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The Green Tensor of the Nonstationary Stokes System in the Half Space

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Abstract: We prove the first ever pointwise estimates of the (unrestricted) Green tensor and the associated pressure tensor of the nonstationary Stokes system in the half-space, for every space dimension greater than one. The force field is not necessarily assumed to be solenoidal. The key is to find a suitable Green tensor formula which maximizes the tangential decay, showing in particular the integrability of Green tensor derivatives. With its pointwise estimates, we show the symmetry of the Green tensor, which in turn improves pointwise estimates. We also study how the solutions converge to the initial data, and the (infinitely many) restricted Green tensors acting on solenoidal vector fields. As applications, we give new proofs of existence of mild solutions of the Navier–Stokes equations in L^q, pointwise decay, and uniformly local L^q spaces in the half-space.

Contents

1. Introduction

This paper considers the Green tensor of the nonstationary Stokes system in the half space. A major goal is to derive its pointwise estimates. Denote $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ and $x^* = (x', -x_n)$ for $x \in \mathbb{R}^n$, $n \geq 2$, and the half space $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n \mid x_n > 0\}$ with boundary $\Sigma = \partial\mathbb{R}^n_+$.

1.1. Background. The nonstationary Stokes system in the half-space $\mathbb{R}^n_+$, $n \geq 2$, reads

$$\left.\begin{aligned} u_t - \Delta u + \nabla\pi &= f \\ \operatorname{div} u &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^n_+ \times (0, \infty), \tag{1.1}$$

with initial and boundary conditions

$$u(\cdot, 0) = u_0; \qquad u(x', 0, t) = 0 \ \text{ on } \ \Sigma \times (0, \infty). \tag{1.2}$$

Here $u = (u_1, \dots, u_n)$ is the velocity, π is the pressure, and $f = (f_1, \dots, f_n)$ is the external force. They are defined for $(x, t) \in \mathbb{R}^n_+ \times (0, \infty)$. The *Green tensor* $G_{ij}(x, y, t)$ and its associated *pressure tensor* $g_j(x, y, t)$ are defined for $(x, y, t) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}$ and $1 \leq i, j \leq n$ so that, for suitable f and u_0, the solution of (1.1) is given by

$$\begin{aligned} u_i(x, t) &= \sum_{j=1}^{n} \int_{\mathbb{R}^n_+} G_{ij}(x, y, t) u_{0,j}(y)\, dy \\ &\quad + \sum_{j=1}^{n} \int_0^t \int_{\mathbb{R}^n_+} G_{ij}(x, y, t - s) f_j(y, s)\, dy\, ds. \end{aligned} \tag{1.3}$$

Another way to write a solution of (1.1) uses the *Stokes semigroup* $e^{-t\mathbf{A}}$, where $\mathbf{A} = -\mathbf{P}\Delta$ is the *Stokes operator*, and $\mathbf{P}$ is the *Helmholtz projection* (see Remark 3.4)

$$u(t) = e^{-t\mathbf{A}}\mathbf{P}u_0 + \int_0^t e^{-(t-s)\mathbf{A}}\mathbf{P}f(s)\, ds. \tag{1.4}$$

We may regard the Green tensor G_{ij} as the kernel of $e^{-t\mathbf{A}}\mathbf{P}$. In using (1.3) and (1.4), we already exclude weird solutions of (1.1) that are unbounded at spatial infinity, and can talk about "the" unique solution in suitable classes. For applications to Navier–Stokes equations,

$$u_t - \Delta u + \nabla\pi = -u \cdot \nabla u, \quad \operatorname{div} u = 0, \quad \text{in } \mathbb{R}^n_+ \times (0, \infty), \tag{NS}$$

with zero boundary condition, a solution of (NS) is called a *mild solution* if it satisfies (1.3) or (1.4) with $f = -u \cdot \nabla u$ and suitable estimates.

The Stokes semigroup $e^{-t\mathbf{A}}$ and the Helmholtz projection $\mathbf{P}$ are only defined in suitable functional spaces. When defined, the image of $\mathbf{P}$ is solenoidal. A vector field $u = (u_1, \ldots, u_n)$ in $\mathbb{R}^n_+$ is called **solenoidal** if

$$\operatorname{div} u = 0, \quad u_n|_\Sigma = 0. \tag{1.5}$$

An equivalent condition for $u \in L^1_{\text{loc}}(\overline{\mathbb{R}^n_+})$ is

$$\int_{\mathbb{R}^n_+} u \cdot \nabla\phi \, dx = 0, \quad \forall \phi \in C^\infty_c(\overline{\mathbb{R}^n_+}). \tag{1.6}$$

For applications to Navier–Stokes equations, although we may assume u_0 is solenoidal, we do not have $\operatorname{div} f = 0$ for $f = -u \cdot \nabla u$. Hence we cannot omit $\mathbf{P}$ in the integral of (1.4).

The initial condition $u(\cdot, 0) = u_0$ in (1.2) is understood by the weak limit

$$\lim_{t \to 0_+} (u(t), w) = (u_0, w), \quad \forall w \in C^\infty_{c,\sigma}(\mathbb{R}^n_+), \tag{1.7}$$

where $C^\infty_{c,\sigma}(\mathbb{R}^n_+) = \{w \in C^\infty_c(\mathbb{R}^n_+; \mathbb{R}^n) : \operatorname{div} w = 0\}$. A strong limit is unavailable unless we further assume u_0 is solenoidal, see Theorem 1.3. This agrees with the expectation that

$$\lim_{t \to 0_+} e^{-t\mathbf{A}}\mathbf{P}u_0 = \mathbf{P}u_0.$$

There are many results for (1.1) in Lebesgue and Sobolev spaces because the Stokes semigroup and the Helmholtz projection are bounded in $L^q(\mathbb{R}^n_+)$, $1 < q < \infty$. Solonnikov [46] expressed the solution u in terms of Oseen and Golovkin tensors (see Sect. 2) and proved estimates of $u_t, \nabla^2 u, \nabla p$ in L^q in $\mathbb{R}^3_+ \times \mathbb{R}_+$, extending the 2D work by Golovkin [13]. Ukai [52] derived an explicit solution formula to (1.1) when $f = 0$ in $\mathbb{R}^n_+$, expressed in terms of Riesz operators and the solution operators for the heat and Laplace equations in $\mathbb{R}^n_+$. It is simpler and different from that of [46] and gives estimates in L^q spaces trivially. Cannone–Planchon–Schonbek [3] extended [52] for nonzero f using pseudo-differential operators. Estimates in borderline L^1 and L^∞ spaces are studied by Desch, Hieber, and Prüss [7]. Koch and Solonnikov [30] derived gradient estimates of u in $L^q_{x,t}$ for $q > 1$ when f is a divergence of some tensor field. These results are applied to the study of (NS) in Lebesgue spaces.

The pointwise behavior of the solutions of (NS) is less studied, as the Helmholtz projection is not bounded in L^∞, and there have been no pointwise estimates for G_{ij} except for two special cases to be explained below. To circumvent this difficulty, many researchers expand explicitly

$$e^{-t\mathbf{A}}\mathbf{P}\partial_k(u_k u)$$

to sums of estimable terms for the study of (NS). See also the literature review for mild solutions later, in particular (1.25). The drawback of this approach is that it does not apply to general nonlinearities $f = f_0(u, \nabla u)$.

The pointwise estimates for G_{ij} and its derivatives will be useful in the following situations:

1. It gives direct estimates of the Navier–Stokes nonlinearity without expanding its Helmholtz projection.

2. It works for general nonlinearities, for example, those considered in Koba [27], and those from the coupling of the fluid velocity with another physical quantity such as

$$f_j = \sum_k \partial_k(b_k b_j), \qquad g_j = -\sum_k \partial_k(\partial_k d \cdot \partial_j d),$$

where f is the coupling with the magnetic field $b : \mathbb{R}^3_+ \times (0, \infty) \to \mathbb{R}^3$ in the *magnetohydrodynamic equations* in the half space $\mathbb{R}^3_+$ with boundary conditions $b_3 = 0$ and $(\nabla \times b) \times e_3 = 0$ (see [15,19,20,34]), and g is the coupling with the orientation field $d : \mathbb{R}^3_+ \times (0, \infty) \to \mathbb{S}^2$ in the *nematic liquid crystal flows* with boundary conditions $\partial_3 d|_\Sigma = 0$ and $\lim_{|x|\to\infty} d = e_3$ (see [16]).
3. It allows to estimate the contribution from a non-solenoidal initial data, e.g., $u_0 \in L^q$ and in particular when $q = 1$, as done by Maremonti [39] for bounded domains.
4. Pointwise estimates are very useful for the study of the local and asymptotic behavior of the solutions of (NS), see e.g. [32] and our companion papers [21,22].

In contrast to the absence in the time-dependent case, pointwise estimates for *stationary* Stokes system in the half-space have been known; See [23] for the literature and the most recent refinement.

We now describe the two special cases of known pointwise estimates for G_{ij}. For the special case of solenoidal vector fields f satisfying (1.5), by using the Fourier transform in x' and the Laplace transform in t of the system (1.1), Solonnikov [47, (3.12)] derived an explicit formula of the *restricted Green tensor* and their pointwise estimates for $n = 3$ (also see [48,49] for $n \ge 2$; The same method is used in [35]). Specifically, he showed that for $u_0 = 0$, and f satisfying (1.5),

$$\begin{aligned} u_i(x,t) &= \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t-s) f_j(y,s)\,dy\,ds, \\ \pi(x,t) &= \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^n_+} \breve{g}_j(x,y,t-s) f_j(y,s)\,dy\,ds, \end{aligned} \tag{1.8}$$

with

$$\begin{aligned} \breve{G}_{ij}(x,y,t) &= \delta_{ij}\Gamma(x-y,t) + G^*_{ij}(x,y,t), \\ G^*_{ij}(x,y,t) &= -\delta_{ij}\Gamma(x-y^*,t) \\ &\qquad - 4(1-\delta_{jn}) \frac{\partial}{\partial x_j} \int_{\Sigma\times[0,x_n]} \frac{\partial}{\partial x_i} E(x-z)\Gamma(z-y^*,t)\,dz, \\ \breve{g}_j(x,y,t) &= 4(1-\delta_{jn})\partial_{x_j}\Big[\int_\Sigma E(x-\xi')\partial_n\Gamma(\xi'-y,t)\,d\xi' \\ &\qquad + \int_\Sigma \Gamma(x'-y'-\xi', y_n, t)\partial_n E(\xi', x_n)\,d\xi' \Big], \end{aligned} \tag{1.9}$$

where $y^* = (y', -y_n)$ for $y = (y', y_n)$, and $E(x)$ and $\Gamma(x,t)$ are the fundamental solutions of the Laplace and heat equations in $\mathbb{R}^n$, respectively. (See Sect. 2. Our $E(x)$ differs from [47] by a sign.) Moreover, G^*_{ij} and $\breve{g}_j$ satisfy the pointwise bound ([49, (2.38), (2.32)]) for $n \ge 2$,

$$|\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q \partial_t^m G_{ij}^*(x,y,t)| \lesssim \frac{e^{-\frac{cy_n^2}{t}}}{t^{m+\frac{q}{2}}(|x^*-y|^2+t)^{\frac{l+n}{2}}(x_n^2+t)^{\frac{k}{2}}}; \tag{1.10}$$
$$|\partial_{x,y'}^l \partial_{y_n}^q \partial_t^m \breve{g}_j(x,y,t)| \lesssim t^{-1-m-\frac{q}{2}}(|x-y^*|^2+t)^{-\frac{n-1+l}{2}} e^{-\frac{cy_n^2}{t}}.$$

His argument is also valid for $n=2$ since the fundamental solution E in (1.9) has a derivative, thus has the scaling property.

Another special case is the pointwise estimate of the Green tensor by Kang [17], but only when the second variable y is zero, or equivalently $y_n=0$,

$$|\partial_x^l \partial_t^m G_{ij}(x,y',t)| \lesssim \frac{1}{t^{m+\frac{1+\alpha}{2}}(|x-y'|^2+t)^{\frac{l+n-2}{2}} x_n^{1-\alpha}}, \tag{1.11}$$

where α is any number with $0<\alpha<1$, and we identify y' with $(y',0)$. Even for $y=0$, this estimate does not seem optimal because we anticipate the symmetry of the Green tensor (see Proposition 1.4).

1.2. Results. The following is our first and key pointwise estimates of the (unrestricted) Green tensor and its derivatives. Even when restricted to $y=0$, it is better than (1.11) by removing the singularity at $x_n=0$. It will be further improved in Theorem 1.5 after we show symmetry.

Proposition 1.1 (First estimates). *Let $n\ge 2$, $x,y\in\mathbb{R}^n_+$, $t>0$, $i,j=1,\dots,n$, and $l,k,q,m\in\mathbb{N}_0$. Let G_{ij} be the Green tensor for the time-dependent Stokes system (1.1) in the half-space $\mathbb{R}^n_+$, and g_j be the associated pressure tensor. We have*

$$|\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q \partial_t^m G_{ij}(x,y,t)| \lesssim \frac{1}{\left(|x-y|^2+t\right)^{\frac{l+k+q+n}{2}+m}} + \frac{LN_{ijkq}^{mn}}{t^m(|x^*-y|^2+t)^{\frac{l+k-k_i+n}{2}}(x_n^2+t)^{\frac{k_i}{2}}(y_n^2+t)^{\frac{q}{2}}}, \tag{1.12}$$

where $k_i=(k-\delta_{in})_+$,

$$\mathrm{LN}_{ijkq}^{mn} := 1+\delta_{n2}\mu_{ik}^m\left[\log(\nu_{ijkq}^m|x'-y'|+x_n+y_n+\sqrt{t})-\log(\sqrt{t})\right], \tag{1.13}$$

with $\mu_{ik}^m = 1-(\delta_{k0}+\delta_{k1}\delta_{in})\delta_{m0}$, and $\nu_{ijkq}^m=\delta_{q0}\delta_{jn}\delta_{k(1+\delta_{in})}\delta_{m0}+\delta_{m>0}$. Also,

$$|\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q g_j(x,y,t)| \lesssim t^{-\frac{1}{2}}\left[\frac{1}{R^{l+q+n}}\left(\frac{1}{x_n^k}+\delta_{k0}\log\frac{R}{x_n}\right)+\frac{1}{R^{k+n-1}y_n^{l+q+1}}\right], \tag{1.14}$$

where $R=|x'-y'|+x_n+y_n+\sqrt{t}\sim|x-y^*|+\sqrt{t}$.

Comments on Proposition 1.1:

1. The numerator LN_{ijkq}^{mn} is a log correction for $n=2$, and equals 1 if $n\ge 3$. The parameters $\mu_{ik}^m,\nu_{ijkq}^m\in\{0,1\}$. For simplicity we may take $\mu_{ik}^m=\nu_{ijkq}^m=1$ for most cases.

2. As we will see in Proposition 3.5, the pressure tensor g contains a delta function supported at $t = 0$. It is not in (1.14) where $t > 0$.
3. The estimate (1.12) of $\partial_t G_{ij}$ is not integrable for $0 < t < 1$. It can be improved using the Green tensor equation (3.1) and estimates of $\Delta_x G_{ij}$ and $\nabla_x g_j$.

With the first estimates, we are able to prove the following theorems on restricted Green tensors, convergence to initial data, and symmetry of the Green tensor. We say a tensor $\bar{G}_{ij}(x, y, t)$ is a *restricted Green tensor* if for any solenoidal u_0, the vector field $u_i(x, t) = \sum_{j=1}^n \int_{\mathbb{R}^n_+} \bar{G}_{ij}(x, y, t) u_{0,j}(y)\, dy$ is a solution of the Stokes system (1.1)–(1.2).

Theorem 1.2 (Restricted Green tensors). *Let $u_0 \in C^1_{c,\sigma}(\overline{\mathbb{R}^n_+})$, i.e., it is a vector field in $C^1_c(\overline{\mathbb{R}^n_+}; \mathbb{R}^n)$ with* div $u_0 = 0$ *and $u_{0,n}|_\Sigma = 0$. Then*

$$\sum_{j=1}^n \int_{\mathbb{R}^n_+} G_{ij}(x, y, t) u_{0,j}(y)\, dy = \sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x, y, t) u_{0,j}(y)\, dy = \sum_{j=1}^n \int_{\mathbb{R}^n_+} \widehat{G}_{ij}(x, y, t) u_{0,j}(y)\, dy$$

as continuous functions in $x \in \mathbb{R}^n_+$ and $t > 0$, where $\breve{G}_{ij}(x, y, t)$ is the restricted Green tensor of Solonnikov given in (1.9)*, and*

$$\widehat{G}_{ij}(x, y, t) = \delta_{ij}\left[\Gamma(x - y, t) - \Gamma(x - y^*, t)\right] - 4\delta_{jn} C_i(x, y, t), \tag{1.15}$$

with $C_i(x, y, t) = \int_0^{x_n} \int_\Sigma \partial_n \Gamma(x - y^* - z, t)\, \partial_i E(z)\, dz'\, dz_n$.

Comments on Theorem 1.2:

1. The last term of $\breve{G}_{ij}$ in (1.9) only acts on the *tangential* components $u_{0,j}$, $j < n$. In contrast, the last term of $\widehat{G}_{ij}$ in (1.15) only acts on the *normal* component $u_{0,n}$. We do not know whether (1.15) has appeared in literature. We will use both $\breve{G}_{ij}$ and $\widehat{G}_{ij}$ in the proof of Lemma 6.1. C_i will be defined in (4.4) with estimates in Remark 5.2.
2. We can get infinitely many restricted Green tensors by adding to $\breve{G}_{ij}$ any tensor T_{ij} that vanishes on all solenoidal vector fields $f = (f_j)$, $\int_{\mathbb{R}^n_+} T_{ij}(x, y, t) f_j(y) dy = 0$, for example, a tensor of the form $T_{ij} = \partial_{y_j} T_i(x, y, t)$ with suitable regularity and decay. We do not need $\sum_i \partial_{x_i} T_{ij}(x, y, t) = 0$ nor $T_{ij}|_{x_n=0} = 0$ since $\int_{\mathbb{R}^n_+} T_{ij}(x, y, t) f_j(y) dy = 0$. In fact, if we denote

$$C_i^\sharp(x, y, t) := \int_0^{x_n} \int_\Sigma \Gamma(x - y^* - z, t) \partial_i E(z)\, dz' dz_n,$$

then we have the (more symmetric) alternative forms:

$$\breve{G}_{ij}(x, y, t) = \delta_{ij}\left[\Gamma(x - y, t) - \Gamma(x - y^*, t)\right] + 4(1 - \delta_{jn}) \partial_{y_j} C_i^\sharp(x, y, t),$$

$$\widehat{G}_{ij}(x, y, t) = \delta_{ij}\left[\Gamma(x - y, t) - \Gamma(x - y^*, t)\right] - 4\delta_{jn} \partial_{y_j} C_i^\sharp(x, y, t) = \breve{G}_{ij}(x, y, t) + \partial_{y_j} 4C_i^\sharp(x, y, t). \tag{1.16}$$

3. In contrast, the unrestricted Green tensor G_{ij} is *unique*: We require it to satisfy the equation $(3.1)_1$, the boundary condition $(3.1)_2$, and the initial condition that the vector field $u_i(x,t) = \sum_{j=1}^n \int_{\mathbb{R}^n_+} G_{ij}(x,y,t)(u_0)_j(y)\,dy$ satisfies $\lim_{t\to 0_+} u(\cdot,t) = \mathbf{P}u_0$ for any initial data u_0 not necessarily solenoidal. Suppose $(\bar G_{ij}(x,y,t), \bar g_j(x,y,t))$ is another pair of unrestricted Green tensor and pressure tensor. For fixed j and y, the difference $u_i(x,t) = (G_{ij} - \bar G_{ij})(x,y,t)$ and its companion pressure $p(x,t) = (g_j - \bar g_j)(x,y,t)$ satisfy the Stokes system (1.1) with zero boundary and initial values. Under bounds such as

$$|u(x,t)| \lesssim \frac{1}{(y_n + |x| + \sqrt{t})^n}, \quad |p(x,t)| \lesssim \frac{1}{\sqrt{t}(y_n + |x| + \sqrt{t})^{n-1}},$$

suggested by Proposition 1.1, we can show $u = 0$ by energy estimate: Testing (1.1) by $u\phi_R$ for some cut-off function $\phi_R(x) = \Phi(x/R)$ and integrating over $t_0 < t < t_1$, sending $R \to \infty$, and then sending $t_0 \to 0_+$. (Also see [36, Theorem 5]). Hence $G_{ij} = \bar G_{ij}$.
4. Theorem 1.2 is extended to $u_0 \in L^p_\sigma$ in Remark 9.2 for $1 \le p \le \infty$. When $p = \infty$ we can only show the first equality, and we need u_0 in the L^∞-closure of C^1_c.

Theorem 1.3 (Convergence to initial data). *Let* $u(x,t) = \sum_{j=1}^n \int_{\mathbb{R}^n_+} G_{ij}(x,y,t)u_{0,j}(y)\,dy$ *for a vector field* u_0 *in* $\mathbb{R}^n_+$. *Let* $\mathbf{P}$ *be the Helmholtz projection in* $\mathbb{R}^n_+$ *to be given in Remark* 3.4.

(a) *If* $u_0 \in C^1_c(\mathbb{R}^n_+)$, *then* $u(x,t) \to (\mathbf{P}u_0)(x)$ *for all* $x \in \mathbb{R}^n_+$, *and uniformly for all* x *with* $x_n \ge \delta$ *for any* $\delta > 0$.
(b) *If* $u_0 \in L^q(\mathbb{R}^n_+)$, $1 < q < \infty$, *then* $u(x,t) \to (\mathbf{P}u_0)(x)$ *in* $L^q(\mathbb{R}^n_+)$.
(c) *If* $u_0 \in C^1_{c,\sigma}(\overline{\mathbb{R}^n_+})$, *i.e., it is a vector field in* $C^1_c(\overline{\mathbb{R}^n_+};\mathbb{R}^n)$ *with* $\operatorname{div} u_0 = 0$ *and* $u_{0,n}|_\Sigma = 0$, *then* $u_0 = \mathbf{P}u_0$ *and* $u(x,t) \to u_0(x)$ *in* $L^q(\mathbb{R}^n_+)$ *for* $1 < q \le \infty$.

In Part (a), the support of u_0 is away from the boundary. In Part (c), the tangential part of u_0 may be nonzero on Σ, and $q = \infty$ is allowed.

Proposition 1.4 (Symmetry of Green tensor). *Let* G_{ij} *be the Green tensor for the Stokes system in the half-space* $\mathbb{R}^n_+$, $n \ge 2$. *Then for* $x, y \in \mathbb{R}^n_+$ *and* $t \ne 0$ *we have*

$$G_{ij}(x,y,t) = G_{ji}(y,x,t), \quad \forall x \ne y \in \mathbb{R}^n_+. \tag{1.17}$$

For the stationary case, the symmetry is known by Odqvist [43, p.358] for $n = 3$ and [23, Lemma 2.1, (2.29)] for $n \ge 2$. We do not know (1.17) for the nonstationary case in the literature. We will prove Proposition 1.4 in Sect. 7, after we have shown Proposition 1.1. It gives an alternative proof of the stationary case for $n \ge 3$, see Remark 3.7.

Although G_{ij} is symmetric by Proposition 1.4, the restricted Green tensors in (1.9) and (1.15) are not. For example, if $i < n$ and $j = n$,

$$\breve G_{in}(x,y,t) = 0, \quad \breve G_{ni}(y,x,t) = -4\int_{\Sigma\times[0,y_n]} \partial_i\partial_n E(y-z)\Gamma(z-x^*,t)\,dz,$$
$$\widehat G_{in}(x,y,t) = -4C_i(x,y,t), \quad \widehat G_{ni}(y,x,t) = 0.$$

By the symmetry of the Green tensor in Proposition 1.4, the estimates in Proposition 1.1 can be improved. Our main estimates are the following:

Theorem 1.5 (Main estimates). *Let $n \ge 2$, $x, y \in \mathbb{R}^n_+$, $t > 0$, $i, j = 1, \ldots, n$, and $l, k, q, m \in \mathbb{N}_0$. We have*

$$|\partial^l_{x',y'}\partial^k_{x_n}\partial^q_{y_n}\partial^m_t G_{ij}(x,y,t)| \lesssim \frac{1}{(|x-y|^2+t)^{\frac{l+k+q+n}{2}+m}} + \frac{\mathrm{LN}^{mn}_{ijkq}+\mathrm{LN}^{mn}_{jiqk}}{t^m(|x^*-y|^2+t)^{\frac{l+k-k_i+q-q_j+n}{2}}(x_n^2+t)^{\frac{k_i}{2}}(y_n^2+t)^{\frac{q_j}{2}}}, \tag{1.18}$$

where LN^{mn}_{ijkq} is given in (1.13), $k_i = (k-\delta_{in})_+$, and $q_j = (q-\delta_{jn})_+$.

Comments on Theorem 1.5:

1. Assume $l + k + q + n \ge 3$. For the cases when $k_i = q_j = 0$ and $m = 0$, the time integrals of the above estimates coincide with the well-known estimates of the stationary Green tensor given in [10, IV.3.52]. We lose tangential spatial decay in other cases.
2. The estimates of the stationary Green tensor mentioned above have been improved by [23]. For example, when there is no normal derivative and $n + l \ge 3$, [23, Theorems 2.4, 2.5] show

$$|\partial^l_{x',y'} G^0_{ij}(x,y)| \lesssim \frac{x_n y_n^{1+\delta_{jn}}}{|x-y|^{n-2+l}\,|x-y^*|^{2+\delta_{jn}}}. \tag{1.19}$$

   (It can be improved using symmetry, but [23] does not have $i = j = n$ case.) The tangential decay rate is better than the normal decay and the whole space case, probably because of the zero boundary condition. Thus (1.18) may have room for improvement. Compare Theorem 1.6.

The following estimates quantify the boundary vanishing of the Green tensor and its derivatives at $x_n = 0$ or $y_n = 0$.

Theorem 1.6 (Boundary vanishing). *Let $n \ge 2$, $x, y \in \mathbb{R}^n_+$, $t > 0$, $i, j = 1, \ldots, n$, and $l, k, q, m \in \mathbb{N}_0$. Let $0 \le \alpha \le 1$. If $k = 0$, we have*

$$\left|\partial^l_{x',y'}\partial^q_{y_n}\partial^m_t G_{ij}(x,y,t)\right| \lesssim \frac{x_n^\alpha}{(|x-y|^2+t)^{\frac{l+q+n}{2}+m}(|x-y^*|^2+t)^{\frac{\alpha}{2}}} + \frac{x_n^\alpha\,\mathrm{LN}}{t^{m+\frac{\alpha}{2}}(|x-y^*|^2+t)^{\frac{l+q-q_j+n}{2}}(y_n^2+t)^{\frac{q_j}{2}}}, \tag{1.20}$$

with $\mathrm{LN} = \sum_{k=0}^1(\mathrm{LN}^{mn}_{ijkq}+\mathrm{LN}^{mn}_{jiqk})(x,y,t)$. If $q = 0$, we have

$$\left|\partial^l_{x',y'}\partial^k_{x_n}\partial^m_t G_{ij}(x,y,t)\right| \lesssim \frac{y_n^\alpha}{(|x-y|^2+t)^{\frac{l+k+n}{2}+m}(|x-y^*|^2+t)^{\frac{\alpha}{2}}} + \frac{y_n^\alpha\,\mathrm{LN}}{t^{m+\frac{\alpha}{2}}(|x-y^*|^2+t)^{\frac{l+k-k_i+n}{2}}(x_n^2+t)^{\frac{k_i}{2}}}, \tag{1.21}$$

with $\mathrm{LN} = \sum_{q=0}^1(\mathrm{LN}^{mn}_{ijkq}+\mathrm{LN}^{mn}_{jiqk})(x,y,t)$.

1.3. Key ideas and the structure of the proof. Let us explain the idea for our key result, Proposition 1.1: The major difficulty is to find a formula for the Green tensor in which each term has good estimates. Our first formula (3.10) with the correction term W_{ij} given by (3.9) is obtained from the definition using the Oseen and Golovkin tensors. The second formula for W_{ij} in Lemma 4.2 is obtained using Poisson's formula for the heat equation to *remove the time integration*. The idea of using Poisson's formula is already in the stationary case of [23,40]. Our final formula for the Green tensor in Lemma 4.3 is obtained by identifying the cancellation of terms in Lemma 4.2, maximizing the tangential decay. We further transform the term $\widehat{H}_{ij}$ in Lemma 4.3 in terms of D_{ijm} in Lemma 5.1, which are integrals over $\Sigma \times [0, x_n]$. For D_{ijm}, we do space partition and integration by parts to estimate their tangential derivatives, and we explore their algebraic properties, e.g., computing their divergence, to *move normal derivatives to tangential derivatives*. These enable us to prove Proposition 1.1.

Maximizing the tangential decay is essential: As seen in Proposition 1.1, normal derivatives do not increase tangential decay, and maximal tangential decay allows us to prove the integrability in y of all derivatives of the Green tensor (uniformly in x). This is used in the proofs of (9.3) of Lemma 9.1 and (9.20) of Lemma 9.4, both relying on the function $H_1 \in L^1$ for H_1 defined in (9.7), for the construction of mild solutions of Navier–Stokes equations. The maximal tangential decay is also used to prove that the Green tensor itself is integrable in y, but with an x_n-dependent constant,

$$\int_{\mathbb{R}^n_+} |G_{ij}(x, y, t)|\, dy \lesssim \ln(e + \frac{x_n}{\sqrt{t}}). \tag{1.22}$$

This is proved in (9.10) of Remark 9.2 using Theorem 1.6, and used to prove an extension of Theorem 1.2 to the L^∞-setting, see Remark 9.2. In this sense, the Green tensor in the half space has a stronger decay than the whole space case. This phenomenon is well known in the stationary case.

Having the first estimates of both Green tensor and its associated pressure tensor in hand, we can investigate restricted Green tensors and initial values, and prove Proposition 1.4 on the symmetry of the Green tensor. Our main estimate Theorem 1.5 is proved using Proposition 1.1 and Proposition 1.4. We then prove the boundary vanishing Theorem 1.6 using the normal derivative estimates of Theorem 1.5.

1.4. Applications. As an application, we will construct mild solutions of the Navier–Stokes equations in the half space in various functional spaces. We will provide other applications in forth coming papers [21] and [22]. Since it is only for illustration, we only consider local-in-time solutions with zero external force. Fujita-Kato [9,25] and Sobolevskii [45] transformed (NS) into an abstract initial value problem using the Stokes semigroup

$$u(t) = e^{-t\mathbf{A}}u_0 - \int_0^t e^{-(t-s)\mathbf{A}}\mathbf{P}\partial_k(u_k u)(s)\, ds, \tag{1.23}$$

whose solution $u(t)$ lies in some Banach spaces and is called a mild solution of (NS). In the whole space setting, there is an extensive literature on the unique existence of mild solutions of (NS). See e.g. [2,8,11,12,24,31,37,42,54] for the most relevant to our study.

For mild solutions of (NS) in the half-space, the unique local and global existence in $L^q(\mathbb{R}^n_+)$ were established by Weissler [53] for $3 \le n < q < \infty$, by Ukai [52] for

$2 \le n \le q < \infty$, and by Kozono [33] for $2 \le n = q$. Canaone-Planchon-Schonbek [3] established unique existence of solutions in $L^\infty L^3$ with initial data in the homogeneous Besov space $\dot{B}^{3/q-1}_{q,\infty}(\mathbb{R}^3_+)$. For mild solutions in weighted L^q spaces, we refer the reader to [28,29].

For solutions with pointwise decay, Crispo-Maremonti [6] proved the local existence of solutions controlled by $(1+|x|)^{-\alpha}(1+t)^{-\beta/2}$, $\alpha+\beta = a \in (1/2, n)$ when $u_0 \in L^\infty(\mathbb{R}^n_+, (1+|x|)^a dx)$ and $n \ge 3$. If $a \in [1, n)$, they further showed the existence is global in time when u_0 is small enough in $L^\infty(\mathbb{R}^n_+, (1+|x|)^a dx)$. The constraints imposed in [6] on a and n are relaxed by Chang-Jin [5] to $a \in (0, n]$ and $n \ge 2$. They proved the existence of mild solutions to (NS) having the same weighted decay estimate as the Stokes solutions if $a \in (0, n]$. Note that for the case $a = n$, the mild solution is local in time because the weighted estimate of solutions to the Stokes system has an additional log factor. They also obtained the weighted decay estimates for $n < a < n+1$ in [4] with an additional condition that $R'_j u_0 \in L^\infty(\mathbb{R}^n_+, (1+|x|)^a dx)$. Regarding solutions whose initial data has no spatial decay, the local existence and uniqueness of strong mild solutions with initial data in L^∞ were established by Bae-Jin [1], improving Solonnikov [50] and Maremonti [38] for continuous initial data. Recently, Maekawa-Miura-Prange [36] studied the analyticity of Stokes semigroup in uniformly local L^q space via the Stokes resolvent problem and constructed mild solutions in such spaces for $q \ge n$.

In the following, Theorems 1.7, 1.8 and 1.10 are already known, while Theorems 1.9 is new. We will provide new proofs using the following solution formula of (NS) with the Green tensor

$$\begin{aligned} u_i(x,t) &= \sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t) u_{0,j}(y)dy \\ &\quad + \sum_{j,k=1}^n \int_0^t \int_{\mathbb{R}^n_+} \partial_{y_k} G_{ij}(x,y,t-s)(u_k u_j)(y,s)dy\,ds. \end{aligned} \tag{1.24}$$

We use the restricted Green tensor $\breve{G}_{ij}$ for the first term and the (unrestricted) Green tensor G_{ij} for the second term. Note that the second term is written as

$$-\sum_{j,k=1}^n \int_0^t \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t-s)(\mathbf{P}\partial_k u_k u)_j(y,s)dy\,ds \tag{1.25}$$

and explicitly computed in [1,6], as the Green tensor G_{ij} was unknown.

For $1 \le q \le \infty$, let

$$L^p_\sigma(\mathbb{R}^n_+) = \left\{ f \in L^p(\mathbb{R}^n_+; \mathbb{R}^n) : \operatorname{div} f = 0,\ f_n(x', 0) = 0 \right\}. \tag{1.26}$$

Theorem 1.7. *Let $2 \le n \le q \le \infty$ and $u_0 \in L^q_\sigma(\mathbb{R}^n_+)$. If $q = \infty$, we also assume u_0 in the L^∞-closure of $C^1_{c,\sigma}(\overline{\mathbb{R}^n_+})$ and $n \ge 3$. There are $T = T(n, q, u_0) > 0$ and a unique mild solution $u(t) \in C([0,T]; L^q)$ of (NS) in the class*

$$\sup_{0<t<T} \left(\|u(t)\|_{L^q} + t^{\frac{n}{2q}} \|u(t)\|_{L^\infty} + t^{1/2} \|\nabla u(t)\|_{L^q} \right) \le C_* \|u_0\|_{L^q}.$$

We can take $T = T(n, q, \|u_0\|_{L^q})$ if $n < q \le \infty$.

This is known in [33,52,53] for $2 \le n \le q < \infty$, and in [1] for $q = \infty$.
For $a \ge 0$, denote

$$Y_a = \left\{ f \in L^\infty_{\text{loc}}(\mathbb{R}^n_+) \;\middle|\; \|f\|_{Y_a} = \sup_{x \in \mathbb{R}^n_+} |f(x)| \langle x \rangle^a < \infty \right\}, \tag{1.27}$$

and

$$Z_a = \left\{ f \in L^\infty_{\text{loc}}(\mathbb{R}^n_+) \;\middle|\; \|f\|_{Z_a} = \sup_{x \in \mathbb{R}^n_+} |f(x)| \langle x_n \rangle^a < \infty \right\}. \tag{1.28}$$

Theorem 1.8. *Let $n \ge 2$ and $0 < a \le n$. For any vector field $u_0 \in Y_a$ with $\operatorname{div} u_0 = 0$ and $u_{0,n}|_\Sigma = 0$, there is a strong mild solution $u \in L^\infty(0, T; Y_a)$ of (NS) for some time interval $(0, T)$. Moreover, the mild solution is unique in the class $L^\infty(\mathbb{R}^n_+ \times (0, T))$.*

Theorem 1.9. *Let $n \ge 2$ and $0 < a \le 1$. For any vector field $u_0 \in Z_a$ with $\operatorname{div} u_0 = 0$ and $u_{0,n}|_\Sigma = 0$, there is a strong mild solution $u \in L^\infty(0, T; Z_a)$ of (NS) for some time interval $(0, T)$. Moreover, the mild solution is unique in the class $L^\infty(\mathbb{R}^n_+ \times (0, T))$.*

Theorem 1.8 corresponds to [5, Theorem 1] and [6, Theorem 2.1]. Theorem 1.9 is new. Its upper bound $a \le 1$ is less than that in Theorem 1.8.

For $1 \le q \le \infty$, denote

$$L^q_{\text{uloc}}(\mathbb{R}^n_+) = \left\{ u \in L^q_{\text{loc}}(\mathbb{R}^n_+) \;\middle|\; \sup_{x \in \mathbb{R}^n_+} \|u\|_{L^q(B_1(x) \cap \mathbb{R}^n_+)} < \infty \right\},$$
$$L^q_{\text{uloc},\sigma}(\mathbb{R}^n_+) = \left\{ u \in L^q_{\text{uloc}}(\mathbb{R}^n_+; \mathbb{R}^n) \;\middle|\; \operatorname{div} u = 0,\ u_{0,n}|_\Sigma = 0 \right\}.$$

Theorem 1.10. *Let $2 \le n \le q \le \infty$ and $u_0 \in L^q_{\text{uloc},\sigma}(\mathbb{R}^n_+)$.*

(a) *If $n < q \le \infty$, and suppose $n \ge 3$ if $q = \infty$, there are $T = T(n, q, \|u_0\|_{L^q_{\text{uloc}}}) > 0$ and a unique mild solution u of (NS) with*

$$\begin{gathered} u(t) \in L^\infty(0, T; L^q_{\text{uloc},\sigma}) \cap C((0, T); W^{1,q}_{\text{uloc},0}(\mathbb{R}^n_+)^n \cap \mathrm{BUC}_\sigma(\mathbb{R}^n_+)), \\ \sup_{0<t<T} \left(\|u(t)\|_{L^q_{\text{uloc}}} + t^{\frac{n}{2q}} \|u(t)\|_{L^\infty} + t^{1/2} \|\nabla u(t)\|_{L^q_{\text{uloc}}} \right) \le C_* \|u_0\|_{L^q_{\text{uloc}}}. \end{gathered} \tag{1.29}$$

(b) *If $n = q$, for any $0 < T < \infty$, there are $\epsilon(T), C_*(T) > 0$ such that if $\|u_0\|_{L^n_{\text{uloc}}} \le \epsilon(T)$, then there is a unique mild solution $u(t)$ of (NS) in the class (1.29).*

This theorem is [36, Propositions 7.1 and 7.2]. Continuity at time zero requires further restrictions on u_0.

In addition to the existence of mild solutions in various spaces, pointwise estimates of the Green tensor is useful for the study of local and asymptotic behavior of solutions. In a forth coming paper [21], we will use the Stokes flows of [18] as the profile to construct solutions of the Navier–Stokes equations in $\mathbb{R}^3_+ \times (0, 2)$ with *finite global energy* such that they are globally bounded with spatial decay but their normal derivatives are unbounded near the boundary due to Hölder continuous boundary fluxes which are not C^1 in time. We will collect other applications in [22].

The rest of this paper is organized as follows. In Sect. 2, we give a few preliminaries and recall the Oseen tensor and the Golovkin tensor. In Sect. 3, we consider the Green tensor and its associated pressure tensor, and derive their first formulas. In Sect. 4, we derive a second formula for the Green tensor which has better estimates. In Sect. 5, we give the first estimates in Proposition 1.1 of the Green tensor and the pressure tensor. In Sect. 6, we study the restricted Green tensors, and how the solutions converge to the initial values. In Sect. 7, we prove the symmetry of the Green tensor in Proposition 1.4. In Sect. 8, the ultimate estimate in Theorem 1.5 is derived from Proposition 1.1 using the symmetry of the Green tensor and the divergence-free condition. We also estimate their vanishing at the boundary, proving Theorem 1.6. In Sect. 9, we prove the key estimates for the construction of mild solutions in various spaces for Theorems 1.7, 1.8, 1.9 and 1.10.

Notation. We denote $\langle \xi \rangle = (|\xi|^2 + 2)^{1/2}$ for any $\xi \in \mathbb{R}^m$, $m \in \mathbb{N}$. We denote $f \lesssim g$ if there is a constant C such that $|f| \le Cg$.

Green tensor	$\cdots$	$G_{ij},\ g_j$
Oseen tensor	$\cdots$	$S_{ij},\ s_j$
Golovkin tensor	$\cdots$	$K_{ij},\ k_j$
Fundamental solution of $-\Delta$	$\cdots$	E
Heat kernel	$\cdots$	Γ
Poisson kernel for heat equation	$\cdots$	P

2. Preliminaries, Oseen and Golovkin Tensors

In this section, we first recall a few definitions and estimates from [46]. We then give two integral estimates. We next recall in Sect. 2.2 the Oseen tensor [44], which is the fundamental solution of the nonstationary Stokes system in $\mathbb{R}^n$. We finally recall in Sect. 2.3 the Golovkin tensor [14], which is the Poisson kernel of the nonstationary Stokes system in $\mathbb{R}^n_+$.

The heat kernel Γ and the fundamental solution E of $-\Delta$ are given by

$$\Gamma(x,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} e^{\frac{-x^2}{4t}} & \text{for } t > 0, \\ 0 & \text{for } t \le 0, \end{cases} \quad \text{and} \quad E(x) = \begin{cases} \frac{1}{n\,(n-2)|B_1|}\,\frac{1}{|x|^{n-2}} & \text{for } n \ge 3, \\ -\frac{1}{2\pi}\,\log|x| & \text{if } n = 2. \end{cases}$$

The Poisson kernel of $-\Delta$ in $\mathbb{R}^n_+$ is $P_0(x) = -2\partial_n E(x)$. We will use [23, (2.32)] for $n \ge 2$,

$$\int_\Sigma E(\xi' - y) P_0(x - \xi')\, d\xi' = E(x - y^*), \quad P_0(x) = -2\partial_n E(x). \tag{2.1}$$

It is because the integral is a harmonic function in x that equals $E(x - y^*)$ when $x_n = 0$, and was first used in Maz′ja–Plamenevskiĭ–Stupjalis [40, Appendix 1] to study the stationary Green tensor for $n = 2, 3$.

We will use the following functions defined in [46, (60)–(61)]:

$$A(x,t) = \int_\Sigma \Gamma(z', 0, t) E(x - z')\, dz' = \int_\Sigma \Gamma(x' - z', 0, t) E(z', x_n)\, dz' \tag{2.2}$$

and

$$B(x,t) = \int_\Sigma \Gamma(x - z', t) E(z', 0)\, dz' = \int_\Sigma \Gamma(z', x_n, t) E(x' - z', 0)\, dz'. \tag{2.3}$$

They are defined only for $n = 3$ in [46] and differ from (2.2)–(2.3) by a factor of 4π. The estimates for A, B, and their derivatives are given in [46, (62, 63)] for $n = 3$. For general case, we can use the same approach and derive the following estimates for $l + n \geq 3$:

$$|\partial_x^l \partial_t^m A(x,t)| \lesssim \frac{1}{t^{m+\frac{1}{2}}(x^2+t)^{\frac{l+n-2}{2}}} \tag{2.4}$$

and

$$|\partial_{x'}^l \partial_{x_n}^k \partial_t^m B(x,t)| \lesssim \frac{1}{(x^2+t)^{\frac{l+n-2}{2}}(x_n^2+t)^{\frac{k+1}{2}+m}}. \tag{2.5}$$

In fact, the last line of [46, page 39] gives

$$|\partial_{x'}^l \partial_{x_n}^k B(x,t)| \lesssim \frac{1}{(x^2+t)^{\frac{l+n-2}{2}} t^{\frac{k+1}{2}}} e^{-\frac{x_n^2}{10t}}. \tag{2.6}$$

Remark 2.1. For $n = 2$, the condition $l \geq 1$ is needed as $A(x,t)$ and $B(x,t)$ grow logarithmically as $|x| \to \infty$. In fact, one may prove for $n = 2$

$$|A(x,t)| + |B(x,t)| \lesssim \frac{1 + |\log(|x_2| + \sqrt{t})| + |\log(|x_1| + |x_2| + \sqrt{t})|}{\sqrt{t}}.$$

2.1. Integral estimates. We now give a few useful integral estimates.

Lemma 2.1. *For positive L, a, d, and k we have*

$$\int_0^L \frac{r^{d-1}\,dr}{(r+a)^k} \lesssim \begin{cases} L^d(a+L)^{-k} & \text{if } k < d, \\ L^d(a+L)^{-d}(1+\log_+ \frac{L}{a}) & \text{if } k = d, \\ L^d(a+L)^{-d}a^{-(k-d)} & \text{if } k > d. \end{cases}$$

Proof. Denote the integral by I. If $a \geq \frac{L}{2}$, then

$$I \lesssim a^{-k}\int_0^L r^{d-1}\,dr \sim L^d a^{-k}.$$

If $a < \frac{L}{2}$, then

$$\begin{aligned} I &= \int_0^a \frac{r^{d-1}\,dr}{(r+a)^k} + \int_a^L \frac{r^{d-1}\,dr}{(r+a)^k} \\ &\lesssim a^{d-k} + \int_a^L r^{d-k-1}\,dr \\ &\lesssim a^{d-k} + \begin{cases} L^{d-k} & \text{if } k < d, \\ \log\frac{L}{a} & \text{if } k = d, \\ a^{d-k} & \text{if } k > d. \end{cases} \end{aligned}$$

For $k < d$,

$$I \lesssim \begin{cases} L^d a^{-k} & \text{if } a \geq \frac{L}{2} \\ L^{d-k} & \text{if } a < \frac{L}{2} \end{cases} \lesssim L^d \max(a,L)^{-k} \lesssim L^d(a+L)^{-k},$$

where we used the fact $2\max(a, L) \geq a + L$. Next, for $k = d$,

$$I \lesssim \begin{cases} \left(\frac{L}{a}\right)^d & \text{if } a \geq \frac{L}{2} \\ 1 + \log\frac{L}{a} & \text{if } a < \frac{L}{2} \end{cases} \lesssim \frac{L^d}{(a+L)^d}(1 + \log_+ \frac{L}{a}),$$

because $a \geq \frac{L}{2}$ implies that $\frac{L}{a} \lesssim 1$. Finally, for $k > d$ we get

$$I \lesssim \begin{cases} L^d a^{-k} & \text{if } a \geq \frac{L}{2} \\ a^{d-k} & \text{if } a < \frac{L}{2} \end{cases} \lesssim a^{-k} \min(a, L)^d \sim \frac{L^d}{(a+L)^d a^{k-d}}. \quad \square$$

Lemma 2.2. *Let* $a > 0$, $b > 0$, $k > 0$, $m > 0$ *and* $k + m > d$. *Let* $0 \neq x \in \mathbb{R}^d$ *and*

$$I := \int_{\mathbb{R}^d} \frac{dz}{(|z| + a)^k (|z - x| + b)^m}.$$

Then, with $R = \max\{|x|,\ a,\ b\} \sim |x| + a + b$,

$$I \lesssim R^{d-k-m} + \delta_{kd} R^{-m} \log\frac{R}{a} + \delta_{md} R^{-k} \log\frac{R}{b} + \mathbb{1}_{k>d} R^{-m} a^{d-k} + \mathbb{1}_{m>d} R^{-k} b^{d-m}.$$

Proof. Decompose I into

$$I = \left(\int_{|z|<2R} + \int_{|z|>2R}\right) \frac{dz}{(|z| + a)^k (|z - x| + b)^m} := I_1 + I_2.$$

For I_2 we have

$$I_2 \lesssim \int_{|z|>2R} \frac{dz}{|z|^k |z|^m} \sim R^{d-k-m}.$$

For I_1 we consider the three cases concerning R: $R = |x|$, $R = a$, and $R = b$.

- If $R = |x|$, we split I_1 into

$$I_1 = \left(\int_{|z|<\frac{R}{2}} + \int_{|z-x|<\frac{R}{2}} + \int_{\substack{\frac{R}{2}<|z|<2R \\ |z-x|>\frac{R}{2}}}\right) \frac{dz}{(|z| + a)^k (|z - x| + b)^m}$$
$$=: I_{1,1} + I_{1,2} + I_{1,3}.$$

By Lemma 2.1 we obtain

$$I_{1,1} \lesssim \int_{|z|<\frac{R}{2}} \frac{dz}{(|z| + a)^k R^m}$$
$$\sim R^{-m} \int_0^{\frac{R}{2}} \frac{r^{d-1}\, dr}{(r + a)^k} \lesssim \begin{cases} R^{d-m}(a + R)^{-k} & \text{if } k < d, \\ R^{-m}\left(1 + \log_+ \frac{R}{a}\right) & \text{if } k = d, \\ R^{-m} a^{-k} \min(a, R)^d & \text{if } k > d \end{cases}$$

since $|z-x| \ge |x| - |z| = R - |z| \ge \frac{R}{2}$. Also by Lemma 2.1,

$$I_{1,2} \lesssim \int_{|z-x|<\frac{R}{2}} \frac{dz}{R^k(|z-x|+b)^m} \sim R^{-k} \int_0^{\frac{R}{2}} \frac{r^{d-1}\,dr}{(r+b)^m}$$
$$\lesssim \begin{cases} R^{d-k}(b+R)^{-m} & \text{if } m<d, \\ R^{-k}\left(1+\log_+ \frac{R}{b}\right) & \text{if } m=d, \\ R^{-k} b^{-m} \min(b,R)^d & \text{if } m>d \end{cases}$$

since $|z| + a \ge |x| - |z-x| = R - |z-x| > \frac{R}{2}$, and

$$I_{1,3} \lesssim \int_{\substack{\frac{R}{2}<|z|<2R \\ |z-x|>\frac{R}{2}}} \frac{dz}{|z|^k|z-x|^m} \lesssim R^{-k-m} \int_{\frac{R}{2}}^{2R} r^{d-1}\,dr \sim R^{d-k-m}.$$

- If $R = a > |x|$,

$$\begin{aligned} I_1 &\le \int_{|z|<2R} \frac{dz}{a^k(|z-x|+b)^m} \\ &\le a^{-k} \int_{|z-x|<3R} \frac{dz}{(|z-x|+b)^m} \\ &= R^{-k} \int_0^{3R} \frac{r^{d-1}\,dr}{(r+b)^m} \\ &\sim I_{1,2}. \end{aligned}$$

- If $R = b > |x|$

$$I_1 \le \int_{|z|<2R} \frac{dz}{(|z|+a)^k b^m} = R^{-m} \int_0^{2R} \frac{r^{d-1}\,dr}{(r+a)^k} \sim I_{1,1}.$$

Combining the above cases, the proof is complete. □

2.2. Oseen tensor. We first recall the Oseen tensor $S_{ij}(x,y,t) = S_{ij}(x-y,t)$, derived by Oseen in [44]. For the Stokes system in $\mathbb{R}^n$:

$$\left.\begin{aligned} v_t - \Delta v + \nabla q &= f \\ v(x,0) = 0, \quad \operatorname{div} v &= 0 \end{aligned}\right\} \text{ in } \mathbb{R}^n \times (0,+\infty), \tag{2.7}$$

with $f(\cdot,t) = 0$ for $t < 0$, the unknown v and q are given by (see e.g. [8] or [46, (46)]):

$$v_i(x,t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^n} S_{ij}(x-y,t-s) f_j(y,s)\,dyds,$$

and

$$\begin{aligned} q(x,t) &= \sum_{j=1}^n \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} s_j(x-y,t-s) f_j(y,s)\,dyds \\ &= -\sum_{j=1}^n \int_{\mathbb{R}^n} \partial_j E(x-y) f_j(y,t)\,dy. \end{aligned}$$

Here (S_{ij}, s_j), the Oseen tensor, is the fundamental solution of the non-stationary Stokes system in $\mathbb{R}^n$, and

$$S_{ij}(x,t) = \delta_{ij}\Gamma(x,t) + \Gamma_{ij}(x,t),$$

$$\Gamma_{ij}(x,t) = \partial_i\partial_j \int_{\mathbb{R}^n} \Gamma(x-z,t)E(z)dz, \tag{2.8}$$

$$s_j(x,t) = -\partial_j E(x)\delta(t). \tag{2.9}$$

In [46, (41), (42), (44)] it is shown that (for $n = 3$, but the general case can be treated in the same way)

$$|\partial_x^l \partial_t^m \Gamma(x,t)| + |\partial_x^l \partial_t^m \Gamma_{ij}(x,t)| + \left|\partial_x^l \partial_t^m S_{ij}(x,t)\right| \lesssim \frac{1}{\left(x^2+t\right)^{\frac{l+n}{2}+m}} \tag{2.10}$$

for $n \geq 2$. It holds for $n = 2$ since we can apply one derivative on E to remove the log.

Remark 2.2. Formally taking the zero time limit of (2.8), we get

$$S_{ij}(x,0_+) = \delta_{ij}\delta(x) + \partial_i\partial_j E(x). \tag{2.11}$$

An exact meaning of (2.11) is given by Lemma 2.3. In other words, the zero time limit of the Oseen tensor is the kernel of the Helmholtz projection $\mathbf{P}_{\mathbb{R}^n}$ in $\mathbb{R}^n$,

$$(\mathbf{P}_{\mathbb{R}^n} u)_i = u_i + \partial_i(-\Delta)^{-1}\nabla \cdot u. \tag{2.12}$$

Lemma 2.3. *Fix $i, j \in \{1, \ldots, n\}$, $n \geq 2$. Suppose $f \in C_c^1(\mathbb{R}^n)$. Let $v(x,t) = \int_{\mathbb{R}^n} S_{ij}(x-y,t) f(y)\,dy$ and $v_0(x) = \delta_{ij} f(x) + \partial_i \int_{\mathbb{R}^n} \partial_j E(x-y) f(y)\,dy$. Then*

$$\lim_{t\to 0_+} \sup_{x\in\mathbb{R}^n} \langle x\rangle^n |v(x,t) - v_0(x)| = 0.$$

Some regularity of f is needed to ensure L^∞ convergence because v_0 may not be continuous if we only assume $f \in C_c^0$. By Lemma 2.3 and approximation, the convergence $v(\cdot,t) \to v_0$ is also valid in $L^q(\mathbb{R}^n)$, $1 < q < \infty$, for $f \in L^q(\mathbb{R}^n)$.

Proof. We first consider $u(x,t) = \int_{\mathbb{R}^n} \Gamma(x-y,t)a(y)\,dy$ for a bounded and uniformly continuous function a. Let $M = \sup |a|$. For any $\varepsilon > 0$, by uniform continuity, there is $r > 0$ such that $|a(x) - a(y)| \leq \varepsilon$ if $|x - y| \leq r$. Using $\int_{\mathbb{R}^n} \Gamma(x-y,t)\,dy = 1$,

$$\begin{aligned}
|u(x,t) - a(x)| &= \left|\left(\int_{B_r(x)} + \int_{B_r^c(x)}\right) \Gamma(x-y,t)[a(y)-a(x)]\,dy\right| \\
&\leq \int_{B_r(x)} \Gamma(x-y,t)\varepsilon\,dy + \int_{B_r^c(x)} \Gamma(x-y,t)2M\,dy \\
&\leq \varepsilon + CM \int_{|z|>r} t^{-n/2} e^{-z^2/4t}\,dz \leq \varepsilon + CMe^{-r^2/8t}.
\end{aligned}$$

This shows $\|u(x,t) - a(x)\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $t \to 0_+$. Suppose furthermore $a \in C_c^0(\mathbb{R}^n)$, $a(y) = 0$ if $|y| > R \geq 1$. Then for $|x| > 2R$,

$$\begin{aligned}
|u(x,t) - a(x)| = |u(x,t)| &\leq \int_{B_R} \Gamma(x-y,t)M\,dy \\
&\leq CMe^{-|x|^2/32t} \int_{\mathbb{R}^n} t^{-n/2} e^{-|x-y|^2/8t}\,dy = CMe^{-|x|^2/32t}.
\end{aligned}$$

We conclude for any $\alpha \ge 0$

$$|u(x,t) - a(x)| \le \frac{o(1)}{(|x|+R)^\alpha}, \quad \forall x \in \mathbb{R}^n, \tag{2.13}$$

where $o(1) \to 0$ as $t \to 0_+$, uniformly in x. (Estimate (2.13) is valid for $n \ge 1$.)

Recall the definition (2.8) of $S_{ij} = \delta_{ij}\Gamma + \Gamma_{ij}$. For $f \in C_c^1(\mathbb{R}^n)$, by (2.13) with $a = f$,

$$v(x,t) - v_0(x) = \frac{o(1)}{(|x|+R)^n} + v_1(x,t),$$

where

$$\begin{aligned} v_1(x,t) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Gamma(x-y-z,t)\partial_j E(z)dz\, \partial_i f(y)\, dy - \int_{\mathbb{R}^n} \partial_j E(x-w)\partial_i f(w)\, dw \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \Gamma(w-y,t)\partial_i f(y)\, dy - \partial_i f(w) \right) \partial_j E(x-w)\, dw. \end{aligned}$$

For the second equality we used $z = x - w$ and Fubini theorem. By (2.13) again with $a = \partial_i f$,

$$|v_1(x,t)| \le \int_{\mathbb{R}^n} \frac{o(1)}{(|w|+R)^{n+1}} |x-w|^{1-n}\, dw \le \frac{o(1)}{R(|x|+R)^{n-1}}.$$

We have used Lemma 2.2 for the second inequality.

We now improve its decay in $|x|$ and assume $|x| > R+1$. Decompose $\mathbb{R}^n = U + V$ where $U = \{w : |w-x| < |x|/2\}$ and $V = U^c$. Integrating by parts in w_i in V, we get

$$\begin{aligned} v_1(x,t) &= \int_U \left(\int_{\mathbb{R}^n} \Gamma(w-y,t)\partial_i f(y)\, dy - \partial_i f(w) \right) \partial_j E(x-w)\, dw \\ &\quad + \int_V \left(\int_{\mathbb{R}^n} \Gamma(w-y,t) f(y)\, dy - f(w) \right) \partial_i \partial_j E(x-w)\, dw \\ &\quad + \int_{\partial V} \left(\int \Gamma(w-y,t) f(y)\, dy - f(w) \right) \partial_j E(x-w)\, n_i\, dS_w \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By (2.13) with $a = \partial_i f$,

$$|I_1| \le \int_U \frac{o(1)}{(|x|+R)^{n+2}} |x-w|^{1-n}\, dw \le \frac{o(1)}{(|x|+R)^{n+1}}.$$

By (2.13) with $a = f$,

$$\begin{aligned} |I_2| &\le \int_V \frac{o(1)}{(|w|+R)^{n+1}} |x|^{-n}\, dw \le \frac{o(1)}{R(|x|+R)^n}, \\ |I_3| &\le \int_{\partial V} \frac{o(1)}{(|x|+R)^{n+1}} |x|^{1-n}\, dS_w \le \frac{o(1)}{(|x|+R)^{n+1}}. \end{aligned}$$

The main term is I_2. This shows the lemma. □

2.3. Golovkin tensor. The Golovkin tensor $K_{ij}(x,t) : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}$ is the Poisson kernel of the nonstationary Stokes system in $\mathbb{R}^n_+$, first constructed by Golovkin [14] for $\mathbb{R}^3_+$. Consider the boundary value problem of the Stokes system in the half-space:

$$\left.\begin{aligned} \hat{v}_t - \Delta \hat{v} + \nabla p = 0 \\ \operatorname{div} \hat{v} = 0 \end{aligned}\right\} \text{ in } \mathbb{R}^n_+ \times (0,\infty), \qquad \hat{v}(x',0,t) = \phi(x',t), \quad \text{on } \Sigma \times (0,\infty). \tag{2.14}$$

We extend $\phi(x',t) = 0$ for $t < 0$. By Solonnikov [46, (82)], the Golovkin tensor $K_{ij}(x,t)$ and its associated pressure tensor k_j are explicitly given by

$$K_{ij}(x,t) = -2\,\delta_{ij}\,\partial_n \Gamma(x,t) - 4\,\partial_j \int_0^{x_n} \int_\Sigma \partial_n \Gamma(z,t)\,\partial_i E(x-z)\,dz'\,dz_n \\ -2\,\delta_{nj}\partial_i E(x)\delta(t), \tag{2.15}$$

$$k_j(x,t) = 2\,\partial_j \partial_n E(x)\delta(t) + 2\,\delta_{nj} E(x)\delta'(t) + \frac{2}{t}\,\partial_j A(x,t), \tag{2.16}$$

where $A(x,t)$ is defined in (2.2). A solution $(\hat{v}, p)$ of (2.14) is represented by ([46, (84)]):

$$\hat{v}_i(x,t) = \sum_{j=1}^n \int_{-\infty}^\infty \int_\Sigma K_{ij}(x-\xi', t-s)\phi_j(\xi',s)\,d\xi'\,ds \tag{2.17}$$

and

$$p(x,t) = 2\sum_{i=1}^n \partial_i \partial_n \int_\Sigma E(x-\xi')\phi_i(\xi',t)\,d\xi' + 2\int_\Sigma E(x-\xi')\partial_t \phi_n(\xi',t)\,d\xi' \\ + \sum_{j=1}^n \partial_i \int_{-\infty}^\infty \int_\Sigma \frac{2}{t-s} A(x-\xi', t-s)[\phi_i(\xi',s) - \phi_i(\xi',t)]\,d\xi'\,ds. \tag{2.18}$$

Note that $\phi_i(\xi',t)$ is subtracted from the last integral to make it integrable. Alternatively, using $(\partial_t - \Delta_{x'})A = (-1/(2t))A$ (since $(\partial_t - \Delta_{x'})\Gamma(x',0,t) = (\partial_t - \Delta_{x'})(4\pi t)^{-1/2}\Gamma_{\mathbb{R}^{n-1}}(x',t) = -(2t)^{-1}\Gamma(x',0,t)$), $p(x,t)$ can also be expressed as [46, (85)]

$$p(x,t) = 2\sum_{i=1}^n \partial_i \partial_n \int_\Sigma E(x-\xi')\phi_i(\xi',t)\,d\xi' + 2\int_\Sigma E(x-\xi')\partial_t \phi_n(\xi',t)\,d\xi' \\ - 4\sum_{i=1}^n (\partial_t - \Delta_{x'}) \int_{-\infty}^\infty \int_\Sigma \partial_i A(x-\xi', t-\tau)\phi_i(\xi',\tau)\,d\xi' d\tau. \tag{2.19}$$

The last term of (2.16) is not integrable (hence not a distribution) and has to be understood in the sense of (2.18) or (2.19). By [46] (for $n = 3$, but the general case can be treated in the same manner), for $n \ge 2$, the Golovkin tensor satisfies, for $i, j = 1, \ldots, n$ and $t > 0$,

$$\left|\partial_{x'}^l \partial_{x_n}^k \partial_t^m K_{ij}(x,t)\right| \lesssim \frac{1}{t^{m+\frac{1}{2}} \left(x^2+t\right)^{\frac{l+n-\sigma}{2}} (x_n^2+t)^{\frac{k+\sigma}{2}}}, \quad \sigma = \delta_{i<n}\delta_{jn}. \tag{2.20}$$

Here $\sigma = 1$ if $i < n = j$ and $\sigma = 0$ otherwise. Specifically, the case $j < n$ is [46, (73)], the case $j = n$ uses $j < n$ case, the formulas for K_{in} on [46, page 47], and [46, (69)].

Remark 2.3. (i) In the proof of [46, (73)], in the equation after [46, (72)], there is at least one x'-derivative acting on B (defined in (2.3)) even if $l = 0$. The same is true for formulas for K_{in} on [46, page 47]. Hence we have estimate (2.20) for all $n \geq 2$ and do not have a log factor for $n = 2$. Compare (2.5) and Remark 2.1.
(ii) Solonnikov [46, pp.46-48] decomposes $\hat{v} = w + w'$ where

$$w_i(x,t) = \sum_{j<n} \iint K_{ij}(x - \xi', t - s)\phi_j(\xi', s)d\xi' ds,$$

$$w_i'(x,t) = \iint K_{in}(x - \xi', t - s)\phi_n(\xi', s)d\xi' ds,$$

and shows that $w_i(x', 0, t) = (1 - \delta_{in})\phi_i(x', t)$ and $w_i'(x', 0, t) = \delta_{in}\phi_i(x', t)$.
(iii) The limit of $\hat{v}(\cdot, t)$ as $t \to 0_+$ depends on the $\lim_{t\to 0_+} \phi(\cdot, t)$. It is in general nonzero unless $\phi(\cdot, t) = 0$ for $0 < t < \delta$. See the following example.

Example 2.4. Let $\rho(\xi', t)$ be any continuous function defined on $\Sigma \times \mathbb{R}$ with suitable decay. Let

$$u(x,t) = \nabla_x h(x,t), \quad h(x,t) = \int_\Sigma -2E(x - \xi')\rho(\xi', t)d\xi'.$$

Let $\hat{v}(x,t)$ be defined by (2.17) with $\phi(x', t) = u(x', 0, t)$. We claim that $\hat{v}(x,t) = u(x,t)$. Note that h is harmonic in x and $u_n|_\Sigma = \rho$ as $-2\partial_n E(x)$ is the Poisson kernel of $-\Delta$ in $\mathbb{R}^n_+$. Since $\operatorname{div} u = 0$ and $\operatorname{curl} u = 0$, by Stein [51] Theorem III.3 on page 65, we have

$$u_n|_\Sigma = \rho, \quad u_i|_\Sigma = R_i'\rho \quad (i < n),$$

where R_j' is the j-th Riesz transform on $\mathbb{R}^{n-1}$, $\widehat{R_j' f}(\xi') = \frac{i\xi_j}{|\xi'|}\hat{f}(\xi')$. By (2.15) and (2.17),

$$\begin{aligned}
\hat{v}_i(x,t) = &-2\int_{-\infty}^{\infty}\int_\Sigma \partial_n\Gamma(x - \xi', t - s)\phi_i(\xi', s)\, d\xi' ds \\
&- 4\sum_{j=1}^{n-1}\int_{-\infty}^{\infty}\int_\Sigma \partial_{x_j}\left(\int_0^{x_n}\int_\Sigma \partial_n\Gamma(z, t-s)\,\partial_i E(x - \xi' - z)\, dz'\, dz_n\right)\phi_j(\xi', s)\, d\xi' ds \\
&- 4\int_{-\infty}^{\infty}\int_\Sigma \partial_{x_n}\left(\int_0^{x_n}\int_\Sigma \partial_n\Gamma(z, t-s)\,\partial_i E(x - \xi' - z)\, dz'\, dz_n\right)\phi_n(\xi', s)\, d\xi' ds \\
&- 2\int_\Sigma \partial_i E(x - \xi')\phi_n(\xi', t)\, d\xi' =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

As $\phi_n = \rho$, $I_4 = u_i(x,t)$ by definition. If $i < n$, since $\phi_j = R_j'\rho$, we can switch derivatives

$$\begin{aligned}
I_2 = -4\sum_{j=1}^{n-1}\int_{-\infty}^{\infty}\int_\Sigma \partial_{x_j}\left(\int_0^{x_n}\int_\Sigma \partial_n\Gamma(z, t-s)\,\partial_j E(x - \xi' - z)\, dz'\, dz_n\right) \\
\phi_i(\xi', s)\, d\xi' ds
\end{aligned}$$

$$= 4 \int_{-\infty}^{\infty} \int_{\Sigma} \partial_{x_n} \left(\int_0^{x_n} \int_{\Sigma} \partial_n \Gamma(z, t-s)\, \partial_n E(x - \xi' - z)\, dz'\, dz_n \right)$$
$$\phi_i(\xi', s)\, d\xi' ds$$
$$+2 \int_{-\infty}^{\infty} \int_{\Sigma} \partial_n \Gamma(x - \xi', t-s) \phi_i(\xi', s)\, d\xi' ds = I_{2a} + I_{2b}.$$

The second equality is [46, (68)]. Note that I_{2b} cancels I_1, and $I_{2a} + I_3 = 0$ because

$$-2 \int_{\Sigma} \partial_i E(x - \xi' - z) \phi_n(\xi', s)\, d\xi' = u_i(x - z, s)$$
$$= -2 \int_{\Sigma} \partial_n E(x - \xi' - z) \phi_i(\xi', s)\, d\xi'.$$

The first equality is by definition of u_i. The second is because $-2\partial_n E$ is the Poisson kernel. Thus $\hat{v}_i(x, t) = u_i(x, t)$ for $i < n$. As they are harmonic conjugates of $\hat{v}_n$ and u_n, and $\hat{v}_n$ and u_n have the same boundary value ρ, we also have $\hat{v}_n(x, t) = u_n(x, t)$. □

3. First Formula for the Green Tensor

In this section, we derive a formula of the Green tensor G_{ij} of the non-stationary Stokes system in the half-space. We decompose $G_{ij} = \tilde{G}_{ij} + W_{ij}$ with explicit $\tilde{G}_{ij}$ given by (3.5), and derive a formula for the remainder term W_{ij}.

For the nonstationary Stokes system in the half-space $\mathbb{R}^n_+$, $n \geq 2$, the Green tensor $G_{ij}(x, y, t)$ and its associated pressure tensor $g_j(x, y, t)$, for each fixed $j = 1, \ldots, n$ and $y \in \mathbb{R}^n_+$, satisfy

$$\partial_t G_{ij} - \Delta_x G_{ij} + \partial_{x_i} g_j = \delta_{ij} \delta_y(x) \delta(t), \quad \sum_{i=1}^n \partial_{x_i} G_{ij} = 0, \quad \text{for } x \in \mathbb{R}^n_+ \text{ and } t \in \mathbb{R},$$
$$G_{ij}(x, y, t)|_{x_n=0} = 0. \tag{3.1}$$

Recall the defining property that solution (u, π) of (1.1)–(1.2) with zero boundary condition is given by (1.3) and

$$\pi(x, t) = \int_{\mathbb{R}^n_+} g(x, y, t) \cdot u_0(y)\, dy + \int_{-\infty}^{\infty} \int_{\mathbb{R}^n_+} g(x, y, t-s) \cdot f(y, s)\, dy\, ds. \tag{3.2}$$

The time interval in (3.2) is the entire $\mathbb{R}$ as we will see in Proposition 3.5 that g contains a delta function in time, cf. (2.9). In contrast, G_{ij} is a function and we can define $G_{ij}(x, y, t) = 0$ for $t \leq 0$ in view of (1.3). Note that $G_{ij}(x, y, 0_+) \neq 0$, see Lemma 3.4.

We now proceed to find a formula for G_{ij}. Let u, π solve (1.1)–(1.2) with zero external force $f = 0$, and non-zero initial data $u(x, 0) = u_0(x)$, in the sense of (1.7). Then

$$u_i(x, t) = \sum_{j=1}^n \int_{\mathbb{R}^n_+} G_{ij}(x, y, t)(u_0)_j(y)\, dy, \tag{3.3}$$

and π is given by (3.2) with $f = 0$. Let $\mathbf{E}u_0$ be an extension of u_0 to $\mathbb{R}^n$ by

$$\mathbf{E}u_0(x', x_n) = (-u_0', u_0^n)(x', -x_n) \ \text{ for } x_n < 0. \tag{3.4}$$

Then $\operatorname{div} \mathbf{E}u_0(x', x_n) = -\operatorname{div} u_0(x', -x_n)$ for $x_n < 0$.

Remark 3.1. If $\operatorname{div} u_0 = 0$ and $u_0^n(x', 0) = 0$, then $\operatorname{div} \mathbf{E}u_0 = 0$ in $\mathcal{D}'(\mathbb{R}^n)$.

Let $\tilde{u}$ be the solution to the homogeneous Stokes system in $\mathbb{R}^n$ with initial data $\mathbf{E}u_0$. Then

$$\begin{aligned}\tilde{u}_i(x,t) &= \sum_{j=1}^n \int_{\mathbb{R}^n} S_{ij}(x-y,t)(\mathbf{E}u_0)_j(y)\,dy \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n_+} (S_{ij}(x-y,t) - \epsilon_j S_{ij}(x-y^*))(u_0)_j(y)\,dy \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n_+} \tilde{G}_{ij}(x,y,t)(u_0)_j(y)\,dy,\end{aligned}$$

where

$$\tilde{G}_{ij}(x,y,t) = S_{ij}(x-y,t) - \epsilon_j S_{ij}(x-y^*,t), \quad \epsilon_j = 1 - 2\delta_{nj}. \tag{3.5}$$

Note that the factor ϵ_j is absent in the second term of Solonnikov's restricted Green tensor (1.9). Eqn. (3.5) is closer to [23, (2.22)].

Lemma 3.1. *We have*

$$S_{ij}(x^*,t) = \epsilon_i \epsilon_j S_{ij}(x,t), \tag{3.6}$$

$$\tilde{G}_{ij}(x,y,t)\big|_{x_n=0} = 2\,\delta_{in}\, S_{nj}(x'-y,t). \tag{3.7}$$

Proof. If $i = j$, then $S_{ii}(x,t)$ is even in all x_k. If $i \neq j$, then $S_{ij}(x,t)$ is odd in x_i and x_j, but even in x_k if $k \neq i, j$. In particular, with $x_k = x_n$, we get (3.6) for all $i, j = 1, \ldots, n$. By (3.6),

$$\tilde{G}_{ij}(x,y,t)\Big|_{x_n=0} = S_{ij}(x'-y,t) - \epsilon_j S_{ij}(x'-y^*,t) = S_{ij}(x'-y,t) - \epsilon_i S_{ij}(x'-y,t)$$

which gives (3.7). □

Let $\hat{u} = u - \tilde{u}|_{\mathbb{R}^n_+}$. Then $\hat{u}$ solves the boundary value problem (2.14) with boundary data $\hat{u}|_{x_n=0} = -\tilde{u}(x,t)|_{x_n=0}$. By the Golovkin formula (2.17),

$$\begin{aligned}\hat{u}_i(x,t) &= \sum_{k=1}^n \int_{-\infty}^{\infty}\int_\Sigma K_{ik}(x-\xi',t-s)(-\tilde{u}_k(\xi',0,s))\,d\xi' ds \\ &= \sum_{k=1}^n \int_{-\infty}^{\infty}\int_\Sigma K_{ik}(x-\xi',t-s)\left(-\sum_{j=1}^n \int_{\mathbb{R}^n_+} \tilde{G}_{kj}(\xi',y,s)(u_0)_j(y)\,dy\right) d\xi' ds \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n_+} \left(-\sum_{k=1}^n \int_{-\infty}^{\infty}\int_\Sigma K_{ik}(x-\xi',t-s)\tilde{G}_{kj}(\xi',y,s)\,d\xi' ds\right)(u_0)_j(y)dy \\ &= \sum_{j=1}^n \int_{\mathbb{R}^n_+} W_{ij}(x,y,t)(u_0)_j(y)dy,\end{aligned}$$

where

$$W_{ij}(x,y,t) = -\sum_{k=1}^n \int_{-\infty}^{\infty}\int_\Sigma K_{ik}(x-\xi',t-s)\tilde{G}_{kj}(\xi',y,s)\,d\xi' ds. \tag{3.8}$$

By Lemma 3.1, we have the following first formula of W_{ij}.

Lemma 3.2 (The first formula of W_{ij}). *For $x, y \in \mathbb{R}^n_+$, $t > 0$, and $i, j = 1, \ldots, n$*

$$W_{ij}(x, y, t) = -2\int_{-\infty}^{\infty}\int_{\Sigma} K_{in}(x - \xi', t - s)S_{nj}(\xi' - y, s)\, d\xi' ds. \tag{3.9}$$

Remark 3.2. Because of $\delta(t)$ in the last term of the formula (2.15) of K_{ij}, when we substitute (2.15) into the right side of (3.9), one of the resulting integrals is spatial only.

As $u = \tilde{u}|_{\mathbb{R}^n_+} + \hat{u}$, the Green tensor G_{ij} has the decomposition

$$\begin{aligned} G_{ij}(x, y, t) &= \tilde{G}_{ij}(x, y, t) + W_{ij}(x, y, t) \\ &= S_{ij}(x - y, t) - \epsilon_j S_{ij}(x - y^*, t) + W_{ij}(x, y, t). \end{aligned} \tag{3.10}$$

All of them are zero for $t \leq 0$.

With the first formula of W_{ij}, we have the scaling property of the Green tensor.

Corollary 3.3. *For $n \geq 2$ the Green tensor G_{ij} obeys the following scaling property*

$$G_{ij}(x, y, t) = \lambda^n G_{ij}(\lambda x, \lambda y, \lambda^2 t).$$

Proof. Note that $\Gamma(\lambda x, \lambda^2 t) = \lambda^{-n}\Gamma(x, t)$ and $\delta(\lambda^2 t) = \lambda^{-2}\delta(t)$. It follows directly from (3.10), (3.9) and the scaling properties of K_{ij} and S_{ij}. □

Remark 3.3. In Lemma 2.1 of the stationary case of [23], the condition $n \geq 3$ is needed for showing the scaling property of $G_{ij}(x, y)$ because the 2D fundamental solution E does not have the scaling property. However, in the nonstationary case we do not have this issue. So the scaling property of the nonstationary Green tensor holds for all dimension $n \geq 2$.

Before we consider the zero time limit of G_{ij}, we consider the Helmholtz projection.

Remark 3.4. (*Helmholtz projection in* $\mathbb{R}^n_+$) For a vector field u in $\mathbb{R}^n_+$, its Helmholtz projection $\mathbf{P}u$ is given by

$$(\mathbf{P}u)_i = u_i - \partial_i p, \tag{3.11}$$

where p satisfies $-\Delta p = -\operatorname{div} u$, and $\partial_n p = u_n$ on $x_n = 0$. Using the Green function of the Laplace equation with Neumann boundary condition, $N(x, y) = E(x - y) + E(x - y^*)$, we have

$$p(x) = -\int_{\mathbb{R}^n_+} N(x, y) \operatorname{div} u(y)\, dy - \int_{\Sigma} u_n(y)N(x, y)\, dS_y. \tag{3.12}$$

Note the unit outer normal $\nu = -e_n$ and $\frac{\partial p}{\partial \nu} = -\partial_n p = -u_n$. The second term is absent in [41, Appendix], [10, (III.1.18)], and [35, Lemma A.3] because they are concerned with L^q bounds of $\mathbf{P}\tilde{u}$ with $\tilde{u} \in L^q$, for which (3.12) is undefined, and they approximate $\tilde{u}$ in L^q by $u \in C^\infty_c(\mathbb{R}^n_+)$, for which the second term in (3.12) is zero. For our purpose, we want pointwise bounds and hence we need to keep the boundary term. Integrating by parts,

$$p(x) = \int_{\mathbb{R}^n_+} \partial_{y_j} N(x, y)u_j(y)\, dy. \tag{3.13}$$

The boundary terms on Σ cancel. Using the definition of $N(x, y)$,

$$\partial_{y_j} N(x, y) = -F_j^y(x), \quad F_j^y(x) := \partial_j E(x - y) + \epsilon_j \partial_j E(x - y^*). \tag{3.14}$$

Thus

$$(\mathbf{P}u)_i(x) = u_i(x) + \partial_i \int_{\mathbb{R}^n_+} F_j^y(x) u_j(y)\, dy. \tag{3.15}$$

We now consider the zero time limit of G_{ij}.

Lemma 3.4. (a) *For $x, y \in \mathbb{R}^n_+$, we have*

$$G_{ij}(x, y, 0_+) = \delta_{ij}\delta(x - y) + \partial_{x_i} F_j^y(x), \tag{3.16}$$

where $F_j^y(x)$ is defined in (3.14), in the sense that, for any $i, j \in \{1, \ldots, n\}$ and $f \in C^1_c(\mathbb{R}^n_+)$, we have

$$\lim_{t \to 0_+} \left(\int_{\mathbb{R}^n_+} G_{ij}(x, y, t) f(y) dy - \delta_{ij} f(x) - \partial_{x_i} \int_{\mathbb{R}^n_+} F_j^y(x) f(y)\, dy \right) = 0, \tag{3.17}$$

for all $x \in \mathbb{R}^n_+$, and uniformly for $x_n \geq \delta$, for any $\delta > 0$.

(b) *Let $u_0 \in C^1_c(\mathbb{R}^n_+; \mathbb{R}^n)$ be a vector field in $\mathbb{R}^n_+$ and let $u(x, t)$ be given by (3.3). Then $u(x, t) \to (\mathbf{P}u_0)(x)$ for all $x \in \mathbb{R}^n_+$, and uniformly for all x with $x_n \geq \delta$ for any $\delta > 0$.*

Note that $\partial_{x_i} F_j^y(x)$ is a distribution since it may produce delta function at y. This lemma shows that the zero time limit of the Green tensor is exactly the Helmholtz projection in $\mathbb{R}^n_+$, given in (3.15). We will show uniform convergence in Lemma 6.1 where we assume $\mathbf{P}u_0 \in C^1_c(\overline{\mathbb{R}^n_+})$, allowing nonzero tangential components of $u_0|_\Sigma$, and show L^q convergence in Lemma 6.2 where we assume $u_0 \in L^q(\mathbb{R}^n_+)$ but do not assume $u_0 = \mathbf{P}u_0$.

Proof. (a) We may extend f to $\mathbb{R}^n$ by setting $f(y) = 0$ for $y_n \leq 0$. Recall that

$$G_{ij}(x, y, t) = \tilde{G}(x, y, t) + W_{ij}(x, y, t), \quad \text{with}$$
$$\tilde{G}(x, y, t) = S_{ij}(x - y, t) - \epsilon_j S_{ij}(x - y^*, t).$$

By (3.6) and Lemma 2.3,

$$\lim_{t \to 0_+} \int_{\mathbb{R}^n_+} \tilde{G}_{ij}(x, y, t) f(y) dy$$
$$= \delta_{ij} f(x) + \partial_i \int_{\mathbb{R}^n_+} \partial_j \big[E(x - y) - \epsilon_j E(x - y^*) \big] f(y)\, dy, \tag{3.18}$$

uniformly in $x \in \mathbb{R}^n_+$. Now we consider the contribution from $W_{ij}(x, y, t)$. By (3.9) and (2.15),

$$W_{ij}(x, y, t) = -2 \int_{-\infty}^{\infty} \int_\Sigma K_{in}(x - \xi', t - s) S_{nj}(\xi' - y, s)\, d\xi' ds$$
$$= W_{ij,1}(x, y, t) + W_{ij,2}(x, y, t),$$

where

$$W_{ij,1}(x,y,t) = -2\int_{-\infty}^{\infty}\int_{\Sigma} \tilde{K}_{in}(x-\xi',t-s)S_{nj}(\xi'-y,s)\,d\xi' ds,$$
$$W_{ij,2}(x,y,t) = 4\int_{\Sigma} \partial_i E(x-\xi')S_{nj}(\xi'-y,t)\,d\xi',$$

and $\tilde{K}_{ij}$ is the sum of the first two terms in the definition (2.15) of K_{ij}. By (2.20), (2.10), change of variable $s=u^2$ and Lemma 2.2,

$$\begin{aligned}
|W_{ij,1}| &\lesssim \int_0^t\int_{\Sigma} \frac{1}{\sqrt{s}(|x-\xi'|+\sqrt{s})^{n-1}(x_n+\sqrt{s})}\,\frac{1}{(|\xi'-y|+\sqrt{t-s})^n}\,d\xi' ds \\
&\le \int_0^t\int_{\Sigma} \frac{1}{\sqrt{s}(x_n+\sqrt{s})|x-\xi'|^{n-1}}\,\frac{1}{|\xi'-y|^n}\,d\xi' ds \\
&= 2\log\left(1+\frac{\sqrt{t}}{x_n}\right)\int_{\Sigma} \frac{1}{|x-\xi'|^{n-1}}\,\frac{1}{|\xi'-y|^n}\,d\xi' \\
&\lesssim \log\left(1+\frac{\sqrt{t}}{x_n}\right)\left\{|x-y^*|^{-n}+|x-y^*|^{-n}\log\frac{|x-y^*|}{x_n}+|x-y^*|^{-(n-1)}y_n^{-1}\right\}.
\end{aligned}$$

From this, one has

$$\lim_{t\to 0_+}\int_{\mathbb{R}^n_+} W_{ij,1}(x,y,t)f(y)dy = 0$$

for all $x\in\mathbb{R}^n_+$, and uniformly for $x_n\ge\delta>0$. On the other hand, by Remark 2.2, $W_{ij,2}(x,y,t)$ for $x_n, y_n>0$ as $t\to 0_+$ formally tends to

$$\begin{aligned}
4\int_{\Sigma} \partial_i E(x-\xi')\partial_n\partial_j E(\xi'-y)\,d\xi' &= -4\partial_{x_i}\partial_{y_j}\int_{\Sigma} E(x-\xi')\partial_n E(\xi'-y)\,d\xi' \\
&= -2\partial_{x_i}\partial_{y_j}\int_{\Sigma} E(\xi'-x)P_0(y-\xi')\,d\xi' \qquad (3.19)\\
&= -2\frac{\partial}{\partial x_i}\frac{\partial}{\partial y_j}E(x-y^*) = 2\epsilon_j\partial_i\partial_j E(x-y^*),
\end{aligned}$$

where $P_0=-2\partial_n E$ and we've used (2.1) for the third equality. It is in the sense of functions since its singularity is at $y=x^*\notin\mathbb{R}^n_+$. Thus

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} W_{ij,2}(x,y,t)f(y)dy - \int_{\mathbb{R}^n_+} 2\epsilon_j\partial_i\partial_j E(x-y^*)f(y)\,dy \\
&= \int_{\mathbb{R}^n} 4\int_{\Sigma} \partial_i E(x-\xi')S_{nj}(\xi'-y,t)f(y)dy \\
&\quad - \int_{\mathbb{R}^n} 4\int_{\Sigma} \partial_i E(x-\xi')\partial_j E(\xi'-y)d\xi'\partial_n f(y)dy \\
&= 4\int_{\Sigma}\left\{\int_{\mathbb{R}^n} S_{nj}(\xi'-y,t)f(y)dy - \int_{\mathbb{R}^n}\partial_j E(\xi'-y)\partial_n f(y)dy\right\}\partial_i E(x-\xi')d\xi'.
\end{aligned}$$

For the first equality we used (3.19) and integrated by parts in y_n in the second integral using $f \in C_c^1(\mathbb{R}^n_+)$. For the second equality we used the Fubini theorem. By Lemmas 2.3 and 2.2, the above is bounded by

$$\lesssim \int_\Sigma \frac{o(1)}{\langle \xi' \rangle^n} \frac{1}{|x-\xi'|^{n-1}} d\xi' \lesssim \frac{o(1)}{\langle x \rangle^{n-1}}.$$

The combination of the above and (3.18) give Part (a).

Part (b) is a consequence of Part (a) and Remark 3.4. □

Finally we derive a formula for the pressure tensor g_j, to be used to estimate g_j in Sect. 5, and show symmetry of G_{ij} in Sect. 7.

Proposition 3.5 (The pressure tensor g_j). *For $x, y \in \mathbb{R}^n_+$, $t \in \mathbb{R}$, and $j = 1, \ldots, n$ we have*

$$g_j(x,y,t) = \widehat{w}_j(x,y,t) - F_j^y(x)\delta(t), \tag{3.20}$$

where $\widehat{w}_j(x,y,t)$ is a function with $\widehat{w}_j(x,y,t) = 0$ for $t \le 0$ and, for $t > 0$,

$$\begin{aligned} \widehat{w}_j(x,y,t) = &-\sum_{i<n} 8 \int_0^t \int_\Sigma \partial_i \partial_n A(\xi', x_n, \tau) \partial_n S_{ij}(x'-y'-\xi', -y_n, t-\tau)\, d\xi' d\tau \\ &+ \sum_{i<n} 4 \int_\Sigma \partial_i E(x-\xi') \partial_n S_{ij}(\xi'-y,t)\, d\xi' \\ &+ 8 \int_\Sigma \partial_n A(\xi', x_n, t) \partial_n \partial_j E(x'-y'-\xi', -y_n)\, d\xi'. \end{aligned} \tag{3.21}$$

Proof. For fixed j, the Green tensor (G_{ij}, g_j) satisfies (3.1) in $\mathbb{R}^n_+$. Let

$$\begin{aligned} \tilde{g}_j(x,y,t) &= s_j(x-y,t) - \epsilon_j s_j(x-y^*,t) \\ &= -\left[\partial_j E(x-y) - \epsilon_j \partial_j E(x-y^*)\right]\delta(t). \end{aligned} \tag{3.22}$$

The pair $(\tilde{G}_{ij}, \tilde{g}_j)$ satisfies in $\mathbb{R}^n$

$$\begin{gathered} (\partial_t - \Delta_x)\tilde{G}_{ij}(x,y,t) + \partial_{x_i}\tilde{g}_j(x,y,t) = \delta_{ij}\delta_y(x)\delta(t) - \epsilon_j\delta_{ij}\delta_{y^*}(x)\delta(t), \\ \textstyle\sum_{i=1}^n \partial_{x_i}\tilde{G}_{ij} = 0. \end{gathered} \tag{3.23}$$

Thus the difference $(W_{ij}, w_j) = (G_{ij}, g_j) - (\tilde{G}_{ij}, \tilde{g}_j)$ solves in $\mathbb{R}^n_+$

$$\begin{cases} (\partial_t - \Delta_x)W_{ij}(x,y,t) + \partial_{x_i} w_j(x,y,t) = 0, \quad \sum_{i=1}^n \partial_{x_i} W_{ij} = 0, \\ W_{ij}(x,y,t)|_{x_n=0} = -2\,\delta_{in} S_{nj}(x'-y,t). \end{cases} \tag{3.24}$$

By (2.19), we have

$$\begin{aligned} w_j(x,y,t) = &-4 \int_\Sigma \partial_n^2 E(x-\xi') S_{nj}(\xi'-y,t)\, d\xi' \\ &- 4 \int_\Sigma E(x-\xi') \partial_t S_{nj}(\xi'-y,t)\, d\xi' \\ &+ 8(\partial_t - \Delta_{x'}) \int_{-\infty}^{\infty} \int_\Sigma \partial_n A(x-\xi', t-\tau) S_{nj}(\xi'-y,\tau)\, d\xi' d\tau \\ = &\ I_1 + I_2 + I_3. \end{aligned} \tag{3.25}$$

Using $(\partial_t - \Delta)S_{ij} + \partial_i s_j = \delta_{ij}\delta(x)\delta(t)$, we have

$$\begin{aligned} I_2 =& -4 \int_\Sigma E(x-\xi')[\Delta S_{nj}(\xi'-y,t) - \partial_n s_j(\xi'-y,t)]\,d\xi' \\ =& -4 \int_\Sigma \Delta_{x'} E(x-\xi') S_{nj}(\xi'-y,t)\,d\xi' \\ & -4 \int_\Sigma E(x-\xi')\partial_n^2 S_{nj}(\xi'-y,t)\,d\xi' \\ & -4\delta(t) \int_\Sigma E(x-\xi')\partial_n\partial_j E(\xi'-y)\,d\xi' \end{aligned} \tag{3.26}$$

The first term of I_2 in (3.26) cancels I_1 since $\Delta E(x-\xi') = 0$, and the last term of (3.26) is $\overline{w}_j(x,y)\delta(t)$ with

$$\begin{aligned} \overline{w}_j(x,y) &= 4\partial_{y_j} \int_\Sigma E(x-\xi')\partial_n E(\xi'-y)\,d\xi' = 2\partial_{y_j} \int_\Sigma E(\xi'-x)P_0(y-\xi')\,d\xi' \\ &= 2\partial_{y_j} E(x-y^*) = -2\epsilon_j \partial_j E(x-y^*) \end{aligned}$$

using (2.1). Note that

$$\tilde{g}_j(x,y,t) + \overline{w}_j(x,y)\delta(t) = -F_j^y(x)\delta(t). \tag{3.27}$$

Using $(\partial_t - \Delta)S_{ij} + \partial_i s_j = \delta_{ij}\delta(x)\delta(t)$ again, we have

$$\begin{aligned} I_3 &= 8(\partial_t - \Delta_{x'}) \int_{-\infty}^{\infty}\int_\Sigma \partial_n A(\xi',x_n,\tau) S_{nj}(x'-y'-\xi',-y_n,t-\tau)\,d\xi' d\tau \\ &= 8 \int_{-\infty}^{\infty}\int_\Sigma \partial_n A(\xi',x_n,\tau)\left[\partial_n^2 S_{nj} - \partial_n s_j\right](x'-y'-\xi',-y_n,t-\tau)\,d\xi' d\tau \\ &= 8 \int_{-\infty}^{\infty}\int_\Sigma \partial_n A(\xi',x_n,\tau)\partial_n^2 S_{nj}(x'-y'-\xi',-y_n,t-\tau)\,d\xi' d\tau \\ &\quad + 8 \int_\Sigma \partial_n A(\xi',x_n,t)\partial_n\partial_j E(x'-y'-\xi',-y_n)\,d\xi'. \end{aligned}$$

Denote $\widehat{w}_j(x,y,t) = w_j(x,y,t) - \overline{w}_j(x,y)\delta(t)$. We conclude

$$\begin{aligned} \widehat{w}_j(x,y,t) =& 8 \int_{-\infty}^{\infty}\int_\Sigma \partial_n A(\xi',x_n,\tau)\partial_n^2 S_{nj}(x'-y'-\xi',-y_n,t-\tau)\,d\xi' d\tau \\ & -4 \int_\Sigma E(x-\xi')\partial_n^2 S_{nj}(\xi'-y,t) \\ & +8 \int_\Sigma \partial_n A(\xi',x_n,t)\partial_n\partial_j E(x'-y'-\xi',-y_n)\,d\xi'. \end{aligned} \tag{3.28}$$

Using $\partial_n^2 S_{nj} = -\sum_{i<n} \partial_i\partial_n S_{ij}$ and integrating by parts in ξ_i the first two terms, we get (3.21) for $\widehat{w}_j(x,y,t)$. Integration by parts is justified since the singularities of the integrands are outside of Σ, and the integrands have sufficient decay as $|\xi'| \to \infty$ by (2.10) and (2.4) even for $n = 2$. This and (3.27) prove the proposition. □

Remark 3.5. (i) Eq. (3.21) is better than (3.28) because its estimate allows more decay in $|x - y^*| + \sqrt{t}$, i.e., in tangential direction. However, it has a boundary singularity at $x_n = 0$; see Remark 7.1.
(ii) With Proposition 3.5, the pressure formula (3.2) in the case $u_0 = 0$ becomes

$$\begin{aligned} \pi(x,t) &= \int_{-\infty}^{\infty} \int_{\mathbb{R}^n_+} g(x,y,t-s) \cdot f(y,s)\, dy\, ds \\ &= \int_0^t \int_{\mathbb{R}^n_+} \widehat{w}(x,y,t-s) \cdot f(y,s)\, dy\, ds - \int_{\mathbb{R}^n_+} F^y_j(x) \cdot f_j(y,t)\, dy. \end{aligned} \tag{3.29}$$

The last term comes from the Helmholtz projection of f at time t (see (3.13)–(3.14)), and corresponds to the pressure formula above (2.8) in the whole space case. The first term of (3.29) shows that $\pi(\cdot, t)$ also depends on the value of f at times $s < t$. There is no such term in the whole space case. This history-dependence property of the pressure in the half space case is well known, see e.g. [52].

Remark 3.6. (*Kernel of Green tensor*) Consider

$$\mathbf{G} = \left\{ u = \nabla h \in C^0(\mathbb{R}^n_+; \mathbb{R}^n),\ \lim_{|x|\to\infty} h(x) = 0 \right\}.$$

If $u_0 \in \mathbf{G}$, then $u(x,t)$ given by (3.3) is identically zero, using integration by parts in (3.3). The whole thing vanishes because $\sum_j \partial_{y_j} G_{ij} = 0$ and $G_{in}|_{y_n=0} = 0$. Thus $\mathbf{G}$ is contained in the kernel of the Green tensor. In fact, it is also inside the kernel of the Helmholtz projection in $L^q(\mathbb{R}^n_+)$, $1 < q < \infty$, if we impose suitable spatial decay on functions in $\mathbf{G}$.

Remark 3.7. (*Relation between stationary and nonstationary Green tensors*) Denote the Green tensor of the stationary Stokes system in the half space as $G^0_{ij}(x,y)$. For $n \geq 3$ we can show

$$\int_{\mathbb{R}} G_{ij}(x,y,t)\, dt = G^0_{ij}(x,y). \tag{3.30}$$

The integral does not converge for $n = 2$. The idea is to decompose $G_{ij}(x,y,t) = \tilde{G}_{ij}(x,y,t) + W_{ij}(x,y,t)$ and show their time integrations converge to corresponding terms in [23, (2.25)]. This relation gives an alternative proof of symmetry $G^0_{ij}(x,y) = G^0_{ji}(y,x)$ for $n \geq 3$ using Proposition 1.4.

4. Revised Formula for the Green Tensor

In this section we derive a second formula for the remainder term W_{ij} which is suitable for pointwise estimate. We also use it to get a new formula for the Green tensor in Lemma 4.3.

We first recall the Poisson kernel $P(x, \xi', t)$ for $\partial_t - \Delta$ in the half-space $\mathbb{R}^n_+$ for $x \in \mathbb{R}^n_+$ and $\xi' \in \Sigma$,

$$P(x,\xi',t) = -2\, \partial_n \Gamma(x - \xi', t). \tag{4.1}$$

The following lemma is based on Poisson's formula, and can be used to *remove the time integration* in the first formula (3.9). It is the time-dependent version of (2.1).

Lemma 4.1. *Let $n \ge 2$. For $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^n$ and $t > 0$,*

$$\int_0^t \int_\Sigma \Gamma(\xi' - y, s) P(x, \xi', t - s)\, d\xi'\, ds = \Gamma(x - y^\sharp, t), \quad y^\sharp = (y', -|y_n|). \tag{4.2}$$

Note that $y^\sharp = y^*$ if $\mathbb{R}^n_+$, and $y^\sharp = y$ if $y \in \mathbb{R}^n_-$.

Proof. First we consider $y \in \mathbb{R}^n_+$. Since $u(x, t) = \Gamma(x - y^*, t)$ satisfies

$$\begin{cases} (\partial_t - \Delta) u(x, t) = 0 & \text{for}(x, t) \in \mathbb{R}^n_+ \times (0, \infty), \\ u(x', t) = \Gamma(x' - y^*, t) = \Gamma(x' - y, t) & \text{for } (x', t) \in \partial\mathbb{R}^n_+ \times (0, \infty), \\ u(x, 0) = \Gamma(x - y^*, 0) = \delta(x - y^*) = 0 & \text{for } x \in \mathbb{R}^n_+, \end{cases}$$

by Poisson's formula for $\partial_t - \Delta$ in $\mathbb{R}^n_+$, we have

$$\int_0^t \int_\Sigma \Gamma(\xi' - y, s) P(x, \xi', t - s)\, d\xi'\, ds = \Gamma(x - y^*, t).$$

For $y \in \mathbb{R}^n_-$, $y^* \in \mathbb{R}^n_+$. Since $\Gamma(\xi' - y, s) = \Gamma(\xi' - y^*, s)$,

$$\begin{aligned} \int_0^t \int_\Sigma \Gamma(\xi' - y, s) P(x, \xi', t - s)\, d\xi'\, ds &= \int_0^t \int_\Sigma \Gamma(\xi' - y^*, s) P(x, \xi', t - s)\, d\xi'\, ds \\ &= \Gamma(x - y^{**}, t) = \Gamma(x - y, t). \end{aligned}$$

The combination of the two cases $y \in \mathbb{R}^n_+$ and $y \in \mathbb{R}^n_-$ gives (4.2). □

With Lemma 4.1 in hand, we are able to derive the second formula for W_{ij}.

Lemma 4.2 (The second formula for W_{ij}). *For $x, y \in \mathbb{R}^n_+$ and $i, j = 1, \ldots, n$,*

$$\begin{aligned} W_{ij}(x, y, t) = &-2\delta_{in}\delta_{nj}\Gamma(x - y^*, t) + 2\delta_{in}\epsilon_j \Gamma_{nj}(x - y^*, t) \\ &- 4\delta_{nj} C_i(x, y, t) - 4H_{ij}(x, y, t) + V_{ij}(x, y, t), \end{aligned} \tag{4.3}$$

where

$$C_i(x, y, t) = \int_0^{x_n} \int_\Sigma \partial_n \Gamma(x - y^* - z, t)\, \partial_i E(z)\, dz'\, dz_n, \tag{4.4}$$

$$H_{ij}(x, y, t) = -\int_{\mathbb{R}^n} \partial_{y_j} C_i(x, y + w, t) \partial_n E(w) dw, \tag{4.5}$$

and

$$V_{ij}(x, y, t) = -2\delta_{in}\Lambda_j(x, y, t) - 4 \int_0^{x_n} \int_\Sigma \partial_{x_n} \Lambda_j(x - z, y, t) \partial_i E(z)\, dz' dz_n. \tag{4.6}$$

Here

$$\Lambda_j(x, y, t) = \partial_{y_n} \partial_{y_j} \int_{w_n < -y_n} G^{ht}(x, y + w, t) E(w)\, dw, \tag{4.7}$$

where $G^{ht}(x, y, t) = \Gamma(x - y, t) - \Gamma(x - y^*, t)$ is the Green function of heat equation in $\mathbb{R}^n_+ \times (0, \infty)$. Note that $C_i(x, y, t)$ is defined in $\mathbb{R}^n_+ \times \mathbb{R}^n \times (0, \infty)$, and y_n is allowed to be negative.

Remark 4.1. We can show that C_i, H_{ij}, and V_{ij} are well defined using Lemma 2.2. The x'- and y'-derivatives are interchangeable for C_i H_{ij}, and V_{ij}: $\partial^l_{x'} C_i(x, y, t) = (-1)^l \partial^l_{y'} C_i(x, y, t)$ and similarly for H_{ij}, and V_{ij}.

Remark 4.2. The formula (4.3) is better than (3.9) because the definitions of the terms on the right side do not involve integration in time. If an integration in time was involved, there might be singularities at $s = 0, t$ when we use the estimates of K_{ij} and S_{ij} in (2.20) and (2.10), respectively. Their estimates would be worse and contain, for example, singularities in x_n for x_n small. The quantity $C_i(x, t)$ studied by Solonnikov [46, (66)] corresponds to our $C_i(x, 0, t)$ with $y = 0$ and he did not study full $C_i(x, y, t)$ with $y \neq 0$ nor $H_{ij}(x, y, t)$.

Remark 4.3. The formula (4.3) corresponds to that of the stationary case in [23, (2.36)]:

$$W_{ij}(x, y) = -\left(\delta_{in} - x_n \partial_{x_i}\right)\left(\delta_{nj} - y_n \partial_{y_j}\right) E(x - y^*).$$

Proof of Lemma 4.2. To obtain (4.3), we use the formulae (2.8) and (2.15) and split the integral of (3.9) into six parts as

$$\int_{-\infty}^{\infty} \int_\Sigma K_{in}(x - \xi', t - s) S_{nj}(\xi' - y, s)\, d\xi'\, ds = I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where

$$\begin{aligned}
I_1 &= -2\,\delta_{in}\delta_{nj} \int_{-\infty}^{\infty} \int_\Sigma \partial_n \Gamma(x - \xi', t - s)\Gamma(\xi' - y, s)\, d\xi' ds,\\
I_2 &= -4\,\delta_{nj} \int_{-\infty}^{\infty} \int_\Sigma \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \partial_n \Gamma(z, t - s)\, \partial_i E(x - \xi' - z)\, dz'\, dz_n \right]\\
&\quad \Gamma(\xi' - y, s)\, d\xi' ds,\\
I_3 &= -2\,\delta_{nj} \int_{-\infty}^{\infty} \int_\Sigma \partial_i E(x - \xi')\delta(t - s)\Gamma(\xi' - y, s)\, d\xi' ds,\\
I_4 &= -2\,\delta_{in} \int_{-\infty}^{\infty} \int_\Sigma \partial_n \Gamma(x - \xi', t - s) \int_{\mathbb{R}^n} \partial_n \partial_j \Gamma(\xi' - y - w, s) E(w)\, dw d\xi' ds,\\
I_5 &= -4 \int_{-\infty}^{\infty} \int_\Sigma \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \partial_n \Gamma(z, t - s)\, \partial_i E(x - \xi' - z)\, dz'\, dz_n \right]\\
&\quad \cdot \left[\int_{\mathbb{R}^n} \partial_n \partial_j \Gamma(\xi' - y - w, s) E(w)\, dw \right] d\xi' ds,\\
I_6 &= -2 \int_{-\infty}^{\infty} \int_\Sigma \partial_i E(x - \xi')\delta(t - s) \int_{\mathbb{R}^n} \partial_n \partial_j \Gamma(\xi' - y - w, s) E(w)\, dw\, d\xi' ds.
\end{aligned}$$

We use Lemma 4.1 to compute I_1, I_2, I_4, I_5. Indeed, we have

$$\begin{aligned}
I_1 &= -2\delta_{in}\delta_{nj} \int_0^t \int_\Sigma \partial_n \Gamma(x - \xi', t - s)\Gamma(\xi' - y, s)\, d\xi' ds\\
&= \delta_{in}\delta_{nj} \int_0^t \int_\Sigma P(x, \xi', t - s)\Gamma(\xi' - y, s)\, d\xi' ds\\
&= \delta_{in}\delta_{nj} \Gamma(x - y^\sharp, t) = \delta_{in}\delta_{nj} \Gamma(x - y^*, t),
\end{aligned}$$

where we used (4.1), Lemma 4.1 and $y \in \mathbb{R}^n_+$. And, by changing the variables and Fubini's theorem, we have

$$
\begin{aligned}
I_2 &= -4\,\delta_{nj} \int_0^t \int_\Sigma \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \partial_n \Gamma(x - \xi' - z, t - s)\, \partial_i E(z)\, dz'\, dz_n \right] \\
&\qquad \Gamma(\xi' - y, s)\, d\xi' ds \\
&= -4\,\delta_{nj} \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \int_0^t \left(\int_\Sigma \partial_n \Gamma(x - \xi' - z, t - s) \right.\right. \\
&\qquad \left.\left. \Gamma(\xi' - y, s)\, d\xi' ds \right) \partial_i E(z)\, dz' dz_n \right].
\end{aligned}
$$

With the aid of (4.1) and Lemma 4.1, we actually get

$$
\begin{aligned}
I_2 &= 2\,\delta_{nj} \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \left(\int_0^t \int_\Sigma P(x - z, \xi', t - s) \Gamma(\xi' - y, s)\, d\xi' ds \right) \partial_i E(z)\, dz' dz_n \right] \\
&= 2\,\delta_{nj} \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \Gamma(x - z - y^\sharp, t)\, \partial_i E(z)\, dz' dz_n \right] \\
&= 2\,\delta_{nj} \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \Gamma(x - z - y^*, t)\, \partial_i E(z)\, dz' dz_n \right] \quad (\text{since } y \in \mathbb{R}^n_+) \\
&= 2\,\delta_{nj} \int_\Sigma \Gamma(x' - z' - y^*, t)\, \partial_i E(z', x_n)\, dz' + 2\,\delta_{nj} C_i(x, y, t),
\end{aligned}
$$

where $C_i(x, y, t)$ is as defined in (4.4).

Moreover, rearranging the integrals and derivatives and using (4.1), we obtain

$$
\begin{aligned}
I_4 &= -2\,\delta_{in} \int_0^t \int_\Sigma \partial_n \Gamma(x - \xi', t - s) \int_{\mathbb{R}^n} \partial_n \partial_j \Gamma(\xi' - y - w, s) E(w)\, dw d\xi' ds \\
&= -2\,\delta_{in} \partial_{y_n} \partial_{y_j} \left[\int_0^t \int_\Sigma \partial_n \Gamma(x - \xi', t - s) \int_{\mathbb{R}^n} \Gamma(\xi' - y - w, s) E(w)\, dw d\xi' ds \right] \\
&= \delta_{in} \partial_{y_n} \partial_{y_j} \left[\int_{\mathbb{R}^n} \left(\int_0^t \int_\Sigma P(x, \xi', t - s) \Gamma(\xi' - (y + w), s) d\xi'\, ds \right) E(w)\, dw \right].
\end{aligned}
$$

Hence, applying Fubini's theorem and Lemma 4.1, we have

$$
\begin{aligned}
I_4 &= \delta_{in} \partial_{y_n} \partial_{y_j} \left[\int_{\mathbb{R}^n} \Gamma(x - (y + w)^\sharp, t) E(w)\, dw \right] \\
&= \delta_{in} \partial_{y_n} \partial_{y_j} \left[\int_{w_n > -y_n} \Gamma(x - (y + w)^*, t) E(w)\, dw \right. \\
&\qquad \left. + \int_{w_n < -y_n} \Gamma(x - y - w, t) E(w)\, dw \right] \\
&= \delta_{in} \partial_{y_n} \partial_{y_j} \left[\int_{\mathbb{R}^n} \Gamma(x - (y + w)^*, t) E(w)\, dw \right. \\
&\qquad \left. + \int_{w_n < -y_n} (\Gamma(x - y - w, t) - \Gamma(x - (y + w)^*, t)) E(w)\, dw \right] \\
&= -\delta_{in} \epsilon_j \Gamma_{nj}(x - y^*, t) + \delta_{in} \partial_{y_n} \partial_{y_j} \int_{w_n < -y_n} G^{ht}(x, y + w, t) E(w)\, dw \\
&= -\delta_{in} \epsilon_j \Gamma_{nj}(x - y^*, t) + \delta_{in} \Lambda_j(x, y, t),
\end{aligned}
$$

where $G^{ht}(x, y, t) = \Gamma(x - y, t) - \Gamma(x - y^*, t)$ is the Green function of heat equation in $\mathbb{R}^n_+ \times (0, \infty)$ and $\Lambda_j(x, y, t)$ is as defined in (4.7).

In addition, by changing the variables, Fubini's theorem and (4.1), we get

$$
\begin{aligned}
I_5 = &- 4 \int_0^t \int_\Sigma \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \partial_n \Gamma(x - \xi' - z, t - s)\, \partial_i E(z)\, dz' dz_n \right] \\
&\cdot \left[\int_{\mathbb{R}^n} \partial_n \partial_j \Gamma(\xi' - y - w, s) E(w)\, dw \right] d\xi' ds \\
= &\, 2\, \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \int_{\mathbb{R}^n} \partial_{y_n} \partial_{y_j} \left(\int_0^t \int_\Sigma P(x - z, \xi', t - s) \Gamma(\xi' - (y + w), s)\, d\xi' ds \right) \right. \\
&\left. \cdot\, \partial_i E(z) E(w)\, dw dz' dz_n \right].
\end{aligned}
$$

Thus, Lemma 4.1 implies

$$
\begin{aligned}
I_5 &= 2\, \partial_{x_n} \left[\int_0^{x_n} \int_\Sigma \int_{\mathbb{R}^n} \partial_{y_n} \partial_{y_j} \Gamma((x - z) - (y + w)^\sharp, t)\, \partial_i E(z) E(w)\, dw dz' dz_n \right] \\
&= 2 \int_\Sigma \int_{\mathbb{R}^n} \partial_{y_n} \partial_{y_j} \Gamma(x' - z' - (y + w)^\sharp, t)\, \partial_i E(z', x_n) E(w)\, dw dz' \\
&\quad + 2 \int_0^{x_n} \int_\Sigma \partial_{x_n} \left(\int_{\mathbb{R}^n} \partial_{y_n} \partial_{y_j} \Gamma((x - z) - (y - w)^\sharp, t) E(w)\, dw \right) \partial_i E(z)\, dz' dz_n \\
&= 2 \int_\Sigma \Gamma_{nj}(x' - z' - y, t)\, \partial_i E(z', x_n)\, dz' + 2 H^\sharp_{ij}(x, y, t),
\end{aligned}
$$

where we've used $\Gamma(x' - z' - (y + w)^\sharp, t) = \Gamma((x' - z' - y) - w, t)$, the functions Γ_{ij} is defined in (2.8), and $H^\sharp_{ij}$ is expanded as

$$
\begin{aligned}
H^\sharp_{ij}(x, y, t) = &\int_0^{x_n} \int_\Sigma \partial_{x_n} \partial_{y_n} \partial_{y_j} \left(\int_{w_n > -y_n} \Gamma((x - z) - (y + w)^*, t) E(w)\, dw \right. \\
&\left. + \int_{w_n < -y_n} \Gamma((x - z) - y - w, t) E(w)\, dw \right) \partial_i E(z)\, dz' dz_n \\
= &\int_0^{x_n} \int_\Sigma \partial_{x_n} \partial_{y_n} \partial_{y_j} \left(\int_{\mathbb{R}^n} \Gamma((x^* - z^* - y) - w, t) E(w)\, dw \right. \\
&\left. + \int_{w_n < -y_n} G^{ht}(x - z, y + w, t) E(w)\, dw \right) \partial_i E(z)\, dz' dz_n \\
= &\, H_{ij}(x, y, t) + \int_0^{x_n} \int_\Sigma \partial_{x_n} \Lambda_j(x - z, y, t) \partial_i E(z)\, dz' dz_n,
\end{aligned}
$$

where H_{ij} is defined in (4.5).

For I_3 and I_6, a direct computation gives

$$
\begin{aligned}
I_3 &= - 2\, \delta_{nj} \int_\Sigma \partial_i E(x - \xi') \Gamma(\xi' - y, t)\, d\xi' \\
&= - 2\, \delta_{nj} \int_\Sigma \Gamma(x' - z' - y^*, t)\, \partial_i E(z', x_n)\, dz',
\end{aligned}
$$

and

$$
\begin{aligned}
I_6 &= -2\int_\Sigma \partial_i E(x-\xi')\int_{\mathbb{R}^n} \partial_n\partial_j \Gamma(\xi'-y-w,t)E(w)\,dwd\xi' \\
&= -2\int_\Sigma \partial_i E(x-\xi')\Gamma_{nj}(\xi'-y,t)\,d\xi' \\
&= -2\int_\Sigma \Gamma_{nj}(x'-z'-y,t)\,\partial_i E(z',x_n)\,dz'.
\end{aligned}
$$

Combining the above computations of $I_1, \cdots, I_6$, and noting that I_3 cancels the first term of I_2 while I_6 cancels the first term of I_5, we get

$$
\begin{aligned}
\sum_{k=1}^{6} I_k = {}& \delta_{in}\delta_{nj}\Gamma(x-y^*,t) + 2\,\delta_{nj}C_i(x,y,t) - \delta_{in}\epsilon_j\Gamma_{nj}(x-y^*,t) \\
& + \delta_{in}\Lambda_j(x,y,t) + 2H^\sharp_{ij}(x,y,t).
\end{aligned}
$$

This completes the proof. □

We now explore a cancellation between C_i and H_{ij} in (4.3), and define

$$
\widehat{H}_{ij}(x,y,t) = H_{ij}(x,y,t) + \delta_{nj}C_i(x,y,t). \tag{4.8}
$$

Then (4.3) becomes

$$
\begin{aligned}
W_{ij}(x,y,t) = {}& -2\delta_{in}\delta_{nj}\Gamma(x-y^*,t) + 2\delta_{in}\epsilon_j\Gamma_{nj}(x-y^*,t) \\
& - 4\widehat{H}_{ij}(x,y,t) + V_{ij}(x,y,t).
\end{aligned} \tag{4.9}
$$

This formula will provide better estimates than summing estimates of individual terms in (4.3). See Remark 5.2 after Proposition 5.5.

We conclude a second formula for the Green tensor.

Lemma 4.3. *The Green tensor satisfies*

$$
\begin{aligned}
G_{ij}(x,y,t) = {}& \delta_{ij}\left[\Gamma(x-y,t) - \Gamma(x-y^*,t)\right] + \left[\Gamma_{ij}(x-y,t) - \epsilon_i\epsilon_j\Gamma_{ij}(x-y^*,t)\right] \\
& -4\widehat{H}_{ij}(x,y,t) + V_{ij}(x,y,t).
\end{aligned} \tag{4.10}
$$

Proof. Recall (3.10) that $G_{ij}(x,y,t) = S_{ij}(x-y,t) - \epsilon_j S_{ij}(x-y^*,t) + W_{ij}(x,y,t)$. By (2.8) and (4.9), we get the lemma. □

5. First Estimates of the Green Tensor

In this section, we first estimate $\widehat{H}_{ij}$, then estimate V_{ij}, and finally prove the Green tensor estimates in Proposition 1.1.

5.1. Estimates of $\widehat{H}_{ij}$.

Lemma 5.1. *For $i, j = 1, \dots, n$, we have*

$$\begin{cases} \widehat{H}_{ij}(x,y,t) = -D_{ijn}(x,y,t) & \text{if } j < n, \\ \widehat{H}_{in}(x,y,t) = \sum_{\beta<n} D_{i\beta\beta}(x,y,t) & \text{if } j = n, \end{cases} \tag{5.1}$$

where for $m = 1, \dots, n$,

$$D_{i\beta m}(x,y,t) = \int_0^{x_n} \int_\Sigma \partial_\beta \Gamma_{mn}(x^* - y - z^*, t)\, \partial_i E(z)\, dz' dz_n. \tag{5.2}$$

Proof. By definition,

$$\begin{aligned} H_{ij}(x,y,t) &= -\int_{\mathbb{R}^n} \partial_{y_j} C_i(x, y+w, t) \partial_n E(w)\, dw \\ &= -\int_{\mathbb{R}^n} \partial_{y_j} \left(\int_0^{x_n} \int_\Sigma \partial_n \Gamma(x - (y+w)^* - z, t) \partial_i E(z)\, dz' dz_n \right) \\ &\quad \partial_n E(w)\, dw. \end{aligned}$$

Integrating by parts in w_n and applying Fubini's theorem give, for $j = 1, \dots, n$,

$$\begin{aligned} H_{ij}(x,y,t) &= \int_0^{x_n} \int_\Sigma \partial_{y_j} \int_{\mathbb{R}^n} \partial_n^2 \Gamma((x^* - y - z^*) - w, t) E(w)\, dw\, \partial_i E(z)\, dz' dz_n \\ &= -\int_0^{x_n} \int_\Sigma \partial_j \Gamma_{nn}(x^* - y - z^*, t) \partial_i E(z)\, dz' dz_n = -D_{ijn}(x,y,t). \end{aligned}$$

This proves (5.1) when $j < n$. For $j = n$, we use the fact that $-\Delta E = \delta$ to obtain

$$\begin{aligned} H_{in}(x,y,t) &= -\int_{\mathbb{R}^n} \partial_{y_n} C_i(x, y+w, t) \partial_n E(w) dw \\ &= -C_i(x,y,t) + \sum_{\beta=1}^{n-1} \int_{\mathbb{R}^n} \partial_{y_\beta} C_i(x, y+w, t) \partial_\beta E(w)\, dw. \end{aligned}$$

Using the same argument above, we get

$$H_{in}(x,y,t) = -C_i(x,y,t) + \sum_{\beta=1}^{n-1} D_{i\beta\beta}(x,y,t).$$

This proves (5.1) when $j = n$. □

The following lemma enables us to change x_n-derivatives to x'-derivatives.

Lemma 5.2. *Let $i, j, m = 1, \dots, n$. For $i < n$,*

$$\partial_{x_n} D_{ijm}(x,y,t) = \partial_{x_i} D_{njm}(x,y,t) + \int_{\mathbb{R}^n} \partial_i \partial_j \partial_m B(x^* - y - w, t) \partial_n E(w)\, dw, \tag{5.3}$$

and for $i = n$,

$$\partial_{x_n} D_{njm}(x,y,t) = -\sum_{\beta=1}^{n-1} \partial_{x_\beta} D_{\beta jm}(x,y,t) - \frac{1}{2} \partial_n \Gamma_{jm}(x^* - y, t). \tag{5.4}$$

Proof. After changing variables, D_{ijm} becomes

$$D_{ijm}(x, y, t) = \int_{-x_n-y_n}^{-y_n} \int_{\Sigma} \partial_j \Gamma_{mn}(z, t)\, \partial_i E(x - y^* - z^*)\, dz' dz_n.$$

For $i < n$ we have

$$D_{ijm}(x, y, t) = \partial_{x_i} \int_{-x_n-y_n}^{-y_n} \int_{\Sigma} \partial_j \Gamma_{mn}(z, t)\, E(x - y^* - z^*)\, dz' dz_n.$$

Hence

$$\begin{aligned}
\partial_{x_n} D_{ijm}(x, y, t) &= \partial_{x_i} \int_{\Sigma} \partial_j \Gamma_{mn}(z', -x_n - y_n, t) E(x' - y' - z', 0)\, dz' \\
&\quad + \partial_{x_i} \int_{-x_n-y_n}^{-y_n} \int_{\Sigma} \partial_j \Gamma_{mn}(z, t) \partial_n E(x - y^* - z^*)\, dz' dz_n \\
&= I + \partial_{x_i} D_{njm}(x, y, t),
\end{aligned}$$

where

$$\begin{aligned}
I = \partial_{x_i} \int_{\Sigma} &\left(\int_{\mathbb{R}^n} \partial_j \partial_m \Gamma(z' - w', -x_n - y_n - w_n, t) \partial_n E(w)\, dw \right) \\
&E(x' - y' - z', 0)\, dz'.
\end{aligned}$$

After changing variables $\xi' = x' - y' - z'$ and applying Fubini theorem,

$$\begin{aligned}
I &= \int_{\mathbb{R}^n} \Big(\partial_{x_i} \int_{\Sigma} \partial_j \partial_m \Gamma(x' - y' - \xi' - w', -x_n \\
&\qquad -y_n - w_n, t) E(\xi', 0)\, d\xi' \Big) \partial_n E(w)\, dw \\
&= \int_{\mathbb{R}^n} \partial_{x_i} \partial_{y_j} \partial_{y_m} B(x^* - y - w, t) \partial_n E(w)\, dw \\
&= \int_{\mathbb{R}^n} \partial_i \partial_j \partial_m B(x^* - y - w, t) \partial_n E(w)\, dw
\end{aligned}$$

using $i < n$ again. This proves (5.3).

For (5.4), we first move normal derivatives in the definition (5.2) of D_{njm} to tangential derivatives. Observe that, using $\partial_j \Gamma_{mn} = \partial_n \Gamma_{jm}$,

$$\begin{aligned}
D_{njm}(x, y, t) &= \lim_{\varepsilon \to 0_+} \left[\int_{\varepsilon}^{x_n} \int_{\Sigma} \partial_{z_n} \Gamma_{jm}(x^* - y - z^*, t) \partial_n E(z)\, dz' dz_n \right] \\
&= \lim_{\varepsilon \to 0_+} \left[\int_{\Sigma} \Gamma_{jm}(x' - y' - z', -y_n, t) \partial_n E(z', x_n)\, dz' \right. \\
&\quad - \int_{\Sigma} \Gamma_{jm}(x' - y' - z', -x_n - y_n + \varepsilon, t) \partial_n E(z', \varepsilon)\, dz' \\
&\quad \left. - \int_{\varepsilon}^{x_n} \int_{\Sigma} \Gamma_{jm}(x^* - y - z^*, t) \partial_n^2 E(z)\, dz' dz_n \right],
\end{aligned}$$

by integration by parts in the z_n-variable. Using the fact that $-\Delta E = \delta$, we obtain

$$D_{njm}(x,y,t) = \partial_{y_j}\partial_{y_m}\int_{\mathbb{R}^n} e^{\frac{-(y_n+w_n)^2}{4t}}\,\partial_n A(x'-y'-w',x_n,t)E(w)\,dw \\ - \partial_{y_j}\partial_{y_m}\int_{\mathbb{R}^n} e^{-\frac{(x_n+y_n+w_n)^2}{4t}}\,\partial_n A(x'-y'-w',0_+,t)E(w)\,dw + J,$$

where

$$J = \sum_{\beta=1}^{n-1}\lim_{\varepsilon\to 0_+}\int_\varepsilon^{x_n}\int_\Sigma \Gamma_{mj}(x^*-y-z^*,t)\partial_\beta^2 E(z)\,dz'dz_n \\ = \sum_{\beta=1}^{n-1}\int_0^{x_n}\int_\Sigma \partial_\beta\Gamma_{mj}(x^*-y-z^*,t)\partial_\beta E(z)\,dz'dz_n,$$

by integration by parts in the z'-variable. Note that

$$\partial_n A(x',0_+,t) = \lim_{\varepsilon\to 0_+}\int_\Sigma \Gamma(x'-z',0,t)\partial_n E(z',\varepsilon)dz' = -\frac{1}{2}\,\Gamma(x',0,t)$$

since $-2\partial_n E(x)$ is the Poisson kernel for the Laplace equation in $\mathbb{R}^n_+$. Using $e^{-\frac{(x_n+y_n+w_n)^2}{4t}}$ $\Gamma(x'-y'-w',0,t) = \Gamma(x^*-y-w,t)$, we get

$$D_{njm}(x,y,t) = \partial_{y_j}\partial_{y_m}\int_{\mathbb{R}^n} e^{\frac{-(y_n+w_n)^2}{4t}}\,\partial_n A(x'-y'-w',x_n,t)E(w)\,dw \\ + \frac{1}{2}\,\Gamma_{mj}(x^*-y,t) \\ + \sum_{\beta=1}^{n-1}\int_0^{x_n}\int_\Sigma \partial_\beta\Gamma_{mj}(x^*-y-z^*,t)\partial_\beta E(z)\,dz'dz_n. \tag{5.5}$$

In this form we have moved normal derivatives in the definition (5.2) of D_{njm} to tangential derivatives. Consequently,

$$\partial_{x_n} D_{njm}(x,y,t) \\ = \partial_{y_j}\partial_{y_m}\int_{\mathbb{R}^n} e^{\frac{-(y_n+w_n)^2}{4t}}\,\partial_n^2 A(x'-y'-w',x_n,t)E(w)\,dw - \frac{1}{2}\,\partial_n\Gamma_{mj}(x^*-y,t) \\ + \sum_{\beta=1}^{n-1}\int_\Sigma \partial_\beta\Gamma_{mj}(x'-y'-z',-y_n,t)\partial_\beta E(z',x_n)\,dz' \\ - \sum_{\beta=1}^{n-1}\int_0^{x_n}\int_\Sigma \partial_n\partial_\beta\Gamma_{mj}(x^*-y-z^*,t)\partial_\beta E(z)\,dz'dz_n \\ = \partial_{y_j}\partial_{y_m}\int_{\mathbb{R}^n} e^{\frac{-(y_n+w_n)^2}{4t}}\,\partial_n^2 A(x'-y'-w',x_n,t)E(w)\,dw - \frac{1}{2}\,\partial_n\Gamma_{mj}(x^*-y,t)$$

$$+\sum_{\beta=1}^{n-1}\partial_{y_j}\partial_{y_m}\int_{\mathbb{R}^n}e^{-\frac{(y_n+w_n)^2}{4t}}\,\partial_\beta^2 A(x'-y'-w',x_n,t)E(w)\,dw$$
$$-\sum_{\beta=1}^{n-1}\partial_{x_\beta}D_{\beta jm}(x,y,t).$$

The first term cancels the third term since $\Delta_x A(x,t) = 0$ for $x_n > 0$. This proves (5.4). □

Remark 5.1. Note that Lemma 5.1 and (5.4) imply

$$\textstyle\sum_{i=1}^n \partial_{x_i}\widehat{H}_{ij}(x,y,t)=\frac{1}{2}\,\epsilon_j\partial_n\Gamma_{nj}(x-y^*,t)-\frac{1}{2}\,\delta_{nj}\partial_n\Gamma(x-y^*,t),$$

which is equivalent to $\sum_{i=1}^n \partial_{x_i}G_{ij}(x,y,t) = 0$ using Lemma 4.3. Since we will use (5.4) to prove (1.12), the property $\sum_{i=1}^n \partial_{x_i}G_{ij}(x,y,t) = 0$ cannot be used to improve (1.12). However, we will use it to prove (1.18).

The following lemma will be used in the x_n-derivative estimate of Proposition 5.5.

Lemma 5.3. *For $B(x,t)$ defined by (2.3), for $l,k\in\mathbb{N}_0$,*

$$\left|\int_{\mathbb{R}^n}\partial_{x'}^{l+1}\partial_{x_n}^k B(x-w,1)\partial_n E(w)\,dw\right|\lesssim\frac{1+\delta_{n2}\log\langle\delta_{k0}|x'|+|x_n|\rangle}{\langle x\rangle^{l+n-1}\langle x_n\rangle^k}. \tag{5.6}$$

Note in (5.6) $\delta_{k0}|x'| = 0$ for $k > 0$.

Recall that $\partial_{x'}^l\partial_{x_n}^k B$ satisfies (2.5)–(2.6) if $l + n \ge 3$, which is invalid if $l = 0$ and $n = 2$.

Proof. We will prove by induction in k. First consider $k = 0$ and full ∂E instead of just $\partial_n E$. Change variables and denote $J = \int_{\mathbb{R}^n}\partial_{w'}^{l+1}B(w,1)\partial E(x-w)\,dw$. By (2.5),

$$|J|\lesssim\int_{\mathbb{R}^n}\frac{dw}{\langle w\rangle^{l+n-1}\langle w_n\rangle|x-w|^{n-1}},$$

which is bounded for all x. We now assume $|x| > 10$ to show its decay. Decompose $\mathbb{R}^n$ to 4 regions: $\mathrm{I} = \{w : |w'| > 2|x|\}$, $\mathrm{II} = \{w : |w'| < 2|x|,\ |w_n| > |x|/2\}$, $\mathrm{III} = \{w : |x|/2 < |w'| < 2|x|,\ |w_n| < |x|/2\}$, and $\mathrm{IV} = \{w : |w'| < |x|/2,\ |w_n| < |x|/2\}$. Decompose

$$J=\left(\int_{\mathrm{I}}+\int_{\mathrm{II}}+\int_{\mathrm{III}}+\int_{\mathrm{IV}}\right)(\partial_{w'}^{l+1}B)(w,1)\partial E(x-w)\,dw=J_1+J_2+J_3+J_4.$$

Using (2.6),

$$|J_1|\lesssim\int_{\mathrm{I}}\frac{e^{-w_n^2/10}}{|w|^{l+n-1}\,|x-w|^{n-1}}\,dw\lesssim\int_{|w'|>2|x|}\frac{e^{-w_n^2/10}}{|w'|^{l+n-1}\,|w'|^{n-1}}\,dw=\frac{C}{|x|^{l+n-1}}.$$

Also by (2.6), and with $z' = x' - w'$,

$$|J_2| \lesssim \int_{\mathrm{II}} \frac{e^{-w_n^2/10}}{|x|^{l+n-1}\,|x-w|^{n-1}}\,dw$$
$$\lesssim \frac{1}{|x|^{l+n-1}} \int_{|w_n|\ge |x|/2} \int_{|z'|<3|x|} \frac{e^{-w_n^2/10}}{(|x_n-w_n|+|z'|)^{n-1}}\,dz'dw_n$$
$$= \frac{1}{|x|^{l+n-1}} \int_{|w_n|\ge |x|/2} \int_0^{3|x|} \frac{r^{n-2}}{(|x_n-w_n|+r)^{n-1}}\,dr\, e^{-w_n^2/10}\,dw_n.$$

By Lemma 2.1, the inner integral is bounded by $1+\log_+ \frac{3|x|}{|x_n-w_n|}$.

$$|J_2| \lesssim \frac{1}{|x|^{l+n-1}} \int_{|w_n|\ge |x|/2} \left(1+\log_+ \frac{3|x|}{|x_n-w_n|}\right) e^{-w_n^2/10}\,dw_n \lesssim \frac{1}{|x|^{l+n-1}}.$$

For J_3, if we have $\partial_n E(x-w) \sim \frac{x_n-w_n}{|x-w|^n}$ in the integrand, using (2.6) and Lemma 2.1,

$$|J_3| \lesssim \int_{\mathrm{III}} \frac{e^{-w_n^2/10}}{|x|^{l+n-1}} \frac{|x_n-w_n|}{(|x'-w'|+|x_n-w_n|)^n}\,dw$$
$$\lesssim \frac{1}{|x|^{l+n-1}} \int_{\mathbb{R}} |x_n-w_n| e^{-w_n^2/10} \int_0^{3|x|} \frac{r^{n-2}}{(|x_n-w_n|+r)^n}\,drdw_n$$
$$\lesssim \frac{1}{|x|^{l+n-1}} \int_{\mathbb{R}} |x_n-w_n| e^{-w_n^2/10} \frac{\min(|x_n-w_n|, 3|x|)^{n-1}}{|x_n-w_n|^n}\,dw_n$$
$$\lesssim \frac{1}{|x|^{l+n-1}}.$$

If we have $\partial_\beta E(x-w)$ with $\beta < n$ in J_3, and if $n \ge 3$, we integrate J_3 by parts in w_β,

$$J_3 = \int_{\mathrm{III}} \partial_{w'}^{l+2} B(w,1) E(x-w)\,dw + \int_\Gamma \partial_{w'}^{l+1} B(w,1) E(x-w)\,dS_w,$$

where $\Gamma = \left\{(w', w_n) \mid |w'| = |x|/2 \text{ or } |w'| = 2|x|,\ |w_n| < |x|/2\right\}$ is the lateral boundary of III. Now using (2.6) and that $|x-w| > c|x|$ on Γ,

$$|J_3| \le \int_{\mathrm{III}} \frac{e^{-w_n^2/10}}{|x|^{l+n}} \frac{1}{|x-w|^{n-2}}\,dw + \int_\Gamma \frac{e^{-w_n^2/10}}{|x|^{l+n-1}} \frac{1}{|x-w|^{n-2}} dS_w$$
$$\lesssim \int_{|w_n|<|x|/2} \frac{e^{-w_n^2/10}}{|x|^{l+n}} \left(\int_{|z'|<3|x|} \frac{dz'}{|z'|^{n-2}}\right) dw_n + \int_\Gamma \frac{e^{-w_n^2/10}}{|x|^{l+n-1}} \frac{1}{|x|^{n-2}}\,dS_w$$
$$\lesssim \frac{1}{|x|^{l+n-1}}.$$

If $\beta < n = 2$, integration by parts does not help. Directly estimating using Lemma 2.1 gives

$$|J_3| \lesssim \frac{1}{|x|^{l+n-1}} \int_{|w_n|<|x|/2} e^{-w_n^2/10} \int_0^{3|x|} \frac{1}{(|x_n-w_n|+r)}\,drdw_n$$
$$\lesssim \frac{1}{|x|^{l+n-1}} \int_{|w_n|<|x|/2} e^{-w_n^2/10} \left(1+\log \frac{3|x|}{|x_n-w_n|}\right) dw_n.$$

If $|x_n| \geq \frac{3}{4}|x|$ so that $|x_n - w_n| \geq \frac{1}{4}|x|$, the integral is of order one. If $|x_n| < \frac{3}{4}|x|$ so that $|x'| \geq c|x|$, the integral is bounded by $\log \langle x' \rangle$. Thus

$$|J_3| \lesssim \frac{1}{|x|^{l+n-1}} \left(1 + \delta_{n2} \log \langle x' \rangle\right).$$

Finally we consider J_4 in region IV. Denote $\Gamma = \{(w', w_n) : |w'| = |x|/2 \geq |w_n|\}$ the lateral boundary of IV. Integrating by parts repeatedly,

$$J_4 = \int_{\mathrm{IV}} B(w,1)\partial_{w'}^{l+1} \partial E(x-w)\, dw + \sum_{p=0}^{l} \int_{\Gamma} \partial_{w'}^{l-p} B(w,1)\, \partial_{w'}^{p} \partial E(x-w) \cdot \chi_p(w) dS_w$$

where χ_p are uniformly bounded functions on Γ depending on multi-index p. By (2.6), that $|x - w| > c|x|$ on IV and Γ, and $|w| > c|x|$ on Γ, and Lemma 2.1,

$$\begin{aligned} |J_4| &\leq \int_{\mathrm{IV}} \frac{e^{-w_n^2/10}}{\langle w \rangle^{n-2}} \frac{1}{|x|^{l+n}}\, dw + \sum_{p=0}^{l} \int_{\Gamma} \frac{e^{-w_n^2/10}}{|x|^{l-p+n-2}} \frac{1}{|x|^{p+n-1}} dS_w \\ &\lesssim \int_{|w_n|<|x|/2} \frac{e^{-w_n^2/10}}{|x|^{l+n}} \left(\int_{|z'|<3|x|} \frac{dz'}{|z'|^{n-2}}\right) dw_n + \int_{\Gamma} \frac{e^{-w_n^2/10}}{|x|^{l+2n-3}}\, dS_w \lesssim \frac{1}{|x|^{l+n-1}}. \end{aligned}$$

If $n = 2$, we do one less step in integration by parts,

$$\begin{aligned} J_4 = &\int_{\mathrm{IV}} \partial_{w'} B(w,1)\partial_{w'}^{l} \partial E(x-w)\, dw \\ &+ \sum_{p=0}^{l-1} \int_{\Gamma} \partial_{w'}^{l-p} B(w,1)\, \partial_{w'}^{p} \partial E(x-w) \cdot \chi_p(w) dS_w \end{aligned}$$

Thus for $n = 2$, by (2.6) and Lemma 2.1,

$$\begin{aligned} |J_4| &\leq \int_{\mathrm{IV}} \frac{e^{-w_n^2/10}}{|w|} \frac{1}{|x|^{l+n-1}}\, dw + \sum_{p=0}^{l-1} \int_{\Gamma} \frac{e^{-w_n^2/10}}{|x|^{l-p+n-2}} \frac{1}{|x|^{p+n-1}} dS_w \\ &\lesssim \int_{|w_n|<|x|/2} \frac{e^{-w_n^2/10}}{|x|^{l+n-1}} \left(\int_0^{|x|/2} \frac{dr}{|w_n| + r}\right) dw_n + \frac{1}{|x|^{l+n-1}} \\ &\lesssim \frac{1}{|x|^{l+n-1}} \left(1 + \int_{|w_n|<|x|/2} e^{-w_n^2/10} \left(1 + \log \frac{|x|}{|w_n|}\right) dw_n\right) \lesssim \frac{\log \langle x \rangle}{|x|^{l+n-1}}. \end{aligned}$$

Unlike $\log \langle x' \rangle$ for J_3, we need $\log \langle x \rangle$ for J_4.

Summing the estimates, we conclude for $k = 0$, for all $x \in \mathbb{R}^n$ and $n \geq 2$,

$$\left| \int_{\mathbb{R}^n} \partial_{x'}^{l+1} B(x-w,1) \partial E(w)\, dw \right| \lesssim \frac{1 + \delta_{n2} \log \langle x \rangle}{\langle x \rangle^{l+n-1}}. \tag{5.7}$$

Suppose now $k \geq 1$ and (5.6) has been proved for all $k' \leq k-1$. Thanks to $-\Delta E = \delta$, we can reduce the order of the x_n-derivative in the integral as

$$J = \int_{\mathbb{R}^n} (\partial_{x'}^{l+1} \partial_{x_n}^k B)(x-w,1)\partial_n E(w)\,dw$$
$$= (\partial_{x'}^{l+1} \partial_{x_n}^{k-1} B)(x,1) - \sum_{\beta_1=1}^{n-1} \int_{\mathbb{R}^n} (\partial_{x'}^{l+1} \partial_{x_n}^{k-1} \partial_{w_{\beta_1}} B)(x-w,1)\partial_{\beta_1} E(w)\,dw.$$

If $k=1$, (5.6) follows from (2.5) and (5.7),

$$|J| \lesssim |\partial_{x'}^{l+1} B(x,1)| + \frac{1+\delta_{n2}\log\langle x\rangle}{\langle x\rangle^{l+n}}$$
$$\lesssim \frac{e^{-x_n^2/10}}{\langle x\rangle^{l+n-1}} + \frac{1+\delta_{n2}\log(|x|+e)}{(|x|+e)^{l+n}} \lesssim \frac{1+\delta_{n2}\log(|x_n|+e)}{\langle x\rangle^{l+n-1}(|x_n|+e)}.$$

In the last inequality we have used that for $m \geq 1$

$$f(t) = t^{-m}\log t \quad \text{is decreasing in } t > e. \tag{5.8}$$

If $k \geq 2$, by integrating by parts, the second term becomes

$$\int_{\mathbb{R}^n} (\partial_{x'}^{l+1} \partial_{x_n}^{k-1} \partial_{w_{\beta_1}} B)(x-w,1)\partial_{\beta_1} E(w)\,dw$$
$$= \int_{\mathbb{R}^n} (\partial_{x'}^{l+1} \partial_{x_n}^{k-2} \partial_{w_{\beta_1}}^2 B)(x-w,1)\partial_n E(w)\,dw.$$

By (5.6) for $k' = k-2$, and (5.8) with $m=2$,

$$|J| \lesssim |\partial_{x'}^{l+1} \partial_{x_n}^{k-1} B(x,1)| + \frac{1+\delta_{n2}\log\langle x\rangle}{\langle x\rangle^{l+n+1}\langle x_n\rangle^{k-2}}$$
$$\lesssim \frac{e^{-x_n^2/10}}{\langle x\rangle^{l+n-1}} + \frac{1+\delta_{n2}\log(|x|+e)}{\langle x\rangle^{l+n+1}(|x_n|+e)^{k-2}} \lesssim \frac{1+\delta_{n2}\log\langle x_n\rangle}{\langle x\rangle^{l+n-1}\langle x_n\rangle^{k}}. \qquad \square$$

Lemma 5.4. *For $B(x,t)$ defined by (2.3), for $l, k \in \mathbb{N}_0$, for $\beta < n$,*

$$\left| \int_{\mathbb{R}^n} \partial_{x'}^{l+1} \partial_{x_n}^k B(x-w,1)\partial_\beta E(w)\,dw \right| \lesssim \frac{1+\delta_{n2}\log\langle \delta_{k\leq 1}|x'|+|x_n|\rangle}{\langle x\rangle^{l+n-\delta_{k0}}\langle x_n\rangle^{(k-1)_+}}. \tag{5.9}$$

Note in (5.9) $\delta_{k\leq 1}|x'| = 0$ for $k > 1$.

Proof. The case $k=0$ is proved in the proof for Lemma 5.3. When $k \geq 1$, we integrate by parts

$$J = \int_{\mathbb{R}^n} \partial_{x'}^{l+1} \partial_{x_n}^k B(x-w,1)\partial_\beta E(w)\,dw = \int_{\mathbb{R}^n} \partial_{x'}^{l+1} \partial_{x_n}^{k-1} \partial_\beta B(x-w,1)\partial_n E(w)\,dw.$$

By Lemma 5.3,

$$|J| \lesssim \frac{1+\delta_{n2}\log\langle \delta_{k\leq 1}|x'|+|x_n|\rangle}{\langle x\rangle^{l+n}\langle x_n\rangle^{k-1}}. \quad \square$$

The following is our estimates of derivatives of D_{ijm}.

Proposition 5.5. *For $x, y \in \mathbb{R}^n_+$, $l, k, q \in \mathbb{N}_0$, $i, m = 1, \dots, n$, and $j < n$, we have*

$$|\partial^l_{x',y'}\partial^k_{x_n}\partial^q_{y_n} D_{ijm}(x,y,1)| \lesssim \frac{1+\mu\delta_{n2}\log\langle \nu|x'-y'|+x_n+y_n\rangle}{\langle x-y^*\rangle^{l+k+n-\sigma}\langle x_n+y_n\rangle^{\sigma}\langle y_n\rangle^q}, \tag{5.10}$$

where $\sigma = (k+\delta_{mn}-\delta_{in}-1)_+$, $\mu = 1-\delta_{k0}-\delta_{k1}\delta_{in}$, and $\nu = \delta_{q0}\delta_{m<n}\delta_{k(1+\delta_{in})}$.

Remark 5.2. By a similar proof, we can show

$$|\partial^l_{x',y'}\partial^k_{x_n}\partial^q_{y_n} C_i(x,y,1)| \lesssim \frac{e^{-\frac{1}{30}y_n^2}}{\langle x-y^*\rangle^{l+n-1}\langle x_n+y_n\rangle^k\langle y_n\rangle^{q+1}}, \tag{5.11}$$

whose decay in x' is not as good as (5.10) since $\partial_n\Gamma$ in the definition of C_i has an additional ∂_n derivative than $\partial_j\Gamma_{mn}$ in the definition of D_{ijm}. This is why formula (4.9) for W_{ij} is preferred than (4.3). It is worth to note that the main term of G^*_{ij} in (1.9) is closely related to $\partial_{y_j}C_i$ (compare (1.16)). Henceforth, their estimates (1.10) and (5.11) are similar.

Proof. • **$\partial_{x',y'}$, ∂_{y_n}-estimate:** Recall the definition (5.2) of D_{ijm}. Changing the variables $w = x - y^* - z$ after taking derivatives, and using $j < n$,

$$\partial^l_{x',y'}\partial^q_{y_n} D_{ijm}(x,y,1) = \int_\Pi \partial^{l+1}_{w'}\partial^q_n \Gamma_{mn}(w,1)\,\partial_i E(x-y^*-w)\,dw$$

up to a sign, where $\Pi = \{w \in \mathbb{R}^n : y_n \le w_n \le x_n + y_n\}$. It is bounded for finite $|x-y^*|$, and to prove the estimate, we may assume $R = |x-y^*| > 100$. Decompose $\Pi = \Pi_1 + \Pi_2$ where

$$\Pi_1 = \Pi \cap \left\{|w| < \tfrac{3}{4}R\right\}, \quad \Pi_2 = \Pi \cap \left\{|w| > \tfrac{3}{4}R\right\}.$$

Integrating by parts in Π_1 with respect to w' iteratively, it equals

$$\begin{aligned} &= \int_{\Pi_1} \left(\partial^q_{w_n}\Gamma_{mn}(w,1)\right)\partial^{l+1}_{w'}\partial_i E(x-y^*-w)\,dw'dw_n \\ &+ \sum_{p=0}^{l}\int_{\Pi\cap\{|w|=\frac{3}{4}R\}} \left(\partial^{l-p}_{w'}\partial^q_{w_n}\Gamma_{mn}(w,1)\right)\partial^p_{w'}\partial_i E\cdot\chi_p(x-y^*-w)\,dS_w \\ &+ \int_{\Pi_2}\left(\partial^{l+1}_{w'}\partial^q_{w_n}\Gamma_{mn}(w,1)\right)\partial_i E(x-y^*-w)\,dw = I_1+I_2+I_3, \end{aligned}$$

where χ_p are bounded functions on the boundary. Estimate (2.10) and Lemma 2.1 imply

$$\begin{aligned} |I_1| &\lesssim \int_{y_n}^{x_n+y_n}\int_{\mathbb{R}^{n-1}} \frac{1}{(|w'|+w_n+1)^{q+n}R^{l+n}}\,dw'dw_n \\ &\lesssim \frac{1}{R^{l+n}}\int_{y_n}^{x_n+y_n}\frac{1}{(w_n+1)^{q+1}}\,dw_n \lesssim \frac{x_n}{R^{l+n}(y_n+1)^q(x_n+y_n+1)}. \end{aligned}$$

For I_2, estimate (2.10) gives

$$|I_2| \lesssim \sum_{p=0}^{l} \int_{|w|=\frac{3}{4}R} \frac{1}{\langle w\rangle^{l+q-p+n}} \frac{1}{|x-y^*-w|^{n+p-1}}\, dS_w$$
$$\lesssim \sum_{p=0}^{l} \frac{1}{R^{l+q-p+n} R^{n+p-1}} R^{n-1} \sim \frac{1}{R^{l+q+n}}$$

Using the estimate (2.10) and Lemma 2.2,

$$\begin{aligned}|I_3| &\lesssim \int_{\Pi_2} \frac{1}{\langle w\rangle^{l+q+n+1}} \frac{1}{|x-y^*-w|^{n-1}}\, dw \\ &\lesssim \frac{1}{R^{l+q+n+1/2}} \int_{y_n}^{x_n+y_n} \int_{\mathbb{R}^{n-1}} \frac{1}{(|w'|+w_n+1)^{1/2}(|x'-y'-w'|+(x_n+y_n-w_n))^{n-1}}\, dw' dw_n \\ &\lesssim \frac{1}{R^{l+q+n+1/2}} \int_{y_n}^{x_n+y_n} \left(R^{-1/2} + R^{-1/2} \log \frac{R}{(x_n+y_n-w_n)}\right) dw_n \\ &\sim \frac{x_n}{R^{l+q+n+1}} \left(1+\log\frac{R}{x_n}\right) \lesssim \frac{1}{R^{l+q+n}},\end{aligned}$$

noting $|x'-y'|+w_n+1+x_n+y_n-w_n \sim R$. Therefore, we conclude that for $i, m = 1, \dots, n$ and $j < n$,

$$|\partial_{x',y'}^l \partial_{y_n}^q D_{ijm}(x, y, 1)| \lesssim \frac{1}{\langle x-y^*\rangle^{l+n} \langle y_n\rangle^q}. \tag{5.12}$$

• $\boldsymbol{\partial_{x_n}}$**-estimate:** Note $j < n$ always. Also note that j and m in D_{ijm} are not changed in (5.3) and (5.4). For $k \ge 1$ and $i < n$, by (5.3) and Lemma 5.3,

$$\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q D_{ijm}(x, y, 1) \lesssim \partial_{x',y'}^{l+1} \partial_{x_n}^{k-1} \partial_{y_n}^q D_{njm}(x, y, 1) + \frac{\mathrm{LN}'}{\langle x-y^*\rangle^{l+n+1-\delta_{mn}} \langle x_n+y_n\rangle^{k+q-1+\delta_{mn}}}, \tag{5.13}$$

where

$$\mathrm{LN}' = 1 + \delta_{n2} \log \langle \nu|x'-y'| + x_n + y_n\rangle, \quad \nu = \delta_{0(k+q-1+\delta_{mn})} = \delta_{k1}\delta_{q0}\delta_{m<n}.$$

For $k \ge 1$ and $i = n$, by (5.4) and (2.10),

$$\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q D_{njm}(x, y, 1) \lesssim \partial_{x',y'}^{l+1} \partial_{x_n}^{k-1} \partial_{y_n}^q D_{\beta jm}(x, y, 1) + \frac{1}{\langle x-y^*\rangle^{l+n+k+q}}, \tag{5.14}$$

where $\beta < n$.The proof of (5.10) is then completed by induction in k using (5.13), (5.14) and the base case (5.12). □

Proposition 5.6. *For $x, y \in \mathbb{R}_+^n$, $t > 0$, $l, k, q, m \in \mathbb{N}_0$, $i, j = 1, \dots, n$, we have*

$$|\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q \partial_t^m \widehat{H}_{ij}(x, y, t)| \lesssim \frac{1 + \mu\, \delta_{n2} \left[\log(\nu|x'-y'| + x_n + y_n + \sqrt{t}) - \log(\sqrt{t})\right]}{t^m (|x^*-y|^2+t)^{\frac{l+k+n-\sigma}{2}} ((x_n+y_n)^2+t)^{\frac{\sigma}{2}} (y_n^2+t)^{\frac{q}{2}}}, \tag{5.15}$$

where $\sigma = (k-\delta_{in}-\delta_{jn})_+$, $\mu = 1-(\delta_{k0}+\delta_{k1}\delta_{in})\delta_{m0}$, *and* $\nu = \delta_{q0}\delta_{jn}\delta_{k(1+\delta_{in})}\delta_{m0}+\delta_{m>0}$.

Proof. From (5.1) and (5.10),

$$|\partial^l_{x',y'}\partial^k_{x_n}\partial^q_{y_n}\widehat{H}_{ij}(x,y,1)|\lesssim\frac{1+\mu\,\delta_{n2}\log\langle\nu|x'-y'|+x_n+y_n\rangle}{\langle x^*-y\rangle^{l+k+n-\sigma}\langle x_n+y_n\rangle^{\sigma}\langle y_n\rangle^q},\tag{5.16}$$

with corresponding σ, μ and ν. Note that $\widehat{H}_{ij}$ satisfies the scaling property

$$\widehat{H}_{ij}(x,y,t)=\frac{1}{t^{\frac{n}{2}}}\widehat{H}_{ij}\left(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}},1\right).\tag{5.17}$$

Therefore, (5.15) can be obtained by differentiating (5.17) in t and using (5.16). Indeed,

$$\begin{aligned}\partial^l_{x',y'}\partial^k_{x_n}\partial^q_{y_n}\partial^m_t\widehat{H}_{ij}(x,y,t)&=\left(\tfrac{\partial}{\partial t}\right)^m\left(t^{-\frac{l+k+q+n}{2}}\partial^l_{X',Y'}\partial^k_{X_n}\partial^q_{Y_n}\widehat{H}_{ij}\left(\tfrac{x}{\sqrt{t}},\tfrac{y}{\sqrt{t}},1\right)\right)\\ &\sim t^{-\frac{l+k+q+n}{2}-m}\left(1+\textstyle\sum_{p=1}^n\frac{x_p}{\sqrt{t}}\partial_{X_p}+\frac{y_p}{\sqrt{t}}\partial_{Y_p}\right)^m\partial^l_{X',Y'}\partial^k_{X_n}\partial^q_{Y_n}\widehat{H}_{ij}\left(\tfrac{x}{\sqrt{t}},\tfrac{y}{\sqrt{t}},1\right).\end{aligned}$$

Here we use $\frac{\partial}{\partial t}$ to indicate a total derivative, and ∂_{X_p} for a partial derivative in that position, e.g., $\frac{\partial}{\partial x}(f(ax,by))=a\partial_X f(ax,by)$. Note that $\frac{x_p}{\sqrt{t}}\partial_{X_p}$ and $\frac{y_p}{\sqrt{t}}\partial_{Y_p}$ do not change the decay estimate no matter $p<n$ or $p=n$, except that we take $\mu=\nu=1$ when $m>0$ for simplicity. This completes the proof of Proposition 5.6. □

5.2. Estimates of V_{ij}.

Lemma 5.7. *Let $V_{ij}(x,y,t)$ be defined by (4.6), $x,y\in\mathbb{R}^n_+$, $t>0$. For $i<n$,*

$$\begin{aligned}V_{ij}(x,y,t)=2\epsilon_j\int_0^{x_n}\int_{\mathbb{R}^n_+}\partial_{x_n}G^{ht}((x_n-z_n)e_n,w,t)\\ \partial_j\partial_i E(w+x'-y^*+z_ne_n)\,dw\,dz_n.\end{aligned}\tag{5.18}$$

For $i=n$,

$$\begin{aligned}V_{nj}(x,y,t)=-2\epsilon_j\sum_{\beta<n}\int_0^{x_n}\int_{\mathbb{R}^n_+}G^{ht}((x_n-z_n)e_n,w,t)\\ \partial_j\partial^2_\beta E(w+x'-y^*+z_ne_n)\,dw\,dz_n.\end{aligned}\tag{5.19}$$

Proof. First of all, by changing variables $\tilde{w}=(y+w)^*$ in definition (4.7),

$$\begin{aligned}\Lambda_j(x,y,t)&=\partial_{y_n}\partial_{y_j}\int_{\mathbb{R}^n_+}G^{ht}(x,\tilde{w}^*,t)E(\tilde{w}^*-y)\,d\tilde{w}\\ &=-\partial_{y_n}\partial_{y_j}\int_{\mathbb{R}^n_+}G^{ht}(x,\tilde{w},t)E(\tilde{w}-y^*)\,d\tilde{w}\\ &=-\partial_{y_j}\int_{\mathbb{R}^n_+}G^{ht}(x,\tilde{w},t)\partial_nE(\tilde{w}-y^*)\,d\tilde{w}.\end{aligned}\tag{5.20}$$

Decompose $V_{ij}(x,y,t)=V_{ij,1}(x,y,t)+V_{ij,2}(x,y,t)$, where $V_{ij,1}(x,y,t)=-2\delta_{in}\Lambda_j(x,y,t)$ and

$$V_{ij,2}(x,y,t)=-4\int_0^{x_n}\int_\Sigma\partial_{x_n}\Lambda_j(x-z,y,t)\partial_iE(z)\,dz'dz_n.\tag{5.21}$$

If $i < n$, integrating by parts,

$$V_{ij,2}(x,y,t) = 4\int_0^{x_n}\int_\Sigma \partial_{z_i}\partial_{x_n}\Lambda_j(x-z,y,t)E(z)\,dz'dz_n$$
$$= -4\partial_{x_i}\int_0^{x_n}\int_\Sigma \partial_{x_n}\Lambda_j(x-z,y,t)E(z)\,dz'dz_n.$$

From the third line of (5.20), changing variable $w = \tilde{w} - x'$ and using $G^{ht}(x, w+p', t) = G^{ht}(x-p', w, t)$ for any $p' \in \Sigma$,

$$\Lambda_j(x,y,t) = -\partial_{y_j}\int_{\mathbb{R}^n_+} G^{ht}(x_n e_n, w, t)\partial_n E(w + x' - y^*)\,dw. \tag{5.22}$$

Using (5.22),

$$V_{ij,2}(x,y,t)$$
$$= 4\partial_{x_i}\int_0^{x_n}\int_\Sigma \partial_{x_n}\left(\partial_{y_j}\int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, w, t)\partial_n E(w + x' - z' - y^*)\,dw\right)$$
$$E(z)\,dz'dz_n$$
$$= 2\partial_{x_i}\int_0^{x_n}\partial_{x_n}\partial_{y_j}\int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, w, t)$$
$$\left(2\int_\Sigma \partial_n E(w + x' - z' - y^*)E(z)\,dz'\right)dw\,dz_n.$$

Using the stationary Poisson formula (2.1), $w + x' - z' - y^* \in \mathbb{R}^n_+$ and $E(z) = E(z' - (z_n e_n))$,

$$\begin{aligned} V_{ij,2}(x,y,t) &= -2\partial_{x_i}\int_0^{x_n}\partial_{x_n}\partial_{y_j}\int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, w, t) \\ &\quad E(w + x' - y^* + z_n e_n)\,dw\,dz_n \\ &= -2\int_0^{x_n}\partial_{y_j}\int_{\mathbb{R}^n_+}\partial_{x_n}G^{ht}((x_n - z_n)e_n, w, t) \\ &\quad \partial_i E(w + x' - y^* + z_n e_n)\,dw\,dz_n. \end{aligned} \tag{5.23}$$

Since $V_{ij,1} = 0$ when $i < n$, we get (5.18).

If $i = n$,

$$V_{ij,2}(x,y,t) = -4\int_0^{x_n}\int_\Sigma \partial_{x_n}\Lambda_j(x-z,y,t)\partial_n E(z)\,dz'dz_n.$$

From the second line of (5.20), changing variable $w = \tilde{w} - x'$ and using $G^{ht}(x, w + p', t) = G^{ht}(x - p', w, t)$ for any $p' \in \Sigma$,

$$\begin{aligned} \Lambda_j(x,y,t) &= -\partial_{y_n}\partial_{y_j}\int_{\mathbb{R}^n_+} G^{ht}(x,\tilde{w},t)E(\tilde{w} - y^*)\,d\tilde{w} \\ &= -\partial_{y_n}\partial_{y_j}\int_{\mathbb{R}^n_+} G^{ht}(x_n e_n, w, t)E(w + x' - y^*)\,dw. \end{aligned} \tag{5.24}$$

Using (5.24),

$$
\begin{aligned}
&V_{ij,2}(x, y, t) \\
&= 4 \int_0^{x_n} \int_\Sigma \partial_{x_n} \left(\partial_{y_n} \partial_{y_j} \int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, w, t) E(w + x' - z' - y^*)\, dw \right) \\
&\quad \partial_n E(z)\, dz' dz_n \\
&= 2 \int_0^{x_n} \partial_{x_n} \partial_{y_n} \partial_{y_j} \int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, w, t) \left(2 \int_\Sigma E(w + x' - z' - y^*) \partial_n E(z)\, dz' \right) dw\, dz_n.
\end{aligned}
$$

By (2.1), one has

$$
\begin{aligned}
2 \int_\Sigma E(w + x' - z' - y^*) \partial_n E(z)\, dz' &= 2 \int_\Sigma E(z' - (w + x' - y^*)) \partial_n E(z_n e_n - z')\, dz' \\
&= -E(w + x' - y^* + z_n e_n)
\end{aligned}
$$

and

$$
\begin{aligned}
V_{ij,2}(x, y, t) &= -2 \int_0^{x_n} \partial_{x_n} \partial_{y_n} \partial_{y_j} \int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, w, t) \\
&\quad E(w + x' - y^* + z_n e_n)\, dw\, dz_n \\
&= -2 \int_0^{x_n} \partial_{y_j} \int_{\mathbb{R}^n_+} \partial_{x_n} G^{ht}((x_n - z_n)e_n, w, t) \partial_n \\
&\quad E(w + x' - y^* + z_n e_n)\, dw\, dz_n.
\end{aligned} \tag{5.25}
$$

Therefore, for all $1 \le i, j \le n$, including $i = n$ or $j = n$, we have (5.23). Integrating (5.23) by parts for $i = n$,

$$
\begin{aligned}
&V_{ij,2}(x, y, t) \\
&= 2\epsilon_j \int_0^{x_n} \int_{\mathbb{R}^n_+} \partial_{x_n} G^{ht}((x_n - z_n)e_n, w, t)\, \partial_j \partial_n E(w + x' - y^* + z_n e_n)\, dw\, dz_n \\
&= -2\epsilon_j \int_0^{x_n} \int_{\mathbb{R}^n_+} \partial_{z_n} G^{ht}((x_n - z_n)e_n, w, t)\, \partial_j \partial_n E(w + x' - y^* + z_n e_n)\, dw\, dz_n \\
&= -2\epsilon_j \int_{\mathbb{R}^n_+} G^{ht}(0, w, t)\, \partial_j \partial_n E(w + x - y^*)\, dw \\
&\quad + 2\epsilon_j \int_{\mathbb{R}^n_+} G^{ht}(x_n e_n, w, t)\, \partial_j \partial_n E(w + x' - y^*)\, dw \\
&\quad + 2\epsilon_j \int_0^{x_n} \int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, w, t)\, \partial_j \partial_n^2 E(w + x' - y^* + z_n e_n)\, dw\, dz_n \\
&= 0 - V_{ij,1}(x, y, t) + 2\epsilon_j \int_0^{x_n} \int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, w, t) \\
&\qquad \sum_{\beta < n} \partial_j \partial_n^2 E(w + x' - y^* + z_n e_n)\, dw\, dz_n.
\end{aligned}
$$

Then (5.19) follows from the above equation and $\partial_n^2 E = -\sum_{\beta<n} \partial_\beta^2 E$, completing the proof of the lemma. □

Proposition 5.8. *For $x, y \in \mathbb{R}^n_+$, $t > 0$, $l, k, q, m \in \mathbb{N}_0$, $i, j = 1, \dots, n$, we have*

$$|\partial^l_{x',y'} \partial^k_{x_n} \partial^q_{y_n} \partial^m_t V_{ij}(x, y, t)| \lesssim \frac{1}{t^m (|x - y^*|^2 + t)^{\frac{l+k-k_i+q+n}{2}} (x_n^2 + t)^{\frac{k_i}{2}}}, \quad k_i = (k - \delta_{in})_+. \tag{5.26}$$

Proof. • $\boldsymbol{\partial_{x',y'}}$, $\boldsymbol{\partial_{y_n}}$**-estimate:** We first estimate $V_{ij}(x, y, 1)$.

For $i = n$ and $j < n$, changing variable in (5.19), it follows that

$$V_{nj}(x, y, 1) = -2 \sum_{\beta < n} \int_0^{x_n} \int_{w_n < x_n - z_n} G^{ht}((x_n - z_n)e_n, (x_n - z_n)e_n - w, 1) \partial_j \partial^2_\beta E(w - x + y^*)\, dw\, dz_n. \tag{5.27}$$

We split the set $A := \{w \in \mathbb{R}^n : w_n < x_n - z_n\}$ into two disjoints sets denoted by

$$A_L = \left\{ w : |w - x + y^*| > \frac{|x - y^*|}{2} \right\} \cap A,$$
$$A_S = \left\{ w : |w - x + y^*| \le \frac{|x - y^*|}{2} \right\} \cap A.$$

For the region on A_L, it is direct that

$$\left| \int_0^{x_n} \int_{A_L} G^{ht}((x_n - z_n)e_n, (x_n - z_n)e_n - w, 1)\, \partial_j \partial^2_\beta E(w - x + y^*)\, dw\, dz_n \right| \le \frac{c}{|x - y^*|^{n+1}} \int_0^{x_n} \int_{A_L} \left| G^{ht}((x_n - z_n)e_n, (x_n - z_n)e_n - w, t) \right| dw\, dz_n. \tag{5.28}$$

On the other hand, on A_S, noting that $|w| > \frac{|x-y^*|}{2}$ and using the integration by parts, we estimate

$$\left| \int_0^{x_n} \int_{A_S} \partial_j \partial^2_\beta G^{ht}((x_n - z_n)e_n, (x_n - z_n)e_n - w, 1)\, E(w - x + y^*)\, dw\, dz_n \right| \le c e^{-c|x-y^*|^2} |x - y^*|^2 x_n \le \frac{c}{|x - y^*|^n}.$$

For $i = n$ and $j = n$, as similarly as the case $j \neq n$, it split the integral as follows:

$$V_{ij}(x, y, 1) = 2 \sum_{\beta < n} \int_0^{x_n} \int_{w_n < x_n - z_n} G^{ht}((x_n - z_n)e_n, (x_n - z_n)e_n - w, 1) \partial_n \partial^2_\beta\, E(w - x + y^*)\, dw\, dz_n$$
$$= 2 \sum_{\beta < n} \int_0^{x_n} \int_{A_L} \cdots dw\, dz_n. + 2 \sum_{\beta < n} \int_0^{x_n} \int_{A_S} \cdots dw\, dz_n.$$

The first term can be treated exactly the same way as (5.28), and thus the detail is skipped. For the second term, we use the integration by parts for only tangential derivatives, which gives

$$\left|2\sum_{\beta<n}\int_0^{x_n}\int_{A_S}\partial_\beta^2 G^{ht}((x_n-z_n)e_n,(x_n-z_n)e_n-w,1)\,\partial_n E(w-x+y^*)\,dw\,dz_n\right|$$
$$\le ce^{-c|x-y^*|^2}|x-y^*|x_n\le\frac{c}{|x-y^*|^n}.$$

For $i<n$, noting that $G^{ht}((x_n-z_n)e_n,w,1)=0$ if $z_n=x_n$, it follows via integration by parts in (5.18) that

$$\begin{aligned}V_{ij}(x,y,1)&=2\epsilon_j\int_{\mathbb{R}^n_+}G^{ht}(x_ne_n,w,1)\,\partial_j\partial_i E(w+x'-y^*)\,dw\\&\quad+2\epsilon_j\int_0^{x_n}\int_{\mathbb{R}^n_+}G^{ht}((x_n-z_n)e_n,w,1)\,\partial_n\partial_j\partial_i\\&\quad E(w+x'-y^*+z_ne_n)\,dw\,dz_n.\end{aligned}\tag{5.29}$$

If $j=n$, then the second term can be treated exactly the same way as the case $i=n$ and $j<n$, since $\partial_n^2\partial_i E(w+x'-y^*+z_ne_n)=-\sum_{k=1}^{n-1}\partial_k^2\partial_i E(w+x'-y^*+z_ne_n)$, and thus it remains to consider the first term. As before, due to change of variables, we rewrite it as

$$\begin{aligned}&-2\int_{\mathbb{R}^n_+}G^{ht}(x_ne_n,w,1)\,\partial_n\partial_i E(w+x'-y^*)\,dw\\&\quad=-2\int_{w_n<x_n}G^{ht}(x_ne_n,x_ne_n-w,1)\,\partial_n\partial_i E(w-x+y^*)\,dw\\&\quad=-2\int_{A_L}\cdots\,dw-2\int_{A_S}\cdots\,dw.\end{aligned}$$

Here we split the integral into two regions A_L and A_S with replacement of $A:=\{w\in\mathbb{R}^n : w_n<x_n\}$. The first term is rather direct that

$$\left|-2\int_{A_L}G^{ht}(x_ne_n,x_ne_n-w,1)\,\partial_n\partial_i E(w-x+y^*)\,dw\right|\le\frac{c}{|x-y^*|^n}.$$

For the second term, since $i<n$, by integration by parts, we have

$$-2\int_{A_S}\cdots\,dw=2\int_{A_S}\partial_{w_i}G^{ht}(x_ne_n,x_ne_n-w,1)\,\partial_n E(w-x+y^*)\,dw.$$

Therefore, we obtain

$$\begin{aligned}\left|2\int_{A_S}\partial_{w_i}G^{ht}(x_ne_n,x_ne_n-w,1)\,\partial_n E(w-x+y^*)\,dw\right|&\le ce^{-c|x-y^*|^2}|x-y^*|\\&\le\frac{c}{|x-y^*|^n}.\end{aligned}$$

If $j < n$, then the first term in (5.29) can be estimated similarly as the boundary term as the case $i < n$, $j = n$, and thus we omit the details. It remains to estimate the second term in (5.29). Using the change of variables and separating the domain, we have

$$
\begin{aligned}
2&\int_0^{x_n}\int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, w, 1)\, \partial_{z_n}\partial_{w_j}\partial_i E(w + x' - y^* + z_n e_n)\, dw\, dz_n \\
&= 2\int_0^{x_n}\int_{\mathbb{R}^n_+} G^{ht}((x_n - z_n)e_n, (x_n - z_n)e_n - w, 1)\, \partial_{z_n}\partial_{w_j}\partial_i E(w - x + y^*)\, dw\, dz_n \\
&= 2\int_0^{x_n}\int_{A_L} \cdots\, dw\, dz_n + 2\int_0^{x_n}\int_{A_S} \cdots\, dw\, dz_n.
\end{aligned}
$$

The first parts is controlled by $|x - y^*|^{-n}$, which can be shown as before, and thus we consider the only the second term. Since $i, j < n$, via the integration by parts, we obtain

$$
\begin{aligned}
&\left| 2\int_0^{x_n}\int_{A_S} \partial_{w_j}\partial_{w_i} G^{ht}((x_n - z_n)e_n, (x_n - z_n)e_n - w, 1)\, \partial_{z_n} E(w - x + y^*)\, dw\, dz_n \right| \\
&\quad \le ce^{-c|x-y^*|^2}|x - y^*|x_n \le \frac{c}{|x - y^*|^n}.
\end{aligned}
$$

Hence, for $i, j = 1, \ldots, n$, we have

$$
|V_{ij}(x, y, 1)| \lesssim \frac{1}{\langle x - y^* \rangle^n}. \tag{5.30}
$$

Any higher tangential derivative can be treated similar way as above. Furthermore, any order of normal derivative for y_n work out as well, with the aid of $\Delta E = 0$. Therefore, we conclude that for $i, j = 1, \ldots, n$,

$$|\partial^l_{x',y'}\partial^q_{y_n} V_{ij}(x,y,1)| \lesssim \frac{1}{\langle x-y^*\rangle^{l+q+n}}. \tag{5.31}$$

• ∂_{x_n}**-estimate:**

For $i<n$, it follows from (5.29) and $G^{ht}((x_n-z_n)e_n,w,1)=0$ if $z_n=x_n$ that

$$\begin{aligned}
&|\partial^k_{x_n} V_{ij}(x,y,1)|\\
&\quad\le \left|\int_{\mathbb{R}^n_+}\partial^k_{x_n}G^{ht}(x_ne_n,w,1)\,\partial_j\partial_i E(w+x'-y^*)\,dw\right|\\
&\qquad+\left|\int_{\mathbb{R}^n_+}\partial_{x_n}G^{ht}(0,w,1)\,\partial^{k-2}_{x_n}\partial_{z_n}\partial_j\partial_i E(w+x-y^*)\,dw\right|+\cdots\\
&\qquad+\left|\int_{\mathbb{R}^n_+}\partial^{k-1}_{x_n}G^{ht}(0,w,1)\,\partial_{z_n}\partial_j\partial_i E(w+x-y^*)\,dw\right|\\
&\qquad+\left|\int_0^{x_n}\int_{\mathbb{R}^n_+}\partial^k_{x_n}G^{ht}((x_n-z_n)e_n,w,1)\,\partial_{z_n}\partial_j\partial_i E(w+x'-y^*+z_ne_n)\,dw\,dz_n\right|\\
&\quad\lesssim e^{-\frac{x_n^2}{8}}|x-y^*|^{-n}+\cdots+|x-y^*|^{-n-k}\lesssim|x-y^*|^{-n}x_n^{-k}.
\end{aligned}$$

For $i=n$, using $G^{ht}((x_n-z_n)e_n,w,1)=0$ if $z_n=x_n$ in (5.19), we deduce

$$\begin{aligned}
\partial_{x_n}V_{nj}(x,y,1) = -2\epsilon_j\sum_{\beta<n}\int_0^{x_n}\int_{w_n>0}\partial_{x_n}G^{ht}((x_n-z_n)e_n,w,1)\,\partial_j\partial^2_\beta\\
E(w+x'-y^*+z_ne_n)\,dw\,dz_n.
\end{aligned}$$

Thus, it is readily to show that

$$|\partial_{x_n}V_{nj}(x,y,1)|\lesssim|x-y^*|^{-n-1}.$$

Similarly, for $k\ge 2$,

$$\begin{aligned}
&\partial^k_{x_n}V_{nj}(x,y,1)\\
&\quad=-2\epsilon_j\sum_{\beta<n}\int_{w_n>0}\partial^{k-1}_{x_n}G^{ht}(x_ne_n,w,t)\,\partial_j\partial^2_\beta E(w+x'-y^*)\,dw\\
&\qquad-2\epsilon_j\sum_{\beta<n}\int_{w_n>0}\partial^{k-2}_{x_n}G^{ht}(x_ne_n,w,t)\,\partial_j\partial^2_\beta\partial_n E(w+x'-y^*)\,dw-\cdots\\
&\qquad-2\epsilon_j\sum_{\beta<n}\int_{w_n>0}\partial_{x_n}G^{ht}(x_ne_n,w,t)\,\partial_j\partial^2_\beta\partial^{k-2}_n E(w+x'-y^*)\,dw\\
&\qquad-2\epsilon_j\sum_{\beta<n}\int_0^{x_n}\int_{w_n>0}\partial_{x_n}G^{ht}((x_n-z_n)e_n,w,t)\,\partial_j\partial^2_\beta\partial^2_n\\
&\qquad E(w+x'-y^*+z_ne_n)\,dw\,dz_n,
\end{aligned}$$

and thus,

$$\begin{aligned}
|\partial^k_{x_n}V_{nj}(x,y,1)| &\lesssim e^{-\frac{x_n^2}{8}}|x-y^*|^{-n-1}+e^{-\frac{x_n^2}{8}}|x-y^*|^{-n-2}+\cdots+|x-y^*|^{-n-k}\\
&\lesssim |x-y^*|^{-n}x_n^{-k}.
\end{aligned}$$

Therefore, we obtain

$$|\partial_{x_n}^k V_{ij}(x,y,1)| \lesssim \frac{1}{\langle x-y^*\rangle^{k-k_i+n} x_n^{k_i}}, \tag{5.32}$$

where $k_i = (k-\delta_{in})_+$.

Finally, Proposition 5.8 follows from (5.31), (5.32), and the scaling property

$$V_{ij}(x,y,t) = \frac{1}{t^{\frac{n}{2}}} V_{ij}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}, 1\right).$$

□

5.3. Proof of Proposition 1.1. We now prove Proposition 1.1.

Proof of Proposition 1.1. We first estimate the Green tensor G_{ij}, which satisfies the formula in Lemma 4.3. By (2.10) and Proposition 5.6, the estimates of G_{ij} is bounded by the sum of $\left(|x-y|^2+t\right)^{-\frac{l+k+q+n}{2}-m}$, those in Proposition 5.6 for $\widehat{H}_{ij}$ and those in Proposition 5.8 for V_{ij}. This shows (1.12).

We now estimate the pressure tensor g_j. Recall the decomposition formula (3.20) that $g_j = -F_j^y(x)\delta(t) + \widehat{w}_j$ in Proposition 3.5. For $t>0$, it suffices to estimate

$$\begin{aligned}
\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q \widehat{w}_j(x,y,t) \sim & -\sum_{i<n} 8 \int_0^t \int_\Sigma \partial_i \partial_n^{k+1} A(\xi', x_n, \tau) \partial_{x'}^l \partial_n^{q+1} \\
& S_{ij}(x'-y'-\xi', -y_n, t-\tau)\, d\xi' d\tau \\
& + \sum_{i<n} 4 \int_\Sigma \partial_{x'}^l \partial_{x_n}^k \partial_i E(x-\xi') \partial_n^{q+1} S_{ij}(\xi'-y,t)\, d\xi' \\
& + 8 \int_\Sigma \partial_n^{k+1} A(\xi', x_n, t) \partial_{x'}^l \partial_n^{q+1} \partial_j E(x'-y'-\xi', -y_n)\, d\xi' \\
=: & \,\mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

We first estimate I. Using (2.4) and (2.10), we get

$$\begin{aligned}
\mathrm{I} &\lesssim \int_0^t \int_\Sigma \frac{1}{\tau^{\frac{1}{2}}(|\xi'|+x_n+\sqrt{\tau})^{k+n}} \frac{1}{(|\xi'-(x'-y')|+y_n+\sqrt{t-\tau})^{l+q+n+1}}\, d\xi' d\tau \\
&= \left(\int_0^{t/2}\int_\Sigma + \int_{t/2}^t \int_\Sigma\right) \{\cdots\}\, d\xi' d\tau =: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

We have

$$|\mathrm{I}_1| \lesssim \int_\Sigma \left(\int_0^{t/2} \frac{1}{\tau^{\frac{1}{2}}(|\xi'|+x_n+\sqrt{\tau})^{k+n}}\, d\tau\right) \frac{1}{(|\xi'-(x'-y')|+y_n+\sqrt{t})^{l+q+n+1}}\, d\xi'.$$

Let

$$R = |x-y^*| + \sqrt{t}.$$

By Lemma 2.1 ($k > d$ case),

$$|\mathrm{I}_1| \lesssim \int_\Sigma \frac{\sqrt{t}}{(|\xi'|+x_n)^{k+n-1}(|\xi'|+x_n+\sqrt{t})} \frac{1}{(|\xi'-(x'-y')|+y_n+\sqrt{t})^{l+q+n+1}} d\xi'$$
$$\lesssim \int_\Sigma \frac{1}{(|\xi'|+x_n)^{k+n-1}} \frac{1}{(|\xi'-(x'-y')|+y_n+\sqrt{t})^{l+q+n+1}} d\xi'.$$

By Lemma 2.2,

$$|\mathrm{I}_1| \lesssim R^{-l-q-k-n-1} + \delta_{k0} R^{-l-q-n-1} \log\left(\frac{R}{x_n}\right) + \mathbb{1}_{k>0} R^{-l-q-n-1} x_n^{-k}$$
$$+ R^{-k-n+1}(y_n+\sqrt{t})^{-l-q-2}.$$

For I_2 and all $n \geq 2$, by Lemma 2.1,

$$|\mathrm{I}_2| \lesssim \int_\Sigma \frac{1}{t^{\frac{1}{2}}(|\xi'|+x_n+\sqrt{t})^{k+n}} \left(\int_{\frac{t}{2}}^t \frac{1}{(|\xi'-(x'-y')|+y_n+\sqrt{t-\tau})^{l+q+n+1}} d\tau\right) d\xi'$$
$$\lesssim \int_\Sigma \frac{1}{t^{\frac{1}{2}}(|\xi'|+x_n+\sqrt{t})^{k+n}}$$
$$\frac{t}{(|\xi'-(x'-y')|+y_n)^{l+q+n-1}(|\xi'-(x'-y')|^2+y_n^2+t)} d\xi'$$
$$\lesssim \int_\Sigma \frac{1}{(|\xi'|+x_n+\sqrt{t})^{k+n}} \frac{1}{(|\xi'-(x'-y')|+y_n)^{l+q+n}} d\xi'.$$

By Lemma 2.2,

$$|\mathrm{I}_2| \lesssim R^{-l-q-k-n-1} + R^{-l-q-n}(x_n+\sqrt{t})^{-k-1} + R^{-k-n} y_n^{-l-q-1}.$$

Now, we estimate II. Using the definition of E, (2.10) and Lemma 2.2, after integrating by parts, we get

$$|\mathrm{II}| \lesssim \int_\Sigma \frac{1}{(|\xi'|+x_n)^{k+n-1}} \frac{1}{(|\xi'-(x'-y')|+y_n+\sqrt{t})^{l+q+n+1}} d\xi',$$

which is similar to I_1. Hence

$$|\mathrm{I}+\mathrm{II}| \lesssim \delta_{k0} R^{-l-q-n-1} \log\left(\frac{R}{x_n}\right) + \mathbb{1}_{k>0} R^{-l-q-n-1} x_n^{-k} + R^{-l-q-n}(x_n+\sqrt{t})^{-k-1}$$
$$+ R^{-k-n+1}(y_n+\sqrt{t})^{-l-q-2} + R^{-k-n} y_n^{-l-q-1}.$$

Using (2.4) and the definition of E, we have

$$|\mathrm{III}| \lesssim \int_\Sigma \frac{1}{t^{\frac{1}{2}}(|\xi'|+x_n+\sqrt{t})^{k+n-1}} \frac{1}{(|\xi'-(x'-y')|+y_n)^{l+q+n}} d\xi'.$$

By Lemma 2.2,

$$|\mathrm{III}| \lesssim t^{-\frac{1}{2}} \left(\delta_{k0} R^{-l-q-n} \log \frac{R}{x_n+\sqrt{t}} + \mathbb{1}_{k>0} R^{-l-q-n}(x_n+\sqrt{t})^{-k} + R^{-k-n+1} y_n^{-l-q-1}\right).$$

We conclude

$$|\partial^l_{x',y'}\partial^k_{x_n}\partial^q_{y_n}\widehat{w}_j(x,y,t)|\lesssim t^{-\frac{1}{2}}\left(\delta_{k0}\frac{1}{R^{l+q+n}}\log\frac{R}{x_n}+\frac{1}{R^{l+q+n}x_n^k}+\frac{1}{R^{k+n-1}y_n^{l+q+1}}\right).$$

This proves estimate (1.14) and completes the proof of Proposition 1.1. □

Remark 5.3. The pressure tensor estimate (1.14) is sufficient for our proof of Proposition 1.4, and can be improved by several ways: One can get alternative estimates by integrating ξ' by parts in all three terms I, II and III to move decay exponents from y_n to x_n. Furthermore, we can rewrite the last term III using integration by parts and $\Delta E = 0$ as

$$\mathrm{III} = \begin{cases} 8\displaystyle\int_\Sigma \partial_j\partial_n A(\xi',x_n,t)\partial_n E(x'-y'-\xi',-y_n)\,d\xi' & \text{if } j<n,\\ \sum_{i<n} 8\int_\Sigma \partial_i\partial_n A(\xi',x_n,t)\partial_i E(x'-y'-\xi',-y_n)\,d\xi' & \text{if } j=n.\end{cases}$$

6. Restricted Green Tensors and Convergence to Initial Data

In this section we first study the restricted Green tensors acting on solenoidal vector fields, showing Theorem 1.2. In addition to the restricted Green tensor $\breve{G}_{ij}$ of Solonnikov given in (1.9), we also identify another restricted Green tensor $\widehat{G}_{ij}$ in (1.15).

We then use them to show the convergence to initial data in pointwise and L^q sense for solenoidal and general u_0 in Lemma 6.1 and Lemma 6.2, respectively. These show Theorem 1.3.

Proof of Theorem 1.2. Suppose div $u_0 = 0$ and $u_{0,n}|_\Sigma = 0$. Let

$$\begin{aligned} u_i^L(x,t) &= \sum_{j=1}^n \int_{\mathbb{R}^n_+} G_{ij}(x,y,t)u_{0,j}(y)\,dy,\\ \breve{u}_i^L(x,t) &= \sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t)u_{0,j}(y)\,dy, \quad \widehat{u}_i^L(x,t)\\ &= \sum_{j=1}^n \int_{\mathbb{R}^n_+} \widehat{G}_{ij}(x,y,t)u_{0,j}(y)\,dy. \end{aligned} \tag{6.1}$$

By Lemma 4.3,

$$\begin{aligned} u_i^L(x,t) = &\int_{\mathbb{R}^n_+}(\Gamma(x-y,t)-\Gamma(x-y^*,t))(u_0)_i(y)\,dy\\ &+\sum_{j=1}^n\int_{\mathbb{R}^n_+}\left(\Gamma_{ij}(x-y,t)-\epsilon_i\epsilon_j\Gamma_{ij}(x-y^*,t)\right)(u_0)_j(y)\,dy\\ &-4\sum_{j=1}^n\int_{\mathbb{R}^n_+}\widehat{H}_{ij}(x,y,t)(u_0)_j(y)\,dy+\sum_{j=1}^n\int_{\mathbb{R}^n_+}V_{ij}(x,y,t)(u_0)_j(y)\,dy\\ =:&\ I_1+I_2+I_3+I_4. \end{aligned} \tag{6.2}$$

Note that I_1 corresponds to the tensor $\delta_{ij}\left[\Gamma(x-y,t)-\Gamma(x-y^*,t)\right]$ in both (1.9) and (1.15). We claim $I_2+I_4=0$. Indeed, since $\Gamma_{ij}(x-y,t)-\epsilon_i\epsilon_j\Gamma_{ij}(x-y^*,t)+V_{ij}(x,y,t)=\partial_{y_j}T_i(x,y,t)$ with

$$\begin{aligned}T_i(x,y,t) = &\int_{\mathbb{R}^n}\partial_{y_i}\left[\Gamma(x-y-w,t)-\Gamma(x-y^*-w,t)\right]E(w)\,dw\\ &+\Bigg[-2\delta_{in}\partial_{y_n}\int_{w_n<-y_n}G^{ht}(x,y+w,t)E(w)\,dw\\ &-4\int_0^{x_n}\int_\Sigma\partial_{x_n}\left(\partial_{y_n}\int_{w_n<-y_n}G^{ht}(x-z,y+w,t)E(w)\,dw\right)dz'dz_n\Bigg],\end{aligned}$$

by (2.8), (4.6) and (4.7),

$$I_2+I_4=\sum_{j=1}^n\int_{\mathbb{R}^n_+}\partial_{y_j}T_i(x,y,t)(u_0)_j(y)\,dy=-\sum_{j=1}^n\int_{\mathbb{R}^n_+}T_i(x,y,t)\partial_{y_j}(u_0)_j(y)\,dy=0.$$

For I_3, by separating the sum over $j<n$ and $j=n$, and using Lemma 5.1,

$$I_3=-4\sum_{j<n}\int_{\mathbb{R}^n_+}(-D_{ijn})(x,y,t)(u_0)_j(y)\,dy-4\int_{\mathbb{R}^n_+}\sum_{\beta<n}D_{i\beta\beta}(x,y,t)(u_0)_n(y)\,dy.$$

Note that

$$\begin{aligned}&\sum_{j<n}\int_{\mathbb{R}^n_+}(-D_{ijn})(x,y,t)(u_0)_j(y)\,dy\\ &\quad=\sum_{j<n}\int_{\mathbb{R}^n_+}\partial_{y_j}\left(\int_0^{x_n}\int_\Sigma\Gamma_{nn}(x^*-y-z^*,t)\partial_iE(z)\,dz'dz_n\right)(u_0)_j(y)\,dy\\ &\quad=-\sum_{j<n}\int_{\mathbb{R}^n_+}\left(\int_0^{x_n}\int_\Sigma\Gamma_{nn}(x^*-y-z^*,t)\partial_iE(z)\,dz'dz_n\right)\partial_{y_j}(u_0)_j(y)\,dy\\ &\quad=\int_{\mathbb{R}^n_+}\left(\int_0^{x_n}\int_\Sigma\Gamma_{nn}(x^*-y-z^*,t)\partial_iE(z)\,dz'dz_n\right)\partial_{y_n}(u_0)_n(y)\,dy\\ &\quad=\int_{\mathbb{R}^n_+}\int_0^{x_n}\int_\Sigma\partial_n\Gamma_{nn}(x^*-y-z^*,t)\partial_iE(z)\,dz'dz_n\,(u_0)_n(y)\,dy\\ &\quad=\int_{\mathbb{R}^n_+}D_{inn}(x,y,t)(u_0)_n(y)\,dy.\end{aligned}$$

Hence

$$I_3=-4\int_{\mathbb{R}^n_+}\sum_{\beta=1}^n D_{i\beta\beta}(x,y,t)\,(u_0)_n(y)\,dy.$$

Since

$$\sum_{\beta=1}^{n} D_{i\beta\beta}(x,y,t) = \int_0^{x_n}\int_\Sigma \sum_{\beta=1}^{n} \partial_\beta \Gamma_{\beta n}(x^*-y-z^*,t)\partial_i E(z)\,dz'dz_n$$
$$= \int_0^{x_n}\int_\Sigma \sum_{\beta=1}^{n} \partial_{y_\beta}^2 \int_{\mathbb{R}^n} \partial_n \Gamma(x^*-y-z^*-w,t)$$
$$E(w)\,dw\,\partial_i E(z)\,dz'dz_n$$
$$= -\int_0^{x_n}\int_\Sigma \partial_n \Gamma(x^*-y-z^*,t)\partial_i E(z)\,dz'dz_n = +C_i(x,y,t),$$

where we used $-\Delta E = \delta$, (6.2) becomes

$$\sum_{j=1}^{n}\int_{\mathbb{R}^n_+} G_{ij}(x,y,t)(u_0)_j(y)\,dy$$
$$= \int_{\mathbb{R}^n_+} (\Gamma(x-y,t)-\Gamma(x-y^*,t))(u_0)_i(y)\,dy - 4\int_{\mathbb{R}^n_+} C_i(x,y,t)\,(u_0)_n(y)\,dy. \tag{6.3}$$

This gives (1.15). On the other hand,

$$I_3 = 4\int_{\mathbb{R}^n_+}\int_0^{x_n}\int_\Sigma \partial_n\Gamma(x^*-y-z^*,t)\partial_i E(z)\,dz'dz_n\,(u_0)_n(y)\,dy$$
$$= 4\int_{\mathbb{R}^n_+}\int_0^{x_n}\int_\Sigma \Gamma(x^*-y-z^*,t)\partial_i E(z)\,dz'dz_n\,\partial_n(u_0)_n(y)\,dy$$
$$= -4\sum_{\beta<n}\int_{\mathbb{R}^n_+}\int_0^{x_n}\int_\Sigma \Gamma(x^*-y-z^*,t)\partial_i E(z)\,dz'dz_n\,\partial_\beta(u_0)_\beta(y)\,dy$$
$$= -4\sum_{\beta<n}\int_{\mathbb{R}^n_+} J_{i\beta}(x,y,t)\cdot(u_0)_\beta(y)\,dy,$$

where for $\beta < n$

$$J_{i\beta} = \partial_{x_\beta}\int_0^{x_n}\int_\Sigma \Gamma(x^*-y-z^*,t)\partial_i E(z)\,dz'dz_n$$
$$= \partial_{x_\beta}\int_0^{x_n}\int_\Sigma \Gamma(z-y^*,t)\partial_i E(x-z)\,dz'dz_n.$$

we conclude that

$$\sum_{j=1}^{n}\int_{\mathbb{R}^n_+} G_{ij}(x,y,t)(u_0)_j(y)\,dy$$
$$= \int_{\mathbb{R}^n_+}(\Gamma(x-y,t)-\Gamma(x-y^*,t))(u_0)_i(y)\,dy \tag{6.4}$$
$$-4\sum_{\beta<n}\int_{\mathbb{R}^n_+} J_{i\beta}(x,y,t)\cdot(u_0)_\beta(y)\,dy,$$

which gives (1.9). This completes the proof of Theorem 1.2. □

Remark 6.1. Similar to Theorem 1.2, we have *restricted pressure tensors*. Let $f \in C_c^1(\overline{\mathbb{R}_+^n} \times \mathbb{R}; \mathbb{R}^n)$ be a vector field in $\mathbb{R}_+^n \times \mathbb{R}$ and $f = \mathbf{P} f$, i.e., $\operatorname{div} f = 0$ and $f_n|_\Sigma = 0$. Then

$$\begin{aligned}
&\sum_{j=1}^n \int_{-\infty}^{\infty} \int_{\mathbb{R}_+^n} g_j(x, y, t-s) f_j(y, s)\, dy\, ds \\
&\quad = \sum_{j=1}^n \int_{-\infty}^{t} \int_{\mathbb{R}_+^n} \breve{g}_j(x, y, t-s) f_j(y, s)\, dy\, ds \\
&\quad = \sum_{j=1}^n \int_{-\infty}^{t} \int_{\mathbb{R}_+^n} \widehat{g}_j(x, y, t-s) f_j(y, s)\, dy\, ds,
\end{aligned}$$

where

$$\breve{g}_j(x, y, t) = (\delta_{jn} - 1)\partial_{y_j} Q(x, y, t), \quad \widehat{g}_j(x, y, t) = \delta_{jn} \partial_{y_j} Q(x, y, t),$$

and

$$Q(x, y, t) = 4 \int_\Sigma \Big[E(x - \xi')\partial_n \Gamma(\xi' - y, t) + \Gamma(x' - y' - \xi', y_n, t)\partial_n E(\xi', x_n) \Big]\, d\xi'.$$

An equivalent formula of $\breve{g}_j$ appeared in Solonnikov [49, (2.4)], but no $\widehat{g}_j$. Both $\breve{g}_j$ and $\widehat{g}_j$ are functions and do not contain delta function in time. Note that $\widehat{g}_j(x, y, t) = \breve{g}_j(x, y, t) - \partial_{y_j} Q(x, y, t)$. We can get infinitely many restricted pressure tensors by adding to them any gradient field $\partial_{y_j} P(x, y, t)$. □

Lemma 6.1. *Let* $u_0 \in C_c^1(\overline{\mathbb{R}_+^n}; \mathbb{R}^n)$ *be a vector field in* $\mathbb{R}_+^n$ *and* $u_0 = \mathbf{P} u_0$, *i.e.,* $\operatorname{div} u_0 = 0$ *and* $u_{0,n}|_\Sigma = 0$. *Then for all* $i = 1, \ldots, n$, *and* $1 < q \le \infty$,

$$\lim_{t \to 0_+} \left\| u_{0,i}(x) - \textstyle\sum_{j=1}^n \int_{\mathbb{R}_+^n} G_{ij}(x, y, t) u_{0,j}(y)\, dy \right\|_{L_x^q(\mathbb{R}_+^n)} = 0. \tag{6.5}$$

Note that the exponent q in (6.5) includes ∞ but not 1.

Proof. Choose $R > 0$ so that $K = \left\{(x', x_n) \in \mathbb{R}^n : |x'| \le R, 0 \le x_n \le R\right\}$ contains the support of u_0. Since u_0 is uniformly continuous with compact support inside K,

$$\left\| \int_{\mathbb{R}_+^n} \Gamma(x - y, t) u_{0,i}(y)\, dy - u_{0,i}(x) \right\|_{L_x^q(\mathbb{R}_+^n)} + \left\| \int_{\mathbb{R}_+^n} \Gamma(x^* - y, t) u_{0,i}(y)\, dy \right\|_{L_x^q(\mathbb{R}_+^n)} \to 0 \tag{6.6}$$

as $t \to 0_+$ for all i. In view of (6.3), to show (6.5), it suffices to show

$$\lim_{t \to 0_+} \sup_{x \in \mathbb{R}_+^n} \| v_i(\cdot, t) \|_{L^q(\mathbb{R}_+^n)} = 0, \tag{6.7}$$

where

$$v_i(x, t) = \int_{\mathbb{R}_+^n} C_i(x, y, t) u_{0,n}(y)\, dy.$$

Note that $|u_{0,n}(y)| \le C y_n$, for some $C > 0$, since $u_{0,n}|_\Sigma = 0$ and $u_0 \in C_c^1(\overline{\mathbb{R}^n_+})$. Using estimate (5.11) for C_i, we have

$$\begin{aligned}|v_i(x,t)| &\le \int_K \frac{e^{-\frac{y_n^2}{30t}}}{\left(y_n+\sqrt{t}\right)\left(|x-y^*|+\sqrt{t}\right)^{n-1}}\,C y_n\,dy \\ &\lesssim \int_K \frac{1}{(|x-y|+\sqrt{t})^{n-1}} e^{-\frac{y_n^2}{30t}}\,dy = \int_{\mathbb{R}^n} f(x-y,t)g(y,t)\,dy,\end{aligned} \tag{6.8}$$

where

$$f(x,t) = \frac{1}{(|x|+\sqrt{t})^{n-1}}, \quad g(x,t) = e^{-\frac{x_n^2}{30t}} \mathbb{1}_K(x).$$

By Young's convolution inequality,

$$\|v_i(\cdot,t)\|_{L^q(\mathbb{R}^n_+)} \lesssim \|(f*g)(\cdot,t)\|_{L^q(\mathbb{R}^n)} \le \|f(\cdot,t)\|_{L^p(\mathbb{R}^n)} \|g(\cdot,t)\|_{L^r(\mathbb{R}^n)}$$

where

$$\frac{1}{p}+\frac{1}{r} = \frac{1}{q}+1, \qquad 1 \le p,q,r \le \infty.$$

We first compute L^p-norm of f. If $p > \frac{n}{n-1}$,

$$\|f(\cdot,t)\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \frac{1}{(|z|+\sqrt{t})^{(n-1)p}}\,dz\right)^{1/p} = C\sqrt{t}^{\frac{n}{p}-(n-1)}.$$

Next, we compute L^r-norm of g. We need $0 \le \frac{1}{p}-\frac{1}{q} < 1$ so that $1 \le r < \infty$.

$$\int_{\mathbb{R}^n} |g|^r \le \int_0^R \int_{B'_R} e^{-\frac{z_n^2}{30t}}\,dz'dz_n = CR^{n-1}\sqrt{t}\int_0^{\frac{R}{\sqrt{t}}} e^{-\frac{u^2}{30}}\,du \lesssim \sqrt{t}.$$

Hence $\|g(\cdot,t)\|_{L^r} \lesssim \sqrt{t}^{\frac{1}{r}}$, and

$$\|(f*g)(\cdot,t)\|_{L^q} \lesssim \sqrt{t}^{\frac{n}{p}-(n-1)+\frac{1}{r}} = \sqrt{t}^{\frac{1}{q}+1+(n-1)\left(\frac{1}{p}-1\right)}.$$

To have vanishing limit when $t \to 0_+$, we require $\frac{1}{q}+1+(n-1)\left(\frac{1}{p}-1\right) > 0$.

When $q \in (\frac{n}{n-1},\infty]$, we can choose $p \in (\frac{n}{n-1}, \min(q, \frac{n-1}{n-2}))$ so that all conditions on p,

$$p > \frac{n}{n-1}, \quad 0 \le \frac{1}{p}-\frac{1}{q} < 1, \quad \frac{1}{q}+1+(n-1)\left(\frac{1}{p}-1\right) > 0$$

are satisfied. This shows (6.7) for all $q \in (\frac{n}{n-1},\infty]$.

For the small q case, let

$$u_i^*(x,t) = \int_{\mathbb{R}^n_+} G_{ij}^*(x,y,t)u_{0,j}(y)\,dy,$$

where G^*_{ij} is given in the (1.9), and is the sum of the last terms of (1.9). It suffices to show

$$\lim_{t\to 0} \left\| u^*_i(x,t) \right\|_{L^q_x(\mathbb{R}^n_+)} = 0.$$

By estimate (1.10), $|G^*_{ij}(x,y,t)| \lesssim e^{-\frac{Cy_n^2}{t}} \left(|x^*-y|^2+t\right)^{-\frac{n}{2}}$. For $1<q<\infty$, using the Minkowski's inequality,

$$\begin{aligned}
\left\| u^*_i(x,t) \right\|_{L^q_x(\mathbb{R}^n_+)} &\lesssim \int_{\mathbb{R}^n_+} \left(\int_{\mathbb{R}^n_+} \left| G^*_{ij}(x,y,t) \right|^q dx \right)^{\frac{1}{q}} |u_0(y)|\, dy \\
&\lesssim \int_{\mathbb{R}^n_+} \left(\int_{\mathbb{R}^n_+} \frac{dx}{\left(|x^*-y|^2+t\right)^{\frac{nq}{2}}} \right)^{\frac{1}{q}} e^{-\frac{Cy_n^2}{t}} |u_0(y)|\, dy \\
&\lesssim \int_0^R \int_{|y'|<R} \frac{1}{(y_n+\sqrt{t})^{\frac{n(q-1)}{q}}} e^{-\frac{Cy_n^2}{t}}\, dy'\, dy_n \\
&\lesssim t^{\frac{1}{2}\left(1-\frac{n(q-1)}{q}\right)} \int_0^{R/\sqrt{t}} \frac{1}{(z_n+1)^{\frac{n(q-1)}{q}}} e^{-Cz_n^2}\, dz_n,
\end{aligned}$$

where $y_n = \sqrt{t} z_n$. Therefore, if $1<q<\frac{n}{n-1}$, then the right hand side goes to zero as $t\to 0_+$.

The case $q=\frac{n}{n-1}$ can be obtained using the previous cases and the Hölder inequality. This finishes the proof of Lemma 6.1. □

Remark 6.2. In the proof of Lemma 6.1, we have used $\widehat{G}_{ij}$ for large q and $\breve{G}_{ij}$ for small q. We do not use $\widehat{G}_{ij}$ for small q because the estimate (6.8) for v_i does not have enough decay in x. We can not use $\breve{G}_{ij}$ for $q=\infty$ because, although the pointwise estimate of $u^*_i(x,t)$ using (1.10) converges to 0 as $t\to 0$ for each $x\in\mathbb{R}^n_+$, it is not uniform in x. In contrast, it is uniform for v_i thanks to $|u_{0,n}(y)|\le Cy_n$.

Lemma 6.2. *Let u_0 be a vector field in $\mathbb{R}^n_+$, $u_0\in L^q(\mathbb{R}^n_+)$, $1<q<\infty$, and let $u_i(x,t)=\sum_{j=1}^n \int_{\mathbb{R}^n_+} G_{ij}(x,y,t)u_{0,j}(y)\,dy$. Then $u(x,t)\to(\mathbf{P}u_0)(x)$ in $L^q(\mathbb{R}^n_+)$.*

This lemma does not assume $u=\mathbf{P}u$, and implies (1.7).

Proof. Since the Helmholtz projection $\mathbf{P}$ is bounded in $L^q(\mathbb{R}^n_+)$, we also have $\mathbf{P}u_0\in L^q(\mathbb{R}^n_+)$. For any $\varepsilon>0$, choose $a=\mathbf{P}a\in C^\infty_c(\overline{\mathbb{R}^n_+};\mathbb{R}^n)$ with $\|a-\mathbf{P}u_0\|_{L^q}\le\varepsilon$. Such a may be obtained by first localizing $\mathbf{P}u_0$ using a Bogovskii map, and then mollifying the extension defined in (3.4) of the localized vector field. Let $v_i(x,t)=\sum_{j=1}^n\int_{\mathbb{R}^n_+} G_{ij}(x,y,t)a_j(y)\,dy$. By Lemma 6.1, there is $t_\varepsilon>0$ such that

$$\|v(\cdot,t)-a\|_{L^q(\mathbb{R}^n)}\le\varepsilon,\quad \forall t\in(0,t_\varepsilon).$$

By L^q estimate (9.1) in Lemma 9.1, $\|u(t)-v(t)\|_{L^q}\le C\,\|\mathbf{P}u_0-a\|_{L^q}\le C\varepsilon$. Hence

$$\|u(t)-\mathbf{P}u_0\|_{L^q}\le\|u(t)-v(t)\|_{L^q}+\|v(t)-a\|_{L^q}+\|a-\mathbf{P}u_0\|_{L^q}\le C\varepsilon$$

for $t\in(0,t_\varepsilon)$. This shows L^q-convergence of $u(t)$ to $\mathbf{P}u_0$. □

Proof of Theorem 1.3. Part (a) is by Lemma 3.4. Part (b) is by Lemma 6.2. Part (c) is by Lemma 6.1. □

7. The Symmetry of the Green Tensor

In this section we prove Proposition 1.4, i.e., the symmetry of the Green tensor of the Stokes system in the half-space,

$$G_{ij}(x, y, t) = G_{ji}(y, x, t), \quad \forall x, y \in \mathbb{R}^n_+, \ \forall t \in \mathbb{R} \setminus \{0\}. \tag{7.1}$$

In the Green tensor formula in Lemma 4.3, this symmetry property is valid for the first three terms but unclear for the last two terms $-\epsilon_i \epsilon_j \Gamma_{ij}(x - y^*, t) - 4\widehat{H}_{ij}(x, y, t)$. To prove it rigorously, we will use its regularity away from the singularity, bounds on spatial decay, and estimates near the singularity from the previous sections. For example, without the pointwise bound in Proposition 1.1, the bound (7.4) is unclear, and it will take extra effort to show their zero limits as $\epsilon \to 0$.

Denote $G^y_{ij}(z, \tau) = G_{ij}(z, y, \tau)$ and $g^y_j(z, \tau) = g_j(z, y, \tau) = \widehat{w}^y_j(z, \tau) - F^y_j(z)\delta(\tau)$ by Proposition 3.5. Equation (3.1) reads: For fixed $j = 1, 2, \cdots, n$ and $y \in \mathbb{R}^n_+$,

$$\partial_\tau G^y_{ij} - \Delta_z G^y_{ij} + \partial_{z_i} g^y_j = \delta_{ij}\delta_y(z)\delta(\tau), \quad \sum_{i=1}^n \partial_{z_i} G^y_{ij} = 0, \quad (z, \tau) \in \mathbb{R}^n_+ \times \mathbb{R}, \tag{7.2}$$

and $G^y_{ij}(z', 0, \tau) = 0$. Denote $U := \mathbb{R}^n_+ \times \mathbb{R}$ and

$$Q^{y,t}_\epsilon = B^y_\epsilon \times (t - \epsilon, \ t + \epsilon).$$

The inward normal ν_z on $\partial Q^{y,t}_\epsilon$ is defined on its lateral boundary as

$$\nu_i(z, \tau) = -\frac{z_i - y_i}{|z - y|}.$$

Lemma 7.1. *For $j = 1, \ldots, n$, $y \in \mathbb{R}^n_+$, $t > 0$, and all $f \in C^\infty(\mathbb{R}^n_+ \times [0, t]; \mathbb{R}^n)$, we have*

$$\begin{aligned} f_j(y, 0) = \lim_{\epsilon \to 0_+} \sum_{k=1}^n \Bigg[&\int_{|z-y|=\epsilon} F^y_j(z) f_k(z, 0)\nu_k \, dS_z \\ &- \int_0^\epsilon \int_{|z-y|=\epsilon} G^y_{kj}(z, \tau)\nabla_z f_k(z, \tau) \cdot \nu_z \, dS_z d\tau \\ &+ \int_0^\epsilon \int_{|z-y|=\epsilon} (\nabla_z G^y_{kj}(z, \tau) \cdot \nu_z) f_k(z, \tau) \, dS_z d\tau \\ &- \int_0^\epsilon \int_{|z-y|=\epsilon} \widehat{w}^y_j(z, \tau) f_k(z, \tau)\nu_k \, dS_z d\tau \Bigg]. \end{aligned} \tag{7.3}$$

Proof. We first assume $f \in C^\infty_c(\mathbb{R}^n_+ \times \mathbb{R}; \mathbb{R}^n)$. By the defining property (7.2) of Green tensor, we have

$$\begin{aligned} f_j(y, 0) = &\sum_{k=1}^n \int_U \Big[G^y_{kj}(z, \tau)(-\partial_\tau f_k(z, \tau) - \Delta_z f_k(z, \tau)) - \widehat{w}^y_j(z, \tau)\partial_{z_k} f_k(z, \tau) \Big] dz \, d\tau \\ &+ \int_{\mathbb{R}^n_+} F^y_j(z) \operatorname{div} f(z, 0) \, dz. \end{aligned}$$

Separating the domain of the first integral, we have

$$
\begin{aligned}
f_j(y,0) = \lim_{\epsilon\to 0_+} \sum_{k=1}^{n} \int_{U\setminus Q_\epsilon^{y,0}} & \Big[G_{kj}^y(z,\tau)(-\partial_\tau f_k(z,\tau) - \Delta_z f_k(z,\tau)) \\
& -\widehat{w}_j^y(z,\tau)\partial_{z_k} f_k(z,\tau)\Big]\, dz\, d\tau \\
& + \int_{\mathbb{R}^n_+} F_j^y \operatorname{div} f(z,0)\, dz \\
= \lim_{\epsilon\to 0_+} \sum_{k=1}^{n} & \left(\int_0^\infty \int_{|z-y|>\epsilon} + \int_\epsilon^\infty \int_{|z-y|<\epsilon} \right) [\cdots]\, dz\, d\tau \\
& + \int_{\mathbb{R}^n_+} F_j^y(z) \operatorname{div} f(z,0)\, dz.
\end{aligned}
$$

Here we have used the fact that $G_{kj}^y(z,\tau) = w_j^y(z,\tau) = 0$ for $\tau < 0$.

Integrating by parts and using $f \in C_c^\infty(\mathbb{R}^n_+ \times \mathbb{R})$, we get

$$
\begin{aligned}
f_j(y,0) = \lim_{\epsilon\to 0_+} \sum_{k=1}^{n} \Bigg[& \int_{|z-y|>\epsilon} G_{kj}^y(z,0_+) f_k(z,0)\, dz \\
& + \int_0^\infty \int_{|z-y|>\epsilon} \partial_\tau G_{kj}^y(z,\tau) f_k(z,\tau)\, dz d\tau \\
& + \int_0^\infty \int_{|z-y|=\epsilon} \Big\{ -G_{kj}^y(z,\tau)\nabla_z f_k(z,\tau) + f_k(z,\tau)\nabla_z G_{kj}^y(z,\tau) \Big\} \cdot \nu_z\, dS_z d\tau \\
& - \int_0^\infty \int_{|z-y|>\epsilon} \Delta_z G_{kj}^y(z,\tau) f_k(z,\tau)\, dz d\tau \\
& - \int_0^\infty \int_{|z-y|=\epsilon} \widehat{w}_j^y(z,\tau) f_k(z,\tau)\nu_k\, dS_z d\tau \\
& + \int_0^\infty \int_{|z-y|>\epsilon} \partial_{z_k} \widehat{w}_j^y(z,\tau) f_k(z,\tau)\, dz d\tau \\
& + \int_{|z-y|<\epsilon} G_{kj}^y(z,\epsilon) f_k(z,\epsilon)\, dz + \int_\epsilon^\infty \int_{|z-y|<\epsilon} \partial_\tau G_{kj}^y(z,\tau) f_k(z,\tau)\, dz d\tau \\
& + \int_\epsilon^\infty \int_{|z-y|=\epsilon} \Big\{ -G_{kj}^y(z,\tau)\nabla_z f_k(z,\tau) + f_k(z,\tau)\nabla_z G_{kj}^y(z,\tau) \Big\} \\
& \cdot (-\nu_z)\, dS_z d\tau - \int_\epsilon^\infty \int_{|z-y|<\epsilon} \Delta_z G_{kj}^y(z,\tau) f_k(z,\tau)\, dz d\tau \\
& - \int_\epsilon^\infty \int_{|z-y|=\epsilon} \widehat{w}_j^y(z,\tau) f_k(z,\tau)(-\nu_k)\, dz d\tau \\
& + \int_\epsilon^\infty \int_{|z-y|<\epsilon} \partial_{z_k} \widehat{w}_j^y(z,\tau) f_k(z,\tau)\, dz d\tau \\
& + \left(\int_{|z-y|<\epsilon} + \int_{|z-y|>\epsilon} \right) F_j^y(z) \partial_{z_k} f_k(z,0)\, dz \Bigg].
\end{aligned}
$$

Note that $\partial_\tau G^y_{kj} - \Delta_z G^y_{kj} + \partial_{z_k} \widehat{w}^y_j = \partial_\tau G^y_{kj} - \Delta_z G^y_{kj} + \partial_{z_k} g^y_j = 0$ for $\tau > 0$ and that $G^y_{kj}(z, 0_+) = \partial_k F^y_j(z)$ if $y \neq z$. Therefore, after combining and integrating by parts the sum of the first term and the last term,

$$
\begin{aligned}
f_j(y,0) = \lim_{\epsilon \to 0_+} \sum_{k=1}^{n} \Bigg[& \int_{|z-y|=\epsilon} F^y_j(z) f_k(z,0) \nu_k \, dS_z \\
& - \int_0^\epsilon \int_{|z-y|=\epsilon} G^y_{kj}(z,\tau) \nabla_z f_k(z,\tau) \cdot \nu_z \, dS_z d\tau \\
& + \int_0^\epsilon \int_{|z-y|=\epsilon} (\nabla_z G^y_{kj}(z,\tau) \cdot \nu_z) f_k(z,\tau) \, dS_z d\tau \\
& - \int_0^\epsilon \int_{|z-y|=\epsilon} \widehat{w}^y_j(z,\tau) f_k(z,\tau) \nu_k \, dS_z d\tau \\
& + \int_{|z-y|<\epsilon} G^y_{kj}(z,\epsilon) f_k(z,\epsilon) \, dz + \int_{|z-y|<\epsilon} F^y_j(z) \partial_{z_k} f_k(z,0) \, dz \Bigg].
\end{aligned}
$$

The last two terms vanish as $\epsilon \to 0_+$ since

$$
\left| \int_{|z-y|<\epsilon} G^y_{kj}(z,\epsilon) f_k(z,\epsilon) \, dz \right| \lesssim \int_0^\epsilon \frac{\|f\|_\infty}{(r+\sqrt{\epsilon})^n} r^{n-1} \, dr \lesssim \epsilon^{\frac{n+1}{2}} \to 0 \ \text{ as } \epsilon \to 0_+ \tag{7.4}
$$

by (1.12) and Lemma 2.1, and

$$
\left| \int_{|z-y|<\epsilon} F^y_j(z) \partial_{z_k} f_k(z,0) \, dz \right| \lesssim \int_0^\epsilon \frac{\|\nabla f\|_\infty}{r^{n-1}} r^{n-1} \, dr \lesssim \epsilon \to 0 \ \text{ as } \epsilon \to 0_+.
$$

Hence (7.3) is valid for all $f \in C^\infty_c(\mathbb{R}^n_+ \times \mathbb{R})$.

If $f \in C^\infty_c(\mathbb{R}^n_+ \times [0,t])$, we can extend it to $\tilde{f} \in C^\infty_c(\mathbb{R}^n_+ \times \mathbb{R})$. Hence (7.3) is valid for all such f. Finally, if $f \in C^\infty(\mathbb{R}^n_+ \times [0,t])$ for some $t > 0$, let $\tilde{f} = f\zeta$ where $\zeta(z,\tau)$ is a smooth cut-off function which equals 1 in $Q^{y,0}_{2\epsilon}$. Then (7.3) is valid for $\tilde{f} \in C^\infty_c(\mathbb{R}^n_+ \times [0,\infty))$ and hence also for f. This completes the proof of the lemma. □

We now prove the symmetry.

Proof of Proposition 1.4. Fix $\Phi \in C^\infty_c(\mathbb{R})$, $\Phi(s) = 1$ for $s \leq 1$, and $\Phi(s) = 0$ for $s \geq 2$. For fixed $x \neq y \in \mathbb{R}^n_+$, $t > 0$, and $i, j = 1, \ldots, n$, by choosing $f_k(z,\tau) = G^x_{ki}(z, t-\tau)\eta^{x,t}(z,\tau)$ in (7.3) of Lemma 7.1, where $\eta^{x,t}$ is a smooth cut-off function defined by

$$
\eta^{x,t}(z,\tau) = 1 - \Phi\left(\frac{|x-z|}{\epsilon}\right) \Phi\left(\frac{|t-\tau|}{\epsilon}\right),
$$

and using that $\eta^{x,t}(z,\tau) = 1$ on $\{(z,\tau) : 0 \le \tau \le \epsilon,\ |z-y| \le \epsilon\}$ for $\epsilon < |x-y|/3$, we obtain

$$\begin{aligned} G^x_{ji}(y,t) = \lim_{\epsilon\to 0_+} \sum_{k=1}^n \Bigg[&\int_{|z-y|=\epsilon} F^y_j(z) G^x_{ki}(z,t)\nu_k \, dS_z \\ &- \int_0^\epsilon \int_{|z-y|=\epsilon} G^y_{kj}(z,\tau)\nabla_z G^x_{ki}(z,t-\tau)\cdot \nu_z \, dS_z d\tau \\ &+ \int_0^\epsilon \int_{|z-y|=\epsilon} (\nabla_z G^y_{kj}(z,\tau)\cdot\nu_z) G^x_{ki}(z,t-\tau)\, dS_z d\tau \\ &- \int_0^\epsilon \int_{|z-y|=\epsilon} \widehat{w}^y_j(z,\tau) G^x_{ki}(z,t-\tau)\nu_k \, dS_z d\tau \Bigg]. \end{aligned} \tag{7.5}$$

Switching y and j in the above identity with x and i, respectively, and changing the variables in τ, we get

$$\begin{aligned} G^y_{ij}(x,t) = \lim_{\epsilon\to 0_+} \sum_{k=1}^n \Bigg[&\int_{|z-x|=\epsilon} F^x_i(z) G^y_{kj}(z,t)\nu_k \, dS_z \\ &- \int_{t-\epsilon}^t \int_{|z-x|=\epsilon} G^x_{ki}(z,t-\tau)\nabla_z G^y_{kj}(z,\tau)\cdot \nu_z \, dS_z d\tau \\ &+ \int_{t-\epsilon}^t \int_{|z-x|=\epsilon} (\nabla_z G^x_{ki}(z,t-\tau)\cdot\nu_z) G^y_{kj}(z,\tau)\, dS_z d\tau \\ &- \int_{t-\epsilon}^t \int_{|z-x|=\epsilon} \widehat{w}^x_i(z,t-\tau) G^y_{kj}(z,\tau)\nu_k \, dS_z d\tau \Bigg]. \end{aligned} \tag{7.6}$$

Denote

$$U^{L,\delta}_\epsilon := \left\{ (\mathbb{R}^n_+ \cap \{|z| < L, Lz_n > 1\}) \times [\delta, t-\delta] \right\} \setminus (Q^{x,t}_\epsilon \cup Q^{y,0}_\epsilon)$$

for $0 < \delta < \epsilon < \min(t, |x-y|)/2$ and $L > 2(|x|+|y|+1)$. Since $G^x_{ki}(z,t-\tau)$ and $G^y_{kj}(z,\tau)$ are smooth in $U^{L,\delta}_\epsilon$, $\left[(\partial_\tau - \Delta_z)G^y_{kj} + \partial_{z_k} g^y_j\right](z,\tau)$ and $\left[(-\partial_\tau - \Delta_z)G^x_{ki}\right]$ $+\partial_{z_k} g^x_i(z,t-\tau)$ vanish in $U^{L,\delta}_\epsilon$, and $g_j(x,y,t) = \widehat{w}_j(x,y,t)$ for $t > 0$,

$$\begin{aligned} 0 &= \sum_{k=1}^n \int_{U^{L,\delta}_\epsilon} G^x_{ki}(z,t-\tau)\left[(\partial_\tau - \Delta_z)G^y_{kj} + \partial_{z_k}\widehat{w}^y_j\right](z,\tau)\, dz\, d\tau \\ &\quad - \sum_{k=1}^n \int_{U^{L,\delta}_\epsilon} G^y_{kj}(z,\tau)\left[(-\partial_\tau - \Delta_z)G^x_{ki} + \partial_{z_k}\widehat{w}^x_i\right](z,t-\tau)\, dz\, d\tau \\ &= \sum_{k=1}^n \left(\int_\delta^\epsilon \int_{|z-y|>\epsilon} + \int_\epsilon^{t-\epsilon} \int_{|z|<L} + \int_{t-\epsilon}^{t-\delta} \int_{|z-x|>\epsilon} \right) [\cdots]\, dz\, d\tau. \end{aligned}$$

By integration by parts, $G^y_{kj}(z', 0, t) = 0$, $G^y_{kj}(z, 0_+) = \partial_k F^y_j(z)$ if $y \neq z$, $\sum_{k=1}^n \partial_{z_k} G^x_{kj} = 0$, and taking limits $L \to \infty$ and $\delta \to 0_+$, ($\varepsilon > 0$ fixed), we get

$$\sum_{k=1}^n \Bigg[-\int_{|z-y|<\epsilon} G^x_{ki}(z, t-\epsilon) G^y_{kj}(z, \epsilon)\, dz - \int_{|z-y|=\epsilon} G^x_{ki}(z, t) F^y_j(z) \nu_k \, dS_z$$
$$+ \int_{|z-x|<\epsilon} G^x_{ki}(z, \epsilon) G^y_{kj}(z, t-\epsilon)\, dz + \int_{|z-x|=\epsilon} G^y_{kj}(z, t) F^x_i(z) \nu_k \, dS_z.$$
$$- \int_0^\epsilon \int_{|z-y|=\epsilon} \Big[G^x_{ki}(z, t-\tau) \nabla_z G^y_{kj}(z, \tau) - G^y_{kj}(z, \tau) \nabla_z G^x_{ki}(z, t-\tau) \Big] \cdot \nu_z \, dS_z d\tau.$$
$$- \int_{t-\epsilon}^t \int_{|z-x|=\epsilon} \Big[G^x_{ki}(z, t-\tau) \nabla_z G^y_{kj}(z, \tau) - G^y_{kj}(z, \tau) \nabla_z G^x_{ki}(z, t-\tau) \Big] \cdot \nu_z \, dS_z d\tau.$$
$$+ \int_0^\epsilon \int_{|z-y|=\epsilon} \Big[G^x_{ki}(z, t-\tau) \widehat{w}^y_j(z, \tau) - G^y_{kj}(z, \tau) \widehat{w}^x_i(z, t-\tau) \Big] \nu_k \, dS_z d\tau.$$
$$+ \int_{t-\epsilon}^t \int_{|z-x|=\epsilon} \Big[G^x_{ki}(z, t-\tau) \widehat{w}^y_j(z, \tau) - G^y_{kj}(z, \tau) \widehat{w}^x_i(z, t-\tau) \Big] \nu_k \, dS_z d\tau \Bigg] = 0. \quad (7.7)$$

Note that the above integrals are over finite regions. We can take limits $\delta \to 0_+$ because in these regions we do not evaluate $G^y_{kj}(z, \tau)$ and $\widehat{w}^y_j(z, \tau)$ at their singularity $(y, 0)$, nor $G^x_{ki}(z, t-\tau)$ and $\widehat{w}^x_i(z, t-\tau)$ at their singularity (x, t). To justify the limits $L \to \infty$, we first need to show that the far-field integrals

$$J_1 = \int_{\mathbb{R}^n_+ \cap \{|z|=L\}} \Big[G^x_{ki}(z, t) F^y_j(z) - G^y_{kj}(z, t) F^x_i(z) \Big] \nu_k \, dS_z$$
$$J_2 = \int_0^t \int_{\mathbb{R}^n_+ \cap \{|z|=L\}} \Big[G^x_{ki}(z, t-\tau) \nabla_z G^y_{kj}(z, \tau) - G^y_{kj}(z, \tau) \nabla_z G^x_{ki}(z, t-\tau) \Big] \cdot \nu_z \, dS_z d\tau$$
$$J_3 = \int_0^t \int_{\mathbb{R}^n_+ \cap \{|z|=L\}} \Big[G^x_{ki}(z, t-\tau) \widehat{w}^y_j(z, \tau) - G^y_{kj}(z, \tau) \widehat{w}^x_i(z, t-\tau) \Big] \nu_k \, dS_z d\tau$$

vanish as $L \to \infty$. By (1.12),

$$|J_1| \lesssim \int_{\mathbb{R}^n_+ \cap \{|z|=L\}} L^{-n} L^{1-n} \, dS_z = C L^{-n} \to 0.$$

For J_2 with $L > 2(|x| + |y| + \sqrt{t})$, the worst estimate of $\nabla_z G^y_{kj}(z, \tau)$ by (1.12) is $L^{-n} (z_n + \sqrt{\tau})^{-1} \log \frac{L}{\sqrt{\tau}}$. Thus

$$|J_2| \lesssim \int_0^t \int_{\mathbb{R}^n_+ \cap \{|z|=L\}} L^{-n} L^{-n} \tau^{-1/2} (\log L + |\log \tau| \mathbb{1}_{\tau<1}) \, dS_z d\tau$$
$$\lesssim \sqrt{t} \, L^{-(n+1)} \log L \to 0.$$

For the integral J_3, by (1.14) with $r = \min(x_n, y_n) > 0$,

$$|J_3| \lesssim \int_0^t \int_{\mathbb{R}^n_+ \cap \{|z|=L\}} L^{-n} \tau^{-1/2} \left[\frac{1}{L^n} \log \frac{L}{z_n} + \frac{1}{L^{n-1} r} \right] dS_z d\tau.$$

Using

$$\int_{|z|=L,\, z_n<1} |\log z_n|\, dS_z = \int_0^1 \int_{|z'|=\sqrt{L^2-z_n^2}} |\log z_n|\, dS_{z'} dz_n \lesssim \int_0^1 L^{n-2}\, |\log z_n|\, dz_n \lesssim L^{n-2},$$

we get

$$|J_3| \lesssim \sqrt{t}\left(L^{-n-1}\log L + L^{-n-2} + L^{-n} r^{-1}\right) \to 0, \quad \text{as } L \to \infty.$$

We also need to show the boundary integrals similar to J_1, J_2 and J_3 at $z_n = 1/L$ (instead of $|z| = L$) vanish as $L \to \infty$. This is clear for J_1 and J_2 as $G^x_{ki}(z', 0, t) = 0$ and the factors F^y_j and $\nabla_z G^y_{kj}$ are bounded near $z_n = 0$. For J_3, estimate (1.14) of the factor $\widehat{w}^y_j$ has a log singularity $\log z_n$, and we use the boundary vanishing estimate (1.20) of G^x_{ki},

$$|J_3| \lesssim \int_0^t \int_{z_n=1/L} \frac{z_n \log\left(e + \frac{|x^*-z|}{\sqrt{t-s}}\right)}{\sqrt{t-\tau}(|z'-x'| + |z_n - x_n| + \sqrt{t-\tau})^n} \cdot \tau^{-\frac{1}{2}} \frac{1}{(|z'-y'| + z_n + y_n + \sqrt{\tau})^n} \log\left(1 + \frac{|z'-y'| + y_n + \sqrt{\tau}}{z_n}\right) dS_z d\tau,$$

which vanishes as $L \to \infty$. Note that the proof of the base case (no derivatives) of (1.20), to be given in Sect. 8, does not rely on the symmetry.

The above show (7.7).

Now take $\epsilon \to 0$. Using (7.5) and (7.6), the identity (7.7) becomes

$$\begin{aligned} \lim_{\epsilon\to 0_+} \sum_{k=1}^n \Bigg[&-\int_{|z-y|<\epsilon} G^x_{ki}(z, t-\epsilon) G^y_{kj}(z, \epsilon)\, dz + \int_{|z-x|<\epsilon} G^x_{ki}(z, \epsilon) G^y_{kj}(z, t-\epsilon)\, dz \\ &- \int_0^\epsilon \int_{|z-y|=\epsilon} G^y_{kj}(z, \tau) \widehat{w}^x_i(z, t-\tau) \nu_k\, dS_z d\tau \\ &+ \int_{t-\epsilon}^t \int_{|z-x|=\epsilon} G^x_{ki}(z, t-\tau) \widehat{w}^y_j(z, \tau) \nu_k\, dS_z d\tau \Bigg] - G^x_{ji}(y, t) + G^y_{ij}(x, t) = 0. \end{aligned} \tag{7.8}$$

The first two terms tend to zero as $\epsilon \to 0_+$ by the same reason as for (7.4). Moreover, since $w^x_i(z, t-\tau)$ is uniformly bounded (independent of ϵ) for $(z, \tau) \in \{(z, \tau) : |z-y| = \epsilon,\ 0 < \tau < \epsilon\}$ by (1.14), we obtain from (1.12) that

$$\begin{aligned} \left| \int_0^\epsilon \int_{|z-y|=\epsilon} G^y_{kj}(z, \tau) \widehat{w}^x_i(z, t-\tau) \nu_k\, dS_z d\tau \right| &\lesssim \int_0^\epsilon \int_{|z-y|=\epsilon} \frac{1}{(|z-y| + \sqrt{\tau})^n}\, dS_z d\tau \\ &\lesssim \int_0^\epsilon \frac{1}{(\epsilon + \sqrt{\tau})^n} \epsilon^{n-1}\, d\tau \lesssim \epsilon + \delta_{n2}\epsilon \log\frac{1}{\epsilon} \to 0 \quad \text{as } \epsilon \to 0_+. \end{aligned} \tag{7.9}$$

Similarly, $\int_{t-\epsilon}^t \int_{|z-x|=\epsilon} G^x_{ki}(z, t-\tau) w^y_j(z, \tau) \nu_k\, dS_z d\tau$ goes to zero as $\varepsilon \to 0_+$. By (7.4) and (7.9), the equation (7.8) turns into

$$-G^x_{ji}(y, t) + G^y_{ij}(x, t) = 0.$$

This completes the proof of Proposition 1.4, i.e., the symmetry (7.1) of the Green tensor. □

Remark 7.1. We can actually show an alternative estimate of $\widehat{w}_j^y(z,\tau)$ which has no singularity as $z_n \to 0_+$ by estimating (3.28) instead of (3.21), cf. Remark 3.5(i). Using it, we don't need the vanishing estimate (1.20). We do not present it in this way since its proof is more involved, in particular in the case $n = 2$.

8. The Main Estimates of the Green Tensor

In this section we prove the main estimates in Theorems 1.5 and 1.6.

Proof of Theorem 1.5. From (1.12) we have that

$$
\begin{aligned}
|\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q \partial_t^m G_{ij}(x,y,t)| \lesssim & \frac{1}{(|x-y|^2+t)^{\frac{l+k+q+n}{2}+m}} \\
& + \frac{\mathrm{LN}_{ijkq}^{mn}}{t^m(|x^*-y|^2+t)^{\frac{l+k-k_i+n}{2}}(x_n^2+t)^{\frac{k_i}{2}}(y_n^2+t)^{\frac{q}{2}}},
\end{aligned} \tag{8.1}
$$

where $k_i = (k-\delta_{in})_+$, and

$$
\begin{aligned}
\mathrm{LN}_{ijkq}^{mn} &:= 1 + \delta_{n2}\mu_{ik}^m \left[\log(\nu_{ijkq}^m|x'-y'| + x_n + y_n + \sqrt{t}) - \log(\sqrt{t})\right], \\
\mu_{ik}^m &= 1 - (\delta_{k0} + \delta_{k1}\delta_{in})\delta_{m0}, \quad \nu_{ijkq}^m = \delta_{q0}\delta_{jn}\delta_{k(1+\delta_{in})}\delta_{m0} + \delta_{m>0}.
\end{aligned}
$$

On the other hand, by the symmetry of the Green tensor (Proposition 1.4) and (1.12),

$$
\begin{aligned}
|\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q \partial_t^m G_{ij}(x,y,t)| &= |\left(\partial_{x',y'}^l \partial_{Y_n}^k \partial_{X_n}^q \partial_t^m G_{ji}\right)(y,x,t)| \\
\lesssim \frac{1}{(|x-y|^2+t)^{\frac{l+k+q+n}{2}+m}} &+ \frac{\mathrm{LN}_{jiqk}^{mn}}{t^m(|x^*-y|^2+t)^{\frac{l+q-q_j+n}{2}}(y_n^2+t)^{\frac{q_j}{2}}(x_n^2+t)^{\frac{k}{2}}},
\end{aligned} \tag{8.2}
$$

where $q_j = (q-\delta_{jn})_+$, ∂_{X_n} denotes the partial derivative in the n-th variable, and ∂_{Y_n} denotes the partial derivative in the $2n$-th variable. The combination of (8.1) and (8.2) gives

$$
\begin{aligned}
|\partial_{x',y'}^l \partial_{x_n}^k \partial_{y_n}^q \partial_t^m G_{ij}(x,y,t)| \lesssim & \frac{1}{(|x-y|^2+t)^{\frac{l+k+q+n}{2}+m}} \\
& + \frac{\mathrm{LN}_{ijkq}^{mn} + \mathrm{LN}_{jiqk}^{mn}}{t^m(|x^*-y|^2+t)^{\frac{l+k-k_i+q-q_j+n}{2}}(x_n^2+t)^{\frac{k_i}{2}}(y_n^2+t)^{\frac{q_j}{2}}}.
\end{aligned}
$$

This shows (1.18) and completes the proof of Theorem 1.5. □

We next show the boundary vanishing of derivatives of G_{ij} at $x_n = 0$ or $y_n = 0$.

Proof of Theorem 1.6. Denote

$$\mathrm{LN} = \textstyle\sum_{k=0}^{1}(\mathrm{LN}_{ijkq}^{mn} + \mathrm{LN}_{jiqk}^{mn})(x, y, t).$$

By $\partial_{x',y'}^{l}\partial_{y_n}^{q}\partial_t^{m} G_{ij}|_{x_n=0} = 0$ and (1.18) with $k = 1$, we have

$$\begin{aligned}
&\left|\partial_{x',y'}^{l}\partial_{y_n}^{q}\partial_t^{m} G_{ij}(x, y, t)\right| \\
&\le \int_0^{x_n} \left|\partial_{x',y'}^{l}\partial_{x_n}\partial_{y_n}^{q}\partial_t^{m} G_{ij}(x', z_n, y, t)\right| dz_n \\
&\lesssim \int_0^{x_n} \left[\frac{1}{(|x'-y'|^2+|z_n-y_n|^2+t)^{\frac{l+q+n+1}{2}+m}}\right. \\
&\quad \left.+\frac{\mathrm{LN}}{t^m(|x'-y'|^2+(z_n+y_n)^2+t)^{\frac{l+q-q_j+n}{2}}(z_n^2+t)^{\frac{1}{2}}(y_n^2+t)^{\frac{q_j}{2}}}\right] dz_n \\
&=: I_1 + I_2.
\end{aligned} \tag{8.3}$$

Above we have used that $\mathrm{LN}_{ijkq}^{mn}(x', z_n, y, t)$ is nondecreasing in z_n.

We first estimate I_1.

Case 1. If $3x_n < y_n$, then $|z_n - y_n| > \frac{1}{2}(x_n + y_n)$ and $z_n + y_n > \frac{1}{4}(x_n + y_n)$ for $0 < z_n < x_n$. Thus, (8.3) gives

$$I_1 \lesssim \frac{x_n}{(|x-y^*|^2+t)^{\frac{l+q+n+1}{2}+m}}.$$

Case 2. If $y_n < 3x_n < \frac{1}{2}\left(|x'-y'| + y_n + \sqrt{t}\right)$, then $x_n + y_n < \frac{4}{3}(|x'-y'| + \sqrt{t})$, which implies $|x-y^*| + \sqrt{t} \lesssim |x'-y'| + \sqrt{t}$. We drop $|z_n - y_n|$ in the integrand of (8.3) to get

$$I_1 \lesssim \frac{x_n}{(|x'-y'|^2+t)^{\frac{l+q+n+1}{2}+m}} \lesssim \frac{x_n}{(|x-y^*|^2+t)^{\frac{l+q+n+1}{2}+m}}.$$

Case 3. If $3x_n > y_n > \frac{1}{2}\left(|x'-y'| + y_n + \sqrt{t}\right)$ or $3x_n > \frac{1}{2}\left(|x'-y'| + y_n + \sqrt{t}\right) > y_n$, then $x_n \approx |x-y^*| + \sqrt{t}$. By (1.18) with $k = 0$,

$$I_1 \lesssim \frac{1}{(|x-y|^2+t)^{\frac{l+q+n}{2}+m}} \lesssim \frac{x_n}{(|x-y|^2+t)^{\frac{l+q+n}{2}+m}(|x-y^*|^2+t)^{\frac{1}{2}}}.$$

Thus, we have

$$I_1 \lesssim \frac{1}{(|x-y|^2+t)^{\frac{l+q+n}{2}+m}} \lesssim \frac{x_n}{(|x-y|^2+t)^{\frac{l+q+n}{2}+m}(|x-y^*|^2+t)^{\frac{1}{2}}}.$$

Next, we estimate I_2.

If $x_n < y_n + \frac{1}{2}\sqrt{t}$, then $|x-y^*|^2+t \approx |x'-y'|^2+y_n^2+t$. We drop z_n in and the integrand of (8.3) to get

$$I_2 \lesssim \frac{x_n\,\mathrm{LN}}{t^{m+\frac{1}{2}}(|x'-y'|^2+y_n^2+t)^{\frac{l+q-q_j+n}{2}}(y_n^2+t)^{\frac{q_j}{2}}} \lesssim \frac{x_n\,\mathrm{LN}}{t^{m+\frac{1}{2}}(|x-y^*|^2+t)^{\frac{l+q-q_j+n}{2}}(y_n^2+t)^{\frac{q_j}{2}}}.$$

If $x_n > y_n + \frac{1}{2}\sqrt{t}$, then $x_n \gtrsim \sqrt{t}$. By (1.18) with $k=0$,

$$I_2 \lesssim \frac{\mathrm{LN}}{t^m(|x-y^*|^2+t)^{\frac{l+q-q_j+n}{2}}(y_n^2+t)^{\frac{q_j}{2}}} \lesssim \frac{x_n\,\mathrm{LN}}{t^{m+\frac{1}{2}}(|x-y^*|^2+t)^{\frac{l+q-q_j+n}{2}}(y_n^2+t)^{\frac{q_j}{2}}}.$$

Combining the above cases, we derive

$$\left|\partial_{x',y'}^l \partial_{y_n}^q \partial_t^m G_{ij}(x,y,t)\right| \lesssim \frac{x_n}{(|x-y|^2+t)^{\frac{l+q+n}{2}+m}(|x-y^*|^2+t)^{\frac{1}{2}}} + \frac{x_n\,\mathrm{LN}}{t^{m+\frac{1}{2}}(|x-y^*|^2+t)^{\frac{l+q-q_j+n}{2}}(y_n^2+t)^{\frac{q_j}{2}}},$$

which is (1.20) for $\alpha = 1$. Since (1.20) also holds for $\alpha = 0$ by (1.18), it holds for all $0 \le \alpha \le 1$. Finally, (1.21) follows from the symmetry. This completes the proof of Theorem 1.6. □

9. Mild Solutions of Navier–Stokes Equations

In this section we apply our linear estimates to the construction of mild solutions of Navier–Stokes equations (NS).

9.1. Mild solutions in L^q. In this subsection we prove Lemma 9.1. It is standard to prove Theorem 1.7 using estimates in Lemma 9.1 and a fixed point argument. We skip the proof of Theorem 1.7.

Lemma 9.1. *Let $n \ge 2$, $1 \le p \le q \le \infty$ and $1 < q$.*

(a) *If $u_0 \in L^p_\sigma(\mathbb{R}^n_+)$ and $\breve{u}_i(x,t) = \sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t)u_{0,j}(y)dy$, then*

$$\|\breve{u}(\cdot,t)\|_{L^q(\mathbb{R}^n_+)} \le Ct^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u_0\|_{L^p(\mathbb{R}^n_+)}, \quad \text{if } u_0 = \mathbf{P}u_0, \tag{9.1}$$

$$L^q\text{-}\lim_{t\to 0_+} t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\breve{u}(\cdot,t) = \begin{cases} 0, & \text{if } 1 \le p < q \le \infty, \\ u_0, & \text{if } 1 < p = q < \infty. \end{cases} \tag{9.2}$$

(9.2)$_2$ *is also valid for $p = q = \infty$ if u_0 in the L^∞-closure of $C^1_{c,\sigma}(\overline{\mathbb{R}^n_+})$.*

(b) *Let $F \in L^p(\mathbb{R}^n_+)$, $a, b \in \mathbb{N}_0$, and $1 \le a+b$. Assume $b \ge 1$ and $n \ge 3$ if $p = q = \infty$. Then*

$$\left\|\int_{\mathbb{R}^n_+} \partial_x^a \partial_y^b G_{ij}(x,y,t)F(y)dy\right\|_{L^q(\mathbb{R}^n_+)} \le Ct^{-\frac{a+b}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|F\|_{L^p(\mathbb{R}^n_+)}. \tag{9.3}$$

Proof. We consider (9.1) and decompose $\breve{u}_i(x,t)$ defined in (6.1) as

$$\breve{u}_i(x,t) = \int_{\mathbb{R}^n_+} \Gamma(x-y,t)u_{0,i}(y)dy + \int_{\mathbb{R}^n_+} G^*_{ij}(x,y,t)u_{0,j}(y)dy$$
$$=: u_i^{heat}(x,t) + u_i^*(x,t).$$

The basic property of heat kernel yields

$$\|u^{heat}\|_{L^q(\mathbb{R}^n_+)} \le Ct^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \|u_0\|_{L^p(\mathbb{R}^n_+)}.$$

By (1.10), $u^*(x,t)$ is bounded by

$$J_t(x) = \int_{\mathbb{R}^n_+} \frac{e^{-\frac{cy_n^2}{t}}}{(|x'-y'|+x_n+y_n+\sqrt{t})^n} |u_0(y)|\,dy$$
$$= \int_0^\infty \frac{1}{(|x'|+x_n+y_n+\sqrt{t})^n} *_\Sigma |u_0(x',y_n)| e^{-\frac{cy_n^2}{t}}\,dy_n,$$

where $*_\Sigma$ indicates convolution over Σ. By Minkowski and Young inequalities,

$$\|J_t(\cdot,x_n)\|_{L^q(\Sigma)} \lesssim \int_0^\infty \left\| \frac{1}{(|x'|+x_n+y_n+\sqrt{t})^n} *_\Sigma |u_0(x',y_n)| \right\|_{L^q(\Sigma)} e^{-\frac{cy_n^2}{t}}\,dy_n$$
$$\lesssim \int_0^\infty \left\| \frac{1}{(|x'|+x_n+y_n+\sqrt{t})^n} \right\|_{L^r(\Sigma)} \cdot$$
$$\|u_0(\cdot,y_n)\|_{L^p(\Sigma)}\, e^{-\frac{cy_n^2}{t}}\,dy_n, \quad \frac{1}{q}+1 = \frac{1}{r}+\frac{1}{p},$$
$$\lesssim \int_0^\infty \frac{1}{(x_n+y_n+\sqrt{t})^{1-(n-1)\left(\frac{1}{q}-\frac{1}{p}\right)}} \cdot \|u_0(\cdot,y_n)\|_{L^p(\Sigma)}\, e^{-\frac{cy_n^2}{t}}\,dy_n.$$

By Minkowski inequality again, (here we need $q>1$)

$$\|J_t\|_{L^q(\mathbb{R}^n_+)} = \big\| \|J_t(\cdot,x_n)\|_{L^q(\Sigma)} \big\|_{L^q(0,\infty)}$$
$$\lesssim \int_0^\infty \left\| \frac{1}{(x_n+y_n+\sqrt{t})^{1-(n-1)\left(\frac{1}{q}-\frac{1}{p}\right)}} \right\|_{L^q(x_n\in(0,\infty))}$$
$$\cdot \|u_0(\cdot,y_n)\|_{L^p(\Sigma)}\, e^{-\frac{cy_n^2}{t}}\,dy_n$$
$$\lesssim \int_0^\infty \frac{1}{(y_n+\sqrt{t})^{1-\frac{1}{q}-(n-1)\left(\frac{1}{q}-\frac{1}{p}\right)}} \cdot \|u_0(\cdot,y_n)\|_{L^p(\Sigma)}\, e^{-\frac{cy_n^2}{t}}\,dy_n. \tag{9.4}$$

By Hölder inequality,

$$\|J_t\|_{L^q(\mathbb{R}^n_+)} \lesssim \|u_0\|_{L^p(\mathbb{R}^n_+)} \left(\int_0^\infty \left(\frac{1}{(y_n+\sqrt{t})^{1-\frac{1}{q}-(n-1)\left(\frac{1}{q}-\frac{1}{p}\right)}} e^{-\frac{cy_n^2}{t}} \right)^{\frac{p}{p-1}} dy_n \right)^{\frac{p-1}{p}}$$
$$\lesssim t^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_{L^p(\mathbb{R}^n_+)}$$

by the change of variables $y_n = \sqrt{t}z$. This proves (9.1).

For (9.2), denote $\sigma = \frac{n}{2}(\frac{1}{p} - \frac{1}{q})$. If $1 \le p < q \le \infty$, then $\sigma > 0$. For any $\varepsilon > 0$, we can choose $b \in L^p_\sigma \cap L^q_\sigma$ with $\|u_0 - b\|_{L^p} \le \varepsilon$. Let $v_i(x,t) = \sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t)b_j(y)dy$. Then by (9.1),

$$t^\sigma \|\breve{u}(\cdot,t)\|_{L^q} \le t^\sigma \|v(\cdot,t)\|_{L^q} + t^\sigma \|\breve{u}(\cdot,t) - v(\cdot,t)\|_{L^q}$$
$$\lesssim t^\sigma \|b\|_{L^q} + \|u_0 - b\|_{L^p}$$

which is less than $C\varepsilon$ for t sufficiently small. This shows $(9.2)_1$.

If $1 < p = q < \infty$, For any $\varepsilon > 0$, there is $M > 0$ such that $\big\| \|u_0(\cdot, y_n)\|_{L^q(\Sigma)} \mathbb{1}_M \big\|_{L^q(0,\infty)} \le \varepsilon$, where $\mathbb{1}_M(y_n) = 1$ if $\|u_0(\cdot,y_n)\|_{L^q(\Sigma)} \ge M$, and $\mathbb{1}_M(y_n) = 0$ otherwise. Then

$$\|u_0(\cdot,y_n)\|_{L^q(\Sigma)} \le M + \|u_0(\cdot,y_n)\|_{L^q(\Sigma)} \mathbb{1}_M.$$

Applying Hölder inequality to (9.4),

$$\|J_t\|_{L^q(\mathbb{R}^n_+)} \lesssim \big\| \|u_0(\cdot,y_n)\|_{L^q(\Sigma)} \mathbb{1}_M \big\|_{L^q(0,\infty)} + \int_0^\infty \frac{1}{(y_n + \sqrt{t})^{1-1/q}} \cdot M e^{-\frac{cy_n^2}{t}}\, dy_n$$
$$\lesssim \varepsilon + M t^{1/2q}$$

which is bounded by $C\varepsilon$ for t sufficiently small. Since $u^{heat}(\cdot,t) \to u_0$ in L^q as $t \to 0_+$, this shows $\breve{u}^L(\cdot,t) \to u_0$ in L^q as $t \to 0_+$. This shows $(9.2)_2$.

If $p = q = \infty$ and u_0 in the L^∞-closure of $C^1_{c,\sigma}(\overline{\mathbb{R}^n_+})$, for any $\varepsilon > 0$, we can choose $b \in C^1_{c,\sigma}(\overline{\mathbb{R}^n_+})$ with $\|u_0 - b\|_{L^\infty} \le \varepsilon$. Let $v_i(x,t) = \sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t)b_j(y)dy$. By Lemma 6.1, $\lim_{t\to 0_+} \|v_i(\cdot,t) - b\|_{L^\infty(\mathbb{R}^n_+)} = 0$. Then by (9.1),

$$\|\breve{u}(\cdot,t) - u_0\|_{L^\infty} \le \|\breve{u}(\cdot,t) - v(\cdot,t)\|_{L^\infty} + \|v(\cdot,t) - b\|_{L^\infty} + \|b - u_0\|_{L^\infty} \lesssim \varepsilon + o(1),$$

which is less than $C\varepsilon$ for t sufficiently small. This shows the remark after $(9.2)_2$.

For (9.3), denote

$$w(x,t) = \int_{\mathbb{R}^n_+} \partial_x^a \partial_y^b G_{ij}(x,y,t) F(y) dy, \quad m = a + b.$$

By Theorem 1.5,

$$|\partial_x^a \partial_y^b G_{ij}(x,y,t)| \lesssim \frac{1}{(|x-y|^2 + t)^{\frac{n+m}{2}}} + \frac{1 + \delta_{n2} \log(1 + \frac{|x^*-y|}{\sqrt{t}})}{(|x^*-y|^2 + t)^{\frac{n}{2}} (x_n^2 + t)^{\frac{a}{2}} (y_n^2 + t)^{\frac{b}{2}}}.$$

Using

$$\frac{\log(e+r)}{(e+r)^n} \le \frac{\log(e+s)}{(e+s)^n}, \quad \forall 0 \le s \le r,$$

we have

$$|\partial_x^a \partial_y^b G_{ij}(x,y,t)| \lesssim \frac{1}{(|x-y|^2 + t)^{\frac{n+m}{2}}} + \frac{\delta_{n2} \log(e + \frac{|x-y|}{\sqrt{t}})}{(|x-y|^2 + t)^{\frac{n}{2}} (x_n^2 + t)^{\frac{a}{2}} (y_n^2 + t)^{\frac{b}{2}}}.$$

Extend $F(y)$ to $y \in \mathbb{R}^n$ by zero for $y_n < 0$. We have

$$\begin{aligned} |w(x,t)| \lesssim & \int_{\mathbb{R}^n} H_t^0(x-y)|F(y)|dy \\ & + \int_{\mathbb{R}^n} H_t(x-y^*)|F(y)| \frac{1}{(y_n^2+t)^{\frac{b}{2}}} dy \frac{1}{(x_n^2+t)^{\frac{a}{2}}} \\ := & \, w_1(x,t) + w_2(x,t), \end{aligned} \tag{9.5}$$

where

$$H_t^0(x) = t^{-\frac{n+m}{2}} H_1^0\left(\frac{x}{\sqrt{t}}\right), \quad H_1^0(x) = \frac{1}{(|x|^2+1)^{\frac{n+m}{2}}} \in L^1 \cap L^\infty(\mathbb{R}^n), \tag{9.6}$$

$$H_t(x) = t^{-\frac{m}{2}} H_1\left(\frac{x}{\sqrt{t}}\right), \quad H_1(x) = \frac{\delta_{n2} \log(e+|x|)}{(|x|^2+1)^{\frac{n}{2}}}. \tag{9.7}$$

By Young's convolution inequality with $\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1$,

$$\|w_1(\cdot,t)\|_{L^q} \lesssim \left\|H_t^0\right\|_{L^r(\mathbb{R}^n)} \|F\|_{L^p} = t^{-\frac{m}{2}+\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \left\|H_1^0\right\|_{L^r(\mathbb{R}^n)} \|F\|_{L^p}.$$

It remains to estimate $w_2(\cdot,t)$.

If $p < q$, we drop the factors $(y_n^2+t)^{-\frac{b}{2}}$ and $(x_n^2+t)^{-\frac{a}{2}}$ in (9.5), $H_t(x-y^*)$ by $H_t(x-y)$, and applying Young's convolution inequality with $\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1$ to get

$$\begin{aligned} \left\| t^{-\frac{m}{2}} \int_{\mathbb{R}^n} H_t(x-y)|F(y)|\,dy \right\|_{L^q} & \lesssim t^{-\frac{m}{2}} \|H_t\|_{L^r(\mathbb{R}^n)} \|F\|_{L^p} \\ & \lesssim t^{-\frac{m}{2}+\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|H_1\|_{L^r(\mathbb{R}^n)} \|F\|_{L^p}. \end{aligned}$$

Note that $H_1 \in L^r$ since $r > 1$ when $p < q$. Thus, we get for $p < q$ that

$$\|w_2(\cdot,t)\|_{L^q} \lesssim t^{-\frac{m}{2}+\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|F\|_{L^p}.$$

If $p = q = \infty$, by the hypotheses $b \geq 1$ and $n \geq 3$ so there is no log term in (9.7). In this case,

$$\begin{aligned} w_2(x,t) & \lesssim \int_{\mathbb{R}^n_+} H_t(x-y^*)|F(y)| \frac{1}{(y_n^2+t)^{\frac{b}{2}}} dy \frac{1}{(x_n^2+t)^{\frac{a}{2}}} \\ & \lesssim \|F\|_{L^\infty} \frac{1}{(x_n+t)^{\frac{a}{2}}} \int_{\mathbb{R}^n_+} \frac{1}{(|x-y^*|^2+t)^{\frac{n}{2}}(y_n^2+t)^{\frac{b}{2}}} dy \\ & \lesssim \|F\|_{L^\infty} \frac{1}{(x_n+t)^{\frac{a}{2}}} \int_0^\infty \frac{1}{(x_n^2+y_n^2+t)^{\frac{1}{2}}(y_n^2+t)^{\frac{b}{2}}} dy_n \\ & \leq \|F\|_{L^\infty} \frac{1}{(x_n+t)^{\frac{a}{2}}} \int_0^\infty \frac{1}{(y_n^2+t)^{\frac{b+1}{2}}} dy \lesssim t^{-\frac{m}{2}} \|F\|_{L^\infty}. \end{aligned}$$

This proves (9.3). □

Remark 9.1. Let $1 \le p < q \le \infty$ and $u_0 \in L^p_\sigma(\mathbb{R}^n_+)$. We claim that

$$u_i^L(x,t) = \int_{\mathbb{R}^n_+} G_{ij}(x,y,t) u_{0,j}(y)\,dy, \quad \widehat{u}_i^L(x,t) = \int_{\mathbb{R}^n_+} \widehat{G}_{ij}(x,y,t) u_{0,j}(y)\,dy, \tag{9.8}$$

are also defined in $L^q(\mathbb{R}^n_+)$ for fixed $t > 0$ and (9.1) holds for u^L and $\widehat{u}^L$:

$$\left\| u^L(\cdot,t) \right\|_{L^q(\mathbb{R}^n_+)} + \left\| \widehat{u}^L(\cdot,t) \right\|_{L^q(\mathbb{R}^n_+)} \le C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^n_+)}. \tag{9.9}$$

Our claim does not include the case $p = q$ as in (9.1). For u^L, this is because $|G_{ij}(x,y,t)| \lesssim (|x-y| + \sqrt{t})^{-n}$ and, by Young's convolution inequality with $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$,

$$\begin{aligned} \left\| u^L(\cdot,t) \right\|_{L^q} &\lesssim \left\| (|x| + \sqrt{t})^{-n} * u_0 \right\|_{L^q} \lesssim \Big(\int_{\mathbb{R}^n} (|x| + \sqrt{t})^{-nr} dx \Big)^{\frac{1}{r}} \|u_0\|_{L^p} \\ &\lesssim t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p} \end{aligned}$$

where we used $q > p$ so that $r > 1$. For $\widehat{u}^L$, by (1.15), we can decompose

$$\begin{aligned} \widehat{u}_i^L(x,t) &= \int_{\mathbb{R}^n_+} \Gamma(x-y,t) u_{0,i}(y) dy \\ &\quad + \int_{\mathbb{R}^n_+} \widehat{G}^*_{ij}(x,y,t) u_{0,j}(y) dy =: u_i^{heat}(x,t) + \widehat{u}_i^*(x,t), \end{aligned}$$

where $\widehat{G}^*_{ij}(x,y,t) = -\delta_{ij}\Gamma(x-y^*,t) - 4\delta_{jn} C_i(x,y,t)$. The first term u^{heat} satisfies (9.9) by the basic property of heat kernel. The second term $\widehat{u}^*(x,t)$ is bounded by

$$|\widehat{u}^*(x,t)| \lesssim \int_{\mathbb{R}^n_+} \frac{e^{-\frac{c y_n^2}{t}}}{(|x'-y'| + x_n + y_n + \sqrt{t})^{n-1} (y_n + \sqrt{t})} |u_0(y)|\,dy$$

using (5.11). Similar to the proof of (9.1), we can first apply Minkowski and Young inequalities in x' (using $q > p$ so that $r > 1$), and then Minkowski and Hölder inequalities in x_n to bound $\|\widehat{u}^*(\cdot,t)\|_{L^q(\mathbb{R}^n_+)}$ by $t^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_0\|_{L^p(\mathbb{R}^n_+)}$. The above shows (9.9) for $1 \le p < q \le \infty$ and $u_0 \in L^p_\sigma(\mathbb{R}^n_+)$.

Remark 9.2. The following extends Theorem 1.2. Assume $u_0 \in L^p_\sigma(\mathbb{R}^n_+)$, $1 \le p < \infty$, and u^L, $\breve{u}^L$ and $\widehat{u}^L$ are defined as in (9.8). There exist $u_0^k \in C^1_{c,\sigma}(\overline{\mathbb{R}^n_+})$ such that $u_0^k \to u_0$ in $L^p(\mathbb{R}^n_+)$ as $k \to \infty$. Let

$$\begin{aligned} u_i^k(x,t) &= \int_{\mathbb{R}^n_+} G_{ij}(x,y,t) u_{0,j}^k(y)\,dy, \quad \breve{u}_i^k(x,t) = \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t) u_{0,j}^k(y)\,dy, \\ \widehat{u}_i^k(x,t) &= \int_{\mathbb{R}^n_+} \widehat{G}_{ij}(x,y,t) u_{0,j}^k(y)\,dy. \end{aligned}$$

They are equal by Theorem 1.2 since $u_0^k \in C^1_{c,\sigma}(\overline{\mathbb{R}^n_+})$. On the other hand, by (9.1) and (9.9),

$$\begin{aligned} &\left\| u^L(t) - u^k(t) \right\|_{L^q} + \left\| \breve{u}^L(t) - \breve{u}^k(t) \right\|_{L^q} + \left\| \widehat{u}^L(t) - \widehat{u}^k(t) \right\|_{L^q} \\ &\le C t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \left\| u_0 - u_0^k \right\|_{L^p} \end{aligned}$$

which vanishes as $k \to \infty$. This shows $u^L(t) = \breve{u}^L(t) = \widehat{u}^L(t)$ in L^q for any $q \in (p, \infty]$ and fixed t.

For $u_0 \in L^\infty_\sigma(\mathbb{R}^n_+)$ and u_0 in the L^∞-closure of $C^1_{c,\sigma}(\overline{\mathbb{R}^n_+})$, we can also show $u^L(t) = \breve{u}^L(t)$ (but we do not know about $\widehat{u}^L(t)$). We use the boundary vanishing (1.20) to get

$$|u^L(x,t) - u^k(x,t)| = \left| \int_{\mathbb{R}^n_+} G_{ij}(x,y,t) \left(u_{0,j}(y) - u^k_{0,j}(y) \right) dy \right| \le C_1 \left\| u_0 - u^k_0 \right\|_{L^\infty}$$

where

$$\begin{aligned} C_1 &= \int_{\mathbb{R}^n_+} \frac{x_n}{(|x-y| + \sqrt{t})^n (x_n + y_n + \sqrt{t})} \, dy \\ &\lesssim \int_0^\infty \frac{x_n}{(|x_n - y_n| + \sqrt{t})(x_n + y_n + \sqrt{t})} \, dy_n \\ &\lesssim \int_0^{2(x_n+\sqrt{t})} \frac{x_n}{(|x_n - y_n| + \sqrt{t})(x_n + \sqrt{t})} \, dy_n \\ &\quad + \int_{2(x_n+\sqrt{t})}^\infty \frac{x_n}{y_n^2} \, dy_n \lesssim \ln(e + \frac{x_n}{\sqrt{t}}). \end{aligned} \tag{9.10}$$

It converges to 0 as $k \to \infty$, and the convergence is uniform in $x_n \le M\sqrt{t}$ for any fixed $t, M > 0$. As $u^k(t) \to \breve{u}^L(t)$ in L^∞ by (9.1), this shows $u^L(x,t) = \breve{u}^L(x,t)$. □

9.2. Mild solutions with pointwise decay. In this subsection we prove Theorems 1.8 and 1.9. We first consider Theorem 1.8. Recall Theorem 1.8 is a direct consequence of [5, Theorem 1] using the estimates in [4, Theorem 1] for $0 < a < n$. For $a = n$, the hypothesis of [5, Theorem 1] is not satisfied: $(1 + |x| + \sqrt{t})^n e^{-tA} u_0 \sim \log(2+t) \notin L^\infty(\mathbb{R}^n_+ \times (0,\infty))$ (see [4, Theorem 1] and (9.11)). Nonetheless, the proof of local existence still works if $\left\| (1 + |x| + \sqrt{t})^n e^{-tA} u_0 \right\|_{L^\infty(\mathbb{R}^n_+ \times (0,T))} \le C(T)$, which is true for $u_0 \in Y_n$. Theorem 1.8 can be proved using the estimates in Lemma 9.2 below and the same iteration argument in [5]. We omit its proof and focus on Lemma 9.2.

Lemma 9.2. *Let $n \ge 2$ and $0 \le a \le n$. For $u_0 \in Y_a$ with* $\operatorname{div} u_0 = 0$ *and $u_{0,n}|_\Sigma = 0$,*

$$\left\| \sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t) u_{0,j}(y) dy \right\|_{Y_a} \le C(1 + \delta_{an} \log_+ t) \left\| u_0 \right\|_{Y_a}. \tag{9.11}$$

For $F \in Y_{2a}$,

$$\left\| \int_{\mathbb{R}^n_+} \partial_{y_p} G_{ij}(x,y,t) F_{pj}(y) \, dy \right\|_{Y_a} \le C t^{-1/2} \left\| F \right\|_{Y_{2a}}. \tag{9.12}$$

The estimate (9.11) is proved in [4, Theorem 1] with space-time decay (see also [6, Theorem 4.2]), whereas (9.12) is not known in [6] and [4] since the pointwise estimates of the Green tensor G_{ij} was not available. Instead, they used (1.25) for the bilinear form in the Duhamel's formula when constructing mild solutions.

Note that $Y_0 = L^\infty$ and $a \le n$ in (9.11) since the decay cannot be faster than the Green tensor. The case $a = 0$ is a special case of (9.1). It is similar to [50, Theorem 1.1] which further assumes continuity. We do not assume any boundary condition on F_{pj}. Also note

$$\left\||u|^2\right\|_{Y_{2a}} = \sup_{x\in\mathbb{R}^n_+} |u(x)|^2\langle x\rangle^{2a} = \|u\|^2_{Y_a}.$$

Proof. If $a = 0$, the lemma follows from (9.1) and (9.3) with $p = q = \infty$. Thus, we consider $a > 0$. For (9.11), write

$$\sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x, y, t)u_{0,j}(y)dy = \int_{\mathbb{R}^n_+} \Gamma(x - y, t)u_{0,i}(y)dy + \int_{\mathbb{R}^n_+} G^*_{ij}(x, y, t)u_{0,j}(y)dy$$
$$=: u_i^{heat}(x, t) + u_i^*(x, t).$$

It is known that for $0 \le a \le n$

$$\left\|u^{heat}\right\|_{Y_a} \lesssim (1 + \delta_{an} \log_+ t)\, \|u_0\|_{Y_a}. \tag{9.13}$$

See e.g. [26, Lemma 1] for $n = 3$ case. Its statement corresponds to $1 \le a \le n$ but its proof also works for $0 \le a < 1$.

For u^* with $|u_0(y)| \lesssim \langle y\rangle^{-a}$, by (1.10), (for both $n \ge 3$ and $n = 2$)

$$|u^*(x, t)| \lesssim J(x) = \int_{\mathbb{R}^n_+} \frac{e^{-\frac{cy_n^2}{t}}}{(|x^* - y|^2 + t)^{\frac{n}{2}}\langle y\rangle^a}\, dy.$$

Suppose $0 < a < n - 1$. By Lemma 2.2,

$$\begin{aligned}
J &\lesssim \int_0^\infty e^{-\frac{cy_n^2}{t}} \int_\Sigma \frac{1}{(|x' - y'| + x_n + y_n + \sqrt{t})^n(|y'| + y_n + 1)^a}\, dy'dy_n \\
&\lesssim \int_0^\infty \left[\frac{1}{(|x| + y_n + \sqrt{t} + 1)^{a+1}} + \frac{e^{-\frac{cy_n^2}{t}}}{(|x| + y_n + \sqrt{t} + 1)^a(x_n + y_n + \sqrt{t})}\right] dy_n \\
&\lesssim \frac{1}{(|x| + \sqrt{t} + 1)^a} + \frac{1}{(|x| + \sqrt{t} + 1)^a}\int_0^\infty \frac{e^{-u^2}}{\left(\frac{x_n}{\sqrt{t}}\right) + 1}\, du \\
&\lesssim \frac{1}{(|x| + \sqrt{t} + 1)^a}.
\end{aligned}$$

This proves

$$\left\|u^*\right\|_{Y_a} \lesssim \|u_0\|_{Y_a}, \quad 0 < a < n - 1.$$

If $a = n - 1$, we have an additional term from Lemma 2.2,

$$\begin{aligned}
&\int_0^\infty e^{-\frac{cy_n^2}{t}} \frac{1}{(|x| + y_n + \sqrt{t} + 1)^n} \log\left(1 + \frac{|x| + \sqrt{t}}{y_n + 1}\right) dy_n \\
&\lesssim \frac{1}{(|x| + \sqrt{t} + 1)^n}\int_0^\infty e^{-\frac{cy_n^2}{t}} \left(\frac{|x| + \sqrt{t}}{y_n + 1}\right)^\varepsilon dy_n,
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^\infty e^{-\frac{cy_n^2}{t}} \left(\frac{|x|+\sqrt{t}}{y_n+1}\right)^\varepsilon dy_n \\
&\le \int_0^{|x|+\sqrt{t}} \left(\frac{|x|+\sqrt{t}}{y_n+1}\right)^\varepsilon dy_n + \int_{|x|+\sqrt{t}}^\infty e^{-\frac{cy_n^2}{t}} \left(\frac{|x|+\sqrt{t}}{y_n+1}\right)^\varepsilon dy_n \\
&\le (|x|+\sqrt{t})^\varepsilon (|x|+\sqrt{t}+1)^{1-\varepsilon} + \frac{(|x|+\sqrt{t})^\varepsilon}{(|x|+\sqrt{t}+1)^\varepsilon} \int_0^\infty e^{-u^2}\sqrt{t}\,du \\
&\le |x|+\sqrt{t}+1.
\end{aligned}$$

So the additional term is bounded by $(|x|+\sqrt{t}+1)^{1-n} = (|x|+\sqrt{t}+1)^{-a}$.

If $n-1<a<n$, we have an additional term from Lemma 2.2,

$$\begin{aligned}
&\int_0^\infty \frac{e^{-\frac{cy_n^2}{t}}}{(|x|+y_n+\sqrt{t}+1)^n (y_n+1)^{a-n+1}} dy_n \\
&\lesssim \frac{1}{(|x|+\sqrt{t}+1)^n} \left[\int_0^{|x|+\sqrt{t}+1} \frac{1}{(y_n+1)^{a-n+1}} dy_n \right. \\
&\quad \left. + \int_{|x|+\sqrt{t}+1}^\infty \frac{e^{-\frac{cy_n^2}{t}}}{(|x|+\sqrt{t}+1)^{a-n+1}} dy_n \right] \\
&\lesssim \frac{1}{(|x|+\sqrt{t}+1)^n} \left[\frac{1}{(|x|+\sqrt{t}+1)^{a-n}} + \frac{1}{(|x|+\sqrt{t}+1)^{a-n+1}} \int_0^\infty e^{-u^2}\sqrt{t}\,du \right] \\
&\lesssim \frac{1}{(|x|+\sqrt{t}+1)^a}.
\end{aligned}$$

If $a=n$, we have the same additional term from Lemma 2.2,

$$\begin{aligned}
&\int_0^\infty \frac{e^{-\frac{cy_n^2}{t}}}{(|x|+y_n+\sqrt{t}+1)^n (y_n+1)} dy_n \lesssim \frac{1}{(|x|+\sqrt{t}+1)^n} \int_0^\infty \frac{e^{-\frac{cy_n^2}{t}}}{y_n+1} dy_n \\
&\lesssim \frac{1}{(|x|+\sqrt{t}+1)^n} \left(\int_0^{\sqrt{t}} \frac{1}{y_n+1} dy_n + \int_{\sqrt{t}}^\infty \frac{e^{-\frac{cy_n^2}{t}}}{y_n} dy_n \right) \\
&\lesssim \frac{1}{(|x|+\sqrt{t}+1)^n} \left(\log(1+\sqrt{t}) + 1 \right).
\end{aligned}$$

We have proved

$$\left\| u^* \right\|_{Y_a} \lesssim \|u_0\|_{Y_a}, \quad 0<a<n; \quad \left\| u^*(t) \right\|_{Y_n} \lesssim \log(2+t) \|u_0\|_{Y_n},$$

and hence (9.11).

We next consider (9.12). For $k=0$ and $l+q=1$, by Proposition 1.1 with $k=k_i=0$ and $q=1$ we have

$$|\partial_{y'}^l \partial_{y_n}^q G_{ij}(x,y,t)| \lesssim \frac{1}{(|x-y|^2+t)^{\frac{n+1}{2}}} + \frac{1}{(|x^*-y|^2+t)^{\frac{n}{2}} (y_n^2+t)^{\frac{1}{2}}}. \tag{9.14}$$

It suffices to show

$$I_1 + I_2 \lesssim t^{-1/2} \frac{1}{\langle x \rangle^a}$$

where

$$I_1 = \int_{\mathbb{R}^n_+} \frac{1}{(|x-y| + \sqrt{t})^{n+1}} \frac{1}{\langle y \rangle^{2a}}\, dy,$$
$$I_2 = \int_{\mathbb{R}^n_+} \frac{1}{(|x^* - y| + \sqrt{t})^n (y_n + \sqrt{t})} \frac{1}{\langle y \rangle^{2a}}\, dy.$$

For I_1, by Lemma 2.2, we have

$$I_1 \le \int_{\mathbb{R}^n} \frac{1}{(|x-y| + \sqrt{t})^{n+1}} \frac{1}{(|y|+1)^{2a}}\, dy$$
$$\lesssim \frac{1}{(|x| + \sqrt{t} + 1)^{2a}\sqrt{t}} + \frac{1}{(|x| + \sqrt{t} + 1)^{n+1}} \left(\mathbb{1}_{2a=n} \log(|x| + \sqrt{t} + 1) + \mathbb{1}_{2a>n} \right)$$

Thus, if $0 < a \le n$,

$$I_1 \lesssim \frac{1}{(|x| + \sqrt{t} + 1)^a \sqrt{t}}. \tag{9.15}$$

For I_2, let $A = x_n + y_n + \sqrt{t}$. We have

$$I_2 \lesssim \int_0^\infty \left(\int_\Sigma \frac{1}{(|x' - y'| + A)^n (|y'| + y_n + 1)^{2a}}\, dy' \right) \frac{dy_n}{y_n + \sqrt{t}}.$$

Let $R = |x'| + A + (y_n + 1) \sim |x| + y_n + 1 + \sqrt{t}$. By Lemma 2.2,

$$I_2 \lesssim \int_0^\infty \left(R^{-2a} A^{-1} + R^{-n} \left(\mathbb{1}_{2a=n-1} \log \frac{R}{y_n + 1} + \frac{\mathbb{1}_{2a>n-1}}{(y_n+1)^{2a+1-n}} \right) \right) \frac{dy_n}{y_n + \sqrt{t}}$$
$$= I_3 + I_4 + I_5.$$

We have

$$I_3 \lesssim \int_0^\infty \frac{dy_n}{(|x| + 1 + \sqrt{t})^{2a} (y_n + \sqrt{t})^2} \lesssim \frac{1}{(|x| + 1 + \sqrt{t})^{2a} \sqrt{t}}.$$

If $2a = n - 1$, for any $0 < \epsilon < a$, we have $n - 1 - \epsilon > a$ and

$$I_4 \lesssim \int_0^\infty \frac{\log(y_n + |x| + 1 + \sqrt{t})}{(y_n + |x| + 1 + \sqrt{t})^n \sqrt{t}} dy_n$$
$$\lesssim \frac{1}{(|x| + 1 + \sqrt{t})^{n-1-\epsilon} \sqrt{t}} \lesssim \frac{1}{(|x| + 1 + \sqrt{t})^a \sqrt{t}}.$$

If $\frac{n-1}{2} < a \le n$,

$$I_5 \lesssim \frac{1}{\sqrt{t}} \left(\int_0^{|x|+1+\sqrt{t}} + \int_{|x|+1+\sqrt{t}}^{\infty} \right) \frac{dy_n}{(y_n + |x| + 1 + \sqrt{t})^n (y_n + 1)^{2a+1-n}}$$
$$\lesssim \frac{1}{\sqrt{t}} \int_0^{|x|+1+\sqrt{t}} \frac{dy_n}{(|x| + 1 + \sqrt{t})^n (y_n + 1)^{2a+1-n}} + \frac{1}{\sqrt{t}} \int_{|x|+1+\sqrt{t}}^{\infty} \frac{dy_n}{y_n^{2a+1}}$$
$$\lesssim \frac{1}{\sqrt{t}} \left(\frac{1}{(|x| + 1 + \sqrt{t})^{2a}} + \frac{\mathbb{1}_{2a=n} \log(|x| + 1 + \sqrt{t})}{(|x| + 1 + \sqrt{t})^{2a}} + \frac{\mathbb{1}_{2a>n}}{(|x| + 1 + \sqrt{t})^n} \right).$$

Thus, if $0 < a \le n$,

$$I_2 \lesssim I_3 + I_4 + I_5 \lesssim \frac{1}{(|x| + \sqrt{t} + 1)^a \sqrt{t}}.$$

This and the I_1 estimate (9.15) show (9.12). □

Remark. In the proof of (9.12), we use Proposition 1.1 instead of Theorem 1.5 to avoid LN since μ_{jq} may be 1 when $q = 1$.

We next consider Theorem 1.9. It can be proved using the same iteration argument in [5] and the estimates in the following.

Lemma 9.3. *Let $n \ge 2$ and $0 \le a \le 1$. For $u_0 \in Z_a$ with* $\operatorname{div} u_0 = 0$ *and $u_{0,n}|_\Sigma = 0$,*

$$\left\| \sum_{j=1}^{n} \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x, y, t) u_{0,j}(y) dy \right\|_{Z_a} \le C(1 + \delta_{a1} \log_+ t) \|u_0\|_{Z_a}. \tag{9.16}$$

For $F \in Z_{2a}$,

$$\left\| \int_{\mathbb{R}^n_+} \partial_{y_p} G_{ij}(x, y, t) F_{pj}(y)\, dy \right\|_{Z_a} \le C t^{-1/2} \|F\|_{Z_{2a}}. \tag{9.17}$$

Our estimates for both inequalities fail for $a > 1$. See Remark 9.3 after the proof.

Proof. If $a = 0$, the lemma follows from (9.1) and (9.3) with $p = q = \infty$. Thus, we only consider $0 < a \le 1$. We may suppose that $\|u_0\|_{Z_a} = 1$ without loss of generality. For (9.16), write

$$\sum_{j=1}^{n} \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x, y, t) u_{0,j}(y) dy = \int_{\mathbb{R}^n_+} \Gamma(x - y, t) u_{0,i}(y) dy + \int_{\mathbb{R}^n_+} G^*_{ij}(x, y, t) u_{0,j}(y) dy$$
$$=: u_i^{heat}(x, t) + u_i^*(x, t).$$

Denote by Γ_k the k-dimensional heat kernel. When $|u_0(y)| \le \langle y_n \rangle^{-a}$, we have

$$|u_i^{heat}(x, t)| \lesssim \int_0^{\infty} \frac{\Gamma_1(x_n - y_n, t)}{(y_n + 1)^a} \int_\Sigma \Gamma_{n-1}(x' - y', t)\, dy' dy_n$$
$$\lesssim \int_0^{\infty} \frac{\Gamma_1(x_n - y_n, t)}{(y_n + 1)^a} dy_n$$
$$\lesssim (1 + \delta_{a=1} \log_+ t)(x_n + 1)^{-a}.$$

We have used the one dimensional version of (9.13) for the last inequality and $0 < a \le 1$.

For u^* with $|u_0(y)| \le \langle y_n \rangle^{-a}$, by (1.10), (for both $n \ge 3$ and $n = 2$) we get

$$|u^*(x,t)| \lesssim J(x,t) = \int_{\mathbb{R}^n_+} \frac{e^{-\frac{cy_n^2}{t}}}{(|x^* - y|^2 + t)^{\frac{n}{2}} \langle y_n \rangle^a} \, dy.$$

For $0 < a < \infty$, we have

$$\begin{aligned} J &\lesssim \int_0^\infty \frac{e^{-\frac{cy_n^2}{t}}}{(y_n+1)^a} \int_\Sigma \frac{1}{(|x'-y'| + x_n + y_n + \sqrt{t})^n} \, dy' dy_n \\ &\lesssim \int_0^\infty \frac{e^{-\frac{cy_n^2}{t}}}{(y_n+1)^a (x_n + y_n + \sqrt{t})} \, dy_n \\ &\lesssim \frac{1}{x_n + \sqrt{t}} \int_0^{x_n + \sqrt{t}} \frac{1}{(y_n+1)^a} \, dy_n + \frac{1}{(x_n + \sqrt{t} + 1)^a} \int_{x_n + \sqrt{t}}^\infty \frac{e^{-c\frac{y_n^2}{t}}}{y_n + \sqrt{t}} \, dy_n. \end{aligned}$$

Using Lemma 2.1 to bound the first integral, we have

$$\begin{aligned} J &\lesssim \frac{1}{x_n + \sqrt{t}} \frac{(x_n + \sqrt{t}) \left(1 + \delta_{a1} \log_+ (x_n + \sqrt{t})\right)}{(1 + x_n + \sqrt{t})^{\min(a,1)}} + \frac{1}{(x_n + \sqrt{t} + 1)^a} \int_0^\infty \frac{e^{-u^2}}{u+1} \, du \\ &\lesssim \frac{1 + \delta_{a1} \log_+ (x_n + \sqrt{t})}{(x_n + \sqrt{t} + 1)^{\min(a,1)}}. \end{aligned}$$

When $a = 1$, we want to improve the above numerator $1 + \delta_{a1} \log_+ (x_n + \sqrt{t})$ to a function of t independent of x_n. It suffices to consider the case $x_n > 10 + \sqrt{t}$. In this case,

$$\begin{aligned} J &\lesssim \frac{1}{x_n} \int_0^\infty \frac{e^{-c\frac{y_n^2}{t}}}{y_n + 1} \, dy_n \lesssim \frac{1}{x_n} \int_0^{\sqrt{t}} \frac{1}{y_n + 1} \, dy_n + \frac{1}{x_n} \int_{\sqrt{t}}^\infty \frac{e^{-c\frac{y_n^2}{t}}}{y_n} \, dy_n \\ &\lesssim \frac{1}{x_n} \log(\sqrt{t} + 1) + \frac{1}{x_n}. \end{aligned}$$

We conclude when $a = 1$, either $x_n > 10 + \sqrt{t}$ or not,

$$J \lesssim \frac{\log(2 + \sqrt{t})}{x_n + \sqrt{t} + 1}.$$

Combining the above estimates of u^{heat} and J, the estimate (9.16) is deduced.

Next, we will show (9.17). For $k = 0$ and $l + q = 1$, by Proposition 1.1 with $k = k_i = 0$ and $q = 1$, we have

$$|\partial_{y'}^l \partial_{y_n}^q G_{ij}(x,y,t)| \lesssim \frac{1}{(|x-y|^2 + t)^{\frac{n+1}{2}}} + \frac{1}{(|x^* - y|^2 + t)^{\frac{n}{2}} (y_n^2 + t)^{\frac{1}{2}}}. \tag{9.18}$$

It suffices to show, for $a > 0$,

$$I_1 + I_2 \lesssim t^{-1/2} \frac{1}{\langle x_n \rangle^a}$$

where

$$I_1 = \int_{\mathbb{R}^n_+} \frac{1}{(|x-y|+\sqrt{t})^{n+1}} \frac{1}{\langle y_n \rangle^{2a}}\, dy,$$
$$I_2 = \int_{\mathbb{R}^n_+} \frac{1}{(|x^*-y|+\sqrt{t})^{n}(y_n+\sqrt{t})} \frac{1}{\langle y_n \rangle^{2a}}\, dy.$$

Indeed, via Lemma 2.2, we have

$$\begin{aligned} I_1 &\lesssim \int_0^\infty \frac{1}{(y_n+1)^{2a}} \int_\Sigma \frac{1}{(|x-y|+\sqrt{t})^{n+1}}\, dy' dy_n \\ &\lesssim \int_0^\infty \frac{1}{(y_n+1)^{2a}(|x_n-y_n|+\sqrt{t})^2}\, dy_n \\ &\lesssim R^{-1-2a} + \delta_{2a=1} R^{-2}\log R + \mathbb{1}_{2a>1} R^{-2} + R^{-2a} t^{-1/2} \\ &\lesssim t^{-1/2} \frac{1}{\langle x_n \rangle^a}, \end{aligned}$$

where $R = x_n + \sqrt{t} + 1$. We have used $a \le 1$ to bound $\mathbb{1}_{2a>1} R^{-2} \lesssim t^{-1/2} \frac{1}{\langle x_n \rangle^a}$. On the other hand,

$$\begin{aligned} I_2 &\lesssim \int_0^\infty \frac{1}{(y_n+1)^{2a}(y_n+\sqrt{t})} \int_\Sigma \frac{1}{(|x^*-y|+\sqrt{t})^n}\, dy' dy_n \\ &\lesssim \int_0^\infty \frac{1}{(y_n+1)^{2a}(y_n+\sqrt{t})(x_n+y_n+\sqrt{t})}\, dy_n. \end{aligned}$$

If $x_n \le 1$, we have

$$I_2 \lesssim \int_0^1 \frac{1}{(y_n+\sqrt{t})^2}\, dy_n + \int_1^\infty \frac{1}{y_n^{2a+1}\sqrt{t}}\, dy_n \lesssim \frac{1}{\sqrt{t}}.$$

If $x_n \ge 1$, using $0 < a \le 1$ we have

$$\begin{aligned} I_2 &\lesssim \int_0^\infty \frac{1}{(y_n+1)^{2a}(\sqrt{t})\, x_n^a (y_n+1)^{1-a}}\, dy_n \\ &= \frac{1}{x_n^a \sqrt{t}} \int_0^\infty \frac{1}{(y_n+1)^{1+a}}\, dy_n = \frac{c}{x_n^a \sqrt{t}}. \end{aligned}$$

Combining the above estimates of I_1 and I_2, we obtain (9.17). □

Remark 9.3. The restriction $a \le 1$ is used for both estimates of u^{heat} and J for (9.16) and for both I_1 and I_2 for (9.17) in the above proof. In fact, J has the lower bound for $t = 1$ and all $a > 0$,

$$J(x,1) \gtrsim \int_{0<y_n<1} \int_\Sigma \frac{dy'\, dy_n}{(|y'|+x_n+1)^n} \gtrsim \frac{1}{1+x_n}.$$

9.3. Mild solutions in $L^q_{\rm uloc}$. In this subsection we prove Lemma 9.4. The estimates in Lemma 9.4 are used by Maekawa, Miura and Prange to construct local in time mild solutions of (NS) in $L^q_{\rm uloc}(\mathbb{R}^n_+)$ in [36, Prop 7.1] for $n < q \le \infty$ and [36, Prop 7.2] for $q = n$. Their same proofs give Theorem 1.10.

Lemma 9.4. *Let $n \ge 2$. Let $1 \le p \le q \le \infty$. For $u_0 \in L^p_{\rm uloc,\sigma}$,*

$$\left\| \sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t) u_{0,j}(y) dy \right\|_{L^q_{\rm uloc}} \le C \left(1 + t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} + \mathbb{1}_{p=q=1} \ln_+ \frac{1}{t} \right) \|u_0\|_{L^p_{\rm uloc}}. \tag{9.19}$$

Let $F \in L^p_{\rm uloc}$, $a, b \in \mathbb{N}_0$ and $1 \le a+b$. Assume $b \ge 1$ and $n \ge 3$ if $p = q = \infty$. Then

$$\left\| \int_{\mathbb{R}^n_+} \partial_x^a \partial_y^b G_{ij}(x,y,t) F(y) dy \right\|_{L^q_{\rm uloc}} \le C t^{-\frac{a+b}{2}} \left(1 + t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \right) \|F\|_{L^p_{\rm uloc}}. \tag{9.20}$$

These estimates correspond to [36, Proposition 5.3] and [36, Theorem 3]. Their proof is based on resolvent estimates in [36, Theorem 1], which does not allow $q = 1$. Thus our estimates for $p = q = 1$ are new. Also note that we do not restrict $a, b \le 1$ as in [36, Theorem 3].

Proof. First consider (9.19). The endpoint case $p = q = \infty$ follows from (9.1). Let $p < \infty$. The formula (1.8) gives

$$\begin{aligned} \sum_{j=1}^n \int_{\mathbb{R}^n_+} \breve{G}_{ij}(x,y,t) u_{0,j}(y) dy &= \int_{\mathbb{R}^n} \Gamma(x-y,t) \mathbb{1}_{y_n>0} u_{0,i}(y) dy \\ &\quad + \int_{\mathbb{R}^n_+} G^*_{ij}(x,y,t) u_{0,j}(y) dy \\ &=: u_i^{heat}(x,t) + u_i^*(x,t). \end{aligned}$$

Since u^{heat} is a convolution with the heat kernel in $\mathbb{R}^n$, it satisfies the estimate in (9.19) by Maekawa–Terasawa [37, (3.18)]. It suffices now to show that $u^*(x,t)$ also satisfies the same estimate. By (1.10), $u^*(x,t)$ is bounded by

$$\begin{aligned} J_t(x) &= \int_{\mathbb{R}^n_+} \frac{e^{-\frac{c y_n^2}{t}}}{(|x'-y'| + x_n + y_n + \sqrt{t})^n} |u_0(y)|\, dy \\ &= \int_0^\infty \frac{1}{(|x'| + x_n + y_n + \sqrt{t})^n} *_\Sigma |u_0(x', y_n)| e^{-\frac{c y_n^2}{t}}\, dy_n, \end{aligned}$$

where $*_\Sigma$ indicates convolution over Σ. Denote

$$Q = [-\tfrac{1}{2}, \tfrac{1}{2}]^{n-1} \subset \Sigma, \quad Q_k = k + Q, \quad k \in \mathbb{Z}^{n-1}.$$

Our goal is to bound

$$\|J_t\|_{L^q(Q_{j'} \times (j_n, j_n+1))}$$

by the right side of (9.19), uniformly for all $j' \in \mathbb{Z}^{n-1}$ and $j_n \in \mathbb{N}_0$. By translation, we may assume $j' = 0$. Decompose

$$J_t(x) = \sum_{k,l\in\mathbb{Z}^{n-1}} \int_0^\infty \frac{\mathbb{1}_{Q_k}(x')}{(|x'| + x_n + y_n + \sqrt{t})^n} *_{\Sigma_{x'}} \left(\mathbb{1}_{Q_l}(x')|u_0(x', y_n)|\right) e^{-\frac{cy_n^2}{t}}\, dy_n.$$

By Minkowski and Young inequalities with $1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}$,

$$\begin{aligned}
&\|J_t(\cdot, x_n)\|_{L^q(Q)} \\
&\lesssim \sum_{k,l\in\mathbb{Z}^{n-1},\, k-l\in 3Q} \int_0^\infty \left\| \frac{\mathbb{1}_{Q_k}(x')}{(|x'| + x_n + y_n + \sqrt{t})^n} *_{\Sigma_{x'}} \left(\mathbb{1}_{Q_l}(x')|u_0(x', y_n)|\right) \right\|_{L^q_{x'}(Q)} \\
&\quad e^{-\frac{cy_n^2}{t}}\, dy_n \\
&\lesssim \sum_{k,l\in\mathbb{Z}^{n-1},\, k-l\in 3Q} \int_0^\infty I_k \cdot \|u_0(\cdot, y_n)\|_{L^p(Q_l)}\, e^{-\frac{cy_n^2}{t}}\, dy_n,
\end{aligned}$$

where

$$I_k = \left\| \frac{1}{(|x'| + x_n + y_n + \sqrt{t})^n} \right\|_{L^r_{x'}(Q_k)}.$$

We have $I_k \approx (1 + |k| + x_n + y_n + \sqrt{t})^{-n}$ when $k \neq 0$, and $I_0 \lesssim (1 + x_n + y_n + \sqrt{t})^{-\frac{n-1}{r}} (x_n + y_n + \sqrt{t})^{-n+\frac{n-1}{r}}$ by Lemma 2.1 .

By Minkowski inequality again with $I = (j_n, j_n + 1)$,

$$\begin{aligned}
\|J_t\|_{L^q(Q\times I)} &= \left\| \|J_t(\cdot, x_n)\|_{L^q(Q)} \right\|_{L^q_{x_n}(I)} \\
&\lesssim \sum_{k\in\mathbb{Z}^{n-1}} \int_0^\infty \|I_k\|_{L^q_{x_n}(I)}\, \|u_0(\cdot, y_n)\|_{L^p(k+4Q)}\, e^{-\frac{cy_n^2}{t}}\, dy_n.
\end{aligned}$$

We have $\|I_k\|_{L^q_{x_n}(I)} \lesssim \frac{1}{(1+|k|+y_n+\sqrt{t})^n}$ except when $k = 0$ and $y_n + \sqrt{t} < 1$. For $k = 0$,

$$\begin{aligned}
\|I_0\|_{L^q_{x_n}(I)} &\lesssim \left\| \frac{1}{(x_n + y_n + \sqrt{t})^{n-\frac{n-1}{r}}} \right\|_{L^q_{x_n}(0,1)} \\
&\lesssim \frac{1}{(y_n + \sqrt{t})^{1+\frac{n-1}{p}-\frac{n}{q}}} + \mathbb{1}_{p=q=1} \ln_+ \frac{1}{y_n + \sqrt{t}},
\end{aligned}$$

using $n - \frac{n-1}{r} = 1 + (n-1)(\frac{1}{p} - \frac{1}{q}) \geq 1$. Thus

$$\|J_t\|_{L^q(Q\times I)} \lesssim \sum_{k\in\mathbb{Z}^{n-1}} \sum_{j=0}^\infty \int_j^{j+1} \frac{1}{(1 + |k| + y_n + \sqrt{t})^n}\, \|u_0(\cdot, y_n)\|_{L^p(k+4Q)}\, e^{-\frac{cy_n^2}{t}}\, dy_n + M,$$

where

$$M = \int_0^1 \|I_0\|_{L^q_{x_n}(I)}\, \|u_0(\cdot, y_n)\|_{L^p(4Q)}\, e^{-\frac{cy_n^2}{t}}\, dy_n.$$

By Hölder inequality with $p' = \frac{p}{p-1}$,

$$\|J_t\|_{L^q(Q\times I)} \lesssim \sum_{k\in\mathbb{Z}^{n-1}} \sum_{j=0}^{\infty} \|u_0\|_{L^p((k+4Q)\times(j,j+1))} \cdot \left\| \frac{1}{(1+|k|+y_n+\sqrt{t})^n} e^{-\frac{cy_n^2}{t}} \right\|_{L^{p'}_{y_n}(j,j+1)} + M$$
$$\lesssim \sum_{k\in\mathbb{Z}^{n-1}} \sum_{j=0}^{\infty} \|u_0\|_{L^p_{\text{uloc}}} \frac{1}{(1+|k|+j+\sqrt{t})^n} e^{-\frac{cj^2}{t}} + M$$
$$\lesssim \|u_0\|_{L^p_{\text{uloc}}} \int_{\mathbb{R}^n_+} \frac{e^{-\frac{cy_n^2}{t}}}{(1+|y|+\sqrt{t})^n} dy + M \lesssim \|u_0\|_{L^p_{\text{uloc}}} + M.$$

Also by Hölder inequality, when $(p, q) \neq (1, 1)$,

$$M \lesssim \|u_0\|_{L^p_{\text{uloc}}} \cdot \left\| \frac{1}{(y_n+\sqrt{t})^{1+\frac{n-1}{p}-\frac{n}{q}}} e^{-\frac{cy_n^2}{t}} \right\|_{L^{p'}_{y_n}(0,1)} \lesssim t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p_{\text{uloc}}},$$

while when $p = q = 1$,

$$M \lesssim \|u_0\|_{L^1_{\text{uloc}}} \cdot \left\| \left(1+\ln_+ \frac{1}{y_n+\sqrt{t}}\right) e^{-\frac{cy_n^2}{t}} \right\|_{L^\infty(0,1)} \lesssim \left(1+\ln_+ \frac{1}{t}\right) \|u_0\|_{L^1_{\text{uloc}}}.$$

We have shown

$$\|J_t\|_{L^q_{\text{uloc}}} \lesssim \left(t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} + \mathbb{1}_{p=q=1} \ln_+ \frac{1}{t}\right) \|u_0\|_{L^p_{\text{uloc}}}. \tag{9.21}$$

This proves (9.19).

For (9.20), denote

$$w(x,t) = \int_{\mathbb{R}^n_+} \partial_x^a \partial_y^b G_{ij}(x,y,t) F(y) dy, \quad m = a+b.$$

By (9.5), $|w(x,t)| \lesssim w_1(x,t) + w_2(x,t)$, where $w_1(x,t) = \int_{\mathbb{R}^n} H_t^0(x-y)|F(y)|dy$ with $H_t^0(x)$ given by (9.6), and $w_2(x,t) = (x_n^2+t)^{a/2} \int_{\mathbb{R}^n} H_t(x-y^*)|F(y)|(y_n^2+t)^{b/2}\, dy$ with $H_t^0(x)$ given by (9.7).

For $w_1(x,t)$, by Maekawa-Terasawa [37, Theorem 3.1] with $\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1$,

$$\|w_1(\cdot,t)\|_{L^q_{\text{uloc}}} \lesssim t^{-\frac{m}{2}} \left(t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \left\|H_1^0\right\|_{L^r(\mathbb{R}^n)} + \left\|H_1^0\right\|_{L^1(\mathbb{R}^n)}\right) \|F\|_{L^p_{\text{uloc}}}.$$

It remains to estimate $w_2(x,t)$. When $p = q = \infty$, noting that $L^\infty_{\text{uloc}} = L^\infty$, (9.20) follows from (9.3). For $p < q$, we drop the factors $(y_n^2+t)^{-\frac{b}{2}}$ and $(x_n^2+t)^{-\frac{a}{2}}$ in (9.5), $H_t(x-y^*)$ by $H_t(x-y)$, and applying Maekawa-Terasawa [37, Theorem 3.1] with $\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1$,

$$\|w_2(\cdot,t)\|_{L^q_{\text{uloc}}} \lesssim t^{-\frac{m}{2}} \left(t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \|H_1\|_{L^r(\mathbb{R}^n)} + \|H_1\|_{L^1(\mathbb{R}^n)}\right) \|F\|_{L^p_{\text{uloc}}}.$$

Note that $H_1 \in L^r$ since $r > 1$ when $p < q$. Thus, we get for $p < q$ that

$$\|w_2(\cdot, t)\|_{L^q_{\mathrm{uloc}}} \lesssim t^{-\frac{m}{2}} \left(t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} + 1 \right) \|F\|_{L^p_{\mathrm{uloc}}}.$$

This shows (9.20). □

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