



Hausdorff Dimension of Julia Sets in the Logistic Family

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Abstract: A closed interval and circle are the only smooth Julia sets in polynomial dynamics. D. Ruelle proved that the Hausdorff dimension of unicritical Julia sets close to the circle depends analytically on the parameter. Near the tip of the Mandelbrot set \mathcal{M} , the Hausdorff dimension is generally discontinuous. Answering a question of J-C. Yoccoz in the conformal setting, we observe that the Hausdorff dimension of quadratic Julia sets depends continuously on c and find explicit bounds at the tip of \mathcal{M} for most real parameters in the the sense of 1-dimensional Lebesgue measure.

1. Introduction

Hénon attractors. There is a natural connection between quadratic maps $f_c(z) = z^2 + c$, $c \geq -2$, and Hénon maps

$$T_{a,b}(x, y) = (x^2 + a + y, bx),$$

$a \geq -2, b \geq 0$. J-C. Yoccoz posed the following question in the strongly dissipative planar setting corresponding to $c \rightarrow -2^+$, [8, 54]. Is it true that the Hausdorff dimension of the Hénon attractor $X_{a,b}$ (the closure of the unstable manifold) of $T_{a,b}$, tends to 1 as $(a, b) \rightarrow (-2, 0)$, for (a, b) in a set A having $(-2, 0)$ as a Lebesgue density point in the quadrant $(-2, +\infty) \times (0, \infty)$? If yes, what is the asymptotic behaviour of $(a, b) \mapsto \dim_{\text{H}}(X_{a,b})$ when $A \ni (a, b) \rightarrow (-2, 0)$?

It is known, by [5, 7], that A can be chosen so that $T_{a,b}$ has a unique SRB invariant measure $\mu = \mu_{a,b}$. By the dimension formula [53],

$$\dim_{\text{H}}(\mu) = h_{\mu} \cdot \left[\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right],$$

λ_1 and λ_2 are the Lyapunov exponents of μ , $\dim_{\text{H}}(\mu)$ is the infimum of the Hausdorff dimension of sets of full μ -measure, and h_{μ} is the metric entropy. By Pesin's formula

h_μ is equal to λ_1 . Clearly, $\lambda_1 + \lambda_2 = \log b$, and $\lambda_1 \sim \log 2$ for (a, b) close to $(-2, 0)$, this means that

$$\dim_{\mathbb{H}}(X_{a,b}) \geq \dim_{\mathbb{H}}(\mu) = 1 + \frac{\lambda_1}{|\lambda_2|} \geq 1 + \frac{C}{|\log b|}$$

for an absolute constant $C > 0$, giving a weak estimate for the lower bound.

Main result. The following theorem answers the logistic family counterpart to J-C. Yoccoz’ questions, obtaining strong bounds on the asymptotic behaviour as c approaches the one-sided density point -2 for a positive measure set of real Collet–Eckmann parameters ($c \in [-2, 0]$ for which $\liminf_{n \rightarrow \infty} \frac{1}{n} \log |(f_c^n)'(c)| > 0$).

Theorem 1. *Let $\dim_{\mathbb{H}}(\mathcal{J}_c)$ denote the Hausdorff dimension of the Julia set of the quadratic map $z \mapsto z^2 + c \in \mathbb{C}$. For some constant $C > 1$,*

$$1 + C^{-1}\sqrt{c+2} \leq \dim_{\mathbb{H}}(\mathcal{J}_c) \leq 1 + C|\log(c+2)|\sqrt{c+2}$$

for all c from a subset of real Collet–Eckmann parameters with the point -2 as a one-sided Lebesgue density point.

Both the upper and the lower bounds are new. The asymptotic lower bound for the Hausdorff dimension holds for all $c > -2$, not just a positive measure set, see Theorem 4.

One of our objectives, in view of Yoccoz’ problem, was to develop methods for the upper estimates of the Hausdorff dimension which generalize to non-analytic and higher dimensional systems where, naturally, technicalities are more challenging. The techniques to prove the lower bound are more anchored in conformal dynamics. A more detailed discussion of our approach can be found in Section 1.3.

Unimodal maps, logistic family and Mandelbrot set. A differentiable map f of the interval I is called unimodal if it has exactly one critical point, $f'(\alpha) = 0$, and maps the boundary of I into itself.

The logistic family $x \mapsto ax(1 - x)$, $a \in (0, 4]$ is both the simplest and the most known model of non-linear dynamics. The family is often studied in the form $z^2 + c$, $c \in [-2, 1/4]$ which is convenient in the complex setting $z, c \in \mathbb{C}$. For a fixed $c \in \mathbb{C}$, the Julia set \mathcal{J}_c is defined as the smallest non-trivial totally invariant compact set, $\mathcal{J}_c = f_c^{-1}(\mathcal{J}_c)$, and is the locus of chaotic dynamics. In the real case, *density of hyperbolicity* implies that the dynamics on $\mathcal{J}_c \cap \mathbb{R}$ is structurally unstable whenever the parameter $c \geq -2$ is not hyperbolic.

The logistic family in the complex parametrization is embedded inside the Mandelbrot set \mathcal{M} . Namely, $[-2, 1/4] = \mathcal{M} \cap \mathbb{R}$, where

$$\mathcal{M} = \{c \in \mathbb{C} : \forall_{n \geq 0} |f_c^n(c)| \leq 2\}.$$

The Mandelbrot set is one of the most studied irregular fractals in science and a vast literature exists in relation to its various remarkable properties, see for example [12].

Even in the simplest quadratic case $f_c(z) = z^2 + c$, the Hausdorff dimension of Julia sets $\dim_{\mathbb{H}}(\mathcal{J}_c)$ is notoriously difficult to estimate. The exact values of $\dim_{\mathbb{H}}(\mathcal{J}_c)$ are known only for $c = 0$ and -2 . M. Shishikura proved in [47] that for a topologically generic set in $\partial\mathcal{M}$, $\dim_{\mathbb{H}}(\mathcal{J}_c) = 2$, while the results of [21, 23, 49] yield that $\dim_{\mathbb{H}}(\mathcal{J}_c) < 2$ for almost all parameters $c \in \mathcal{M}$ with respect to the harmonic measure. Another consequence of [47] is that the dimension function $c \in \partial\mathcal{M} \mapsto \dim_{\mathbb{H}}(\mathcal{J}_c)$ is discontinuous at every $c \in \partial\mathcal{M}$ such that $\dim_{\mathbb{H}}(\mathcal{J}_c) < 2$.

At parabolic parameters such as $c = 1/4$, the dimension function may be discontinuous, as observed in [19], due to parabolic implosion [16, 18]. This direction of research was advanced further in [25, 28] in relation to A. Douady’s longstanding project to understand oscillations of $\dim_{\mathbb{H}}(\mathcal{J}_c)$ near a parabolic bifurcation.

Certain continuity properties of $c \in \mathcal{M} \mapsto \dim_{\mathbb{H}}(J_c)$ were obtained in [33, 34] and in [21].

Even restricting to $c \in \partial\mathcal{M} \cap \mathbb{R}$, the dimension function is discontinuous at -2 , as detailed in Proposition 1.1. We shall study the asymptotic behaviour at -2 of a large set of real parameters with respect to Lebesgue measure.

1.1. Real parameters.

Typical parameters. By the work of Benedicks and Carleson, [5, Theorem 1] and the summary on [5, page 87], there exist positive constants $\Omega' \in (0, 1)$ and $\Omega \in (0, \log 2)$ such that the set of non-hyperbolic parameters

$$\hat{\mathcal{A}} := \{c \in \mathcal{M} \cap \mathbb{R} : \forall_{n \geq 1} |(f_c^n)'(c)| \geq \Omega' e^{\Omega n}\} \tag{1}$$

has full one-sided Lebesgue density at -2 (see also Collet and Eckmann [13]). Fix such a set $\hat{\mathcal{A}}$ of uniform *Collet–Eckmann* parameters. This was refined further by J.-C. Yoccoz in [55] to obtain an explicit bound on Ω in terms of $c + 2$. Our Proposition 1.4 will show that even if one took $\Omega' = 1$ and Ω arbitrarily close to $\log 2$, $\hat{\mathcal{A}}$ would have positive measure with -2 as a one-sided density point.

We shall be interested in the dynamics as we approach the parameter -2 . For $c > -2$, we shall call c , and the associated dynamics, *typical*, if $c \in \hat{\mathcal{A}}$.

Non-hyperbolic dynamics. By the work of M. Jakobson [26], for $c > -2$, a typical parameter exhibits chaotic dynamics on $[c, f_c(c)]$ and the asymptotic distribution of almost every orbit is given by an invariant measure absolutely continuous with respect to 1-dimensional Lebesgue measure. A complex counterpart of M. Jakobson theorem follows from [5, 21]: for typical $c > -2$, there is a continuous conformal measure on \mathcal{J}_c , with respect to which almost every orbit distributes according to the unique invariant probabilistic measure, equivalent to the conformal one.

Strong phase transition at the tip of the Mandelbrot set. If $c < -2$ then f_c is hyperbolic and the half-line $(-\infty, -2]$ is a hyperbolic geodesic in $\hat{\mathbb{C}} \setminus \mathcal{M}$ landing at -2 . The estimates of [33] and [21] imply that

$$\lim_{c \rightarrow -2^-} \dim_{\mathbb{H}}(J_c) = \dim_{\mathbb{H}}(J_{-2}).$$

Applying real methods, a precise Hölder estimate was obtained in [20] for $c \in (-\infty, -2]$,

$$C^{-1} \sqrt{|c + 2|} \leq 1 - \dim_{\mathbb{H}}(\mathcal{J}_c) \leq C \sqrt{|c + 2|}$$

where $C > 1$ is a universal constant. The result is further discussed in Section 1.4.

By [45], the dimension function $c \in \mathcal{M} \cap \mathbb{R} \mapsto \dim_{\mathbb{H}}(J_c)$ is real analytic for $c < -2$. However, at the unfolding parameter $c = -2$ a strong bifurcation takes place, the relation $c \mapsto \dim_{\mathbb{H}}(\mathcal{J}_c)$ enters a discontinuous regime and its oscillations at the right-hand side of -2 are extreme.

Proposition 1.1. *The dimension function $c \in \mathcal{M} \cap \mathbb{R} \mapsto \dim_{\mathbb{H}}(\mathcal{J}_c)$ is discontinuous at -2 . Moreover,*

$$\limsup_{c \rightarrow -2} \dim_{\mathbb{H}}(J_c) = \sup_{c \in \mathcal{M} \cap \mathbb{R}} \dim_{\mathbb{H}}(J_c) > 1 = \liminf_{c \rightarrow -2^+} \dim_{\mathbb{H}}(J_c).$$

Proof. G. Levin and M. Zinsmeister, [30], showed that $\dim_{\mathbb{H}}(\mathcal{J}_c) > \frac{4}{3}$ for a (non-connected) open set of real parameters with -2 in its boundary, showing discontinuity.

Let \hat{c} be a non-zero parameter in $(-2, 1/4]$. Given $c \in [-2, -1]$, denote by $p(c)$ the orientation-preserving and $q(c)$ the orientation-reversing fixed points of f_c , so $0 < -q(c) < p(c)$. For each $n \geq 5$, denote by I_n the parameter set of $c \in \mathbb{R}$ for which

$$f_c(0) = c < q(c) < f_c^n(0) < -q(c) < f_c^{n-1}(0) < f_c^{j+1}(0) < f_c^j(0)$$

for all $j = 2, 3, \dots, n - 3$. I_n contains an interval of parameters J_n for which f_c^n has a symmetric closed forward-invariant subinterval $W_n(c) \subset (q(c), -q(c))$. The set of maps $\{f_c^n\}_{c \in J_n}$ restricted to $W_n(c)$ forms a full unimodal family, see [15, Section II.4], so for some $c_n \in J_n$, $f_{c_n}^n$ restricted to $W_n(c)$ is conjugate to $f_{\hat{c}}$ restricted to $[-p(\hat{c}), p(\hat{c})]$. We fix such a sequence $(c_n)_{n \geq 5}$. Standard arguments (reprised later in the paper) show that $c_n + 2 \sim 4^{-n} \sim p(c_n) + c_n$ and that there is a topological disc V'_n containing c_n of diameter $\sim 4^{-n}$ mapped univalently by $f_{c_n}^{n-1}$ onto $U_n = \mathbb{D}_{[q(c_n), -q(c_n)]}$, the disc whose diameter is the line segment $[q(c_n), -q(c_n)]$. Then $V_n := f_{c_n}^{-1}(V'_n)$ is a topological disc of diameter $\sim 2^{-n}$ mapped by $f_{c_n}^n$ as a two-to-one branched cover onto U_n .

The orbit of 0 under $f_{c_n}^n$ lies in V_n and $f_{c_n}^n : V_n \rightarrow U_n$ is a *quadratic-like* map and conjugate as such to $f_{\hat{c}}$. See [17] for properties of polynomial-like mappings. Observe that, as $c_n \rightarrow -2$ when $n \rightarrow \infty$, $U_n \rightarrow \mathbb{D}_{[-1,1]}$. As $0 \in V_n$ and $\text{diam}(V_n) \sim 2^{-n}$, the modulus of $(U_n \setminus V_n)$ grows $\sim n$. There exist K_n -quasiconformal maps $h_n : \mathbb{C} \rightarrow \mathbb{C}$ such that for every $z \in V_n$,

$$h_n \circ f_{c_n}^n = f_c \circ h_n$$

and $\lim_{n \rightarrow \infty} K_n = 1$, see [51, Section 11]. In particular,

$$\liminf_{n \rightarrow \infty} \dim_{\mathbb{H}}(J_{c_n}) \geq \dim_{\mathbb{H}}(J_{\hat{c}}).$$

By A. Zdunik’s theorem [56], $\dim_{\mathbb{H}}(J_{\hat{c}}) > 1$, from which the limsup estimate of Proposition 1.1 follows. The liminf estimate is a direct consequence of Theorem 2 or [2].

□

To the best of our knowledge, the value of

$$\sup_{c \in \mathcal{M} \cap \mathbb{R}} \dim_{\mathbb{H}}(J_c) \tag{2}$$

remains unknown. It is unknown whether the supremum is strictly less than 2.

In [2], examples were given of sequences of Julia sets J_c , $c \in \mathcal{M} \cap \mathbb{R}$, of infinitely renormalizable quadratic polynomials f_c for which $\dim_{\mathbb{H}}(J_c) \rightarrow 1$ when c tends to -2 .

Theorem 1, which we discuss in Section 1.3, estimates the dimension for c near -2 for a *large* set of parameters. A weaker, preliminary result, Theorem 2, asserts that continuity of the dimension function $c \mapsto \dim_{\mathbb{H}}(\mathcal{J}_c)$ holds for typical parameters $c > -2$.

Theorem 2. *The dimension function*

$$c \in \hat{\mathcal{A}} \mapsto \dim_{\text{H}}(\mathcal{J}_c)$$

is continuous at every parameter $c \in \hat{\mathcal{A}}$.

Proof. This theorem follows from the fact that every map $f_c, c \in \hat{\mathcal{A}}$ is uniformly summable in the sense of [21] and by invoking Theorem 12 of [21]. \square

In particular, Theorem 2 answers the first part of Yoccoz’ question in the quadratic setting and is a starting point for the research presented in the current paper. A more general version of Theorem 2 formally follows from [6], namely the limit exists in the sense of Theorem 2 along any C^2 -curve ending at -2 .

1.2. Unfolding families. It is known, by [56], that $\dim_{\text{H}}(\mathcal{J}_c) > 1$ for all $c \in \mathcal{M} \setminus \{0, -2\}$. At $c = 0$ or -2 , the Julia set is respectively the unit circle or the segment $[-2, 2]$. We can take a one-parameter family traversing either -2 or 0 to understand how an analytic set unfolds into a fractal. A natural quantity to capture the unfolding is $\dim_{\text{H}}(\mathcal{J}_c)$.

Structurally stable case. D. Ruelle proved in [45] that the function $c \in \mathcal{M} \mapsto \dim_{\text{H}}(\mathcal{J}_c)$ is real analytic at every hyperbolic $c \in \mathbb{C}$ and its asymptotic behaviour at $c = 0$ is given by

$$\dim_{\text{H}}(\mathcal{J}_c) = 1 + \frac{|c|^2}{4 \log 2} + O(|c|^3). \tag{3}$$

The methods used in [45] are an early and innovative application of thermodynamical formalism.

Logistic family. Our objective is to understand how non-linear attractors unfold within one-parameter families of bifurcating maps. The logistic family at -2 is a prototype example of bifurcating dynamics. Our research shows that one of the most important averaging parameters of the system, the Hausdorff dimension $\dim_{\text{H}}(\mathcal{J}_c)$, for typical parameters c , shows almost Hölder dependence on c , Theorem 3 and 4.

Additionally, the lower bound of $\dim_{\text{H}}(\mathcal{J}_c)$ is valid for all parameters $c \in \mathbb{R}$ from a vicinity of -2 .

1.3. Methods and statement of the results. Uniform and explicit estimates of the dimension function $c \mapsto \dim_{\text{H}}(\mathcal{J}_c)$ in bifurcating families are a major technical challenge. Lower bounds on the Hausdorff dimension are usually more difficult to obtain. We prove, in Section 6, a uniform lower bound for all parameters by constructing an induced Cantor repeller and applying Sinai-Ruelle-Bowen methods of thermodynamical formalism. A careful combinatorial and probabilistic argument yields the final result, Theorem 4.

The upper bound is, strangely, more involved. The Hausdorff dimension of Julia sets is strongly discontinuous in the upward direction due to ongoing bifurcations. It is enough that a Julia set grows locally in one scale or another and a similar growth is inherited in most scales and points due to invariance and dynamics. This phenomena is well-known in complex dynamics and leads to discontinuities in the dimension function $c \mapsto \dim_{\text{H}}(\mathcal{J}_c)$ [19]. The upper estimates rely on the technique of the Poincaré series,

[21], statistical properties of conformal densities, the theory of β -numbers from [24], and the parameter exclusion constructions of [5,55].

Theory of β -numbers and dynamics. P. Jones [29] introduced the technique of β -numbers to study geometry of planar sets from the viewpoint of the theory of L^2 functions. One of the outcomes was quantification of the intuition that sets which wiggle in most scales must be metrically large.

Definition 1.2. Let $K \subseteq \mathbb{R}^d$ with $d \geq 2$, $x \in K$ and $r > 0$. We define $\beta_K(x, r)$ by

$$\beta_K(x, r) := \inf_L \sup_{z \in K \cap B(x,r)} \frac{\text{dist}(z, L)}{r},$$

where the infimum is taken over all lines L in \mathbb{R}^d .

A bounded set K is called *uniformly wiggly* if

$$\beta_\infty(K) := \inf_{x \in K} \inf_{r \leq \text{diam } K} \beta(x, r) > 0.$$

[9, Theorem 1.1] states that if $K \subset \mathbb{C}$ is a continuum and $\beta_\infty(K) > 0$ then $\dim_H(K) \geq 1 + c\beta_\infty^2(K)$, where c is a universal constant.

In dynamical systems one cannot expect that generic systems be uniformly hyperbolic and that the geometry of invariant fractals can be controlled at every scale, as is required in [9]. However, there are numerous results showing that non-uniformly hyperbolic systems are typical in ambient parameter spaces; examples include the logistic family [4,26], rational maps [1,46], quadratic polynomials $z^2 + c$ with $c \in \partial\mathcal{M}$ [23,49], Hénon family [5].

To provide tools to study attractors/repellers of non-uniformly hyperbolic systems, [24, Theorem 1] states that if a continuum $K \subset \mathbb{C}$ is the union of two subsets $K = W \cup E$, $\mathcal{H}^1(E) < \infty$ and $\mathcal{H}^1(W) > 0$, then

$$\dim_H(K) \geq 1 + C \inf_{x \in W} \liminf_{r \rightarrow 0} \frac{\int_r^{\text{diam } K} \beta_K^2(x, t) \frac{dt}{t}}{-\log r}, \tag{4}$$

where C is a universal constant and \mathcal{H}^1 is the 1-Hausdorff measure. If the infimum in (4) is bigger than β_0^2 then we say that continuum K is *mean wiggly* with the parameter $\beta_0 \geq 0$.

It follows from [24,40] that Julia sets of quadratic polynomials \mathcal{J}_c are uniformly mean wiggly with β_0 comparable to $\sqrt{c+2}$, $c \in \hat{\mathcal{A}}$. This yields an ad hoc estimate for $c \in \hat{\mathcal{A}}$,

$$\dim_H(\mathcal{J}_c) \geq 1 + C|c + 2|$$

that is much weaker than the estimate given by Theorem 4. The explanation lies in the fact that (4) is valid for all continua including self-similar curves of von Koch type where the length in a given scale r in terms of $\beta := \beta(x, r)$ is, by the Pythagorean theorem, at least a constant multiple of

$$r\sqrt{1 + \beta^2} \sim r(1 + \beta^2/2).$$

The situation is very different for Julia sets \mathcal{J}_c , $c \in \hat{\mathcal{A}}$ which have a very particular property that they contain, at most points and scales $r > 0$, a ‘‘cross’’ of length $\sim r(1 + \beta)$, that corresponds to the lower estimate Theorem 4.

Surprisingly, according to [24, Theorem 2], estimate (4) can be almost inversed

$$\dim_{\text{H}}(W) \leq 1 + C' \sup_{x \in W} \liminf_{r \rightarrow 0} \frac{\int_r^{\text{diam}K} \beta_K^2(x, t) \frac{dt}{t}}{-\log r}, \tag{5}$$

where $C' > 0$ is a universal constant. If the supremum in (5) is smaller than β_0^2 then W is *almost flat* with $\beta_0 \geq 0$, see Fact 5.1. In the non-uniformly hyperbolic setting, of which $c \in \hat{\mathcal{A}}$ is a particular case, when K is a Julia set, W can be usefully chosen in (5) so that $\dim_{\text{H}}(K) = \dim_{\text{H}}(W)$. Indeed, the unique invariant conformal density has the property that it gives positive mass only to sets of the Hausdorff dimension $\dim_{\text{H}}(K)$, [21]. An additional advantage of such a choice is that every point from W is typical with respect to the dynamics and the Birkhoff averages converge. This turns out to be crucial for efficient estimates of the integral quantity

$$\liminf_{r \rightarrow 0} \frac{\int_r^{\text{diam}K} \beta_K^2(x, t) \frac{dt}{t}}{-\log r}$$

which is affected by the density of scales at x where $\beta_K(x, r)$ is large.

Passages to the large scale. The frequency with which points go univalently to the large scale will play a rôle in our analysis. With this in mind, we introduce the following terminology.

Definition 1.3. Given $z \in \mathbb{C}$, $r > 0$ and $f = f_c$, we denote by

$$\mathcal{U}(z, r)$$

the set of positive integers n for which, for some neighbourhood U of z , the map $f^n : U \rightarrow B(f^n(z), r)$ is biholomorphic. We call U a *level- n univalent pullback* of $B(f^n(z), r)$ containing z , or to z .

More generally, if W is connected and V is a connected component of $f^{-n}(W)$, we call V a *level- n pullback* of W . If W is simply connected then, as f^n is a polynomial, so is V .

Passages of the critical value to the large scale. In particular, the frequency with which the critical value goes univalently to the large scale has importance for us.

Proposition 1.4. *Given $\delta > 0$, there exist constants $\kappa, c_0 > -2$ and a set of parameters $\mathcal{A} \subseteq \hat{\mathcal{A}} \cap [-2, c_0]$ which has -2 as a one-sided Lebesgue density point and the following properties. For every $c \in \mathcal{A}$, $n \in \mathbb{N}$, there exists*

$$n^* \in \mathcal{U}(c, 1/2) \cap [n, \max(n(1 + \tau(c + 2)), n - \log(c + 2))],$$

where $\tau(\epsilon) = \exp(-\kappa \sqrt{-\log \epsilon})$.

For every n ,

$$|(f_c^{n^*})'(c)| \geq e^{n(\log 2 - \delta)}.$$

This is essentially due to Yoccoz [55] and we prove the proposition in Section 2.4. There is a set \mathcal{A}_{Yoc} of *strongly regular* parameters defined by Yoccoz. We shall show that \mathcal{A} can be taken in the form $\mathcal{A}_{\text{Yoc}} \cap [-2, c_0]$. We fix such a set \mathcal{A} , for $\delta = 1/10$. Remark that $c_0 > -2$ may be further decreased and the conclusion of the proposition above remains valid.

Conformal and absolutely continuous invariant measures. It is known [21] that every map $f_c, c \in \hat{\mathcal{A}}$, has a unique probabilistic invariant absolutely continuous invariant measure σ_c with respect to the unique *geometric measure* $\nu_c, \nu_c(\mathcal{J}_c) = 1$,

$$\nu_c(f_c(U)) = \int_U |f'_c(z)|^{\dim_{\text{H}}(\mathcal{J}_c)} d\nu_c$$

for any Borel set U on which f is injective, see Definition 3.1.

We shall eventually apply Birkhoff’s ergodic theorem; to do so, we require new, strong estimates on the distribution of σ_c near c (or equivalently, near 0). The following two propositions are proven in Section 3, see the *diagonal estimate* on page 32 and the *general upper bound* on page 33.

Let $\sigma_c(r) := \sigma_c(B(c, r))$ and $\epsilon = c + 2$.

Proposition 1.5. *Given $t_0 > 1$, there exists $C > 1$ such that, if $c \in \mathcal{A}$ and $c + 2 = \epsilon$ is sufficiently small, then for $r = a\epsilon, a \geq \epsilon^{t_0}$,*

$$\sigma_c(a\epsilon) \leq C a^{1/2} \sqrt{\epsilon} . \tag{6}$$

For smaller scales there is a weaker estimate.

Proposition 1.6. *There exists $C > 1$ such that, if $c \in \mathcal{A}$ and ϵ is sufficiently small, then for $r = a\epsilon > 0$,*

$$\sigma_c(a\epsilon) \leq C \max(a^{1/2}, a^{2/5}) \sqrt{\epsilon |\log \epsilon|} . \tag{7}$$

Upper estimates of $\dim_{\text{H}}(\mathcal{J}_c)$. Large scale geometric parameters for typical Julia sets $\mathcal{J}_c, c \in \mathcal{A}$, like global flatness [24], as well as regularity of the invariant conformal density, enter into the upper bound of Theorem 3. This surprising observation can be explained through the ergodic and β -numbers theories.

By [3], the Julia set \mathcal{J}_c is contained in the horizontal strip $|\Im(z)| \leq 2\sqrt{\epsilon}, \epsilon = |c + 2|$, and hence β s in the large scale are uniformly comparable to $\sqrt{\epsilon}$.

Scales of Julia sets \mathcal{J}_c bigger than $\sqrt{\epsilon}$ are directly affected by a large scale geometry which becomes flat when c tends to -2 ; scales smaller than $\sqrt{\epsilon}$ are wiggly with a frequency that can be calculated by the Birkhoff ergodic theory. A typical orbit that approaches c at the distance ϵ suffers $|\log \epsilon|$ corresponding scales where the wiggleness (that is, the beta number) is bigger than $\sqrt{\epsilon}$. Hence, the density of the scales with wiggleness bigger than $\sqrt{\epsilon}$ at the initial point of the orbit is at least $\sigma_c(\epsilon) |\log \epsilon|$. Taking into account (5), the above argument suggests that the upper bound for $\dim_{\text{H}}(\mathcal{J}_c)$ may not be better than $1 + C\sigma_c(\epsilon) |\log \epsilon|$.

Theorem 3. *There exists $\kappa^* > 0$ and $c_0 > -2$ such that for all parameters $c \in \mathcal{A} \cap (-2, c_0]$,*

$$\dim_{\text{H}}(\mathcal{J}_c) \leq 1 + \kappa^* |\log(c + 2)| (c + 2)^{\frac{1}{2}} .$$

Lower estimate of $\dim_{\text{H}}(\mathcal{J}_c)$. The lower estimate for $\dim_{\text{H}}(\mathcal{J}_c)$ is based on completely different techniques from the upper estimate.

Cantor repellers, Definition 6.1, form a well-known class of mappings that was studied intensively in the context of thermodynamical formalism [10,45,48] and the relations between harmonic and Hausdorff measures [11,31,32,36] both from ergodic and probabilistic perspectives.

A more general class of functions than Cantor repellers is constituted by *box mappings* [22], see Section 6. An important property of box mappings is that they have a non-trivial inner dynamics through *inducing* constructions proposed by M. Jakobson [26] for real unimodal maps and by J.-C. Yoccoz [55] for unicritical polynomials. In the current paper, we will apply a few simple inducing steps to canonically-defined box mappings to construct a family φ_c of Cantor repellers for every c close enough to -2 .

A novelty of our thermodynamical approach lies in proving uniform estimates of the Hausdorff dimension of Julia sets \mathcal{J}_{φ_c} , initially based on the dimension of the Julia set of φ_c restricted to the real line, and then using a further branch of φ_c to improve the lower bound. The thermodynamical foundations of this approach come from [44], we refer the reader for a more detailed discussion to [42].

Theorem 4. *There exists $\kappa_* > 0$ and $c_0 > -2$ such that for every $c \in (-2, c_0]$,*

$$\dim_{\text{H}}(\mathcal{J}_c) \geq 1 + \kappa_*(c + 2)^{\frac{1}{2}}.$$

Notation and uniform constants. We will use the notation $A \lesssim B$ to indicate that A/B is bounded from above by a uniform constant (with respect to $c \in \mathcal{A}$, where \mathcal{A} is fixed following Proposition 1.4). Similarly, we define $A \gtrsim B$. Finally, $A \sim B \Leftrightarrow A \lesssim B$ and $A \gtrsim B$, in which case we say that A and B are *comparable*.

Given $a < b$, we write $\llbracket a, b \rrbracket := [a, b] \cap \mathbb{Z}$.

1.4. Hyperbolic estimates along ray $(-\infty, -2]$. Asymptotic estimates at -2 for $c < -2$ are easier than these for $c > -2$ as they correspond to hyperbolic dynamics. They were originally proven in [20], we revisit the estimates giving different proofs based on both real and complex methods.

Lower bound. An ad hoc estimate for $\dim_{\text{H}}(J_c)$, $c < -2$, comes from Manning’s formula (see [11], [36]) for the Hausdorff dimension of the harmonic measure ω_c with a base point at ∞ ,

$$\dim_{\text{H}}(\mathcal{J}_c) \geq \dim_{\text{H}}(\omega_c) = \frac{\log 2}{\log 2 + G_c(c)}, \tag{8}$$

where G_c is the Green function for $\mathbb{C} \setminus \mathcal{J}_c$; G_c coincides at c with the Green function for $\hat{\mathbb{C}} \setminus \mathcal{M}$.

Since \mathcal{M} contains the interval $[-2, 0]$ which has logarithmic capacity $1/2$ [35, Corollary 9.9], $G_c(c) \leq 2\sqrt{|c+2|}$ and

$$\dim_{\text{H}}(\mathcal{J}_c) \geq \frac{\log 2}{\log 2 + 2\sqrt{|c+2|}} \geq 1 - \frac{2\sqrt{|c+2|}}{\log 2}.$$

Upper bound via real methods. The estimate that $G_c(c) \leq 2\sqrt{|c+2|}$, for $c < -2$, can be almost inversed due to Tan Lei’s result [52] about the conformal similarity between

\mathcal{M} and J_{-2} which implies that asymptotically for any positive ε , $G_c(c) \geq |c + 2|^{\frac{1}{2} + \varepsilon}$. It was proved by purely real methods in [20] that the ad hoc estimate is indeed precise. For the convenience of the reader we will present a short proof of the upper estimate of [20], $1 - C'\sqrt{|c + 2|} \geq \dim_{\text{H}}(\mathcal{J}_c)$.

Let

$$\epsilon = |c + 2| \tag{9}$$

and suppose that $c \in (-3, -2)$. Let p_c be the repelling fixed point of f_c at which f_c preserves the orientation on the real line. Let $\pm y$ denote the points $f^{-1}(-p_c)$, so $y \sim \sqrt{\epsilon}$. Denote by I the interval $[-p_c, p_c]$. Since $c < -2$, $f_c^n(c)$ tends to $+\infty$ and the Julia set is

$$\bigcap_{n \geq 0} f^{-n}(I).$$

If A is a connected component of $f^{-n}(I)$, then there are at most two iterates k ($0 \leq k \leq n - 1$) for which $f^k(\partial A) \cap \{\pm y\} \neq \emptyset$, one of which equals $n - 1$. Denote the other by k_0 . Hence f^n on A can be decomposed as

$$f^n = f \circ g_1 \circ f \circ g_2,$$

(or $f^n = f \circ g_1$ if k_0 does not exist) where g_1 and $g_2 = f^{k_0}$ have uniformly bounded distortion, using the Koebe distortion theorem for g_2 (a neighbourhood of A is mapped diffeomorphically by g_2 onto I and $g_2(A)$ is far from ∂I) and uniform boundedness of $\sum_{j=k_0+1}^{n-1} |f_c^j(A)|$ together with a standard argument using

$$\sum_{j=k_0+1}^{n-1} \int_{f_c^j(x)}^{f_c^j(y)} \log |Df_c(s)| ds$$

for g_1 , noting $-p_c \in f \circ g_2(A)$.

Let $E_j = \{x \in f^j(A) : f^{n-j}(x) \in [-y, y]\}$ and remark that $f^j(E_0) = E_j$ for $j \leq n$. The measure of $E_n = [-y, y]$ and E_{n-1} are both $\sim \sqrt{\epsilon}$. By bounded distortion (of g_1), $|E_{k_0+1}|/|f^{k_0+1}(A)| \sim \sqrt{\epsilon}$. Moreover, E_{k_0+1} is well inside $f^{k_0+1}(A)$ in the sense that each component of $f^{k_0+1}(A) \setminus E_{k_0+1}$ has length uniformly comparable to $|f^{k_0+1}(A)|$.

Pulling back once gives $|E_{k_0}|/|f^{k_0}(A)| \sim \sqrt{\epsilon}$ and then bounded distortion of g_2 gives $|E_0|/|A| \sim \sqrt{\epsilon}$.

Consequently, the length of $f^{-n}(I)$ is smaller than $(1 - C\sqrt{\epsilon})^n$, where C is a uniform constant. Since $f^{-n}(I)$ has 2^n components A , using Hölder's inequality with $p = \frac{1}{1-\alpha}$ and $q = \frac{1}{\alpha}$,

$$\sum |A|^\alpha = \sum 1 \cdot |A|^\alpha \leq 2^{(1-\alpha)n} (1 - C\sqrt{\epsilon})^{\alpha n} \lesssim 1$$

provided $\alpha \geq 1 - C'\sqrt{\epsilon}$ for some small uniform constant $C' > 0$. Therefore $\dim_{\text{H}}(\mathcal{J}_c) \leq \alpha$.

1.5. *Structure of the paper.* In the following section we recall general properties of Collet–Eckmann maps and then give properties of Yoccoz’ strongly regular parameters, those comprising the set \mathcal{A} .

Section 3 contains estimates on the conformal and absolutely continuous invariant measures. We decompose forward orbits depending on the passages near the critical point and use ergodicity to obtain estimates on the density of blocks of different types.

In Section 4, density of blocks of the forward orbit are translated into densities of scales of preimages. Section 5 is dedicated to the proof of the upper bound of the Hausdorff dimension using the β numbers theory.

The last section presents the proof of the lower bound of the Hausdorff dimension by constructing a Cantor repeller inside the Julia set.

2. Collet–Eckmann Parameters

2.1. *Technical sequences.* We shall consider maps $f = f_c$ with Collet–Eckmann parameter $c \in \hat{\mathcal{A}}$, see formula (1), with associated constants Ω, Ω' . We follow closely the formalism introduced in [21, Lemma 2.2] in order to make precise references to estimates of [21]. As in [21], we introduce three positive sequences $(\alpha_n), (\gamma_n), (\delta_n)$ as follows.

Definition 2.1. For $n \geq 1$, set

- $\delta_n = \frac{1}{8n^2},$
- $\gamma_n = 64 \frac{e^{n\Omega/4}}{1 - e^{-\Omega/4}},$
- $\alpha_n = (1 - e^{-\Omega/4}) \sqrt{\delta_n \Omega'} e^{n\Omega/4} / 64.$

The growth of the derivative of f^n will be given in terms of γ_n , the corresponding distortion will be bounded by δ_n , and various constants will be controlled through α_n .

Lemma 2.2. *The sequences $(\alpha_n), (\gamma_n),$ and (δ_n) satisfy*

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n &= \infty, \\ \sum_n \gamma_n^{-1} &< 1/64, \\ \sum_n \delta_n &< 1/2 \end{aligned}$$

and, for every $c \in \hat{\mathcal{A}},$

$$|(f_c^n)'(c)| \geq \alpha_n^2 \gamma_n^2 / \delta_n.$$

Proof. The bound is a direct consequence of Definitions (1) and 2.1. □

2.2. *Constants and scales.* A scale around the critical point 0 of $f_c(z) = z^2 + c$ is given in terms of a fixed number $R' \ll 1$ as in [21] where the general case of rational maps was studied. We will refer to objects which stay away from the critical points at distance R' and are comparable in size to R' as the objects of the *large scale*. The proper choice of R' is crucial in obtaining uniform estimates based on the Poincaré series technique of [21].

The following conditions define R' , compare the conditions (i – iv) from [21, Section 2.3].

- (i) fix τ so that $\alpha_k \geq 10^3$ for all $k \geq \tau$; let $R > 0$ be so small that the first return time of 0 to $\{|z| < \sqrt{R}\}$ by f_c is at least τ ;
- (ii) R' satisfies the condition $0 < R' \leq R \inf_n (\alpha_n)^2 / 10^3$.

Note that $R' \ll R$ as $\alpha_1 < 1$. The constants R and R' above can be fixed uniformly for $c \in \hat{\mathcal{A}}$. Indeed, if $|f_c^n(c)| < \sqrt{R}$ for some $n < \tau$ and $c \in \hat{\mathcal{A}}$, then by (1)

$$\Omega' e^{\tau\Omega} \leq |(f_c^\tau)'(c)| < 4^\tau \sqrt{R},$$

so R can be chosen explicitly

$$R := (\Omega')^2 4^{-2\tau} e^{\tau\Omega}.$$

2.3. Uniform contraction. Replacing R' by a smaller constant if necessary, the following backward contraction results hold.

Proposition 2.3. *There exist $\xi \in (0, 1)$ and $C > 0$ such that, for all $c \in \hat{\mathcal{A}}$, for every $n \in \mathbb{N}$, for every ball $B(z, r)$, $z \in \mathcal{J}_c$ with $r \leq R'$, the diameter of every component of $f_c^{-n}(B(z, r))$ is smaller than ξ^n . Moreover, if W_n is a component of $f_c^{-n}(B(z, r))$, $r \leq R'$, then*

$$\text{diam } W_n < C \xi^n \sqrt{r}. \tag{10}$$

For all $n \geq 0$, $z \in \mathcal{J}_c$ and V_n a connected component of $f_c^{-n}(B(z, 2R'))$

$$\text{diam } V_n < \frac{1}{1000}. \tag{11}$$

The constant ξ is called the *backward contraction factor*, see [40]. For a given c , these estimates are known to hold; we must check that the estimates are uniform for $c \in \hat{\mathcal{A}}$. The proof occupies the remainder of this subsection.

In [21], a decomposition of orbits into three types was introduced. Types 1 and 3 correspond to pieces of the orbit shadowed closely by the corresponding pieces of the critical orbit. For the reader familiar with the work of [5], types 1 and 3 correspond to the bound period, the formal definition can be found in [21]. Our focus will be on type 2 preimages which correspond to the free period in [5].

Second type. A piece of a backward orbit is of the second type if there exists a neighbourhood of size R' which can be pulled back univalently along the backward orbit. Type two preimages yield expansion along pieces of orbits of a uniformly bounded length L . In this setting, type 2 corresponds to pieces of backward orbits which stay at a definite distance from the critical points. The following lemma shows that L can be chosen uniformly in $c \in \hat{\mathcal{A}}$ so that any univalent pullback of length at least L of a large scale ball yields a definite backward contraction.

Lemma 2.4. *Given $\lambda > 1$, there exist $L > 1$ such that the following holds for all $c \in \hat{\mathcal{A}}$, $n \geq L$ and $z \in \mathbb{C}$ with $\text{dist}(z, \mathcal{J}_c) \leq R'/2$. If the ball $B(z, R')$ can be pulled back univalently along a sequence $f_c^{-n}(z), \dots, f_c^{-1}(z)$, z of preimages of z , then*

$$|(f_c^n)'(f_c^{-n}(z))| > \lambda.$$

Proof. Observe that by its definition (1), \hat{A} is a compact set. Also, by the Collet–Eckmann condition (1) for parameters $c \in \hat{A}$, the only Fatou component of f_c is its basin of attraction of infinity $A_c(\infty)$. If a point $z \in A_c(\infty)$, then there exists $\varepsilon > 0$ and, for all $z' \in B(z, \varepsilon)$ and $c' \in B(c, \varepsilon)$, $z' \in A_{c'}(\infty)$.

Let $(c_k) \subset \hat{A}$, $(z_k) \subset \mathbb{C}$ and a strictly increasing sequence $(N_k) \subset \mathbb{N}$ such that for all $k \geq 0$, $z_k \in J_{c_k}$ and the ball $B(z_k, R'/2)$ can be pulled back univalently by f_{c_k} along a backward orbit of z_k of length N_k .

By compactness, we may assume that there exists $c \in \hat{A}$ and $z \in \overline{B(0, 2)}$ such that

$$c = \lim_{k \rightarrow \infty} c_k \text{ and } z = \lim_{k \rightarrow \infty} f_{c_k}^{-N_k}(z_k).$$

As $c \in \hat{A}$, $z_k \in J_{c_k}$, $z \notin A_c(\infty)$ so $z \in J_c$.

For any $\varepsilon > 0$, there exists an N such that $f_c^N(B(z, \varepsilon))$ contains neighbourhoods of both fixed points of f_c , so for large k , $f_{c_k}^n(B(z, \varepsilon))$ contains both fixed points of f_{c_k} for all $n \geq N$. Therefore $f_{c_k}^n(B(z, \varepsilon))$ is not contained in a ball of radius R' for $n \geq N$. Consequently, $B(z, \varepsilon)$ is not contained in a pullback of $B(z_k, R'/2)$ by $f_{c_k}^{N_k}$ for large k . Upon choosing ε appropriately, the result follows by the Koebe 1/4 theorem. \square

We fix L provided by the previous lemma for $\lambda = 100$.

Contraction of preimages. The Collet–Eckmann condition implies the existence of the backward contraction factor ξ , via [21, Propositions 2.1 and 7.2], which depends only on the constant Ω from (1), see also the condition *ExpShrink* and the main result in [39]. Indeed in the proof of [21, Proposition 7.1], we have that ξ^{-1} can be taken as $\inf_n (\tilde{\omega}_n)^{1/n}$, where

$$\tilde{\omega}_n := \inf \{ \gamma_{k_1} \dots \gamma_{k_l} \omega_m / 16 : k_1 + \dots + k_l + m = n \}$$

and ω_n was defined in the proof of Proposition 2.1 of [21],

$$\omega_n := \inf \left\{ K \lambda^{k_0} \prod_{j \geq 1} \gamma'_{k_j} : k_0 + k_1 + k_2 + \dots \in [n - L, n) \right\},$$

where $K > 0$ is the expansion yielded by type 2 preimages of the length $l \in [0, L)$, λ is the average expansion of a type 2 block and

$$\gamma'_n := \inf \left\{ \prod_j \gamma_{i_j} : i_0 + i_1 + i_2 + \dots = n \right\}.$$

This shows that $\xi < 1$ and depends solely on Ω and Ω' in (1).

Remark 7.1 of [21] implies the stronger uniform estimate (10). By eventually shrinking R' , we directly obtain (11). This completes the proof of Proposition 2.3.

2.4. *Visits to the large scale for strongly regular parameters.* For $c \in [-2, c_0]$, $c_0 + 2 > 0$ and small, we denote by q_c the orientation-reversing fixed point of f_c and p_c the orientation-preserving one. By direct computation

$$p_c = 2 - \frac{\epsilon}{3} + O(\epsilon^2) \text{ and } c + p_c = \frac{2}{3}\epsilon + O(\epsilon^2). \tag{12}$$

The other fixed point, q_c , is in a small neighbourhood of -1 . We have $-2 < -p_c < c < q_c < 0$.

J.-C. Yoccoz in [55] defines a set of *strongly regular* parameters, which we shall denote by \mathcal{A}_{Yoc} , and shows that -2 is a one-sided density point for \mathcal{A}_{Yoc} . Let $c \in \mathcal{A}_{\text{Yoc}}$. Just prior to Definition 3.7 in [55], a sequence of times $(N_k)_k$ called *regular returns* is defined. By the definition of regular returns, $f_c^{N_k-1}(c) \in (q_c, -q_c)$ and a neighbourhood of c is mapped by $f_c^{N_k-1}$ diffeomorphically onto the open interval \widehat{A} of [55] (defined such that $f_c(\partial\widehat{A}) = \{-q_c\}$), which contains a $\frac{1}{2}$ -neighbourhood of $A = [q_c, -q_c]$ if c is close to -2 . The number $M = N_1$ depends on c , increasing as $c \rightarrow -2$, $\log \epsilon = \log |c + 2| \sim -2M \log 2$ in [55, Proposition 3.1(3)].

If $\epsilon = c + 2$ is close to 0 then the critical orbit (starting with $f_c(c)$) is “trapped” for some time, comparable to M , in a vicinity of p_c .

Let g_c be the inverse of the restriction of f_c to $\{\Re(z) > 0\}$. Observe that g_c is univalent on $B(p_c, 3)$ and $g_c(B(p_c, 3)) \subseteq B(p_c, 3)$. Therefore, g_c^k is also univalent on $B(p_c, 3)$ for all $k > 0$. As $R' \ll 1$, the distortion of g_c^k is bounded on $B(p_c, 20R')$ by a constant very close to 1 and by 2 on $B(p_c, 1/10)$.

A return $N_k, k \geq 1$ is *simple* if for all $N_k < n < N_{k+1}$,

$$f_c^n(0) \notin A.$$

Another characterisation of simple returns is [55, Lemma 3.6],

$$N_{k+1} - N_k < M - 1.$$

By Remark 3.9 in [55], all returns with $N_k < 2^{\sqrt{M}}M$ are simple.

Let us denote $c_k := f_c^{N_k}(0)$. Assume that N_k is a simple return and let $m = N_{k+1} - N_k$. Then by the distortion bounds above (and few iterates outside $B(p_c, 1/10)$), we have that for all $j \in \llbracket 0, m - 1 \rrbracket$

$$|(f_c^j)'(f_c(c_k))| \sim |2p_c|^j \text{ and } f_c(c_k) + p_c \sim |2p_c|^{-m}.$$

By estimate (12), for all $j \in \llbracket 0, \epsilon^{-1} \rrbracket$,

$$|2p_c|^j = 4^j \left(1 - \frac{\epsilon}{6} + O(\epsilon^2)\right)^j \sim 4^j.$$

As $m < M - 1 \sim |\log \epsilon| \ll \epsilon^{-1}$, combining the previous two estimates, for all $j \in \llbracket 0, m - 1 \rrbracket$,

$$|(f_c^j)'(f_c(c_k))| \sim 4^j \text{ and } f_c(c_k) + p_c \sim 4^{-m}. \tag{13}$$

As $N_1 = M$ and $m \leq M - 2$, by estimate (12)

$$-p_c < c < -f_c(c) < f_c(c_k) \text{ and } |f_c(c) + c| > \frac{5}{2}|c + p_c| > \epsilon.$$

Pulling back once by f_c , for all $j \in \llbracket 1, m \rrbracket$ we have

$$\sqrt{\epsilon} < |c_k| \sim 2^{-m} \text{ and } |(f_c^j)'(c_k)| \sim 2^{2j-m}. \tag{14}$$

Lemma 2.5. *Given $\delta > 0$, there exists $c_0 > -2$ such that, if $c \in [-2, c_0]$ and the first k return times N_1, \dots, N_k of c to A are simple, then*

$$\left| (f_c^j)'(c) \right| \geq e^{j(\log 2 - \delta)}$$

for all $j = 1, \dots, N_k$.

Proof. Let $C_* > 4$ be a universal multiplicative constant which makes (14) hold. Choose N large enough that $e^{N\delta} > C_*^2$. Let ϵ be small enough that $e^{M\delta} > C_*^{2N}$. Then

$$4^M = 2^M e^{M \log 2} > 2^M e^{M(\log 2 - \delta)} C_*^{2N}.$$

By the conjugacy between f_{-2} and the full tent map, if $x \in [-1, 1]$ and $f_{-2}^n(x) \in B(0, 1.1)$ then (see [55, Section 2.2]), for all $n \geq 1$,

$$\left| (f_{-2}^n)'(x) \right| \geq 2^n \frac{1}{2}.$$

By continuity, if c is close enough to -2 , if $x, f_c^n(x) \in A$, and $n \leq N$ then

$$\left| (f_c^n)'(x) \right| \geq 2^n \frac{1}{4}. \tag{15}$$

We have $N_1 = M$ and

$$\left| (f_c^j)'(c) \right| \geq \max(2^j, 4^j / C_*)$$

for $j \leq M - 1$.

Now we examine times N_{k_j} where $k_0 = 1$ and k_{j+1} is the minimal $k' > k_j$ with $N_{k'} - N_{k_j} > N$. Let $m_j = N_{j+1} - N_j \geq 2$ and $m = N_{k_{j+1}} - N_{k_j}$. We have that $m - m_{k_{j+1}-1} \leq N$. Thus we can split up the iterates from c_{k_j} until $c_{k_{j+1}}$ into one sequence of simple returns with combined length at most N , and one single simple return. Applying (15) and (14), we obtain

$$\left| (f_c^m)'(c_{k_j}) \right| \geq 2^m \frac{1}{4C_*}.$$

As $e^{N\delta} > C_*^2$ and $m > N$,

$$\left| (f_c^m)'(c_{k_j}) \right| \geq e^{m(\log 2 - \delta)}.$$

This proves the lemma at all times N_{k_j} .

For intermediate times, we use $k_{j+1} - k_j < N$ and the initial growth estimate

$$\left| (f_c^{M-1})'(c) \right| \geq 4^M / C_*^2 > 2^M e^{M(\log 2 - \delta)}$$

and apply (14) repeatedly. □

Proof of Proposition 1.4. With the notation introduced above, let $m = N_{k+1} - N_k$ for some $k \geq 1$. From [55, equation (3.2)] if $m > M$,

$$m \leq 2^{-\sqrt{M}}(N_k + m). \tag{16}$$

If we set $\tau(\epsilon) = \exp(-\kappa \sqrt{\log 1/\epsilon})$, for κ small enough, then $m \leq N_k \tau(\epsilon)$, which yields the bound on n^* in Proposition 1.4 if $c_0 + 2$ is small enough and $c \in \mathcal{A}_{\text{Yoc}}$.

Now we prove the estimate on the growth of the derivative along the critical orbit of $c \in \mathcal{A}_{\text{Yoc}}$. By [55, Proposition 3.10] with $g_{B(k)}$ satisfying $f_c^{N_k-1} \circ g_{B(k)} = \text{Id}_{\hat{\Delta}}$ and $x = c_k$,

$$\left| \log \left(|(f_c^{N_k-1})'(c)| \frac{h_c(c)}{h_c(c_k)} \right) - (N_k - 1) \log 2 \right| < CM^{-1}N_k,$$

where $\log h_c(c) \sim M \log 2$ and $h_c(c_k) \sim 1$; C is a universal constant. In particular, at each such time,

$$\left| \frac{1}{N_k - 1} \log |(f_c^{N_k-1})'(c)| - \log 2 \right| < \frac{C}{M} \frac{N_k}{N_k - 1} + \frac{C' M}{N_k - 1},$$

where $C' > 0$ is an universal constant. The returns being regular, the Koebe distortion theorem gives a lower bound $C_1 > 0$ to the derivative of f_c^m at c_k . Hence for any $1 \leq j \leq m$, $|(f_c^j)'(c_k)| \geq 4^{-m} C_1$. We also deduce that

$$\begin{aligned} \left| \frac{\log |(f_c^{N_k+j-1})'(c)|}{N_k + j - 1} - \log 2 \right| &< \frac{C}{M} \frac{N_k}{N_k - 1} + \frac{C' M}{N_k - 1} \\ &+ \left| \frac{1}{N_k + j - 1} \log \frac{|(f_c^j)'(c_k)|}{2^j} \right|. \end{aligned}$$

The last term above is bounded by $4m/N_k$ and allows us to treat the case $N_k > 2^{\sqrt{M}-2}M$. In this case,

$$4m/N_k \leq 2^{4-\sqrt{M}}(1 - 2^{2-\sqrt{M}}) \ll M^{-1},$$

which is immediate if $m \leq M$, while for $m > M$ we use (16). As the first non-simple return happens at a time at least $2^{\sqrt{M}}M$ and simple returns have return time bounded by M , there is a regular return N_k with

$$2^{\sqrt{M}-2}M < N_k < 2^{\sqrt{M}-1}M.$$

Consequently, for all $n \geq 2^{\sqrt{M}-1}M$,

$$\left| \frac{1}{n} \log |(f_c^n)'(c)| - \log 2 \right| < \frac{C_2}{M} \sim \frac{1}{\log \frac{1}{\epsilon}}, \tag{17}$$

where C_2 is a universal constant. Given $\delta > 0$, for large M , $C_2/M < \delta$. To finish the proof, note that regular return times up to some $N_k > 2^{\sqrt{M}-1}M$ are simple and apply Lemma 2.5. \square

Let

$$k_0 := \max\{k > 0 : N_k < 2^{\sqrt{M}} M\}.$$

For each $k \in \llbracket 1, k_0 \rrbracket$, let ϵ_k be maximal such that

$$B(c, \epsilon_k) \subseteq f_c^{1-N_k}(B(c_k, R')).$$

By bounded distortion and Schwarz lemma

$$\epsilon_k |(f_c^{N_k-1})'(c)| \in (R'/2, R') \tag{18}$$

and $\epsilon_1 \sim \epsilon_{R'} \sim \epsilon$.

By estimates (14), for all $k \in \llbracket 2, k_0 \rrbracket$

$$\sqrt{\epsilon} \lesssim 2^{N_{k-1}-N_k} \sim \frac{\epsilon_k}{\epsilon_{k-1}} \sim |c_{k-1}|. \tag{19}$$

Let us denote

$$r_c := \epsilon_{k_0}.$$

Lemma 2.6. *For any $t > 1$, if $\epsilon = c + 2$ is sufficiently small and $c \in \mathcal{A}$, then*

$$r_c < \epsilon^t.$$

Proof. By choice of \mathcal{A} following Proposition 1.4, taking $\delta = 1/10$ and $c \in \mathcal{A}$, for all $n \geq 0$,

$$|(f_c^n)'(c)| \geq e^{n(\log 2 - \delta)}.$$

Combine this with bound (18) to get

$$r_c < R' e^{(1-N_{k_0})(\log 2 - \delta)},$$

while bound (13) for $k = 0$ gives $\epsilon \geq e^{-2M \log 2 - C}$, for some uniform constant $C > 0$.

By inequality (16),

$$N_{k_0} \geq M(2^{\sqrt{M}} - 2) > 1 + t \frac{2M \log 2 + C}{\log 2 - \delta},$$

if M is large. The above bounds on ϵ and r_c are sufficient to conclude. □

3. Statistical Methods

3.1. *Conformal measure of disks centred at c .* Conformal or *Sullivan-Patterson* measures are dynamical analogues of Hausdorff measures in dynamical systems.

Definition 3.1. Let $f_c = z^2 + c$ be a rational map with the Julia set \mathcal{J}_c . A Borel measure ν supported on J is called *conformal with an exponent p* (or *p -conformal*) if for every Borel set U on which f_c is injective one has

$$\nu(f_c(U)) = \int_U |f'_c(z)|^p \, d\nu .$$

As observed in [50], the set of pairs (p, ν) with p -conformal measure ν is compact (in the weak-* topology). Hence, there exists a conformal measure with the *minimal exponent*

$$\delta_{\text{conf}}(c) := \inf\{p : \exists \text{ a } p\text{-conformal measure on } \mathcal{J}_c\}.$$

The minimal exponent $\delta_{\text{conf}}(c)$ is also called the *conformal dimension* of \mathcal{J}_c .

The following fact, proven in [21] (Theorems 3 and 7), explains basic properties of conformal measures for Collet–Eckmann parameters. Several claims of the following two facts were known (in the Collet–Eckmann setting) earlier, see [38].

Fact 3.2. *Let $c \in \mathcal{A}$. Then there is a unique, ergodic, non-atomic, and probabilistic conformal measure ν_c for f_c , with exponent*

$$\delta_{\text{conf}}(c) = \dim_{\text{H}} \mathcal{J}_c.$$

Moreover,

$$\dim_{\text{H}}(\nu_c) = \dim_{\text{H}} \mathcal{J}_c .$$

Preliminaries. By compactness, the eventually onto property and Definition 3.1, we have

$$\nu_0 := \inf\{\nu_c(B(x, R'/2)) : c \in \mathcal{A} \text{ and } x \in [-p_c, p_c]\} > 0.$$

Therefore, whenever a branch of f_c^{-n} is univalent on some $B(x, R')$, $x \in \mathcal{J}_c$, there is a constant $C_\nu > 1$ depending only on ν_0 and R' , such that if W is the corresponding connected component of $f_c^{-n}(B(x, R'/2))$

$$C_\nu^{-1}(\text{diam } W)^{\delta_{\text{conf}}(c)} \leq \nu_c(W) \leq C_\nu(\text{diam } W)^{\delta_{\text{conf}}(c)}. \tag{20}$$

Moreover, by the bounded distortion of f_c^n on W , there is a universal constant $K_2 > 1$ such that if $W \ni y := f_c^{-n}(x)$, $r := K_2^{-1} \text{diam } W$ and $r' := K_2 \text{diam } W$,

$$\nu_c(B(y, r)) \leq C_\nu K_2^{\delta_{\text{conf}}(c)} r^{\delta_{\text{conf}}(c)}$$

and

$$C_\nu^{-1} K_2^{-\delta_{\text{conf}}(c)} r'^{\delta_{\text{conf}}(c)} \leq \nu_c(B(y, r')).$$

We apply this now near p_c . As p_c is a repelling fixed point, any ball of radius at least comparable to (in particular, at least equal to R' times) the distance from its centre to

p_c gets mapped by an iterate of f_c with bounded distortion to the large scale. Applying this and using conformality, we obtain that, for all $0 < r < 2$ (and $c \in \mathcal{A}$)

$$v_c(B(p_c, r)) \sim r^{\delta_{\text{conf}}(c)},$$

and for all $\epsilon R' \leq r \leq 2$

$$v_c(r) \sim r^{\delta_{\text{conf}}(c)}, \tag{21}$$

where $v_c(r) := v_c(B(c, r)) = v_c(B(-c, r))$ and we used the \sim notation for uniform constants from page 12.

As an immediate consequence, for all $x \in [-p_c, p_c] \setminus (-\sqrt{\epsilon R'}, \sqrt{\epsilon R'})$,

$$v_c(B(x, |x|/2)) \lesssim |x|^{\delta_{\text{conf}}(c)}. \tag{22}$$

Recall the definition of k_0 and ϵ_k before bound (18). For all $k \in \llbracket 1, k_0 \rrbracket$, by the bounded distortion of $f_c^{N_k-1}$,

$$v_c(\epsilon_k) \sim \epsilon_k^{\delta_{\text{conf}}(c)}. \tag{23}$$

General estimates. We want to obtain a sharp upper bound for $v_c(r)$ for all $r > 0$. We have to distinguish two cases depending on the range of r .

Lemma 3.3. *There exists a uniform constant $S \geq 1$ such that for every $0 < r \leq r_c = \epsilon_{k_0}$ and $c \in \mathcal{A}$,*

$$r^{\delta_{\text{conf}}(c)(1+S\tau(\epsilon))} \lesssim v_c(r) \lesssim r^{\delta_{\text{conf}}(c)(1-S\tau(\epsilon))},$$

where $\tau(\epsilon) = \exp(-\kappa \sqrt{\log 1/\epsilon})$ comes from Proposition 1.4.

Proof. We show the upper bound. Denote by U_k the level $(N_k - 1)$ univalent pullback of $B(c_k, R'/2)$ to c . Let $k \geq k_0$ be maximal with $U_k \supset B(c, r)$. With this choice,

$$v_c(r) \leq v_c(U_k) \sim \text{diam}(U_k)^{\delta_{\text{conf}}(c)}.$$

Choose $k' \geq k + 1$ minimal with $N_{k'} - N_k \geq L$ (the constant L was fixed after the proof of Lemma 2.4). Then $U_{k'} \subset U_k$ by Lemma 2.4 and

$$\text{diam}(U_{k'}) \lesssim r \lesssim \text{diam}(U_k).$$

It remains to bound the ratio of $\text{diam}(U_k)$ and $\text{diam}(U_{k'})$. By the definitions of k_0 before bound (18) and of $\tau(\epsilon)$ in Proposition 1.4,

$$N_{k'} \leq (N_k + L)(1 + \tau(\epsilon)).$$

By bounded distortion,

$$\frac{\text{diam}(U_{k'})}{\text{diam}(U_k)} \gtrsim \frac{\text{diam}(f_c^{N_k-1}(U_{k'}))}{R'} \gtrsim 4^{-(N_{k'}-N_k)} \gtrsim 4^{-N_k \tau(\epsilon)}.$$

Now $4^{-N_k} = \xi^{SN_k}$, where $S = -\log 4 / \log \xi$ and ξ is the backward contraction factor of (10). Meanwhile, $\xi^{N_k} > r$ by choice of k . Hence

$$\frac{\text{diam}(U_{k'})}{\text{diam}(U_k)} \gtrsim r^{S\tau(\epsilon)}.$$

We conclude that

$$\begin{aligned} \nu_c(r) &\leq \nu_c(U_k) \lesssim \text{diam}(U_k)^{\delta_{\text{conf}}(c)} \\ &\lesssim \text{diam}(U_{k'})^{\delta_{\text{conf}}(c)} r^{-\delta_{\text{conf}}(c)S\tau(\epsilon)} \lesssim r^{\delta_{\text{conf}}(1-S\tau(\epsilon))}. \end{aligned}$$

The lower bound is proved by a similar method. □

Sharp estimates for intermediate scales. For scales larger than r_c , we need first to estimate $\nu_c(B(0, 2\sqrt{\epsilon}))$. For any $0 < r < r'$, let us denote $A(r, r') := B(0, r') \setminus B(0, r)$ the annulus around the critical point of given radii. Observe that by Definition 3.1

$$\nu_c(A(r_1, r_2)) \leq 2^{1-\delta_{\text{conf}}(c)} r_1^{-\delta_{\text{conf}}(c)} \nu_c(r_2^2). \tag{24}$$

This bound will allow us to transfer estimates near c to estimates for annuli around 0. The times N_k will allow us to transfer estimates near 0 to estimates around c . We do so repeatedly in the following bootstrapping argument.

Lemma 3.4. *There exists a constant $C > 0$ such that for all $c \in \mathcal{A}$ and $r \geq \sqrt{\epsilon}$*

$$\nu_c(B(0, r)) \leq Cr^{\delta_{\text{conf}}(c)}.$$

Proof. First, using bounds (21) and (24), if $r \geq \sqrt{\epsilon_1} \gtrsim \sqrt{\epsilon R'}$

$$\nu_c(A(r, 2r)) \lesssim r^{\delta_{\text{conf}}(c)},$$

which, as $\delta_{\text{conf}}(c) \geq 1$, sums to

$$\nu_c(A(\sqrt{\epsilon_1}, r)) \lesssim \sum_{n \geq 1} (2^{-n}r)^{\delta_{\text{conf}}(c)} \leq r^{\delta_{\text{conf}}(c)}. \tag{25}$$

By bounds (24) and (19), we have

$$\nu_c(A(\sqrt{\epsilon_{k+1}}, \sqrt{\epsilon_k})) \lesssim \epsilon_{k+1}^{-\delta_{\text{conf}}(c)/2} \epsilon_k^{\delta_{\text{conf}}(c)} \lesssim \epsilon^{-\delta_{\text{conf}}(c)/4} \epsilon_k^{\delta_{\text{conf}}(c)/2}.$$

Summing up, together with bound (25), we obtain

$$\nu_c(A(\sqrt{r_c}, \sqrt{\epsilon_1})) \lesssim \epsilon^{\delta_{\text{conf}}(c)/4}. \tag{26}$$

For $r \leq \frac{1}{2}\sqrt{r_c}$, by Lemma 3.3 and bound (24)

$$\nu_c(A(r, 2r)) \lesssim r^{\delta_{\text{conf}}(c)(1-S\tau(\epsilon))},$$

which sums to

$$\nu_c(B(0, \sqrt{r_c})) \lesssim r_c^{\delta_{\text{conf}}(c)(1-S\tau(\epsilon))}, \tag{27}$$

a negligible quantity compared to the estimate we have for $\nu_c(A(\sqrt{r_c}, \sqrt{\epsilon}))$, as $r_c \ll \epsilon^2$ by Lemma 2.6 and $\tau(\epsilon) \ll 1$.

Combining (25)-(27), we obtain that for all $r \in [\epsilon^{1/4}, 2]$,

$$\nu_c(B(0, r)) \lesssim r^{\delta_{\text{conf}}(c)}. \tag{28}$$

In order to extend the range of validity to the desired $r \in [\sqrt{\epsilon}, 2]$, it suffices to improve the exponent $\delta_{\text{conf}}(c)/4$ in (26) to $\delta_{\text{conf}}(c)/2$. This task occupies the remainder of the proof.

Let

$$\Lambda := \left\{ r \in (0, 2] : \nu_c(r) \lesssim r^{\delta_{\text{conf}}(c)} \right\}.$$

We have already proven in (23) that $\epsilon_k \in \Lambda$ for all $k \in \llbracket 1, k_0 \rrbracket$ and estimate (21) shows that $[\epsilon R', 2] \subseteq \Lambda$. We need to show that $[\epsilon_{k+1}, \epsilon_k] \in \Lambda$. For each $k \in \llbracket 1, k_0 - 1 \rrbracket$, we distinguish three cases:

1. $|c_k| \geq R'$;
2. $|c_k| \in (\epsilon^{1/4}, R')$;
3. $|c_k| \leq \epsilon^{1/4}$.

The first case is the easiest to treat, as $\epsilon_{k+1} \sim \epsilon_k$ by equation (19), so $[\epsilon_{k+1}, \epsilon_k] \subseteq \Lambda$.

To treat case 2, we pull back $B(0, 2r) \supset B(c_k, r)$ by $f_c^{1-N_k}$ for all $r \in [|c_k|, R']$ and, using estimate (28), we get that

$$\llbracket |c_k| \epsilon_k, \epsilon_k \rrbracket \subseteq \Lambda.$$

By estimate (19), we have that $|c_k| \epsilon_k \sim \epsilon_{k+1}$ so $[\epsilon_{k+1}, \epsilon_k] \subseteq \Lambda$.

For case 3, we again pull back $B(0, 2r) \supset B(c_k, r)$ by $f_c^{1-N_k}$, this time for all $r \in [\epsilon^{1/4}, R']$, to obtain

$$[\epsilon^{1/4} \epsilon_k, \epsilon_k] \subseteq \Lambda.$$

By estimate (19), $\epsilon^{1/4} \epsilon_k \lesssim \sqrt{\epsilon_k \epsilon_{k+1}}$ so $[\sqrt{\epsilon_k \epsilon_{k+1}}, \epsilon_k] \subseteq \Lambda$.

For $r \in [(\epsilon_{k+1} \epsilon_k)^{1/4}, \sqrt{\epsilon_k}/2]$, as $4r^2 \in \Lambda$, we bound $\nu_c(A(r, 2r))$ by (24) and sum up to get

$$\nu_c(A((\epsilon_{k+1} \epsilon_k)^{1/4}, \sqrt{\epsilon_k})) \lesssim \epsilon_k^{\delta_{\text{conf}}(c)/2}.$$

We use estimate (24) for $A(\sqrt{\epsilon_{k+1}}, (\epsilon_{k+1} \epsilon_k)^{1/4})$ to obtain the same upper bound for its measure, thus

$$\nu_c(A(\sqrt{\epsilon_{k+1}}, \sqrt{\epsilon_k})) \lesssim \epsilon_k^{\delta_{\text{conf}}(c)/2}.$$

Summing for $k \in \llbracket 1, k_0 - 1 \rrbracket$ provides the desired estimate. □

Lemma 3.5. *Uniformly in $c \in \mathcal{A}$, for all $r_c \leq r \leq 2$,*

$$\nu_c(r) \lesssim r^{\delta_{\text{conf}}(c)}.$$

Proof. Remark that, using the notations from the proof of Lemma 3.4, it is enough to show that $\Lambda \supset [r_c, 2]$. As its conclusion strengthens bound (28), we can replace $\epsilon^{1/4}$ by $\sqrt{\epsilon}$ in the definition and proof of the three cases treated there. By the bound (14), $\sqrt{\epsilon} < |c_k|$, so the third case becomes void. The same argument, when $\sqrt{\epsilon}$ is substituted for $\epsilon^{1/4}$ proves that in the other two cases, for all $k \in \llbracket 1, k_0 - 1 \rrbracket$, $[\epsilon_{k+1}, \epsilon_k] \subseteq \Lambda$. As $\epsilon \sim \epsilon_1$, bound (21) completes the proof. □

For each $c \in \mathcal{A}$, $\epsilon = c + 2$ let us define

$$\eta(\epsilon) := 1 - S\tau(\epsilon), \tag{29}$$

where S and $\tau(\epsilon)$ are given by Lemma 3.3.

Proposition 3.6. *There exists a uniform constant $C > 1$ so that for every $\zeta \in (1, 2\eta(\epsilon))$, $c \in \mathcal{A}$ and $r \in (r_c, R')$*

$$C^{-1} \int_{B(c,r)} |x - c|^{-\delta_{\text{conf}}(c)\zeta/2} d\nu_c(x) \leq \frac{r^{1-\zeta/2}}{2 - \zeta} + \frac{r_c^{\eta(\epsilon)-\zeta/2}}{2\eta(\epsilon) - \zeta}. \tag{30}$$

Proof. We put n_c to be the smallest n such that $r/2^n \leq r_c$, hence $n_c \sim \log \frac{r}{r_c}$.

Define $A_n = \{x \in \mathbb{C} : 2^{-(n+1)} \leq |x - c|/r < 2^{-n}\}$, $n \in \mathbb{N}$, and set $I_r := \int_{B(c,r)} |x - c|^{-\delta_{\text{conf}}(c)\zeta/2} d\nu_c$. Splitting $B(c, r)$ into annuli A_n , $n \geq 0$, and using that ν_c is non-atomic measure by Fact 3.2,

$$I_r \leq 4r^{-\delta_{\text{conf}}(c)\zeta/2} \left(\sum_{n=0}^{n_c} 2^{n\delta_{\text{conf}}(c)\zeta/2} \nu_c(A_n) + \sum_{n=n_c}^{+\infty} 2^{n\delta_{\text{conf}}(c)\zeta/2} \nu_c(A_n) \right)$$

For all $n \leq n_c$, we use the sharp estimate of Lemma 3.5,

$$\nu(A_n) \leq \nu(B(c, 2^{-n}r)) \lesssim r^{\delta_{\text{conf}}(c)} 2^{-n\delta_{\text{conf}}(c)}$$

while for all $n \geq n_c$, the general estimate of Lemma 3.3 gives

$$\nu(A_n) \lesssim r^{\delta_{\text{conf}}(c)\eta(\epsilon)} 2^{-n\delta_{\text{conf}}(c)\eta(\epsilon)}.$$

Recall $\delta_{\text{conf}}(c) > 1$ for $c \in \mathcal{A}$ and $\zeta/2 < \eta(\epsilon) < 1$. Also, $r2^{-n_c} \leq r_c$.

$$\begin{aligned} I_r &\lesssim r^{1-\zeta/2} \sum_{n=0}^{n_c} 2^{-n(1-\zeta/2)} + r^{\eta(\epsilon)-\zeta/2} \sum_{n=n_c}^{+\infty} 2^{-n(\eta(\epsilon)-\zeta/2)} \\ &\lesssim \frac{r^{1-\zeta/2}}{1 - 2^{-(1-\zeta/2)}} + (r2^{-n_c})^{\eta(\epsilon)-\zeta/2} \sum_{n=0}^{+\infty} 2^{-n(\eta(\epsilon)-\zeta/2)} \\ &\lesssim \frac{r^{1-\zeta/2}}{2 - \zeta} + \frac{r_c^{\eta(\epsilon)-\zeta/2}}{1 - 2^{-(\eta(\epsilon)-\zeta/2)}} \lesssim \frac{r^{1-\zeta/2}}{2 - \zeta} + \frac{r_c^{\eta(\epsilon)-\zeta/2}}{2\eta(\epsilon) - \zeta}. \end{aligned}$$

□

3.2. Absolutely continuous invariant measures. Absolutely continuous invariant measures can exist only with respect to conformal measures without atoms at critical points. This necessary condition is satisfied for the geometric measures of $f_c(z) = z^2 + c$, $c \in \mathcal{A}$.

We refer to Theorem 4 in [21] for the following result.

Fact 3.7. *Let $c \in \mathcal{A}$. Then f_c has a unique absolutely continuous invariant probabilistic measure σ_c with respect to the conformal measure ν_c from Fact 3.2. Moreover, σ_c is ergodic, exact, and has positive entropy and Lyapunov exponent.*

To prove the following proposition, we will need to control distortion; we will use the method of *shrinking neighbourhoods*, introduced in [37], see also [41]. With our technical sequence (δ_n) of Definition 2.1, let $\Delta_n := \prod_{k \leq n} (1 - \delta_k)$. Let $B(z, r)$ be the ball of radius r around a point z and $\{f^{-n}(z)\}$ be a sequence of preimages of z . We define U_n and U'_n as the connected components of $f^{-n}(B(z, r\Delta_n))$ and $f^{-n}(B(z, r\Delta_{n+1}))$, respectively, which contain $f^{-n}(z)$. Clearly,

$$f(U_{n+1}) = U'_n \subset U_n .$$

If U_k , for $1 \leq k \leq n$, do not contain critical points then the distortion of $f^n : U'_n \rightarrow B(z, r\Delta_{n+1})$ is bounded (by the Koebe distortion theorem) by a power of $\frac{1}{\delta_{n+1}}$, multiplied by an absolute constant.

Since $\sum_n \delta_n < \frac{1}{2}$, one also has $\prod_n (1 - \delta_n) > \frac{1}{2}$, and hence always $B(z, r/2) \subset B(z, r\Delta_n)$.

Let $\rho_c(x) = \frac{d\sigma_c}{dv_c}(x)$ be the Radon-Nikodym derivative of σ_c with respect to v_c .

Proposition 3.8. *There exists a uniform constant $C > 0$ so that for every $0 < r \leq \text{diam } J_c$, $\zeta \in (1, 2)$ and $c \in \mathcal{A}$,*

$$\int_{B(c,r)} \rho_c(x)^\zeta dv_c \leq C \int_{B(c,r)} |x - c|^{-\delta_{\text{conf}}(c)\zeta/2} dv_c(x). \tag{31}$$

In particular, by Proposition 3.6, the densities $\rho_c(x)$ of σ_c with respect to v_c are uniformly L^ζ -integrable. Just L^ζ -integrability was proven before in the non-uniform setting in [21] and [43] for large classes of rational functions.

Proof. By splitting the integral into two integrals, one over the set where $\rho_c(x) \leq |x - c|^{-\delta_{\text{conf}}(c)/2}$ and one over its complement in $B(c, r)$, it suffices to show the proposition when ζ is close to 2.

A starting point is a general upper estimate of $\rho_c(x)$ obtained in [21, Proposition 10.1]. Assume that $c \in \mathcal{A}$. Let us set

$$\hat{\Delta}_k(x) := \left(\text{dist} \left(f_c^k(0), x \right) \right)^{-\delta_{\text{conf}}(c)/2}$$

and

$$g_c(x) := \sum_{k=1}^{\infty} \gamma_k^{-\delta_{\text{conf}}(c)} \hat{\Delta}_k(x) ,$$

where γ_k is defined in Definition 2.1. The [21, Proposition 10.1] asserts that there exists a positive constant K so that, for all $c \in \mathcal{A}$ and every $x \notin \bigcup_{n=0}^{\infty} f_c^n(c)$,

$$\rho_c(x) < K g_c(x) . \tag{32}$$

The sequence γ_k (defined in Definition 2.1), independent of $c \in \mathcal{A}$, tends exponentially fast to ∞ .

Let $\zeta \in [3/2, 2)$ be an arbitrary number. We use the Hölder inequality similarly to the proof of [21, Corollary 10.1], for positive sequences \mathbf{x}, \mathbf{y} ,

$$\begin{aligned} \|\mathbf{x}\mathbf{y}\|_1 &\leq \|\mathbf{x}^{1-1/\zeta}\|_{\frac{1}{1-1/\zeta}} \cdot \|\mathbf{x}^{1/\zeta}\mathbf{y}\|_\zeta \\ &\lesssim \|\mathbf{x}^{1/\zeta}\mathbf{y}\|_\zeta = \|\mathbf{x}\mathbf{y}^\zeta\|_1^{1/\zeta*} , \end{aligned}$$

provided \mathbf{x} is geometrically decreasing.

With this model, we deduce from (32) that

$$\begin{aligned} \int_{B(c,r)} \rho_c(x)^\zeta d\nu_c &\leq K^\zeta \int_{B(c,r)} \left(\sum_{k=1}^\infty \gamma_k^{-\delta_{\text{conf}}(c)} \hat{\Delta}_k(x) \right)^\zeta d\nu_c \\ &\lesssim \sum_{k=1}^\infty \gamma_k^{-\delta_{\text{conf}}(c)} \int_{B(c,r)} \hat{\Delta}_k^\zeta(x) d\nu_c. \end{aligned} \tag{33}$$

Suppose that $r \leq R'/6$ and let $k > 1$. If $f_c^k(0) \notin B(c, 2r)$ then we have a direct bound

$$\int_{B(c,r)} \hat{\Delta}_k^\zeta(x) d\nu_c \leq \int_{B(c,r)} \hat{\Delta}_1^\zeta(y) d\nu_c(y),$$

because for all $x \in B(c, r)$, $|x - c| \leq |x - f_c^k(0)|$.

If $f_c^k(0) \in B(c, 2r)$, put $B_k := B(f_c^k(0), 3r)$ and let

$$I_{B_k} := \int_{B_k} \hat{\Delta}_k^\zeta(x) d\nu_c \geq \int_{B(c,r)} \hat{\Delta}_k^\zeta(x) d\nu_c.$$

We estimate I_{B_k} using shrinking neighbourhoods. Given $x \in B_k \setminus \bigcup_{n=1}^k f_c^n(0)$ let $r(x) > 0$ be minimal such that some shrinking neighbourhood U_m , $m = m(x) \leq k$, for $B(x, r(x))$ hits the critical point ($0 \in \partial U_m$). By construction, $r(x) < 2 \text{dist}(f_c^k(0), x) \leq 6r \leq R'$ and $r(x)$ is comparable with $\text{dist}(f_c^{m(x)}(0), x)$. We can write $B_k \setminus \bigcup_{n=1}^k f_c^n(0) = \bigcup_{m=1}^k E_m$, where $E_m = \{x \in B_k : m(x) = m\}$. For some m , E_m may be empty.

By Lemma 2.3 of [21], $\text{dist}(c, f^{-m+1}(x)) \leq 6r\gamma_{m-1}^{-1} < r$, as $\gamma_{m-1} > 64$ by Definition 2.1. Hence, for all $m \leq k$,

$$f_c^{-m+1}(E_m) \subset B(c, r).$$

We obtain an upper bound of I_{B_k} solely in terms of $\int_{B(c,r)} \hat{\Delta}_1^\zeta(y) d\nu_c(y)$, as follows. We change variables in the integrals (34) and use the fact that the distortion of the inverse branch of f_c^{m-1} mapping $f_c^m(0)$ to c is controlled on $B(x, r(x))$, for $x \in E_m$, by the technique of shrinking neighbourhoods.

$$\begin{aligned} I_{B_k} &\leq \sum_{m=1}^k \int_{E_m} \hat{\Delta}_m^\zeta(x) d\nu_c(x) \\ &\lesssim \sum_{m=1}^k \int_{f_c^{-m+1}(E_m)} (\delta_m)^{-2\delta_{\text{conf}}(c)\zeta} \frac{\hat{\Delta}_1^\zeta(y)}{|(f^{m-1})'(y)|^{-\delta_{\text{conf}}(c)(1-\zeta/2)}} d\nu_c(y) \\ &\lesssim \sum_{m=1}^k (\delta_m^{-4} 4^{(1-\zeta/2)m})^{\delta_{\text{conf}}(c)} \int_{B(c,r)} \hat{\Delta}_1^\zeta(y) d\nu_c(y), \end{aligned} \tag{34}$$

where the last line follows from $\zeta < 2$ and $\sup_{x \in \mathcal{J}_c} |f_c'(x)| \leq 4$.

Provided ζ is sufficiently close to 2,

$$\sum_{k=1}^{\infty} \gamma_k^{-\delta_{\text{conf}}(c)} \sum_{m=1}^k (\delta_m^{-4} 4^{(1-\zeta/2)m})^{\delta_{\text{conf}}(c)} \tag{35}$$

is uniformly bounded. Invoking (33), we see that $\int_{B(c,r)} \rho_c(x)^\zeta d\nu_c$ can be bounded by

$$\sum_{k=1}^{\infty} \gamma_k^{-\delta_{\text{conf}}(c)} I_{B_k} \lesssim \int_{B(c,r)} \hat{\Delta}_1^\zeta(x) d\nu_c(x). \tag{36}$$

This completes the case $r \leq R'/6$.

Let $B'_k := B(f_c^k(0), R'/2)$ for all $k > 1$. For $r \geq R'/6$ we have

$$\int_{B(c,r)} \hat{\Delta}_k^\zeta(x) d\nu_c \leq \int_{\mathcal{J}_c} \hat{\Delta}_k^\zeta(x) d\nu_c \lesssim I_{B'_k} + (R')^{-\delta_{\text{conf}}(c)} \lesssim I_{B'_k},$$

because $I_{B'_k} \geq \int_{B'_k} d\nu_c \geq \nu_0$. Therefore, from (33) and (36),

$$\int_{B(c,r)} \rho_c(x)^\zeta d\nu_c \lesssim \sum_{k=1}^{\infty} \gamma_k^{-\delta_{\text{conf}}(c)} I_{B'_k} \lesssim \int_{B(c,R'/6)} \hat{\Delta}_1^\zeta(x) d\nu_c(x).$$

□

Using the the estimate on $\nu_c(r)$ from Lemma 3.5 and bounds (30) and (31), we obtain good control on $\sigma_c(r) := \sigma_c(B(c, r))$.

Proposition 3.9. *There exists $C > 1$ such that*

for every $c \in \mathcal{A}$ and every $r_c^{1/2} \leq r \leq R'$,

$$\sigma_c(r) \leq C r^{1/2}.$$

Proof. It suffices to show the proposition for $\epsilon = c + 2$ small. Then $\eta(\epsilon)$ defined by (29) is greater than $1 - \frac{1}{8}$.

Given $\zeta \in (1, 2\eta(\epsilon))$, the Hölder inequality, together with inequality (31), gives

$$\begin{aligned} \sigma_c(r) &= \int_{B(c,r)} \rho_c(x) d\nu_c \leq \left(\int_{B(c,r)} \rho_c(x)^\zeta d\nu_c \right)^{1/\zeta} \left(\int_{B(c,r)} d\nu_c \right)^{1-1/\zeta} \\ &\lesssim (I_r)^{1/\zeta} \nu_c(r)^{1-1/\zeta}, \end{aligned} \tag{37}$$

where $I_r = \int_{B(c,r)} |x - c|^{-\delta_{\text{conf}}(c)\zeta/2} d\nu_c(x)$.

By inequality (30),

$$I_r \lesssim \frac{r^{1-\zeta/2}}{2-\zeta} + \frac{r_c^{\eta(\epsilon)-\zeta/2}}{2\eta(\epsilon)-\zeta}.$$

Put $\zeta = 3/2$. If $r > r_c^{1/2}$, then

$$\frac{r^{1-\zeta/2}}{2-\zeta} \geq 2r_c^{1/8} \geq 2r_c^{\eta(\epsilon)-\zeta/2} \geq 8 \frac{r_c^{\eta(\epsilon)-\zeta/2}}{2\eta(\epsilon)-\zeta},$$

so $I_r \lesssim r^{1/4}$. Using the estimate of Lemma 3.5 in (37), we obtain

$$\sigma_c(r) \lesssim r^{1/6} r^{\delta_{\text{conf}}(c)/3} \lesssim r^{1/2}.$$

□

Key “diagonal” estimate. Fix $t_0 \gg 1$. By Lemma 2.6, $r_c < \epsilon^{2(1+t_0)}$, if ϵ is sufficiently small. Proposition 3.9 implies that, for $r = a\epsilon$, $a \geq \epsilon^{t_0}$,

$$\sigma_c(a\epsilon) \lesssim a^{1/2} \sqrt{\epsilon}.$$

For the scales smaller than $r_c^{1/2}$ we have a slightly weaker estimate. Recall that $\tau(\epsilon) = \exp(-\kappa \sqrt{\log 1/\epsilon})$ is the function from Proposition 1.4 and S the constant from Lemma 3.3.

Proposition 3.10. *There exists $C > 1$ such that for every function $u(\epsilon) \geq S\tau(\epsilon)$ with $\lim_{\epsilon \rightarrow 0} u(\epsilon) = 0$, every $c \in \mathcal{A}$ and $r \leq R'$,*

$$\sigma_c(r) \leq C u(\epsilon)^{-1/2} \nu_c(B(c, r))^{(1-8u(\epsilon))/2}.$$

Proof. We apply the Hölder inequality with the exponents $\zeta \in (1, 2\eta(\epsilon))$, $\eta(\epsilon)$ is the function from Proposition 3.6, and $\zeta' > 0$, $1/\zeta + 1/\zeta' = 1$. Similarly to (37),

$$\begin{aligned} \sigma_c(r) &= \int_{B(c,r)} \rho_c(x) d\nu_c \leq \left(\int_{B(c,r)} d\nu_c \right)^{1/\zeta'} \left(\int_{\mathcal{J}_c} \rho_c(x)^\zeta d\nu_c \right)^{1/\zeta} \\ &\lesssim (2\eta(\epsilon) - \zeta)^{-1/\zeta} \nu_c(B(c, r))^{(1-1/\zeta)}. \end{aligned} \tag{38}$$

Put $\zeta = 2\eta(\epsilon) - 2u(\epsilon)$. We can assume that $S\tau(\epsilon) + u(\epsilon) < 1/4$. Then

$$1/2 + 4u(\epsilon) \geq 1/\zeta \geq 1/2 + u(\epsilon).$$

By (38),

$$\begin{aligned} \sigma_c(r) &\lesssim u(\epsilon)^{-1/\zeta} \nu_c(r)^{1-1/\zeta} \\ &\lesssim u(\epsilon)^{-1/2} e^{-2u(\epsilon) \log u(\epsilon)} \nu_c(r)^{(1-8u(\epsilon))/2} \\ &\lesssim u(\epsilon)^{-1/2} \nu_c(r)^{(1-8u(\epsilon))/2}. \end{aligned}$$

□

Corollary 3.11. *There exists a uniform constant $C > 1$ such that for all $r \leq R'$ and $c \in \mathcal{A}$,*

$$\sigma_c(r) \leq C \sqrt{|\log \epsilon|} r^{\frac{1}{2} \left(1 - \frac{10}{|\log \epsilon|}\right)}.$$

Proof. As $\dim_{\mathbb{H}}(\mathcal{J}_c) = \delta_{\text{conf}}(c) \geq 1$, we may combine the estimate for $\sigma_c(r)$ with the upper bound in Lemma 3.3 to obtain

$$\begin{aligned} \sigma_c(r) &\leq C u(\epsilon)^{-1/2} r^{\dim_{\mathbb{H}}(\mathcal{J}_c)(1-S\tau(\epsilon))(1-8u(\epsilon))/2} \\ &\leq C u(\epsilon)^{-1/2} r^{1/2-5u(\epsilon)}. \end{aligned}$$

By choosing $u(\epsilon) = |\log(\epsilon)|^{-1}$ and $c_0 > -2$ such that for all $c \in \mathcal{A} \cap (-2, c_0]$, $S\tau(\epsilon) \leq u(\epsilon)$, we obtain the claim of Corollary 3.11. □

General upper bound for σ_c near c . Assuming that $\epsilon < e^{-50}$, Corollary 3.11 implies for $r = a\epsilon$, $a > 0$,

$$\sigma_c(a\epsilon) \lesssim \max(a^{1/2}, a^{2/5}) \sqrt{\epsilon |\log \epsilon|}.$$

4. Orbital Estimates

Recall the definition of M , the first return time of 0 to $(q_c, -q_c)$. Furthermore, $\mathcal{U}(z, r)$ is the set of times n for which there is a level- n univalent pullback of $B(f_c^n(z), r)$ containing z , see Definition 1.3.

Lemma 4.1. *There exists $C > 1$ such that the following holds. If $|z| \in [\sqrt{10\epsilon}, 1/1000)$, let*

$$e(z) = \min\{k \geq 2 : |f_c^k(z) - p_c| \geq 1/20\}.$$

Then $2 \leq e(z) < M$. There exist domains $\hat{W} \supset W \ni z$ such that

- $f_c^k(\hat{W}) \subset B(p_c, 1/2)$ for $k = 2, \dots, e(z)$;
- \hat{W} is a level- $e(z)$ univalent pullback of $B(f_c^{e(z)}(z), 1/1000)$;
- if V is a level- k pullback, not necessarily univalent, of $B(f_c^k(z), R')$ and $k > e(z)$, then $0 \notin V$;
- W is a level- $e(z)$ univalent pullback of $B(f_c^{e(z)}(z), R'/2)$;
- $B(z, |z|/C) \subset W \subset B(z, C|z|)$.

Proof. Observe that $|f_c^{e(z)}(z) - p_c| < 1/5$ and that $|(f_c^k)'(-p_c)| = (2p_c)^k$. Let $z_n := f_c^n(z)$. By choice of R' in Subsection 2.2, $B(z_k, 4R') \subset B(p_c, 1/10)$ for $k = 2, \dots, e(z) - 1$. From formula (12), $|c + p_c| < \epsilon$. Recall the discussion following (12) about the dynamics and distortion bounds for g_c , the inverse of the restriction of f_c to $\{\Re(z) > 0\}$. It follows that $|f_c^k(c) - p_c| < |z_k - p_c|$ for $k = 0, \dots, e(z)$, so $e(z) \leq M - 1$.

As $g_c^{e(z)-1}(B(z_{e(z)}, 1/20))$ cannot contain p_c ,

$$\text{diam } g_c^{e(z)-1}(B(z_{e(z)}, 1/1000)) < |f_c(z) + p_c|/4 < |f_c(z) - c|/2. \tag{39}$$

Let \hat{W} be the connected component of $f_c^{-2}(g_c^{e(z)-2}(B(z_{e(z)}, 1/1000)))$ containing z . The third point follows from (11).

The topological conditions of the lemma define W uniquely. The distortion on W is uniformly bounded as $R'/2 < 1/1000$. The derivative of $f_c^{e(z)-1}$ on $f_c(W)$ is comparable in modulus to $|f_c(z) + p_c|^{-1} \sim |f_c(z) - c|^{-1} = |z|^{-2}$. Consequently, $\text{diam}(W)$ is comparable to $|z|$ and we obtain the final claim. \square

We fix $K \geq 4$ satisfy $\xi^K < C^{-1}/4$, where ξ, C are the constants of (10).

Lemma 4.2. *Given $z \in \mathcal{J}_c$ with $0 < |z| < 1$ and $n \geq K \lceil \log |z| \rceil$, if $V \ni z$ is a domain with*

$$f_c^n(V) = B(f_c^n(z), R'),$$

then $0 \notin V$.

Proof. By (10) and choice of K , $\text{diam}(V) \leq 4^{\log |z|} < |z|$. \square

Assume that $x \in \mathcal{J}_c \setminus \bigcup_{n \geq 0} f_c^{-n}(0)$ and define the sets of integers

$$E_1 = E_1(x) := \bigcup \lceil \lceil n + 1, n + K \lceil \lceil \log |x_n| \rceil \rceil \rceil, \\ E = E(x) := \bigcup \lceil \lceil n + 1, n + 3K \lceil \lceil \log |x_n| \rceil \rceil \rceil,$$

where the unions are taken over n for which $|x_n| \in (0, \sqrt{10\epsilon})$.

Let $E' = E'(x)$ denote the union of $\llbracket n + 1, n + e(x_n) \rrbracket$ over n for which $|x_n| \in [\sqrt{10\epsilon}, 1/1000)$. Let $G = \mathbb{N} \setminus (E \cup E')$. We summarise useful properties in the following lemma.

Lemma 4.3. *The sets satisfy $G \subset \mathbb{N} \setminus (E_1 \cup E') \subset \mathcal{U}(x, R')$. Each interval $\llbracket a, b \rrbracket \subset E'$ has $|b - a| < M - 1$. A maximal interval $\llbracket a, b \rrbracket \subset \mathbb{N} \setminus G$ with $b - a \geq M$ has*

$$\#\llbracket a, b \rrbracket \cap E \geq \frac{b - a}{2}. \tag{40}$$

If $n \in G$ and $n > 2K \lceil \log \sqrt{10\epsilon} \rceil$, there exists

$$m = m(n) \in \llbracket n - 2K \lceil \log \sqrt{10\epsilon} \rceil, n - K \lceil \log \sqrt{10\epsilon} \rceil \rrbracket \tag{41}$$

such that $m \in \mathcal{U}(x, R')$. For $k \in \llbracket m, n \rrbracket$,

$$|x_k - c| \geq 10\epsilon. \tag{42}$$

Proof. The first two statements follow from (11) and Lemmas 4.1 and 4.2. For (40), note that $\llbracket a, b \rrbracket$ necessarily intersects E and therefore $b - a \geq 3M$, while maximal intervals in E' have length bounded by $M - 1$.

To show (41), the interval of integers $\llbracket n - 2K \lceil \log \sqrt{10\epsilon} \rceil, n - K \lceil \log \sqrt{10\epsilon} \rceil \rrbracket$ has length at least M and is contained in $\mathbb{N} \setminus E_1$ and hence contains a number m in $\mathbb{N} \setminus (E_1 \cup E')$; necessarily $m \in \mathcal{U}(x, R')$. The final statement follows from $\llbracket m, n \rrbracket \cap E_1 = \emptyset$. \square

Estimates based on ergodicity. Let p_0 be maximal with $e^{-p_0} \geq 10\epsilon$. Set $\chi_p := \chi_{B(c, e^{-p})}$, the characteristic (or indicator) function of the ball $B(c, e^{-p})$.

Let X denote the set of points $x \in \mathcal{I}_\infty \cup \bigcup_{n \geq 0} f_c^{-n}(0)$ which satisfy, for each function h of the form

- $\chi_p, p = 1, \dots, p_0,$
- $h_p = 3Kp\chi_p, p = 1, \dots, 2p_0,$
- and $h_\infty = \sum_{p=2p_0}^\infty 3Kp\chi_p,$

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} h(f_c^k(x)) = \int h d\sigma_c. \tag{43}$$

By Birkhoff’s ergodic theorem, $\sigma_c(X) = 1$.

Fix a point $x = x_0 \in X$ and consider the sets of integers $E_1 = E_1(x), E = E(x), E' = E'(x)$ as before. The asymptotic upper density of a set of integers Q in \mathbb{N} is defined by

$$d(Q) := \limsup_{N \rightarrow +\infty} \frac{1}{N} \#\{Q \cap [1, N]\}. \tag{44}$$

We compute upper bounds for the asymptotic upper density of E using the definition of p_0 and inequalities (6) and (7):

$$d(E) \leq 3K \sum_{p=p_0}^{2p_0} pC_1 e^{-p/2} + 3K \sum_{p=2p_0+1}^{\infty} pC_1 e^{-(p-p_0)2/5} \sqrt{\epsilon |\log \epsilon|} \tag{45}$$

$$\begin{aligned} &\leq 6K C_1 p_0 \left(\sqrt{e^{-p_0}} + e^{-2p_0/5} \sqrt{\epsilon |\log \epsilon|} \sum_{p \geq 1} p e^{-2p/5} \right) \\ &\leq C_2 \epsilon^{1/2} |\log \epsilon|, \end{aligned} \tag{46}$$

where $C_1, C_2 > 0$ are uniform constants; in particular, while E depends on $x \in X$, the bound (45) holds for all E and ϵ under consideration. Subsequent bounds will similarly work for all considered ϵ and all $x \in X$.

Density of iterates versus density of scales. We want to translate the density of $G = \mathbb{N} \setminus (E \cup E')$ in \mathbb{N} into the density of the corresponding scales at $x \in X$. Given $n \in G$, let $n' > n$ denote the next smallest element of G . If $n' - n = 1$ or $n' - n \geq M$, let $j_n = 1$. Otherwise, $n + 1 \in E'$ and $|x_n| \in [\sqrt{10}\epsilon, 1/1000)$. Let j_n be minimal with $R'2^{-j_n} \leq |x_n|$. We consider

$$\mathcal{N} = \{(n, j) : n \in G, 1 \leq j \leq j_n\}.$$

For $n \in G$, denote by $r_{n,1}$ the maximal radius with $f_c^n(B(x, r_{n,1})) \subset B(x_n, R'/2)$. For $(n, j) \in \mathcal{N}$, let $r_{n,j} = 2^{-j+1}r_{n,1}$. By bounded distortion, if $n' - n < M$,

$$\text{diam}(f_c^n(B(x, r_{n,j_n}))) \sim |x_n|.$$

By Lemma 4.1, r_{n,j_n} is comparable to $r_{n',1}$ if $n' - n < M$. Therefore we can fix $k_* \geq 1$ so that

$$2^{k_*} r_{n',1} > r_{n,j_n}$$

for all such pairs $n, n', n' - n < M$.

Let Q denote the set of integers k for which there exists $(n, j) \in \mathcal{N}$ with $r_{n,j} \subset I_k$, where $I_k = (2^{-(k+k_*)}, 2^{-k}]$.

Define a function $\pi : G \mapsto \mathbb{N}$, $\pi(n)$ is the smallest integer k such that $r_{n,1} \in I_k$. By (10), for large $n, n \lesssim \pi(n)$.

Suppose $n, n' \in G$. Then $\pi(n') - \pi(n) \leq C_3(n' - n)$ (since the set $W \ni x_n$ mapped univalently by $f_c^{n'-n}$ to $B(x_{n'}, R'/2)$ contains $B(x_n, 4^{-n'+n}R'/2)$). We shall apply this estimate whenever $\llbracket n + 1, n' - 1 \rrbracket$ contains a component of E , recalling (40). On the other hand, if $\llbracket n, n' \rrbracket$ is a subset of $\mathbb{N} \setminus E$, then $\llbracket \pi(n), \pi(n') \rrbracket \subset Q$, by choice of k_* .

Therefore $d(\mathbb{N} \setminus Q) \lesssim d(E) \lesssim \epsilon^{1/2} |\log \epsilon|$.

We need additional estimates on the density of $G_j := \{n \in G : (n, j) \in \mathcal{N}\}$, when $j \geq 2$. If $n \in G_j$ then, crudely, $|x_n| < C_4 2^{-j}$. This latter condition happens with frequency bounded by $\sigma_c(B(0, C_4 2^{-j})) \lesssim 2^{-j}$. Hence $d(G_j) \lesssim 2^{-j}$ and, as $n \lesssim \pi(n)$,

$$d(\pi(G_j)) \lesssim 2^{-j}. \tag{47}$$

5. Proof of Theorem 3

Recall Definition 1.2 of $\beta(x, r)$ -numbers for $K \subseteq \mathbb{R}^d$ with $d \geq 2$. Observe that in general, for $x \in K$ and $0 < r < r'$,

$$\beta_K(x, r) \leq \frac{r'}{r} \beta_K(x, r'). \tag{48}$$

Hence

$$\beta_K(x, 2^{-k-k_*}) \leq 2^{k_*} \inf_{r \in I_k} \beta_K(x, r).$$

Almost flat sets. Let $0 = \beta_0 < \beta_1 < \dots < \beta_n = 1$ and $\mathcal{F} = \{(d_i, \beta_i) : d_i \in [0, 1] \text{ for all } i = 1, \dots, n\}$. A set $K \subset \mathbb{R}^d$ is *almost flat* at a point $x \in K$ with respect to \mathcal{F} if there exist a partition $\sqcup_{i \in \llbracket 0, n \rrbracket} Q_i = \mathbb{N}$ such that for all $i \in \llbracket 0, n \rrbracket$ and $m \in Q_i$

$$\beta(x, 2^{-m}) \leq \beta_i \text{ and } d(Q_i) \leq d_i,$$

recalling the definition (44) of the upper density $d(Q)$ of a subset Q of \mathbb{N} . The following fact is a direct corollary of [24, Theorem 2].

Fact 5.1. *Suppose that the set $K \subset \mathbb{R}^d$, $d \geq 2$, is almost flat at every point $x \in K$ with respect to a given family \mathcal{F} as above. Then,*

$$\dim_H(K) \leq 1 + C'_d \sum_{i=1}^n d_i \beta_i^2,$$

where for each $d \geq 2$, $C'_d > 0$ is a universal constant.

Geometric estimates. The upper bound of Theorem 3 will follow from the estimates of Section 4 and from Fact 5.1 about the Hausdorff dimensions of almost flat sets. An initial geometric estimate comes from [3, Proposition 2] stated as Fact 5.2.

Fact 5.2. *If a quadratic Julia set \mathcal{J}_c is connected and $c \neq -2$ then*

$$\mathcal{J}_c \subset E_4 := \{z \in \mathbb{C} : |z - c| + |z + c| < 4\}.$$

Let $\epsilon = c + 2$, $c > -2$, be close to 0. Then the Julia set \mathcal{J}_c is contained in the horizontal strip $B(\mathbb{R}, 2\sqrt{\epsilon})$.

Lemma 5.3. *There exists a constant $C > 0$ such that for all $c \in \mathcal{A}$,*

$$\mathcal{J}_c \subset \{z \in \mathbb{C} : |z| \leq p_c \text{ and } |\Im(z)| \leq C\sqrt{\epsilon}(p_c - |\Re(z)|)\}.$$

Proof. Consider the rectangle

$$H_\epsilon := \{z : \Re(z) < 5/4, \Im(z) < 2\sqrt{\epsilon}\}.$$

It contains $\mathcal{J}_c \cap \{z : \Re(z) < 5/4\}$.

Let $\mathcal{J}_c^+ := \{z \in \mathcal{J}_c : \Re(z) \geq 0\}$. Using the definition of the map g_c on page 18, we can see that

$$\mathcal{J}_c^+ \subset \{p_0\} \cup \bigcup_{n \geq 0} g_c^n(\overline{H_\epsilon}).$$

Then \mathcal{J}_c^+ is included in a half cone at p_c with aperture comparable to $\sqrt{\epsilon}$. As $\mathcal{A} \subseteq \mathbb{R}$, \mathcal{J}_c is symmetric w.r.t. both axes. □

Combining Lemma 5.3 with (42) and formulas (12), we obtain the following.

Corollary 5.4. *For $c \in \mathcal{A}$ sufficiently close to -2 and $x \in X$, if $n \in G$ and $m(n)$ is given by (41), then*

$$\Re(x_k) \geq c + \epsilon,$$

for all $k \in \llbracket m(n), n \rrbracket$.

In order to recover estimates of β numbers at the large scale via univalent pullbacks, we will employ the following version of Koebe’s Theorem.

Lemma 5.5. *Let $g : \mathbb{D} \rightarrow \mathbb{C}$ be univalent satisfying $g(0) = 0$ and $g'(0) = 1$. Then for all $z \in \mathbb{D}$,*

$$|g(z) - z| \leq |z|^2 \frac{2 - |z|}{(1 - |z|)^2}.$$

Proof. Let $g(z) = z + \sum_{k \geq 2} a_k z^k$. By the Bieberbach conjecture, proven by de Branges [14], we know that for all $k \geq 2$, $|a_n| \leq k$. Thus it is enough to compute a bound for

$$\left| \sum_{k \geq 2} a_k z^k \right| \leq \sum_{k \geq 2} k |z|^k = |z|^2 \frac{2 - |z|}{(1 - |z|)^2}.$$

□

Corollary 5.6. *Let $g : \mathbb{D} \rightarrow \mathbb{C}$ be univalent, $g(0) = 0$, $g'(0) = 1$ and $|z| = r \leq 1/6$. Then*

$$|g(z) - z| \leq 3r^2.$$

Corollary 5.7. *Let $g : \mathbb{D} \rightarrow \mathbb{C}$ be univalent, $g(0) = 0$ and $g'(0) = 1$, and let $r \leq 1/6$. Let $Y \subset B(0, r)$ and L a line through 0 with $g(Y) \subset B(L, \rho)$, for some $\rho > 0$. Then $Y \subset B(L, \rho + 3r^2)$.*

Proof. Suppose $z \in B(0, r)$ with $\text{dist}(z, L) \geq \rho + 3r^2$. By Corollary 5.6 and the triangle inequality, $\text{dist}(g(z), L) \geq \rho + 3r^2 - 3r^2$. □

We set $\beta(z, r) := \beta_{\mathcal{J}_c}(z, r)$. The real line is a good comparator when estimating β -numbers for c near -2 . For $x \in X$, we have a corresponding set of neighbourhoods of x .

Lemma 5.8. *There exists $C > 1$ such that, given $(n, j) \in \mathcal{N}$,*

$$\beta(x, r_{n,j}) \leq C 2^j \epsilon^{1/2}.$$

Proof. It is enough to show this when $j = 1$, and then apply (48).

Given $n \in G$, let $m = m(n) \in \mathcal{U}(x, R')$ be given by (41). Each $x_k, k \in \llbracket m(n), n \rrbracket$ has real part at least $c + \epsilon$ by Corollary 5.4. The imaginary part of x_n is bounded by $2\epsilon^{1/2}$, so x_n is very near the centre of the large-scale line segment $B(x_n, R'/2) \cap \mathbb{R}$. Consequently (as $m > 2$ and $n \in \mathcal{U}(x, R')$), $B(x_n, R'/2) \cap \mathbb{R} \subset (c, \infty)$.

Consider the univalent pullback W_k of $W_0 := B(x_n, R'/2)$ by f_c^k to x_{n-k} . Note that $c \notin W_k$. As $W_0 \cap \mathbb{R} \subset (c, \infty)$, $f_c(W_1 \cap \mathbb{R}) = W_0 \cap \mathbb{R}$ and $W_1 \cap \mathbb{R}$ is a line

segment. By bounded distortion, x_{n-1} lies in the vertical strip with real part $W_1 \cap \mathbb{R}$, so $W_1 \cap \mathbb{R} \subset (c, \infty)$.

Repeating this argument for $k = 2, \dots, n - m$, we obtain that

$$f_c^{n-m}(W_{n-m} \cap \mathbb{R}) = B(x_n, R'/2) \cap \mathbb{R}.$$

Let $W := W_{n-m}$. We deduce that $J_c \cap W$ is very close to the real axis: by bounded distortion, if $w \in J_c \cap W$, then

$$|\Im(w)| \lesssim \frac{\text{diam}(W)}{R'} \sup_{z \in J_c} |\Im(z)| \lesssim \text{diam}(W) \epsilon^{1/2}. \tag{49}$$

By the definitions, $f_c^m(B(x, r_{n,1})) \subset W$. Since $n - m > K|\log \epsilon|/2$,

$$\text{diam}(W) < \sqrt{\epsilon}.$$

By choice of m , there is a level- m univalent pullback $\hat{W} \ni x$ of $B(x_m, R')$. By bounded distortion, for a uniform constant $C_1 > 1$,

$$B(x, r_{n,1} \text{diam}(W)^{-1}/C_1) \subset \hat{W}.$$

This gives us the modulus needed to apply Corollary 5.7.

From (49), for $z \in f_c^m(B(x, r_{n,1})) \cap J_c$, the distance from z to the horizontal line passing through x_m is bounded, $\lesssim \text{diam}(W)\epsilon^{1/2}$. Applying Corollary 5.7 (and some affine transformations, with $r = C_1 \text{diam}(W)$),

$$\beta(x, r_{n,1}) \lesssim \epsilon^{1/2} + 3\epsilon C_1^2 \lesssim \epsilon^{1/2}.$$

□

Recall our set $Q \subset \mathbb{N}$ of controlled scales. We have that, for some uniform constant $C_2 > 0$,

$$d(\mathbb{N} \setminus Q) \leq C_2 \epsilon^{1/2} |\log \epsilon|.$$

For each scale $k \in Q$, we choose some $r_{n,j} \in I_k$ and set $\zeta(k) := j$. Then $d(\{k : \zeta(k) = j\}) \leq k_* d(\pi(G_j))$. By Lemma 5.8, we obtain an associated beta number $\beta(x, r_{n,j}) \leq C_2^j \epsilon^{1/2}$. Hence

$$\beta(x, 2^{-k-k_*}) \leq 2^{k_*} C_2^j \epsilon^{1/2} =: C_3 2^j \epsilon^{1/2}.$$

We set

$$\beta_j := \min(1, C_3 2^j \epsilon^{1/2}).$$

We associate $\beta = 1$ to $\mathbb{N} \setminus Q$ (shifted by k_*), set $d_1 = 1$ and

$$d_j := k_* d(\pi(G_j)) \lesssim 2^{-j}.$$

Thus $d_j \beta_j^2 \leq C_4 2^j \epsilon$, where $C_4 > 0$ is a uniform constant. Writing $j_* := \max\{j_n : n \in G\}$, then $2^{-j_*} \sim \sqrt{\epsilon}$ and, summing over j we obtain

$$\sum_{j=1}^{j_*} d_j \beta_j^2 \leq C_4 2^{j_*+1} \epsilon \lesssim \epsilon^{1/2}.$$

The upper bound on $d(\mathbb{N} \setminus Q)$ and applying Fact 5.1 completes the proof of Theorem 3.

6. Induced Cantor Repellers and the Lower Bound

We construct an induced Cantor repeller in Proposition 6.2. In Proposition 6.9, we determine a lower bound on the Hausdorff dimension of its Julia set. As its Julia set is contained in the Julia set of f_c , this will prove Theorem 4.

6.1. Preliminaries.

Cantor repellers and inducing

Definition 6.1. Suppose that D_1, \dots, D_n is a collection of open and non-degenerate topological disks with pairwise disjoint closures compactly contained in a topological disk $D \subset \mathbb{C}$. A map $\varphi : \bigcup_{i=1}^n D_i \mapsto D$ which is biholomorphic onto D on every D_i , $1 \leq i \leq n$, is called a *Cantor repeller*.

If φ preserves the real line and each branch domain D_i is symmetric with respect to \mathbb{R} then φ is a *real* Cantor repeller. With respect to a map f , if there are integers n_i such that $\varphi|_{D_i} = f^{n_i}$, we say that φ is *induced* (by f).

Every $f_c(z) = z^2 + c$, $c \in [-2, 0)$ has two fixed points $p, q \in [-2, 2]$, $0 < -q < p$, and is unimodal on $[-p, p]$. The non-empty interval $U = (q, -q)$ is called a *fundamental inducing interval*. U is a *regularly returning set*, that is, $\forall n > 0$, $f_c^n(\partial U) \cap U = \emptyset$.

Let $\phi = \phi_c$ be the first return map to U (under the unimodal map $f := f_c$, restricted to \mathbb{R}), defined on

$$\mathcal{D}_\phi := \{x \in U : \exists n > 0, f^n(x) \in U\}$$

by the formula $\phi(x) := f^{n(x)}(x)$ where $n(x) := \min\{n > 0 : f^n(x) \in U\}$. As U is regularly returning, the function $n(x)$ is continuous and locally constant on \mathcal{D}_ϕ . For c close to -2 , the set $\{x \in U : n(x) = 2\}$ has two connected components, d_q adjacent to q and d_{-q} adjacent to $-q$. We define another regularly returning interval

$$V = U \setminus \overline{d_q \cup d_{-q}}. \tag{50}$$

For a Borel subset $X \subset \mathbb{R}$, we denote by $|X|$ its Lebesgue measure. Given an interval $W \subset \mathbb{R}$, we denote by \mathbb{D}_W the disc in \mathbb{C} with diameter W .

Proposition 6.2. *There exist $C, \alpha > 0$ and $c^* > -2$ such that for every $c \in (-2, c^*)$, there is a Cantor repeller $\varphi : \mathcal{D} \mapsto \mathbb{D}_V$ induced by f_c with range \mathbb{D}_V , with the following properties, $\epsilon = c + 2$,*

- each branch of φ is extensible as a univalent map onto \mathbb{D}_U ,
- the map φ is defined on $\mathcal{D} \subset \mathbb{D}_V$,

$$|\mathcal{D} \cap \mathbb{R}| \geq |V|(1 - C\epsilon^{3/4}), \tag{51}$$

- there is exactly one component W of \mathcal{D} for which ϕ restricted to $\mathcal{D} \setminus W$ is a real Cantor repeller, and

$$\text{diam } W \geq C^{-1}\sqrt{\epsilon}, \tag{52}$$

- for every $x \in \mathcal{D}$, $2 \leq |\varphi'(x)|$,

- for all $t > 0$,

$$|\{x \in \mathbb{R} : |\varphi'(x)| > e^t\}| < C e^{-\alpha t}.$$

By *extensibility* above, we mean that the branch of ϕ is the restriction of a biholomorphic map between a larger domain and \mathbb{D}_U . By the Koebe distortion theorem, the distortion of any branch of ϕ is bounded by a constant depending only on the modulus of $\mathbb{D}_U \setminus \mathbb{D}_V$.

Corollary 6.3. *With φ, α as above, the number of branches ζ for which $\inf |\zeta'(z)| \in [e^n, e^{n+1}]$ is bounded, for some uniform constant C' , by $C' e^{n(1-\alpha)}$.*

Proof. Given $n \geq 1$ and such a branch ζ , $\sup |\zeta'(z)| \leq e^{n+C_1}$ for some uniform constant C_1 . If there are N such branches, the Lebesgue measure of the real line intersected with the union of the domains is at least $N e^{-n-C_1}$ but is bounded by $C e^{-\alpha n}$. Hence

$$N \leq C e^{-\alpha n + n + C_1}.$$

□

The estimates are essentially real and require us to study the dynamics of $f_c : \mathbb{R} \rightarrow \mathbb{R}$. To obtain the estimates, we carry out some fundamental inducing steps to canonically-defined *box mappings*. The reader familiar with such inducing schemes can skip to the next key estimate, Lemma 6.8.

Box mappings. Consider a finite sequence of compactly nested open intervals around a point $0 \in b_0 \subset b_1 \cdots \subset b_k$. Let $\phi : \mathcal{D} \mapsto \mathbb{R}$ be a real-valued C^1 map defined on some open and bounded set $\mathcal{D} \subset b_k \subset \mathbb{R}$ satisfying the following:

- ϕ has at most one local extremum which, if it exists, is at 0;
- if $0 \in \mathcal{D}$, then b_0 is a connected component of \mathcal{D} ;
- for every $i = 0, \dots, k$, we have that $\partial b_i \cap \mathcal{D} = \emptyset$;
- for every connected component d of \mathcal{D} there exists $0 \leq i \leq k$ so that $\phi : d \mapsto b_i$ is proper.

The map ϕ is a *box mapping* and the intervals b_i are called boxes. If ϕ has a local extremum, it has a *central branch* $\psi := \phi|_{b_0}$ and b_0 is called the *central domain*. All other branches $\zeta_d := \phi|_d$ are monotone.

A box map ϕ is *induced* by a unimodal map $f_c(x) = x^2 + c$ if each branch of ϕ coincides on its domain with an iterate of f_c . We shall construct box mappings with up to four boxes $b \subset Z \subset V \subset U$, where Z is an interval to be defined and b is the central domain, should such exist.

6.2. Exponential tails. We say that a map g has *exponential tails* if there are $C, \theta > 0$ such that $|\{g^t \geq e^t\}| < C e^{-t\theta}$ for all $t \geq 0$. A family of maps $(g_\epsilon)_{\epsilon \in A}$ has *uniform exponential tails* if all g_ϵ have exponential tails, with constants C, θ independent of $\epsilon \in A$.

Lemma 6.4. *Let $I \subset [-2, 2]$ be an interval containing the domains of maps h with uniform exponential tails. Given a*

compact family of non-zero real polynomial maps f defined on \mathbb{R} and a family of expanding diffeomorphisms $g : Y \rightarrow I$ with uniformly bounded distortion,

- $h \circ f$ has uniform exponential tails;
- $h \circ g$ has uniform exponential tails.

Proof. There are $C, \theta > 0$ such that $|\{|h'| \geq e^t\}| < Ce^{-t\theta}$ for all $t \geq 0$ and C_1, d such that $|f'| < C_1$ on $f^{-1}([-2, 2])$ and $|f^{-1}(A)| \leq C_1|A|^{1/d}$ for every Borel subset $A \subset I$. If $|(h \circ f)'(x)| \geq e^t$ then $|h'(f(x))| \geq C_1^{-1}e^t$. Hence

$$\begin{aligned} |\{x : |(h \circ f)'(x)| \geq e^t\}| &\leq \left| f^{-1} \left(\{y : |h'(y)| \geq C_1^{-1}e^t\} \right) \right| \\ &\leq C_1 C_1^{1/d} e^{-\theta(t - \log C_1)/d}, \end{aligned}$$

from which the first estimate follows.

The second estimate is straightforward. □

In particular, pulling back via quadratic maps does not destroy exponential tails. The following lemma will be used to show that the composition of well-behaved maps with uniform exponential tails will have uniform exponential tails.

Lemma 6.5. *Given $K, C, \theta > 0$ there are $C', \kappa > 0$ for which the following holds. Let X be an open interval and $H : X \rightarrow [1, \infty)$. Let g be a function, defined on an open set $Y \subset \mathbb{R}$, which maps each branch domain of g diffeomorphically onto X with distortion bounded by K and with $|g'| \geq 1$. Let*

$$B_t = \{x : H(x) > e^t\}; \quad A_s = \{x : |g'(x)| > e^s\}.$$

If

$$|B_t| \leq Ce^{-t\theta} |X| \text{ and } |A_s| \leq Ce^{-s\theta}$$

for all $s, t \geq 0$, then

$$|\{x \in Y : |H(g(x))g'(x)| > e^t\}| \leq C'e^{-t\kappa}$$

for all $t \geq 0$.

Proof. Let W_s denote the union of all branches of g which contain a point x with $|g'(x)| \in [e^s, e^{s+1})$. Crudely, $W_s \subset A_{s-K}$ and $W_s \cap A_{s+K} = \emptyset$. Then

$$\begin{aligned} |\{x \in W_s : |H(g(x))g'(x)| > e^t\}| &\leq |A_{s-K}| K \frac{|B_{t-s-K}|}{|X|} \\ &\leq C^2 K e^{2K\theta} e^{-s\theta} e^{-(t-s)\theta} \\ &= C^2 K e^{2K\theta} e^{-t\theta}. \end{aligned}$$

Now sum over integers $s \geq 0$ to obtain

$$|\{x : |H(g(x))g'(x)| > e^t\}| \leq C^2 K e^{2K\theta} e^{-t\theta} (t + 1) + |A_{t-K}|,$$

from which the result follows. □

6.3. *Initial inducing.* We shall construct successive box mappings with up to four boxes $b \subset Z \subset V \subset U$, where Z is an interval to be defined and b is the central domain of the box mapping, should such exist. We shall call branches mapping monotonically onto V or U *long* and other branches will be called *short*.

Definition 6.6. A diffeomorphism $\zeta : d \rightarrow W$ between intervals d and W is said to *extend* (to map) *over* an interval $\hat{W} \supset W$ if there are $\hat{d} \supset d$ and an analytic diffeomorphism $\hat{\zeta} : \hat{d} \rightarrow \hat{W}$ whose restriction to d coincides with ζ . We say ζ is *extensible* over \hat{W} and call \hat{d} the *extension domain*.

Note that if ζ is a restriction of f_c^j for some j , then so is $\hat{\zeta}$.

We consider the first return map $\phi : \mathcal{D}_\phi \mapsto U$ of $f_c(z) = z^2 + c$ to the fundamental inducing interval $U = (q, -q)$. If ϕ has no central domain, we can just take ϕ_* to be the first return map to V and skip to Section 6.5. Henceforth, to avoid unnecessary caveats, we assume that the central branch $\psi : b \rightarrow U$ exists.

We denote by ζ_l and ζ_r the two branches adjacent to the central branch; these branches are monotone. Denote by Z the smallest interval containing the domains of ζ_l, ψ, ζ_r . Put $\hat{U} := (-\gamma, \gamma)$, where $f_c(\gamma) = -q$ (this \hat{U} is the same interval as \hat{A} of Section 2.4). Components of the following fact are well-known or follow by elementary arguments, noting that $|c + p| \sim \epsilon = c + 2$.

Fact 6.7. *There exist $C > 0$ and $c^* > -2$ such that for every $c \in (-2, c^*)$, the first return map ϕ for f_c to U is a box mapping with boxes b, U . Every monotone branch, except possibly ζ_l and ζ_r , extends over \hat{U} . Additionally,*

- (i) $|U| \leq C \text{dist}(U, \partial \hat{U})$,
- (ii) $|Z| \leq C\sqrt{\epsilon}$,
- (iii) ϕ has only finitely many monotone branches and for every $x \in \mathcal{D}_\phi \setminus Z$,

$$3 \leq |\phi'(x)| \leq C/\sqrt{\epsilon},$$

- (iv) each branch ζ_l, ψ, ζ_r , with domain d say, can be represented as $f^i \circ f$ and there is an interval $W \supset f(d)$ such that $f^i : W \rightarrow \hat{U}$ is a diffeomorphism onto \hat{U} with $\inf_W |(f^i)'| \geq C^{-1}\epsilon^{-1}$,
- (v) ϕ has uniform exponential tails.

Indeed, (i) is trivial.

Let $r \in (0, -q)$ satisfy $\hat{U} = (f(r), -f(r))$, and let $g = f|_{[r,p]}$. The intervals $g^{-k}([r, p])$ decrease geometrically so g^k has uniformly (in k and in c) bounded distortion on its domain, noting that r is far from 0 for c close to -2 . Then

$$\text{dist}(g^{-k}(\hat{U}), p) \sim |g^{-k}(\hat{U})| \sim |g^{-k}(U)|. \tag{53}$$

Together with $|c + p| \sim \epsilon$, one readily deduces (ii), (iv) and the upper bound of (iii). The lower bound of (iii) follows from the estimate, see [55, Page 5],

$$|(f_{-2}^i)'(x)| = 2^n \frac{h(x)}{h(f_{-2}^i(x))} \in [2^i \sqrt{3/4}, 2^i \sqrt{4/3}]$$

provided that $x, f_{-2}^i(x) \in (-1, 1) = U$; h denotes the conjugacy between f_{-2} and the full tent map. For i small, the estimate transfers to other c by continuity; for large return time i , the estimates on distortion and on $|g^{-k}(U)|$ kick in, with $k + 2 = i$.

It remains to show (v). A point $x \in [-q, p]$ has initial orbit satisfying

$$x > f_c(x) > \dots > f_c^k(x) \in [q, -q],$$

where k is the first entry time to $[q, -q] = \bar{U}$. Let $s > 2$ and let X_s denote the set of points x in $(-q, p)$ where the first entry of x to U happens with derivative greater than s but the same is not true of $f_c(x)$. Then $|X_s| \leq |U|/s$. Moreover, $X = \cup_k g^{-k}(X_s)$ has measure bounded by $2|U|/s$ (estimated via a geometric series). If $x \in U$ and $|\phi'(x)| > 2s$, then $-f_c(x) \in X$. Hence the set of points in U with $|\phi'(x)| > 2s$ is contained in a set of measure $\sqrt{2|U|/s}$, showing (v).

6.4. Inducing. In the inducing process, we shall pre-emptively use *boundary refinement*, applying the following map h_V , to avoid creating long non-extensible branches in the pull-back step. The technique of the boundary refinement was proposed in [27] in the quest to prove the so-called starting condition for unimodal maps [22]. Denote by h_V the first entry map for f_c from U to the interval V (defined in (50)). It is a box mapping with two boxes V, U , is defined almost everywhere on U , coincides with the identity map on V , and all its branches are long, diffeomorphic onto V and extensible over U with extension domains contained in U . We remark that h_V has exponential tails, as f_c restricted to $[-p, p] \setminus V$ is uniformly expanding.

We now describe a process which transforms the first return map ϕ to U into a new box mapping ϕ_* with up to three boxes $b_* \subset Z \subset V$.

Postcritical filling.

Let

$$\phi_0(x) = \begin{cases} x & \text{if } x \in Z, \\ \phi(x) & \text{if } x \in U \setminus Z. \end{cases}$$

We construct ϕ_j algorithmically for $j = 1, 2, \dots$ and denote the resulting limit map by ϕ_∞ . If $\psi(0)$ does not belong to the domain of a long branch of ϕ_{j-1} , let $\phi_j = \phi_{j-1}$, so $\phi_\infty = \phi_{j-1}$ and the process stops.

Otherwise, $\psi(0)$ belongs to the domain $d_{P,j}$ of a long branch $\zeta_{P,j}$ of ϕ_{j-1} . We modify ϕ_{j-1} on $d_{P,j}$ to obtain ϕ_j . Set

$$\phi_j(x) = \begin{cases} \phi_{j-1}(x) & \text{if } x \in U \setminus d_{P,j}, \\ \phi_0 \circ \phi_{j-1}(x) & \text{if } x \in d_{P,j}. \end{cases}$$

If $x \notin Z$ then $|\phi'_0(x)| > 3$. By induction, $|\phi'_j(x)| \geq 3^j$ for $x \in d_{P,j}$. By the construction, ϕ_∞ is a box mapping with long branches mapping over U and short branches, whose domains all lie in $d_{P,1}$, mapping over Z . The total length of the short branches is bounded by

$$|Z| \sum_{j \geq 0} 3^{-j} \lesssim \sqrt{\epsilon}. \tag{54}$$

Applying appropriate translations T_j , one can view the branch domains $d_{P,j}$ as pairwise disjoint and the collection of branches $\zeta_{P,j}$ as having uniform exponential tails; then apply Lemma 6.5 (with $H = |\phi'_0|$ and $g = \{\zeta_{P,j} \circ T_j\}$) to deduce that ϕ_∞ has uniform exponential tails.

Pull-back by ϕ . We transform the initial box mapping ϕ into ϕ_* by pull-back:

$$\phi_*(x) = \begin{cases} h_V \circ \phi_\infty \circ \phi(x) & \text{if } x \in Z, \\ h_V \circ \phi(x) & \text{if } x \in V \setminus Z. \end{cases}$$

Thus defined, ϕ_* is a box mapping. Recall that h_V coincides with the identity map on V and that all branches of h_V are long, diffeomorphic onto V and extensible over U . On $V \setminus Z$, branches of $h_V \circ \phi$ are long and extensible over U . By construction of ϕ_∞ , the domain of a long branch ζ of ϕ_∞ does not contain $\psi(0)$. Each branch of $h_V \circ \zeta$ is long, mapping onto V , and is extensible over U with extension domain contained in the domain of ζ and, therefore, not containing $\psi(0)$. The short branches of $h_V \circ \phi_\infty$ coincide with those of ϕ_∞ . Hence ϕ_* has two types of monotone branches, short ones mapping over Z and long ones mapping over V , extensible over U . The short branches and the possible central branch of ϕ_* have total length

$$\lesssim \epsilon^{3/4}$$

by (54) and Fact 6.7 (iv). Via Lemma 6.5, $h_V \circ \phi$ restricted to $V \setminus Z$ and $h_V \circ \phi_\infty$ on U have uniform exponential tails. Consequently, using Fact 6.7 (iv) and Lemma 6.4, ϕ_* has uniform exponential tails.

6.5. Construction of Cantor repeller. We finish the construction of the Cantor repeller φ of Proposition 6.2 for a map $f = f_c$ with c in the domain of Fact 6.7. Because $f^n(\partial V) \cap U = \emptyset$ for all $n \geq 1$, no two branches of ϕ_* are adjacent.

Let $\hat{\phi}_*$ denote ϕ_* with its short branches removed. Define $\tilde{\varphi}$ by retaining from the K th iterate $\hat{\phi}_*^K$ a finite number of branches contained in V whose domains have union whose measure is at least $|V| - C_1\epsilon^{3/4}$, where $K \geq 1$ is large enough to ensure that the map φ of the following paragraph satisfies $|\varphi'| > 2$, via the uniform distortion bound. Inherited from the same property for ϕ_* , no two branches of $\tilde{\varphi}$ are adjacent.

Consider a branch ζ of $\tilde{\varphi}$ with domain $I_\zeta \subset \mathbb{R}$. Its inverse extends univalently to a map ζ^{-1} defined on $\mathbb{C} \setminus (\mathbb{R} \setminus U)$. Let $D_\zeta := \zeta^{-1}(\mathbb{D}_V)$, so $D_\zeta \cap \mathbb{R} = I_\zeta$. Because of negative Schwarzian, these inverse branches have the property of contracting Poincaré disks so

$$D_\zeta \subset \mathbb{D}_{I_\zeta}$$

(see [22, Fact 2.1.2]) and thus their images D_ζ have pairwise-disjoint closures. We still need to add one imaginary branch.

Let g denote the restriction of f to the right half-plane $\{\Re(z) > 0\}$, so g is invertible. The sets $g^{-k}(\mathbb{D}_U)$ accumulate geometrically on the fixed point p , just as in (53):

$$\text{dist} \left(g^{-k}(\mathbb{D}_U), p \right) \sim \text{diam} \left(g^{-k}(\mathbb{D}_U) \right) \sim \text{diam} \left(g^{-k-2}(\mathbb{D}_U) \right).$$

There exists $m \geq 1$ such that $f^2(0) \in \overline{g^{-m}(\mathbb{D}_U)}$; for this m , $f^2(0) \notin g^{-k}(\mathbb{D}_U)$ for $k > m$. As $p - f^2(0) \sim \epsilon$,

$$\epsilon \sim \text{diam} \left(g^{-m-2}(\mathbb{D}_U) \right) \sim \text{diam} \left(g^{-m-2}(\mathbb{D}_V) \right).$$

Let W be one of the two connected components of $f^{-1}(-g^{-m-2}(\mathbb{D}_V))$. As $f(W) \cap \mathbb{R} \subset (-p, c)$, $W \cap \mathbb{R} = \emptyset$. The diameter satisfies $\text{diam } W \sim \sqrt{\epsilon}$. Set $\varphi = f^{m+3}$ on W .

Branches of $\tilde{\varphi}$ are contained in $f^{-1}(-\cup_{j=1}^{m+1} g^{-j}(\mathbb{D}_U))$ and so have domains whose closures are disjoint from \overline{W} .

The inverse of $\varphi|_W$ similarly extends univalently to $\mathbb{C} \setminus (\mathbb{R} \setminus U)$. Hence the distortion of iterates of φ is uniformly bounded independently of ϵ .

The map φ satisfies all claims of Proposition 6.2.

6.6. *Thermodynamical formalism.* Let $\varphi : \mathcal{D} \mapsto \mathbb{C}$ be a Cantor repeller. Its Julia set

$$\mathcal{J}_\varphi = \bigcap_{n \geq 0} \varphi^{-n}(\mathbb{C})$$

is a fully invariant Cantor set. It is well known that φ has an absolutely continuous invariant probabilistic measure σ with respect to the $\dim_{\mathbb{H}}(\mathcal{J}_\varphi)$ -conformal probabilistic measure ν , σ is an ergodic Gibbs measure with the Hölder potential $\mathcal{G}(x) = -\dim_{\mathbb{H}}(\mathcal{J}_\varphi) \log |\varphi'(x)|$.

The Hausdorff dimension of \mathcal{J}_φ is the unique solution of the equation $P_\varphi(t) = 0$, where the pressure function $t \in \mathbb{R} \mapsto P(t)$ defined by

$$P_\varphi(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{y \in \varphi^{-n}(0)} |(\varphi^n)'(y)|^{-t}$$

is an analytic and strictly convex function on \mathbb{R} , see [42].

The density $h(x) = \frac{d\sigma}{d\nu}(x) > 0$ is a bounded measurable function and is a fixed point of the Perron-Frobenius operator for φ that acts on continuous functions $g : \mathcal{J}_\varphi \mapsto \mathbb{R}$,

$$\mathcal{L}_\varphi(g)(x) = \sum_{y \in \varphi^{-1}(x)} g(y) \exp(\mathcal{G}(y)),$$

and h is the limit of

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathcal{L}_\varphi^k(1)(x)$$

in $L^1(\nu)$ topology. From the uniform distortion bound for branches of iterates of φ one obtains upper and positive lower bounds on h . The invariant measure σ is a fixed point of the dual operator $\mathcal{L}_\varphi^*(g)$ that acts on the space of Borel measures [44].

By the Birkhoff ergodic theorem, almost surely with respect to ν ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |\varphi'(\varphi^k(x))| = \int_{\mathcal{J}_\varphi} \log |\varphi'(x)| d\sigma. \tag{55}$$

Families of Cantor repellers. Let us return to our one-parameter family $\varphi_\epsilon, \epsilon = c + 2 > 0$, of Cantor repellers given by Proposition 6.2 and its real counterpart family $\tilde{\varphi}_\epsilon, \epsilon > 0$, defined by restricting φ_ϵ to the real line. To simplify notation, we write $\tilde{\mathcal{J}}_\epsilon$ for $\mathcal{J}_{\tilde{\varphi}_\epsilon}$, the Julia set of the real counterpart, and \mathcal{D}_n for the domain of $\tilde{\varphi}_\epsilon^n$, with connected components denoted by d . Let

$$Q(n, t) = \sum_{d \subset \mathcal{D}_n} |d|^t.$$

Lemma 6.8. *There is a uniform constant $K > 0$ such that, for any $\rho \in (0, 1)$, the following holds for all $\epsilon > 0$ small enough.*

- $\dim_{\mathbb{H}}(\tilde{\mathcal{J}}_\epsilon) \geq 1 - K\epsilon^{3/4}$,
- $Q(n, 1 + \rho\sqrt{\epsilon}) \geq K^{-1} (1 - 4K\rho\sqrt{\epsilon})^n$ for all $n \geq 1$.

Proof. From (51) and bounded distortion, there exists a uniform C' (note $|V| > 1$) such that, for $n \geq 1$,

$$\sum_{d \subset \mathcal{D}_n} |d| \geq (1 - C'\epsilon^{3/4})^n.$$

If we put $t = 1 - K\epsilon^{3/4}$ and $K > 2C'/\log 2$, then

$$\begin{aligned} Q(n, t) &= \sum_{d \subset \mathcal{D}_n} |d|^{1-K\epsilon^{3/4}} \geq \frac{(1 - C'\epsilon^{3/4})^n}{\max_{d \subset \mathcal{D}_n} |d|^{K\epsilon^{3/4}}} \\ &\geq \frac{(1 - C'\epsilon^{3/4})^n}{2^{-K\epsilon^{3/4}n}} \geq (1 + (K \log 2 - 2C')\epsilon^{3/4})^n. \end{aligned}$$

As

$$\sum_{y \in \tilde{\varphi}_\epsilon^{-n}(0)} |(\varphi_\epsilon^n)'(y)|^{-t} \sim Q(n, t),$$

the pressure function $P_{\tilde{\varphi}_\epsilon}(t) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Q(n, t)$ is positive, which implies the dimension estimate.

The map $\tilde{\varphi}_\epsilon$ has an invariant probabilistic measure σ_ϵ with respect to the $\dim_{\mathbb{H}}(\tilde{\mathcal{J}}_\epsilon)$ -conformal measure ν_ϵ , both supported on \mathbb{R} . From the uniformly bounded distortion of iterates of φ_ϵ , there is a uniform bound on the densities with respect to each other. By bounded distortion and conformality of ν_ϵ , the Lyapunov exponent

$$\chi_\epsilon = \int \log |\tilde{\varphi}'_\epsilon| d\sigma_\epsilon \lesssim \sum_{\zeta} \inf |\zeta'|^{-\dim_{\mathbb{H}}(\tilde{\mathcal{J}}_\epsilon)} \log \inf |\zeta'|,$$

where the sum is over branches of $\tilde{\varphi}_\epsilon$. Applying Corollary 6.3, we deduce that

$$\chi_\epsilon < \sum_{n \geq 0} C' e^{(1-\alpha)n} e^{-n(1-K\epsilon^{3/4})} (n+1) \leq K',$$

for uniform constants $C', K' > 0$. We redefine $K := \max(K, K')$.

For the final estimate, we make use of the Chebyshev Inequality,

$$\sigma_\epsilon(\{x : \log |(\tilde{\varphi}_\epsilon^n)'(x)| \geq 2nK\}) \leq \frac{n\chi_\epsilon}{2nK} < \frac{nK}{2nK} = \frac{1}{2}.$$

Let I_n denote the collection of connected components of \mathcal{D}_n which contain at least one point of $\{x : \log |(\varphi_\epsilon^n)'(x)| < 2nK\}$. Then

$$\sum_{d \in I_n} \nu_\epsilon(d) \gtrsim \sum_{d \in I_n} \sigma_\epsilon(d) > \frac{1}{2}.$$

By bounded distortion, for $d \in I_n$, $|d| \gtrsim e^{-2nK}$. Moreover,

$$\sigma_\epsilon(d) \sim v_\epsilon(d) \sim |d|^{\dim_H(\tilde{\mathcal{J}}_\epsilon)}.$$

Then

$$\begin{aligned} Q(n, 1 + \rho\sqrt{\epsilon}) &\geq \sum_{d \in I_n} |d|^{\dim_H(\tilde{\mathcal{J}}_\epsilon) + \rho\sqrt{\epsilon} + K\epsilon^{3/4}} \gtrsim \sum_{d \in I_n} v_\epsilon(d) |d|^{2\rho\sqrt{\epsilon}} \\ &\gtrsim \frac{1}{2} e^{-4K\rho\sqrt{\epsilon}n} \\ &\geq \frac{1}{2} (1 - 4K\rho\sqrt{\epsilon})^n. \end{aligned}$$

□

For $t \in [0, 2]$, by bounded distortion there is a constant $C > 1$ such that

$$\inf_{z \in \mathbb{D}_V} \sum_{y \in \tilde{\varphi}_\epsilon^{-n}(z)} |(\tilde{\varphi}_\epsilon^n)'(y)|^{-t} > C^{-1} Q(n, t). \tag{56}$$

We now turn our attention to complex estimates.

Proposition 6.9. *There is a universal constant $\rho > 0$ such that, for all small $\epsilon > 0$, the Hausdorff dimension of the Julia set \mathcal{J}_φ of the Cantor repeller $\varphi = \varphi_\epsilon$ supplied by Proposition 6.2 satisfies*

$$\dim_H(\mathcal{J}_\varphi) \geq 1 + \rho\sqrt{\epsilon}.$$

Proof. Let us denote the complex branch of φ by $\zeta : W \rightarrow \mathbb{D}_V$.

We wish to estimate $\sum_{y \in \varphi^{-n}(0)} |(\varphi^n)'(y)|^{-t}$. We can decompose the set $\varphi^{-n}(0)$ into a disjoint union of sets

$$S_\alpha = \zeta^{-j_k} \circ \tilde{\varphi}_\epsilon^{-j_{k-1}} \circ \dots \circ \zeta^{-j_2} \circ \tilde{\varphi}_\epsilon^{-j_1} \circ \zeta^{-j_0}(0)$$

indexed by words $\alpha = j_0 \dots j_k$ for which $0 \leq j_0, j_k \leq n, 1 \leq j_1, \dots, j_{k-1} \leq n$ and $\sum_i j_i = n$. Note that, for small $\epsilon > 0$,

$$\epsilon^{\frac{1}{2}\rho\sqrt{\epsilon}} > \frac{1}{2}.$$

Then, with $t = 1 + \rho\sqrt{\epsilon}$ and $\kappa = 4K\rho$, using estimates (52) from Proposition 6.2, (56) and Lemma 6.8,

$$\begin{aligned} \sum_{y \in S_\alpha} |(\varphi^n)'(y)|^{-t} &\geq (C^{-1}\sqrt{\epsilon})^{t(j_0+j_2+\dots+j_k)} C^{-1} Q(j_1, t) \dots C^{-1} Q(j_{k-1}, t) \\ &\geq C^{-1} K^{-1} (2^{-1} C^{-2t} K^{-1} \sqrt{\epsilon})^{j_0+j_2+\dots+j_k} (1 - \kappa\sqrt{\epsilon})^n. \end{aligned}$$

Each word α corresponds to a path in a binomial tree, starting by descending j_0 left branches, then j_1 right branches, then j_2 left branches and so on. Summing over α , we obtain (reversing the binomial expansion)

$$\sum_{y \in \varphi^{-n}(0)} |(\varphi^n)'(y)|^{-t} \geq C^{-1} K^{-1} (1 + 2^{-1} C^{-3} K^{-1} \sqrt{\epsilon})^n (1 - \kappa\sqrt{\epsilon})^n$$

which is greater than 1 for large n , if we choose $\rho < (8C^3K^2)^{-1}$. Hence the dimension is at least $t = 1 + \rho\sqrt{\epsilon}$ for all small $\epsilon > 0$. □

Corollary 6.10. *Theorem 4 holds.*

Proof. Simply note that $\mathcal{J}_c \supset \mathcal{J}_\varphi$ so $\dim_{\mathbb{H}}(\mathcal{J}_c) \geq \dim_{\mathbb{H}}(\mathcal{J}_\varphi)$. □

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