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Representations of the Yangians Associated with Lie Superalgebras $\mathfrak{osp}(1|2n)$

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Abstract: We give a complete description of the finite-dimensional irreducible representations of the Yangians associated with the orthosymplectic Lie superalgebras $\mathfrak{osp}_{1|2n}$. The representations are classified in terms of their highest weights and are parameterized by n-tuples of monic polynomials in one variable. The arguments rely on explicit constructions of a family of elementary modules of the Yangian for $\mathfrak{osp}_{1|2}$. We show that a wide class of irreducible representations of this Yangian can be produced by taking tensor products of the elementary modules.

1. Introduction

The Yangians form a remarkable family of quantum groups with a deep and substantive representation theory and numerous connections in mathematical physics. According to the original definition of Drinfeld [10], the Yangian $\mathrm{Y}(\mathfrak{a})$ associated with a simple Lie algebra $\mathfrak{a}$ is a canonical deformation of the universal enveloping algebra $\mathrm{U}(\mathfrak{a}[u])$ in the class of Hopf algebras; see also [8, Ch. 12] for more details on their basic properties. The Yangians admit at least three different presentations, as shown in [11,12], including the R-matrix presentation going back to the work of Faddeev's school; see e.g. [21,26]. However, the equivalence of the presentations in the classical types have only been proved more recently; see [5,18,20].

It is the R-matrix approach which turned out to be more suitable for the introduction of the super-versions of the Yangians as given by Nazarov [24,25] in the case of Lie superalgebra $\mathfrak{gl}_{m|n}$. It was followed by a Drinfeld-type presentation (analogous to [12]) obtained by Gow [17]. The orthosymplectic Yangians $\mathrm{Y}(\mathfrak{osp}_{M|2n})$ were introduced by Arnaudon *et al.* [1] with the use of the R-matrix originated in [28]. In the subsequent work [2], a Drinfeld-type presentation of the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2})$ was produced, the double Yangian was constructed and its universal R-matrix was calculated in an explicit form. Applications of the orthosymplectic Yangians to spin chain models were discussed in [3].

More recently, linear and quadratic L-operators with values in the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|2n})$ were investigated in [13, 15].

The finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{a})$ were classified by Drinfeld [12]. The arguments rely on the work of Tarasov [27] on the particular case of $\mathrm{Y}(\mathfrak{sl}_2)$, where the classification was carried over in the language of monodromy matrices within the quantum inverse scattering method; see [22, Sec. 3.3] for a detailed adapted exposition of these results. This description of the representations of the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$, along with some other low rank cases, should also play an essential role in the classification of the finite-dimensional irreducible representations of the Yangians associated with simple Lie superalgebras. It was already used in the work of Zhang [29], where the finite-dimensional irreducible representations of $\mathrm{Y}(\mathfrak{gl}_{m|n})$ were classified. However, the general classification problem for the orthosymplectic Yangians still remains open.

Our goal in this paper is to describe finite-dimensional irreducible representations of the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2n})$. The description relies on the basic case $n = 1$, the extension to arbitrary values on n is then carried over by using some reduction properties of the representations with respect to the shift $n \mapsto n-1$.

To describe the results in more detail, recall that according to [1], the Yangian $\mathrm{Y}(\mathfrak{osp}_{M|2n})$ can be considered as a quotient of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{M|2n})$ by an ideal generated by central elements. A standard argument shows that every finite-dimensional irreducible representation of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ is a highest weight representation. It is isomorphic to the irreducible quotient $L(\lambda(u))$ of the Verma module $M(\lambda(u))$ associated with an $(n+1)$-tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_{n+1}(u))$ of formal series $\lambda_i(u) \in 1+u^{-1}\mathbb{C}[[u^{-1}]]$. The tuple is called the *highest weight* of the representation. The key step in the classification is to find the conditions on the highest weight for the representation $L(\lambda(u))$ to be finite-dimensional.

Main Theorem. *Every finite-dimensional irreducible representation of the algebra* $\mathrm{X}(\mathfrak{osp}_{1|2n})$ *is isomorphic to* $L(\lambda(u))$ *for a certain highest weight* $\lambda(u)$. *The representation* $L(\lambda(u))$ *is finite-dimensional if and only if*

$$\frac{\lambda_{i+1}(u)}{\lambda_i(u)} = \frac{P_i(u+1)}{P_i(u)}, \qquad i = 1, \dots, n, \tag{1.1}$$

for some monic polynomials $P_i(u)$ *in* u. *The finite-dimensional irreducible representations of the Yangian* $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ *are in a one-to-one correspondence with the* n*-tuples of monic polynomials* $(P_1(u), \dots, P_n(u))$.

This description is quite similar to the classification results of [12]. The monic polynomials occurring therein are called the *Drinfeld polynomials* of the representation.

The required necessary conditions are derived by induction from those for the associated actions of the Yangians $\mathrm{Y}(\mathfrak{gl}_2)$ and $\mathrm{X}(\mathfrak{osp}_{1|2})$ on the respective cyclic spans of the highest vector of $L(\lambda(u))$. An essential step in the proof of the Main Theorem is the analysis of the *elementary modules* $L(\alpha, \beta)$ over $\mathrm{X}(\mathfrak{osp}_{1|2})$ associated with the highest weights of the form

$$\lambda_1(u) = \frac{u+\alpha}{u+\beta}, \qquad \lambda_2(u) = 1, \tag{1.2}$$

for arbitrary complex numbers α and β. The corresponding *small Verma module* $M(\alpha, \beta)$ turns out to be irreducible if and only if $\beta - \alpha$ and $\beta - \alpha + 1/2$ are not nonnegative integers. The elementary modules $L(\alpha, \beta)$ are the irreducible quotients of $M(\alpha, \beta)$ and so they split into three families, according to these conditions. The module $L(\alpha, \beta)$ is finite-dimensional if and only if $\beta - \alpha \in \mathbb{Z}_+$. In this case, when regarded as an $\mathfrak{osp}_{1|2}$-module, $L(\alpha, \beta)$ decomposes into the direct sum

$$L(\alpha, \beta) \cong \bigoplus_{p=0}^{\lfloor \frac{\beta-\alpha}{2} \rfloor} V(\beta - \alpha - 2p),$$

where $V(\mu)$ denotes the $2\mu + 1$-dimensional $\mathfrak{osp}_{1|2}$-module with the highest weight $\mu \in \mathbb{Z}_+$. In particular,

$$\dim L(\alpha, \beta) = \binom{\beta - \alpha + 2}{2}.$$

We construct a basis of each small Verma module $M(\alpha, \beta)$ and give explicit formulas for the action of the generators of $\mathrm{X}(\mathfrak{osp}_{1|2})$. This leads to a corresponding description of all elementary modules. We show that, up to twisting by a multiplication automorphism of $\mathrm{X}(\mathfrak{osp}_{1|2})$, every finite-dimensional irreducible representation of this algebra is isomorphic to a subquotient of the tensor product module of the form

$$L(\alpha_1, \beta_1) \otimes \ldots \otimes L(\alpha_k, \beta_k). \tag{1.3}$$

The final step in the description of the $\mathrm{X}(\mathfrak{osp}_{1|2})$-modules is to investigate irreducibility conditions for such tensor products.

In the case of the Yangian $\mathrm{Y}(\mathfrak{sl}_2)$, an irreducibility criterion for tensor products of evaluation modules was given by Chari and Pressley [6]; see also [22, Ch. 3]. Such tensor products exhaust all finite-dimensional irreducible $\mathrm{Y}(\mathfrak{sl}_2)$-modules. This property turns out not to extend to representations of the Yangian for $\mathfrak{osp}_{1|2}$; see Example 5.19 below. A wide class of irreducible modules over $\mathrm{X}(\mathfrak{osp}_{1|2})$ can still be constructed explicitly via tensor products of the form (1.3); see Theorem 5.15.

The proof of the Main Theorem will be completed in Sect. 6, where we will rely on Proposition 4.1 to establish necessary conditions for the $\mathrm{X}(\mathfrak{osp}_{1|2n})$-module $L(\lambda(u))$ to be finite-dimensional. The sufficiency of these conditions is verified by constructing the *fundamental representations* of the Yangian $\mathrm{X}(\mathfrak{osp}_{1|2n})$; cf. [4,7].

It is well-known (see, e.g., [9,23]), that the finite-dimensional irreducible representations of the Lie superalgebras $\mathfrak{osp}_{M|2n}$ are significantly more complicated for general values $M > 1$. Therefore, some additional methods need to be developed to obtain a classification of the representations of the Yangians associated with $\mathfrak{osp}_{M|2n}$.

2. Definitions and Preliminaries

For any integer $n \geqslant 1$ introduce the involution $i \mapsto i' = 2n - i + 2$ on the set $\{1, 2, \ldots, 2n + 1\}$. Consider the $\mathbb{Z}_2$-graded vector space $\mathbb{C}^{1|2n}$ over $\mathbb{C}$ with the basis $e_1, e_2, \ldots, e_{2n+1}$, where the vectors e_i and $e_{i'}$ with $i = 1, \ldots, n$ are odd and the vector e_{n+1} is even. We set

$$\bar{\imath} = \begin{cases} 1 & \text{for} \quad i = 1, \ldots, n, n', \ldots, 1', \\ 0 & \text{for} \quad i = n + 1. \end{cases}$$

The endomorphism algebra $\mathrm{End}\, \mathbb{C}^{1|2n}$ gets a $\mathbb{Z}_2$-gradation with the parity of the matrix unit e_{ij} found by $\bar{\imath} + \bar{\jmath} \mod 2$.

We will consider even square matrices with entries in $\mathbb{Z}_2$-graded algebras, their (i, j) entries will have the parity $\bar{\imath}+\bar{\jmath} \mod 2$. The algebra of even matrices over a superalgebra $\mathcal{A}$ will be identified with the tensor product algebra $\operatorname{End}\mathbb{C}^{1|2n} \otimes \mathcal{A}$, so that a matrix $A = [a_{ij}]$ is regarded as the element

$$A = \sum_{i,j=1}^{2n+1} e_{ij} \otimes a_{ij}(-1)^{\bar{\imath}\,\bar{\jmath}+\bar{\jmath}} \in \operatorname{End}\mathbb{C}^{1|2n} \otimes \mathcal{A}.$$

We will use the involutive matrix *super-transposition* t defined by $(A^t)_{ij} = A_{j'i'}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\jmath}}\theta_i\theta_j$, where we set

$$\theta_i = \begin{cases} 1 & \text{for} \quad i = 1, \dots, n+1, \\ -1 & \text{for} \quad i = n+2, \dots, 2n+1. \end{cases}$$

This super-transposition is associated with the bilinear form on the space $\mathbb{C}^{1|2n}$ defined by the anti-diagonal matrix $G = [\delta_{ij'}\theta_i]$. We will also regard t as the linear map

$$t : \operatorname{End}\mathbb{C}^{1|2n} \to \operatorname{End}\mathbb{C}^{1|2n}, \qquad e_{ij} \mapsto e_{j'i'}(-1)^{\bar{\imath}\bar{\jmath}+\bar{\imath}}\theta_i\theta_j. \tag{2.1}$$

In the case of multiple tensor products of the endomorphism algebras, we will indicate by t_a the map (2.1) acting on the a-th copy of $\operatorname{End}\mathbb{C}^{1|2n}$.

A standard basis of the general linear Lie superalgebra $\mathfrak{gl}_{1|2n}$ is formed by elements E_{ij} of the parity $\bar{\imath}+\bar{\jmath} \mod 2$ for $1 \leqslant i, j \leqslant 2n+1$ with the commutation relations

$$[E_{ij}, E_{kl}] = \delta_{kj}\, E_{il} - \delta_{il}\, E_{kj}(-1)^{(\bar{\imath}+\bar{\jmath})(\bar{k}+\bar{l})}.$$

We will regard the orthosymplectic Lie superalgebra $\mathfrak{osp}_{1|2n}$ associated with the bilinear form defined by G as the subalgebra of $\mathfrak{gl}_{1|2n}$ spanned by the elements

$$F_{ij} = E_{ij} - E_{j'i'}(-1)^{\bar{\imath}\,\bar{\jmath}+\bar{\imath}}\,\theta_i\theta_j.$$

Introduce the permutation operator P by

$$P = \sum_{i,j=1}^{2n+1} e_{ij} \otimes e_{ji}(-1)^{\bar{\jmath}} \in \operatorname{End}\mathbb{C}^{1|2n} \otimes \operatorname{End}\mathbb{C}^{1|2n}$$

and set

$$Q = P^{t_1} = P^{t_2} = \sum_{i,j=1}^{2n+1} e_{ij} \otimes e_{i'j'}(-1)^{\bar{\imath}\bar{\jmath}}\,\theta_i\theta_j \in \operatorname{End}\mathbb{C}^{1|2n} \otimes \operatorname{End}\mathbb{C}^{1|2n}.$$

The *R-matrix* associated with $\mathfrak{osp}_{1|2n}$ is the rational function in u given by

$$R(u) = 1 - \frac{P}{u} + \frac{Q}{u-\kappa}, \qquad \kappa = -n - 1/2.$$

This is a super-version of the R-matrix originally found in [28]. The R-matrices produced in that paper are known to extend to the Brauer algebra so that the Yang–Baxter equation can be verified by taking a suitable Brauer algebra representation in tensor products of the $\mathbb{Z}_2$-graded spaces; cf. [13,16].

Following [1], we define the *extended Yangian* $\mathrm{X}(\mathfrak{osp}_{1|2n})$ as a $\mathbb{Z}_2$-graded algebra with generators $t_{ij}^{(r)}$ of parity $\bar{\imath} + \bar{\jmath} \mod 2$, where $1 \leqslant i, j \leqslant 2n+1$ and $r = 1, 2, \dots,$ satisfying certain quadratic relations. In order to write them down, introduce the formal series

$$t_{ij}(u) = \delta_{ij} + \sum_{r=1}^{\infty} t_{ij}^{(r)} u^{-r} \in \mathrm{X}(\mathfrak{osp}_{1|2n})[[u^{-1}]] \tag{2.2}$$

and combine them into the matrix $T(u) = [t_{ij}(u)]$ so that

$$T(u) = \sum_{i,j=1}^{2n+1} e_{ij} \otimes t_{ij}(u)(-1)^{\bar{\imath}\,\bar{\jmath}+\bar{\jmath}} \in \mathrm{End}\,\mathbb{C}^{1|2n} \otimes \mathrm{X}(\mathfrak{osp}_{1|2n})[[u^{-1}]].$$

Consider the algebra $\mathrm{End}\,\mathbb{C}^{1|2n} \otimes \mathrm{End}\,\mathbb{C}^{1|2n} \otimes \mathrm{X}(\mathfrak{osp}_{1|2n})[[u^{-1}]]$ and introduce its elements $T_1(u)$ and $T_2(u)$ by

$$T_1(u) = \sum_{i,j=1}^{2n+1} e_{ij} \otimes 1 \otimes t_{ij}(u)(-1)^{\bar{\imath}\,\bar{\jmath}+\bar{\jmath}}, \qquad T_2(u) = \sum_{i,j=1}^{2n+1} 1 \otimes e_{ij} \otimes t_{ij}(u)(-1)^{\bar{\imath}\,\bar{\jmath}+\bar{\jmath}}.$$

The defining relations for the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ take the form of the *RTT-relation*

$$R(u-v)\, T_1(u)\, T_2(v) = T_2(v)\, T_1(u)\, R(u-v). \tag{2.3}$$

As shown in [1], the product $T(u-\kappa)\, T^{\,t}(u)$ is a scalar matrix with

$$T(u-\kappa)\, T^{\,t}(u) = c(u)\, 1, \tag{2.4}$$

where $c(u)$ is a series in u^{-1}. All its coefficients belong to the center $\mathrm{ZX}(\mathfrak{osp}_{1|2n})$ of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ and generate the center.

The *Yangian* $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ is defined as the subalgebra of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ which consists of the elements stable under the automorphisms

$$t_{ij}(u) \mapsto f(u)\, t_{ij}(u) \tag{2.5}$$

for all series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. We have the tensor product decomposition

$$\mathrm{X}(\mathfrak{osp}_{1|2n}) = \mathrm{ZX}(\mathfrak{osp}_{1|2n}) \otimes \mathrm{Y}(\mathfrak{osp}_{1|2n}). \tag{2.6}$$

The Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ is isomorphic to the quotient of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ by the relation $c(u) = 1$.

A more explicit form of the defining relations (2.3) can be written with the use of super-commutator in terms of the series (2.2) as follows:

$$\begin{aligned}
[t_{ij}(u), t_{kl}(v)] &= \frac{1}{u-v}\big(t_{kj}(u)\, t_{il}(v) - t_{kj}(v)\, t_{il}(u)\big)(-1)^{\bar{\imath}\,\bar{\jmath}+\bar{\imath}\,\bar{k}+\bar{\jmath}\,\bar{k}} \\
&\quad - \frac{1}{u-v-\kappa}\Big(\delta_{ki'} \sum_{p=1}^{2n+1} t_{pj}(u)\, t_{p'l}(v)(-1)^{\bar{\imath}+\bar{\imath}\,\bar{\jmath}+\bar{\jmath}\,\bar{p}}\, \theta_i\theta_p \\
&\quad - \delta_{lj'} \sum_{p=1}^{2n+1} t_{k\,p'}(v)\, t_{ip}(u)(-1)^{\bar{\jmath}+\bar{p}+\bar{\imath}\,\bar{k}+\bar{\jmath}\,\bar{k}+\bar{\imath}\,\bar{p}}\, \theta_j\theta_p\Big).
\end{aligned} \tag{2.7}$$

The mapping $t_{ij}(u) \mapsto t_{ij}(-u)$ defines an anti-automorphism of $\mathrm{X}(\mathfrak{osp}_{1|2n})$, while each of the mappings

$$t_{ij}(u) \mapsto t_{ij}(u+a), \quad a \in \mathbb{C}, \tag{2.8}$$

and $t_{ij}(u) \mapsto t_{i'j'}(u)\,\theta_i\theta_j$ defines an automorphism. Consider their composition to define the anti-automorphism

$$\omega : t_{ij}(u) \mapsto t_{i'j'}(-u+1/2)\,\theta_i\theta_j. \tag{2.9}$$

The universal enveloping algebra $\mathrm{U}(\mathfrak{osp}_{1|2n})$ can be regarded as a subalgebra of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ via the embedding

$$F_{ij} \mapsto \frac{1}{2}\big(t_{ij}^{(1)} - t_{j'i'}^{(1)}(-1)^{\bar{j}+\bar{i}\bar{j}}\,\theta_i\theta_j\big)(-1)^{\bar{i}}. \tag{2.10}$$

This fact relies on the Poincaré–Birkhoff–Witt theorem for the orthosymplectic Yangian which was pointed out in [1,2]. It states that the associated graded algebra for $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ is isomorphic to $\mathrm{U}(\mathfrak{osp}_{1|2n}[u])$. A detailed proof of the theorem can be given by extending the arguments of [4, Sec. 3] to the super case with the use of the vector representation recalled below in (6.2).

The extended Yangian $\mathrm{X}(\mathfrak{osp}_{1|2n})$ is a Hopf algebra with the coproduct defined by

$$\Delta : t_{ij}(u) \mapsto \sum_{k=1}^{2n+1} t_{ik}(u) \otimes t_{kj}(u). \tag{2.11}$$

For the image of the series $c(u)$ we have $\Delta : c(u) \mapsto c(u) \otimes c(u)$ and so the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ inherits the Hopf algebra structure from $\mathrm{X}(\mathfrak{osp}_{1|2n})$.

3. Gaussian Generators for $\mathrm{X}(\mathfrak{osp}_{1|2})$

A Drinfeld-type presentation of the Yangian for $\mathfrak{osp}_{1|2}$ was given in [2] with the use of the Gauss decomposition of the matrix $T(u)$. We will use some calculations produced therein and derive consistency relations for the Gaussian generators.

Apply the Gauss decomposition to the generator matrix $T(u)$ for $\mathrm{X}(\mathfrak{osp}_{1|2})$,

$$T(u) = F(u)\,H(u)\,E(u), \tag{3.1}$$

where $F(u)$, $H(u)$ and $E(u)$ are uniquely determined matrices of the form

$$F(u) = \begin{bmatrix} 1 & 0 & 0 \\ f_{21}(u) & 1 & 0 \\ f_{31}(u) & f_{32}(u) & 1 \end{bmatrix}, \qquad E(u) = \begin{bmatrix} 1 & e_{12}(u) & e_{13}(u) \\ 0 & 1 & e_{23}(u) \\ 0 & 0 & 1 \end{bmatrix},$$

and $H(u) = \mathrm{diag}\big[h_1(u), h_2(u), h_3(u)\big]$. Explicit formulas for the entries of the matrices $F(u)$, $H(u)$ and $E(u)$ can be written with the use of the Gelfand–Retakh quasideterminants [14]; cf. [20, Sec. 4]. In particular, we have

$$h_1(u) = t_{11}(u), \qquad h_2(u) = \begin{vmatrix} t_{11}(u) & t_{12}(u) \\ t_{21}(u) & \boxed{t_{22}(u)} \end{vmatrix}, \qquad h_3(u) = \begin{vmatrix} t_{11}(u) & t_{12}(u) & t_{13}(u) \\ t_{21}(u) & t_{22}(u) & t_{23}(u) \\ t_{31}(u) & t_{32}(u) & \boxed{t_{33}(u)} \end{vmatrix},$$

whereas

$$e_{12}(u) = h_1(u)^{-1}\, t_{12}(u), \qquad e_{23}(u) = h_2(u)^{-1} \begin{vmatrix} t_{11}(u) & t_{13}(u) \\ t_{21}(u) & \boxed{t_{23}(u)} \end{vmatrix},$$

and

$$f_{21}(u) = t_{21}(u)\, h_1(u)^{-1}, \qquad f_{32}(u) = \begin{vmatrix} t_{11}(u) & t_{12}(u) \\ t_{31}(u) & \boxed{t_{32}(u)} \end{vmatrix}\, h_2(u)^{-1}.$$

Proposition 3.1. *The following relations for the Gaussian generators hold:*

$$e_{12}(u) = -e_{23}(u+1/2), \qquad f_{21}(u) = f_{32}(u+1/2), \tag{3.2}$$

and

$$h_1(u)\, h_3(u+1/2) = h_2(u)\, h_2(u+1/2). \tag{3.3}$$

Moreover,

$$c(u) = h_1(u)\, h_1(u+1)^{-1}\, h_2(u+1)\, h_2(u+3/2). \tag{3.4}$$

Proof. The argument is quite similar to the proof of the corresponding relations for the Gaussian generators of $\mathrm{Y}(\mathfrak{o}_3)$ given in [19]; see also [20, Sec. 5.3]. We will outline a few key steps.

By inverting the matrices on both sides of (3.1), we get

$$T(u)^{-1} = E(u)^{-1}\, H(u)^{-1}\, F(u)^{-1}.$$

On the other hand, relation (2.4) implies $T^{t}(u) = c(u)\, T(u-\kappa)^{-1}$. Hence, by equating the (i, j) entries with $i, j = 2, 3$ in this matrix relation, we derive

$$\begin{aligned} h_1(u) &= c(u)\, h_3(u-\kappa)^{-1}, \\ h_1(u)\, e_{12}(u) &= -c(u)\, e_{23}(u-\kappa)\, h_3(u-\kappa)^{-1}, \\ f_{21}(u)\, h_1(u) &= c(u)\, h_3(u-\kappa)^{-1}\, f_{32}(u-\kappa), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &h_2(u) + f_{21}(u)\, h_1(u)\, e_{12}(u) \\ &\qquad = c(u)\big(h_2(u-\kappa)^{-1} + e_{23}(u-\kappa)\, h_3(u-\kappa)^{-1}\, f_{32}(u-\kappa)\big). \end{aligned} \tag{3.6}$$

Calculating as in [2,19], we verify that the coefficients of the series $h_1(u)$, $h_2(u)$ and $h_3(u)$ pairwise commute. Furthermore, we get

$$h_1(u)\, e_{12}(u) = e_{12}(u+1)\, h_1(u) \qquad \text{and} \qquad h_1(u)\, f_{21}(u+1) = f_{21}(u)\, h_1(u)$$

which together with relations (3.5) imply the first two desired identities, where we replaced κ by its value $-3/2$. They imply that relation (3.6) can be written in the form

$$h_2(u) - c(u)\, h_2(u-\kappa)^{-1} = -\big[e_{12}(u+1), f_{21}(u)\big]\, h_1(u). \tag{3.7}$$

As a final step, use one more relation between the Gaussian generators,

$$\big[e_{12}(u), f_{21}(v)\big] = \frac{h_1(u)^{-1}\, h_2(u) - h_1(v)^{-1}\, h_2(v)}{u-v},$$

so that eliminating $c(u)$ from (3.7) we come to (3.3). Relation (3.4) follows by eliminating $h_3(u)$ from the first relation in (3.5) with the use of (3.3). $\square$

Observe that the coefficients of the series $e_{12}(u)$ and $f_{21}(u)$ are stable under all automorphisms (2.5) and so belong to the subalgebra $\mathrm{Y}(\mathfrak{osp}_{1|2})$ of $\mathrm{X}(\mathfrak{osp}_{1|2})$. Together with the coefficients of the series $h(u) = h_1(u)^{-1}h_2(u)$ they generate the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2})$, and the defining relations for these generators are given in [2] in a slightly different setting.

4. Highest Weight Representations

The following reduction property for representations of the extended Yangians $\mathrm{X}(\mathfrak{osp}_{1|2n})$ will be frequently used; cf. [4, Lemma 5.13]. For an $\mathrm{X}(\mathfrak{osp}_{1|2n})$-module V set

$$V^+ = \{\eta \in V \mid t_{1j}(u)\,\eta = 0 \quad \text{for} \quad j > 1 \quad \text{and} \quad t_{i\,1'}(u)\,\eta = 0 \quad \text{for} \quad i < 1'\}. \tag{4.1}$$

Proposition 4.1. *The subspace V^+ is stable under the action of the operators $t_{ij}(u)$ subject to $2 \leqslant i, j \leqslant 2n$. Moreover, the assignment $\bar{t}_{ij}(u) \mapsto t_{i+1,j+1}(u)$ for $1 \leqslant i, j \leqslant 2n-1$ defines a representation of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$ on V^+, where the $\bar{t}_{ij}(u)$ denote the respective generating series for $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$.*

Proof. Suppose that $2 \leqslant k, l \leqslant 2n$ and $j > 1$. For any $\eta \in V^+$ apply (2.7) to get

$$t_{1j}(u)\,t_{kl}(u)\,\eta = \frac{1}{u-v-\kappa}\,\delta_{lj'}\,(-1)^{\bar{j}+\bar{k}+\bar{j}\,\bar{k}}\,\theta_j\,t_{k\,1'}(v)\,t_{11}(u)\,\eta.$$

Another application of (2.7) yields

$$t_{k\,1'}(v)\,t_{11}(u)\,\eta = -[t_{11}(u), t_{k\,1'}(v)]\,\eta = \frac{1}{u-v-\kappa}\,t_{k\,1'}(v)\,t_{11}(u)\,\eta,$$

implying $t_{1j}(u)\,t_{kl}(u)\,\eta = 0$. A similar calculation shows that $t_{i\,1'}(u)\,t_{kl}(u)\,\eta = 0$ for $i < 1'$ thus proving the first part of the proposition.

Now suppose that $2 \leqslant i, j, k, l \leqslant 2n$. By (2.7) the super-commutator $[t_{ij}(u), t_{kl}(v)]$ of the operators in V^+ equals

$$\begin{aligned}
&\frac{1}{u-v}\big(t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u)\big)(-1)^{\bar{\imath}\,\bar{j}+\bar{\imath}\,\bar{k}+\bar{j}\,\bar{k}}\\
&\quad - \frac{1}{u-v-\kappa}\Big(\delta_{ki'}\sum_{p=2}^{2n} t_{pj}(u)\,t_{p'l}(v)(-1)^{\bar{\imath}+\bar{\imath}\,\bar{j}+\bar{j}\,\bar{p}}\,\theta_i\theta_p\\
&\quad - \delta_{lj'}\sum_{p=2}^{2n} t_{k\,p'}(v)\,t_{ip}(u)(-1)^{\bar{j}+\bar{p}+\bar{\imath}\,\bar{k}+\bar{j}\,\bar{k}+\bar{\imath}\,\bar{p}}\,\theta_j\theta_p\Big)
\end{aligned}$$

plus the additional terms

$$-\frac{1}{u-v-\kappa}\Big(\delta_{ki'}\,t_{1j}(u)\,t_{1'l}(v)(-1)^{\bar{\imath}+\bar{\imath}\,\bar{j}+\bar{j}}\,\theta_i + \delta_{lj'}\,t_{k\,1'}(v)\,t_{i1}(u)(-1)^{\bar{j}+\bar{\imath}\,\bar{k}+\bar{j}\,\bar{k}+\bar{\imath}}\,\theta_j\Big).$$

To transform these terms, use (2.7) again to get the relations

$$\begin{aligned}
t_{1j}(u)\,t_{1'l}(v) = {}& \frac{1}{u-v-\kappa-1}\sum_{p=2}^{2n} t_{pj}(u)\,t_{p'l}(v)(-1)^{\bar{j}+\bar{j}\,\bar{p}}\,\theta_p\\
&- \frac{1}{u-v-\kappa-1}\,\delta_{lj'}\,t_{1'1'}(v)t_{11}(u)\,\theta_j
\end{aligned}$$

and

$$t_{k\,1'}(v)\,t_{i\,1}(u) = [t_{i\,1}(u)\,t_{k\,1'}(v)](-1)^{\bar{\imath}+\bar{k}+\bar{\imath}\,\bar{k}} = \frac{1}{u-v-\kappa-1}\,\delta_{ki'}\,t_{11}(u)\,t_{1'1'}(v)(-1)^{\bar{\imath}}\,\theta_i$$

$$- \frac{1}{u-v-\kappa-1}\sum_{p=2}^{2n} t_{k\,p'}(v)\,t_{ip}(u)(-1)^{\bar{\imath}+\bar{p}+\bar{\imath}\,\bar{p}}\,\theta_p.$$

Now combine the expressions together and observe that the actions of the operators $t_{11}(u)$ and $t_{1'1'}(v)$ in V^+ commute. Taking into account the change of the value $\kappa \mapsto \kappa+1$ for the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$, we find that the formula for the super-commutator $[t_{ij}(u), t_{kl}(v)]$ agrees with the defining relations of $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$. □

Remark 4.2. The reduction property of Proposition 4.1 should be related to a super-version of the embedding theorem for the orthogonal and symplectic Yangians proven in [20, Thm 3.1]. The arguments of that paper should apply to the super-case to lead to a Drinfeld-type presentation of the Yangians $\mathrm{Y}(\mathfrak{osp}_{1|2n})$ extending the work [2]. □

A representation V of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ is called a *highest weight representation* if there exists a nonzero vector $\xi \in V$ such that V is generated by ξ,

$$\begin{aligned} t_{ij}(u)\,\xi &= 0 && \text{for} \quad 1 \leqslant i < j \leqslant 2n+1, \qquad \text{and} \\ t_{ii}(u)\,\xi &= \lambda_i(u)\,\xi && \text{for} \quad i = 1, \dots, 2n+1, \end{aligned} \tag{4.2}$$

for some formal series

$$\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]. \tag{4.3}$$

The vector ξ is called the *highest vector* of V.

Remark 4.3. In terms of the Drinfeld presentation of the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2})$ given in [2], the highest vector conditions take the form $e_{12}(u)\,\xi = 0$ and $h(u)\,\xi = \mu(u)\,\xi$ for a certain series $\mu(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. According to the Main Theorem, the irreducible highest weight representation of $\mathrm{Y}(\mathfrak{osp}_{1|2})$ associated with $\mu(u)$ is finite-dimensional if and only if

$$\mu(u) = \frac{P(u+1)}{P(u)}$$

for some monic polynomial $P(u)$ in u. □

Proposition 4.4. *The series $\lambda_i(u)$ associated with a highest weight representation V satisfy the consistency conditions*

$$\lambda_i(u)\,\lambda_{i'}(u+n-i+1/2) = \lambda_{i+1}(u)\,\lambda_{(i+1)'}(u+n-i+1/2) \tag{4.4}$$

for $i = 1, \dots, n$. Moreover, the coefficients of the series $c(u)$ act in the representation V as the multiplications by scalars determined by $c(u) \mapsto \lambda_1(u)\,\lambda_{1'}(u+n+1/2)$.

Proof. To prove the first part, we will use the induction on n and begin with the case $n = 1$. The quasideterminant formulas for the Gaussian generators $h_i(u)$ given in Sect. 3 imply that the conditions (4.2) in the above definition can be replaced with $h_i(u)\,\xi = \lambda_i(u)\,\xi$ for $i = 1, 2, 3$. Hence, relation (3.3) of Proposition 3.1 implies the consistency condition (4.4) in the case $n = 1$.

Now suppose that $n \geqslant 2$ and introduce the subspace V^+ by (4.1). The vector ξ belongs to V^+, and applying Proposition 4.1 we find that the cyclic span $\mathrm{X}(\mathfrak{osp}_{1|2n-2})\,\xi$ is a highest weight submodule with the highest weight $(\lambda_2(u), \dots, \lambda_{2'}(u))$. By the induction hypothesis, this implies conditions (4.4) with $i = 2, \dots, n$. Furthermore, using the defining relations (2.7), we get

$$t_{12}(u)\, t_{1'2'}(v)\, \xi = \frac{1}{u - v - \kappa} \Big(t_{12}(u)\, t_{1'2'}(v) - \lambda_1(u)\, \lambda_{1'}(v) + \lambda_2(u)\, \lambda_{2'}(v) \Big)\, \xi$$

and so

$$(u - v - \kappa - 1)\, t_{12}(u)\, t_{1'2'}(v)\, \xi = \big(-\lambda_1(u)\, \lambda_{1'}(v) + \lambda_2(u)\, \lambda_{2'}(v) \big)\, \xi.$$

Setting $v = u - \kappa - 1 = u + n - 1/2$ we obtain (4.4) for $i = 1$. Finally, the last part of the proposition is obtained by using the expression for $c(u)$ implied by taking the $(1', 1')$ entry in the matrix relation (2.4). □

As Proposition 4.4 shows, the series $\lambda_i(u)$ in (4.2) with $i > n+1$ are uniquely determined by the first $n+1$ series. The corresponding $(n+1)$-tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_{n+1}(u))$ will be called the *highest weight* of V.

Given an arbitrary $(n + 1)$-tuple $\lambda(u) = (\lambda_1(u), \dots, \lambda_{n+1}(u))$ of formal series of the form (4.3), introduce the series $\lambda_i(u)$ with $i = n + 2, \dots, 2n + 1$ to satisfy the consistency conditions (4.4). Define the *Verma module* $M(\lambda(u))$ as the quotient of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ by the left ideal generated by all coefficients of the series $t_{ij}(u)$ with $1 \leqslant i < j \leqslant 2n + 1$, and $t_{ii}(u) - \lambda_i(u)$ for $i = 1, \dots, 2n + 1$. As in [4, Prop. 5.14], the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ implies that the Verma module $M(\lambda(u))$ is nonzero, and we denote by $L(\lambda(u))$ its irreducible quotient. It is clear that the isomorphism class of $L(\lambda(u))$ is determined by $\lambda(u)$.

Proposition 4.5. *Every finite-dimensional irreducible representation of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ is isomorphic to $L(\lambda(u))$ for a certain highest weight $\lambda(u) = (\lambda_1(u), \dots, \lambda_{n+1}(u))$.*

Proof. The argument is essentially the same as for the proof of the corresponding counterparts of the property for the Yangians associated with Lie algebras; cf. [4, Thm 5.1], [22, Sec. 3.2]. We online some key steps.

Suppose that V is a finite-dimensional irreducible representation of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ and introduce its subspace V^0 by

$$V^0 = \{\eta \in V \mid t_{ij}(u)\, \eta = 0, \qquad 1 \leqslant i < j \leqslant 2n + 1\}.$$

First we note that V^0 is nonzero, which follows by considering the set of weights of V, regarded as an $\mathfrak{osp}_{1|2n}$-module defined via the embedding (2.10). This set is finite and hence contains a maximal weight with respect to the standard partial ordering on the set of weights of V. A weight vector with this weight belongs to V^0.

Furthermore, we show that V^0 is stable under the action of all operators $t_{ii}(u)$. This follows by straightforward calculations similar to those used in the proof of Proposition 4.1, relying on the defining relations (2.7). In a similar way, we verify that all the operators $t_{ii}(u)$ with $i = 1, \dots, 2n + 1$ form a commuting family of operators on V^0. Hence they have a simultaneous eigenvector $\xi \in V^0$. Since the representation V is irreducible, the submodule $\mathrm{X}(\mathfrak{osp}_{1|2n})\,\xi$ must coincide with V thus proving that V is a highest weight module.

By considering the $\mathfrak{osp}_{1|2n}$-weights of V we can also conclude that the highest vector ξ of V is determined uniquely, up to a constant factor. □

Proposition 4.5 yields the first part of the Main Theorem. We will first complete the proof of the theorem in the case $n = 1$. Section 5 will be devoted to this particular case.

5. Representations of the Yangian $\mathrm{X}(\mathfrak{osp}_{1|2})$

For $n = 1$ the series $\lambda_3(u)$ is uniquely determined by $\lambda_1(u)$ and $\lambda_2(u)$ by (4.4), and so we will normally parameterize the highest weights of $\mathrm{X}(\mathfrak{osp}_{1|2})$-modules by arbitrary pairs of formal series $\lambda(u) = (\lambda_1(u), \lambda_2(u))$, omitting $\lambda_3(u)$.

5.1. Rationality conditions.

Proposition 5.1. *If the module $L(\lambda(u))$ is finite-dimensional, then*

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{(u+\alpha_1)\dots(u+\alpha_k)}{(u+\beta_1)\dots(u+\beta_k)}$$

for $k \in \mathbb{Z}_+$ and certain complex numbers α_i, β_i.

Proof. We follow the proof of a similar property for the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$; see [22, Prop. 3.3.1]. By twisting the action of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{1|2})$ on the space $L(\lambda(u))$ by the automorphism (2.5) with $f(u) = \lambda_2(u)^{-1}$, we get an $\mathrm{X}(\mathfrak{osp}_{1|2})$-module isomorphic to $L(\mu(u), 1)$ for the series $\mu(u) = \lambda_1(u)/\lambda_2(u)$. Let ξ denote the highest vector of $L(\mu(u), 1)$. Since this representation is finite-dimensional, the vectors $t_{21}^{(i)}\xi \in L(\mu(u), 1)$ with $i \geqslant 1$ are linearly dependent,

$$\sum_{i=1}^{m} c_i\, t_{21}^{(i)}\xi = 0$$

with $c_i \in \mathbb{C}$, assuming $c_m \neq 0$. Apply the operators $t_{12}^{(r)}$ for all $r \geqslant 1$ to the linear combination on the left hand side and take the coefficient of ξ. Since $t_{12}(u)\xi = 0$, we get from the defining relations (2.7) that

$$t_{12}(u)\,t_{21}(v)\,\xi = \frac{1}{u-v}\big(t_{22}(u)\,t_{11}(v) - t_{22}(v)\,t_{11}(u)\big)\xi = -\frac{\mu(u)-\mu(v)}{u-v}\,\xi.$$

Hence, writing

$$\mu(u) = 1 + \mu^{(1)}u^{-1} + \mu^{(2)}u^{-2} + \dots, \qquad \mu^{(i)} \in \mathbb{C},$$

we derive $t_{12}^{(r)}\,t_{21}^{(i)}\xi = \mu^{(r+i-1)}\,\xi$. Therefore, for all $r \geqslant 1$ we have the relations

$$\sum_{i=1}^{m} c_i\,\mu^{(r+i-1)} = 0.$$

They imply that for some coefficients b_i with $b_m = c_m$ we have

$$\mu(u)\,(c_1 + c_2\,u + \dots + c_m\,u^{m-1}) = (b_1 + b_2\,u + \dots + b_m\,u^{m-1})$$

so that $\mu(u)$ can be written as a rational function in u, as required. □

We will use the name *elementary module* for the module $L(\lambda(u))$ with

$$\lambda_1(u) = \frac{u+\alpha}{u+\beta} \qquad \text{and} \qquad \lambda_2(u) = 1 \tag{5.1}$$

and denote it by $L(\alpha, \beta)$. The Hopf algebra structure on the extended Yangian $\mathrm{X}(\mathfrak{osp}_{1|2})$ allows us to regard tensor products of the form

$$L = L(\alpha_1, \beta_1) \otimes \ldots \otimes L(\alpha_k, \beta_k) \tag{5.2}$$

as $\mathrm{X}(\mathfrak{osp}_{1|2})$-modules. Let $\xi^{(i)}$ denote the highest vector of $L(\alpha_i, \beta_i)$.

Proposition 5.2. *The* $\mathrm{X}(\mathfrak{osp}_{1|2})$*-module* $L(\lambda(u))$ *with*

$$\lambda_1(u) = \frac{(u+\alpha_1)\ldots(u+\alpha_k)}{(u+\beta_1)\ldots(u+\beta_k)} \qquad \text{and} \qquad \lambda_2(u) = 1 \tag{5.3}$$

is isomorphic to the irreducible quotient of the submodule of L*, generated by the tensor product of the highest vectors* $\xi^{(1)} \otimes \ldots \otimes \xi^{(k)}$.

Proof. The coproduct formula (2.11) implies that the cyclic span $\mathrm{X}(\mathfrak{osp}_{1|2})(\xi^{(1)} \otimes \ldots \otimes \xi^{(k)})$ is a highest weight module with the highest weight $(\lambda_1(u), \lambda_2(u))$ which implies the claim. □

We will need to find the conditions for the elementary modules to be finite-dimensional and establish some sufficient conditions for the module L in (5.2) to be irreducible.

5.2. Small Verma modules. Note that by twisting the action of the extended Yangian in a highest weight module with the highest weight (5.1) by the shift automorphism (2.8) with $a = -\beta$, we get the corresponding module whose highest weight is found by shifting $\alpha \mapsto \alpha - \beta$ and $\beta \mapsto 0$. We will now assume that $\beta = 0$. Let $\alpha \in \mathbb{C}$ and consider the Verma module $M(\lambda(u))$ with

$$\lambda_1(u) = \frac{u+\alpha}{u}, \qquad \lambda_2(u) = 1, \qquad \lambda_3(u) = \frac{u-1/2}{u+\alpha-1/2}. \tag{5.4}$$

Let K be the submodule of $M(\lambda(u))$ generated by all vectors of the form

$$t_{21}^{(r)}\xi \quad \text{for} \quad r \geqslant 2 \qquad \text{and} \qquad \big(t_{31}^{(r)} + (\alpha - 1/2)\, t_{31}^{(r-1)}\big)\xi \quad \text{for} \quad r \geqslant 3, \tag{5.5}$$

where ξ denotes the highest vector of the Verma module. Introduce the *small Verma module* $M(\alpha)$ as the quotient $M(\lambda(u))/K$. We will keep the notation ξ for the image of the highest vector of the Verma module in the quotient. More general small Verma modules of the form $M(\alpha, \beta)$ corresponding to the highest weights (5.1) are then obtained by twisting the modules $M(\alpha)$ by suitable automorphisms (2.8).

Proposition 5.3. *The module* $M(\alpha)$ *is spanned by the vectors*

$$t_{31}^{(2)\,r}\, t_{21}^{(1)\,s}\, \xi, \qquad r, s \in \mathbb{Z}_+. \tag{5.6}$$

Proof. By the Poincaré–Birkhoff–Witt theorem for the extended Yangian, the Verma module $M(\lambda(u))$ has the basis

$$t_{31}^{(k_1)} \dots t_{31}^{(k_p)} t_{21}^{(l_1)} \dots t_{21}^{(l_q)} \xi, \tag{5.7}$$

where $k_1 \geqslant \dots \geqslant k_p \geqslant 1$ and $l_1 > \dots > l_q \geqslant 1$. Hence, the induction on the length of the monomial in (5.7) reduces the argument to the verification of the property that the span of the vectors (5.6) is stable under the action of the generators $t_{31}^{(k)}$ and $t_{21}^{(l)}$.

The defining relations (2.7) imply that $[t_{31}^{(k)}, t_{31}^{(m)}] = 0$ and $[t_{31}^{(k)}, t_{21}^{(1)}] = 0$ for all k, m. Therefore, for $k \geqslant 2$ in $M(\lambda(u))$ we have

$$t_{31}^{(k)} t_{31}^{(2)\,r} t_{21}^{(1)\,s} \xi \equiv (-\alpha + 1/2)^{k-2} t_{31}^{(2)\,r+1} t_{21}^{(1)\,s} \xi \mod K.$$

The property is also clear for $k = 1$ because $t_{31}^{(1)} = 2t_{21}^{(1)\,2}$. Furthermore, since

$$[t_{21}^{(l)}, t_{31}^{(2)}] = t_{21}^{(1)} t_{31}^{(l)} - t_{31}^{(1)} t_{21}^{(l)}$$

and $[t_{21}^{(l)}, t_{21}^{(1)}] = t_{31}^{(l)}$, the property for the generators $t_{21}^{(l)}$ easily follows too. □

We will regard $M(\alpha)$ as an $\mathfrak{osp}_{1|2}$-module via the embedding (2.10). We get the weight space decomposition

$$M(\alpha) = \bigoplus_{p=0}^{\infty} M(\alpha)_{-\alpha-p},$$

where we define the weight subspaces of an arbitrary $\mathfrak{osp}_{1|2}$-module V by

$$V_\gamma = \{v \in V \mid F_{11}\, v = \gamma\, v\}. \tag{5.8}$$

Proposition 5.3 implies that

$$\dim M(\alpha)_{-\alpha-p} \leqslant \lfloor p/2 \rfloor + 1. \tag{5.9}$$

For all values $i, j \in \{1, 2, 3\}$ set $T_{ij}(u) = u\,(u + \alpha - 1/2)\, t_{ij}(u)$. We will regard the coefficients of these Laurent series in u as operators in $M(\alpha)$.

Proposition 5.4. *All operators $T_{ij}(u)$ on the small Verma module $M(\alpha)$ are polynomials in u.*

Proof. Calculating modulo K, we get

$$t_{21}(u)\xi = u^{-1} t_{21}^{(1)} \xi \qquad \text{and} \qquad t_{31}(u)\xi = \Big(u^{-1} t_{31}^{(1)} + \frac{1}{u(u + \alpha - 1/2)} t_{31}^{(2)}\Big)\xi$$

so that the claim holds for the action of the operators $T_{21}(u)$ and $T_{31}(u)$ on ξ. By acting on the vectors (5.6) of the spanning set, we note that the operator $T_{31}(u)$ commutes with $t_{31}^{(2)}$ and $t_{21}^{(1)}$, while for the operator $T_{21}(u)$ we have the relations

$$[T_{21}(u), t_{31}^{(2)}] = t_{21}^{(1)} T_{21}(u) - t_{31}^{(1)} T_{31}(u) \qquad \text{and} \qquad [T_{21}(u), t_{21}^{(1)}] = T_{31}(u).$$

Hence the property for the operators $T_{21}(u)$ and $T_{31}(u)$ follows by an obvious induction.

As a next step, consider the relations for the series $T_{11}(u)$ implied by (2.7):

$$[T_{11}(u), t_{21}^{(1)}] = T_{21}(u), \qquad [T_{11}(u), t_{31}^{(1)}] = 2\,T_{31}(u)$$

and

$$[T_{11}(u), t_{31}^{(2)}] = T_{31}(u)\big(2u + 1/2 + t_{11}^{(1)}\big) - 2\,t_{31}^{(1)}\,T_{11}(u) - t_{21}^{(1)}\,T_{21}(u).$$

Together with the relation

$$T_{11}(u)\,\xi = (u + \alpha - 1/2)(u + \alpha)\,\xi \tag{5.10}$$

they imply the claim for the operator $T_{11}(u)$. For the remaining operators the property follows from the relations

$$[t_{12}^{(1)}, T_{21}(u)] = T_{11}(u) - T_{22}(u), \quad [t_{21}^{(1)}, T_{22}(u)] = T_{32}(u) - T_{21}(u)$$

and

$$[t_{12}^{(1)}, T_{11}(u)] = T_{12}(u), \quad [t_{23}^{(1)}, T_{32}(u)] = T_{33}(u) - T_{22}(u), \quad [t_{23}^{(1)}, T_{33}(u)] = -T_{23}(u),$$

which are consequences of (2.7). □

For any $r, s \in \mathbb{Z}_+$ introduce vectors of the small Verma module $M(\alpha)$ by setting

$$\begin{aligned}\xi_{rs} &= T_{21}(-\alpha - r + 3/2) \dots T_{21}(-\alpha - 1/2)\,T_{21}(-\alpha + 1/2)\\ &\quad\times\; T_{21}(-\alpha - s + 1) \dots T_{21}(-\alpha - 1)\,T_{21}(-\alpha)\,\xi.\end{aligned}$$

We would like to show that under certain additional conditions the vectors ξ_{rs} form a basis of $M(\alpha)$; see Theorem 5.8 and Corollary 5.9 below. This will require a few lemmas where the action of the operators $T_{ij}(u)$ on these vectors is calculated.

Lemma 5.5. *In the module $M(\alpha)$ we have*

$$T_{11}(u)\,\xi_{rs} = (u + \alpha + r - 1/2)(u + \alpha + s)\,\xi_{rs}.$$

Proof. The formula holds for $\xi_{00} = \xi$ by (5.10). The defining relations (2.7) give

$$T_{11}(u)\,T_{21}(v) = \frac{u - v + 1}{u - v}\,T_{21}(v)\,T_{11}(u) - \frac{1}{u - v}\,T_{21}(u)\,T_{11}(v),$$

which implies the desired formula by an obvious induction. □

Lemma 5.6. *In the module $M(\alpha)$ for all $r \leqslant s + 1$ we have*

$$T_{21}(u)\,\xi_{rs} = \frac{(-1)^{r+1}\,(s - r + 1)(2u + 2\alpha + 2r - 1)}{(s + 1)(2s - 2r + 1)}\,\xi_{r,s+1} + \frac{2\,(u + \alpha + s)}{2s - 2r + 1}\,\xi_{r+1,s}.$$

Proof. By the definition of the vectors ξ_{rs} we have $T_{21}(-\alpha-r+1/2)\,\xi_{rs}=\xi_{r+1,s}$. Next we point out the following relation for generators of $\mathrm{X}(\mathfrak{osp}_{1|2})$:

$$(u-v-1/2)\,t_{21}(u)t_{21}(v)+(u-v+1/2)\,t_{21}(v)t_{21}(u)=t_{31}(v)t_{11}(u)-t_{31}(u)t_{11}(v).$$

It is derived by calculating the commutators $[t_{21}(u),t_{21}(v)]$ and $[t_{11}(u),t_{31}(v)]$ by (2.7) and eliminating the term $t_{11}(u)t_{31}(v)$. By Lemma 5.5 we have $T_{11}(u)\,\xi_{rs}=0$ for $u=-\alpha-r+1/2$ and $u=-\alpha-s$. Hence, we come to the relation

$$\begin{aligned}(r-s-1)\,T_{21}(-\alpha-s)\,T_{21}(-\alpha-r+1/2)\,\xi_{rs}\\ =-(r-s)\,T_{21}(-\alpha-r+1/2)\,T_{21}(-\alpha-s)\,\xi_{rs}.\end{aligned}$$

Since $T_{21}(-\alpha-s)\,\xi_{0s}=\xi_{0,s+1}$, applying the relation repeatedly, we get the formula

$$T_{21}(-\alpha-s)\,\xi_{rs}=\frac{(-1)^r\,(s-r+1)}{s+1}\,\xi_{r,s+1} \tag{5.11}$$

which is valid for all $r\leqslant s+1$. Finally, using the Lagrange interpolation formula

$$T_{21}(u)=\frac{u+\alpha+r-1/2}{r-s-1/2}\,T_{21}(-\alpha-s)-\frac{u+\alpha+s}{r-s-1/2}\,T_{21}(-\alpha-r+1/2),$$

we get the relation in the lemma. □

Lemma 5.7. *In the module $M(\alpha)$ for all $r\leqslant s$ we have*

$$\begin{aligned}T_{12}(u)\,\xi_{rs}=&-\frac{r\,(s-r+1)(2\alpha+2r-3)(u+\alpha+s)}{2\,(2s-2r+1)}\,\xi_{r-1,s}\\ &+\frac{(-1)^{r+1}\,s\,(2s+1)(\alpha+s-1)(2u+2\alpha+2r-1)}{4\,(2s-2r+1)}\,\xi_{r,s-1}.\end{aligned}$$

Proof. By Proposition 5.4, the operator $T_{12}(u)$ is a polynomial in u of degree one. As in the proof of Lemma 5.6, it will be sufficient to calculate the action of the operator for two different values $u=-\alpha-r+1/2$ and $u=-\alpha-s$, and then apply the Lagrange interpolation formula.

Recall from Sect. 3 that the coefficients of the series $h_1(u)$ and $h_2(u)$ pairwise commute. Set $d(u)=h_1(u)h_2(u+1)$. Using the defining relations (2.7), we can also write this series in the form

$$d(u)=t_{22}(u)t_{11}(u+1)+t_{12}(u)t_{21}(u+1).$$

The coefficients of the series $c(u)$ act by scalar multiplication in the small Verma module. The scalars are found from (3.4) and given by

$$c(u)\mapsto\frac{(u+1)\,(u+\alpha)}{u\,(u+\alpha+1)}. \tag{5.12}$$

On the other hand, by Lemma 5.5, the coefficients of the series $h_1(u)=t_{11}(u)$ act on each vector ξ_{rs} as multiplications by scalars depending on r and s. Hence the same property holds for the coefficients of $d(u)$ whose action is uniquely determined by the relation

$$d(u)\,d(u+1/2)=c(u)\,h_1(u+1/2)\,h_1(u+1)$$

implied by (3.4). Therefore, the action is found by

$$d(u) \mapsto \frac{(u+1/2)\,(u+\alpha)}{u\,(u+\alpha+1/2)}\, h_1(u+1/2).$$

For the corresponding polynomial operator

$$D(u) = T_{22}(u)\, T_{11}(u+1) + T_{12}(u)\, T_{21}(u+1) \tag{5.13}$$

we then have

$$D(u) = (u+1)(u+\alpha-1/2)\, T_{11}(u+1/2). \tag{5.14}$$

For any $r, s \in \mathbb{Z}_+$ we find from (5.13) by applying Lemma 5.5 that

$$\begin{aligned} D(-\alpha-r-1/2)\,\xi_{rs} &= T_{12}(-\alpha-r-1/2)\, T_{21}(-\alpha-r+1/2)\,\xi_{rs} \\ &= T_{12}(-\alpha-r-1/2)\,\xi_{r+1,s}. \end{aligned}$$

Hence using (5.14) and replacing r by $r-1$ we find

$$T_{12}(-\alpha-r+1/2)\,\xi_{rs} = -\frac{1}{4}\, r(s-r+1)(2\alpha+2r-3)\,\xi_{r-1,s},$$

which holds for $r \geqslant 1$. To extend this formula to the case $r = 0$ use Lemma 5.5 and relations

$$[T_{12}(u)\, T_{21}(v)] = \frac{1}{u-v}\, \big(T_{22}(u)\, T_{11}(v) - T_{22}(v)\, T_{11}(u)\big) \tag{5.15}$$

implied by (2.7) to derive by induction on s that $T_{12}(-\alpha+1/2)\,\xi_{0s} = 0$.

Similarly, taking $u = -\alpha - s - 1$ in (5.13) and (5.14), we get by using (5.11) that

$$T_{12}(-\alpha-s)\,\xi_{rs} = \frac{1}{4}\,(-1)^r\, s\,(2s+1)(\alpha+s-1)\,\xi_{r,s-1},$$

which holds for $r < s$. This formula extends to the case $r = s$ by applying relation (5.15) and taking into account Lemma 5.5. □

Theorem 5.8. *Suppose that $-\alpha \notin \mathbb{Z}_+$ and $-\alpha + 1/2 \notin \mathbb{Z}_+$. Then the $\mathrm{X}(\mathfrak{osp}_{1|2})$-module $M(\alpha)$ is irreducible. Moreover, the vectors ξ_{rs} with $r \leqslant s$ form a basis of $M(\alpha)$ and $\xi_{rs} = 0$ for $r > s$.*

Proof. We start by showing that all vectors ξ_{rs} with $0 \leqslant r \leqslant s$ are nonzero in $M(\alpha)$. The conditions on α and Lemma 5.7 imply that it is sufficient to verify that $\xi \neq 0$; the vector ξ_{rs} would then also have to be nonzero, because the application of suitable operators $T_{12}(v)$ to ξ_{rs} gives the vector ξ with a nonzero coefficient.

The relation $\xi = 0$ in $M(\alpha)$ would mean that ξ, as an element of the Verma module $M(\lambda(u))$ with the highest weight given in (5.4), belongs to the submodule K. That is, ξ is a linear combination of vectors of the form

$$x_r\, t_{21}^{(r)}\,\xi \quad \text{for} \quad r \geqslant 2 \qquad \text{and} \qquad y_r\, \big(t_{31}^{(r)} + (\alpha-1/2)\, t_{31}^{(r-1)}\big)\,\xi \quad \text{for} \quad r \geqslant 3,$$

with $x_r, y_r \in \mathrm{X}(\mathfrak{osp}_{1|2})$. The elements x_r and y_r must have the respective $\mathfrak{osp}_{1|2}$-weights 1 and 2 as eigenvectors of the operator F_{11}. Write these elements as linear combinations of the vectors of the Poincaré–Birkhoff–Witt basis of $\mathrm{X}(\mathfrak{osp}_{1|2})$ by using any ordering on the generators consistent with the increasing $\mathfrak{osp}_{1|2}$-weights. The right-most generators occurring in each basis monomial will have positive $\mathfrak{osp}_{1|2}$-weights. On the other hand, calculating in the Verma module $M(\lambda(u))$ we find

$$t_{12}(u)\big(t_{21}(v)-v^{-1}t_{21}^{(1)}\big)\,\xi=\frac{1}{u-v}\big(t_{22}(u)\,t_{11}(v)-t_{22}(v)\,t_{11}(u)\big)\,\xi \\ -v^{-1}\big(t_{11}(u)-t_{22}(u)\big)\,\xi=0,$$

as the coefficient of ξ equals

$$\frac{1}{u-v}\Big(\frac{v+\alpha}{v}-\frac{u+\alpha}{u}\Big)-\alpha\,u^{-1}v^{-1}=0.$$

Now combine the second family of generators of the submodule K given in (5.5) into the generating series

$$t_{31}(v)-v^{-1}\,t_{31}^{(1)}-\frac{1}{v(v+\alpha-1/2)}\,t_{31}^{(2)}$$

which can be written as the anti-commutator of $t_{21}^{(1)}$ with the series

$$t_{21}(v)-v^{-1}\,t_{21}^{(1)}-\frac{1}{v(v+\alpha-1/2)}t_{21}^{(2)}$$

whose coefficients are also generators of K. Working first with one part of the anti-commutator and using the previous calculation we get

$$t_{12}(u)\,t_{21}^{(1)}\big(t_{21}(v)-v^{-1}\,t_{21}^{(1)}\big)\,\xi=\big(t_{11}(u)-t_{22}(u)\big)\big(t_{21}(v)-v^{-1}\,t_{21}^{(1)}\big)\,\xi.$$

By the previous argument, the coefficients of this series vanish under the action of the coefficients of the series $t_{12}(w)$. Turning to the second part of the anti-commutator, we find that the expression

$$t_{12}(u)\Big(t_{21}(v)-v^{-1}\,t_{21}^{(1)}-\frac{1}{v(v+\alpha-1/2)}\,t_{21}^{(2)}\Big)\,t_{21}^{(1)}\,\xi$$

equals

$$-\Big(t_{21}(v)-v^{-1}\,t_{21}^{(1)}-\frac{1}{v(v+\alpha-1/2)}t_{21}^{(2)}\Big)\,t_{12}(u)\,t_{21}^{(1)}\,\xi \tag{5.16}$$

plus

$$\frac{1}{u-v}\big(t_{22}(u)\,t_{11}(v)-t_{22}(v)\,t_{11}(u)\big)\,t_{21}^{(1)}\,\xi-v^{-1}\big(t_{11}(u)-t_{22}(u)\big)\,t_{21}^{(1)}\,\xi \\ -\frac{1}{v(v+\alpha-1/2)}\big((u+t_{22}^{(1)})\,t_{11}(u)-t_{22}(u)(u+t_{11}^{(1)})\big)\,t_{21}^{(1)}\,\xi.$$

The expression (5.16) vanishes under the action of the coefficients of the series $t_{12}(w)$, so we only need to transform the second expression. We will do this modulo terms of the form $x_r t_{21}^{(r)}\,\xi$ with $r\geqslant 2$ which were already considered above. Note the commutators

$$[t_{11}(u),t_{21}^{(1)}]=t_{21}(u),\qquad [t_{22}(u),t_{21}^{(1)}]=t_{21}(u)-t_{32}(u).$$

Using the second relation in (3.2) and writing the Gaussian generators in terms of the $t_{ij}(u)$, we find

$$t_{21}(u)\,t_{22}(u+1/2)\,\xi = t_{32}(u)\,t_{11}(u+1/2)\,\xi.$$

Since $t_{21}(u)\xi \equiv u^{-1}\,t_{21}^{(1)}\,\xi$, we derive that $t_{32}(u)\xi \equiv (u+\alpha-1/2)^{-1}\,t_{21}^{(1)}\,\xi$. Therefore, the expression in question is then simplified by using relations

$$t_{11}(u)\,t_{21}^{(1)}\,\xi \equiv u^{-1}\,t_{21}^{(1)}\,\xi \qquad \text{and} \qquad t_{22}(u)\,t_{21}^{(1)}\,\xi \equiv \frac{u^2+(\alpha-1/2)(u+1)}{u(u+\alpha-1/2)}\,t_{21}^{(1)}\,\xi$$

and thus verifying that it reduces to zero. This completes the proof that $\xi \not\equiv 0 \mod K$.

As a next step, observe that since the vectors ξ_{rs} with $0 \leqslant r \leqslant s$ are nonzero in $M(\alpha)$, they are eigenvectors for the operator $T_{11}(u)$, whose eigenvalues are distinct as polynomials in u. Hence the vectors are linearly independent. The number of those vectors of the $\mathfrak{osp}_{1|2}$-weight $-\alpha-p$ equals $\lfloor p/2\rfloor+1$, which together with the inequality (5.9) proves that they form a basis of the weight space $M(\alpha)_{-\alpha-p}$. Thus, all vectors ξ_{rs} with $0 \leqslant r \leqslant s$ form a basis of $M(\alpha)$. Any vector ξ_{rs} with $r > s$ cannot be nonzero, because otherwise it would be an eigenvector for the operator $T_{11}(u)$ whose eigenvalue does not occur among those of the vectors in $M(\alpha)$.

Finally, we prove the irreducibility of $M(\alpha)$. As we noted in the beginning of the proof, the application of suitable operators $T_{12}(v)$ to an arbitrary basis vector ξ_{rs} yields the highest vector ξ with a nonzero coefficient. This implies that any nonzero submodule of $M(\alpha)$ must contain ξ and so coincide with $M(\alpha)$. □

Corollary 5.9. *For any $\alpha \in \mathbb{C}$ the vectors ξ_{rs} with $0 \leqslant r \leqslant s$ form a basis of $M(\alpha)$.*

Proof. Consider the vector space $\widetilde{M}(\alpha)$ with basis elements $\widetilde{\xi}_{rs}$ labelled by $r, s \in \mathbb{Z}_+$ with $0 \leqslant r \leqslant s$. Note that the coefficients of the series $t_{11}(u), t_{12}(u), t_{21}(u)$ and $c(u)$ generate the algebra $\mathrm{X}(\mathfrak{osp}_{1|2})$. Define the action of the generators $t_{11}^{(r)}, t_{21}^{(r)}$ and $t_{12}^{(r)}$ of $X(\mathfrak{osp}_{1|2})$ in $\widetilde{M}(\alpha)$ by using the formulas of Lemmas 5.5, 5.6 and 5.7, where the vectors ξ_{rs} with $r \leqslant s$ are respectively replaced with $\widetilde{\xi}_{rs}$, while all vectors ξ_{rs} with $r > s$ are replaced by 0. Also, let the coefficients of the series $c(u)$ act in $\widetilde{M}(\alpha)$ by scalar multiplication defined by (5.12). By Theorem 5.8, this assignment endows the space $\widetilde{M}(\alpha)$ with a $X(\mathfrak{osp}_{1|2})$-module structure for all $-\alpha \notin \mathbb{Z}_+$ and $-\alpha+1/2 \notin \mathbb{Z}_+$. Since the matrix elements of the generators in the basis depend polynomially on α, the same formulas define a representation of $X(\mathfrak{osp}_{1|2})$ in $\widetilde{M}(\alpha)$ for all values of α by continuity.

The formulas for the action of the generators in the basis $\widetilde{\xi}_{rs}$ show that for any $\alpha \in \mathbb{C}$ there is an $X(\mathfrak{osp}_{1|2})$-module epimorphism $\pi : M(\lambda(u)) \to \widetilde{M}(\alpha)$ defined by $\xi \mapsto \widetilde{\xi}_{00}$, where the highest weight $\lambda(u)$ of the Verma module is given by (5.4). Moreover, the submodule K of $M(\lambda(u))$ is contained in the kernel of π which gives rise to an epimorphism $\bar{\pi} : M(\alpha) \to \widetilde{M}(\alpha)$ with $\xi_{rs} \mapsto \widetilde{\xi}_{rs}$. By taking into account the dimensions of the respective $\mathfrak{osp}_{1|2}$-weight components, we conclude from (5.9) that $\bar{\pi}$ is an isomorphism. □

As was pointed out in the proof of Corollary 5.9, for any $\alpha \in \mathbb{C}$ the vectors (5.6) form a basis of $M(\alpha)$, and (5.9) is in fact an equality: $\dim M(\alpha)_{-\alpha-p} = \lfloor p/2\rfloor + 1$.

5.3. Elementary modules. The elementary modules $L(\alpha)$ can be regarded as the irreducible quotients of $M(\alpha)$. We would like to describe the structure of $L(\alpha)$ for the values of α which do not satisfy the assumptions of Theorem 5.8; that is, $-\alpha \in \mathbb{Z}_+$ or $-\alpha + 1/2 \in \mathbb{Z}_+$.

Proposition 5.10. *Suppose that $-\alpha = k \in \mathbb{Z}_+$. The linear span J of all basis vectors ξ_{rs} of $M(-k)$ with $s > k$ is an $\mathrm{X}(\mathfrak{osp}_{1|2})$-submodule. The module $L(-k)$ is isomorphic to the quotient $M(-k)/J$, and the vectors $\xi_{rs} \mod J$ with $0 \leqslant r \leqslant s \leqslant k$ form its basis.*

Proof. The formula of Lemma 5.7 gives

$$T_{12}(u)\, \xi_{r,k+1} = \frac{1}{2}\, r\, (k - r + 2)(u + 1)\, \xi_{r-1,k+1}$$

for all $r \leqslant k+1$. This implies that the subspace J of $M(-k)$ is invariant under the action of $\mathrm{X}(\mathfrak{osp}_{1|2})$. Furthermore, the formula of Lemma 5.7 also shows that the quotient $M(-k)/J$ is irreducible and hence isomorphic to $L(-k)$. □

Proposition 5.11. *Suppose that $-\alpha + 1/2 = k \in \mathbb{Z}_+$. The linear span I of all basis vectors ξ_{rs} of $M(-k+1/2)$ with $r > k$ is an $\mathrm{X}(\mathfrak{osp}_{1|2})$-submodule. The module $L(-k+1/2)$ is isomorphic to the quotient $M(-k+1/2)/I$, and the vectors $\xi_{rs} \mod I$ with $0 \leqslant r \leqslant k$ form its basis.*

Proof. The formula of Lemma 5.7 now gives

$$T_{12}(u)\, \xi_{k+1,s} = \frac{(-1)^k}{4}\, s\, (2s + 1)(u + 1)\, \xi_{k+1,s-1}$$

for all $s \geqslant k + 1$. Recalling that $\xi_{rs} = 0$ for $r > s$ we conclude that the subspace I of $M(-k+1/2)$ is invariant under the action of $\mathrm{X}(\mathfrak{osp}_{1|2})$. Furthermore, Lemma 5.7 implies that the quotient $M(-k+1/2)/I$ is irreducible and hence isomorphic to $L(-k+1/2)$. □

Corollary 5.12. *We have the following criteria.*

1. *The $\mathrm{X}(\mathfrak{osp}_{1|2})$-module $M(\alpha)$ is irreducible if and only if $-\alpha \notin \mathbb{Z}_+$ and $-\alpha + 1/2 \notin \mathbb{Z}_+$.*
2. *The $\mathrm{X}(\mathfrak{osp}_{1|2})$-module $L(\alpha)$ is finite-dimensional if and only if $-\alpha = k \in \mathbb{Z}_+$. Moreover,*

$$\dim L(-k) = \binom{k+2}{2}.$$

Proof. All parts are immediate from Theorem 5.8 and Propositions 5.10 and 5.11. □

As the above description of the elementary modules shows, they admit bases formed by $\mathfrak{osp}_{1|2}$-weight vectors. Accordingly, we can define their *characters* by using formal exponents of a variable q and using the definition (5.8) of $\mathfrak{osp}_{1|2}$-weight subspaces. Namely, we set

$$\mathrm{ch}\, V = \sum_{\gamma} \dim V_{-\gamma}\, q^{\gamma}.$$

For any given $\mu \in \mathbb{C}$ we will denote by $V(\mu)$ the irreducible highest weight module over $\mathfrak{osp}_{1|2}$ generated by a nonzero vector ξ such that $F_{11}\xi = \mu\, \xi$ and $F_{12}\xi = 0$. The module $V(\mu)$ is finite-dimensional if and only if $\mu \in \mathbb{Z}_+$. In that case, $\dim V(\mu) = 2\mu + 1$. The character of $V(\mu)$ is found by

$$\operatorname{ch} V(\mu) = \frac{q^{-\mu}}{1-q} \qquad \text{and} \qquad \operatorname{ch} V(\mu) = \frac{q^{-\mu} - q^{\mu+1}}{1-q}$$

for $\mu \notin \mathbb{Z}_+$ and $\mu \in \mathbb{Z}_+$, respectively.

Corollary 5.13. *1. The character of $M(\alpha)$ is given by*

$$\operatorname{ch} M(\alpha) = \frac{q^{\alpha}}{(1-q)(1-q^2)}.$$

2. For $-\alpha = k \in \mathbb{Z}_+$ we have

$$\operatorname{ch} L(-k) = q^{-k} \frac{(1-q^{k+1})(1-q^{k+2})}{(1-q)(1-q^2)}.$$

3. For $-\alpha + 1/2 = k \in \mathbb{Z}_+$ we have

$$\operatorname{ch} L(-k+1/2) = q^{-k+1/2} \frac{1-q^{2k+2}}{(1-q)(1-q^2)}.$$

Proof. The formulas follow by evaluating the dimensions of the weight subspaces. □

In terms of the characters of the $\mathfrak{osp}_{1|2}$-modules, we can write the above formulas as

$$\operatorname{ch} L(-k) = \sum_{p=0}^{\lfloor k/2 \rfloor} \operatorname{ch} V(k-2p)$$

and

$$\operatorname{ch} L(-k+1/2) = \sum_{p=0}^{k} \operatorname{ch} V(k-1/2-2p).$$

Finite-dimensional modules over the Lie superalgebras $\mathfrak{osp}_{1|2n}$ are known to be completely reducible; see e.g. [9, Sec. 2.2.5]. The formulas for the action of the generator F_{12} of $\mathfrak{osp}_{1|2}$ in the basis ξ_{rs} of $L(-k)$ show that there are singular vectors of the weights $k, k-2$, etc., to imply the direct sum decomposition

$$L(-k) \cong \bigoplus_{p=0}^{\lfloor k/2 \rfloor} V(k-2p).$$

Corollary 5.14. *The restriction of the module $L(\alpha)$ to the Lie superalgebra $\mathfrak{osp}_{1|2}$ is irreducible if and only if $\alpha = 0, -1$ or $1/2$.* □

Corollary 5.14 shows that the $\mathfrak{osp}_{1|2}$-modules $V(0)$, $V(1)$ and $V(-1/2)$ can be extended to $\mathrm{X}(\mathfrak{osp}_{1|2})$. The Yangian action on the three-dimensional vector representation $V(1) = \mathbb{C}^{1|2}$ which gives rise to $L(-1)$, comes from the replacement of $T(u)$ in the RTT-relation (2.3) by a transposed R-matrix $R(u)$; cf. [2]. This construction of the vector representation extends to all values of n with the explicit formula for the action given in (6.2) below.

5.4. Tensor product modules. We will now use the results of the previous sections to complete the proof of the Main Theorem in the case $n = 1$. Recall that the elementary modules of the form $L(\alpha, \beta)$ and small Verma modules $M(\alpha, \beta)$ are associated with the highest weights of the form (1.2). They can be obtained by twisting the respective modules $L(\alpha)$ and $M(\alpha)$ with the shift automorphisms (2.8). Corollary 5.12(2) implies that the module $L(\alpha, \beta)$ is finite-dimensional if and only if $\beta - \alpha \in \mathbb{Z}_+$.

For the highest weight of the form (5.3), the existence of a monic polynomial $P_1(u)$ satisfying (1.1) is equivalent to the condition that the parameters $\beta_1, \dots, \beta_k$ can be renumbered in such a way that all differences $\beta_i - \alpha_i$ with $i = 1, \dots, k$ belong to $\mathbb{Z}_+$. If this condition holds, then the tensor product module (5.2) is finite-dimensional and so is its irreducible subquotient $L(\lambda(u))$. This thus proves that the conditions of the Main Theorem are sufficient for the irreducible highest weight module to be finite-dimensional. In the rest of this section, we will show that the conditions are also necessary.

By the results of Sect. 5.2, each small Verma module $M(\alpha, \beta)$ has the basis ξ_{rs} parameterized by $r, s \in \mathbb{Z}_+$ with $r \leqslant s$ and the generators of the extended Yangian $\mathrm{X}(\mathfrak{osp}_{1|2})$ act by the rules implied by Lemmas 5.5, 5.6 and 5.7. For all $i, j \in \{1, 2, 3\}$ we now introduce the operators $T_{ij}(u) = (u + \alpha - 1/2)\,(u + \beta)\,t_{ij}(u)$, and the formulas take the following form, where the vectors ξ_{rs} with $r > s$ are equal to zero:

$$T_{11}(u)\,\xi_{rs} = (u + \alpha + r - 1/2)(u + \alpha + s)\,\xi_{rs}$$

together with

$$T_{21}(u)\,\xi_{rs} = \frac{(-1)^{r+1}\,(s - r + 1)(2u + 2\alpha + 2r - 1)}{(s + 1)(2s - 2r + 1)}\,\xi_{r,s+1} + \frac{2\,(u + \alpha + s)}{2s - 2r + 1}\,\xi_{r+1,s}$$

and

$$\begin{aligned} T_{12}(u)\,\xi_{rs} = &-\frac{r\,(s - r + 1)(2\alpha - 2\beta + 2r - 3)(u + \alpha + s)}{2\,(2s - 2r + 1)}\,\xi_{r-1,s} \\ &+ \frac{(-1)^{r+1}\,s\,(2s + 1)(\alpha - \beta + s - 1)(2u + 2\alpha + 2r - 1)}{4\,(2s - 2r + 1)}\,\xi_{r,s-1}. \end{aligned}$$

The coefficients of the series $c(u)$ act on $M(\alpha, \beta)$ by scalar multiplication, with the scalars found from (3.4) and given by

$$c(u) \mapsto \frac{(u + \alpha)\,(u + \beta + 1)}{(u + \alpha + 1)\,(u + \beta)}.$$

By Corollary 5.12(1), the $\mathrm{X}(\mathfrak{osp}_{1|2})$-module $M(\alpha, \beta)$ is irreducible if and only if $\beta - \alpha \notin \mathbb{Z}_+$ and $\beta - \alpha + 1/2 \notin \mathbb{Z}_+$. In the cases where $M(\alpha, \beta)$ is reducible, the above formulas for the action of $T_{ij}(u)$ extend to the irreducible quotients $L(\alpha, \beta)$ with the assumption that the vectors ξ_{rs} belonging to the maximal proper submodule of $M(\alpha, \beta)$ are understood as equal to zero.

Our argument will rely on certain sufficient conditions for the tensor product of the form (5.2) to be irreducible as an $\mathrm{X}(\mathfrak{osp}_{1|2})$-module. To state the conditions we will use a notation involving multisets of complex numbers $\{z_1, \dots, z_l\}$. For such a multiset we will write $\{z_1, \dots, z_l\}_+$ to denote the multiset formed by all elements z_i which belong to $\mathbb{Z}_+$.

Theorem 5.15. *Suppose that for each $h = 1, \dots, k - 1$ the following holds:*

1. *If the multiset* $\{\beta_h-\alpha_i,\ \beta_i-\alpha_h \mid i=h,\dots,k\}_+$ *is not empty, then* $\beta_h-\alpha_h$ *is a minimal element of the multiset* $\{\beta_h-\alpha_i,\ \beta_i-\alpha_h,\ \beta_h-\alpha_i+1/2,\ \beta_i-\alpha_h+1/2 \mid i=h,\dots,k\}_+$.
2. *If the multiset* $\{\beta_h-\alpha_i,\ \beta_i-\alpha_h \mid i=h,\dots,k\}_+$ *is empty and the multiset* $\{\beta_h-\alpha_i+1/2,\ \beta_i-\alpha_h+1/2 \mid i=h,\dots,k\}_+$ *is not empty, then* $\beta_h-\alpha_h+1/2$ *is a minimal element of this multiset.*

Then the $\mathrm{X}(\mathfrak{osp}_{1|2})$*-module* L *defined in* (5.2) *is irreducible.*

Proof. We let $\xi_{rs}^{(l)}$ denote the basis vectors of the module $L(\alpha_l,\beta_l)$ with the highest vector $\xi^{(l)}$. Proposition 5.4 implies that all operators

$$T_{ij}(u)=\prod_{l=1}^{k}(u+\alpha_l-1/2)\,(u+\beta_l)\,t_{ij}(u)$$

acting in the module L are polynomials in u.

As a first step, we will show by induction on k that any vector $\zeta\in L$ satisfying the condition $T_{12}(u)\,\zeta=0$ is proportional to $\xi^{(1)}\otimes\ldots\otimes\xi^{(k)}$. The case $k=1$ is clear so we will suppose that $k\geqslant 2$. We may assume that such a vector ζ is an $\mathfrak{osp}_{1|2}$-weight vector and write

$$\zeta=\sum_{r,s}\xi_{rs}^{(1)}\otimes\zeta_{rs},\qquad \zeta_{rs}\in L(\alpha_2,\beta_2)\otimes\ldots\otimes L(\alpha_k,\beta_k).$$

The sum is finite and taken over the pairs $r\leqslant s$ with the condition that the $\xi_{rs}^{(1)}$ are basis vectors of $L(\alpha_1,\beta_1)$. Let p be the maximal sum $r+s$ for which there are nonzero elements ζ_{rs} in the expression. By taking the coefficient of $\xi_{rs}^{(1)}$ with $r+s=p$ in the relation $T_{12}(u)\,\zeta=0$, we get $T_{12}(u)\,\zeta_{rs}=0$. By the induction hypothesis, ζ_{rs} is proportional to the vector $\xi'=\xi^{(2)}\otimes\ldots\otimes\xi^{(k)}$. Furthermore, the defining relations (2.7) give

$$T_{12}(u)\,T_{11}(v)=\frac{u-v-1}{u-v}\,T_{11}(v)\,T_{12}(u)+\frac{1}{u-v}\,T_{11}(u)\,T_{12}(v).$$

Hence, for any value of v, the vector $T_{11}(v)\,\zeta$ is also annihilated by the operator $T_{12}(u)$. Note that the basis vectors $\xi_{rs}^{(1)}$ are eigenvectors for the operator $T_{11}(v)$ with distinct eigenvalues, as polynomials in v. This implies that by taking a suitable value of v, we can find a linear combination of the vectors $T_{11}(v)^m\,\zeta$ with $m=0,1,\ldots$ to get an $\mathfrak{osp}_{1|2}$-weight vector ζ of the form

$$\zeta=\xi_{r_0s_0}^{(1)}\otimes\xi'+\sum_{r+s<p}\xi_{rs}^{(1)}\otimes\zeta_{rs},\tag{5.17}$$

with $r_0+s_0=p$ such that $T_{12}(u)\,\zeta=0$.

Next we will show that the condition $r_0<s_0$ is impossible in such a vector. Indeed, if this condition holds, consider the coefficient of the vector $\xi_{r_0,s_0-1}^{(1)}\otimes\xi'$ in the relation $T_{12}(u)\,\zeta=0$. This coefficient can only arise from the terms

$$T_{12}(u)\,\xi_{r_0,s_0}^{(1)}\otimes T_{22}(u)\,\xi'\pm T_{11}(u)\,\xi_{r_0,s_0-1}^{(1)}\otimes T_{12}(u)\,\zeta_{r_0,s_0-1}$$

with the sign depending on the parity of the vector $\xi^{(1)}_{r_0,s_0-1}$. The $\mathfrak{osp}_{1|2}$-weight condition implies that

$$\zeta_{r_0,s_0-1} = \sum_{l=2}^{k} c_l\, \xi^{(2)} \otimes \ldots \otimes \xi^{(l)}_{01} \otimes \ldots \otimes \xi^{(k)}$$

for some constants $c_l \in \mathbb{C}$. We have

$$T_{12}(u)\, \zeta_{r_0,s_0-1} = \sum_{l=2}^{k} \pm c_l\, T_{11}(u)\, \xi^{(2)} \otimes \ldots \otimes T_{12}(u)\, \xi^{(l)}_{01} \otimes \ldots \otimes T_{22}(u)\, \xi^{(k)}.$$

By using the formulas for the action of the operators $T_{ij}(u)$ and equating the coefficient in question to zero, we get

$$\begin{aligned} &b\,(u+\alpha_1+r_0-1/2)\prod_{i=2}^{k}(u+\alpha_i-1/2)(u+\beta_i) \\ &\quad + (u+\alpha_1+r_0-1/2)(u+\alpha_1+s_0-1)\sum_{l=2}^{k} b_l \prod_{i=2}^{l-1}(u+\alpha_i-1/2)(u+\alpha_i) \\ &\quad \times (u+\alpha_l-1/2)\prod_{i=l+1}^{k}(u+\alpha_i-1/2)(u+\beta_i) = 0, \end{aligned}$$

where b_l are some constants, while b is a nonzero constant, because of the condition $s_0 \leqslant \beta_1 - \alpha_1$ in the case $\beta_1 - \alpha_1 \in \mathbb{Z}_+$ implied by Proposition 5.10. By cancelling the common factors and setting $u = -\alpha_1 - s_0 + 1$ we get

$$\prod_{i=2}^{k}(\beta_i - \alpha_1 - s_0 + 1) = 0.$$

It follows from this relation that the multiset $\{\beta_i - \alpha_1 \mid i = 1, \ldots, k\}_+$ is not empty, because $\beta_i - \alpha_1 = s_0 - 1 \in \mathbb{Z}_+$ for some $i \in \{2, \ldots, k\}$. By assumption (1) of the theorem, we have $\beta_1 - \alpha_1 \in \mathbb{Z}_+$ and $\beta_1 - \alpha_1 \leqslant \beta_i - \alpha_1$. However, this makes a contradiction, as by Proposition 5.10 we must have $s_0 \leqslant \beta_1 - \alpha_1$.

Excluding the condition $r_0 < s_0$ in (5.17), we show next that the condition $r_0 = s_0 \geqslant 1$ is impossible either. If this condition holds, consider the coefficient of the vector $\xi^{(1)}_{r_0-1,r_0} \otimes \xi'$ in the relation $T_{12}(u)\, \zeta = 0$. This coefficient can only arise from the terms

$$T_{12}(u)\, \xi^{(1)}_{r_0,r_0} \otimes T_{22}(u)\, \xi' \pm T_{11}(u)\, \xi^{(1)}_{r_0-1,r_0} \otimes T_{12}(u)\, \zeta_{r_0-1,r_0}.$$

By the $\mathfrak{osp}_{1|2}$-weight condition,

$$\zeta_{r_0-1,r_0} = \sum_{l=2}^{k} c_l\, \xi^{(2)} \otimes \ldots \otimes \xi^{(l)}_{01} \otimes \ldots \otimes \xi^{(k)}$$

for some constants $c_l \in \mathbb{C}$. Calculating as in the previous case, we now come to the relation

$$b\,(u+\alpha_1+r_0)\prod_{i=2}^{k}(u+\alpha_i-1/2)(u+\beta_i)$$
$$+\,(u+\alpha_1+r_0-3/2)(u+\alpha_1+r_0)\sum_{l=2}^{k}b_l\prod_{i=2}^{l-1}(u+\alpha_i-1/2)(u+\alpha_i)$$
$$\times\,(u+\alpha_l-1/2)\prod_{i=l+1}^{k}(u+\alpha_i-1/2)(u+\beta_i)=0,$$

where b_l are some constants, while b is a nonzero constant. The latter property holds because of the condition $r_0 \leqslant \beta_1-\alpha_1+1/2$ in the case $\beta_1-\alpha_1+1/2 \in \mathbb{Z}_+$ implied by Proposition 5.11. Cancel the common factors and set $u=-\alpha_1-r_0+3/2$ to get

$$\prod_{i=2}^{k}(\beta_i-\alpha_1-r_0+3/2)=0.$$

This means that for some $i \in \{2,\dots,k\}$ we have $\beta_i-\alpha_1+1/2=r_0-1 \in \mathbb{Z}_+$. If the multiset $\{\beta_1-\alpha_j,\ \beta_j-\alpha_1 \mid j=1,\dots,k\}_+$ is not empty, then by assumption (1) of the theorem, we have $\beta_1-\alpha_1 \in \mathbb{Z}_+$ and $\beta_1-\alpha_1 \leqslant \beta_i-\alpha_1+1/2$. This is impossible because by Proposition 5.10 we must have $r_0 \leqslant \beta_1-\alpha_1$. Hence assumption (2) of the theorem for $h=1$ should apply, and we have $\beta_1-\alpha_1+1/2 \in \mathbb{Z}_+$ together with the inequality

$$\beta_1-\alpha_1+1/2 \leqslant \beta_i-\alpha_1+1/2.$$

This makes a contradiction, as by Proposition 5.11 we must have $r_0 \leqslant \beta_1-\alpha_1+1/2$.

We have thus showed that any vector $\zeta \in L$ with $T_{12}(u)\,\zeta=0$ is proportional to $\xi^{(1)}\otimes\xi'$. By looking at the set of $\mathfrak{osp}_{1|2}$-weights of any nonzero submodule of L we derive that such a submodule must contain a nonzero vector ζ with $T_{12}(u)\,\zeta=0$, and so contain the vector $\xi^{(1)}\otimes\xi'$. It remains to prove this vector is cyclic in L.

Consider the vector space L^* dual to L which is spanned by all linear maps $\sigma: L\to\mathbb{C}$ satisfying the condition that the linear span of the vectors $\eta\in L$ such that $\sigma(\eta)\neq 0$, is finite-dimensional. Equip L^* with an $\mathrm{X}(\mathfrak{osp}_{1|2})$-module structure by setting

$$(x\,\sigma)(\eta)=\sigma(\omega(x)\,\eta)\quad\text{for}\quad x\in\mathrm{X}(\mathfrak{osp}_{1|2})\quad\text{and}\quad \sigma\in L^*,\ \eta\in L,\tag{5.18}$$

where ω is the anti-automorphism of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2})$ defined in (2.9). It is easy to verify that L^* is isomorphic to the tensor product module

$$L(-\beta_1,-\alpha_1)\otimes\dots\otimes L(-\beta_k,-\alpha_k).\tag{5.19}$$

Moreover, the highest vector of the module $L(-\beta_i,-\alpha_i)$ can be identified with the dual basis vector $\xi^{(i)*}$. Suppose now that the submodule $N=\mathrm{X}(\mathfrak{osp}_{1|2})(\xi^{(1)}\otimes\dots\otimes\xi^{(k)})$ of L is proper and consider its annihilator

$$\operatorname{Ann}N=\{\rho\in L^*\mid \rho(\eta)=0\quad\text{for all}\quad \eta\in N\}.\tag{5.20}$$

Then $\operatorname{Ann}N$ is a nonzero submodule of L^*, which does not contain the vector $\xi^{(1)*}\otimes\dots\otimes\xi^{(k)*}$. However, this contradicts the claim verified in the first part of the proof, because the conditions on the parameters α_i and β_i stated in the theorem will remain satisfied after we replace each α_i by $-\beta_i$ and each β_i by $-\alpha_i$. □

Proposition 5.16. *Suppose that the* $\mathrm{X}(\mathfrak{osp}_{1|2})$*-module* $L(\lambda(u))$ *with the highest weight* (5.3) *is finite-dimensional. Then for any nonnegative integers* $l_1, \dots, l_k$ *and* $m_1, \dots, m_k$ *the module* $L(\lambda^+(u))$ *with the highest weight*

$$\lambda_1^+(u) = \frac{(u+\alpha_1-l_1)\dots(u+\alpha_k-l_k)}{(u+\beta_1+m_1)\dots(u+\beta_k+m_k)} \quad \text{and} \quad \lambda_2^+(u) = 1 \tag{5.21}$$

is also finite-dimensional.

Proof. The highest weight module $L(\lambda^+(u))$ is isomorphic to an irreducible subquotient of the finite-dimensional module

$$L(\lambda(u)) \otimes L(\alpha_1-l_1, \alpha_1) \otimes \dots \otimes L(\alpha_k-l_k, \alpha_k) \otimes L(\beta_1, \beta_1+m_1) \\ \otimes \dots \otimes L(\beta_k, \beta_k+m_k)$$

and hence is finite-dimensional. □

We now return to proving the Main Theorem in the case $n = 1$. Let the irreducible highest weight module $L(\lambda(u))$ with the highest weight (5.3) be finite-dimensional. To argue by contradiction, suppose that it is impossible to renumber the parameters $\beta_1, \dots, \beta_k$ in such a way that all differences $\beta_i - \alpha_i$ with $i = 1, \dots, k$ belong to $\mathbb{Z}_+$. By Proposition 5.16, all modules $L(\lambda^+(u))$ with the highest weight of the form (5.21) are also finite-dimensional. It is possible to choose nonnegative integers l_i and m_i to ensure that the assumptions of Theorem 5.15 are satisfied by the shifted parameters $\alpha_i' = \alpha_i - l_i$ and $\beta_i' = \beta_i + m_i$, after a possible renumbering. This can be done by induction, beginning with the multiset

$$\{\beta_1 - \alpha_i,\ \beta_i - \alpha_1 \mid i = 1, \dots, k\}$$

and renumbering the parameters α_i and β_i, if necessary, to ensure that $\beta_1 - \alpha_1$ is a minimal element of the multiset

$$\{\beta_1 - \alpha_i,\ \beta_i - \alpha_1 \mid i = 1, \dots, k\}_+ \tag{5.22}$$

if it is nonempty. Then assumption (1) of the theorem for $h = 1$ is achieved by suitable shifts $\alpha_i \mapsto \alpha_i - l_i$ and $\beta_i \mapsto \beta_i + m_i$ for $i = 2, \dots, k$. If the multiset (5.22) is empty, then assumption (2) for $h = 1$ is achieved by a suitable renumbering of the parameters α_i and β_i. Then we continue in the same way to consider the multisets for $h = 2$, etc. As a result, by Theorem 5.15, the module $L(\lambda^+(u))$ is isomorphic to the tensor product of the corresponding elementary modules. Since it is finite-dimensional, all new differences $\beta_i' - \alpha_i'$ must be nonnegative integers due to Corollary 5.12 (2).

This argument implies, that all the differences $\beta_i - \alpha_i$ of the original parameters may be assumed to be integers. Moreover, we can apply some shifts as given in Proposition 5.16, to further suppose that $\beta_i - \alpha_i \in \mathbb{Z}_+$ for $i = 1, \dots, k-1$, while $\alpha_k - \beta_k \in 1 + \mathbb{Z}_+$, and that it is impossible to renumber the parameters to make all the differences $\beta_i - \alpha_i$ nonnegative integers.

Now consider all the parameters α_i and β_i which belong to the $\mathbb{Z}$-coset in $\mathbb{C}$ containing α_k and β_k. Renumbering them, if necessary, suppose that they correspond to $i = d + 1, \dots, k$ for some $d \in \{0, 1, \dots, k-1\}$. After a further renumbering to satisfy the assumptions of Theorem 5.15, we obtain that the $\mathrm{X}(\mathfrak{osp}_{1|2})$-module

$$L^{(2)} = L(\alpha_{d+1}, \beta_{d+1}) \otimes \dots \otimes L(\alpha_k, \beta_k)$$

is irreducible. Similarly, by applying suitable shifts of Proposition 5.16 to the remaining parameters α_i, β_i with $i = 1, \dots, d$, and possible relabelling, we may assume that they satisfy the assumptions of Theorem 5.15 and so the $\mathrm{X}(\mathfrak{osp}_{1|2})$-module

$$L^{(1)} = L(\alpha_1, \beta_1) \otimes \dots \otimes L(\alpha_d, \beta_d)$$

is also irreducible. If the tensor product $L = L^{(1)} \otimes L^{(2)}$ turns out to be irreducible, then we arrive at a contradiction, because the module $L(\alpha_k, \beta_k)$ is infinite-dimensional. So we will suppose that L is not irreducible and denote by μ the $\mathfrak{osp}_{1|2}$-weight of the vector $\xi^{(1)} \otimes \dots \otimes \xi^{(k)}$. Consider the multiset

$$\{\beta_i - \alpha_j + 1/2 \mid 1 \leqslant i \leqslant d, \quad d+1 \leqslant j \leqslant k\}_+ \tag{5.23}$$

and let p_0 denote its minimal element, if the multiset is nonempty, or set $p_0 = +\infty$ otherwise.

Lemma 5.17. *The $\mathfrak{osp}_{1|2}$-weight component $N_{\mu-p}$ of the cyclic span*

$$N = \mathrm{X}(\mathfrak{osp}_{1|2})(\xi^{(1)} \otimes \dots \otimes \xi^{(k)})$$

coincides with $L_{\mu-p}$ for all $0 \leqslant p \leqslant 2\,p_0$.

Proof. Equip L^* with an $\mathrm{X}(\mathfrak{osp}_{1|2})$-module structure by using (5.18). By considering the annihilator Ann N, as defined by (5.20), it will be sufficient to show that any vector $\zeta \in L^*$ of the $\mathfrak{osp}_{1|2}$-weight $\mu - p$ with the property $t_{12}(u)\,\zeta = 0$ is proportional to the vector $\xi^{(1)*} \otimes \dots \otimes \xi^{(k)*}$. As before, we will identify L^* with the tensor product module (5.19) and denote by $\check{\xi}^{(i)}$ the highest vector of the elementary module $L(-\beta_i, -\alpha_i)$. We will now follow the first part of the proof of Theorem 5.15 to derive by a reverse induction on $l \in \{1, \dots, k\}$, beginning with $l = k$, that any vector

$$\zeta^{(l)} \in L(-\beta_l, -\alpha_l) \otimes \dots \otimes L(-\beta_k, -\alpha_k)$$

of the $\mathfrak{osp}_{1|2}$-weight $\mu - p$ with the property $t_{12}(u)\,\zeta^{(l)} = 0$ is proportional to $\check{\xi}^{(l)} \otimes \dots \otimes \check{\xi}^{(k)}$. This is clear for the values $l = d+1, \dots, k$, because the assumptions of Theorem 5.15 are satisfied by the corresponding parameters.

Now suppose that $l \in \{1, \dots, d\}$ and repeat the argument of the first part of the proof of Theorem 5.15 to come to the expression

$$\zeta^{(l)} = \check{\xi}^{(l)}_{r_0 s_0} \otimes \xi' + \sum_{r+s<p} \check{\xi}^{(l)}_{rs} \otimes \zeta_{rs},$$

analogous to (5.17), where $r_0 + s_0 = p$ and $\xi' = \check{\xi}^{(l+1)} \otimes \dots \otimes \check{\xi}^{(k)}$. Arguing as in that proof, we find that the condition $r_0 < s_0$ is impossible, leading to the only possibility that $r_0 = s_0 \geqslant 1$. In this case, with our conditions of the parameters, we must have $p = 2r_0$ and

$$\beta_l - \alpha_j + 1/2 = r_0 - 1 \tag{5.24}$$

for some $d+1 \leqslant j \leqslant k$. Since $r_0 - 1 \in \mathbb{Z}_+$, relation (5.24) implies that p_0 has a finite value and $r_0 > p_0$. This makes a contradiction, because $p = 2r_0 \leqslant 2p_0$ by the assumption, thus completing the proof of the lemma. ◻

For any $s \in \mathbb{Z}_+$ set $\eta_s = \xi^{(1)} \otimes \ldots \otimes \xi^{(k-1)} \otimes \xi^{(k)}_{0s} \in L(\alpha_1, \beta_1) \otimes \ldots \otimes L(\alpha_k, \beta_k)$.

Lemma 5.18. *In the tensor product module, for any $s \in \mathbb{Z}_+$ we have*

$$T_{21}(-\alpha_k - s)\,\eta_s = \prod_{i=1}^{k-1} (\beta_i - \alpha_k - s)(\alpha_i - \alpha_k - s - 1/2)\,\eta_{s+1}, \tag{5.25}$$

$$T_{12}(u)\,\eta_s = \frac{s}{2}\,(\beta_k - \alpha_k - s + 1) \prod_{i=1}^{k-1} (u + \alpha_i - 1/2)(u + \alpha_i)\,\eta_{s-1},$$

and

$$T_{11}(u)\,\eta_s = (u + \alpha_k - 1/2)(u + \alpha_k + s) \prod_{i=1}^{k-1} (u + \alpha_i - 1/2)(u + \alpha_i)\,\eta_s.$$

Proof. All relations are immediate from the coproduct rule (2.11) and the formulas for the action of the generators of the extended Yangian in the basis ξ_{rs} of the elementary module $L(\alpha, \beta)$, which were recalled in the beginning of this section. In particular, for (5.25) we take into account the relations $T_{11}(-\alpha_k - s)\,\xi^{(k)}_{0s} = 0$ and $T_{21}(-\alpha_k - s)\,\xi^{(k)}_{0s} = \xi^{(k)}_{0,s+1}$ in $L(\alpha_k, \beta_k)$. □

Observe that the numerical coefficient on the right hand side of (5.25) is nonzero for any values of s outside the multisets

$$\{\beta_i - \alpha_k \mid i = d+1, \ldots, k-1\}_+ \quad \text{and} \quad \{\alpha_i - \alpha_k - 1/2 \mid i = 1, \ldots, d\}_+. \tag{5.26}$$

On the other hand, recalling that p_0 is the minimal element of the multiset (5.23) when it is nonempty, note that we can use the shifts of the parameters α_i, β_i with $i = 1, \ldots, d$ as in Proposition 5.16 to keep the assumptions of Theorem 5.15 satisfied. The module $L^{(1)}$ with the shifted parameters remains irreducible, while we can make the value of p_0 arbitrarily large. It will be sufficient to make p_0 large enough for the elements of both multisets in (5.26) not to exceed $2p_0$, noting that the elements of the second multiset can only decrease after the shifts $\alpha_i \mapsto \alpha_i - l_i$ for $i = 1, \ldots, d$.

The $\mathfrak{osp}_{1|2}$-weight of the vector η_s equals $\mu - s$, and hence, by Lemma 5.17, all vectors η_s with $s \leqslant 2p_0$ belong to the cyclic span $N = \mathrm{X}(\mathfrak{osp}_{1|2})\,\eta_0$. This property extends to all values $s \in \mathbb{Z}_+$ by relation (5.25) of Lemma 5.18, because the numerical coefficient of η_{s+1} does not vanish for $s > 2p_0$. The remaining two relations of Lemma 5.18 imply that the images of the vectors η_s in the irreducible quotient $L(\lambda(u))$ of N are linearly independent. Hence, $L(\lambda(u))$ is infinite-dimensional, as it contains an infinite family of linearly independent vectors. This contradiction completes the proof of the second part of the Main Theorem for $n = 1$. The last part concerning representations of the Yangian $\mathrm{Y}(\mathfrak{osp}_{1|2})$ is immediate from the decomposition (2.6); cf. [4, Sec. 5.3].

Comparing the irreducibility conditions with those for the evaluation modules over the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$ (see e.g. [22, Sec. 3.3]), note that it is not possible, in general, to renumber the parameters of the given highest weight (5.3) to satisfy the assumptions of Theorem 5.15. In fact, not every module $L(\lambda(u))$ is isomorphic to a tensor product module of the form (5.2), as illustrated by the following example.

Example 5.19. To describe the $\mathrm{X}(\mathfrak{osp}_{1|2})$-module $L(\lambda(u))$ with

$$\lambda_1(u) = \frac{(u-1)(u-5/2)}{u\,(u-3/2)}, \qquad \lambda_2(u) = 1,$$

consider the tensor product $L = L(-1,0) \otimes L(-5/2,-3/2)$ of two three-dimensional modules. Note that its parameters do not satisfy the assumptions of Theorem 5.15. The module L turns out to have a proper submodule K which is generated by the vector

$$\zeta = \xi^{(1)}_{11} \otimes \xi^{(2)} + 3\,\xi^{(1)}_{01} \otimes \xi^{(2)}_{01} - \xi^{(1)} \otimes \xi^{(2)}_{11}.$$

The submodule K is one-dimensional, isomorphic to a highest weight module $L(\mu(u))$ with the components

$$\mu_1(u) = \mu_2(u) = \frac{(u-1/2)(u-5/2)}{(u-3/2)^2}.$$

The module $L(\lambda(u))$ is isomorphic to the quotient L/K with $\dim L(\lambda(u)) = 8$ and so does not admit a tensor product decomposition of the form (5.2). □

To conclude this section, we note that by analysing submodules of reducible small Verma modules $M(\alpha,\beta)$, we can obtain explicit constructions of some modules $L(\lambda(u))$ beyond the elementary modules. In particular, for any $k \in \mathbb{Z}_+$ the submodule of $M(-k)$ generated by the vector $\xi_{0,k+1}$ is isomorphic to the highest weight module $L(\lambda(u))$ with

$$\lambda_1(u) = \frac{u+1}{u} \qquad \text{and} \qquad \lambda_2(u) = \frac{(u+1/2)(u-k-1)}{u(u-k-1/2)}.$$

The vectors ξ_{rs} with $r \leqslant s$ and $s > k$ form its basis, and the action of the generators is described in Sect. 5.2. The character of $L(\lambda(u))$, as defined in Sect. 5.3, is found by

$$\mathrm{ch}\, L(\lambda(u)) = \frac{q + q^2 - q^{k+3}}{(1-q)(1-q^2)}.$$

6. Proof of the Main Theorem: General Case

We will complete the proof of the Main Theorem by the induction on n taking the case $n = 1$ considered in Sect. 5 as the induction base. Suppose that $n \geqslant 2$. Recall that the Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ for the general linear Lie algebra $\mathfrak{gl}_n$ is defined as a unital associative algebra with countably many generators $t^{(1)\circ}_{ij},\ t^{(2)\circ}_{ij}, \dots$ where $1 \leqslant i, j \leqslant n$, and the defining relations

$$(u-v)\,[t^{\circ}_{ij}(u), t^{\circ}_{kl}(v)] = t^{\circ}_{kj}(u)\, t^{\circ}_{il}(v) - t^{\circ}_{kj}(v)\, t^{\circ}_{il}(u)$$

written in terms of the series

$$t^{\circ}_{ij}(u) = \delta_{ij} + t^{(1)\circ}_{ij} u^{-1} + t^{(2)\circ}_{ij} u^{-2} + \dots \in \mathrm{Y}(\mathfrak{gl}_n)[[u^{-1}]];$$

see [22] for a detailed exposition of the algebraic structure and representation of these algebras. The Yangian $\mathrm{Y}(\mathfrak{gl}_n)$ can be regarded as a subalgebra of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ via the embedding

$$\mathrm{Y}(\mathfrak{gl}_n) \hookrightarrow \mathrm{X}(\mathfrak{osp}_{1|2n}), \qquad t^{\circ}_{ij}(u) \mapsto t_{ij}(-u) \quad \text{for} \quad 1 \leqslant i, j \leqslant n. \tag{6.1}$$

The cyclic span $\mathrm{Y}(\mathfrak{gl}_n)\,\xi \subset L(\lambda(u))$ is a highest weight module over $\mathrm{Y}(\mathfrak{gl}_n)$. Its highest weight is the n-tuple $(\lambda_1(-u), \dots, \lambda_n(-u))$. If $\dim L(\lambda(u)) < \infty$, the corresponding conditions for finite-dimensional highest weight representations of $\mathrm{Y}(\mathfrak{gl}_n)$ must be satisfied; see [22, Sec. 3.4]. This implies conditions (1.1) of the Main Theorem for $i = 1, \dots, n-1$.

Furthermore, by Proposition 4.1, the subspace $L(\lambda(u))^+$ is a module over the extended Yangian $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$. The vector ξ generates a highest weight $\mathrm{X}(\mathfrak{osp}_{1|2n-2})$-module with the highest weight $(\lambda_2(u), \dots, \lambda_{n+1}(u))$. Since this module is finite-dimensional, conditions (1.1) hold for $i = 2, \dots, n$ by the induction hypothesis. This completes the proof of the necessity of the conditions.

Now suppose that conditions (1.1) hold and derive that the corresponding module $L(\lambda(u))$ is finite-dimensional. The n-tuple of Drinfeld polynomials $(P_1(u), \dots, P_n(u))$ determines the highest weight $\lambda(u)$ up to a simultaneous multiplication of all components $\lambda_i(u)$ by a series $f(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. This operation corresponds to twisting the action of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ on $L(\lambda(u))$ by the automorphism (2.5). Hence, it suffices to prove that a particular module $L(\lambda(u))$ corresponding to a given set of Drinfeld polynomials is finite-dimensional.

Suppose that $L(\nu(u))$ and $L(\mu(u))$ are the irreducible highest weight modules with the highest weights

$$\nu(u) = \big(\nu_1(u), \dots, \nu_{n+1}(u)\big) \qquad \text{and} \qquad \mu(u) = \big(\mu_1(u), \dots, \mu_{n+1}(u)\big).$$

By the coproduct rule (2.11), the cyclic span $\mathrm{X}(\mathfrak{osp}_{1|2n})(\xi \otimes \xi')$ of the tensor product of the respective highest vectors of $L(\nu(u))$ and $L(\mu(u))$ is a highest weight module with the highest weight

$$\big(\nu_1(u)\,\mu_1(u), \dots, \nu_{n+1}(u)\,\mu_{n+1}(u)\big).$$

This observation implies that the cyclic span corresponds to the set of Drinfeld polynomials $(P_1(u)\,Q_1(u), \dots, P_n(u)\,Q_n(u))$, where the $P_i(u)$ and $Q_i(u)$ are the Drinfeld polynomials for $L(\nu(u))$ and $L(\mu(u))$, respectively. Therefore, we only need to establish the sufficiency of conditions (1.1) for the *fundamental representations* of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ associated with the n-tuples of Drinfeld polynomials such that $P_j(u) = 1$ for all $j \neq i$ and $P_i(u) = u + b$ for a certain $i \in \{1, \dots, n\}$ and $b \in \mathbb{C}$; cf. [7]. Moreover, it is enough to take one particular value of $b \in \mathbb{C}$; the general case will then follow by twisting the action of the algebra $\mathrm{X}(\mathfrak{osp}_{1|2n})$ in such representations by automorphisms of the form (2.8).

Consider the vector representation of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ on $\mathbb{C}^{1|2n}$ defined by

$$t_{ij}(u) \mapsto \delta_{ij} + u^{-1}\,e_{ij}(-1)^{\bar{i}} - (u+\kappa)^{-1}\,e_{j'i'}(-1)^{\bar{i}\bar{j}}\,\theta_i\theta_j. \tag{6.2}$$

The homomorphism property follows from (2.3) by applying the standard transposition to one copy of $\operatorname{End}\mathbb{C}^{1|2n}$ in the Yang–Baxter equation satisfied by $R(u)$. Now use the coproduct (2.11) and suitable automorphisms (2.8) to equip the tensor product space $(\mathbb{C}^{1|2n})^{\otimes k}$ with the action of $\mathrm{X}(\mathfrak{osp}_{1|2n})$ by setting

$$t_{ij}(u) \mapsto \sum_{a_1,\dots,a_{k-1}=1}^{2n+1} t_{ia_1}(u) \otimes t_{a_1a_2}(u-1) \otimes \dots \otimes t_{a_{k-1}j}(u-k+1), \tag{6.3}$$

where the generators act in the respective copies of the vector space $\mathbb{C}^{1|2n}$ via the rule (6.2). For the values $k = 1, \ldots, n$ introduce the vectors

$$\xi_k = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn} \sigma \cdot e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)} \in (\mathbb{C}^{1|2n})^{\otimes k}.$$

Now verify that each vector ξ_k has the properties

$$t_{ij}(u)\, \xi_k = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant n+1 \tag{6.4}$$

and

$$t_{ii}(u)\, \xi_k = \begin{cases} \dfrac{u-k}{u-k+1}\, \xi_k & \text{for} \quad i = 1, \ldots, k, \\ \xi_k & \text{for} \quad i = k+1, \ldots, n+1. \end{cases} \tag{6.5}$$

The expression for the vector ξ_k involves only tensor products of the basis vectors e_i with $i \leqslant n$. This implies that for the application of the operators $t_{ij}(u)$ with $1 \leqslant i \leqslant j \leqslant n$ to ξ_k we may restrict the sum in formula (6.3) to the values $a_p \in \{1, \ldots, n\}$.

By using the embedding (6.1), we may regard the cyclic span $\mathrm{Y}(\mathfrak{gl}_n)\, \xi_k$ as a $\mathrm{Y}(\mathfrak{gl}_n)$-module. Moreover, this module is isomorphic to $A^{(k)}(\mathbb{C}^n)^{\otimes k}$, where $A^{(k)}$ is the anti-symmetrization operator. It is well-known that this $\mathrm{Y}(\mathfrak{gl}_n)$-module is isomorphic to the evaluation module $L(1, \ldots, 1, 0, \ldots, 0)$ (with k ones) twisted by a shift automorphism $u \mapsto u + k - 1$; see e.g. [22, Sec. 6.5]. This yields formulas (6.4) and (6.5) with $1 \leqslant i \leqslant j \leqslant n$. They are easily verified directly for the remaining generators.

Formulas (6.5) show that the corresponding set of Drinfeld polynomials for the highest weight module $\mathrm{X}(\mathfrak{osp}_{1|2n})\, \xi_k$ has the form $P_i(u) = 1$ for $i \neq k$, while $P_k(u) = u - k$. This completes the proof of the second part of the Main Theorem concerning conditions (1.1). The last part follows from the decomposition (2.6) as in [4, Sec. 5.3].

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