Communications in
**Mathematical
Physics**

# Regularity of Minimizers for a Model of Charged Droplets

**Guido de Philippis**[1]**, Jonas Hirsch**[2]**, Giulia Vescovo**[3]

[1] Courant Institute of Mathematical Sciences, New York University, New York, NY, USA.
   E-mail: guido@cims.nyu.edu
[2] Mathematisches Institut, Universität Leipzig, Augustus Platz 10, 04109 Leipzig, Germany.
   E-mail: hirsch@math.uni-leipzig.de
[3] SISSA, Via Bonomea 265, 34136 Trieste, Italy. E-mail: gvescovo@sissa.it; giulia.vescovo88@gmail.com

**Abstract:** We investigate properties of minimizers of a variational model describing the shape of charged liquid droplets. The model, proposed by Muratov and Novaga, takes into account the regularizing effect due to the screening of free counterionions in the droplet. In particular we prove partial regularity of minimizers, a first step toward the understanding of further properties of minimizers.

## 1. Introduction

*1.1. Background and description of the model.* In this paper we investigate the regularity of minimizers for a variational model describing the shape of charged liquid droplets. Roughly speaking, the shape of a charged liquid droplet is determined by the competition between an "aggregating" term, due to surface tension forces, and to a "disaggregating" term due to the repulsion effect between charged particles.

Several models proposed in the literature are based on this principle. Among them, one of the simplest and most used assumes that charged droplets are stationary points of the following free energy:

$$P(E) + \frac{Q^2}{\mathcal{C}(E)}. \tag{1.1}$$

Here, $E \subset \mathbb{R}^3$ corresponds to the volume occupied by the droplet, $P(E)$ is its perimeter, $Q$ is the total charge and

$$\frac{1}{\mathcal{C}(E)} := \inf \left\{ \frac{1}{4\pi} \iint \frac{d\mu(x)d\mu(y)}{|x - y|} : \text{spt } \mu \subset E, \mu(E) = 1 \right\}, \tag{1.2}$$

takes into account the repulsive forces between charged particles. Note that $\mu$ can be though as a (normalized) density of charges and that $\mathcal{C}(E)$ is the classical Newtonian

capacity of the set $E$. In particular one assumes that the optimal shapes are given by the following variational problem:

$$\min_{|E|=V} \left\{ P(E) + \frac{Q^2}{\mathcal{C}(E)} \right\}. \tag{1.3}$$

Heuristically, one expects the perimeter term to dominate for small values of the charge $Q$ thus forcing the droplet to have a spherical or almost spherical shape, while the repulsion term should become dominant for large values of $Q$, thus leading to the formation of singularities and/or to the ill-posedness of (1.3). This heuristics is confirmed by the perturbative analysis of (1.1) around a spherical shape. This computation, performed for the first time by Lord Rayleigh in 1882, [22], shows that the spherical droplet is linearly stable only for $Q$ smaller than a critical threshold. This is known as the Rayleigh criterion.

The transition from a stable to an unstable behavior of spherical droplets has also been verified experimentally, starting from the work of Zeleny at the beginning of 1900 [29] (in a slightly different context). More precisely, it has been observed that a spherical droplet exposed to an electric field, remains stable until the total charge is below a critical value $Q_c > 0$, while, as soon as $Q$ exceeds $Q_c$ the droplet changes its appearance and the surface start to develop singularities, the so called Taylor's cones, [26]. Whenever $Q > Q_c$ a very thin steady jet composed by small but highly charged little balls is formed, [9,10,23,28].

In spite of the interest of (1.3) in applications, a rigorous mathematical study of this model has been only performed in the last years, mostly thanks to the work of Goldman, Muratov, Novaga and Ruffini, see [15–17,20,21] and references therein.

The starting point of their analysis is the following remarkable and somehow disappointing observation: *Problem* (1.3) *is always ill-posed*. More precisely, in [15], it is shown that

$$\inf_{|E|=V} \left\{ P(E) + \frac{Q^2}{\mathcal{C}(E)} \right\} = P(B^V),$$

where $B^V$ is the ball of volume $V$. Since $B^V$ is a competitor for the variational problem , this clearly implies that there are no minimizers of (1.3).

The above equality is obtained by constructing a minimizing sequence consisting of a ball of roughly volume $V$ together with several balls with vanishing perimeter and volumes and very high charge escaping at infinity. Hence, on the mathematical side, the phenomena observed by Zeleny appears for *every value of the charge*. Let us also remark that ill-posedness of (1.3) is shown also if one assumes that all the set involved in the minimization problem are a-priori bounded, [15, Theorem 1.3].

It then becomes natural to investigate the local minimality of the ball, at least for "small" perturbations and small values of $Q$. In [15, Theorems 1.4 & 1.7] the linear stability of the ball in the small charge regime, is upgraded to local minimality in a sufficiently strong topology. On the other hand, Muratov and Novaga showed that the ball is *never* a local minimizer of (1.3) under (smooth) perturbation which are small in $L^\infty$, [20, Theorem 2]. We also refer the reader to [17] where well-posedness is recovered under suitable geometric restrictions and to [21] for the case of "flat" droplets.

The main phenomena driving to the ill-posedness of (1.3) is the possibility of concentrating a high charge on small volumes. In order to avoid this situation, in [20], Muratov and Novaga proposed as a possible regularization mechanism the finite screening length

in the conducting liquid , by introducing the entropic effects associated with the presence of free ions in the liquid, see also [8,27] for a related model. They suggested to consider the following *Debye-Hückel-type free energy* (in every dimension)

$$\mathcal{F}(E, u, \rho) := P(E) + Q^2 \left\{ \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K \int_E \rho^2 \, dx \right\}. \qquad (1.4)$$

Here

$$a_E(x) := \mathbf{1}_{E^c} + \beta \mathbf{1}_E,$$

where $\mathbf{1}_F$ is the characteristic function of a set $F$ and $\beta > 1$ is the permittivity of the droplet. The (normalized) density of charge $\rho \in L^2(\mathbb{R}^n)$ satisfies

$$\rho \mathbf{1}_{E^c} = 0 \quad \text{and} \quad \int \rho = 1, \qquad (1.5)$$

and the electrostatic potential $u$ is such that $\nabla u \in L^2(\mathbb{R}^n)$ and

$$-\operatorname{div}\left(a_E \nabla u\right) = \rho \quad \text{in } \mathcal{D}'(\mathbb{R}^n). \qquad (1.6)$$

$K > 0$ is a physical constant related to the model.[1]

The variational model proposed in [20], where one assumes *a-priori* that all the sets involved are contained in a fixed (large) ball $B_R$, is the following

$$\min \left\{ \mathcal{F}(E, u, \rho) : |E| = V, E \subset B_R, (u, \rho) \in \mathcal{A}(E) \right\}, \qquad (1.7)$$

where we have set

$$\mathcal{A}(E) := \left\{ (u, \rho) \in D^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n) : u \text{ and } \rho \text{ satisfy (1.6) and (1.5)} \right\}, \qquad (1.8)$$

and

$$D^1(\mathbb{R}^n) = \overline{C_c^\infty(\mathbb{R}^n)}^{\mathring{W}^{1,2}(\mathbb{R}^n)} \qquad \|\varphi\|_{\mathring{W}^{1,2}(\mathbb{R}^n)} = \|\nabla \varphi\|_{L^2(\mathbb{R}^n)}.$$

Note that the class of admissible couples $\mathcal{A}(E)$ is non-empty only if $n \geq 3$, for this reason *this assumption will be in force throughout the whole paper*, see also Remark 2.2.

Thanks to the a-priori boundedness assumption $E \subset B_R$, existence of a minimizer in the class of sets of finite perimeter can be easily shown, see [20, Theorem 3].

Note that the presence of the $L^2$ norm of $\rho$ in the energy is exactly what prevents the concentration of charges. Indeed, if one assumes that $\beta = 1$ so that (1.6) reduces to

$$-\Delta u = \rho,$$

---

[1] Actually in [20], the energy (1.4) is written as

$$\sigma P(E) + Q^2 \left\{ \frac{\beta_0}{2} \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K \int_E \rho^2 \, dx \right\},$$

for suitable parameters $\sigma$ and $\beta_0$ and the relation (1.6) is replaced by $-\beta_0 \operatorname{div}\left(a_E \nabla u\right) = \rho$. However it is easy to see that the parameters $\sigma$ and $\beta_0$ can be absorbed in $Q$ and $K$, see also the discussion below.

then the minimization problem (1.7) can be written, in dimension $n = 3$, as

$$\min_{|E|=V, E \subset B_R} P(E) + Q^2 \min \left\{ \frac{1}{4\pi} \iint \frac{\rho(x)\rho(y)dxdy}{|x-y|} + K \int \rho^2 \text{ s.t.} \rho \mathbf{1}_{E^c} = 0, \int \rho = 1 \right\},$$

which should be compared with (1.2) and (1.3). In view of this we also note that, on the mathematical ground, the variational problem (1.7) can also be considered as an "interpolation" between the classical Ohta-Kawasaki problem, and the free-interface problems arising in optimal design studied for instance in [3,6,12,18].

### 1.2. Main results.

Once existence of a minimizers of (1.7) is obtained it is natural to investigate their qualitative and quantitative properties, also to understand to which extent the predictions of the model agree with the observed phenomenology. In particular the following questions arise, compare with [20]:

- Is every minimizers smooth, at least outside a small singular set?
- Which is the structure of (possible) singularities of minimizers? Do they agree with Taylor's cones[2].?
- Is it possible to show existence/non-existence of minimizers removing the a-priori confinement assumption?
- Can one show that for small values of the total charge the minimizers of (1.7) are balls in agreement with experimental observations?

In this paper we address the question of regularity of minimizers. Our main result is the following partial regularity theorem:

**Theorem 1.1.** Let $n \geq 3$ and $B \geq 1$. Then there exists $\eta = \eta(n, B) > 0$ with the following property: if $E$ is a minimizer of (1.7) with $1 \leq \beta \leq B$ then there exists a closed set $\Sigma_E \subset \partial E$ such that $\mathcal{H}^{n-1-\eta}(\Sigma_E) = 0$ and $\partial E \backslash \Sigma_E$ is a $C^{1,\vartheta}$ manifold for all $\vartheta \in (0, 1/2)$.

As it is customary in Geometric Measure Theory, the proof Theorem 1.1 is based on an $\varepsilon$-regularity result which is interesting on its own. In order to keep track of the various dependence on the parameters let us first fix some notation which will be useful also in the sequel. For $E \subset \mathbb{R}^n$ we define

$$\mathcal{G}_{\beta,K}(E) := \inf_{(u,\rho) \in \mathcal{A}(E)} \left\{ \int_{\mathbb{R}^n} a_E |\nabla u|^2 + K \int_E \rho^2 \right\}, \tag{1.9}$$

where the set of admissible pairs $\mathcal{A}(E)$ is defined in (1.8) (if the dependence on the parameter is not relevant we will simply write $\mathcal{G}$). Since

$$(u, \rho) \in \mathcal{A}(E) \implies \left( \lambda^{2-n} u\left(\frac{\cdot}{\lambda}\right), \lambda^{-n} \rho\left(\frac{\cdot}{\lambda}\right) \right) \in \mathcal{A}(\lambda E),$$

one has

$$\mathcal{G}_{\beta,\lambda^2 K}(\lambda E) = \lambda^{2-n} \mathcal{G}_{\beta,K}(E).$$

Setting

$$\mathcal{F}_{\beta,K,Q}(E) := P(E) + Q^2 \mathcal{G}_{\beta,K}(E),$$

---

[2] Note that this is possible only if $\beta$ is large compared to 1, see the discussion at the end of this introduction and Remark 4.6.

one gets

$$\mathcal{F}_{\beta,K,Q}(E) = \lambda^{1-n}\mathcal{F}_{\beta,K\lambda^2,Q\lambda^{\frac{2n-3}{2}}}(\lambda E).$$

In particular, by replacing $K$ and $Q$ with $K(\omega_n/V)^{\frac{2}{n}}$ and $Q(\omega_n/V)^{1-\frac{3}{2n}}$ we can assume that $V = |B_1| =: \omega_n$. Namely, for $R \geq 1$ we will consider the following problem

$$\min\left\{\mathcal{F}_{\beta,K,Q}(E) : |E| = |B_1|, \ E \subset B_R\right\}. \qquad (\mathcal{P}_{\beta,K,Q,R})$$

Furthermore, given a set of finite perimeter $E$ we define the *spherical excess* at a point $x \in \partial E$ and at scale $r > 0$ as

$$\mathbf{e}_E(x,r) := \inf_{\nu \in \mathbb{S}^{n-1}} \frac{1}{r^{n-1}} \int_{\partial^* E \cap B_r(x)} \frac{|\nu_E(y) - \nu|^2}{2} d\mathcal{H}^{n-1}(y),$$

where, $\partial^* E$ is the reduced boundary of $E$, $\nu_E$ is the measure-theoretic unit normal to $\partial E$, see [19], and $B_r(x)$ is the ball of center $x$ and radius $r$. We also define the *normalized Dirichlet energy* as

$$D_E(x,r) := \frac{1}{r^{n-1}} \int_{B_r(x)} |\nabla u_E|^2 \, dy,$$

where $u_E$ is the minimizer in (1.9), whose existence and uniqueness can be easily proved, see Proposition 2.3 below. With these conventions, the $\varepsilon$-regularity results can be stated as follows, see also Theorem 8.1 below for a slightly more precise statement,

**Theorem 1.2.** *Given $n \geq 3$, $A > 0$ and $\vartheta \in (0, 1/2)$, there exits $\varepsilon_{\mathrm{reg}} = \varepsilon_{\mathrm{reg}}(n, A, \vartheta) > 0$ such that if $E$ is minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ with $Q + \beta + K + \frac{1}{K} \leq A$, $x \in \partial E$ and*

$$r + \mathbf{e}_E(x,r) + Q^2 D_E(x,r) \leq \varepsilon_{\mathrm{reg}},$$

*then $E \cap B(x, r/2)$ coincides with the epi-graph of a $C^{1,\vartheta}$ function. In particular $\partial E \cap B(x, r/2)$ is a $C^{1,\vartheta}$ $(n-1)$-dimensional manifold.*

Let us conclude this section with some remarks:

First beside its intrinsic interest, combining Theorem 1.2 with the analysis of the linearized energy around a ball one can show show that the balls uniquely minimize $(\mathcal{P}_{\beta,K,Q,R})$ for small value of $Q$. This will be addressed in a forthcoming paper.

Second we note that the dimension of the singular set in Theorem 1.1 depends only on the gap between the two permittivity constants and not on the other parameters appearing in the model. On the other hand the "regularity scale" in Theorem 1.2 depends on all the parameters involved. A similar fact has been observed in the context of free interfaces models in [6, 12].

Finally, it seems reasonable to expect that $C^{1,\vartheta}$ regularity of $\partial E$ can be upgraded to $C^\infty$ smoothness by some bootstrap argument. We leave this interesting question open.

*1.3. Strategy of the proof and structure of the paper.* Though the energy we are considering has a certain similarity with those studied in optimal design problems, the fact that the minimization problem in (1.9) is performed only among admissible pairs $(u, \rho) \in \mathcal{A}(E)$ makes very difficult to make local perturbations. In particular, problem ($\mathcal{P}_{\beta, K, Q, R}$) has (a priori) no local scaling invariance. For this reason in Sect. 2 we study carefully the energy $\mathcal{G}(E)$ and its minimizers $(u_E, \rho_E)$. Moreover we establish boundedness of $u_E$ and $\rho_E$ .

In order to study the regularity of minimizers one needs to perform local variations and hope that these give localized (or almost localized) changes of the energy. This is not completely obvious due to the presence of a volume constraint and to the non-local character of $\mathcal{G}$. As it is well known the volume constraint can be relaxed into a "perturbed" minimality property of minimizers. In order to have estimates uniform in the structural parameters it will be important to have this "perturbed" minimality property uniform in the class of minimizers. In Sect. 3 we start studying how the energy varies according to a flow of diffeomorphism, which will be important in performing small volume adjustments and we establish the Euler Lagrange equations for minimizers. In Sect. 4 we prove the perturbed minimality property and we study the behavior of the energy under local perturbations. In Sect. 5 we prove the compactness of the class of minimizers in the $L^1$ topology, which though not used in the proof of our main results is interesting by its own.

The next step consists in establishing local perimeter and volume estimates for the minimizers of ($\mathcal{P}_{\beta, K, Q, R}$). Usually these estimates are easily obtained by combining minimality with local isoperimetric inequalities. Here, due to the non-local character of the energy term $\mathcal{G}(E)$ and the absence of a natural scaling invariance of the problem, more refined arguments are required. In particular we will first show that the energy $\mathcal{G}$ is monotone by set inclusion. This implies that $E$ is an outer minimizer for the perimeter and leads to upper perimeter bounds and lower density estimates for $E^c$. Estimating the density of $E$ is instead more complicated and requires to perform an inductive argument showing that if $E$ has small relative measure in a ball $B_r(x)$, then the Dirichlet energy of $u_E$ decays enough to preserve this information at smaller scales, leading to a contradiction. In doing this, higher integrability of the gradient of minimizers of $\mathcal{G}$ plays a key role. Local density estimates are obtained in Sect. 6 together with the boundedness of $D_E(x, r)$. This fact combined with the local density and perimeter estimates allow somehow to recover the scaling invariance of the problem.

The main step of the proof of Theorems 1.1 and 1.2 is the decay of the excess established in Sect. 7. Once the local scaling invariance of the problem is recovered, the proof Theorem 1.2 follows the classical De Giorgi's idea of harmonic approximation. Namely we will show that in the regime of small excess and small normalized Dirichlet energy, $\partial E$ can be well approximated by the graph of a function with "small" Laplacian. This leads to the decay of the excess which, thanks to the higher integrability of $\nabla u_E$, in turn also implies the decay of the normalized Dirichlet energy and eventually allows to conclude the proof.

In Sect. 8 we prove Theorems 1.1 and 1.2. Theorem 1.2 will be an immediate consequence of Theorem 7.1 (see also Theorem 8.1 for a more quantitative version). Theorem 1.1 is proved by following the strategy of [12] where one combines the the $\varepsilon$-regularity result with the higher integrability of the $\nabla u_E$ and the classical regularity theory for minimal surfaces.

Let us remark that most of the above described difficulties arises only in the case when $\beta$ is relatively large compared to 1. Indeed in the regime $\beta - 1 \ll 1$, Cordes estimates,

see [4], imply that $\nabla u_E$ belongs to $L^p$ with $p$ large. In this case Hölder inequality immediately gives that the energy term $\mathcal{G}$ is lower order with respect to the perimeter at small scales. $E$ will then be an *ω-minimizer* of the perimeter and the regularity theory follows for instance from [25], see Remark 4.6. In particular in this case one obtains full regularity in $n = 3$, thus excluding the formation of Taylor's cone singularities. This phenomena was already observed in [24] for a different model of charged droplets.

## 2. Properties of Minimizers of $\mathcal{G}$

In this section we start establishing some basic properties of minimizers of $\mathcal{G}$. We start with the following easy lemma. Here and in the following let $2^* := 2n/(n-2)$ (recall that we are always working with $n \geq 3$).

**Lemma 2.1.** *Let $n \geq 3$, $\beta > 1$ and let $A : \mathbb{R}^n \to \mathrm{Sym}_n(\mathbb{R}^n)$ be a symmetric matrix valued function such that*

$$\mathrm{Id} \leq A(x) \leq \beta \, \mathrm{Id} \quad \text{for all} x \in \mathbb{R}^n.$$

*Then for every $\rho \in L^{(2^*)'}$ (i.e. the dual of $L^{2^*}$) there exists a unique $u \in D^1(\mathbb{R}^n)$ such that*

$$-\mathrm{div}(A\nabla u) = \rho.$$

*Proof.* Recall that for $u \in D^1(\mathbb{R}^n)$ one has the following Sobolev inequality

$$\|u\|_{2^*} \leq S(n)\|\nabla u\|_{L^2}.$$

In particular by the assumptions on $\rho$ and $A$ the energy

$$\mathcal{E}(v) := \frac{1}{2} \int_{\mathbb{R}^n} A\nabla v \cdot \nabla v \, dx - \int_{\mathbb{R}^n} \rho v \, dx,$$

is finite. By Young's inequality $\mathcal{E}(v)$ is bounded from below by

$$\|\nabla v\|_{L^2}^2 - C(n)\|\rho\|_{(2^*)'}^2.$$

Hence, the direct method of the calculus of variations imply the existence of a unique minimizer which is the desired solution. Furthermore for the solution $u$ we have

$$\min_{v \in D^1(\mathbb{R}^n)} \mathcal{E}(v) = \mathcal{E}(u) = -\frac{1}{2} \int_{\mathbb{R}^n} A\nabla u \nabla u \, dx = -\frac{1}{2} \int_{\mathbb{R}^n} \rho u \, dx.$$

$\square$

*Remark 2.2.* In dimension $n = 2$ the above lemma is easily seen to be false, indeed even for a smooth and compactly supported $\rho$, the solution of

$$-\Delta u = \rho,$$

does not in general satisfy $\nabla u \in L^2$.

By the above lemma , if $|E| < \infty$ the couple $(u, \rho)$ defined by

$$\rho = \frac{\mathbf{1}_E}{|E|}, \qquad -\operatorname{div}(a_E \nabla u) = \rho,$$

is admissible, $(u, \rho) \in \mathcal{A}(E)$. By testing the equation by $u$ and using the Sobolev embedding, we then get

$$\int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx = \fint_E u \, dx \leq \left( \fint_E u^{2^*} \, dx \right)^{\frac{1}{2^*}} \leq \frac{S(n)}{|E|^{\frac{1}{2^*}}} \|\nabla u\|_2.$$

In particular (recall $\beta > 1$)

$$\mathcal{G}(E) \leq \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K \int_{\mathbb{R}^n} \rho^2 \, dx \leq C(n, \beta, K, 1/|E|). \tag{2.1}$$

**Proposition 2.3.** *Let $E \subset \mathbb{R}^n$ be a set of finite measure. Then there exists a unique pair $(u_E, \rho_E) \in \mathcal{A}(E)$ minimizing $\mathcal{G}_{\beta, K}(E)$. Moreover*

$$u_E + K \rho_E = \mathcal{G}_{\beta, K}(E) \quad \text{in } E \tag{2.2}$$

*and*

$$0 \leq u_E \leq \mathcal{G}_{\beta, K}(E) \quad \text{and} \quad 0 \leq K \rho_E \leq \mathcal{G}_{\beta, K}(E) \mathbf{1}_E. \tag{2.3}$$

*In particular, $\rho_E \in L^p$ for all $p \in [1, \infty]$ and*

$$\|\rho_E\|_p \leq C(n, \beta, K, 1/|E|). \tag{2.4}$$

*Proof.* Existence of a minimizer is an immediate application of the direct method in the calculus of variations. Uniqueness follows from the convexity of the admissible set $\mathcal{A}(E)$ and of the strict convexity of the energy

$$(u, \rho) \mapsto \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K \int_{\mathbb{R}^n} \rho^2 \, dx.$$

Let now $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that

$$\psi \mathbf{1}_{E^c} = 0, \qquad \int_{\mathbb{R}^n} \psi \, dx = 0. \tag{2.5}$$

Let $v \in D^1(\mathbb{R}^n)$ be the solution of

$$-\operatorname{div}(a_E \nabla v) = \psi. \tag{2.6}$$

If $(u_E, \rho_E)$ is the minimizing pair then $(v_\varepsilon, \rho_\varepsilon) = (u_E + \varepsilon v, \rho_E + \varepsilon \psi) \in \mathcal{A}(E)$ is admissible. Hence, by taking the derivative with respect to $\varepsilon$ of its energy we get

$$0 = \int_{\mathbb{R}^n} a_E \nabla u_E \nabla v \, dx + K \int_{\mathbb{R}^n} \rho_E \psi \, dx \overset{(2.6)}{=} \int_{\mathbb{R}^n} (u_E + K \rho_E) \psi \, dx.$$

Since this holds for all $\psi$ satisfying (2.5) we get that $u_E + K \rho_E = C$ in $E$. By multiplying this equation by $\rho_E$, integrating and by recalling the identity

$$\int u_E \rho_E = \int a_E |\nabla u_E|^2,$$

we obtain

$$C = C \int \rho_E = \int u_E \rho_E + K \int \rho_E^2 = \int a_E |\nabla u_E|^2 + K \int \rho_E^2 = \mathcal{G}(E),$$

which proves (2.2). In particular $u_E$ solves

$$- \operatorname{div}(a_E \nabla u_E) = \frac{\mathcal{G}(E) - u_E}{K} \mathbf{1}_E. \tag{2.7}$$

By testing the above with $(\mathcal{G}(E) - u_E)_- = - \min\{0, \mathcal{G}(E) - u_E\}$ we obtain

$$0 = \int_{\{\mathcal{G}(E) < u_E\}} a_E |\nabla u_E|^2 \, dx + \int_{\{\mathcal{G}(E) < u_E\}} \frac{(\mathcal{G}(E) - u_E)^2}{K} \, dx,$$

which implies the second half of the first inequality in (2.3). Testing (2.7) with $u_- = - \min\{0, u\}$ we obtain the first half. The second inequality in (2.3) follows now from the first and (2.2). Inequality (2.4) follows from (2.1). $\qquad\square$

We establish now the monotonicity of $\mathcal{G}$ with respect to set inclusion. We start from the following lemma.

**Lemma 2.4.** *Let $A$, $B : \mathbb{R}^n \to \operatorname{Sym}_n(\mathbb{R}^n)$ two symmetric matrix valued functions such that $\operatorname{Id} \leq A(x) \leq B(x)$ for all $x \in \mathbb{R}^n$. If $\rho \in L^{(2^*)'}(\mathbb{R}^n)$ and $u, v \in D^1(\mathbb{R}^n)$ are the unique solutions of*

$$- \operatorname{div}(A \nabla u) = \rho \quad \text{and} \quad - \operatorname{div}(B \nabla v) = \rho, \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{2.8}$$

*then*

$$2 \int_{\mathbb{R}^n} (B - A) \nabla v \cdot \nabla v \, dx + \int_{\mathbb{R}^n} B \nabla v \cdot \nabla v \, dx \leq \int_{\mathbb{R}^n} A \nabla u \cdot \nabla u \, dx. \tag{2.9}$$

*In particular*

$$\int_{\mathbb{R}^n} B \nabla v \cdot \nabla v \, dx \leq \int_{\mathbb{R}^n} A \nabla u \cdot \nabla u \, dx.$$

*Proof.* Let $\mathcal{E}_A$ and $\mathcal{E}_B$ be the following functionals defined on $D^1(\mathbb{R}^n)$:

$$\mathcal{E}_A(w) := \frac{1}{2} \int_{\mathbb{R}^n} A \nabla w \cdot \nabla w \, dx - \int_{\mathbb{R}^n} \rho w \, dx,$$

$$\mathcal{E}_B(w) := \frac{1}{2} \int_{\mathbb{R}^n} B \nabla w \cdot \nabla w \, dx - \int_{\mathbb{R}^n} \rho w \, dx.$$

Hence $\mathcal{E}_A(w) \leq \mathcal{E}_B(w)$ for every $w \in D^1(\mathbb{R}^n)$. Since the solutions of (2.8) are minimizers of these energies, compare with Lemma 2.1, we have

$$\mathcal{E}_A(u) = \min_{D^1(\mathbb{R}^n)} \mathcal{E}_A \leq \min_{D^1(\mathbb{R}^n)} \mathcal{E}_B = \mathcal{E}_B(v).$$

Then

$$-\frac{1}{2} \int_{\mathbb{R}^n} A \nabla u \cdot \nabla u \, dx = \mathcal{E}_A(u) \leq \mathcal{E}_B(v) = -\frac{1}{2} \int_{\mathbb{R}^n} B \nabla v \cdot \nabla v \, dx,$$

and

$$-\frac{1}{2} \int_{\mathbb{R}^n} B \nabla v \cdot \nabla v \, dx = \int_{\mathbb{R}^n} B \nabla v \cdot \nabla v \, dx - \int_{\mathbb{R}^n} \rho v \, dx$$

$$= \int_{\mathbb{R}^n} (B - A) \nabla v \cdot \nabla v \, dx + \int_{\mathbb{R}^n} A \nabla v \cdot \nabla v \, dx - \int_{\mathbb{R}^n} \rho v \, dx$$

$$\geq \int_{\mathbb{R}^n} (B - A) \nabla v \cdot \nabla v \, dx - \frac{1}{2} \int_{\mathbb{R}^n} A \nabla u \cdot \nabla u \, dx,$$

concluding the proof. $\qquad\square$

The following corollary is an immediate consequence of the above lemma.

**Corollary 2.5.** *Let $E \subset F \subset \mathbb{R}^n$ be two sets of finite measure. Then*

$$\mathcal{G}_{\beta,K}(E) \geq \mathcal{G}_{\beta,K}(F).$$

*Proof.* Let $(u_E, \rho_E)$ be the optimal pair for $E$ and let $v$ be a solution of

$$-\operatorname{div}(a_F \nabla v) = \rho_E.$$

Then $(v, \rho_E)$ is admissible in the minimization problem defining $\mathcal{G}_{\beta,K}(F)$, hence

$$\mathcal{G}_{\beta,K}(F) \leq \int_{\mathbb{R}^n} a_F |\nabla v|^2 \, dx + K \int_{\mathbb{R}^n} \rho_E^2 \, dx$$

$$\leq \int_{\mathbb{R}^n} a_E |\nabla u_E|^2 \, dx + K \int_{\mathbb{R}^n} \rho_E^2 \, dx = \mathcal{G}_{\beta,K}(E),$$

where the last inequality follows from Lemma 2.4. $\qquad\square$

We conclude this section by proving the continuity of $\mathcal{G}$ under $L^1$ convergence. Recall that given two sets $E$ and $F$, $E \Delta F := (E \cup F) \setminus (E \cap F)$ is their symmetric difference.

**Proposition 2.6.** *Let $\{E_h\}$ be a sequence of sets with $|E_h| =: V_h \to V > 0$ when $h \to \infty$ and let $E$ be such that*

$$\lim_{h \to \infty} |E_h \Delta E| = 0,$$

*so that in particular $|E| = V$. Assume that $\beta_h \to \beta$ and that $K_h \to K$ when $h \to \infty$. Then*

$$\lim_{h \to \infty} \mathcal{G}_{\beta_h, K_h}(E_h) = \mathcal{G}_{\beta,K}(E).$$

*Moreover, $\nabla u_{E_h}$ and $\rho_{E_h}$ converge in $L^2$ to $\nabla u_E$ and $\rho_E$ respectively.*

*Proof.* Note that by (2.1)

$$\sup_h \mathcal{G}_{\beta_h, K_h}(E_h) < +\infty. \tag{2.10}$$

Thus

$$\sup_h \int_{\mathbb{R}^n} |\nabla v_{E_h}|^2 \, dx + \int_{\mathbb{R}^n} \rho_{E_h}^2 \, dx < \infty.$$

Moreover

$$a_h := a_{E_h} \xrightarrow{L^2} a_E = \mathbf{1}_{E^c} + \beta \mathbf{1}_E.$$

In particular, if $(u_h, \rho_h) = (u_{E_h}, \rho_{E_h})$ is the minimizing pair for $\mathcal{G}_{\beta_h, K_h}(E_h)$, then up to subsequence there exists $(u, \rho)$ such that

$$\nabla u_h \xrightarrow{L^2} \nabla u, \quad a_h \nabla u_h \xrightarrow{L^2} a_E \nabla u, \qquad \rho_h \xrightarrow{L^2} \rho.$$

Since $(u_h, \rho_h)$ are in $\mathcal{A}(E_h)$, one immediately deduces that $(u, \rho) \in \mathcal{A}(E)$ and thus, by lower semicontinuity,

$$\mathcal{G}_{\beta, K}(E) \leq \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx + K \int_{\mathbb{R}^n} \rho^2 \, dx \leq \liminf_{h \to \infty} \int_{\mathbb{R}^n} a_h |\nabla u_h|^2 \, dx + K_h \int_{\mathbb{R}^n} \rho_h^2 \, dx.$$

To prove the opposite inequality we take $(u_E, \rho_E)$ to be the minimizing pair for $\mathcal{G}_{\beta, K}(E)$ and we define $(w_h, \widetilde{\rho}_h) \in \mathcal{A}(E_h)$ as

$$\widetilde{\rho}_h = \sigma_h \, \rho_E \mathbf{1}_{E_h}, \qquad -\operatorname{div}(a_h \nabla w_h) = \widetilde{\rho}_h,$$

where $\sigma_h \to 1$ is such that $\int_{\mathbb{R}^n} \widetilde{\rho}_h \, dx = 1$. Since

$$-\operatorname{div}(a_h \nabla(u_E - w_h)) = -\operatorname{div}((a_h - a_E)\nabla u_E) + \rho_E(\mathbf{1}_E - \sigma_h \mathbf{1}_{E_h}),$$

by testing with $u_E - w_h$ and by exploiting the Sobolev embedding we obtain

$$\|\nabla(u_E - w_h)\|_2^2 \leq \int_{\mathbb{R}^n} a_h(\nabla u_E - \nabla w_h) \cdot (\nabla u_E - \nabla w_h) \, dx$$

$$= \int_{\mathbb{R}^n} (a_h - a_E)\nabla u_E \cdot \nabla(u_E - w_h) \, dx + \int_{\mathbb{R}^n} \rho_E(\mathbf{1}_E - \sigma_h \mathbf{1}_{E_h})\rho_E(u_E - w_h) \, dx$$

$$\leq \|(a_h - a_E)\nabla u_E\|_2 \|\nabla(u_E - w_h)\|_2 + S(n)\|\rho_E(\mathbf{1}_E - \sigma_h \mathbf{1}_{E_h})\|_2 \|\nabla(u_E - w_h)\|_2.$$

Then Young's inequality implies that $\|\nabla(u_E - w_h)\|_2 \to 0$. Since also $\|\widetilde{\rho}_h - \rho_E\|_2 \to 0$ and $(w_h, \widetilde{\rho}_h) \in \mathcal{A}(E_h)$, we get that

$$\limsup_{h \to \infty} \mathcal{G}_{\beta_h, K_h}(E_h) \leq \lim_{h \to \infty} \int_{\mathbb{R}^n} a_h |\nabla w_h|^2 \, dx + K_h \int_{\mathbb{R}^n} \widetilde{\rho}_h^2 \, dx$$

$$= \int_{\mathbb{R}^n} a_E |\nabla u_E|^2 \, dx + K \int_{\mathbb{R}^n} \rho^2 \, dx = \mathcal{G}_{\beta, K}(E).$$

The strong convergence of $\nabla u_{E_h}$ and $\rho_{E_h}$ is now a simple consequence of the convergence of energies. □

## 3. Small Volume Adjustments and Euler Lagrange Equations

In this section we show how to adjust the volume of a given set without increasing too much its energy which will be instrumental both to prove compactness of the class of minimizers in Sect. 5 and to get rid of the volume constraint in studying regularity of solutions of $(\mathcal{P}_{\beta,K,Q,R})$, see Sect. 4. The "adjustment" lemma will be proved with the aid of a deformation via a family of diffeomorphism close to the identity. Though not needed in the sequel we also establishes the Euler Lagrange equations associated to $(\mathcal{P}_{\beta,K,Q,R})$. We start with the following lemma.

**Lemma 3.1.** *For every $\eta \in C_c^1(B_R; \mathbb{R}^n)$ there exists $t_0 = t_0(\text{dist}(\text{spt }\eta, \partial B_R) > 0$ such that $\{\varphi_t\}_{|t| \leq t_0}$ defined by $\varphi_t(x) := x + t\,\eta(x)$ is a family of diffeomorphisms of $B_R$ into itself. Moreover for some set $E \subset B_R$ let $(u, \rho)$ be a solution of*

$$- \text{div}(a_E \nabla u) = \rho.$$

*Then setting*

$$u_t := u \circ \varphi_t^{-1} \quad and \quad \widetilde{\rho}_t := \det(\nabla \varphi_t^{-1})\, \rho \circ \varphi_t^{-1},$$

*we have*

$$- \text{div}\left(a_{E_t}\, A_t \nabla u_t\right) = \widetilde{\rho}_t, \tag{3.1}$$

*where $\|A_t - \text{Id}\|_\infty = O(t)$ and the implicit constant depends only on $\|\nabla \eta\|_\infty$.*

*Proof.* The proof of the first part of the Lemma is straightforward. For the second we see that for $\psi \in C_c^\infty$, by change of variables $x = \varphi_t(y)$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \widetilde{\rho}_t(x)\, \psi(x)\, dx &= \int_{\mathbb{R}^n} \rho(y)\psi(\varphi_t(y)) \det(\nabla \varphi_t^{-1})(\varphi_t(y)) \det(\nabla \varphi_t(y))\, dy \\
&= \int_{\mathbb{R}^n} a_E(y)\nabla u(y) \cdot \nabla(\psi \circ \varphi_t)(y)\, dy \\
&= \int_{\mathbb{R}^n} a_E(y)\nabla u(y) \cdot \left(\nabla \varphi_t(y)\right)^T \nabla \psi(\varphi_t(y))\, dy \\
&= \int_{\mathbb{R}^n} a_{E_t} \nabla u \circ \varphi_t^{-1} \left(\nabla \varphi_t \circ \varphi_t^{-1}\right)^T \nabla \psi \det \nabla \varphi_t^{-1}\, dx \\
&= \int_{\mathbb{R}^n} a_{E_t} \left(\nabla \varphi_t^{-1}\right)^{-T} \nabla u_t \cdot \left(\nabla \varphi_t^{-1}\right)^{-T} \nabla \psi \det \nabla \varphi_t^{-1}\, dx \\
&= \int_{\mathbb{R}^n} a_{E_t}\, A_t\, \nabla u_t \cdot \nabla \psi\, dx.
\end{aligned}
$$

Where we have used the equality $\nabla \varphi \circ \varphi_t^{-1} = (\nabla \varphi_t^{-1})^{-1}$ and for a matrix $N$ we denoted by $N^T$ its transpose and we set $N^{-T} = (N^{-1})^T$. Hence $u_t$ is a solution of (3.1) with

$$A_t = \det \nabla \varphi_t^{-1} \left(\nabla \varphi_t^{-1}\right)^{-T} \left(\nabla \varphi_t^{-1}\right)^{-1}.$$

By an explicit computation we see that $A_t$ satisfies the desired bound. $\qquad\square$

We now show how the energy $\mathcal{G}$ changes by the effect of a family of diffeomorphism.

**Lemma 3.2.** *Let $E \subseteq B_M$ be a measurable set and let $\{\varphi_t\}_{|t| \leq t_0}$ be a family of diffeomorphisms as in Lemma 3.1. Then*

$$\mathcal{G}_{\beta,K}(E_t) \leq (1 + O(t)) \, \mathcal{G}_{\beta,K}(E), \tag{3.2}$$

*where $E_t := \varphi_t(E)$ and the implicit constant depends only on $\|\nabla \eta\|_\infty$. Moreover*

$$\mathcal{G}_{\beta,K}(E_t) \leq \mathcal{G}_{\beta,K}(E)$$
$$+ t \left( \int_{\mathbb{R}^n} a_E \left( |\nabla u_E|^2 \operatorname{div} \eta - 2\nabla u_E \cdot \nabla \eta \, \nabla u_E \right) - K \int_{\mathbb{R}^n} \rho_E^2 \operatorname{div} \eta \right) + O(t^2). \tag{3.3}$$

*Proof.* Let $(u_E, \rho_E) \in \mathcal{A}(E)$ be a the optimal pair for $\mathcal{G}_{\beta,K}(E)$. By Lemma 3.1 $u_t = u_E \circ \varphi_t^{-1}$ solves (3.1) with $\widetilde{\rho}_t = \rho_E \circ \varphi_t^{-1} \det(\nabla \varphi_t^{-1})$. Let $v_t$ be the solution of

$$- \operatorname{div} \left( a_{E_t} \nabla v_t \right) = \widetilde{\rho}_t \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n). \tag{3.4}$$

*Step 1*: We start by proving the following estimate

$$\int_{\mathbb{R}^n} a_{E_t} \left( |\nabla v_t|^2 - |\nabla u_t|^2 \right) dx \leq O(t) \int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \, dx, \tag{3.5}$$

where the implicit constant depends only on $\|\nabla \eta\|_\infty$. In order to prove (3.5) we claim that

$$\left( \int_{\mathbb{R}^n} a_{E_t} |\nabla(u_t - v_t)|^2 \, dx \right)^{1/2} \leq O(t) \left( \int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \right)^{1/2}. \tag{3.6}$$

Indeed assuming that (3.6) holds true and using that $|a|^2 - |b|^2 = 2b \cdot (a - b) + |a - b|^2$ for every $a, b \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} a_{E_t} \left( |\nabla v_t|^2 - |\nabla u_t|^2 \right) dx = 2 \int_{\mathbb{R}^n} a_{E_t} \nabla u_t \cdot \nabla(v_t - u_t) \, dx$$
$$+ \int_{\mathbb{R}^n} a_{E_t} |\nabla(u_t - v_t)|^2 \, dx. \tag{3.7}$$

We estimate the first term in the right hand side of (3.7). By (3.6), we find that

$$\int_{\mathbb{R}^n} a_{E_t} \nabla u_t \cdot \nabla(v_t - u_t) \, dx \leq \left( \int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}^n} a_{E_t} |\nabla(u_t - v_t)|^2 \, dx \right)^{1/2}$$
$$\leq O(t) \int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \, dx. \tag{3.8}$$

By (3.7) and (3.8), we have:

$$\int_{\mathbb{R}^n} a_{E_t} \left( |\nabla v_t|^2 - |\nabla u_t|^2 \right) dx \leq O(t) \int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \, dx + O(t^2) \int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \, dx,$$

which proves (3.5).

Let us now prove (3.6). By testing (3.1) and (3.4) with $v_t - u_t$ we get

$$\int_{\mathbb{R}^n} a_{E_t} \nabla v_t \cdot \nabla(v_t - u_t) \, dx = \int_{\mathbb{R}^n} \widetilde{\rho}_t \, (v_t - u_t) \, dx = \int_{\mathbb{R}^n} a_{E_t} A_t \, \nabla u_t \cdot \nabla(v_t - u_t) \, dx$$
$$= \int_{\mathbb{R}^n} a_{E_t} \, (A_t - \operatorname{Id}) \, \nabla u_t \cdot \nabla(v_t - u_t) \, dx + \int_{\mathbb{R}^n} a_{E_t} \nabla u_t \cdot \nabla(v_t - u_t) \, dx.$$

Rearranging terms and recalling that $|A_t - \mathrm{Id}\,| = O(t)$, this gives

$$\int_{\mathbb{R}^n} a_{E_t} |\nabla(v_t - u_t)|^2 \, dx \leq O(t) \int_{\mathbb{R}^n} |\nabla v_t - \nabla u_t||\nabla u_t|,$$

which, by Young's inequality, implies (3.6).

*Step 2*: By changing variables:

$$\int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \, dx = \int_{\mathbb{R}^n} |(\nabla \varphi_t)^{-T} \nabla u|^2 \det \nabla \varphi_t.$$

Moreover

$$\nabla \phi_t = \mathrm{Id} + t \nabla \eta + o(t) \qquad \text{and} \qquad \det \nabla \phi_t = 1 + t \, \mathrm{div}\, \eta + o(t),$$

which gives

$$\int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \, dx = (1 + O(t)) \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx, \tag{3.9}$$

and the more precise equality

$$\int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \, dx = \int_{\mathbb{R}^n} a_E |\nabla u|^2 \, dx$$

$$+ t \int_{\mathbb{R}^n} a_E \left( \mathrm{div}\, \eta \, |\nabla u_E|^2 - 2\nabla u_E \cdot \nabla \eta \, \nabla u_E \right) + o(t). \tag{3.10}$$

In the same way we get

$$\int_{E_t} \widetilde{\rho}_t^2 \, dx = \int_E \frac{\rho^2}{\det \nabla \varphi_t} \, dx = (1 + O(t)) \int_E \rho^2 \, dx. \tag{3.11}$$

Furthermore, since $\det \nabla \varphi_t = 1 + t \, \mathrm{div}\, \eta + o(t)$, we also get

$$\int_{E_t} \widetilde{\rho}_t^2 \, dx = \int_E \rho^2 \, dx - t \int_E \rho_E^2 \, \mathrm{div}\, \eta \, dx + o(t). \tag{3.12}$$

*Step 4:* Since, by its definition

$$\int_{\mathbb{R}^n} \widetilde{\rho}_t \, dx = 1, \qquad \widetilde{\rho}_t \mathbf{1}_{E_t^c} = 0,$$

and $v_t$ solves (3.4), we see that $(v_t, \widetilde{\rho}_t) \in \mathcal{A}(E_t)$. Hence, by combining (3.5), (3.9) and (3.11) we obtain

$$\mathcal{G}_{\beta,K}(E_t) \leq \int_{\mathbb{R}^n} a_{E_t} |\nabla v_t|^2 \, dx + K \int_{\mathbb{R}^n} \widetilde{\rho}_t^2 \, dx$$

$$\leq (1 + O(t)) \int_{\mathbb{R}^n} a_{E_t} |\nabla u_t|^2 \, dx + K \int_{\mathbb{R}^n} \widetilde{\rho}_t^2 \, dx$$

$$\leq (1 + O(t)) \left( \int_{\mathbb{R}^n} a_E |\nabla u_E|^2 \, dx + K \int_E \rho_E^2 \, dx \right) = (1 + O(t)) \, \mathcal{G}_{\beta,K}(E),$$

which proves (3.2). The proof of (3.3) is obtained by combining the above argument with (3.10) and (3.12). $\qquad\qquad\square$

By combining the Taylor expansion of the perimeter, [19, Theorem 17.8],

$$P(\varphi_t(E)) = P(E) + t \int_{\partial^* E} \operatorname{div}_E \eta \, d\mathcal{H}^{n-1} + o(t), \qquad \operatorname{div}_E \eta = \operatorname{div} \eta - \nu_E \cdot \nabla \eta \, \nu_E,$$

with (3.3) we obtain the Euler Lagrange equations for minimizers of $(\mathcal{P}_{\beta,K,Q,R})$ whose proof is left to the reader.

**Corollary 3.3.** *Let $E$ be a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, then*

$$\int_{\partial^* E} \operatorname{div}_E \eta \, d\mathcal{H}^{n-1} + Q^2 \int_{\mathbb{R}^n} a_E \Big( |\nabla u_E|^2 \operatorname{div} \eta - 2 \nabla u_E \cdot \nabla \eta \, \nabla u_E \Big) dx$$

$$- Q^2 K \int_{\mathbb{R}^n} \rho_E^2 \operatorname{div} \eta \, dx = 0,$$

*for all $\eta \in C_c^1(B_R; \mathbb{R}^n)$ with $\int_E \operatorname{div} \eta \, dx = 0$.*

The next series of results are modeled after [2] and allow to do small volume adjustments without increasing too much the perimeter, see also [19, Chapter 17]. The first lemma is elementary.

**Lemma 3.4.** *Let $E \subseteq \mathbb{R}^n$ be a set of finite perimeter and let $U$ be an open set such that $P(E, U) > 0$. Then there exist $\gamma = \gamma(E) > 0$ and a vector field $\eta_E \in C_c^1(U; \mathbb{R}^n)$ with $\|\eta\|_{C^1} \leq 1$ such that*

$$\int_E \operatorname{div} \eta_E \, dx \geq \gamma(E) > 0.$$

*Proof.* Since

$$P(E, U) = \sup \left\{ \int_E \operatorname{div} \eta \, dx \ : \ \eta \in C_c^1(U; \mathbb{R}^n), \ \|\eta\|_\infty \leq 1 \right\},$$

we find a vector field $\widetilde{\eta} \in C_c^1(U; \mathbb{R}^n)$ with $\|\widetilde{\eta}\|_\infty \leq 1$ such that

$$\int_E \operatorname{div} \widetilde{\eta} \, dx \geq P(E, U)/2.$$

Taking $\eta = \widetilde{\eta}/\|\widetilde{\eta}\|_{C^1}$ we obtain the desired conclusion. $\qquad\square$

In order to have uniform controls on the constants involved in our regularity theory, we need to enforce the above lemma in the following one. Note that this time the constants depend only on the upper bound on the perimeter, in particular they do not depend on the radius $R$ in $(\mathcal{P}_{\beta,K,Q,R})$.

**Lemma 3.5.** *For every $P > 0$ there exist two constants $\bar{\gamma} = \bar{\gamma}(n, P) > 0$ and $\bar{\delta} = \bar{\delta}(n, P) > 0$ such that if $R \in (1, \infty)$ and $E \subset B_R$ satisfies*

$$\frac{|B_1|}{2} \leq |E| \leq \frac{3|B_1|}{2}, \qquad P(E) \leq P, \tag{3.13}$$

*then there exists a vector field $\eta \in C_c^1(B_{R-\bar{\delta}}; \mathbb{R}^n)$ with $\|\eta\|_{C^1} \leq 1$ such that*

$$\int_E \operatorname{div} \eta \, dx \geq \bar{\gamma}.$$

*Proof.* Let us argue by contradiction: assume that there exist a sequence of radii $R_k$ and a sequence of sets $E_k$ satisfying (3.13) such that

$$\int_{E_k} \operatorname{div} \eta \, dx \to 0 \qquad \text{for all } \eta \in C^1_c(B_{R_k - \bar{\delta}}; \mathbb{R}^n), \text{with} \|\eta\|_{C^1} \leq 1, \qquad (3.14)$$

where $\bar{\delta} = \bar{\delta}(n, P)$ is a small constant to be fixed later only in dependence of $n$ and $P$. By [19, Remark 29.11] there exist points $y_k \in \mathbb{R}^n$ and a constant $\delta_1 = \delta_1(n, P)$ such that

$$|E_k \cap B_1(y_k)| \geq 2\delta_1.$$

Then by taking $z_k \in E_k \cap B_1(y_k) \subset B_{R_k} \cap B_1(y_k)$ we get

$$|E_k \cap B_2(z_k)| \geq 2\delta_1 \qquad z_k \in B_{R_k}.$$

Let us now detail the proof in the case in which, up to subsequences, $R_k \to \infty$ and $\partial B_{R_k - 1} \cap B_2(z_k) \neq \emptyset$. The other cases are actually simpler and we explain how to modify the argument at the end of the proof. We first note that since $\partial B_{R_k - 1} \cap B_2(z_k) \neq \emptyset$, we can take $x_k \in \partial B_{R_k}$ such that

$$|E_k \cap B_4(x_k)| \geq |E_k \cap B_2(y_k)| \geq 2\delta_1 \qquad x_k \in \partial B_{R_k}.$$

Now a simple geometric argument ensures that

$$\lim_{\delta \to 0} \sup_k |B_4(x_k) \cap (B_{R_k} \setminus B_{R_k - \delta})| \to 0.$$

In particular we can choose $\delta_2 = \delta_2(n, P)$ such that

$$|E_k \cap B_4(x_k) \cap B_{R_k - \delta_2}| \geq \delta_1. \qquad (3.15)$$

Let us now assume that, up to subsequences and a possible rotation of coordinates

$$F_k := \left(E_k \cap B_4(x_k) \cap B_{R_k - \delta_2}\right) - x_k \to F, \quad \frac{x_k}{R_k} \to e_1,$$

where the first limit exits due to our assumption on the perimeters. In particular

$$B_4 \cap B_{R_k - \delta_2}(-x_k) \to \widehat{B} = B_4(0) \cap \{x_1 < -\delta_2\},$$

and $F \subset \widehat{B}$. Note that by (3.15), $F \neq \emptyset$ and, since $|F_k| \leq 3|B_1|/2$, $|\widehat{B} \setminus F| > 0$. In particular, $P(F, \widehat{B}) > 0$. By Lemma 3.4, we can find a constant $\gamma = \gamma(F) > 0$ and vector field $\eta_F \in C^1_c(\widehat{B}; R^n)$ with $\|\eta\|_{C^1} \leq 1$ such that

$$\gamma \leq \int_F \operatorname{div} \eta_F \, dx.$$

For $k$ large, the vector field $\eta_k(\cdot) = \eta_F(\cdot + x_k)$ satisfies: $\eta_k \in C^1_c(B_{R_k - \delta_2/2}; \mathbb{R}^n)$, $\|\eta\|_{C^1} \leq 1$ and contradicts (3.14) with $\bar{\delta} = \delta_2/2$.

Let us conclude by explaining how to modify the proof in the case in which either $B_2(z_k) \cap \partial B_{R_k - 1} = \emptyset$ or $R_k \to \bar{R} < \infty$. In the first case instead one argue as above by considering the sets $F_k := \left(E_k \cap B_2(z_k)\right) - z_k$ and by noticing that the vector fields $\eta_k(\cdot) = \eta_F(\cdot + y_k)$ with $F := \lim F_k$ are compactly supported in $B_{R_k - 1/2}$. In the second case one can simply reproduce the above argument.  □

The next proposition will be crucial in removing the volume constraint and in making comparison estimates for minimizers of $(\mathcal{P}_{\beta,K,Q,R})$.

**Proposition 3.6.** *For every $P > 0$ there exist constants $\bar{\sigma} = \bar{\sigma}(n, P) > 0$ and $C = C(n)$ such that if $R \in (1, \infty)$ and $E \subset B_R$ satisfies*

$$\frac{|B_1|}{2} \leq |E| \leq \frac{3|B_1|}{2}, \qquad P(E) \leq P,$$

*then for all $\sigma \in (-\bar{\sigma}, \bar{\sigma})$ there exists $F_\sigma \subset B_R$ such that*

$$|F_\sigma| = |E| + \sigma \quad \text{and} \quad |\mathcal{F}_{\beta,K,Q}(F_\sigma) - \mathcal{F}_{\beta,K,Q}(E)| \leq C|\sigma|\,\mathcal{F}_{\beta,K,Q}(E).$$

*Proof.* By Lemma 3.5 we can find $\bar{\gamma} = \bar{\gamma}(n, P) > 0$, $\bar{\delta} = \bar{\delta}(n, P) > 0$ and a vector field $\eta \in C^1_c\big(B_{R-\bar{\delta}}; \mathbb{R}^n\big)$ with $\|\eta\|_{C^1} \leq 1$ such that

$$\bar{\gamma} \leq \int_E \operatorname{div} \eta \, dx. \tag{3.16}$$

Define a family of diffeomorphisms $\varphi_t := \operatorname{Id} + t\,\eta$ and note that, since $\operatorname{dist}(\operatorname{spt}(\eta), \partial B_R) \geq \bar{\delta}(n, P)$, they send $B_R$ into itself for $|t| \leq t_0(n, P)$. By Taylor expansion

$$|E_t| = |E| + t \int_E \operatorname{div} \eta \, dx + O(t^2)|E|, \tag{3.17}$$

and

$$P(E_t) = P(E) + t \int_{\partial^* G} \operatorname{div}_E \eta \, d\mathcal{H}^{n-1} + O(t^2)P(E),$$

where the implicit constant is universal since $\|\nabla \eta\|_\infty \leq 1$. Moreover

$$\mathcal{G}_{\beta,K}(E_t) \leq (1 + C|t|)\mathcal{G}_{\beta,K}(E), \tag{3.18}$$

where $E_t = \varphi_t(E)$ and the constant in (3.18) depends only on $\|\nabla \eta\|_\infty \leq 1$. Hence we can find $t_1 = t_1(n, P) > 0$ such that

$$\big||E_t| - |E|\big| \geq |t|\frac{\bar{\gamma}}{2} \qquad \text{(by(3.16))}, \tag{3.19a}$$

and

$$|\mathcal{F}_{\beta,K,Q}(E_t) - \mathcal{F}_{\beta,K,Q}(E)| \leq C|t|\mathcal{F}_{\beta,K,Q}(E). \tag{3.19b}$$

for every $|t| \leq t_1$. By equations (3.19a) and (3.19b) we get

$$|\mathcal{F}_{\beta,K,Q}(E_t) - \mathcal{F}_{\beta,K,Q}(E)| \leq C\mathcal{F}_{\beta,K,Q}(E)\big||E_t| - |E|\big|.$$

Let $g(t) := |E_t|$ and note that thanks to (3.17) and (3.16), $g$ is increasing in a neighborhood of 0. Take $\bar{\sigma} > 0$ such that $(|E| - \bar{\sigma}, |E| + \bar{\sigma}) \subseteq g\big((-t_1, t_1)\big)$. Then for every $|\sigma| \leq \bar{\sigma}$ there exists $t_\sigma > 0$ such that $|E_{t_\sigma}| = |E| + \sigma$. Setting $F_\sigma = E_{t_\sigma}$ we obtain the desired conclusion. $\qquad \square$

## 4. Λ-Minimality and Local Variations

In order to study the regularity of minimizers it will be convenient to understand what is the behavior under small perturbations in balls. In this section we start by removing the volume constraint by showing that minimizers are $\Lambda$-minimizer of $\mathcal{F}$ under small perturbations. In order to keep track of the dependence of the parameters in Theorem 1.2, it will be important that this "almost"-minimality depends only on the structural parameter of the problem. We thus start by fixing the following convention, which will be in force throughout all the rest of the paper:

**Convention 4.1** (*Universal constants*). Given $A > 0$, we say that

- the parameters $\beta$, $K$, $Q$ with $\beta \geq 1$ are *controlled by $A$* if

$$\beta + K + \frac{1}{K} + Q \leq A.$$

- A constant is *universal* if it depends only on the dimension $n$ and on $A$.
- For two positive quantities $X$ and $Y$, we will write $X \lesssim Y$ if there exists a universal constant $C$ such that $X \leq CY$ and we write $X \gtrsim Y$ if $Y \lesssim X$.

Note in particular that universal constants *do not depend* on the size of the container where the minimization problem is solved. Moreover we also remark here the following elementary fact: since $B_1$ is always a competitor for $(\mathcal{P}_{\beta,K,Q,R})$, if $E$ is a minimizer then

$$P(E) \leq \mathcal{F}_{\beta,K,Q}(E) \leq \mathcal{F}_{\beta,K,Q}(B_1) \leq C(n, A), \tag{4.1}$$

whenever $\beta$, $K$, $Q$ are controlled by $A$.

Let us now introduce the following perturbed minimality condition.

**Definition 4.2** (($\Lambda, \bar{r}$)-*minimizer*). We say that $E$ is a ($\Lambda, \bar{r}$)-minimizer of the energy $\mathcal{F}$ if there exist constants $\Lambda > 0$ and $\bar{r} > 0$ such that for every ball $B_r(x) \subseteq \mathbb{R}^n$ with $r \leq \bar{r}$ we have

$$\mathcal{F}_{\beta,K,Q}(E) \leq \mathcal{F}_{\beta,K,Q}(F) + \Lambda |E \Delta F| \quad \text{whenever} E \Delta F \subset B_{\bar{r}}(x). \tag{4.2}$$

*Remark 4.3.* Note that if $E$ is ($\bar{\Lambda}, \bar{r}$)-minimizer than it is also a ($\bar{\Lambda}_1, \bar{r}_1$)-minimizer whenever $\bar{\Lambda}_1 \geq \bar{\Lambda}$ and $\bar{r}_1 \leq \bar{r}$. Hence there is no loss of generality in assuming that $\bar{r} \leq 1$.

We can now establish the desired $\Lambda$-minimality property for minimizers of $(\mathcal{P}_{\beta,K,Q,R})$.

**Proposition 4.4.** *Let $A > 0$ and let $\beta$, $K$, $Q$ with $\beta \geq 1$ be controlled by $A$ and let $R \geq 1$. Then there exist $\Lambda_1, \bar{r}_1 > 0$ universal such that all minimizers $(\mathcal{P}_{\beta,K,Q,R})$ satisfy*

$$\mathcal{F}_{\beta,K,Q}(E) \leq \mathcal{F}_{\beta,K,Q}(F) + \Lambda_1 |E \Delta F|,$$

*whenever $F \subset B_R$ and $E \Delta F \subset B_r(x_0)$, $r \leq \bar{r}_1$.*

*Proof.* Clearly we can suppose that

$$\mathcal{F}_{\beta,K,Q}(F) \le \mathcal{F}_{\beta,K,Q}(E) \lesssim 1,$$

since otherwise the result is trivial. In particular $P(F)$ is bounded by a universal constant $P$. Let $\bar{\sigma}$ and $C$ be the parameters in Proposition 3.6 associated to $P$. If $\bar{r}_1$ is chosen small enough we have

$$|E \Delta F| \le \omega_n \bar{r}_1^n \ll \bar{\sigma}.$$

Moreover, since $|E| = |B_1|$, $|F| \in (|B_1|/2, 3|B_1|/2)$. Hence we can apply Proposition 3.6 to $F$ to obtain a set $\widetilde{F} \subset B_R$ such that $|\widetilde{F}| = |B_1|$ and

$$\mathcal{F}_{\beta,K,Q}(E) \le \mathcal{F}_{\beta,K,Q}(\widetilde{F}) \le \left(1 + C\big||\widetilde{F}| - |F|\big|\right) \mathcal{F}_{\beta,K,Q}(F), \tag{4.3}$$

where the first inequality is due to the minimality of $E$. Since $\mathcal{F}_{\beta,K,Q}(F) \lesssim 1$ and

$$\big||\widetilde{F}| - |F|\big| = \big||E| - |F|\big| \le |F \Delta E|,$$

we obtain the conclusion for a suitable universal constant $\Lambda_1$.

$\square$

We conclude this section by establishing the following "local" minimality properties of minimizers $(\mathcal{P}_{\beta,K,Q,R})$. Note that in (ii) below we are not requiring $F$ to be contained in $B_R$.

**Proposition 4.5.** *Let $A > 0$, and let $\beta$, $K$, $Q$ be controlled by $A$ and $R \ge 1$. Then there exist universal constants $\Lambda_2$ and $\bar{r}_2$ such that given a minimizer $E$ of $(\mathcal{P}_{\beta,K,Q,R})$ we have that it satisfies the following two properties:*

*(i) for every set of finite perimeter $F \subseteq E$ with $E \backslash F \subset B_r(x)$ and $r \le \bar{r}_2$ it holds:*

$$P(E) \le P(F) + \Lambda_2|E \backslash F| + \Lambda_2 Q^2 \int_{E \backslash F} |\nabla u_E|^2 \, dx, \tag{4.4}$$

*where $u_E$ the minimizer in (1.9).*

*(ii) For every set of finite perimeter $F \supseteq E$ with $F \backslash E \subset B_r(x)$ and $r \le \bar{r}_2$ it holds:*

$$P(E) \le P(F) + \Lambda_2|F \backslash E|. \tag{4.5}$$

*In particular, if $u_E$ is the minimizer in (1.9) then*

$$P(E) \le P(F) + \Lambda_2|E \Delta F| + \Lambda_2 Q^2 \int_{E \Delta F} |\nabla u_E|^2 \, dx, \tag{4.6}$$

*whenever $F \Delta E \subset B_r(x)$ with $r \le \bar{r}_2$.*

*Proof.* We start proving (i). Let $E$ be a minimizer and $(u_E, \rho_E)$ be the minimizing pair for $\mathcal{G}(E)$. Let $F \subseteq E$ be such that $E \backslash F \subset B_r(x)$ with $r \le \bar{r}_2 \le \bar{r}_1$ where $\bar{r}_1$ is the constant defined in Proposition 4.4, by possibly choosing $\bar{r}_2$ smaller, we can assume that

$$|F| \ge \frac{|E|}{2} = \frac{|B_1|}{2}. \tag{4.7}$$

Let us set

$$\rho = (\rho_E + \lambda_F)\,\mathbf{1}_F \qquad \text{where} \quad \lambda_F = \frac{\int_{E \setminus F} \rho_E\,dx}{|F|},$$

and let $u$ be the solution of

$$-\operatorname{div}(a_F \nabla u) = \rho.$$

Note that $(u, \rho) \in \mathcal{A}(F)$ and thus, by using the $\Lambda$-minimality of $E$ established in Proposition 4.4,

$$P(E) + Q^2\left(\int_{\mathbb{R}^n} a_E |\nabla u_E|^2\,dx + K \int \rho_E^2\,dx\right) \le P(F) + Q^2\left(\int_{\mathbb{R}^n} a_F |\nabla u|^2\,dx + K \int \rho^2\,dx\right)$$
$$+ \Lambda_1 |E \setminus F|.$$

Item (ii) will then follow if we can prove

$$\int_{\mathbb{R}^n} \left(\rho^2 - \rho_E^2\right) dx \lesssim |E \setminus F|, \tag{4.8}$$

and

$$\int_{\mathbb{R}^n} \left(a_F\,|\nabla u_F|^2 - a_E |\nabla u|^2\right) dx \lesssim |E \setminus F| + \int_{E \setminus F} |\nabla u_E|^2\,dx. \tag{4.9}$$

To prove (4.8) we estimate

$$\int_{\mathbb{R}^n} (\rho^2 - \rho_E^2)\,dx = -\int_{E \setminus F} \rho_E^2\,dx + \int_F (\lambda_F^2 + 2\rho_E\,\lambda_F)\,dx$$
$$\le -\int_{E \setminus F} \rho_E^2\,dx + \frac{|E \setminus F|}{|F|} \int_{E \setminus F} \rho_E^2\,dx + 2\|\rho_E\|_\infty \int_{E \setminus F} \rho_E\,dx$$
$$\le 2\|\rho_E\|_\infty^2 |E \setminus F|,$$

where in the first inequality we have used (4.7) and the definition of $\lambda_F$. By (2.4), $\|\rho_E\|_\infty \lesssim 1$ and this concludes the proof of (4.8).

Let us now prove (4.9). First note that

$$\int_{\mathbb{R}^n} \left(a_F|\nabla u|^2 - a_E|\nabla u_E|^2\right) dx = \int_{\mathbb{R}^n} a_F(|\nabla u|^2 - |\nabla u_E|^2)\,dx$$
$$+ \int_{\mathbb{R}^n} (a_F - a_E)\,|\nabla u_E|^2\,dx. \tag{4.10}$$

Testing the equations satisfied by $u_E$ and $u$ with $u_E$ and $u$ respectively and subtracting the result we obtain also

$$\int_{\mathbb{R}^n} \left(a_F|\nabla u|^2 - a_E|\nabla u_E|^2\right) dx = \int_{R^n} u\rho\,dx - \int_{\mathbb{R}^n} u_E \rho_E\,dx. \tag{4.11}$$

Subtracting (4.10) from two times (4.11) we get

$$\int_{\mathbb{R}^n} \left( a_F |\nabla u|^2 - a_E |\nabla u_E|^2 \right) dx = \int_{\mathbb{R}^n} a_F \left( |\nabla u_E|^2 - |\nabla u|^2 \right) dx$$
$$+ \int_{\mathbb{R}^n} (a_E - a_F) |\nabla u_E|^2 \, dx \qquad (4.12)$$
$$+ 2 \int_{\mathbb{R}^n} u \rho \, dx - 2 \int_{\mathbb{R}^n} u_E \rho_E \, dx.$$

Moreover,

$$\int_{\mathbb{R}^n} a_F \left( |\nabla u_E|^2 - |\nabla u|^2 \right) dx = 2 \int_{\mathbb{R}^n} a_F \nabla u \cdot (\nabla u_E - \nabla u) \, dx$$
$$+ \int_{\mathbb{R}^n} a_F |\nabla u_E - \nabla u|^2 \, dx$$
$$= 2 \int_{\mathbb{R}^n} \rho (u_E - u) \, dx \qquad (4.13)$$
$$+ \int_{\mathbb{R}^n} a_F |\nabla u_F - \nabla u|^2 \, dx.$$

Combining (4.12) and (4.13) we then obtain:

$$\int_{\mathbb{R}^n} \left( a_F |\nabla u|^2 - a_E |\nabla u_E|^2 \right) dx = 2 \int_{\mathbb{R}^n} (\rho - \rho_E) u_E \, dx + \int_{\mathbb{R}^n} a_F |\nabla u - \nabla u_E|^2 \, dx$$
$$+ \int_{\mathbb{R}^n} (a_E - a_F) |\nabla u_E|^2 \, dx. \qquad (4.14)$$

We start to estimate the first term in the right hand side of (4.14). By using Proposition 2.3 and by arguing as in the proof of (4.8) the first term can be easily estimated as

$$\int_{\mathbb{R}^n} (\rho - \rho_E) u_E \, dx \lesssim |E \backslash F|.$$

To estimate the second term in the right hand side of (4.14), we write

$$- \operatorname{div} \left( a_F (\nabla u - \nabla u_E) \right) = \rho - \rho_E + \operatorname{div} \left( (a_F - a_E) \nabla u_E \right).$$

Therefore

$$\int_{\mathbb{R}^n} a_F |\nabla u - \nabla u_E|^2 \, dx = \int_{\mathbb{R}^n} (\rho - \rho_E) (u - u_E) \, dx$$
$$+ \int_{\mathbb{R}^n} (a_E - a_F) \nabla u_E \cdot (\nabla u - \nabla u_E) \, dx$$
$$\leq \| \rho - \rho_E \|_{(2^*)'} \| u - u_E \|_{2^*}$$
$$+ \left( \int_{\mathbb{R}^n} (a_F - a_E)^2 |\nabla u_E|^2 \, dx \right)^{\frac{1}{2}} \| \nabla u - \nabla u_E \|_2.$$

By the Sobolev embedding and Young inequality (and recalling that $1 \leq a_F \leq \beta$), the above inequality immediately imply

$$\int_{\mathbb{R}^n} a_F |\nabla u - \nabla u_E|^2 \, dx \lesssim \int_{\mathbb{R}^n} (a_F - a_E)^2 |\nabla u_E|^2 \, dx + \| \rho - \rho_E \|_{(2^*)'}^2.$$

By the definition of $\rho$, the second term is $\lesssim |E \setminus F|$ (note that $2/(2^*)' \geq 1$) while the first one is less than

$$\beta^2 \int_{E \setminus F} |\nabla u_E|^2 \, dx.$$

Since also the third term in (4.14) can be estimated by the above integral, this concludes the proof of (4.9).

Let us now prove (ii). Let $F \supseteq E$, note that $P(F \cap B_R) \leq P(F)$ and that $(F \cap B_R) \setminus E \subset F \setminus E$. Hence if we can prove (i) for subsets of $B_R$ we will get it for all sets. Let us then assume that $E \subseteq F \subseteq B_R$. By $\Lambda$-minimality of $E$

$$\mathcal{F}_{\beta,K,Q}(E) \leq \mathcal{F}_{\beta,K,Q}(F) + \Lambda |F \setminus E|.$$

Since, by Lemma 2.4, $\mathcal{G}_{\beta,K}(E) \geq \mathcal{G}_{\beta,K}(F)$ the conclusion follows. □

*Remark 4.6.* We record here the following simple consequence of (4.6). Assume that $|\nabla u_E|^2 \in L^p$, then (4.6) and Hölder inequality imply that for $F$ such that $F \Delta E \subset B_r(x)$ with $r \leq \bar{r}_2$,

$$P(E) \leq P(F) + \Lambda_2 |E \Delta F| + \Lambda_2 Q^2 \int_{E \Delta F} |\nabla u|^2 \, dx$$

$$\leq P(F) + \Lambda_2 |B_r| + \Lambda_2 Q^2 |B_r|^{1-\frac{1}{p}} \|\nabla u_E\|_{2p}^2 \leq P(F) + C r^{n-\frac{n}{p}}.$$

In particular if $p > n$, then $n - \frac{n}{p} > n - 1$ and thus $E$ is a $\omega$ minimizers of the perimeter in the sense of [25]. Hence $\partial E$ is a $C^1$ manifold outside a singular closed set $\Sigma$ of dimension at most $(n - 8)$. Note that by Cordes estimate, [4], the assumption $|\nabla u_E|^2 \in L^p$ with $p > n$ is satisfied wherever $\beta - 1 \ll 1$. In particular, in this regime, Taylor cones singularities are excluded in $\mathbb{R}^3$.

## 5. Compactness of Minimizers

In this section we prove that the class of minimizers of $(\mathcal{P}_{\beta,K,Q,R})$ is a compact subset of $L^1$, this is not really necessary in the proof of the main result, but we believe it can be interesting by its own.

**Proposition 5.1.** *Let $K_h$, $Q_h \in \mathbb{R}$, $\beta_h \geq 1$ and $R_h \geq 1$ be such that*

$$K_h \to K > 0, \quad \beta_h \to \beta \geq 1, \quad R_h \to R \geq 1, \quad Q_h \to Q \geq 0,$$

*when $h \to \infty$. For every $h \in \mathbb{N}$ let $E_h$ be a minimizer of $(\mathcal{P}_{\beta_h,K_h,Q_h,R_h})$. Then, up to a non relabelled subsequence, there exists a set of finite perimeter $E$ such that*

$$\lim_{h \to \infty} |E \Delta E_h| = 0. \tag{5.1}$$

*Moreover $E$ is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$ and*

$$\mathcal{F}_{\beta,K,Q}(E) = \lim_{h \to \infty} \mathcal{F}_{\beta_h,K_h,Q_h}(E_h), \qquad \lim_{h \to \infty} P(E_h) = P(E).$$

*Proof.* Since if $R_h = 1$ for all $h$ the problem is trivial (recall that $|E_h| = |B_1|$) we can assume that $R_h$ and $R$ are strictly bigger than one. Moreover $B_1$ is always an admissible competitor and thus,

$$\limsup_h \mathcal{F}_{\beta_h, K_h, Q_h}(E_h) \leq \limsup_h \mathcal{F}_{\beta_h, K_h, Q_h}(B_1) = C(n, K, Q, \beta).$$

In particular the perimeters of $E_h$ are uniformly bounded and since all the sets are included in, say, $B_{2R}$ there exists a non relabelled subsequence and set $E \subset B_R$ such that (5.1) hold true. Since the perimeter is lower-semicontinuous and, by Proposition 2.6, $\mathcal{G}$ is continuous we also get that

$$\mathcal{F}_{\beta, K, Q}(E) \leq \liminf_h \mathcal{F}_{\beta_h, K_h, Q_h}(E_h). \tag{5.2}$$

We now show that $E$ is a minimizer. For let $F \subset B_R$ with $|F| = |B_1|$. Since $R_h \to R$, we can find $\lambda_h \to 1$ such that $F_h := \lambda_h F \subset B_{R_h}$. Clearly, $F_h \to F$, $|F_h| = |F| + o(1)$ and $P(F_h) = P(F) + o(1)$. Thus

$$\mathcal{F}_{\beta, K, Q}(F) = \mathcal{F}_{\beta_h, K_h, Q_h}(F_h) + o(1). \tag{5.3}$$

By Proposition 3.6 applied to $F_h$ we can find sets $\widetilde{F}_h \subset B_{R_h}$ such that $|\widetilde{F}_h| = |B_1|$ and

$$\mathcal{F}_{\beta_h, K_h, Q_h}(\widetilde{F}_h) = \mathcal{F}_{\beta_h, K_h, Q_h}(F_h) + o(1) = \mathcal{F}_{\beta, K, Q}(F) + o(1).$$

where in the last equality we have used (5.3). By minimality of $E_h$ we get

$$\mathcal{F}_{\beta_h, K_h, Q_h}(E_h) \leq \mathcal{F}_{\beta_h, K_h, Q_h}(\widetilde{F}_h) = \mathcal{F}_{\beta, K, Q}(F) + o(1),$$

which combined with (5.2) implies the minimality of $E$. By choosing $E = F$ we also deduce the convergence of the energies and, by Proposition 2.6, this implies the convergence of the perimeters. □

## 6. Decay of the Dirichlet Energy and Density Estimates

*6.1. Decay of the Dirichlet energy.* Following [12], in this subsection we establish an almost Lipschitz decay for the Dirichlet energy of $u_E$ in certain regimes. Namely when the set or the complement almost fill a ball or when the set is very close to an half space.

We start by recalling the following higher integrability lemma for solution of (1.6) The proof can be found for instance in [14].

**Lemma 6.1.** *Let $E$ be set of finite measure and let $(u, \rho) \in \mathcal{A}(E)$. Then there exists $C = C(n, \beta)$ and $p = p(n, \beta) > 1$ such that for all balls $B_r(x) \subset \mathbb{R}^n$*

$$\fint_{B_r(x)} |\nabla u|^{2p} \, dx \leq C \left\{ \left( \fint_{B_{2r}(x)} |\nabla u|^2 \, dx \right)^p + r^{2p} \fint_{B_{2r}(x)} \rho^{2p} \, dx \right\}. \tag{6.1}$$

*Furthermore, the constants $C$ and $p$ depends only on an upper bound for $\beta$.*

We start with the following elementary lemma where the optimal decay is obtained in some limiting situations.

**Lemma 6.2.** *Let $\beta \geq 1$ and $\rho \in L^\infty(\mathbb{R}^n)$. Then here exists a dimensional constant $C = C(n)$ such that:*

*(i) if $v \in W^{1,2}(B_r(x))$ is a solution of*

$$-\Delta v = \rho,$$

*then for all $\lambda \in (0, 1)$*

$$\fint_{B_{\lambda r}(x)} |\nabla v|^2 \, dx \leq C \fint_{B_r(x)} |\nabla v|^2 \, dx + \frac{C}{\lambda^n} r^2 \|\rho\|_\infty^2. \tag{6.2}$$

*(ii) If $v \in W^{1,2}(B_r(x))$ is a solution of*

$$-\operatorname{div}(a_H \nabla v) = \rho, \qquad a_H = \beta \mathbf{1}_H + \mathbf{1}_{H^c},$$

*where $H := \left\{ y \in \mathbb{R}^n : (y - x) \cdot e \leq 0 \right\}$ for some $e \in \mathbb{S}^{n-1}$. Then for all $\lambda \in (0, 1)$*

$$\fint_{B_{\lambda r}(x)} a_H |\nabla v|^2 \, dx \leq C \fint_{B_r(x)} a_H |\nabla v|^2 \, dx + \frac{C}{\lambda^n} r^2 \|\rho\|_\infty^2. \tag{6.3}$$

*Proof.* We just prove point (ii) since (i) is a particular case (and well known). By scaling and translating, we can assume without loss of generality that $x = 0$ and $r = 1$. Let $w$ be the solution of

$$\begin{cases} -\operatorname{div}(a_H \nabla w) = 0 & \text{in } B_1 \\ w = v & \text{on } \partial B_1, \end{cases}$$

so that $u = v - w$ solves

$$\begin{cases} -\operatorname{div}(a_H \nabla u) = \rho & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

By multiplying the last equation by $u$, applying Poincaré inequality we obtain

$$\int_{B_1} a_H |\nabla u|^2 \, dx \leq \|\rho\|_2 \|u\|_2 \leq C(n) \|\rho\|_\infty \|\nabla u\|_2 \leq C(n) \|\rho\|_\infty \left( \int_{B_1} a_H |\nabla u|^2 \, dx \right)^{\frac{1}{2}}.$$

where we have used that $a_H \geq 1$. Hence

$$\int_{B_1} a_H |\nabla v - \nabla w|^2 \, dx = \int_{B_1} a_H |\nabla u|^2 \, dx \leq C \|\rho\|_\infty^2. \tag{6.4}$$

Moreover, by [12, Lemma 2.3],

$$\fint_{B_\lambda} a_H |\nabla w|^2 \, dx \leq \fint_{B_1} a_H |\nabla w|^2 \, dx.$$

Hence,

$$\begin{aligned} \fint_{B_\lambda} a_H |\nabla v|^2 \, dx &\leq 2 \fint_{B_\lambda} a_H |\nabla w|^2 \, dx + 2 \fint_{B_\lambda} a_H |\nabla v - \nabla w|^2 \, dx \\ &\leq 2 \fint_{B_1} a_H |\nabla w|^2 \, dx + 2 \fint_{B_\lambda} a_H |\nabla v - \nabla w|^2 \, dx \\ &\leq 4 \fint_{B_1} a_H |\nabla v|^2 \, dx + 2 \fint_{B_\lambda} a_H |\nabla v - \nabla w|^2 \, dx + 4 \fint_{B_1} a_H |\nabla v - \nabla w|^2 \, dx. \end{aligned} \tag{6.5}$$

which together with (6.4) concludes the proof. $\qquad \square$

As in [12], we now exploit the higher integrability recalled in Lemma 6.1 to obtain an "almost version" of the above decay.

**Proposition 6.3** (Decay of Dirichlet energy). *Let $\beta \geq 1$ then there exists a constant $C = C(n, \beta)$ with the following property: if $E \subset \mathbb{R}^n$, $u$ and $\rho$ satisfy*

$$- \operatorname{div}(a_E \nabla u) = \rho, \qquad a_E = \beta \mathbf{1}_E + \mathbf{1}_{E^c},$$

*then for all $\lambda \in \left(0, \frac{1}{2}\right)$ there exists $\varepsilon_0 = \varepsilon_0(\lambda, \beta) > 0$ such that*

*(i) if*

$$\text{either} \quad \frac{|E \cap B_r(x)|}{|B_r(x)|} \leq \varepsilon_0 \quad \text{or} \quad \frac{|B_r(x) \setminus E|}{|B_r(x)|} \leq \varepsilon_0,$$

*then*

$$\fint_{B_{\lambda r}(x)} |\nabla u|^2 \, dx \leq C \fint_{B_r(x)} |\nabla u|^2 \, dx + \frac{Cr^2}{\lambda^n} \|\rho\|_\infty^2.$$

*(ii) If*

$$\frac{|(E \Delta H) \cap B_r(x)|}{|B_r(x)|} \leq \varepsilon_0,$$

*where $H := \left\{ y \in \mathbb{R}^n : (y - x) \cdot e \leq 0 \right\}$ for some $e \in \mathbb{S}^{n-1}$, then*

$$\fint_{B_{\lambda r}(x)} |\nabla u|^2 \, dx \leq C \fint_{B_r(x)} |\nabla u|^2 \, dx + \frac{Cr^2}{\lambda^n} \|\rho\|_\infty^2.$$

*Moreover the constants $C$ and $\varepsilon_0$ can be chosen to depend only on un upper bound on $\beta$.*

*Proof.* We detail the proof of item (ii). Item (i) can be obtained in a similar way and we sketch the argument at the end of the proof. Without loss of generality, by translating, we can assume $x = 0$. Let $\lambda \in (0, 1/2)$ be given and let $v$ the solution of

$$\begin{cases} - \operatorname{div}(a_H \nabla v) = \rho & \text{in } B_{r/2} \\ v = u & \text{on } \partial B_{r/2}. \end{cases}$$

where $a_H = \beta \mathbf{1}_H + \mathbf{1}_{H^c}$. In particular, $w = (u - v) \in W_0^{1,2}(B_{r/2})$ and

$$- \operatorname{div}(a_H \nabla w) = - \operatorname{div}((a_E - a_H) \nabla u).$$

By testing the above equation with $w$ and using Young inequality we get

$$\int_{B_{r/2}} |\nabla u - \nabla v|^2 \, dx \leq \int_{B_{r/2}} (a_E - a_H)^2 |\nabla u|^2 \, dx \leq \beta^2 \int_{(E \Delta H) \cap B_{r/2}} |\nabla u|^2 \, dx.$$

By the higher integrability lemma there exists $p > 1$ such that

$$\left( \fint_{B_{r/2}} |\nabla u|^{2p} \, dx \right)^{\frac{1}{p}} \leq C \fint_{B_r} |\nabla u|^2 \, dx + C \, r^{n+2} \|\rho\|_\infty^2. \tag{6.6}$$

Hence by exploiting Hölder inequality with exponent $p$ we have

$$
\begin{aligned}
\int_{(E \Delta H) \cap B_{r/2}} |\nabla u|^2 \, dx &\leq \left| (E \Delta H) \cap B_{r/2} \right|^{1-\frac{1}{p}} \left( \int_{B_{r/2}} |\nabla u|^{2p} \, dx \right)^{\frac{1}{p}} \\
&\leq c(2, p) \, |B_r| \left( \frac{|(E \Delta H) \cap B_r|}{|B_r|} \right)^{1-\frac{1}{p}} \left( \fint_{B_{r/2}} |\nabla u|^{2p} \, dx \right)^{\frac{1}{p}} \\
&\leq C \, \varepsilon_0^{1-\frac{1}{p}} \left\{ \int_{B_r} |\nabla u|^2 \, dx + r^{n+2} \|\rho\|_\infty^2 \right\}.
\end{aligned}
\tag{6.7}
$$

Therefore, (recall $r < 1$) the above estimates yield

$$
\int_{B_{r/2}} |\nabla w|^2 \, dx \leq C \left\{ (\beta - 1)^2 \varepsilon_0^{1-\frac{1}{p}} \int_{B_r} |\nabla u|^2 \, dx + r^{n+2} \|\rho\|_\infty^2 \right\}. \tag{6.8}
$$

Since the decay estimate (6.3) apply to $v$, we can argue as in the proof of (6.5) to obtain

$$
\fint_{B_{\lambda r}} |\nabla u|^2 \, dx \leq C \fint_{B_r} |\nabla u|^2 \, dx + \frac{C \varepsilon_0^{1-\frac{1}{p}}}{\lambda^n} \int_{B_r} |\nabla u|^2 \, dx + \frac{C \|\rho\|_\infty^2}{\lambda^n}.
$$

Choosing $\varepsilon_0 = \varepsilon_0(n, \lambda) \ll \lambda$ sufficiently small we conclude the proof of (ii). The proof of (i) can be obtained in the same way by comparing $u$ to a solution of $-\Delta u = \rho$ (or $-\beta \Delta u = \rho$) and by using (6.2). $\qquad\square$

*6.2. Density estimates.* In this section we establish scaling invariant upper and lower bounds for the perimeter and for the measure of a minimizer in balls. We also establish a universal upper bound for the normalized Dirichlet energy of the minimizer of $u_E$. We start with the following proposition which is a simple consequence of the outward minimizing property of $E$ established in Proposition 4.5 (ii).

**Proposition 6.4.** *Let $A > 0$, and let $\beta, K, Q$ be controlled by $A$ and $R \geq 1$. Then there exist universal constants $C_o$ and $r_o$ such that, if $E$ is a minimizer of $(\mathcal{P}_{\beta, K, Q, R})$, $r \in (0, r_o)$, then[3]*

$$
P(E, B_r(x)) \leq C_o r^{n-1} \quad \text{for all } x \in \partial E \text{ and } r \in (0, r_o), \tag{6.9}
$$

*and*

$$
\frac{|B_r(x) \setminus E|}{|B_r(x)|} \geq \frac{1}{C_o} \quad \text{for all } x \in E^c \text{ and } r \in (0, r_o), \tag{6.10}
$$

---

[3] Here and in the sequel we will always work with the representative of $E$ such that

$$
\partial E = \left\{ x : \frac{|B_r(x) \setminus E|}{|B_r(x)|} \cdot \frac{|B_r(x) \cap E|}{|B_r(x)|} > 0 \quad \text{for all } r > 0 \right\},
$$

see [19, Proposition 12.19].

*Proof.* We let $\Lambda_2$ and $\bar{r}_2$ be the constants appearing in Proposition 4.5. We take $r_0 \leq \bar{r}_2$. For $r \leq r_0$, we plug $F = E \cup B_r(x)$ in (4.5) and we obtain, after simple manipulations,

$$P(E, B_r(x)) \leq \mathcal{H}^{n-1}(\partial B_r(x) \backslash E) + \Lambda_2 |E \backslash B_r(x)| \leq n\omega_n r^{n-1} + \Lambda_2 \omega_n r^n.$$

Hence, assuming that $\Lambda_2 r_0 \leq 1$, we immediately get $P(E, B_r(x)) \lesssim r^{n-1}$. To obtain the lower density bound for $E^c$ we set $m(r) := |B_r(x) \backslash E|$ and we use the isoperimetric inequality to deduce

$$\begin{aligned}
m(r)^{\frac{n-1}{n}} = |B_r(x) \backslash E|^{\frac{n-1}{n}} &\lesssim P(E \backslash B_r(x)) \\
&= P(E, B_r(x)) + \mathcal{H}^{n-1}(\partial B_r(x) \backslash E) \\
&\lesssim \mathcal{H}^{n-1}(\partial B_r(x) \backslash E) + |E \backslash B_r(x)| \\
&\lesssim m'(r) + m(r),
\end{aligned}$$

where we have used that, by co-area formula $m'(r) = \mathcal{H}^{n-1}(\partial B_r(x) \backslash E)$. If we choose $r_0$ such that $Cm(r)^{\frac{1}{n}} \leq C(n\omega_n)^{\frac{1}{n}} r_0 \leq 1/2$ where $C$ is the implied universal constant in the above estimate, we obtain

$$m(r)^{\frac{n-1}{n}} \lesssim m'(r).$$

Since $x \in \partial E$, $m(r) > 0$ for all $r > 0$ then the above inequality implies that

$$\frac{\mathrm{d}}{\mathrm{d}r} m(r)^{\frac{1}{n}} \gtrsim 1 \qquad \text{for all } r \in (0, r_0).$$

Hence $m(r) \gtrsim r^n$ and this concludes the proof. $\qquad\square$

The next lemma establishes a universal bound on the normalized Dirichlet integral.

**Lemma 6.5.** *Let $A > 0$, and let $\beta, K, Q$ be controlled by $A$ and $R \geq 1$. Then there exists a universal constant $C_e$ such that, if $E$ is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, then for all $x \in \overline{B}_R$,*

$$Q^2 D_E(x, r) = \frac{Q^2}{r^{n-1}} \int_{B_r(x)} |\nabla u|^2 \, dy \leq C_e. \tag{6.11}$$

*Proof.* The estimates is clearly true if $r \geq r_0$ where $r_0 = r_0(n, A)$, indeed recall that

$$Q^2 \int_{\mathbb{R}^n} |\nabla u_E|^2 \, dy \leq \mathcal{F}_{\beta,K,Q}(E) \lesssim 1.$$

Hence we can assume that $r \leq r_0 \ll 1$. We claim the following: there exist constants $\lambda = \lambda(n, A) \in (0, 1/2)$, $C = C(n, A)$ and $r_0 = r_0(n, A)$ such that

(a) If $x \in \partial B_R$ and $r \leq r_0$, then

$$Q^2 D_E(x, \lambda r) \leq \frac{1}{2} Q^2 D_E(x, r) + C. \tag{6.12}$$

(b) If $x \in B_R$ and $r \leq \min\{\text{dist}(x, \partial B_R), r_0/2\}$, then

$$Q^2 D_E(x, \lambda r) \leq \frac{1}{2} Q^2 D_E(x, r) + C. \tag{6.13}$$

Let $\varepsilon \ll 1$ to be fixed and let $r_0 = r_0(\varepsilon) \ll \bar{r}_1$ where $\bar{r}_1$ is the constant in Proposition 4.4 and such that the following holds true

$$x \in \partial B_R \quad \text{and} \quad r \leq r_0 \quad \implies \quad \frac{|(B_R \cap B_r(x)) \Delta H_x|}{|B_r(x)|} \leq \varepsilon, \qquad (6.14)$$

where $H_x := \{y : (y - x) \cdot x \leq 0\}$ is the supporting half space of $B_R$ at $x$. Note that since the curvatures of $\partial B_R$ are universally bounded (recall that $R \geq 1$), this can be achieved by choosing $r_0$ small only in dependence of $\varepsilon$.

Let now $x \in \overline{B}_R$ and $r \leq r_0$ be a radius satisfying either condition (a) (if $x \in \partial B_R$) or condition (b) (if $x \in B_R$) above. Let $(u_E, \rho_E)$ be the minimizers for $\mathcal{G}(E)$ and consider

$$F = (E \cup B_r(x)) \cap B_R.$$

We define $u$ to be the solution of

$$- \operatorname{div}(a_F \nabla u) = \rho_E. \qquad (6.15)$$

Note that $(u, \rho_E) \in \mathcal{A}(F)$ since $F \supset E$. Hence, by Proposition 4.4,

$$P(E) + Q^2 \left( \int_{\mathbb{R}^n} a_E |\nabla u_E|^2 \, dx + K \int_{\mathbb{R}^n} \rho_E^2 \, dx \right)$$

$$\leq P(F) + Q^2 \left( \int_{\mathbb{R}^n} a_F |\nabla u|^2 \, dx + K \int_{\mathbb{R}^n} \rho_E^2 \, dx \right) + \Lambda_1 |F \backslash E|$$

$$\leq P(E \cup B_r(x)) + Q^2 \left( \int_{\mathbb{R}^n} a_F |\nabla u|^2 \, dx + K \int_{\mathbb{R}^n} \rho_E^2 \, dx \right) + \Lambda_1 |B_r(x)|,$$

where we have used that $F \backslash E \subset B_r(x)$ and that $P(F) \leq P(E \cup B_r(x))$, by the convexity of $B_R$. Rearranging the terms we get

$$Q^2 \left( \int_{\mathbb{R}^n} a_E |\nabla u_E|^2 \, dx - \int_{\mathbb{R}^n} a_F |\nabla u|^2 \, dx \right) \leq P(E \cup B_r(x)) - P(E) + \Lambda_1 |B_r(x)| \lesssim r^{n-1}.$$

Recall now that $u_E$ solves

$$- \operatorname{div}(a_E \nabla u_E) = \rho_E,$$

and we use (2.9) in Lemma 2.4 to infer that

$$\int (a_F - a_E) |\nabla u|^2 \, dx \leq \int_{\mathbb{R}^n} a_E |\nabla u_E|^2 \, dx - \int_{\mathbb{R}^n} a_F |\nabla u|^2 \, dx \lesssim \frac{r^{n-1}}{Q^2}. \qquad (6.16)$$

Since

$$- \operatorname{div}(a_E \nabla (u_E - u)) = - \operatorname{div}((a_F - a_E) \nabla u),$$

by testing with $u_E - u$ and by Young inequality we get

$$Q^2 \int_{\mathbb{R}^n} |\nabla u_E - \nabla u|^2 \, dx \leq Q^2 \int (a_F - a_E)^2 |\nabla u|^2 \, dx \lesssim r^{n-1},$$

where the last inequality follows from (6.16).

We want now apply Proposition 6.3 to $u$. Note that since

$$F \cap B_r(x) = B_r(x) \cap B_R,$$

then the assumption are satisfied both in case (a) (thanks to (6.14)) and in case (b) (since $B_r(x) \subset B_R$). Hence, given $\lambda \in (0, 1/2)$, we have:

$$
\frac{1}{(\lambda r)^{n-1}} \int_{B_{\lambda r}(x)} |\nabla u_E|^2 \, dy \leq \frac{2}{(\lambda r)^{n-1}} \int_{B_{\lambda r}(x)} |\nabla u - \nabla u_E|^2 \, dy + \frac{2}{(\lambda r)^{n-1}} \int_{B_{\lambda r}(x)} |\nabla u|^2 \, dy
$$

$$
\leq \frac{2}{(\lambda r)^{n-1}} \int_{B_{\lambda r}(x)} |\nabla u - \nabla u_E|^2 \, dy + \frac{C\lambda}{r^{n-1}} \int_{B_r(x)} |\nabla u|^2 \, dy + \frac{Cr^2 \|\rho_E\|_\infty}{\lambda^{n-1}}
$$

$$
\leq \frac{C}{\lambda^{n-1}} \frac{1}{r^{n-1}} \int_{B_r(x)} |\nabla u - \nabla u_E|^2 \, dy + \frac{C\lambda}{r^{n-1}} \int_{B_r(x)} |\nabla u_E|^2 \, dy + \frac{Cr^2 \|\rho_E\|_\infty}{\lambda^{n-1}},
$$

(6.17)

for a constant $C = C(n, A)$ provided $\varepsilon$ (and thus $r_0$) is chosen sufficiently small. Since by (2.4) $\|\rho_E\|_\infty \lesssim 1$, we deduce from (6.17) that

$$
Q^2 D_E(x, \lambda r) \leq C\lambda \, Q^2 D_E(x, r) + \frac{C(n, A)}{\lambda^{n-1}}.
$$

(6.18)

Now choosing $\lambda = \lambda(n, A)$ such that $C\lambda = 1/2$ we conclude the proof of the claim. Note that this fixes $\varepsilon$ and thus $r_0$ as functions depending only on $n$ and $A$.

To conclude the proof we have to show that (a) and (b) above implies that

$$
S := \sup_{y \in \overline{B}_R} \sup_{0 < s \leq r_0} Q^2 D_E(y, s) \leq C(n, A).
$$

We first assume that $S < +\infty$ and show that we can bound it by a universal constant. Let $\bar{y} \in \overline{B}_R$ and $\bar{s} \in (0, r_0)$ be such that

$$
\frac{3S}{4} \leq Q^2 D_E(\bar{y}, \bar{s}).
$$

Let us distinguish a few cases:

• *Case 1:* $\bar{y} \in \partial B_R$. If $\bar{s} \leq \lambda r_0$, (6.12) implies that

$$
\frac{3S}{4} \leq Q^2 D_E(\bar{y}, \bar{s}) \leq \frac{1}{2} Q^2 D_E\left(\bar{y}, \frac{\bar{s}}{\lambda}\right) + C \leq \frac{1}{2} S + C,
$$

and we are done. On the other end if $\bar{s} \geq \lambda r_0$, then

$$
\frac{3S}{4} \leq Q^2 D_E(\bar{y}, \bar{s}) \leq \frac{Q^2}{(\lambda r_0)^{n-1}} \int_{\mathbb{R}^n} |\nabla u_E|^2
$$

$$
\leq \frac{1}{(\lambda r_0)^{n-1}} \mathcal{F}_{\beta, K, Q}(E) \leq C(n, A).
$$

• *Case 2:* $\bar{y} \in B_R$. If $\bar{s} \leq \lambda \min\{\text{dist}(\bar{y}, \partial B_R), r_0/2\}$, we can use (6.13) and we argue as in the first part of Case 1. If $\bar{s} \geq \lambda r_0/2$ we argue instead as in the second part of Case 1 to conclude. We are thus left to consider the case

$$
\lambda \, \text{dist}(\bar{x}, \partial B_R) \leq \bar{s} \leq \lambda r_0/2.
$$

In this case $B_{\bar{s}}(\bar{y}) \subset B_{r_0}(\bar{y})$, $\bar{y} \in \partial B_R$ and

$$
\frac{3S}{4} \leq Q^2 D_E(\bar{y}, \bar{s}) \leq \frac{1}{2} S + C.
$$

Thus we are done.

To show that one can actually assume that $S < +\infty$ one can consider

$$S_\delta = \sup_{y \in \overline{B}_R} \sup_{\delta \leq s \leq r_0} Q^2 D_E(y, s) \leq C(n, A)\delta^{1-n},$$

and argue as above to show that $S_\delta \leq C(n, A)$. Letting $\delta \to 0$ we conclude the proof. $\square$

We are now ready to complete the proof of density and perimeter estimates.

**Proposition 6.6.** *Let $A > 0$, and let $\beta$, $K$, $Q$ be controlled by $A$ and $R \geq 1$. Then there exist universal constants $C_i$ and $\bar{r}_i$ such that, if $E$ is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, then*

$$P(E, B_r(x)) \geq \frac{r^{n-1}}{C_i} \quad \text{for all } x \in \partial E \text{ and } r \in (0, \bar{r}_i), \tag{6.19}$$

*and*

$$\frac{|B_r(x) \cap E|}{|B_r(x)|} \geq \frac{1}{C_i} \quad \text{for all } x \in E \text{ and } r \in (0, \bar{r}_i), \tag{6.20}$$

*Proof.* We start showing the validity of (6.19) and we divide the proof in few steps.
● *Step 1*: We claim that for every $\lambda \in (0, 1/4)$, there exist $\varepsilon_1 = \varepsilon_1(\lambda, , A), C_1 = C_1(n, A)$ and and $\bar{r} = \bar{r}(n, A, \lambda)$

$$P(E, B_r(x)) \leq \varepsilon r^{n-1} \quad \varepsilon \leq \varepsilon_1 \quad r \leq \bar{r},$$

then,

$$P(E, B_{\lambda r}(x)) + Q^2 \int_{B_{\lambda r}(x)} |\nabla u_E|^2 \, dy$$
$$\leq C_1 \lambda^n \left( P(E, B_r(x)) + Q^2 \int_{B_r(x)} |\nabla u_E|^2 \, dy + r^n \right). \tag{6.21}$$

For the ease of notation let us assume that $x = 0$. Let $\lambda \in (0, 1/4)$ be fixed. By the relative isoperimetric inequality

$$\left( \min \left\{ \frac{|E \cap B_r|}{|B_r|}, \frac{|B_r \setminus E|}{|B_r|} \right\} \right)^{\frac{n-1}{n}} \leq C(n) \frac{P(E, B_r)}{r^{n-1}} \lesssim \varepsilon.$$

By (6.10) and by choosing $\varepsilon_1, \bar{r} \ll 1$ we get

$$\frac{|E \cap B_r|}{|B_r|} \leq C(n) \left( \frac{P(E, B_r)}{r^{n-1}} \right)^{\frac{1}{n-1}} \frac{P(E, B_r)}{r^{n-1}} \lesssim \varepsilon^{\frac{1}{n-1}} \frac{P(E, B_r)}{r^{n-1}}. \tag{6.22}$$

Let us choose $t \in (\lambda r, 2\lambda r)$ such that

$$\mathcal{H}^{n-1}(E \cap \partial B_t) \leq \int_{\lambda r}^{2\lambda r} \mathcal{H}^{n-1}(E \cap \partial B_s) \, ds$$
$$\leq \frac{|E \cap B_{2\lambda r}|}{\lambda r} \leq C(n, \lambda)\varepsilon^{\frac{1}{n-1}} P(E, B_r). \tag{6.23}$$

By testing (4.5) with $F = E \setminus B_t$ we obtain

$$P(E, B_t) \leq \mathcal{H}^{n-1}(E \cap \partial B_t) + \Lambda_2 |E \cap B_t| + \Lambda_2 Q^2 \int_{E \cap B_t} |\nabla u_E|^2 \, dx, \quad (6.24)$$

which, together with (6.23) and recalling that $t \in (\lambda r, 2\lambda r)$, implies that

$$P(E, B_{\lambda r}) + Q^2 \int_{B_{\lambda r}} |\nabla u_E|^2 \, dx$$

$$\leq C(n, \lambda) \varepsilon^{\frac{1}{n-1}} P(E, B_r) + (\Lambda_2 + 1) Q^2 \int_{B_{2\lambda r}} |\nabla u_E|^2 \, dx + \Lambda_2 |B_{2\lambda r}|. \quad (6.25)$$

If we now choose $\varepsilon_1 = \varepsilon_1(\lambda) \ll 1$, (6.22) allows to apply Proposition 6.3 (i). Hence by also choosing $\bar{r} \ll \lambda$ we deduce that

$$\int_{B_{2\lambda r}} |\nabla u_E|^2 \, dx \leq C(n, A)\lambda^n \left( \int_{B_r} |\nabla u_E|^2 \, dx + \frac{\bar{r}^2}{\lambda^n} r^n \right)$$

$$\leq C(n, A)\lambda^n \left( \int_{B_r} |\nabla u_E|^2 \, dx + r^n \right), \quad (6.26)$$

where we have used that, by (2.4), $\|\rho_E\|_\infty \lesssim 1$. By gathering equations (6.25) and (6.26) we then get

$$P(E, B_{\lambda r}) + Q^2 \int_{B_{\lambda r}} |\nabla u_E|^2 \, dx$$

$$\leq C(n, \lambda) \varepsilon^{\frac{1}{n-1}} P(E, B_r) + C(n, A)\lambda^n \left( Q^2 \int_{B_r} |\nabla u_E|^2 \, dx + r^n \right).$$

If we choose $\varepsilon_1 = \varepsilon_1(n, A, \lambda) \ll 1$ such that $C(n, \lambda) \varepsilon^{\frac{1}{n-1}} \leq \lambda^n$ the above inequality implies (6.21).

• *Step* 2: We now prove the validity of (6.19). By density it is enough to prove it at all $x \in \partial^* E$. Again we set coordinates so that $x = 0$. Let us choose $\lambda = \lambda(n, A) \in (0, 1/4)$ such that $C_1 \lambda \leq 1/2$ where $C_1$ is the constant appearing in (6.21) and let $\bar{r}$ and $\varepsilon_1$ be the corresponding constants (which now depend only on $A$ and $n$). We claim that

$$P(E, B_r) + Q^2 \int_{B_r} |\nabla u_E|^2 \, dx \geq \frac{\varepsilon_1}{2} r^{n-1} \quad \text{for all } r \leq \min\{r_1, \varepsilon_1/2\}. \quad (6.27)$$

Indeed otherwise, by (6.21) and the choice of $\lambda$

$$P(E, B_{\lambda r}) + Q^2 \int_{B_{\lambda r}} |\nabla u_E|^2 \, dx$$

$$\leq \frac{\lambda^{n-1}}{2} \left( P(E, B_r) + Q^2 \int_{B_r} |\nabla u_E|^2 \, dx + \frac{\varepsilon_1}{2} r^{n-1} \right) \leq \frac{\varepsilon_1}{2} (\lambda r)^{n-1}.$$

We can thus iterate the above estimate and deduce that

$$\liminf_{r \to 0} \frac{P(E, B_r)}{r^{n-1}} = 0,$$

in contradiction with the assumption that $0 \in \partial^* E$. Let now $\bar{\lambda} \ll \varepsilon_1$ to be chosen where $\varepsilon_1$ is the constant obtained above. Let $\varepsilon_2$ and $r_2$ be the constants corresponding to $\bar{\lambda}$ in Step 1. We claim that if we choose $\bar{\lambda}$ small enough depending only on $n$ and $A$ then

$$P(E, B_r) \geq \varepsilon_2 r^{n-1} \qquad \text{for all } r \leq r_3, \tag{6.28}$$

where $r_3 \ll \min\{r_2, r_1\}$ will depend only on $n$ and $A$. Indeed otherwise we can apply Step 1, (6.9), and Lemma 6.5 to get

$$P(E, B_{\bar{\lambda}r}) + Q^2 \int_{B_{\bar{\lambda}r}} |\nabla u_E|^2 \, dx \leq C(n, A)\bar{\lambda}^n \left( P(E, B_r) + Q^2 \int_{B_r} |\nabla u_E|^2 \, dx + r^n \right)$$

$$\leq \bar{C}(n, A)\bar{\lambda}(\bar{\lambda}r)^{n-1},$$

where $\varepsilon_2 \ll \varepsilon_1$ and $r_2 \ll r_1$ are universal constants. If $\bar{\lambda}$ is chosen so that $\bar{C}(n, A)\bar{\lambda} \leq \varepsilon_1/4$ this contradicts (6.27) and thus proves (6.19) with $c_i \leq \varepsilon_2$.

• *Step 3*: We now prove the validity of (6.20). Assume indeed that

$$\frac{|E \cap B_r|}{|B_r|} \leq \varepsilon_4 \qquad \text{for } r \leq r_4,$$

with $\varepsilon_4, r_4 \ll 1$ to be fixed only in term of $n$ and $A$. Then, by (6.1) and (6.11), for all $s \in (r/4, r/2)$

$$Q^2 \int_{B_s} |\nabla u_E|^2 \, dx \leq Q^2 |E \cap B_s|^{1-\frac{1}{p}} \left( \int_{B_s} |\nabla u_E|^{2p} \, dx \right)^{\frac{1}{p}}$$

$$\lesssim Q^2 \left( \frac{|E \cap B_r|}{|B_r|} \right)^{1-\frac{1}{p}} \int_{B_{2s}} |\nabla u_E|^{2p} \, dx \lesssim \varepsilon_4^{1-\frac{1}{p}} r^{n-1} \lesssim \varepsilon_4^{1-\frac{1}{p}} s^{n-1}. \tag{6.29}$$

Moreover, by the co-area formula, there exists $s \in (r/4, r/2)$ such that

$$\mathcal{H}^{n-1}(E \cap B_s) \leq \fint_{r/4}^{r/2} \mathcal{H}^{n-1}(E \cap B_t) dt \leq \frac{4|E \cap B_r|}{r} \lesssim \varepsilon_4 r^{n-1} \lesssim \varepsilon_4 s^{n-1}. \tag{6.30}$$

By testing (4.4) with $E \setminus B_s$ we get

$$P(E, B_s) \leq \mathcal{H}^{n-1}(E \cap B_s) + \Lambda_2 |B_s| + Q^2 \Lambda_2 \int_{B_s} |\nabla u_E|^2 \, dx,$$

which together with (6.29) and (6.30) and provided $r_4 \ll \varepsilon_4 \ll 1$ implies

$$P(E, B_s) \leq C\varepsilon_4^{1-\frac{1}{p}} s^{n-1},$$

for a suitable universal constant $C$. Choosing $\varepsilon_4$ small with respect to $\varepsilon_2$ we get

$$P(E, B_s) \leq \varepsilon_2 s^{n-1},$$

in contradiction with (6.28).

$\square$

## 7. Decay of the Excess

In this section we prove Theorem 1.2. First of all we recall the following definitions.

**Notation.** Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and $x \in \mathbb{R}^n$.

- The *spherical excess* of $E$ at the point $x \in \partial E$ is

$$\mathbf{e}_E(x, r) := \inf_{\nu \in \mathbb{S}^{n-1}} \frac{1}{r^{n-1}} \int_{\partial^* E \cap B_r(x)} \frac{|\nu_E(y) - \nu|^2}{2} d\mathcal{H}^{n-1}(y),$$

where $\partial^* E$ is the *reduced boundary* and $\nu_E$ is the *measure-theoretic outer unit normal to $E$*.

- The *normalized Dirichlet energy* of $E$ at a point $x$ is

$$D_E(x, r) = \frac{1}{r^{n-1}} \int_{B_r(x)} |\nabla u_E|^2 \, dy,$$

where $u_E$ minimizes (1.9).

Since the seminal works of De Giorgi and Almgren, [1,5] the proof of Theorem 1.2 is based on an excess decay theorem, namely:

**Theorem 7.1** (Excess improvement)*. Let $A > 0$, and let $\beta$, $K$, $Q$ be controlled by $A$ and $R \geq 1$. There exists a universal constant $C_{\mathrm{dec}} > 0$ such that for all $\lambda \in (0, 1/4)$ there exists $\varepsilon_{\mathrm{dec}} = \varepsilon_{\mathrm{dec}}(n, A, \lambda) > 0$ satisfying the following: if $E$ is a minimizer of $(\mathcal{P}_{\beta, K, Q, R})$ and*

$$x \in \partial E, \quad r + Q^2 D_E(x, r) + \mathbf{e}_E(x, r) \leq \varepsilon_{\mathrm{dec}},$$

*then*

$$Q^2 D_E(x, \lambda r) + \mathbf{e}_E(x, \lambda r) \leq C_{\mathrm{dec}} \lambda \Big( \mathbf{e}_E(x, r) + Q^2 D_E(x, r) + r \Big). \tag{7.1}$$

As it is customary, the proof of the above theorem is based on an "harmonic approximation" technique. More precisely we will go through the following steps:

(i) In the small excess regime, the boundary of $E$ can be well approximated by the graph of a Lipschitz function $f$ with Dirichlet energy bounded by the excess.

(ii) If the excess and the normalized Dirichlet of $u_E$ are small, then $f$ is almost harmonic.

(iii) Almost harmonicity of $f$ implies closeness to an harmonic function $g$ in the $L^2$ topology. By classical estimates for harmonic functions, an $L^2$ type of excess of $g$ decays. This in turn implies the decay of the *flatness* $\mathbf{f}$ of $E$, see (7.16) below for the definition.

(iv) Via a Caccioppoli type inequality, the decay of the flatness can be transferred to the decay of the excess.

(v) Via Proposition 6.3 (ii), the decay of the excess implies the decay of the normalized Dirichlet energy.

Usually, Step (i) is obtained by reproducing at most points and at all scale an height type bound for $\partial E$ in the small excess regime and it thus relies on the scaling invariance of the problem studied. Step (ii) and (iv) are obtained by simple comparison arguments and Step (iii) is based on a compactness argument together with the classical regularity theory for harmonic functions.

In our situation the problem does not enjoy of a nice scaling behaviour, due to the global constraint $\int_{\mathbb{R}^n} \rho_E \, dx = 1$. However, the local estimates obtained in the previous section are exactly what we need to carry on the proof of Step (i), see Lemma 7.2 below. Since beside this fact, most of the proofs of the needed lemmas are almost verbatim adaptation of those present in the literature, we will not detail all of them and we will just focus on the key points and on the main differences.

### 7.1. Lipschitz approximation.

In this subsection we prove the Lipschitz approximation lemma. Let us first fix a few notation that will be useful through all the section.

**Notation** Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, $x \in \mathbb{R}^n$, $z \in \mathbb{R}^{n-1}$, $v \in \mathbb{S}^{n-1}$ and $r > 0$.

- We call $\mathbf{p}^v(x) := x - (x \cdot v) \, v$ and $\mathbf{q}^v(x) := (x \cdot v) \, v$, respectively, the *orthogonal projection* onto the plane $v^\perp$ and the *projection* on $v$. For simplicity we denote $\mathbf{p}(x) := \mathbf{p}^{e_n}(x)$ and $\mathbf{q}(x) := \mathbf{q}^{e_n}(x) = x_n$.
- We define the *cylinder* with center at $x$ and radius $r$ with respect to the direction $v$ as

$$\mathbf{C}(x, r, v) := \left\{ y \in \mathbb{R}^n \; : \; |\mathbf{p}^v(y - x)| < r \, , |\mathbf{q}^v(y - x)| < r \right\}.$$

We write $\mathbf{C}_r(x) := \mathbf{C}(x, r, e_n)$, $\mathbf{C}_r := \mathbf{C}(0, r, e_n)$ and $\mathbf{C} := \mathbf{C}_1$.
- We denote the $(n-1)$-dimensional *disk* centered at $z$ and of radius $r$ by

$$\mathbf{D}_r(z) := \left\{ y \in \mathbb{R}^{n-1} : |y - z| < r \right\}.$$

For simplicity we write $\mathbf{D}_r = \mathbf{D}(0, r)$ and $\mathbf{D} = \mathbf{D}(0, 1)$.
- The *cylindrical excess* in a direction $v \in \mathbb{S}^{n-1}$ is defined as

$$\mathbf{e}_E(x, r, v) = \frac{1}{r^{n-1}} \int_{\mathbf{C}(x,r,v) \cap \partial^* E} \frac{|v_E(y) - v|^2}{2} \, d\mathcal{H}^{n-1}(y) \, ,$$

so that

$$\mathbf{e}_E(x, r) \leq \inf_{v \in \mathbb{S}^{n-1}} \mathbf{e}_E(x, r, v). \tag{7.2}$$

The following height bound is crucial in the sequel. Note that it does not require any minimality property on $E$, only the validity of inequality (7.3) at all scales.

**Lemma 7.2** *Let $C > 0$. Then there exists an increasing function $\omega_C : (0, 1) \to \mathbb{R}$ with $\omega_C(0^+) = 0$ depending only on $C$ such that if $E \subseteq \mathbb{R}^n$ is of finite perimeter in $\mathbf{C}(x, 2r)$ satisfying the following properties:*

*(i) $x \in \partial E$,*
*(ii) for all $y \in \partial E$ and $s$ such that $B_s(y) \subset \mathbf{C}(x, 2r)$*

$$\frac{1}{C} \leq \frac{|E \cap B_s(y))|}{|B_s(y)|} \leq \left(1 - \frac{1}{C}\right), \qquad P(E, B_s(y)) \leq C s^{n-1}, \tag{7.3}$$

*then*

$$\mathbf{e}_E(x, 2r, e_n) \leq t \quad \Longrightarrow \quad \sup_{y \in \mathbf{C}(x,r) \cap \partial E} |\mathbf{q}(y - x)| \leq \omega_C(t)r, \tag{7.4}$$

$$\left| \{y \in \mathbf{C}(x, r) \cap E : \mathbf{q}(y - x) > \omega_C(t)r\} \right| = 0, \tag{7.5}$$

$$\left| \{y \in \mathbf{C}(x, r) \backslash E : \mathbf{q}(y - x) < -\omega_C(t)r\} \right| = 0. \tag{7.6}$$

*Remark 7.3* Note that the (7.3) and the relative isoperimetric inequality imply

$$P(E, B_s(y)) \geq \frac{s^{n-1}}{C'}$$

for all $y \in \partial E$ and $s$ such that $B_s(y) \subset \mathbf{C}(x, 2r)$ and for a suitable constant $C' = C'(C, n)$.

*Proof* Note that the assumptions are scaling and translation invariant, hence we can assume that $x = 0$ and $r = 1$. For every $t \in (0, 1)$ let

$$\mathcal{M}_t := \left\{\text{sets of finite perimeter satisfying } \mathbf{e}_E(0, 2, e_n) \leq t, \text{ (i) and (ii) with } x = 0 \text{ and } r = 1\right\}.$$

One can easily show that this is a compact class of sets (with respect to the $L^1$ topology). Moreover $\mathcal{M}_{t_1} \subset \mathcal{M}_{t_2}$ for $t_1 \leq t_2$. For every $E \subseteq \mathbb{R}^n$ let us call

$$h_E := \sup_{x \in \mathbf{C} \cap \partial E} |\mathbf{q}x|,$$

$$g_E := \inf \left\{s \in [0, 1] : |\{x \in \mathbf{C} \cap E : \mathbf{q}x > s\}| = 0\right\} \quad \text{and} \tag{7.7}$$

$$f_E := \inf \left\{s \in [0, 1] : |\{x \in \mathbf{C} \backslash E : \mathbf{q}x < -s\}| = 0\right\}.$$

Define the functions $\omega_1, \omega_2, \omega_3 : (0, 1) \to \mathbb{R}$ as

$$\omega_1(t) := \sup_{E \in \mathcal{M}_t} h_E, \quad \omega_2(t) := \sup_{E \in \mathcal{M}_t} g_E \quad \text{and} \quad \omega_3(t) := \sup_{E \in \mathcal{M}_t} f_E. \tag{7.8}$$

Let $\omega_C := \max\{\omega_1, \omega_2, \omega_3\}$. Notice that $\omega_C$ is increasing since it is the maximum of increasing functions and by definition it satisfies (7.4), (7.5), and (7.6). Let us prove that $\omega_C(0^+) = 0$. Assume by contradiction that $\lim_{t \to 0^+} \omega_C(t) > 0$ then there exist a sequence $t_k \searrow 0$ and $L > 0$ such that $\omega_C(t_k) > L$ for all $k$. We now distinguish three cases.

*Case 1*: Up to subsequences $\omega_C(t_k) = \omega_1(t_k)$ for every $k \in \mathbb{N}$. For every $k$ there exists $E_k \in \mathcal{M}_{t_k}$ such that $h_{E_k} \geq L$. By (7.3) up to subsequences there exists a set of finite perimeter $E \subseteq \mathbb{R}^n$ such that $E_k \cap \mathbf{C}_r \to E \cap \mathbf{C}_r$ whenever $r < 2$ and

$$\lim_{k \to +\infty} \mathbf{e}_{E_k}(0, 2, e_n) = 0. \tag{7.9}$$

Now take $\mathbf{C}_s \subset \mathbf{C}_r \subset \mathbf{C}_2$ with $s > 1$. By the lower semicontinuity of the excess we obtain that $\mathbf{e}_E(0, s, e_n) = 0$. Moreover let $\{x_k\}_{k \in \mathbb{N}}$ be a sequence such that $x_k \in \partial E_k \cap \mathbf{C}$ and let us assume that $x_k \to x$. By (ii) one easily deduce that

$$\min\left\{|E \cap B_s(x))|, |B_s(x) \backslash E|\right\} \geq \frac{|B_s(x)|}{C},$$

which implies that $x \in \partial E$ (recall that we are working with the representative of $E$ such that $\partial E = \operatorname{spt} D\mathbf{1}_E$). This in particular implies that $0 \in \partial E$. Since $\mathbf{e}_E(0, s, e_n) = 0$ we

get $E = \{x : \mathbf{q}x \leq 0\}$ in $\mathbf{C}_2$. Let $x_k \in \partial E_k$ be such that $|\mathbf{q}x_k| \geq L$ and let $\bar{x}$ be such that, up subsequence, $x_k \to \bar{x}$, clearly $|\mathbf{q}\bar{x}| \geq L$, however, by the previous discussion, $\bar{x} \in \partial E = \{x : \mathbf{q}x = 0\}$, a contradiction.

*Case 2*: Up to subsequences $\omega_C(t_k) = \omega_2(t_k)$ for every $k \in \mathbb{N}$. Hence for every $k$ there exists $E_k \in \mathcal{M}_{t_k}$ such that $g_{E_k} \geq L$. Note that if $\ell \in (0, L)$ then

$$\left| \{x \in \mathbf{C} \cap E_k : \mathbf{q}x > \ell\} \right| > 0 \qquad \text{for all } k \in \mathbb{N}. \tag{7.10}$$

Hence (7.10) implies that, up to extracting a subsequence,
either

$$\text{there exists } \ell \in (0, L) \text{ such that for } k \text{ there exists } x_k \in \mathbf{C} \cap \partial E_k \cap \{\mathbf{q}x > \ell\}, \tag{7.11}$$

or

$$\mathbf{1}_{E_k \cap \{\mathbf{q}x > 0\}} \longrightarrow \mathbf{1}_{\{\mathbf{q}x > 0\}} \text{ in } L^1(\mathbf{C}). \tag{7.12}$$

Indeed if by contradiction (7.11) does not hold then for every $j \gg 1$ there exists $k_j \in \mathbb{N}$ such that $\mathbf{q}x \leq \frac{1}{j}$ for every $x \in \mathbf{C} \cap \partial E_{k_j}$. By (7.10), since $\{\mathbf{q}x > \frac{1}{j}\}$ is connected, then necessarily $\mathbf{C} \cap E_{k_j} \supseteq \{\mathbf{q}x > \frac{1}{j}\}$. By letting $j \to +\infty$ we get (7.12).

By arguing as in Case 1, we have that $E_k \to \{\mathbf{q}x \leq 0\}$, hence (7.12) cannot hold. Hence (7.11) holds, which is again in contradiction with Case 1.

*Case 3*: Up to subsequences $\omega_C(t_k) = \omega_3(t_k)$ for every $k \in \mathbb{N}$. This case can be ruled out by arguing as in Case 2 (or by working with $E^c$ which satisfies the same assumption of $E$). Therefore $\omega_C$ is the required function. $\square$

Once the "qualitative" height bound has been established, one can repeat verbatim the proof of the Lipschitz approximation in [19, Theorem 2.37] to deduce that in the small excess regime $\partial E$ is mostly covered by the graph of a Lipschitz function. Note that in the cited reference one has an explicit formula for $\omega_C$ (namely $\omega_C(t) \lesssim t^{1/(n-1)}$) however this plays at all no role in the proof, see also [7, Lemma 4.3]. Note also that the lower perimeter bound $P(E, B_s(y)) \gtrsim s^{n-1}$ needed in the proof is ensured by Remark 7.3.

**Lemma 7.4** (Lipschitz approximation I). *Let $C > 0$. Then there exist $\varepsilon_L = \varepsilon_L(n, C) > 0$ and $C_L = C_L(n, C) > 0$ with the following property: if $E$ is a set of finite perimeter in $\mathbf{C}(x, 4r)$ satisfying $x \in \partial E$,*

$$\frac{1}{C} \leq \frac{|E \cap B_s(y)|}{|B_s(y)|} \leq \left(1 - \frac{1}{C}\right), \qquad P(E, B_s(y)) \leq Cs^{n-1},$$
$$\text{for all } y \in \partial E \cap \mathbf{C}(x, 2r) \text{ such that } B_s(y) \subset \mathbf{C}(x, 2r),$$

*and*

$$\mathbf{e}_E(x, 2r, e_n) \leq \varepsilon_L,$$

*then there exists a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ with*

$$\text{Lip}(f) \leq 1, \quad \frac{1}{r^{n-1}} \int_{\mathbf{D}_r} |\nabla f|^2 \leq C_L \mathbf{e}_E(x, 2r, e_n), \quad \frac{\|f\|_\infty}{r} \leq \omega_C\big(\mathbf{e}_E(x, 2r, e_n)\big),$$
$$\tag{7.13}$$

*such that, defining* $\Gamma_f := x + \{(z, f(z)) : z \in \mathbf{D}_r,$

$$\frac{\mathcal{H}^{n-1}\big((\partial E \cap \mathbf{C}(x, r, e_n))\Delta\Gamma_f\big)}{r^{n-1}} \leq C_L \mathbf{e}_E(x, 2r, e_n), \tag{7.14}$$

*where $\omega_C$ is the function in Lemma 7.2.*

Note that if $E$ is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, the assumption of the Lipschitz approximation lemma are satisfied with some universal constant $C$ by (6.9) and (6.19). Hence we can cover most of its boundary by the graph of a Lipschitz function $f$. Moreover a simple comparison argument implies that the laplacian of $f$ is small in a suitable negative norm. More precisely we have the following:

**Proposition 7.5** (Lipschitz approximation II). *Let $A > 0$, and let $\beta$, $K$, $Q$ be controlled by $A$ and $R \geq 1$. Then there exist universal constants $\varepsilon_{\mathrm{lip}}$, $C_{\mathrm{lip}}$ and a "universal" increasing function (i.e. depending only on n and A) $\omega_{\mathrm{lip}}$ with $\omega_{\mathrm{lip}}(0+) = 0$ such that if $E$ is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, $x \in \partial E$ and*

$$r + \mathbf{e}_E(x, 2r, e_n) \leq \varepsilon_{\mathrm{lip}},$$

*then there exists a function $f$ satisfying (7.13) and (7.14) with $C_L$ and $\omega_C$ replaced by $C_{\mathrm{lip}}$ and $\omega_{\mathrm{lip}}$ respectively. Moreover*

$$\frac{1}{r^{n-1}}\left|\int_{\mathbf{D}_r} \nabla f \cdot \nabla \varphi \, dz\right| \leq C_{\mathrm{lip}} \|\nabla \varphi\|_\infty \left(\mathbf{e}_E(x, 2r, e_n) + r + Q^2 D_E(x, 2r)\right), \tag{7.15}$$

*for every $\varphi \in C_c^1(\mathbf{D}_r)$.*

*Proof* Upper and lower volume and perimeter estimates established in (6.10), (6.20) and (6.19) ensure that in every cylinder $\mathbf{C}(x, 4r)$ centered at $x \in \partial E$, $E$ satisfies the assumption of Lemma 7.4 with a universal constant $C = C(n, A)$, provided $r$ is smaller than a universal radius $\bar{r}$. This proves the first part of the proposition. The second part follows by plugging in (4.6) $F := \psi_t(E)$, $\psi_t(x) = x + t\varphi(\mathbf{p}x)e_n$, and by performing the same computations done in [19, Proof of Theorem 23.7]. □

*7.2. The Caccioppoli inequality.* By (7.15) one will deduce that under the assumption of Theorem 7.1, there exists a harmonic function $h : \mathbf{D}_r \to \mathbb{R}$ which is close to $f$ in $L^2$. This closeness, together with the regularity theory for harmonic function will allow to deduce the decay of an $L^2$ type excess of $f$ and thus for $E$. In order to pass from the $L^2$ excess to the classical one, one needs to establish a Caccioppoli type inequality. To this end we recall the following definition.

**Definition 7.6** Given a set $E$ we define the *flatness* of $E$ at the point $x \in \mathbb{R}^n$, at the scale $r > 0$ with respect to the direction $\nu \in \mathbb{S}^{n-1}$ as

$$\mathbf{f}_E(x, r, \nu) := \frac{1}{r^{n-1}} \inf_{h \in \mathbb{R}} \int_{\mathbf{C}(x,r,\nu) \cap \partial^* E} \frac{|\nu \cdot (y - x) - h|^2}{r^2} \, d\mathcal{H}^{n-1}(y). \tag{7.16}$$

**Proposition 7.7** (Caccioppoli inequality). *Let $A > 0$, and let $\beta$, $K$, $Q$ be controlled by $A$ and $R \geq 1$. Then there exist universal constants $\varepsilon_{\text{cac}}$, and $C_{\text{cac}}$ such that if $E$ is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, $x \in \partial E$, and*

$$r + \mathbf{e}_E(x, 4r, e_n) \leq \varepsilon_{\text{cac}},$$

*then*

$$\mathbf{e}_E(x, r, e_n) \leq C_{\text{cac}}\Big(\mathbf{f}_E(x, 2r, e_n) + r + Q^2 D_E(x, 2r)\Big). \tag{7.17}$$

*Proof* The proof can be obtained by verbatim repeating the arguments of [19, Chapter 24] and using (4.6) instead of the perimeter minimality in the comparison estimate of [19, Equation 24.48].

□

*7.3. Dirichlet improvement.* We now show that in the small excess regime there is fixed scale decay of the Dirichlet energy.

**Proposition 7.8** (Decay of the Dirichlet energy). *Let $A > 0$, and let $\beta$, $K$, $Q$ be controlled by $A$ and $R \geq 1$. There exists a universal constant $C_{\text{dir}} > 0$ such that for all $\lambda \in (0, 1/2)$ there exists $\varepsilon_{\text{dir}} = \varepsilon_{\text{dir}}(n, A, \lambda)$ satisfying the following: if $E$ is a minimizer of $(\mathcal{P}_{\beta,K,Q,R})$, $x \in \partial E$ and*

$$r + \mathbf{e}_E(x, r, e_n) \leq \varepsilon_{\text{dir}}, \tag{7.18}$$

*then*

$$D_E(x, \lambda r) \leq C_{\text{dir}}\lambda\Big(D_E(x, r) + r\Big).$$

*Proof* By (6.9) and (6.19) we have that if $r$ is universally small we can apply Lemma 7.2 to $E$ in $\mathbf{C}(x, r)$ to obtain a universal modulus of continuity $\omega$ such that for $H = \{y : \mathbf{q}(y - x) \leq 0\}$,

$$\frac{|(E \Delta H) \cap B_{r/2}(x)|}{|B_{r/2}|} \leq \omega(\varepsilon_{\text{dir}}).$$

By Proposition 6.3 (ii) (applied in $B_{r/2}(x)$) and the above inequality, for all $\lambda \in (0, 1/2)$ we can choose $\varepsilon_{\text{dir}} = \varepsilon_{\text{dir}}(n, A, \lambda)$ sufficiently small such that

$$D_E(x, \lambda r) \leq C(n, A)\lambda\left(D_E\left(x, \frac{r}{2}\right) + \frac{r^3}{\lambda^n}\right)$$

$$\leq C(n, A)\lambda\left(D_E(x, r) + \frac{\varepsilon_{\text{dir}}^2 r}{\lambda^n}\right) \leq C(n, A)\lambda\big(D_E(x, r) + r\big),$$

where in the first inequality we have also exploited (2.4) and in the second the obvious inequality $D_E(x, r/2) \leq 2^{n-1} D_E(x, r)$. This concludes the proof.                □

*7.4. Excess improvement.* In this section we prove Theorem 7.1.

*Proof (Proof of Theorem 7.1)* We claim that there exists a universal constant $C_{\text{exc}}$ such that for all $\lambda \in (0, 1/8)$ there exists $\varepsilon_{\text{exc}} = \varepsilon_{\text{exc}}(n, A, \lambda)$ satisfying the following: for all minimizers of $(\mathcal{P}_{\beta, K, Q, R})$ with $\beta$, $K$, $Q$ controlled by $A$ and $R \geq 1$ if $x \in \partial E$ the following holds

$$\mathbf{e}_E(x, r) + Q^2 D_E(x, r) + r \ le \varepsilon_{\text{exc}} \implies \mathbf{e}_E(x, \lambda r)$$
$$\leq C_{\text{exc}} \lambda \Big( \mathbf{e}_E(x, r) + Q^2 D_E(x, r) + r \Big).$$

Note that the above claim, combined with Proposition 7.8 immediately implies the conclusion of the theorem. Let us assume hence that there exist $\lambda \in (0, 1/8)$ a sequence of minimizers $E_k \subset B_{R_k}$ with parameters $\beta_k$, $K_k$, $Q_k$ controlled by $A$, radii $r_k$ and points $x_k \in \partial E_k$ such that

$$\varepsilon_k = \mathbf{e}_{E_k}(x_k, r_k) + Q^2 D_E(x_k, r_k) + r_k \to 0$$

but

$$\mathbf{e}_{E_k}(x_k, \lambda r_k) \geq C_{\text{exc}} \lambda \varepsilon_k \tag{7.19}$$

for a suitable universal constant $C_{\text{exc}}$. Note that up to translating and rotating we can assume that $x_k = 0$ and that

$$\mathbf{e}_{E_k}(0, r_k) = \mathbf{e}_{E_k}(0, r_k, e_n).$$

We apply Proposition 7.5 to each $E_k$. Hence, there exists a sequence of 1-Lipschitz functions $f_k : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\frac{\mathcal{H}^{n-1}\left(\mathbf{C}_{\frac{r_k}{2}} \cap \partial E_k \Delta \Gamma_{f_k}\right)}{r_k^{n-1}} \leq 2^{n-1} C_{\text{lip}} \varepsilon_k, \tag{7.20a}$$

$$\frac{1}{r_k^{n-1}} \int_{\mathbf{D}_{\frac{r_k}{2}}} |\nabla f_k|^2 \, dz \leq 2^{n-1} C_{\text{lip}} \varepsilon_k, \tag{7.20b}$$

$$\|f_k\|_\infty \leq \omega(\varepsilon_k) r_k, \tag{7.20c}$$

$$\left| \fint_{\mathbf{D}_{\frac{r_k}{2}}} \nabla f_k \cdot \nabla \varphi \, dz \right| \leq 2^{n-1} C_{\text{lip}} \|\nabla \varphi\|_\infty \varepsilon_k \quad \text{for all } \varphi \in C_c^1(\mathbf{D}_{\frac{r_k}{2}}). \tag{7.20d}$$

Let us set

$$g_k := \frac{f_k^{r_k} - m_k}{\sqrt{\varepsilon_k}} \quad \text{where} \quad m_k := \fint_{\mathbf{D}_{\frac{r_k}{2}}} f_k^{r_k} \, dz, \quad \text{and} \quad f_k^{r_k}(z) := \frac{f_k(r_k z)}{r_k}.$$

By the Poincaré–Wirtinger inequality and (7.20b),

$$\sup_k \|g_k\|_{W^{1,2}(\mathbf{D}_{\frac{1}{2}})} \leq 2^{n-1} C_{\text{lip}}. \tag{7.21}$$

Hence there exists $g$ in $W^{1,2}(\mathbf{D}_{\frac{1}{2}})$ such that $g_k \rightharpoonup g$ weakly in $W^{1,2}(\mathbf{D}_{\frac{1}{2}})$ to some $g$ and strongly in $L^2(\mathbf{D}_{\frac{1}{2}})$. Moreover, by (7.20d), for all $\varphi \in C_c^1(\mathbf{D}_{\frac{1}{2}})$

$$
\begin{aligned}
\left| \int_{\mathbf{D}_{\frac{1}{2}}} \nabla g \cdot \nabla \varphi \, dz \right| &= \lim_{k \to +\infty} \frac{1}{\sqrt{\varepsilon_k}} \left| \int_{\mathbf{D}_{\frac{1}{2}}} \nabla f_k^{r_k} \cdot \nabla \varphi \, dz \right| \\
&= \lim_{k \to +\infty} \frac{1}{\sqrt{\varepsilon_k}} \left| \fint_{\mathbf{D}_{\frac{r_k}{2}}} \nabla f_k \cdot \nabla \varphi_{r_k} \, dz \right| = 0,
\end{aligned}
\tag{7.22}
$$

where $\varphi_{r_k}(z) = r_k \varphi(z/r_k) \in C_c^1(\mathbf{D}_{\frac{r_k}{2}})$ satisfies $\|\nabla \varphi_{r_k}\|_\infty = \|\nabla \varphi\|_\infty$. Hence $g$ is harmonic. By the mean value property and (7.21)

$$
\sup_{\mathbf{D}_{1/4}} |\nabla^2 g|^2 \le C(n) \int_{\mathbf{D}_{\frac{1}{2}}} |\nabla g|^2 \, dz \le C(n, A).
$$

By Taylor expansion,

$$
|g(z) - g(0) - \nabla g(0) \cdot z| \le C(n, A)|z|^2 \qquad \text{for all } z \in \mathbf{D}_{\frac{1}{4}}.
\tag{7.23}
$$

If $2\lambda \in (0, 1/4)$ we can integrate the above inequality to get

$$
\fint_{\mathbf{D}_{2\lambda}} |g(z) - g(0) - \nabla g(0) \cdot z|^2 \, dz \le C(n, A)\lambda^4.
$$

Recall that, by the mean value property of harmonic functions, for every $r \le \frac{1}{2}$ we have

$$
(g)_r := \fint_{\mathbf{D}_r} g \, dz = g(0) \quad \text{and} \quad (\nabla g)_r = \nabla g(0).
$$

Hence,

$$
\begin{aligned}
\lim_{k \to +\infty} \fint_{\mathbf{D}_{2\lambda}} |g_k(z) - (g_k)_{2\lambda} - (\nabla g_k)_{2\lambda} \cdot z|^2 \, dz &= \fint_{\mathbf{D}_{2\lambda}} |g(z) - (g)_{2\lambda} - (\nabla g)_{2\lambda} \cdot z|^2 dz \\
&= \fint_{\mathbf{D}_{2\lambda}} |g(z) - g(0) - \nabla g(0) \cdot z|^2 \, dz \\
&\le C(n, A)\lambda^4.
\end{aligned}
$$

which, by the definition of $g_k$ and changing variables implies

$$
\lim_{k \to +\infty} \frac{1}{\varepsilon_k (\lambda r_k)^{n+1}} \int_{\mathbf{D}_{2\lambda r_k}} \left| f_k(z) - (f_k)_{2\lambda r_k} - (\nabla f_k)_{2\lambda r_k} \cdot z \right|^2 \, dz \le C(n, A)\lambda^2.
\tag{7.24}
$$

Let us define

$$
v_k := \frac{\left( -(\nabla f_k)_{2\lambda r_k}, 1 \right)}{\sqrt{1 + |(\nabla f_k)_{2\lambda r_k}|^2}} \qquad h_k := \frac{(f_k)_{2\lambda r_k}}{\sqrt{1 + |(\nabla f_k)_{2\lambda r_k}|^2}},
$$

and note that, by (7.20b), Jensen inequality and (7.20c)

$$|v_k - e_n|^2 \leq C\left(\fint_{\mathbf{D}_{\lambda r_k}} |\nabla f_k| \, dz\right)^2 \leq C(n, A, \lambda)\varepsilon_k \quad \text{and} \quad |h_k| \leq C\omega(\varepsilon_k)r_k.$$

(7.25)

Since the $f_k$'s are 1-Lipschitz, (7.24) implies

$$\limsup_{k\to+\infty} \frac{1}{\varepsilon_k(\lambda r_k)^{n+1}} \int_{\Gamma_{f_k}\cap \mathbf{C}_{2\lambda r_k}} |v_k \cdot x - h_k|^2 \, d\mathcal{H}^{n-1}(x)$$

$$\leq \lim_{k\to+\infty} \frac{\sqrt{2}}{\varepsilon_k(\lambda r_k)^{n+1}} \int_{\mathbf{D}_{2\lambda r_k}} \left| f_k(z) - (f_k)_{2\lambda r_k} - (\nabla f_k)_{2\lambda r_k} \cdot z \right|^2 \, dz \leq C(n, A)\lambda^2.$$

and thus

$$\limsup_{k\to+\infty} \frac{1}{\varepsilon_k(\lambda r_k)^{n+1}} \int_{\Gamma_{f_k}\cap\partial E_k\cap\mathbf{C}_{2\lambda r_k}} |v_k \cdot x - h_k|^2 \, d\mathcal{H}^{n-1}(x) \leq C(n, A)\lambda^2.$$

(7.26)

On the other hand, (7.20a), Lemma 7.2 and (7.25) imply

$$\frac{1}{\varepsilon_k(\lambda r_k)^{n+1}} \int_{(\partial E_k\setminus\Gamma_{f_k})\cap\mathbf{C}_{2\lambda r_k}} |v_k \cdot x - h_k|^2 \, d\mathcal{H}^{n-1}(x)$$

$$\leq C(n, A, \lambda) \frac{\mathcal{H}^{n-1}\left((\partial E_k \Delta \Gamma_{f_k}) \cap \mathbf{C}_{r_k}\right)}{\varepsilon_k r_k^{n-1}} \left(|v_k - e_n|^2 + \sup_{x\in\partial E_k\cap\mathbf{C}_{r_k}} \frac{|\mathbf{q}x|}{r_k^2} + \frac{|h_k|^2}{r_k^2}\right)$$

$$\leq C(n, A, \lambda) \frac{\mathcal{H}^{n-1}\left((\partial E_k \Delta \Gamma_{f_k}) \cap \mathbf{C}_{r_k}\right)}{\varepsilon_k r_k^{n-1}} \left(\varepsilon_k + \omega(\varepsilon_k)\right) = o(1).$$

(7.27)

Combining (7.26) and (7.27) we deduce that

$$\limsup_{k\to\infty} \frac{\mathbf{f}_{E_k}(0, 2\lambda r_k, v_k)}{\varepsilon_k}$$

$$\leq \limsup_{k\to\infty} \frac{1}{\varepsilon_k(\lambda r_k)^{n+1}} \int_{\partial E_k\cap\mathbf{C}_{2\lambda r_k}} |v_k \cdot x - h_k|^2 \, d\mathcal{H}^{n-1}(x) \leq C(n, A)\lambda^2.$$

(7.28)

On the other hand, by the perimeter density estimates (6.9) and (7.25)

$$\mathbf{e}_{E_k}(0, 4\lambda r_k, v_k) \leq \frac{1}{(4\lambda r_k)^{n-1}} \int_{\partial E_k\cap\mathbf{C}_{4\lambda r_k}} \frac{|v_{E_k} - v_k|^2}{2} \, d\mathcal{H}^{n-1}$$

$$\leq C(n, \lambda)\left(\mathbf{e}_{E_k}(0, r_k, e_n) + |e_n - v_k|^2 \frac{P(E, B_{r_k})}{r_k^{n-1}}\right) = o(1).$$

Hence we can apply Proposition 7.7 in $B_{4\lambda r_k}$ to get that

$$\mathbf{e}_{E_k}(0, \lambda r_k) \leq \mathbf{e}_{E_k}(0, \lambda r_k, v_k)$$

$$\leq C_{\text{cac}}\left(\mathbf{f}_{E_k}(0, 2\lambda r_k, v_k) + Q^2 D_{E_k}(0, 2\lambda r_k) + \lambda r_k\right),$$

(7.29)

where in the first inequality we have used (7.2). Furthermore, by Proposition 7.8 applied in $B_{r_k}$ we have

$$Q^2 D_{E_k}(0, 2\lambda r_k) \leq C_{\text{dir}}\lambda(Q^2 D_{E_k}(0, r_k) + Q^2 r_k) \leq C(n, A)\lambda\varepsilon_k. \qquad (7.30)$$

Combining (7.28), (7.29) and (7.30) we thus infer that

$$\limsup_{k\to\infty} \frac{\mathbf{e}_{E_k}(0, \lambda r_k)}{\varepsilon_k} \leq C(n, A)\lambda,$$

in contradiction with (7.19) if $C_{\text{exc}}$ is chosen big enough depending only on $n$ and $A$. $\square$

## 8. Proof of Theorems 1.1 and 1.2

In this section we prove our main theorems. Theorem 1.2 is an immediate consequence of the following slightly more general statement.

**Theorem 8.1** *Let $A > 0$ $\vartheta \in (0, 1)$, and let $\beta$, $K$, $Q$ be controlled by $A$ and $R \geq 1$. There exist constants $C_{\text{reg}}(n, A, \theta) > 0$ and $\varepsilon_{\text{reg}} = \varepsilon_{\text{reg}}(n, A, \theta) > 0$ such that, if $E$ is a minimizer of $(\mathcal{P}_{\beta, K, Q, R})$, $x \in \partial E$ $r > 0$ and $\nu \in S^{n-1}$ satisfy*

$$r + Q^2 D_E(x, 2r) + \mathbf{e}_E(x, 2r, \nu) \leq \varepsilon_{\text{reg}},$$

*then there exists a $C^{1,\vartheta}$ function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ with [4]*

$$f(0) = 0, \quad |\nabla f(0) - \nu|^2 + r^\vartheta [\nabla f]_{\vartheta/2}^2 \leq C_{\text{reg}}(r + Q^2 D_E(x, 2r) + \mathbf{e}_E(x, 2r, \nu)),$$

*such that*

$$E \cap B_r(x) = \left\{ y \in B_r(x) : \nu \cdot (y - x) \leq f(\mathbf{p}^\nu(y - x)) \right\}.$$

*Proof* Given $\vartheta \in (0, 1)$ we fix $\bar{\lambda} \in (0, 1/8)$ be such that

$$C_{\text{dec}}\bar{\lambda} + \bar{\lambda} \leq \bar{\lambda}^\vartheta, \qquad (8.1)$$

and we let $\bar{\varepsilon}$ be the corresponding $\varepsilon_{\text{dec}}$ in Theorem 7.1. Note that $\bar{\varepsilon}$ depends only on $n$, $A$ and $\vartheta$. We now choose $\varepsilon_{\text{reg}}$ so that for all $y \in \partial E \cap B_r(x)$

$$r + Q^2 D_E(y, r) + \mathbf{e}_E(y, r) \leq r + Q^2 D_E(y, r, \nu) + \mathbf{e}_E(y, r, \nu)$$
$$\leq 2^{n-1}(r + Q^2 D_E(x, 2r, \nu) + \mathbf{e}_E(x, 2r, \nu)) \leq 2^{n-1}\varepsilon_{\text{reg}} \leq \bar{\varepsilon}.$$

Hence we can apply Theorem 7.1 and (8.1) to deduce that for all $y \in \partial E \cap B_{r/2}(x)$,

$$\bar{\lambda}r + Q^2 D_E(y, \bar{\lambda}r) + \mathbf{e}_E(y, \bar{\lambda}r) \leq \bar{\lambda}^\vartheta(r + Q^2 D_E(y, r) + \mathbf{e}_E(y, r)).$$

---

[4] Here

$$[\nabla f]_{\vartheta/2} := \sup_{x \neq y} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^{\frac{\vartheta}{2}}}.$$

Iterating we get

$$\mathbf{e}_E(y, \bar{\lambda}^k r) \le \bar{\lambda}^{k\vartheta}\left(r + Q^2 D_E(y, r) + \mathbf{e}_E(y, r)\right),$$

which implies

$$\mathbf{e}_E(y, s) \le C(\vartheta)\left(\frac{s}{r}\right)^{\vartheta}\left(r + Q^2 D_E(y, r) + \mathbf{e}_E(y, r)\right) \quad \text{for all } s \le r.$$

By classical arguments this together with the density estimates (6.9) and (6.19), implies that for all $y \in B_r(x) \cap \partial E$ there exists $\nu_y$ such that

$$\mathbf{e}_E(y, s/2, \nu_y) = C(n, \vartheta, A)\left(\frac{s}{r}\right)^{\vartheta}\left(r + Q^2 D_E(y, r) + \mathbf{e}_E(y, r)\right) \quad \text{for all } s \le r.$$

and

$$|\nu_y - \nu|^2 \le C(n, A)\left(r + Q^2 D_E(y, 2r, \nu) + \mathbf{e}_E(y, r)\right),$$

The last two display yield the desired conclusion, see for instance [19, Theorem 26.3] or [13, Theorem 4.8]. □

We can now prove Theorem 1.1 by following the arguments in [12].

*Proof (Proof of Theorem 1.1)* By Theorem 1.2, if we set

$$\Sigma_E = \left\{x \in \partial E : \limsup_{r \to 0} \mathbf{e}_E(x, r) + D_E(x, r) > 0\right\}$$

then $\partial E \setminus \Sigma_E$ is a $C^{1,\vartheta}$ manifold for all $\vartheta \in (0, 1/2)$. Hence we will conclude the proof if we show that

$$\mathcal{H}^{n-1-\eta}(\Sigma_E) = 0,$$

for some $\eta = \eta(n, B) > 0$. Recall that by Lemma 6.1, $|\nabla u_E|^{2p} \in L^1_{\text{loc}}$ for some $p = p(n, B) > 1$ By Hölder inequality,

$$\Sigma_E^1 = \left\{x : \limsup_{r \to 0} D_E(x, r) > 0\right\} \subset \left\{x : \limsup_{r \to 0} \frac{1}{r^{n-p}} \int_{B(x,r)} |\nabla u_E|^{2p} > 0\right\},$$

hence, by [11, Theorem 2.10], $\mathcal{H}^{n-p}(\Sigma_E^1) = 0$. We now show that

$$\mathcal{H}^{\alpha}(\Sigma_E \setminus \Sigma_E^1) = 0$$

for all $\alpha > n - 8$ which clearly concludes the proof. Let us fix $\alpha > n - 8$ and assume the contrary. By [13, Proposition 11.3], there will be a point

$$x \in \Sigma_E^2 := \left\{x \in \partial E : \limsup_{r \to 0} \mathbf{e}_E(x, r) > 0, \lim_{r \to 0} D_E(x, r) = 0\right\},$$

and a sequence $r_k \to 0$ such that

$$\limsup_{k \to \infty} \frac{\mathcal{H}^{\alpha}_{\infty}(\Sigma_E^2 \cap B(x, r_k))}{r_k^{\alpha}} \ge c(\alpha) > 0.$$

where $\mathcal{H}^s_\infty$ is the infinity Hausdorff pre-measure. Let us set $E_k = (E - x)/r_k$ and note that by (6.9) and the above equation

$$P(E_k, B_s) \lesssim s^{n-1},$$

for all $s > 0$ and

$$\limsup_{k \to \infty} \mathcal{H}^\alpha_\infty(\Sigma^2_{E_k} \cap B_1) \geq c(\alpha), \tag{8.2}$$

where $\Sigma^2_{E_k} = (\Sigma^2_E - x)/r_k$. Up to subsequences, $E_k \to F$. We claim that $F$ is a local minimizer of the perimeter. Indeed let $G \Delta F \Subset B_s$. By averaging we can choose $t \in (s, 2s)$ such that

$$\mathcal{H}^{n-1}((E_k \Delta G) \cap \partial B_t) = \mathcal{H}^{n-1}((E_k \Delta F) \cap \partial B_t) \leq \frac{|(E_k \Delta F) \cap B_{2s}|}{s} = \sigma_k \to 0.$$

With this choice, we define $G_k := (x + r_k G) \cap B_{tr_k}(x) \cup (E \setminus B_{tr_k}(x))$ and we note that $E \Delta G_k \Subset B_{2sr_k}(x)$. Hence by (4.6) and classical computations

$$P(F, B_t) - P(G, B_t) \leq \limsup_{k \to \infty} \frac{P(E_k, B_{tr_k}(x)) - P(G_k, B_{tr_k}(x))}{r_k^{n-1}}$$

$$\lesssim \limsup_{k \to \infty} \sigma_k + s^n r_k + s^{n-1} D_E(x, sr_k) = 0,$$

which implies the desired minimality property. Moreover, by using $G = F$ we also deduce that $P(E_k, B_s) \to P(F, B_s)$ for almost all $s > 0$.

Let now $\Sigma_F$ be the singular set of $F$, and recall that, by the regularity theory for set of minimal perimeter [19, Part III], $\mathcal{H}^\alpha(\Sigma_F) = \mathcal{H}^\alpha_\infty(\Sigma_F) = 0$. Hence by the definition of Hausdorff measure, for all $\delta > 0$ there exists an open set $U_\delta$ such that

$$\Sigma_F \cap B_2 \subset U_\delta \quad \text{and} \quad \mathcal{H}^\alpha_\infty(U_\delta) \leq \delta.$$

We claim that there exists $k = k_\delta > 0$ such that $\Sigma^2_{E_k} \cap B_1 \subset U_\delta$ which will be in contradiction with (8.2) if $\delta$ is chosen small enough. Assume the claim is false, hence there is a sequence of points $\Sigma^2_{E_k} \cap B_1 \ni y_k \to \bar{y} \in \overline{B}_1$ with $\text{dist}(\bar{y}, \Sigma_F) > 0$. It is easy to see that, by the lower perimeter estimates (6.19), $\bar{y} \in \partial F$. Hence by regularity, for all $\varepsilon > 0$ there exists $r > 0$ such that

$$\mathbf{e}_F(\bar{y}, r) \leq \varepsilon.$$

By perimeter convergence, this implies that, for $k$ large

$$\mathbf{e}_E(x + r_k y_k, r r_k) = \mathbf{e}_{E_k}(y_k, r) \leq \mathbf{e}_F(\bar{y}, r) + \varepsilon \leq 2\varepsilon.$$

Choosing $\varepsilon \ll 1$ we can apply Theorem 1.2 to deduce that $x + r_k y_k \notin \Sigma^2_E$, i.e. $y_k \notin \Sigma^2_{E_k}$. This final contradiction concludes the proof. $\qquad \square$

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

## References

1. Almgren, F.J.J.: Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. Ann. Math. **87**, 321–391 (1968)
2. Almgren, F.J.J.: *Existence and Regularity Almost Everywhere of Solutions to Elliptic Variational Problems with Constraints*. Mem. Amer. Math. Soc., 4 (1976), pp. viii+199
3. Ambrosio, L., Buttazzo, G.: An optimal design problem with perimeter penalization. Calc. Var. Partial Differ. Equ. **1**, 55–69 (1993)
4. Cordes, H.O.: Über die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen. Math. Ann. **131**, 278–312 (1956)
5. De Giorgi, E.: Frontiere Orientate di Misura Minima, Seminario di Matematica della Scuola Normale Superiore di Pisa. Editrice Tecnico Scientifica, Pisa (1960)
6. De Philippis, G., Figalli, A.: A note on the dimension of the singular set in free interface problems. Differ. Integral Equ. **28**, 523–536 (2015)
7. De Philippis, G., Maggi, F.: Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. Arch. Ration. Mech. Anal. **216**, 473–568 (2015)
8. Deserno, M.: Rayleigh instability of charged droplets in the presence of counterions. Eur. Phys. J. E **6**, 163–168 (2001)
9. Doyle, A., Moffett, D., Vonnegut, B.: Behavior of evaporating electrically charged droplets. J. Colloid Sci. **19**, 136–143 (1964)
10. Duft, D., Achtzehn, T., Müller, R., Huber, B.A., Leisner, T.: Rayleigh jets from levitated microdroplets. Nature **421**, 128 (2003)
11. Evans, L.C., Gariepy, R.F.: Measure Theory and Fine Properties of Functions, Textbooks in Mathematics. CRC Press, Boca Raton (2015)
12. Fusco, N., Julin, V.: On the regularity of critical and minimal sets of a free interface problem. Interfaces Free Bound. **17**, 117–142 (2015)
13. Giusti, E.: Minimal Surfaces and Functions of Bounded Variation, vol. 80 of Monographs in Mathematics. Birkhäuser Verlag, Basel (1984)
14. Giusti, E.: Direct Methods in the Calculus of Variations. World Scientific Publishing Co., Inc., River Edge, NJ (2003)
15. Goldman, M., Novaga, M., Ruffini, B.: Existence and stability for a non-local isoperimetric model of charged liquid drops. Arch. Ration. Mech. Anal. **217**, 1–36 (2015)
16. Goldman, M., Novaga, M., Ruffini, B.: On minimizers of an isoperimetric problem with long-range interactions under a convexity constraint. Anal. PDE **11**, 1113–1142 (2018)
17. Goldman, M., Ruffini, B.: Equilibrium shapes of charged droplets and related problems: (mostly) a review. Geom. Flows **2**, 94–104 (2017)
18. Lin, F.-H.: Variational problems with free interfaces. Calc. Var. Partial Differ. Equ. **1**, 149–168 (1993)
19. Maggi, F.: *Sets of Finite Perimeter and Geometric Variational Problems: an Introduction to Geometric Measure Theory*, vol. 135 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, (2012)
20. Muratov, C.B., Novaga, M.: *On Well-Posedness of Variational Models of Charged Drops*. Proc. A., 472 (2016)
21. Muratov, C.B., Novaga, M., Ruffini, B.: On equilibrium shape of charged flat drops. Commun. Pure Appl. Math. **71**, 1049–1073 (2018)
22. Rayleigh, J.: On the equilibrium of liquid conducting masses charged with electricity. Lond Edinb Dublin Philos Mag J Sci **14**, 184–186 (1882)
23. Richardson, C.B., Pigg, A.L., Hightower, R.L.: On the stability limit of charged droplets. Proc. R. Soc. Lond. Ser. A Math. Phys. Sci. **422**, 319–328 (1989)
24. Stone, H.A., Lister, J.R., Brenner, M.P.: Drops with Conical Ends in Electric and Magnetic Fields (1998)

25. Tamanini, I.: Boundaries of Caccioppoli sets with Hölder-continuous normal vector. J. Reine Angew. Math. **334**, 27–39 (1982)
26. Taylor, G.: Disintegration of water drops in electric field. Proc. R. Soc. Lond. Ser. A Math. Phys. Sci. **280**, 383–397 (1964)
27. Thaokar, R.M., Deshmukh, S.D.: Rayleigh instability of charged drops and vesicles in the presence of counterions. Phys. Fluids **22**, 034107 (2010)
28. Wilson, C.T.R., Taylor, G.I.: The bursting of soap-bubbles in a uniform electric field. Math. Proc. Camb. Philos. Soc. **22**, 728–730 (1925)
29. Zeleny, J.: Instability of electrified liquid surfaces. Phys. Rev. **10**, 1–6 (1917)