand the connection between entanglement wedge reconstruction and the equivalence of bulk and boundary relative entropies in infinite dimensional Hilbert spaces, using Tomita–Takesaki theory. Tomita–Takesaki theory provides us with the relative modular operator which is used to define the relative entropy. In this section, we review properties of the relative modular operator and the definition of the relative entropy, following [14,16,23].

Definition 3.1. A vector $|\Psi\rangle \in \mathcal{H}$ is said to be *cyclic* with respect to a von Neumann algebra M when the set of vectors $\mathcal{O}|\Psi\rangle$ for $\mathcal{O} \in M$ is dense in \mathcal{H} .

Definition 3.2. A vector $|\Psi\rangle \in \mathcal{H}$ is *separating* with respect to a von Neumann algebra M when zero is the only operator in M that annihilates $|\Psi\rangle$. That is, $\mathcal{O}|\Psi\rangle = 0 \implies \mathcal{O} = 0$ for $\mathcal{O} \in M$.

Given a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ and a vector $|\Psi\rangle \in \mathcal{H}$, we may define a map $e_\Psi : M \rightarrow \mathcal{H} : \mathcal{O} \mapsto \mathcal{O}|\Psi\rangle$. \mathcal{H} is the closure of the image of e_Ψ iff $|\Psi\rangle$ is cyclic with respect to M . Also, $\ker e_\Psi = \{0\}$ ¹⁰ iff $|\Psi\rangle$ is separating with respect to M .

Definition 3.3. Let $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}$ and M be a von Neumann algebra. The *relative Tomita operator* is the operator $S_{\Psi|\Phi}$ that acts as

$$S_{\Psi|\Phi} |x\rangle := |y\rangle$$

for any sequence $\{\mathcal{O}_n\} \in M$ such that the limits $|x\rangle = \lim_{n \rightarrow \infty} \mathcal{O}_n |\Psi\rangle$ and $|y\rangle = \lim_{n \rightarrow \infty} \mathcal{O}_n^\dagger |\Phi\rangle$ both exist.

The relative Tomita operator $S_{\Psi|\Phi}$ is well-defined on a dense subset of the Hilbert space if and only if $|\Psi\rangle$ is cyclic and separating with respect to M .¹¹ Note that $S_{\Psi|\Phi}$ is a closed operator.

Theorem 3.4 ([23], page 94). *Let $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}$ both be cyclic and separating with respect to a von Neumann algebra M . Let $S_{\Psi|\Phi}$ and $S'_{\Psi|\Phi}$ be the relative Tomita operators defined with respect to M and its commutant M' respectively. Then*

$$S_{\Psi|\Phi}^\dagger = S'_{\Psi|\Phi}, \quad S'_{\Psi|\Phi}^\dagger = S_{\Psi|\Phi}. \tag{3.1}$$

Definition 3.5. Let $S_{\Psi|\Phi}$ be a relative Tomita operator and $|\Psi\rangle \in \mathcal{H}$ be cyclic and separating with respect to a von Neumann algebra M . The *relative modular operator* is

$$\Delta_{\Psi|\Phi} := S_{\Psi|\Phi}^\dagger S_{\Psi|\Phi}.$$

¹⁰ In other words, e_Ψ is an injective map.

¹¹ $S_{\Psi|\Phi}$ is well-defined if and only if $\lim_{n \rightarrow \infty} \mathcal{O}_n |\Psi\rangle = 0 \implies \lim_{n \rightarrow \infty} \mathcal{O}_n^\dagger |\Phi\rangle = 0$. See footnote 14 of [14] for a proof of why this is true. $S_{\Psi|\Phi}$ is densely defined because $|\Psi\rangle$ is cyclic with respect to M .

If $|\Phi\rangle$ is replaced with $\mathcal{O}'|\Phi\rangle$, where $\mathcal{O}' \in M'$ is unitary, then the relative modular operator remains unchanged [14]:

$$\Delta_{\Psi|\Phi} = \Delta_{\Psi|\mathcal{O}'\Phi}. \tag{3.2}$$

Definition 3.6 ([16]). Let $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}$ and $|\Psi\rangle$ be cyclic and separating with respect to a von Neumann algebra M . Let $\Delta_{\Psi|\Phi}$ be a relative modular operator. The *relative entropy* with respect to M of $|\Psi\rangle$ is

$$\mathcal{S}_{\Psi|\Phi}(M) = -\langle \Psi | \log \Delta_{\Psi|\Phi} | \Psi \rangle.$$

Note that the relative entropy $\mathcal{S}_{\Psi|\Phi}(M)$ is nonnegative and it vanishes precisely when $|\Phi\rangle = \mathcal{O}'|\Psi\rangle$ for a unitary $\mathcal{O}' \in M'$.

Definition 3.7. Let M be a von Neumann algebra on \mathcal{H} and $|\Psi\rangle$ be a cyclic and separating vector for M . The *Tomita operator* S_Ψ is

$$S_\Psi := S_{\Psi|\Psi},$$

where $S_{\Psi|\Psi}$ is the relative modular operator defined with respect to M . The *modular operator* $\Delta_\Psi = S_\Psi^\dagger S_\Psi$ and the *antiunitary operator* J_Ψ are the operators that appear in the polar decomposition of S_Ψ such that

$$S_\Psi = J_\Psi \Delta_\Psi^{1/2}.$$

Theorem 3.8 (Tomita–Takesaki [26]). *Let M be a von Neumann algebra on \mathcal{H} and let $|\Psi\rangle$ be a cyclic and separating vector for M . Then*

- $J_\Psi M J_\Psi = M'$.
- $\Delta_\Psi^{it} M \Delta_\Psi^{-it} = M \quad \forall t \in \mathbb{R}$.

Theorem 3.8 is important because it allows us to interpret Δ_Ψ as the operator that generates a modular flow on M . Suppose that the Hilbert space \mathcal{H} factorizes as $\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$. For concreteness, we may intuitively think of \mathcal{H}_ℓ as a Hilbert space that corresponds to the left Rindler wedge of Minkowski space, while \mathcal{H}_r corresponds to the right Rindler wedge.

For a given state $|\Psi\rangle \in \mathcal{H}$, if we define

$$\rho := |\Psi\rangle \langle \Psi|, \quad \rho_\ell := \text{Tr}_r \rho, \quad \rho_r := \text{Tr}_\ell \rho, \tag{3.3}$$

then the reduced density matrices ρ_ℓ and ρ_r generate a modular flow on operators that act on \mathcal{H}_ℓ and \mathcal{H}_r , respectively. The modular operator Δ_Ψ corresponding to the von Neumann algebra that acts nontrivially on \mathcal{H}_ℓ is then given by

$$\Delta_\Psi = \rho_\ell \otimes \rho_r^{-1}. \tag{3.4}$$

When $|\Psi\rangle$ is the vacuum and \mathcal{H}_ℓ and \mathcal{H}_r correspond to Rindler wedges, we have that

$$\rho_\ell = e^{-2\pi K_\ell}, \quad \rho_r = e^{-2\pi K_r}, \tag{3.5}$$

where K_ℓ and K_r are the boost generators that act respectively on the left and right wedges (see Fig. 1). The modular operator Δ_Ψ is then given by

$$\Delta_\Psi = e^{-2\pi(K_\ell - K_r)}. \tag{3.6}$$

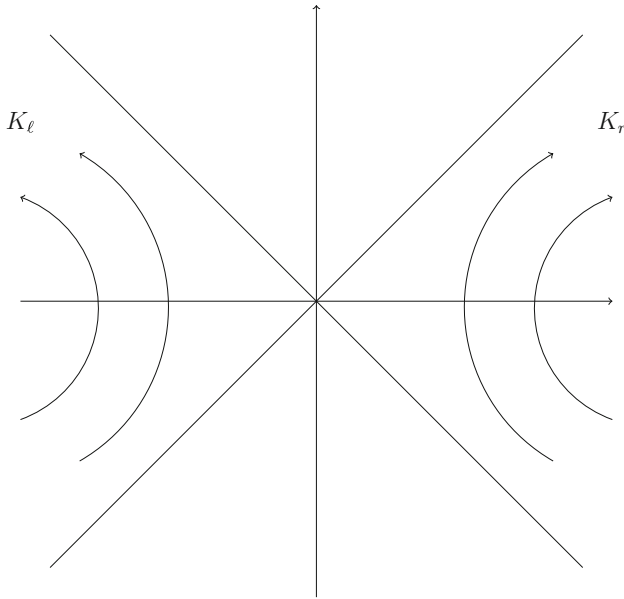


Fig. 1. Two Rindler wedges in Minkowski space. The generators K_r and K_ℓ correspond to boosts, as shown

In this context, Theorem 3.8 states that the modular flow maps operators in a Rindler wedge to operators in the same Rindler wedge. Thus, the algebraically defined modular flow in Theorem 3.8 has a geometric interpretation. This is an example of *modular covariance*, which is the property that the modular flow is a spacetime symmetry. The unitary group generated by the modular operator associated with the vacuum state implements the Lorentz boosts that leave the Rindler wedges invariant. The antiunitary operator J corresponds to the operator CRT , where C denotes charge conjugation, R is a reflection that maps one wedge into the other, and T is time reversal [27].

One of the findings of [10] is that bulk modular flow is dual to boundary modular flow. As an intermediate step in proving the equivalence of bulk and boundary entropies, we will also show that the bulk and boundary modular operators act on the code subspace in the same way. This is further evidence that the definitions and theorems of Tomita–Takesaki theory are relevant for understanding bulk reconstruction.

4. Proof of Theorem 1.2

This section contains the proof of Theorem 1.2. In Lemma 4.1, we show that cyclic and separating states in \mathcal{H}_{code} are mapped to cyclic and separating states in \mathcal{H}_{phys} . In Lemma 4.2, we relate operators in M_{phys} to operators in M_{code} . In Sect. 4.1, we consider Theorem 1.2 in a special case where the relative Tomita operators are bounded. In Sect. 4.2, we prove Theorem 1.2 in full generality.

Lemma 4.1. *Under the assumptions of Theorem 1.2, for every $|\Psi\rangle \in \mathcal{H}_{code}$ that is cyclic and separating with respect to M_{code} , $u|\Psi\rangle$ is cyclic and separating with respect to M_{phys} .*

Proof. Let $|\Omega\rangle$ be defined as in Theorem 1.2. We will first show that $u|\Psi\rangle$ is cyclic with respect to M_{phys} . That is, we can act on $u|\Psi\rangle$ with an operator in M_{phys} to get a state arbitrarily close to any state in \mathcal{H}_{phys} . Given any $|\Phi\rangle \in \mathcal{H}_{phys}$ and $\epsilon > 0$, we need to choose $\mathcal{P} \in M_{phys}$ such that $\| |\Phi\rangle - \mathcal{P}u|\Psi\rangle \| < \epsilon$. Choose $\hat{\mathcal{P}} \in M_{phys}$ such that $\| \hat{\mathcal{P}}u|\Omega\rangle - |\Phi\rangle \| < \frac{\epsilon}{2}$ and $\hat{\mathcal{P}} \neq 0$. Since $|\Psi\rangle$ is cyclic with respect to M_{code} , choose $\mathcal{O} \in M_{code}$ such that $\| |\mathcal{O}|\Psi\rangle - |\Omega\rangle \| < \frac{\epsilon}{2\|\hat{\mathcal{P}}\|}$. Let $\tilde{\mathcal{O}} \in M_{phys}$ be an operator that satisfies $\tilde{\mathcal{O}}u|\Theta\rangle = u\mathcal{O}|\Theta\rangle \forall |\Theta\rangle \in \mathcal{H}_{code}$. Then, note that

$$|\Phi\rangle - \hat{\mathcal{P}}\tilde{\mathcal{O}}u|\Psi\rangle = |\Phi\rangle - \hat{\mathcal{P}}u\mathcal{O}|\Psi\rangle = |\Phi\rangle - \hat{\mathcal{P}}u|\Omega\rangle - \hat{\mathcal{P}}u(\mathcal{O}|\Psi\rangle - |\Omega\rangle). \tag{4.1}$$

By the triangle inequality,

$$\| |\Phi\rangle - \hat{\mathcal{P}}\tilde{\mathcal{O}}u|\Psi\rangle \| \leq \| |\Phi\rangle - \hat{\mathcal{P}}u|\Omega\rangle \| + \|\hat{\mathcal{P}}\| \cdot \| \mathcal{O}|\Psi\rangle - |\Omega\rangle \|. \tag{4.2}$$

By choosing $\mathcal{P} = \hat{\mathcal{P}}\tilde{\mathcal{O}}$, we see that $u|\Psi\rangle$ is cyclic with respect to M_{phys} . The same logic shows that $u|\Psi\rangle$ is cyclic with respect to M'_{phys} and hence separating for M_{phys} . \square

Lemma 4.2. *Under the assumptions of Theorem 1.2, for any $\mathcal{P} \in M_{phys}$, $u^\dagger\mathcal{P}u \in M_{code}$.*

Proof. Choose any $\mathcal{O}' \in M'_{code}$. For any $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_{code}$, we have that

$$\begin{aligned} \langle \Psi | u^\dagger \mathcal{P} u \mathcal{O}' | \Phi \rangle &= \langle \Psi | u^\dagger \mathcal{P} \tilde{\mathcal{O}}' u | \Phi \rangle = \langle \Psi | u^\dagger \tilde{\mathcal{O}}' \mathcal{P} u | \Phi \rangle = \langle \tilde{\mathcal{O}}'^\dagger u \Psi | \mathcal{P} u | \Phi \rangle \\ &= \langle u \mathcal{O}'^\dagger \Psi | \mathcal{P} u | \Phi \rangle = \langle \Psi | \mathcal{O}' u^\dagger \mathcal{P} u | \Phi \rangle. \end{aligned} \tag{4.3}$$

Hence, $u^\dagger\mathcal{P}u \in M''_{code} = M_{code}$. \square

4.1. Special case of bounded relative Tomita operator. We will first prove Theorem 1.2 in the special case where the relative Tomita operators with respect to M_{code} and M_{phys} , denoted respectively by $S_{\Psi|\Phi}^c$ and $S_{\Psi|\Phi}^p$, are bounded operators. In this special case, we do not have to keep track of their domains. The proof of the general case is similar, but technically more complicated.

For any $\mathcal{O} \in M_{code}$,

$$uS_{\Psi|\Phi}^c\mathcal{O}|\Psi\rangle = u\mathcal{O}^\dagger|\Phi\rangle = \tilde{\mathcal{O}}^\dagger u|\Phi\rangle = S_{u\Psi|u\Phi}^p\tilde{\mathcal{O}}u|\Psi\rangle = S_{u\Psi|u\Phi}^p u\mathcal{O}|\Psi\rangle, \tag{4.4}$$

Hence

$$(uS_{\Psi|\Phi}^c - S_{u\Psi|u\Phi}^p)\mathcal{O}|\Psi\rangle = 0. \tag{4.5}$$

$(uS_{\Psi|\Phi}^c - S_{u\Psi|u\Phi}^p)$ is a bounded operator that annihilates a dense subspace of \mathcal{H}_{code} , since $|\Psi\rangle$ is cyclic with respect to M_{code} . It follows from the fact that the kernel of $(uS_{\Psi|\Phi}^c - S_{u\Psi|u\Phi}^p)$ is closed that

$$uS_{\Psi|\Phi}^c = S_{u\Psi|u\Phi}^p. \tag{4.6}$$

Likewise, for any $\mathcal{P} \in M_{phys}$,

$$u^\dagger S_{u\Psi|u\Phi}^P \mathcal{P} |u\Psi\rangle = u^\dagger \mathcal{P}^\dagger u |\Phi\rangle = S_{\Psi|\Phi}^c u^\dagger \mathcal{P} |u\Psi\rangle, \tag{4.7}$$

$$\left(u^\dagger S_{u\Psi|u\Phi}^P - S_{\Psi|\Phi}^c u^\dagger\right) \mathcal{P} |u\Psi\rangle = 0. \tag{4.8}$$

As $u |\Psi\rangle$ is cyclic with respect to M_{phys} by assumption, we have that

$$u^\dagger S_{u\Psi|u\Phi}^P = S_{\Psi|\Phi}^c u^\dagger, \quad S_{u\Psi|u\Phi}^{P\dagger} u = u S_{\Psi|\Phi}^{c\dagger}. \tag{4.9}$$

Equations (4.6) and (4.9) imply that the subspace $\text{Im } u$ is mapped to itself under $S_{u\Psi|u\Phi}^P$ and $S_{u\Psi|u\Phi}^{P\dagger}$. Thus, the subspace $\text{Im } u$ is mapped to itself under $\Delta_{u\Psi|u\Phi}^P$. From the fact that $\Delta_{u\Psi|u\Phi}^P$ is self-adjoint and bounded, it follows that the subspace $(\text{Im } u)^\perp$ is mapped to itself under $\Delta_{u\Psi|u\Phi}^P$. Equations (4.6) and (4.9) also imply that

$$\Delta_{\Psi|\Phi}^c = u^\dagger \Delta_{u\Psi|u\Phi}^P u. \tag{4.10}$$

Note that $\Delta_{u\Psi|u\Phi}^P$ and $\Delta_{\Psi|\Phi}^c$ are positive, self-adjoint, bounded operators. Thus, we may use the spectral theorem to study them. We will apply the spectral theorem to $(\Delta_{u\Psi|u\Phi}^P)|_{\text{Im } u}$ and $(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)^\perp}$ separately.¹² We write

$$(\Delta_{u\Psi|u\Phi}^P)|_{\text{Im } u} = \int_{\mathbb{R}} \lambda d(P_\lambda^{\text{Im } u}), \quad (\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)^\perp} = \int_{\mathbb{R}} \lambda d(P_\lambda^{(\text{Im } u)^\perp}). \tag{4.11}$$

For a Borel set $\Omega \subset \mathbb{R}$, the projections $P_\Omega^{\text{Im } u}$ and $P_\Omega^{(\text{Im } u)^\perp}$ commute with uu^\dagger because uu^\dagger is the projection onto $\text{Im } u$. The spectral decomposition of $\Delta_{u\Psi|u\Phi}^P$ is given by

$$\Delta_{u\Psi|u\Phi}^P = \int_{\mathbb{R}} \lambda d(P_\lambda^P). \tag{4.12}$$

By the uniqueness of the spectral decomposition, we have that $P_\Omega^P = P_\Omega^{\text{Im } u} + P_\Omega^{(\text{Im } u)^\perp}$. Thus, P_Ω^P commutes with uu^\dagger . Let Ω_1 and Ω_2 be two Borel sets. Then

$$u^\dagger P_{\Omega_1}^P uu^\dagger P_{\Omega_2}^P u = u^\dagger P_{\Omega_1}^P P_{\Omega_2}^P u. \tag{4.13}$$

One can then check that the family of projections $u^\dagger P_\Omega^P u = u^\dagger P_\Omega^{\text{Im } u} u$ is a projection valued measure on \mathcal{H}_{code} . We will now show that this is the projection valued measure associated with $\Delta_{\Psi|\Phi}^c$. From Eq. (4.10), it follows that for any $|\Theta\rangle \in \mathcal{H}_{code}$, we have that

$$\Delta_{\Psi|\Phi}^c |\Theta\rangle = u^\dagger \Delta_{u\Psi|u\Phi}^P u |\Theta\rangle = \int_{\mathbb{R}} \lambda d(u^\dagger P_\lambda^P u |\Theta\rangle). \tag{4.14}$$

¹² $(\Delta_{u\Psi|u\Phi}^P)|_{\text{Im } u}$ denotes the restriction of $\Delta_{u\Psi|u\Phi}^P$ to the closed subspace $\text{Im } u$.

By the uniqueness of the spectral decomposition of $\Delta_{\Psi|\Phi}^c$, we conclude that $u^\dagger P_\Omega^P u$ is the projection valued measure associated with $\Delta_{\Psi|\Phi}^c$. It follows that

$$\begin{aligned} -\langle \Psi | \log(\Delta_{\Psi|\Phi}^c) | \Psi \rangle &= -\int_0^\infty \log(\lambda) d(\langle \Psi | u^\dagger P_\lambda^P u | \Psi \rangle) \\ &= -\int_0^\infty \log(\lambda) d(\langle u \Psi | P_\lambda^P | u \Psi \rangle) = -\langle u \Psi | \log(\Delta_{u\Psi|u\Phi}^P) | u \Psi \rangle. \end{aligned} \tag{4.15}$$

The same logic can be applied to the commutant algebras M'_{code} and M'_{phys} . Hence,

$$\mathcal{S}_{\Psi|\Phi}(M_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M_{phys}), \quad \mathcal{S}_{\Psi|\Phi}(M'_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M'_{phys}). \tag{4.16}$$

4.2. General proof of Theorem 1.2.

Lemma 4.3. *Let $S_{\Psi|\Phi}^c$ denote the relative Tomita operator defined with respect to M_{code} . Let $S_{u\Psi|u\Phi}^P$ denote the relative Tomita operator defined with respect to M_{phys} . Let $S_{\Psi|\Phi}^{c'}$ and $S_{u\Psi|u\Phi}^{P'}$ denote the relative Tomita operators defined with respect to M'_{code} and M'_{phys} . Then $uS_{\Psi|\Phi}^c = S_{u\Psi|u\Phi}^P u$ and $uS_{\Psi|\Phi}^{c'} = S_{u\Psi|u\Phi}^{P'} u$.*

Proof. $D(S_{\Psi|\Phi}^c)$ is given by all $|x\rangle \in \mathcal{H}_{code}$ that may be written as

$$|x\rangle = \lim_{n \rightarrow \infty} \mathcal{O}_n | \Psi \rangle \tag{4.17}$$

for some sequence $\{\mathcal{O}_n\} \in M_{code}$ such that the limit

$$|y\rangle := \lim_{n \rightarrow \infty} \mathcal{O}_n^\dagger | \Phi \rangle \tag{4.18}$$

exists. By definition, $S_{\Psi|\Phi}^c |x\rangle := |y\rangle$. Given $|x\rangle$ and $|y\rangle$ defined as above, it follows that

$$u |x\rangle = \lim_{n \rightarrow \infty} \tilde{\mathcal{O}}_n u | \Psi \rangle, \quad u |y\rangle = \lim_{n \rightarrow \infty} \tilde{\mathcal{O}}_n^\dagger u | \Phi \rangle. \tag{4.19}$$

Hence, $u |x\rangle \in D(S_{u\Psi|u\Phi}^P)$. It follows that for all $|x\rangle \in D(S_{\Psi|\Phi}^c)$,

$$uS_{\Psi|\Phi}^c |x\rangle = S_{u\Psi|u\Phi}^P u |x\rangle,$$

which means that $S_{u\Psi|u\Phi}^P u$ is an extension of $uS_{\Psi|\Phi}^c$. To see that $S_{u\Psi|u\Phi}^P u$ is not a proper extension, suppose $|w\rangle \in D(S_{u\Psi|u\Phi}^P)$. Then $u |w\rangle \in D(S_{\Psi|\Phi}^c)$, meaning that there exists a sequence $\{\mathcal{P}_n\} \in M_{phys}$ such that

$$u |w\rangle = \lim_{n \rightarrow \infty} \mathcal{P}_n u | \Psi \rangle, \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{P}_n^\dagger u | \Psi \rangle \text{ exists.} \tag{4.20}$$

We may also write $|w\rangle = \lim_{n \rightarrow \infty} u^\dagger \mathcal{P}_n u | \Psi \rangle$. From Lemma 4.2, $u^\dagger \mathcal{P}_n u \in M_{code}$. Hence, $|w\rangle \in D(S_{\Psi|\Phi}^c)$; so we may write $uS_{\Psi|\Phi}^c = S_{u\Psi|u\Phi}^P u$ as an operator equality because the operators on both sides have the same domain and act the same way on vectors in the domain. The same logic establishes that $uS_{\Psi|\Phi}^{c'} = S_{u\Psi|u\Phi}^{P'} u$. \square

Lemma 4.4. Let $\Delta_{u\Psi|u\Phi}^P := S_{u\Psi|u\Phi}^{P\dagger} S_{u\Psi|u\Phi}^P$ be the relative modular operator associated with $S_{u\Psi|u\Phi}^P$. Then,

- $\Delta_{u\Psi|u\Phi}^P$ maps the vector space $(\text{Im } u) \cap D(\Delta_{u\Psi|u\Phi}^P)$ into $(\text{Im } u)$, and $(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)}$ is densely defined on $(\text{Im } u)$.
- $\Delta_{u\Psi|u\Phi}^P$ maps the vector space $(\text{Im } u)^\perp \cap D(\Delta_{u\Psi|u\Phi}^P)$ into $(\text{Im } u)^\perp$, and $(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)^\perp}$ is densely defined on $(\text{Im } u)^\perp$.

Proof. Let $|x\rangle \in D(S_{u\Psi|u\Phi}^P)$. Then there exists a sequence $\{\mathcal{P}_n\} \in M_{phys}$ such that

$$|x\rangle = \lim_{n \rightarrow \infty} \mathcal{P}_n |u\Psi\rangle, \text{ and } \lim_{n \rightarrow \infty} \mathcal{P}_n^\dagger |u\Phi\rangle \text{ exists.} \tag{4.21}$$

Then $u^\dagger |x\rangle \in D(S_{\Psi|\Phi}^c)$. We may write

$$S_{\Psi|\Phi}^c u^\dagger |x\rangle = u^\dagger S_{u\Psi|u\Phi}^P |x\rangle. \tag{4.22}$$

The fact that $u^\dagger |x\rangle \in D(S_{\Psi|\Phi}^c)$ and Lemma 4.3 together imply that $uu^\dagger |x\rangle \in D(S_{u\Psi|u\Phi}^P)$.

We may uniquely decompose $|x\rangle$ into the sum

$$|x\rangle = |a\rangle + |b\rangle \tag{4.23}$$

where $|a\rangle \in \text{Im } u$ and $|b\rangle \in (\text{Im } u)^\perp$. We know that $|a\rangle = uu^\dagger |x\rangle \in D(S_{u\Psi|u\Phi}^P)$. As $D(S_{u\Psi|u\Phi}^P)$ is a vector space, this implies that $|b\rangle \in D(S_{u\Psi|u\Phi}^P)$.

It follows from the above that

$$D(S_{u\Psi|u\Phi}^P) = \text{Im } u \cap D(S_{u\Psi|u\Phi}^P) \oplus (\text{Im } u)^\perp \cap D(S_{u\Psi|u\Phi}^P). \tag{4.24}$$

From Eq. (4.22),

$$uu^\dagger S_{u\Psi|u\Phi}^P |b\rangle = 0, \tag{4.25}$$

which means that $S_{u\Psi|u\Phi}^P$ maps the vector space $(\text{Im } u)^\perp \cap D(S_{u\Psi|u\Phi}^P) \rightarrow (\text{Im } u)^\perp$.

From Lemma 4.3 we may write, for all $|x\rangle \in D(S_{u\Psi|u\Phi}^P)$,

$$u S_{\Psi|\Phi}^c u^\dagger |x\rangle = S_{u\Psi|u\Phi}^P uu^\dagger |x\rangle. \tag{4.26}$$

It follows from $uu^\dagger |x\rangle = |a\rangle$ that

$$u S_{\Psi|\Phi}^c u^\dagger |x\rangle = S_{u\Psi|u\Phi}^P |a\rangle. \tag{4.27}$$

It follows from $u^\dagger |b\rangle = 0$ that

$$u S_{\Psi|\Phi}^c u^\dagger |a\rangle = S_{u\Psi|u\Phi}^P |a\rangle, \tag{4.28}$$

which means that $S_{u\Psi|u\Phi}^P$ maps the vector space $(\text{Im } u) \cap D(S_{u\Psi|u\Phi}^P) \rightarrow (\text{Im } u)$.

We will now show that $(\text{Im } u) \cap D(S_{u\Psi|u\Phi}^P)$ is dense in $(\text{Im } u)$. Given any $|A\rangle \in (\text{Im } u)$, choose $|X\rangle \in \mathcal{H}_{phys}$ such that $uu^\dagger |X\rangle = |A\rangle$. Next, choose a sequence $\{|x_n\rangle\} \in D(S_{u\Psi|u\Phi}^P)$ that converges to $|X\rangle$. We then have that $\lim_{n \rightarrow \infty} uu^\dagger |x_n\rangle = |A\rangle$. Since $|x_n\rangle \in D(S_{u\Psi|u\Phi}^P)$, we know from earlier that $uu^\dagger |x_n\rangle \in D(S_{u\Psi|u\Phi}^P)$. Hence, $(\text{Im } u) \cap D(S_{u\Psi|u\Phi}^P)$ is dense in $(\text{Im } u)$. The same logic shows that $(\text{Im } u)^\perp \cap D(S_{u\Psi|u\Phi}^P)$ is dense in $(\text{Im } u)^\perp$.

Furthermore, $(S_{u\Psi|u\Phi}^P)|_{(\text{Im } u)}$ is a closed operator because $(\text{Im } u)$ is a closed subspace.

We can apply all of the above logic to the commutant algebras. To summarize,

- $S_{u\Psi|u\Phi}^P$ maps the vector space $(\text{Im } u)^\perp \cap D(S_{u\Psi|u\Phi}^P) \rightarrow (\text{Im } u)^\perp$, and $(S_{u\Psi|u\Phi}^P)|_{(\text{Im } u)^\perp}$ is closed and densely defined on $(\text{Im } u)^\perp$.
- $S_{u\Psi|u\Phi}^P$ maps the vector space $(\text{Im } u) \cap D(S_{u\Psi|u\Phi}^P) \rightarrow (\text{Im } u)$, and $(S_{u\Psi|u\Phi}^P)|_{(\text{Im } u)}$ is closed and densely defined on $(\text{Im } u)$.
- $S_{u\Psi|u\Phi}^{P'}$ maps the vector space $(\text{Im } u)^\perp \cap D(S_{u\Psi|u\Phi}^{P'}) \rightarrow (\text{Im } u)^\perp$, and $(S_{u\Psi|u\Phi}^{P'})|_{(\text{Im } u)^\perp}$ is closed and densely defined on $(\text{Im } u)^\perp$.
- $S_{u\Psi|u\Phi}^{P'}$ maps the vector space $(\text{Im } u) \cap D(S_{u\Psi|u\Phi}^{P'}) \rightarrow (\text{Im } u)$, and $(S_{u\Psi|u\Phi}^{P'})|_{(\text{Im } u)}$ is closed and densely defined on $(\text{Im } u)$.

It directly follows that the above statements also hold for the adjoints $S_{u\Psi|u\Phi}^{P\dagger}$ and $S_{u\Psi|u\Phi}^{P'\dagger}$. Recall that $\Delta_{u\Psi|u\Phi}^P = S_{u\Psi|u\Phi}^{P\dagger} S_{u\Psi|u\Phi}^P$. We may compute $(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)}$ from $(S_{u\Psi|u\Phi}^P)|_{(\text{Im } u)}$ and $(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)^\perp}$ from $(S_{u\Psi|u\Phi}^P)|_{(\text{Im } u)^\perp}$. In particular, $(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)}$ is given by

$$(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)} = (S_{u\Psi|u\Phi}^{P\dagger}|_{(\text{Im } u)})(S_{u\Psi|u\Phi}^P|_{(\text{Im } u)}) = (S_{u\Psi|u\Phi}^P|_{(\text{Im } u)})^\dagger(S_{u\Psi|u\Phi}^P|_{(\text{Im } u)}). \tag{4.29}$$

It follows that $(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)}$ is densely defined and self-adjoint on $(\text{Im } u)$. The same logic can be applied to $(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)^\perp}$. \square

Having established Lemmas 4.1–4.4, we can now prove Theorem 1.2, which shows that entanglement wedge reconstruction implies the equivalence of bulk and boundary relative entropies.

Theorem 1.2. *Let $u : \mathcal{H}_{code} \rightarrow \mathcal{H}_{phys}$ be an isometry between two Hilbert spaces. Let M_{code} and M_{phys} be von Neumann algebras on \mathcal{H}_{code} and \mathcal{H}_{phys} respectively. Let M'_{code} and M'_{phys} respectively be the commutants of M_{code} and M_{phys} . Suppose that*

- *There exists some state $|\Omega\rangle \in \mathcal{H}_{code}$ such that $u|\Omega\rangle \in \mathcal{H}_{phys}$ is cyclic and separating with respect to M_{phys} .*
- *$\forall \mathcal{O} \in M_{code} \forall \mathcal{O}' \in M'_{code}, \exists \tilde{\mathcal{O}} \in M_{phys} \exists \tilde{\mathcal{O}}' \in M'_{phys}$ such that*

$$\forall |\Theta\rangle \in \mathcal{H}_{code} \quad \begin{cases} u\mathcal{O}|\Theta\rangle = \tilde{\mathcal{O}}u|\Theta\rangle, & u\mathcal{O}'|\Theta\rangle = \tilde{\mathcal{O}}'u|\Theta\rangle, \\ u\mathcal{O}^\dagger|\Theta\rangle = \tilde{\mathcal{O}}^\dagger u|\Theta\rangle, & u\mathcal{O}'^\dagger|\Theta\rangle = \tilde{\mathcal{O}}'^\dagger u|\Theta\rangle. \end{cases}$$

Then, for any $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_{code}$ with $|\Psi\rangle$ cyclic and separating with respect to M_{code} ,

- *$u|\Psi\rangle$ is cyclic and separating with respect to M_{phys} and M'_{phys} ,*
- *$\mathcal{S}_{\Psi|\Phi}(M_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M_{phys}), \quad \mathcal{S}_{\Psi|\Phi}(M'_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M'_{phys}),$*

where $\mathcal{S}_{\Psi|\Phi}(M)$ is the relative entropy.

Proof. $\Delta_{u\Psi|u\Phi}^P$ and $\Delta_{\Psi|\Phi}^c$ are positive, densely defined, self-adjoint operators that are generically unbounded. Thus, we may use the spectral theorem to study them. We will apply the spectral theorem to $(\Delta_{u\Psi|u\Phi}^P)|_{\text{Im } u}$ and $(\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)^\perp}$ separately. We write

$$(\Delta_{u\Psi|u\Phi}^P)|_{\text{Im } u} = \int_{\mathbb{R}} \lambda d(P_\lambda^{\text{Im } u}), \quad (\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)^\perp} = \int_{\mathbb{R}} \lambda d(P_\lambda^{(\text{Im } u)^\perp}). \tag{4.30}$$

For a Borel set $\Omega \subset \mathbb{R}$, the projections $P_\Omega^{\text{Im } u}$ and $P_\Omega^{(\text{Im } u)^\perp}$ commute with uu^\dagger because uu^\dagger is the projection onto $\text{Im } u$. The spectral decomposition of $\Delta_{u\Psi|u\Phi}^P$ is given by

$$\Delta_{u\Psi|u\Phi}^P = \int_{\mathbb{R}} \lambda d(P_\lambda^P). \tag{4.31}$$

By the uniqueness of the spectral decomposition, we have that $P_\Omega^P = P_\Omega^{\text{Im } u} + P_\Omega^{(\text{Im } u)^\perp}$. Thus, P_Ω^P commutes with uu^\dagger . Let Ω_1 and Ω_2 be two Borel sets. Then

$$u^\dagger P_{\Omega_1}^P uu^\dagger P_{\Omega_2}^P u = u^\dagger P_{\Omega_1}^P P_{\Omega_2}^P u. \tag{4.32}$$

One can then check that the family of projections $u^\dagger P_\Omega^P u = u^\dagger P_\Omega^{\text{Im } u} u$ is a projection valued measure on \mathcal{H}_{code} . We will now show that this is the projection valued measure associated with $\Delta_{\Psi|\Phi}^c$. From Lemma 4.3 we have that

$$u S_{\Psi|\Phi}^c u^\dagger = S_{u\Psi|u\Phi}^P uu^\dagger = (S_{u\Psi|u\Phi}^P)|_{(\text{Im } u)}. \tag{4.33}$$

We may take the adjoint of the above equation to obtain

$$u S_{\Psi|\Phi}^c u^\dagger = (S_{u\Psi|u\Phi}^P)^\dagger|_{(\text{Im } u)}, \tag{4.34}$$

from which it follows that

$$u \Delta_{\Psi|\Phi}^c u^\dagger = (\Delta_{u\Psi|u\Phi}^P)|_{(\text{Im } u)}, \quad \Delta_{\Psi|\Phi}^c = u^\dagger \Delta_{u\Psi|u\Phi}^P u. \tag{4.35}$$

For any $|\Theta\rangle \in D(\Delta_{\Psi|\Phi}^c)$, we have that

$$\Delta_{\Psi|\Phi}^c |\Theta\rangle = u^\dagger \Delta_{u\Psi|u\Phi}^P u |\Theta\rangle = \int_{\mathbb{R}} \lambda d(u^\dagger P_\lambda^P u |\Theta\rangle). \tag{4.36}$$

By the uniqueness of the spectral decomposition of $\Delta_{\Psi|\Phi}^c$, we conclude that $u^\dagger P_\Omega^P u$ is the projection valued measure associated with $\Delta_{\Psi|\Phi}^c$.

It follows that

$$\begin{aligned} -\langle \Psi | \log(\Delta_{\Psi|\Phi}^c) | \Psi \rangle &= -\int_0^\infty \log(\lambda) d(\langle \Psi | u^\dagger P_\lambda^P u | \Psi \rangle) \\ &= -\int_0^\infty \log(\lambda) d(\langle u \Psi | P_\lambda^P | u \Psi \rangle) = -\langle u \Psi | \log(\Delta_{u\Psi|u\Phi}^P) | u \Psi \rangle. \end{aligned} \tag{4.37}$$

The same logic can be applied to the commutant algebras M'_{code} and M'_{phys} . Hence,

$$\mathcal{S}_{\Psi|\Phi}(M_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M_{phys}), \quad \mathcal{S}_{\Psi|\Phi}(M'_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M'_{phys}). \tag{4.38}$$

□

5. Proof of Theorem 1.1

Theorem 1.1. *Let $u : \mathcal{H}_{code} \rightarrow \mathcal{H}_{phys}$ be an isometry between two Hilbert spaces. Let M_{code} and M_{phys} be von Neumann algebras on \mathcal{H}_{code} and \mathcal{H}_{phys} respectively. Let M'_{code} and M'_{phys} respectively be the commutants of M_{code} and M_{phys} . Suppose that the set of cyclic and separating vectors with respect to M_{code} is dense in \mathcal{H}_{code} . Also suppose that if $|\Psi\rangle \in \mathcal{H}_{code}$ is cyclic and separating with respect to M_{code} , then $u|\Psi\rangle$ is cyclic and separating with respect to M_{phys} . Then the following two statements are equivalent:*

1. Bulk reconstruction $\forall \mathcal{O} \in M_{code} \forall \mathcal{O}' \in M'_{code}, \exists \tilde{\mathcal{O}} \in M_{phys} \exists \tilde{\mathcal{O}}' \in M'_{phys}$ such that

$$\forall |\Theta\rangle \in \mathcal{H}_{code} \begin{cases} u\mathcal{O}|\Theta\rangle = \tilde{\mathcal{O}}u|\Theta\rangle, & u\mathcal{O}'|\Theta\rangle = \tilde{\mathcal{O}}'u|\Theta\rangle, \\ u\mathcal{O}^\dagger|\Theta\rangle = \tilde{\mathcal{O}}^\dagger u|\Theta\rangle, & u\mathcal{O}'^\dagger|\Theta\rangle = \tilde{\mathcal{O}}'^\dagger u|\Theta\rangle. \end{cases}$$

2. Relative entropy equals bulk relative entropy For any $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_{code}$ with $|\Psi\rangle$ cyclic and separating with respect to M_{code} ,

$$\mathcal{S}_{\Psi|\Phi}(M_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M_{phys}), \text{ and } \mathcal{S}_{\Psi|\Phi}(M'_{code}) = \mathcal{S}_{u\Psi|u\Phi}(M'_{phys}),$$

where $\mathcal{S}_{\Psi|\Phi}(M)$ is the relative entropy.

Proof. Given the proof of Theorem 1.2, we only need to show that statement 2 implies statement 1. Let $|\Phi\rangle \in \mathcal{H}_{code}$ be cyclic and separating with respect to M_{code} , and let $U \in M_{code}$ and $U' \in M'_{code}$ be unitary operators. We can easily see that

$$0 = \mathcal{S}_{\Phi|U'\Phi}(M_{code}) = \mathcal{S}_{u\Phi|uU'\Phi}(M_{phys}). \tag{5.1}$$

Due to Theorem 2.26, this implies that

$$\Delta_{u\Phi|uU'\Phi}^P |u\Phi\rangle = |u\Phi\rangle, \tag{5.2}$$

where $\Delta_{u\Phi|uU'\Phi}^P = S_{u\Phi|uU'\Phi}^{P^\dagger} S_{u\Phi|uU'\Phi}^P$ and $S_{u\Phi|uU'\Phi}^P$ is the relative modular operator defined with respect to M_{phys} . It follows that for any $\mathcal{P} \in M_{phys}$,

$$\begin{aligned} \langle uU'\Phi|\mathcal{P}uU'\Phi\rangle &= \langle S_{u\Phi|uU'\Phi}^P u\Phi|S_{u\Phi|uU'\Phi}^P \mathcal{P}^\dagger u\Phi\rangle \\ &= \langle \mathcal{P}^\dagger u\Phi|S_{u\Phi|uU'\Phi}^{P^\dagger} S_{u\Phi|uU'\Phi}^P u\Phi\rangle = \langle u\Phi|\mathcal{P}|u\Phi\rangle. \end{aligned} \tag{5.3}$$

This implies that

$$\langle \Phi|U'^\dagger u^\dagger \mathcal{P}uU' - u^\dagger \mathcal{P}u|\Phi\rangle = 0. \tag{5.4}$$

We now use the assumption that cyclic and separating vectors with respect to M_{code} are dense in \mathcal{H}_{code} . For any $|\Psi\rangle \in \mathcal{H}_{code}$, choose a sequence $\{|\Phi_n\rangle\} \in \mathcal{H}_{code}$ such that each $|\Phi_n\rangle$ is cyclic and separating with respect to M_{code} , and $|\Psi\rangle = \lim_{n \rightarrow \infty} |\Phi_n\rangle$. Then,

$$\langle \Psi|U'^\dagger u^\dagger \mathcal{P}uU' - u^\dagger \mathcal{P}u|\Psi\rangle = \lim_{n \rightarrow \infty} \langle \Phi_n|U'^\dagger u^\dagger \mathcal{P}uU' - u^\dagger \mathcal{P}u|\Phi_n\rangle = 0. \tag{5.5}$$

Hence, this implies that the operators that are measured in the limit itself is zero, i.e. $U'^\dagger u^\dagger \mathcal{P}uU' - u^\dagger \mathcal{P}u = 0$. This then gives the following identity involving the isometry u , an arbitrary operator $\mathcal{P} \in M_{phys}$, and a unitary operator $U' \in M'_{code}$:

$$u^\dagger \mathcal{P}uU' = U' u^\dagger \mathcal{P}u. \tag{5.6}$$

The same logic can be applied to the commutant algebras; thus, for any $\mathcal{P}' \in M'_{phys}$, $U \in M_{code}$ with U unitary, we have a similar relation:

$$u^\dagger \mathcal{P}' u U = U u^\dagger \mathcal{P}' u. \tag{5.7}$$

Another consequence of Eq. (5.3) is that for any $\mathcal{P}_1, \mathcal{P}_2 \in M_{phys}$, we have that

$$\langle \mathcal{P}_1 u U' \Phi | \mathcal{P}_2 u U' \Phi \rangle = \langle \mathcal{P}_1 u \Phi | \mathcal{P}_2 u \Phi \rangle. \tag{5.8}$$

Naturally, we define a linear map $X'^{\Phi U'} : \mathcal{H}_{phys} \rightarrow \mathcal{H}_{phys}$. We define $X'^{\Phi U'}$ by

$$X'^{\Phi U'} \mathcal{P} u |\Phi\rangle := \mathcal{P} u U' |\Phi\rangle \quad \forall \mathcal{P} \in M_{phys}. \tag{5.9}$$

Then we see that $X'^{\Phi U'}$ is densely defined. From Eq. (5.8), we see that $X'^{\Phi U'}$ preserves the norm of all vectors in its domain. Hence, $X'^{\Phi U'}$ may be uniquely extended to a bounded operator, which is unitary. By definition, $X'^{\Phi U'}$ commutes with all operators in M_{phys} ; hence, we deduce that $X'^{\Phi U'} \in M'_{phys}$. (The superscripts on $X'^{\Phi U'}$ remind us that it depends on the choice of $|\Phi\rangle$ and U' and that it is in the commutant of M_{phys} .)

Next, we use Eqs. (5.7) and (5.9) with $\mathcal{P}' = X'^{\Phi U'}$. We find that

$$u^\dagger X'^{\Phi U'} u U |\Phi\rangle = U u^\dagger X'^{\Phi U'} u |\Phi\rangle = U u^\dagger u U' |\Phi\rangle = U U' |\Phi\rangle = U' U |\Phi\rangle. \tag{5.10}$$

The first equality follows from Eq. (5.7), the second equality follows from (5.9), the third equality follows from the fact that $u^\dagger u$ is the identity on \mathcal{H}_{code} , and the last equality follows because $U \in M_{code}$ and $U' \in M'_{code}$. Recall that U is an arbitrary unitary operator in M_{code} . We now need Theorem 2.14, which states that any operator in M_{code} may be written as a linear combination of four unitary operators in M_{code} [23]. The above equation implies that for any $\mathcal{O} \in M_{code}$, we have that

$$(u^\dagger X'^{\Phi U'} u - U') \mathcal{O} |\Phi\rangle = 0. \tag{5.11}$$

Note that $(u^\dagger X'^{\Phi U'} u - U')$ is a bounded operator, so its kernel is closed. Recall that $|\Phi\rangle$ is cyclic with respect to M_{code} . Since any vector in the Hilbert space may be written as $\lim_{n \rightarrow \infty} \mathcal{O}_n |\Phi\rangle$ for some sequence of operators $\{\mathcal{O}_n\} \in M_{code}$, it follows that $(u^\dagger X'^{\Phi U'} u - U')$ annihilates every vector in \mathcal{H}_{code} . In other words,

$$u^\dagger X'^{\Phi U'} u = U'. \tag{5.12}$$

Choose an arbitrary $|\Psi\rangle \in \mathcal{H}_{code}$ with $\langle \Psi | \Psi \rangle = 1$. We may uniquely write $X'^{\Phi U'} u |\Psi\rangle$ as

$$X'^{\Phi U'} u |\Psi\rangle = |a\rangle + |b\rangle, \tag{5.13}$$

where $|a\rangle \in \text{Im } u$, and $|b\rangle \in (\text{Im } u)^\perp$. Note that $X'^{\Phi U'}$ is unitary; hence, we can decompose as

$$\langle u \Psi | X'^{\Phi U'} u |\Psi\rangle = 1 = \langle a | a \rangle + \langle b | b \rangle. \tag{5.14}$$

Next, note that

$$u^\dagger |a\rangle = u^\dagger (|a\rangle + |b\rangle) = u^\dagger X'^{\Phi U'} u |\Psi\rangle = U' |\Psi\rangle. \tag{5.15}$$

Hence,

$$\langle a|a \rangle = \langle u^\dagger a|u^\dagger a \rangle = \langle U'\Psi|U'\Psi \rangle = 1. \tag{5.16}$$

This implies that $\langle b|b \rangle = 0$; hence $|b \rangle = 0$. Hence, $X'^{\Phi U'}$ maps the vector space $\text{Im } u$ to itself. We may then use Eq. (5.12) to find that

$$X'^{\Phi U'} u = uu^\dagger X'^{\Phi U'} u = uU'. \tag{5.17}$$

Next, we define a linear map $X'^{(U'\Phi)(U'^\dagger)} : \mathcal{H}_{phys} \rightarrow \mathcal{H}_{phys}$. We define $X'^{(U'\Phi)(U'^\dagger)}$ by

$$X'^{(U'\Phi)(U'^\dagger)} \mathcal{P}uU'|\Phi \rangle := \mathcal{P}u|\Phi \rangle \quad \forall \mathcal{P} \in M_{phys}. \tag{5.18}$$

It is easy to see that $U'|\Phi \rangle$ is cyclic and separating with respect to M_{code} given that $|\Phi \rangle$ is cyclic and separating with respect to M_{code} and that $U' \in M'_{code}$ is unitary. It follows that $X'^{(U'\Phi)(U'^\dagger)}$ is densely defined and uniquely extends to a bounded operator, which is unitary. Since Eq. (5.17) is true for any $|\Phi \rangle \in \mathcal{H}_{code}$ that is cyclic and separating with respect to M_{code} and any unitary $U' \in M'_{code}$,

$$X'^{(U'\Phi)(U'^\dagger)} u = uU'^\dagger. \tag{5.19}$$

This relation can be used to see that for any $\mathcal{P} \in M_{phys}$,

$$X'^{(U'\Phi)(U'^\dagger)} X'^{\Phi U'} \mathcal{P}u|\Phi \rangle = \mathcal{P}u|\Phi \rangle. \tag{5.20}$$

Thus, we deduce that the two operators we defined are adjoints of each other:

$$(X'^{\Phi U'})^\dagger = X'^{(U'\Phi)(U'^\dagger)}. \tag{5.21}$$

We have thus shown that for every unitary operator $U' \in M'_{code}$, there exists a unitary operator $X' \in M'_{phys}$ such that

$$X'u = uU', \text{ and } X'^\dagger u = uU'^\dagger. \tag{5.22}$$

The same logic applies to show that for every unitary operator $U \in M_{code}$, there exists a unitary operator $X \in M_{phys}$ such that

$$Xu = uU, \text{ and } X^\dagger u = uU^\dagger. \tag{5.23}$$

We conclude the proof by noting that any operator in a von Neumann algebra M may be written as a linear combination of four unitary operators in M (Theorem 2.14). \square

Our proof provides an explicit formula for reconstructing an operator in M_{code} as an operator in M_{phys} . Given $\mathcal{O} \in M_{code}$, we define the operator $\tilde{\mathcal{O}} \in M_{phys}$ by

$$\tilde{\mathcal{O}}\mathcal{P}'u|\Phi \rangle := \mathcal{P}'u\mathcal{O}|\Phi \rangle \quad \forall \mathcal{P}' \in M'_{phys}, \tag{5.24}$$

where $|\Phi \rangle \in \mathcal{H}_{code}$ is a fiducial state that is cyclic and separating with respect to M_{code} and M'_{code} . This formula follows from writing \mathcal{O} as a linear combination of four unitary operators in M_{code} and using Eq. (5.9) on each unitary operator. The arguments in our

proof then establish that $\tilde{\mathcal{O}}u = u\mathcal{O}$. Note that $\tilde{\mathcal{O}}$ does not depend on the choice of the fiducial state $|\Phi\rangle$. To see this, we define $\tilde{\mathcal{O}}_\star \in M_{phys}$ by

$$\tilde{\mathcal{O}}_\star \mathcal{P}'u |\Phi_\star\rangle := \mathcal{P}'u\mathcal{O} |\Phi_\star\rangle \quad \forall \mathcal{P}' \in M'_{phys}, \tag{5.25}$$

where $|\Phi_\star\rangle \in \mathcal{H}_{code}$ is a different fiducial state. Since $\tilde{\mathcal{O}}u |\Phi_\star\rangle = u\mathcal{O} |\Phi_\star\rangle$, it follows that

$$\tilde{\mathcal{O}}\mathcal{P}'u |\Phi_\star\rangle = \mathcal{P}'\tilde{\mathcal{O}}u |\Phi_\star\rangle = \mathcal{P}'u\mathcal{O} |\Phi_\star\rangle = \tilde{\mathcal{O}}_\star \mathcal{P}'u |\Phi_\star\rangle \quad \forall \mathcal{P}' \in M'_{phys}. \tag{5.26}$$

Hence, $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}_\star$ are equal because they are both bounded operators that act the same way on a dense subspace of \mathcal{H}_{phys} .

6. Discussion

In this section, we discuss the physical implications of Theorem 1.1. In particular, we explain in physical settings the validity of the technical assumptions of the theorem. In Sect. 6.1, we motivate our use of von Neumann algebras by explaining how they arise in quantum field theory, with an approach inspired by [15]. In Sect. 6.2, we summarize reasons why Theorem 1.1 is only approximately applicable to quantum gravity. In Sect. 6.3, we summarize the Reeh–Schlieder theorem. In Sect. 6.4, we use the Reeh–Schlieder theorem to physically motivate the assumptions of Theorem 1.1. In Sect. 6.5, we compare Theorem 1.1 with previous work on finite-dimensional error correction [13].

6.1. Von Neumann algebras in quantum field theory. Quantum field theories are characterized by algebras of operators acting on a Hilbert space \mathcal{H} . For every open region in spacetime, there is an associated algebra [15]. We will assume that there is a unique ground state $|\Omega\rangle \in \mathcal{H}$. The closure of the set of states obtained by acting on $|\Omega\rangle$ with all operators in the algebra associated with the entire spacetime is defined to be the vacuum superselection sector, \mathcal{H}_0 . By definition, each superselection sector of the theory is an invariant subspace of this algebra.

Theories with lagrangian descriptions have a notion of an elementary field. Given an open region of spacetime \mathcal{U} , we can define an associated operator algebra $\mathcal{A}(\mathcal{U})$ by smearing the elementary fields with functions supported only in \mathcal{U} .¹³ The operator algebra $\mathcal{A}(\mathcal{U})$ generically contains unbounded operators. Given $\mathcal{A}(\mathcal{U})$, we may obtain a von Neumann algebra $M(\mathcal{U})$, which only consists of bounded operators, as follows [15]. For every unbounded operator (which we assume to be closed) in $\mathcal{A}(\mathcal{U})$, we may perform a polar decomposition to obtain a partial isometry and a self-adjoint positive operator, which is canonically associated with a set of projections by the spectral theorem. The von Neumann algebra $M(\mathcal{U})$ is generated by the set of all spectral projections and partial isometries associated with the operators in $\mathcal{A}(\mathcal{U})$.¹⁴ We assume that the operators

¹³ Assuming that the time-slice axiom [15] holds, $\mathcal{A}(\mathcal{U})$ should really be associated with the domain of dependence of \mathcal{U} , as operators in the domain of dependence are related to operators in \mathcal{U} via an equation of motion. Note that the time-slice axiom does not hold for generalized free fields [28], which we consider in Sect. 6.2.

¹⁴ If a subalgebra \mathcal{S} of bounded operators contains the identity and is closed under hermitian conjugation, then its double commutant, \mathcal{S}'' , is the von Neumann algebra generated by \mathcal{S} . Von Neumann algebras are naturally associated with causally complete subregions [14,23].

in $\mathcal{A}(\mathcal{U})$ may be approximated by operators in $M(\mathcal{U})$. As shown in [14], the Reeh–Schlieder theorem implies that states with bounded energy-momentum are cyclic with respect to $\mathcal{A}(\mathcal{U})$ for any open subregion of spacetime \mathcal{U} . We assume that this is also true for $M(\mathcal{U})$.

6.2. Approximate entanglement wedge reconstruction. Throughout the paper, we have used von Neumann algebras to denote subregions in the bulk and the boundary. In AdS/CFT, the boundary theory is a quantum field theory, so the discussion in Sect. 6.1 directly applies. However, the bulk theory is a theory of quantum gravity (string theory). For states with a semi-classical bulk dual, the bulk theory may be effectively described using quantum field theory on an asymptotically AdS background that might contain black holes. The applicability of quantum field theory motivates us to use von Neumann algebras to describe operators associated with covariantly defined subregions in the bulk, like the entanglement wedge of a boundary subregion.¹⁵ Since entanglement wedges are causally complete, they naturally have an associated von Neumann algebra.

Since the long-distance bulk physics is only approximately described by quantum field theory, we need a generalization of Theorem 1.1 that relates the approximate bulk reconstruction to the approximate equivalence of relative entropies between the boundary and the bulk [31–33]. We want to note that our formulation of bulk reconstruction in Theorem 1.1 is exact in the sense that correlation functions of operators in M_{code} exactly equal correlation functions computed on the boundary with the corresponding operators in M_{phys} .

To be more precise, Theorem 1.1 is only valid for certain choices of the code subspace. If the code subspace consists of states with semi-classically distinct geometries, it is not clear how von Neumann algebras can be associated with subregions in a state independent way. For Theorem 1.1 to be relevant, we could choose \mathcal{H}_{code} to be a subspace describing long wavelength modes in quantum field theory on a fixed background and the entanglement wedge to be the classical minimal area surface corresponding to a boundary subregion. To order G_N^0 , the bulk dual of entanglement entropy is given by the bulk entanglement entropy of the entanglement wedge plus a local integral on the minimal area surface [34]. This was used to relate the bulk and boundary modular hamiltonians [10]. Since the bulk and boundary modular hamiltonians only differ by operators localized on the minimal surface, the bulk and boundary relative entropies are equivalent up to $\mathcal{O}(G_N)$ corrections [10]. The bulk dual of relative entropy beyond order G_N^0 involves bulk modular hamiltonians evaluated with respect to different bulk surfaces [31].¹⁶ Since the formula for the bulk dual of relative entropy in Theorem 1.1 is only valid to order G_N^0 , the two main statements in Theorem 1.1 can only be true in quantum gravity in an approximate sense. Theorem 4 of [35] proves that in the case of finite-dimensional von Neumann algebras, the approximate equivalence of bulk and boundary relative entropies implies approximate bulk reconstruction. Furthermore, [32] proves that entanglement wedge reconstruction can be exact to all orders in perturbation theory, under the assumptions that the code subspace is at most polynomially large in

¹⁵ Associating a set of operators with a subregion in the bulk is highly nontrivial due to nonlocal effects in the bulk [29]. This is addressed in [30], which studies information measures for sets of operators that are not closed under multiplication. We do not consider this subtlety in our analysis.

¹⁶ It will be interesting to generalize equation (5.4) in [31] to an expression that uses infinite-dimensional von Neumann algebras.

G_N^{-1} and the difference of bulk and boundary relative entropies is nonperturbatively small (the latter assumption is a result of [31]).¹⁷

It is possible for both statements in Theorem 1.1 to be exactly true in the limit $G_N \rightarrow 0$. In this case, the AdS/CFT duality relates a $(d + 1)$ -dimensional quantum field theory in AdS and a d -dimensional generalized free field theory, for which all connected n -point correlation functions vanish when $n \geq 3$.¹⁸ We may set $\mathcal{H}_{code} = \mathcal{H}_{phys}$ because every state in the boundary theory has a geometric dual. The case where the bulk theory is a free scalar is studied in [28]. The authors of [28] work in Poincaré coordinates, which has d -dimensional Minkowski space as its conformal boundary. They argue that in the boundary generalized free field theory, the algebra associated with the domain of dependence of any ball-shaped region in a spatial slice of Minkowski space is equal to the algebra associated with the causal wedge in the bulk.¹⁹ This statement is expressed in equation (5.7) of [28]. This implies that M_{code} and M_{phys} are isomorphic, i.e. $M_{code} = M_{phys}$, which means that the bulk and boundary relative entropies are equal.

6.3. The Reeh–Schlieder theorem. In the previous subsection, we explained how we use von Neumann algebras to approximately characterize bulk physics. Before we physically motivate the assumption in Theorem 1.1 that the set of cyclic and separating vectors with respect to M_{code} is dense in \mathcal{H}_{code} , we outline the conclusions of the Reeh–Schlieder theorem. Our discussion of the Reeh–Schlieder theorem follows the spirit of [14].

For the purposes of presenting the Reeh–Schlieder theorem, we restrict ourselves to quantum field theory in d -dimensional Minkowski space. Let P^μ be the energy-momentum operator. Each component of P^μ is a self-adjoint operator with its own set of spectral projections. Let S_Λ be the subset of momentum space defined by

$$S_\Lambda = \{p^\mu : |p^\mu| < \Lambda \quad \forall \mu \in \{0, 1, \dots, d - 1\}\}$$

for some cutoff energy Λ . Using the spectral projections of each P^μ , we may construct a projection operator Π_{S_Λ} that projects onto the subspace of states with energy-momentum in S_Λ . As P^μ is defined by smearing the local operator $T^{0\mu}$ (where $T^{\mu\nu}$ is the stress tensor) over an entire spatial slice,²⁰ Π_{S_Λ} leaves each superselection sector invariant. Furthermore, for every $|\Psi\rangle \in \mathcal{H}$,

$$\lim_{\Lambda \rightarrow \infty} \Pi_{S_\Lambda} |\Psi\rangle = |\Psi\rangle.$$

Thus, the set of states of bounded energy-momentum in a given superselection sector is dense in that superselection sector.

The Reeh–Schlieder theorem may be applied to states of bounded energy-momentum. Let $|\Xi\rangle$ denote such a state. Let Σ denote a spatial slice. Given an open proper subregion $\mathcal{V} \subset \Sigma$, let $\mathcal{U}_\mathcal{V}$ be a small neighborhood in spacetime containing \mathcal{V} . The Reeh–Schlieder theorem tells us that the closure of the set of states obtained by acting on $|\Xi\rangle$ with

¹⁷ In certain contexts, the entanglement wedge reconstruction proposal must be nonperturbatively approximate (see [32,33]). The approximate reconstruction with nonperturbative gravity corrections is rigorously formulated and analyzed in [36], that further expands the scope of our algebraic framework.

¹⁸ The fact that all correlation functions may be expressed in terms of two-point functions arises from large- N factorization in the boundary CFT.

¹⁹ This statement is also true for conformal transformations of such regions. For these boundary regions, the causal wedge is the same as the entanglement wedge [37].

²⁰ Technically, a spatial slice is not an open subregion of spacetime.

operators in the algebra $\mathcal{A}(\mathcal{U}_\gamma)$ is equal to the closure of the set of states obtained by acting on $|\Xi\rangle$ with all local operators, which is the superselection sector of $|\Xi\rangle$.

Let us restrict our attention to a single superselection sector. Then $|\Xi\rangle$ is cyclic with respect to $\mathcal{A}(\mathcal{U}_\gamma)$ and $M(\mathcal{U}_\gamma)$. Since \mathcal{V} is a proper subregion of Σ , the Reeh–Schlieder theorem may also be applied to the subregion $\mathcal{U}_{\mathcal{V}'}$, where \mathcal{V}' is the complement of the closure of \mathcal{V} in Σ . The result is that $|\Xi\rangle$ is also separating with respect to $M(\mathcal{U}_\gamma)$ [14]. Thus, in quantum field theory in Minkowski space restricted to a single superselection sector, the fact that the set of states of bounded energy-momentum is dense implies that the set of cyclic and separating vectors with respect to $M(\mathcal{U}_\gamma)$ is dense.

6.4. Physical motivation for the assumptions of Theorem 1.1. We now use the Reeh–Schlieder theorem to understand the assumptions in Theorem 1.1 in a physical context. Without loss of generality, we assume that the bulk-to-boundary isometry u in Theorem 1.1 maps \mathcal{H}_{code} into a single superselection sector of \mathcal{H}_{phys} . That is, the code subspace lies within a single superselection sector. If this is not the case, then we can decompose \mathcal{H}_{code} into orthogonal subspaces that each are mapped into different superselection sectors of the boundary theory, and we can study Theorem 1.1 separately for each orthogonal subspace.

In Theorem 1.1, we assume that the set of cyclic and separating vectors with respect to M_{code} is dense in \mathcal{H}_{code} . If the bulk theory was quantum field theory in Minkowski space, then the discussion in Sect. 6.3 directly applies. However, the discussion in Sect. 6.3 does not directly imply this because the bulk theory is only approximately described by quantum field theory and the background spacetime is asymptotically AdS. In [38], a version of the Reeh–Schlieder theorem is proved for free scalar fields in global AdS. The theorem is valid for the vacuum state of the field quantized in global AdS, the vacuum state of the field quantized in any causal wedge, and finite-energy excitations of these vacua. If we choose to ignore the gravitational backreaction in the bulk and treat the bulk theory as a local QFT, then the Reeh–Schlieder theorem [20] applied to the bulk implies that the set of cyclic and separating vectors with respect to M_{code} , where M_{code} is associated with an entanglement wedge, is dense in the bulk vacuum superselection sector \mathcal{H}_0 .²¹ This is because the Reeh–Schlieder theorem guarantees the existence of a single cyclic and separating vector (namely the vacuum), which in turn implies the existence of a dense set of cyclic and separating vectors. In particular, note that [39, 40] proves that the set $G(M_{code})$, defined as

$$G(M_{code}) := \left\{ \mathcal{O} \in M_{code} : \mathcal{O}^{-1} \in M_{code} \right\}, \quad (6.1)$$

obeys the property that $G(M_{code})|\Omega\rangle$ is dense in \mathcal{H}_{code} , where $|\Omega\rangle$ is the vacuum state. Furthermore, if $\mathcal{O} \in G(M_{code})$, then it is straightforward to show that $\mathcal{O}|\Omega\rangle$ is cyclic and separating with respect to M_{code} . Hence, the assumption that the set of cyclic and separating vectors with respect to M_{code} is dense in \mathcal{H}_{code} follows from the Reeh–Schlieder theorem.

In a generic local quantum field theory, the von Neumann algebra of a type III₁ factor²² is associated with a causal subregion of the spacetime. When M_{code} and M'_{code} are type III₁ factors, the assumption of Theorem 1.1 that cyclic and separating states with respect to M_{code} are dense in \mathcal{H}_{code} also follows from a result of Connes–Størmer, which is presented below.

²¹ If \mathcal{H}_0 is a proper subset of \mathcal{H}_{code} , we should redefine \mathcal{H}_{code} to be \mathcal{H}_0 for Theorem 1.1 to apply.

²² The definition of a type III₁ factor is given in [41].

Theorem 6.1. (Connes–Størmer [21]) *A factor M is of type III_1 if and only if the action of its unitary group on its state space by inner automorphisms is topologically transitive in the norm topology.*

Let us elaborate on the special case of type III_1 factors. Let $|\Psi\rangle$ be a cyclic and separating vector with respect to M . The above theorem implies that the set of vectors that can be written as $UU'|\Psi\rangle$, where $U \in M$ and $U' \in M'$ are both unitary operators, is dense in \mathcal{H} . Given that $|\Psi\rangle$ is cyclic and separating with respect to M , $UU'|\Psi\rangle$ is also cyclic and separating. The existence of one cyclic and separating vector $|\Psi\rangle$ in Theorem 1.1 guarantees, for a factor of type III_1 , that the set of cyclic and separating vectors with respect to M_{code} is dense in \mathcal{H}_{code} .

It would be interesting to investigate the plausibility of the assumption that the set of cyclic and separating states with respect to M_{code} is dense in \mathcal{H}_{code} when \mathcal{H}_{code} contains black hole microstates. For a sufficiently large boundary subregion, the entanglement wedge of M_{code} will contain the black hole, and the operators in M_{code} correspond to local operators associated with the field degrees of freedom outside of the black hole as well as operators that act on the black hole microstates, whose description involves quantum gravity at the Planck scale. In quantum field theory, it is possible to generate the whole Hilbert space by acting on the vacuum with operators in a small subregion because the vacuum is highly entangled. It would be interesting to understand how the presence of a black hole changes the structure of spacetime entanglement outside the horizon. Holographic tensor network models suggest that entanglement wedge reconstruction is possible in the presence of a black hole [42]; operators outside the black hole can in fact be “pushed through” the black hole tensor [32]. However, tensor network models of holography involve finite dimensional Hilbert spaces and thus cannot capture the pattern of entanglement that makes the Reeh–Schlieder theorem work.

Finally, we address the assumption in Theorem 1.1 that for all states $|\Psi\rangle \in \mathcal{H}_{code}$ that are cyclic and separating with respect to M_{code} , $u|\Psi\rangle$ is cyclic and separating with respect to M_{phys} . In [38], the Reeh–Schlieder theorem holds for the vacuum of global AdS, implying that the vacuum is cyclic and separating with respect to the local operator algebra associated with a bulk subregion. The image of the bulk vacuum state under the bulk-to-boundary isometry is the boundary vacuum state, which is cyclic and separating with respect to the local operator algebras associated with boundary subregions. Likewise, finite-energy excited states in the bulk map to states in the boundary CFT of bounded energy-momentum, which are also cyclic and separating. This supports the assumption of Theorem 1.1 that the cyclic and separating states with respect to M_{code} map to the cyclic and separating states with respect to M_{phys} .

6.5. Finite-dimensional quantum error correction. In this section, we explain Theorem 1.1 in the context of previous work on finite-dimensional error correction [12, 13, 43]. First, we interpret the assumption that cyclic and separating vectors with respect to M_{code} map to cyclic and separating vectors with respect to M_{phys} in the case that \mathcal{H}_{code} and \mathcal{H}_{phys} are finite dimensional. As discussed in [13], a finite dimensional M_{code} induces a decomposition of the code subspace,

$$\mathcal{H}_{code} = \oplus_{\alpha} \mathcal{H}_{a_{\alpha}} \otimes \mathcal{H}_{\bar{a}_{\alpha}}, \tag{6.2}$$

such that any $\mathcal{O} \in M_{code}$ may be written in block-diagonal form:

$$\mathcal{O} = \begin{pmatrix} \mathcal{O}_{a_1} \otimes I_{\bar{a}_1} & 0 & \cdots \\ 0 & \mathcal{O}_{a_2} \otimes I_{\bar{a}_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{6.3}$$

In the setup of [13], \mathcal{H}_{phys} may be written in the factorized form $\mathcal{H}_{phys} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ where each factor corresponds to a boundary subregion and its complement. Let M_{phys} induce the factorization $\mathcal{H}_{phys} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ such that operators in M_{phys} act trivially on $\mathcal{H}_{\bar{A}}$. As [13] points out, subalgebra codes with complementary recovery are especially relevant for AdS/CFT as they display a Ryu–Takayanagi formula with a nontrivial area operator. For such codes, an orthonormal basis of $\mathcal{H}_{a_\alpha} \otimes \mathcal{H}_{\bar{a}_\alpha}$ may be written as

$$u |\alpha, ij\rangle_{code} = U_A U_{\bar{A}} \left(|\alpha, i\rangle_{A_1^\alpha} |\alpha, j\rangle_{\bar{A}_1^\alpha} |\chi_\alpha\rangle_{A_2^\alpha \bar{A}_2^\alpha} \right), \tag{6.4}$$

for a decomposition of \mathcal{H}_A given by

$$\mathcal{H}_A = \oplus_\alpha (\mathcal{H}_{A_1^\alpha} \otimes \mathcal{H}_{A_2^\alpha}) \oplus \mathcal{H}_{A_3}, \tag{6.5}$$

and similarly for $\mathcal{H}_{\bar{A}}$. Also,

$$\dim \mathcal{H}_{A_1^\alpha} = \dim \mathcal{H}_{a_\alpha} \text{ and } \dim \mathcal{H}_{\bar{A}_1^\alpha} = \dim \mathcal{H}_{\bar{a}_\alpha}.$$

For each α , i and j are indices that denote basis vectors in \mathcal{H}_{a_α} and $\mathcal{H}_{\bar{a}_\alpha}$ respectively. We have explicitly included u , the isometry from the code subspace to the physical Hilbert space. $U_A, U_{\bar{A}}$ are unitary matrices that act on $\mathcal{H}_A, \mathcal{H}_{\bar{A}}$, and $|\chi\rangle_{A_2^\alpha \bar{A}_2^\alpha}$ is a state that depends on the specific code under consideration. It is important that in the state $|\chi\rangle_{A_2^\alpha \bar{A}_2^\alpha}$, subsystems A_2^α and \bar{A}_2^α are entangled. If $|\chi\rangle_{A_2^\alpha \bar{A}_2^\alpha}$ were a factorized state for every α , then it would not be possible to express $|\alpha, ij\rangle_{code}$ as in (6.4) for arbitrary choices of the factorization $\mathcal{H}_{phys} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. That is, the code would not be useful for studying bulk reconstruction for arbitrary choices of boundary subregions. Furthermore, equation (5.26) of [13] would imply that the area operator vanishes.

We now discuss the implications of Theorem 1.1 for the state $|\chi_\alpha\rangle_{A_2^\alpha \bar{A}_2^\alpha}$. Let us assume that $\dim \mathcal{H}_A = \dim \mathcal{H}_{\bar{A}}$ and that for every α , $\dim \mathcal{H}_{a_\alpha} = \dim \mathcal{H}_{\bar{a}_\alpha}$. Otherwise, there do not exist any cyclic and separating vectors with respect to M_{code} or M_{phys} . A vector in \mathcal{H}_{phys} is cyclic and separating with respect to M_{phys} if and only if it has maximal Schmidt number with respect to $\mathcal{H}_{phys} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. The assumption that cyclic and separating vectors with respect to M_{code} map to cyclic and separating vectors with respect to M_{phys} implies that $|\chi\rangle_{A_2^\alpha \bar{A}_2^\alpha}$ must have maximal Schmidt number with respect to the factorization $\mathcal{H}_{A_2^\alpha} \otimes \mathcal{H}_{\bar{A}_2^\alpha}$ and that $\dim \mathcal{H}_{A_2^\alpha} = \dim \mathcal{H}_{\bar{A}_2^\alpha}$. To see why, note that a cyclic and separating vector $|\Phi\rangle \in \mathcal{H}_{code}$ with respect to M_{code} may be written as

$$|\Phi\rangle = \sum_{\alpha, i, j} c_{ij}^\alpha |\alpha, ij\rangle_{code}, \tag{6.6}$$

where c_{ij}^α is a full-rank square matrix for each α . Using Eq. (6.4) to map $|\Phi\rangle$ to $u |\Phi\rangle \in \mathcal{H}_{phys}$, we see that if $|\chi_{\hat{\alpha}}\rangle_{A_2^{\hat{\alpha}} \bar{A}_2^{\hat{\alpha}}}$ does not have maximal Schmidt number for some $\hat{\alpha}$, then we can annihilate $u |\Phi\rangle$ with an operator that, up to conjugation by U_A , acts as the identity on $\mathcal{H}_{\bar{A}}$, annihilates \mathcal{H}_{A_3} , annihilates $\mathcal{H}_{A_1^\alpha} \otimes \mathcal{H}_{A_2^\alpha}$ for $\alpha \neq \hat{\alpha}$, and acts nontrivially on $\mathcal{H}_{A_1^{\hat{\alpha}}} \otimes \mathcal{H}_{A_2^{\hat{\alpha}}}$. This implies that $u |\Phi\rangle$ is not separating with respect

to M_{phys} , which contradicts the assumption. Another consequence of the assumption is that \mathcal{H}_{A_3} and $\mathcal{H}_{\bar{A}_3}$ must be trivial. Previous work on finite-dimensional error correction [13,43] has highlighted the crucial role of entanglement in bulk reconstruction. We have shown that the Reeh–Schlieder theorem suggests that cyclic and separating vectors with respect to M_{code} are mapped via the bulk-to-boundary isometry to vectors that are cyclic and separating with respect to M_{phys} . In the context of finite-dimensional subalgebra codes, this implies that the area term in the Ryu–Takayangi formula cannot vanish.

Our proof of entanglement wedge reconstruction in Theorem 1.1 is constructive. Given a bulk operator $\mathcal{O} \in M_{code}$, Eq. (5.24) provides an explicit formula for a boundary operator $\tilde{\mathcal{O}} \in M_{phys}$. In order to understand our formula in the finite dimensional case, we use the decomposition $\mathcal{H}_A = \oplus_{\alpha} (\mathcal{H}_{A_1^{\alpha}} \otimes \mathcal{H}_{A_2^{\alpha}})$ (and similarly for $\mathcal{H}_{\bar{A}}$) and let $|\Phi\rangle$ (defined in Eq. (6.6)) be our fiducial state. The action of $\mathcal{O} \in M_{code}$ on a code subspace basis vector is

$$\mathcal{O}|\Phi\rangle = \sum_{\alpha,i,\hat{i},j} c_{ij}^{\alpha} \langle \hat{i}|\mathcal{O}_{a_{\alpha}}|i\rangle |\alpha, \hat{i}j\rangle_{code}, \tag{6.7}$$

where $\mathcal{O}_{a_{\alpha}}$ is defined in Eq. (6.3). By Eq. (6.4) we then have that

$$u|\Phi\rangle = \sum_{\alpha,i,j} c_{ij}^{\alpha} U_A U_{\bar{A}} \left(|\alpha, i\rangle_{A_1^{\alpha}} |\alpha, j\rangle_{\bar{A}_1^{\alpha}} |\chi_{\alpha}\rangle_{A_2^{\alpha} \bar{A}_2^{\alpha}} \right), \tag{6.8}$$

$$u\mathcal{O}|\Phi\rangle = \sum_{\alpha,i,\hat{i},j} c_{ij}^{\alpha} \langle \hat{i}|\mathcal{O}_{a_{\alpha}}|i\rangle U_A U_{\bar{A}} \left(|\alpha, \hat{i}\rangle_{A_1^{\alpha}} |\alpha, j\rangle_{\bar{A}_1^{\alpha}} |\chi_{\alpha}\rangle_{A_2^{\alpha} \bar{A}_2^{\alpha}} \right). \tag{6.9}$$

Equation (5.24) then defines $\tilde{\mathcal{O}} \in M_{phys}$ by

$$\begin{aligned} &\tilde{\mathcal{O}} \mathcal{P}' U_A U_{\bar{A}} \sum_{\alpha,i,j} c_{ij}^{\alpha} \left(|\alpha, i\rangle_{A_1^{\alpha}} |\alpha, j\rangle_{\bar{A}_1^{\alpha}} |\chi_{\alpha}\rangle_{A_2^{\alpha} \bar{A}_2^{\alpha}} \right) \\ &:= \sum_{\alpha,i,\hat{i},j} c_{ij}^{\alpha} \langle \hat{i}|\mathcal{O}_{a_{\alpha}}|i\rangle \mathcal{P}' U_A U_{\bar{A}} \left(|\alpha, \hat{i}\rangle_{A_1^{\alpha}} |\alpha, j\rangle_{\bar{A}_1^{\alpha}} |\chi_{\alpha}\rangle_{A_2^{\alpha} \bar{A}_2^{\alpha}} \right), \end{aligned} \tag{6.10}$$

where $\mathcal{P}' \in M_{phys}$ can be any operator that acts as the identity on \mathcal{H}_A . With a suitable choice of \mathcal{P}' , we may show that for any α, i, j ,

$$\begin{aligned} &\tilde{\mathcal{O}} U_A U_{\bar{A}} \left(|\alpha, i\rangle_{A_1^{\alpha}} |\alpha, j\rangle_{\bar{A}_1^{\alpha}} |\chi_{\alpha}\rangle_{A_2^{\alpha} \bar{A}_2^{\alpha}} \right) \\ &= U_A U_{\bar{A}} \left(\sum_{\hat{i}} \langle \hat{i}|\mathcal{O}_{a_{\alpha}}|i\rangle |\alpha, \hat{i}\rangle_{A_1^{\alpha}} |\alpha, j\rangle_{\bar{A}_1^{\alpha}} |\chi_{\alpha}\rangle_{A_2^{\alpha} \bar{A}_2^{\alpha}} \right). \end{aligned} \tag{6.11}$$

Thus, Theorem 1.1 along with the reconstruction formula in Eq. (5.24) is an appropriate infinite-dimensional generalization of the finite-dimensional subalgebra codes with complementary recovery studied in [13].

6.6. Outlook for holographic relative entropy. The entanglement wedge reconstruction proposal is an example of bulk reconstruction. It asserts that for holographic theories, local operators in the entanglement wedge of a boundary subregion A can be written in terms of CFT operators localized on A [10,12,42,44]. Assuming that the operators in M_{code} and M'_{code} in Theorem 1.1 lie respectively in an entanglement wedge and

its complement, Theorem 1.1 establishes entanglement wedge reconstruction from the equivalence of bulk and boundary relative entropies and vice versa. Thus, it has been suggested that the entanglement wedge is “dual” to its corresponding boundary subregion [10]. Another interesting result of [10] is that bulk modular flow is dual to boundary modular flow, which we have captured in Eq. (4.35). The bulk and boundary modular operators act on the code subspace in the same way.

Quantum error correction in finite dimensional Hilbert spaces has been crucially used to argue for the entanglement wedge reconstruction proposal [12, 13]. When \mathcal{H}_{code} and \mathcal{H}_{phys} are finite-dimensional, Theorem 1.1 has parallels to Theorem 1.1 of [13]. In Theorem 1.1, we assume that cyclic and separating vectors with respect to M_{code} are dense in \mathcal{H}_{code} , which is essentially a bulk version of the Reeh–Schlieder Theorem [38]. We also assume that cyclic and separating states with respect to M_{code} map to cyclic and separating states with respect to M_{phys} , the algebra corresponding to a boundary subregion. These assumptions guarantee that the subalgebra codes studied in [13] have a nonzero area operator [13]. Defines relative entropy in the boundary theory as $S(\rho, \sigma) = \text{Tr} \rho(\log \rho - \log \sigma)$. The definition of relative entropy we use in the bulk and boundary is appropriate for infinite-dimensional Hilbert spaces and reduces to the aforementioned formula in the finite-dimensional case [16]. Thus, we have shown that the relative entropy formula in [16] naturally describes the holographic relative entropy in quantum field theory to order G_N^0 .

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