



Existence and Uniqueness for Plane Stationary Navier–Stokes Flows with Compactly Supported Force

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Abstract: We study the stationary Navier–Stokes equations in the whole plane with a compactly supported force term and with a prescribed constant spatial limit. Prior to this work, existence of solutions to this problem was only known under special symmetry and smallness assumptions. In the paper we solve the key difficulties in applying Leray’s *invading domains method* and, as a consequence, prove the existence of D -solutions in the whole plane for arbitrary compactly supported force. The boundary condition at infinity are verified in two different scenarios: (I) the limiting velocity is sufficiently large with respect to the external force, (II) both the total integral of force and the limiting velocity vanish. Hence, our method produces large class of new solutions with prescribed spatial limits. Moreover, we show the uniqueness of D -solutions to this problem in a perturbative regime. The main tools here are two new estimates for general Navier–Stokes solutions, which have rather simple forms. They control the difference between mean values of the velocity over two concentric circles in terms of the Dirichlet integral in the annulus between them.

1. Introduction

1.1. Motivation and main results. We study the stationary Navier–Stokes equations in the whole plane \mathbb{R}^2 driven by a force term:

$$\begin{cases} -\Delta \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{w} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{w} = 0, \\ \mathbf{w}(z) \rightarrow \mathbf{w}_\infty = \lambda \mathbf{e}_1 \text{ as } |z| \rightarrow \infty. \end{cases} \quad (1.1)$$

Here, $\lambda \in \mathbb{R}_{\geq 0}$ is a physical parameter specifying the “boundary” condition at spatial infinity. One major mathematical challenge in the study of the stationary Navier–Stokes equations is the existence of solutions in two-dimensional unbounded domains along with the characterization of their asymptotic behaviours. In the case when the domain Ω

is exterior, *i.e.*, $\Omega = \mathbb{R}^2 \setminus \overline{U}$ with U bounded, one often takes $\mathbf{f} = 0$ and homogeneous Dirichlet boundary condition on $\partial\Omega$, so that the system describes stationary flows past rigid obstacles. For comparison, we also present such an exterior domain problem here,

$$\begin{cases} -\Delta \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} + \nabla p = 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{w} = 0 & \text{in } \Omega, \\ \mathbf{w}|_{\partial\Omega} = 0, \\ \mathbf{w}(z) \rightarrow \mathbf{w}_\infty = \lambda \mathbf{e}_1 & \text{as } |z| \rightarrow \infty. \end{cases} \tag{1.2}$$

In this paper, we focus on another physically important case where the unbounded domain is simply the whole plane \mathbb{R}^2 . The aim of this paper is to tackle some key difficulties in the \mathbb{R}^2 case and establish new existence and uniqueness results.

Our starting point is to apply Leray’s invading domains method proposed in 1933 [18]. Leray’s original idea was intended for the exterior domain problem (1.2) — one first solves the system on large bounded domains $\Omega_k := \Omega \cap B_{R_k}$ ($k = 1, 2, 3, \dots$) with an increasing sequence of radii $R_k \rightarrow \infty$, and then takes the limit $k \rightarrow \infty$. The boundary condition (1.2)₄ at infinity will now be imposed on the outer boundary.

$$\begin{cases} -\Delta \mathbf{w}_k + (\mathbf{w}_k \cdot \nabla)\mathbf{w}_k + \nabla p_k = \mathbf{0} & \text{in } \Omega_k, \\ \nabla \cdot \mathbf{w}_k = 0 & \text{in } \Omega_k, \\ \mathbf{w}_k = \mathbf{0} & \text{on } \partial\Omega, \\ \mathbf{w}_k = \mathbf{w}_\infty & \text{for } |z| = R_k. \end{cases} \tag{1.3}$$

In [18], Leray showed that the above invading domain problems produce a sequence of solutions with uniformly bounded Dirichlet integral, *i.e.*,

$$\int_{\Omega} |\nabla \mathbf{w}_k|^2 \leq C,$$

with C independent of k . Since \mathbf{w}_k vanishes on $\partial\Omega$, by Sobolev embedding the Dirichlet integral controls local L^p norms of \mathbf{w} for any $p < \infty$. Hence, weak limits of \mathbf{w}_k are well-defined, and one obtains solutions in Ω with bounded total Dirichlet integral.¹ These solutions are now referred to as Leray’s solutions. Whether Leray’s solutions achieve the desired spatial limit \mathbf{w}_∞ has been open for nearly 90 years.² Recently, it was settled in the small Reynolds number case [17]. The classical papers of Gilbarg–Weinberger [8,9], Amick [1] and many recent works, *e.g.*, [5,13,14,20] are devoted to the asymptotic properties of Leray’s solutions, or more generally, of D -solutions in exterior domains. In particular, the second author with Pileckas and Russo showed that D -solutions in exterior domains always have uniform spatial limits [13]. For more backgrounds and details on the exterior domain problem, we refer the readers to the papers [15,16] and the book of Galdi [6].

However, Leray’s original arguments left out the whole plane case. Although it is easy to formulate the invading domain problems here which appear similar to (1.3),

$$\begin{cases} -\Delta \mathbf{w}_k + (\mathbf{w}_k \cdot \nabla)\mathbf{w}_k + \nabla p_k = \mathbf{f} & \text{in } B_{R_k}, \\ \nabla \cdot \mathbf{w}_k = 0 & \text{in } B_{R_k}, \\ \mathbf{w}_k = \mathbf{w}_\infty & \text{for } |z| = R_k. \end{cases} \tag{1.4}$$

¹ Solutions of bounded Dirichlet integral are often called D -solutions.

² This major open problem is only for two dimensions. The main difficulty in 2d is that the Dirichlet energy alone is not sufficient to control the behaviour of functions at infinity. In three dimensions, Leray’s solutions do achieve the correct limiting values.

there are new essential obstructions arising in the analysis, as already pointed out in [12]. Below, we summarize all the key obstructions that must be tackled. Note that (a) and (b) are inherent to the whole plane problem (1.1), while (c) are shared for both (1.1) and (1.2).

- (a) It is difficult to prove that the solutions \mathbf{w}_k to (1.4) have uniformly bounded Dirichlet integrals. It should be mentioned that, when the total force $\mathcal{F} = \int_{\mathbb{R}^2} \mathbf{f}$ vanishes, such a difficulty is absent by the following simple argument. We can write $\mathbf{f} = \nabla \cdot \mathbb{F}$ for some tensor $\mathbb{F} \in L^2(\mathbb{R}^2)$ (see [12, Lemma 3.6] or Lemma 2 below). Then, we test (1.4)₁ with $\mathbf{w}_k - \lambda \mathbf{e}_1$ and integrate by parts to get the energy equality

$$\int_{B_{R_k}} |\nabla \mathbf{w}_k|^2 = \int_{B_{R_k}} \mathbb{F} : \nabla \mathbf{w}_k \tag{1.5}$$

where $\mathbb{F} : \nabla \mathbf{w}_k$ stands for $\sum_{i,j} \mathbb{F}_{ij} \partial_j w_{k,i}$. Using Hölder’s inequality we get a nice bound on the Dirichlet integral,

$$\int_{B_{R_k}} |\nabla \mathbf{w}_k|^2 \leq \int_{\mathbb{R}^2} |\mathbb{F}|^2.$$

However, for a general \mathbf{f} , such a direct estimate is unavailable.

- (b) Moreover, even if the uniform boundedness in (1) is assumed, it is not clear whether the local L^p norms of \mathbf{w}_k are uniformly bounded since the Dirichlet integral controls only the derivative of \mathbf{w}_k and the condition (1.4)₃ is imposed on distant outer boundaries.
- (c) If both (a) and (b) are solved, one may define \mathbf{w}_L as the weak limit of some subsequence of \mathbf{w}_k . Then there is still one more difficult question, that is,

$$\text{do we have } \lim_{|z| \rightarrow \infty} \mathbf{w}_L(z) = \mathbf{w}_\infty? \tag{1.6}$$

Now, we are ready to introduce the main results of the paper for the systems (1.1) and (1.4). We shall assume that the force \mathbf{f} is compactly supported, *i.e.*, $\text{supp } \mathbf{f} \subseteq B_R$ for some $R > 0$. Hence, the theories for D -solutions in exterior domains without force can be applied in our situation. We also assume a minimal regularity of \mathbf{f} , that is, $\mathbf{f} \in H^{-1}(B_{2R})$, see Section 2.1.

First of all, fortunately we were able to completely solve (a) and (b), that is, we proved the uniform estimates $\|\mathbf{w}_k\|_{L^q(B_1)} + \|\nabla \mathbf{w}_k\|_{L^2(B_{R_k})} \leq C$ (see Theorem 12). As an immediate corollary, the Leray solutions \mathbf{w}_L are well-defined as weak limits of \mathbf{w}_k . Hence we have constructed solutions to (1.1)_{1,2} for arbitrary force \mathbf{f} in the whole plane. Second, we study (c) and justify the identity (1.6) for each of the following two scenarios.

- (I) the limiting speed λ is large enough with respect to the norm of force $\|\mathbf{f}\|_{H^{-1}(B_{2R})}$ (see Theorem 15);
- (II) the total integral of force and the limiting speed λ are both zero (see Theorem 16).

Third, we prove the uniqueness of D -solutions to (1.1) in a perturbative regime (see Theorem 17). We emphasize that the conditions in Theorem 15 and Theorem 17 are very different in nature. In particular, for fixed R , the condition (7.1) implies that $\|\mathbf{f}\|_{H^{-1}(B_{2R})}$ is smaller than an absolute constant independent of λ , while scenario (I) allows arbitrarily large external force as long as λ is chosen even larger.

With the help of Steps 4-5 in our proof of Theorem 17, one may extend the fixed-point methods in [4] and [6, Section XII.5], to construct perturbative solutions to (1.1) with

$\lambda \neq 0$. The main idea is to view (1.1) as a perturbation of the linear Oseen system with external force \mathbf{f} , and the nonlinear solution will be found close to the Oseen solution in the Banach space X defined in (7.41). Such a completely perturbative scheme would require a condition stricter than scenario (I) and milder than that of Theorem 17. Theorem 17 can be viewed as a weak-strong uniqueness theorem, in the sense that the perturbative solutions are unique in the broader class of D -solutions.

1.2. Main tools: Two new estimates for general Navier–Stokes solutions. The main tools for our research here are two new estimates on the difference between mean values of the velocity over two concentric circles in terms of the Dirichlet integral in the annulus between them (see Theorems 7 and 9). They are also among the main contributions of this paper. As well-known, it is not possible in general to control the growth of a function through its Dirichlet integral in large planar domains.³ Nevertheless, the special structures of the Navier–Stokes system create the possibility to derive such very general estimates, whose forms turn out to be rather clear and simple. The first estimate claims that

$$|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| \leq C_*(1 + \mu)\sqrt{D}, \tag{1.7}$$

where

$$\mu = \frac{1}{r_1 m}, \quad m := \max\{|\bar{\mathbf{w}}(r_1)|, |\bar{\mathbf{w}}(r_2)|\}, \quad D := \int_{r_1 \leq |z| \leq r_2} |\nabla \mathbf{w}|^2 \tag{1.8}$$

and C_* is some universal positive constant (does not depend on \mathbf{w} , r_i , etc.) The proof of it is based on classical methods in [1, 9] as well as the recent progress in [15]. Namely, by [9] the pressure is under control of the Dirichlet integral. So, assuming that the estimate (1.7) fails, we obtain the existence of two concentric levels sets of the Bernoulli pressure $\Phi = p + \frac{1}{2}|\mathbf{w}|^2$, and the difference of the values of Φ on these level sets is much bigger than $m(1 + \mu)\sqrt{D}$. Recall, that

$$\nabla \Phi = -\nabla^\perp \omega + \omega \mathbf{w}^\perp, \tag{1.9}$$

where $\omega = \partial_2 w_1 - \partial_1 w_2$ is the corresponding vorticity. By results of [9], the line integrals of the first term $\nabla \omega$ are small with respect to D , moreover, the variation of *the direction* of the velocity \mathbf{w} between our Φ -level sets is under control of the Dirichlet integral as well. The crucial fact is that the vorticity ω does not change sign between levels sets of Φ (it was proved in [15] based on the elegant ideas of [1]). So *the direction* of $\nabla \Phi$ is almost constant between the concentric level sets of Φ , a contradiction.

The proof of the second (asymptotic) estimate crucially relies on the recent results in [17] concerning solutions to the Euler system in the unit disk with constant velocity on the boundary which are produced by a blow-down procedure (see Section 4.1 of the present paper). Namely, it turns out that the pressure satisfies the classical Neumann boundary conditions, so the absolute value of the pressure for the considered Euler solutions is under control of the Dirichlet integrals as well.

Both estimates are invariant with respect to the natural rescaling of the stationary Navier–Stokes system.

³ For instance, the function $f(z) = (\ln(|z|))^\frac{1}{3}$ has a finite Dirichlet integral in $\mathbb{R}^2 \setminus B_2$ but diverges at infinity.

1.3. Open problem and discussions. The key open problem for (1.1) is to prove existence of solutions in \mathbb{R}^2 given arbitrary force (with sufficient decay and regularity) and arbitrary λ , that is, to remove the constraints in our Theorems 15 and 16.

There are a few works on the construction of solutions to (1.1) using different approaches from ours. In [12], the first author and Wittwer proposed a modified invading domains method which could, for zero total force, produce infinitely many solutions parametrized by their mean values in the unit disc. The spatial limits of their solutions are not clear, *i.e.*, the condition (1.1)₃ is hard to verify. In [25], Yamazaki constructed solutions with $\lambda = 0$ and explicit decay rates under special symmetry and smallness assumptions on \mathbf{f} . In [10, Section 3], the first author also considered the $\lambda = 0$ case and proved existence for \mathbf{f} from a set of codimension three. All three constructions mentioned here work under zero total force assumption. The study for $\lambda \neq 0$ case seems rather limited prior to this work, apart from the possibility of constructing perturbative solutions which we have mentioned earlier.

For $\lambda = 0$, the precise asymptotic behaviour of solutions is of particular interest and difficulty, see, *e.g.*, the discussions in [10, Section 5]. Theorem 16 constructs a large class of solutions converging to 0 at infinity without giving explicit rates. It would be of great value to prove uniform decay estimates for these solutions. Note that in three dimensions, the Liouville conjecture for D -solutions in whole space \mathbb{R}^3 with zero spatial limit remains an important open problem where the lack of decay estimates is the key obstacle, see, *e.g.*, [6, Remark X.9.4] and [21].

For comparison, we mention that the existence problem for the exterior problem (1.2) with arbitrary λ is also open, and is listed by Yudovich as one of “eleven great problems of mathematical hydrodynamics” [24]. See the work of Galdi [7] for a conditional result on the existence of solutions with large λ .

Note, in conclusion, that the very recent paper [2] on non-uniqueness of the classical Leray–Hopf solutions to the non-stationary forced Navier–Stokes system demonstrates the importance and fruitfulness of the studying of such forced equations in the whole space.

1.4. Organization of the paper. In Section 2, we introduce the notations, function spaces to be used throughout the paper, along with some useful lemmas for stationary Navier–Stokes solutions. In Section 3, we state and prove the first basic estimate for solutions on an arbitrary annulus domain. In Section 4, we state and prove the second basic (asymptotic) estimate for a sequence of solutions on enlarging annulus domains. In Section 5, we show the uniform bounds for the invading domain solutions \mathbf{w}_k based on the first basic estimate. As a corollary, we prove existence of Leray’s solutions \mathbf{w}_L . In Section 6, we justify (1.6) in two scenarios based on the second basic estimate. Finally, in Section 7, we state and prove the uniqueness result in a perturbative regime.

2. Preliminaries

2.1. Notations. Throughout the paper, \mathbf{w}_k will be the invading domain solutions to (1.4) for some sequence of radii $R_k \rightarrow \infty$.

We use C to denote constants that are independent of k , R , λ , \mathbf{f} , etc. The exact values of C may change from line to line.

We always assume that $\text{supp } \mathbf{f} \subset B_R$ and $\mathbf{f} \in H^{-1}(B_{2R})$. Definition and some properties of the H^{-1} space are summarized below.

2.2. *Function spaces.* We use standard notations for Sobolev spaces $H^1(\Omega) = W^{1,2}(\Omega)$. Let $\dot{H}_0^1(\Omega)$ be the completion of $C_c^\infty(\Omega)$ in $H^1(\Omega)$ as usual, with norm given by

$$\|\varphi\|_{\dot{H}_0^1(\Omega)}^2 = \|\nabla\varphi\|_{L^2(\Omega)}^2. \tag{2.1}$$

We recall the following elementary fact:

Lemma 1. *For any function $\varphi \in H^1(B_R)$ and for every $q \in (1, \infty)$ the inequality*

$$\|\varphi\|_{L^q(B_R)} \leq C \left(|\bar{\varphi}(R)| + \|\nabla\varphi\|_{L^2(B_R)} \right) R^{2/q} \tag{2.2}$$

holds, where $\bar{\varphi}(R)$ is the average of φ on the circle S_R , and C is a constant depending only on q .

Proof. By scaling, the statement can be reduced to the case $R = 1$. Then, the estimate follows from [19, Theorem 1.1.16]. \square

The H^{-1} norm of a (scalar valued) distribution ψ in B_{2R} is defined as

$$\|\psi\|_{H^{-1}(B_{2R})} := \sup_{\varphi \in C_0^\infty(B_{2R}), \|\varphi\|_{\dot{H}_0^1(B_{2R})} = 1} \langle \psi, \varphi \rangle. \tag{2.3}$$

Let $\sigma > 0$. It is easy to check, that for the function $\varphi_\sigma(z) := \sigma^3 \varphi(\sigma z)$ we have the following scaling identity:

$$\|\varphi_\sigma\|_{H^{-1}(B_{2R/\sigma})} = \sigma \|\varphi\|_{H^{-1}(B_{2R})}. \tag{2.4}$$

Let χ be a smooth cut-off function satisfying $\chi \equiv 1$ on B_R and $\text{supp } \chi \subset B_{2R}$, and define the *total force* $\mathcal{F} := \langle \mathbf{f}, \chi \rangle$. Since χ can be extended up to the function from $C_0^\infty(B_{2R})$ satisfying $|\nabla\chi| \leq C \frac{1}{R}$, clearly, we have

$$|\mathcal{F}| \leq C \|\mathbf{f}\|_{H^{-1}(B_{2R})}, \tag{2.5}$$

where C is some universal constant (does not depend on R). For simplicity, we will formally write $\mathcal{F} = \int_{\mathbb{R}^2} \mathbf{f}$ although it is possible that $\mathbf{f} \notin L^1_{\text{loc}}$. Next, we state a result in the spirit of [12, Lemma 3.6]:

Lemma 2. *Under the above notations, if $\mathcal{F} = 0$, then there exists a tensor $\mathbb{F} \in L^2(\mathbb{R}^2)$ such that $\text{supp } \mathbb{F} \subset B_{2R}$, $\mathbf{f} = \nabla \cdot \mathbb{F}$ on \mathbb{R}^2 , and*

$$\|\mathbb{F}\|_{L^2} \leq C \|\mathbf{f}\|_{H^{-1}(B_{2R})}$$

with C independent of R .

Proof. Let us define \mathbb{F} to be $\nabla \mathbf{g}$ with $\mathbf{g} \in H^1(B_{2R})$ solving

$$\begin{cases} \Delta \mathbf{g} = \mathbf{f}, & \text{in } B_{2R}, \\ \partial_n \mathbf{g} = 0, & \text{on } \partial B_{2R}. \end{cases}$$

Such \mathbf{g} exists uniquely up to adding constants since we assumed $\mathcal{F} = 0$. We extend \mathbb{F} by 0 outside B_{2R} . Then using $\partial_n \mathbf{g} = 0$ on ∂B_{2R} , one can check that $\nabla \cdot \mathbb{F} = \mathbf{f}$ holds in the sense of distributions not only on B_{2R} , but also on \mathbb{R}^2 . \square

The following change-of-domain lemma will be useful in Sections 6 and 7.

Lemma 3. *Suppose that $\text{supp } f \subset B_R$ and $f \in H^{-1}(B_{2R})$. Then for any $a \geq 2$,*

$$\|f\|_{H^{-1}(B_{aR})} \leq C \left(1 + \left(\ln \frac{a}{2}\right)^{\frac{1}{2}}\right) \|f\|_{H^{-1}(B_{2R})} \tag{2.6}$$

with C independent of a and R . If in addition $\mathcal{F} = 0$, then

$$\|f\|_{H^{-1}(B_{aR})} \leq C \|f\|_{H^{-1}(B_{2R})}.$$

Proof. Fix the cut-off function χ as above. For any $\varphi \in \dot{H}_0^1(B_{aR})$, we have $\varphi\chi \in \dot{H}_0^1(B_{2R})$ and $\langle f, \varphi \rangle = \langle f, \varphi\chi \rangle$. Notice that

$$\begin{aligned} \|\varphi\chi\|_{\dot{H}_0^1(B_{2R})} &\leq \|\chi\nabla\varphi\|_{L^2} + \|\varphi\nabla\chi\|_{L^2} \\ &\leq \|\varphi\|_{\dot{H}_0^1(B_{aR})} + \frac{C}{R} \|\varphi\|_{L^2(B_{2R})} \\ &\stackrel{(2.2)}{\leq} \|\varphi\|_{\dot{H}_0^1(B_{aR})} + C \left(|\bar{\varphi}(2R)| + \|\nabla\varphi\|_{L^2(B_{2R})}\right) \\ &\leq C \|\varphi\|_{\dot{H}_0^1(B_{aR})} \left(1 + \left(\ln \frac{a}{2}\right)^{\frac{1}{2}}\right) \end{aligned}$$

In the last step we applied the estimate (2.7) below to control $|\bar{\varphi}(2R)| = |\bar{\varphi}(2R) - \bar{\varphi}(aR)|$. Now, a standard duality argument proves the lemma.

If in addition $\mathcal{F} = 0$, then the corresponding claim follows from Lemma 2, and the fact that for any $a \geq 2$, we have

$$\|\nabla \cdot \mathbb{F}\|_{H^{-1}(B_{aR})} \leq \|\mathbb{F}\|_{L^2}.$$

□

Lemma 4. *For any $\phi \in H^1(\Omega_{\rho_1, \rho_2})$, where $\Omega_{\rho_1, \rho_2} = \{z \in \mathbb{R}^2 : 0 < \rho_1 < |z| < \rho_2\}$, we have*

$$|\bar{\phi}(\rho_2) - \bar{\phi}(\rho_1)| \leq \frac{1}{\sqrt{2\pi}} \left(\int_{\Omega_{\rho_1, \rho_2}} |\nabla\phi|^2\right)^{\frac{1}{2}} \cdot \left(\ln \frac{\rho_2}{\rho_1}\right)^{\frac{1}{2}} \tag{2.7}$$

For the proof of Lemma 4, see, e.g., [13, Lemma 2.1].

2.3. Properties of D -solutions. We present two important lemmas for general D -solutions to the Navier–Stokes equations. They have been very useful in many previous studies on the Navier–Stokes exterior problem, see, for instance, [1, 9, 13, 15, 16]. Lemma 5 was proved in [9, Lemma 4.1], and Lemma 6 was proved in [9, Theorem 4, page 399]. Let \mathbf{w} be a D -solution to the Navier–Stokes equations in some ring $\Omega_{r_1, r_2} = \{z \in \mathbb{R}^2 : 0 < r_1 < |z| < r_2\}$, and p be the corresponding pressure.

Lemma 5 [9]. *Denote by $\bar{p}(r)$ the average of p over the circle S_r . Then for any $r_1 < \rho_1 \leq \rho_2 < r_2$, we have*

$$|\bar{p}(\rho_2) - \bar{p}(\rho_1)| \leq \frac{1}{4\pi} \int_{\Omega_{\rho_1, \rho_2}} |\nabla\mathbf{w}|^2. \tag{2.8}$$

Denote by $\bar{\mathbf{w}}(r)$ the average of \mathbf{w} over the circle S_r . Note, that the direct estimates for $\bar{\mathbf{w}}(r)$ are not so good:

$$|\bar{\mathbf{w}}(\rho_2) - \bar{\mathbf{w}}(\rho_1)| \leq \frac{1}{\sqrt{2\pi}} \left(\int_{\Omega_{\rho_1, \rho_2}} |\nabla \mathbf{w}|^2 \right)^{\frac{1}{2}} \cdot \left(\ln \frac{\rho_2}{\rho_1} \right)^{\frac{1}{2}} \tag{2.9}$$

(see (2.7)). Nevertheless, the direction of $\bar{\mathbf{w}}(r)$ is still under control of the Dirichlet integral:

Lemma 6. [9] *Denote by $\bar{\mathbf{w}}$ the average of \mathbf{w} over the circle S_r and let $\varphi(r) \in [0, 2\pi]$ (modulo 2π) be the argument of the complex number associated with the vector $\bar{\mathbf{w}}(r) = (\bar{w}_1(r), \bar{w}_2(r))$, i.e., $\bar{\mathbf{w}}(r) = |\bar{\mathbf{w}}(r)| (\cos \varphi(r), \sin \varphi(r))$. Assume also that $|\bar{\mathbf{w}}(r)| \geq \sigma > 0$ for some constant σ and for all $r \in (r_1, r_2)$. Then for any $r_1 < \rho_1 \leq \rho_2 < r_2$, we have*

$$|\varphi(\rho_2) - \varphi(\rho_1)| \leq \frac{1}{4\pi\sigma^2} \int_{\Omega_{\rho_1, \rho_2}} \left(\frac{1}{r} |\nabla \omega| + |\nabla \mathbf{w}|^2 \right). \tag{2.10}$$

Here $\omega = \partial_2 w_1 - \partial_1 w_2$ is the vorticity of \mathbf{w} .

3. The First Basic Estimate for the Velocity

We have already mentioned, that the usual estimates for plane functions with finite Dirichlet integral are not efficient to compare the difference of mean values of the function over two circles whose radii are very different (see (2.9)). But the special structure of the plane Navier–Stokes system allows to improve the issue. Here and in the next section we formulate and prove two important estimates for the velocity vector, which will be the main tool for future research in the paper. Note, that they are valid not only for the solutions in the whole plane (considered above), but in much more general situations.

Theorem 7. *Let \mathbf{w} be the D -solution to the Navier–Stokes system*

$$\begin{cases} \Delta \mathbf{w} - (\mathbf{w} \cdot \nabla) \mathbf{w} - \nabla p = \mathbf{0}, \\ \nabla \cdot \mathbf{w} = 0 \end{cases} \tag{3.1}$$

in the annulus type domain $\Omega_{r_1, r_2} = \{z \in \mathbb{R}^2 : r_1 \leq |z| \leq r_2\}$. Then

$$|\bar{\mathbf{w}}(r_1) - \bar{\mathbf{w}}(r_2)| \leq C_*(1 + \mu) \sqrt{D(r_1, r_2)}, \tag{3.2}$$

where

$$\mu = \frac{1}{r_1 m}, \quad m := \max\{|\bar{\mathbf{w}}(r_1)|, |\bar{\mathbf{w}}(r_2)|\}, \quad D(r_1, r_2) := \int_{\Omega_{r_1, r_2}} |\nabla \mathbf{w}|^2 \tag{3.3}$$

and C_* is some universal positive constant (does not depend on \mathbf{w} , r_i , etc.)

In order to prove the theorem, first of all we need to obtain the corresponding estimate for the absolute value of the velocity.

Lemma 8. *Under conditions and notation of Theorem 7 the estimate*

$$\left| |\bar{\mathbf{w}}(r_2)| - |\bar{\mathbf{w}}(r_1)| \right| \leq C_0(1 + \mu)\sqrt{D(r_1, r_2)}, \tag{3.4}$$

holds with some universal positive constant C_0 (not depending on \mathbf{w}, r_i , etc.)

Proof. If $|\bar{\mathbf{w}}(r_2)| = |\bar{\mathbf{w}}(r_1)|$, we have zero in the left hand side, and there is nothing to prove. For definiteness, assume that

$$|\bar{\mathbf{w}}(r_2)| > |\bar{\mathbf{w}}(r_1)|$$

(in the case of opposite inequality the arguments are the same). Let us make several simplifications. First of all, it is sufficient to consider the case

$$m = |\bar{\mathbf{w}}(r_2)| = \max_{r \in [r_1, r_2]} |\bar{\mathbf{w}}(r)|. \tag{3.5}$$

Indeed, if the last assumption is not fulfilled, we can take

$$r'_2 \in [r_1, r_2] : |\bar{\mathbf{w}}(r'_2)| = \max_{r \in [r_1, r_2]} |\bar{\mathbf{w}}(r)|,$$

and then consider the interval $[r_1, r'_2]$ instead of $[r_1, r_2]$. By construction,

$$D(r_1, r'_2) < D(r_1, r_2), \quad |\bar{\mathbf{w}}(r_2)| - |\bar{\mathbf{w}}(r_1)| < |\bar{\mathbf{w}}(r'_2)| - |\bar{\mathbf{w}}(r_1)|, \quad \mu(r_1, r'_2) < \mu(r_1, r_2).$$

So if we prove the required estimate (3.4) for r_1, r'_2 , then it implies the required estimate for initial pair r_1, r_2 , etc. So below we assume that (3.5) is fulfilled.

Further, it is sufficient to consider only the case when

$$|\bar{\mathbf{w}}(r)| \geq \frac{1}{6}m \quad \forall r \in [r_1, r_2]. \tag{3.6}$$

Indeed, if the last assumption is not fulfilled, we can take

$$r'_1 = \max \left\{ r \in [r_1, r_2] : |\bar{\mathbf{w}}(r)| \leq \frac{1}{6}m \right\},$$

and then consider the interval $[r'_1, r_2]$ instead of $[r_1, r_2]$. By construction,

$$|\bar{\mathbf{w}}(r)| \geq \frac{1}{6}m \quad \forall r \in [r'_1, r_2], \tag{3.7}$$

$$D(r'_1, r_2) < D(r_1, r_2), \quad |\bar{\mathbf{w}}(r_2)| - |\bar{\mathbf{w}}(r_1)| \leq m = \frac{6}{5} \left(|\bar{\mathbf{w}}(r_2)| - |\bar{\mathbf{w}}(r'_1)| \right).$$

So if we prove the required estimate (3.4) for r'_1, r_2 , then it implies the required estimate for initial pair r_1, r_2 , etc. So below we assume that (3.6) is fulfilled.

Now take and fix some small number $\varepsilon_0 \in (0, \frac{1}{100})$ (the exact value of ε_0 will be specified below). Obviously, it is sufficient to consider only the case when

$$(1 + \mu)\sqrt{D(r_1, r_2)} \leq \varepsilon_0 m. \tag{3.8}$$

Indeed, if the opposite inequality were valid, then the required estimate (3.4) is fulfilled automatically with the fixed constant $C_0 = \frac{1}{\varepsilon_0}$, and again there are nothing to prove. So below we assume that (3.8) is fulfilled as well.

Also we can assume without loss of generality that

$$r_2 > 1000r_1$$

(otherwise the required estimate (3.4) follows from (2.9)). Further proof splits into several steps. Our general strategy is as follows: first of all, we obtain some uniform estimates for pressure, Bernoulli pressure $\Phi = p + \frac{1}{2}|\mathbf{w}|^2$ and velocity in the suitable annulus type domain; then we deduce some improved estimates for the integral $\int |\nabla\omega|^2$, and finally we obtain the required estimate (3.4) using contradiction argument and level sets of Bernoulli pressure.

STEP 1. Using standard estimates (“self-improvement of regularity”) for stationary Stokes system, one can prove that for any $\rho \in [r_1, \frac{1}{5}r_2]$

$$\left(\int_{\frac{3}{2}\rho \leq r \leq \frac{9}{2}\rho} |\nabla^2 \mathbf{w}|^{\frac{4}{3}} + |\nabla p|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq C m \rho^{\frac{1}{2}} D(\rho, 5\rho)^{\frac{1}{2}} (1 + \mu) \tag{3.9}$$

(see, e.g., [16, Lemma 8 and Appendix II] for the detailed calculations concerning this estimate).

STEP 2. Now we would like to obtain the uniform estimates for pressure. Using (3.9), it is easy to see, that there exist two “good” radii $\tilde{r}_1 \in (\frac{3}{2}r_1, 2r_1)$, $\tilde{r}_2 \in (\frac{1}{2}r_2, \frac{9}{10}r_2)$ such that for $i = 1, 2$ and for $D := D(r_1, r_2)$ we have

$$\left\{ \begin{array}{l} \int_{S_{\tilde{r}_i}} |\nabla \mathbf{w}|^2 ds \leq \frac{CD}{\tilde{r}_i}, \\ \int_{S_{\tilde{r}_i}} |\nabla \mathbf{w}| ds \leq (2\pi\tilde{r}_i)^{\frac{1}{2}} \left(\int_{S_{\tilde{r}_i}} |\nabla \mathbf{w}|^2 ds \right)^{\frac{1}{2}} \leq CD^{\frac{1}{2}}, \\ \int_{S_{\tilde{r}_i}} |\nabla p|^{\frac{4}{3}} ds \leq C \frac{1}{\tilde{r}_i^{\frac{2}{3}}} m^{\frac{4}{3}} D^{\frac{2}{3}} (1 + \mu)^{\frac{4}{3}}, \\ \int_{S_{\tilde{r}_i}} |\nabla p| ds \leq (2\pi\tilde{r}_i)^{\frac{1}{4}} \left(\int_{S_{\tilde{r}_i}} |\nabla p|^{\frac{4}{3}} ds \right)^{\frac{3}{4}} \leq C m D^{\frac{1}{2}} (1 + \mu). \end{array} \right. \tag{3.10}$$

Without loss of generality we may assume that $\bar{p}(\tilde{r}_1) = 0$. Then by (2.8) we have

$$|\bar{p}(r)| \leq CD \leq Cm\sqrt{D} \quad \text{for any } r \in [r_1, r_2], \tag{3.11}$$

thus the estimates (3.10₄) and the trivial inequality

$$|p(z)| \leq \bar{p}(r) + \int_{S_r} |\nabla p| ds \quad \forall z \in S_r,$$

imply the corresponding pointwise bound for the pressure:

$$|p| \leq Cm(1 + \mu)\sqrt{D} \quad \text{on } S_{\tilde{r}_i}, i = 1, 2. \tag{3.12}$$

From the main estimate (2.8), using “good circles” technique, it is very easy to deduce from (3.9) and (3.12), that

$$|p| \leq Cm(1 + \mu)\sqrt{D} \quad \text{in } \Omega(6r_1, \frac{1}{6}r_2) \tag{3.13}$$

(see Appendix A for details).

STEP 3. Denote $m_0 = |\bar{\mathbf{w}}(r_1)|$. Recall, that $m = |\bar{\mathbf{w}}(r_2)| > m_0$. From (2.9) we have

$$|\bar{\mathbf{w}}(r_i)| - C\sqrt{D} \leq |\bar{\mathbf{w}}(\tilde{r}_i)| \leq |\bar{\mathbf{w}}(r_i)| + C\sqrt{D}, \quad i = 1, 2. \tag{3.14}$$

In particular,

$$|\bar{\mathbf{w}}(\tilde{r}_1)| \leq m_0 + C\sqrt{D}, \tag{3.15}$$

$$|\bar{\mathbf{w}}(\tilde{r}_1)|^2 \stackrel{(3.8)}{\leq} m_0^2 + Cm\sqrt{D}. \tag{3.16}$$

Analogously,

$$m - C\sqrt{D} \leq |\bar{\mathbf{w}}(\tilde{r}_2)| \leq m + C\sqrt{D}, \tag{3.17}$$

$$m^2 - Cm\sqrt{D} \leq |\bar{\mathbf{w}}(\tilde{r}_2)|^2 \leq m^2 + Cm\sqrt{D}. \tag{3.18}$$

Furthermore,

$$|\mathbf{w}| \leq |\bar{\mathbf{w}}(\tilde{r}_1)| + \int_{S_{\tilde{r}_1}} |\nabla \mathbf{w}| ds \stackrel{(3.10_2)}{\leq} m_0 + C\sqrt{D} \quad \text{on } S_{\tilde{r}_1}, \tag{3.19}$$

$$|\mathbf{w}|^2 \stackrel{(3.8)}{\leq} m_0^2 + Cm\sqrt{D} \quad \text{on } S_{\tilde{r}_1}. \tag{3.20}$$

Analogously,

$$m - C\sqrt{D} \leq |\mathbf{w}| \leq m + C\sqrt{D} \quad \text{on } S_{\tilde{r}_2}, \tag{3.21}$$

$$m^2 - Cm\sqrt{D} \leq |\mathbf{w}|^2 \leq m^2 + Cm\sqrt{D} \quad \text{on } S_{\tilde{r}_2}. \tag{3.22}$$

Estimates (3.12), (3.20), (3.22) imply the following bound for the Bernoulli pressure $\Phi = p + \frac{1}{2}|\mathbf{w}|^2$:

$$\Phi \leq \frac{1}{2}m_0^2 + Cm(1 + \mu)\sqrt{D} \quad \text{on } S_{\tilde{r}_1}, \tag{3.23}$$

$$\Phi \leq \frac{1}{2}m^2 + Cm(1 + \mu)\sqrt{D} \quad \text{on } S_{\tilde{r}_2}, \tag{3.24}$$

$$\Phi \geq \frac{1}{2}m^2 - Cm(1 + \mu)\sqrt{D} \quad \text{on } S_{\tilde{r}_2} \tag{3.25}$$

for some universal constant C (does not depend on \mathbf{w} , D , m , μ , etc.). By one-sided maximum principle for the Bernoulli pressure (see, e.g., [9]),

$$\Phi \leq \frac{1}{2}m^2 + Cm(1 + \mu)\sqrt{D} \quad \text{in } \Omega(\tilde{r}_1, \tilde{r}_2). \tag{3.26}$$

In particular, by (3.13)

$$|\mathbf{w}|^2 \leq m^2 + Cm(1 + \mu)\sqrt{D} \stackrel{(3.8)}{\leq} C m^2 \quad \text{in } \Omega(6r_1, \frac{1}{6}r_2), \tag{3.27}$$

in other words,

$$|\mathbf{w}| \leq C m \quad \text{in } \Omega(6r_1, \frac{1}{6}r_2). \tag{3.28}$$

The last inequality gives us the possibility to obtain the improved estimates for $\nabla\omega$.

STEP 4. We prove that for any $\rho \in [6r_1, \frac{1}{30}r_2]$

$$\int_{2\rho \leq r \leq 4\rho} |\nabla\omega|^2 \leq C \frac{1}{\rho} m D(\rho, 5\rho)(1 + \mu). \tag{3.29}$$

Indeed, for $D_\rho := D(\rho, 5\rho)$ there exist two ‘‘good’’ radii $\rho_1 \in (\rho, 2\rho)$, $\rho_2 \in (4\rho, 5\rho)$ such that for $i = 1, 2$, we have

$$\int_{S_{\rho_i}} \omega^2 ds \leq \frac{CD_\rho}{\rho}, \tag{3.30}$$

and

$$\left[\partial_r \int_{S_r} \omega^2 ds \right]_{r=\rho_2} \leq \frac{CD_\rho}{\rho^2}, \quad - \left[\partial_r \int_{S_r} \omega^2 ds \right]_{r=\rho_1} \leq \frac{CD_\rho}{\rho^2}, \tag{3.31}$$

The proof of (3.30)–(3.31) is based on some elementary real analysis and is given in the Appendix B for convenience.

Note that (3.28) imply

$$\max_{z \in S_{\rho_i}} |\mathbf{w}(z)| \leq C m. \tag{3.32}$$

It is well-known that ω satisfies the vorticity equation

$$-\Delta\omega + \mathbf{w} \cdot \nabla\omega = 0. \tag{3.33}$$

A standard energy estimate in the domain $\{\rho_1 \leq r \leq \rho_2\}$ for the above equation, together with the bounds (3.30)–(3.32), gives

$$\begin{aligned} \int_{\rho_1 \leq r \leq \rho_2} |\nabla\omega|^2 &= \int_{S_{\rho_2}} \left(\omega \partial_r \omega - \mathbf{w} \cdot \mathbf{e}_r \frac{\omega^2}{2} \right) ds \\ &\quad - \int_{S_{\rho_1}} \left(\omega \partial_r \omega - \mathbf{w} \cdot \mathbf{e}_r \frac{\omega^2}{2} \right) ds \\ &\leq C \frac{1}{\rho} D_\rho \left(m + \frac{1}{\rho} \right) \leq C \frac{1}{\rho} m D_\rho (1 + \mu). \end{aligned}$$

STEP 5. Applying (3.29) for $\rho = 6r_1, 12r_1, \dots$, etc., we obtain finally that

$$\int_{12r_1 \leq r \leq \frac{1}{8}r_2} r |\nabla \omega|^2 \leq CmD(1 + \mu). \tag{3.34}$$

STEP 6. We need a pair of “good circles” with the same properties as in Step 2–3, but now inside the annulus $\Omega(12r_1, \frac{1}{8}r_2)$ (in order to use estimates (3.34)). So, repeating the arguments of these steps, we obtain, that there exists $\tilde{\rho}_1 \in (12r_1, 24r_1)$, $\tilde{\rho}_2 \in (\frac{1}{16}r_2, \frac{1}{8}r_2)$ such that

$$\Phi \leq \frac{1}{2}m_0^2 + Cm(1 + \mu)\sqrt{D} \quad \text{on } S_{\tilde{\rho}_1}, \tag{3.35}$$

$$\Phi \geq \frac{1}{2}m^2 - Cm(1 + \mu)\sqrt{D} \quad \text{on } S_{\tilde{\rho}_2} \tag{3.36}$$

for some universal constant C (does not depend on \mathbf{w}, D, m, μ , etc.). Therefore,

$$\min_{z \in S_{\tilde{\rho}_2}} \Phi(z) \geq \max_{z \in S_{\tilde{\rho}_1}} \Phi(z) + \frac{1}{2} \left(m^2 - m_0^2 - Cm(1 + \mu)\sqrt{D} \right) \tag{3.37}$$

Let $\varphi(r)$ be the direction of the vector $\bar{\mathbf{w}}(r) = (\bar{w}_1(r), \bar{w}_2(r))$, i.e., $\bar{\mathbf{w}}(r) = |\bar{\mathbf{w}}(r)|(\cos \varphi(r), \sin \varphi(r))$. Without loss of generality we may assume, that $\varphi(\tilde{\rho}_2) = 0$. Then from the inequality (3.34) we have

$$\int_{12r_1 \leq r \leq \frac{1}{8}r_2} \frac{1}{r} |\nabla \omega| \leq \left(CmD(1 + \mu) \right)^{\frac{1}{2}} \sqrt{\frac{\pi}{6r_1}} \leq Cm(1 + \mu)\sqrt{D}, \tag{3.38}$$

consequently, from the formula (2.10) and assumptions (3.6), (3.8) we obtain that

$$|\varphi(r)| \leq C' \varepsilon_0 \quad \forall r \in [\tilde{\rho}_1, \tilde{\rho}_2] \tag{3.39}$$

with some universal constant C' (does not depend on \mathbf{w}, D, m, μ , etc.).

Without loss of generality we may assume that all the constants C in the inequalities (3.37)–(3.38) are the same. Now we are ready for the key step of the proof.

STEP 7. We claim that the inequality

$$m - m_0 \leq 5C(1 + \mu)\sqrt{D} \tag{3.40}$$

holds for sufficiently small ε_0 , where C is the same as in (3.37)–(3.38).

Indeed, suppose this claim fails, than

$$m - m_0 > 5C(1 + \mu)\sqrt{D}, \tag{3.41}$$

in particular, from (3.37) we have

$$\min_{z \in S_{\tilde{\rho}_2}} \Phi(z) > \max_{z \in S_{\tilde{\rho}_1}} \Phi(z) + 2Cm(1 + \mu)\sqrt{D}. \tag{3.42}$$

Now, we are in a position to apply the methods of [15] based on level set structures of Φ to obtain a contradiction when ε_0 is sufficiently small. For a reader’s convenience, we recall the main ideas of the proof in [15] adapted for the present paper.

From (3.42) it follows immediately, that there are two closed regular level sets S'_k and S''_k of Φ such that:

- (i) $\Phi|_{S'} \equiv t', \quad \Phi|_{S''} \equiv t'', \quad t'' - t' > 2Cm(1 + \mu)\sqrt{D}$;
- (ii) S', S'' are smooth closed curves (homeomorphic to the circle) surrounding the origin, both of them lie between circles $S_{\tilde{\rho}_1}$ and $S_{\tilde{\rho}_2}$.
- (iii) The vorticity $\omega(z)$ does not change sign between the curves S', S'' ; for definiteness, we can assume without loss of generality that

$$\omega(z) > 0 \quad \text{for all } z \text{ between } S' \text{ and } S'' \tag{3.43}$$

(for the last assertion, see [15, Step 6]).

Using (3.39), it is not difficult to prove for ε_0 small enough, that there exists a unit vector $\tilde{\mathbf{e}} = (\cos \tilde{\theta}, \sin \tilde{\theta})$ such that the segment $L = \{\rho\tilde{\mathbf{e}} : \rho \in [\tilde{\rho}_1, \tilde{\rho}_2]\}$ satisfies the following properties:

- (iv) $\mathbf{w}^\perp(z) \cdot \tilde{\mathbf{e}} < 0$ for any $z \in L$, where we denote $(a, b)^\perp = (-b, a)$;
- (v) $\int_L |\nabla\omega| ds < 2Cm(1 + \mu)\sqrt{D}$.

Now take two points $A \in L \cap S'$ and $B \in L \cap S''$ such that the line segment $[A, B]$ lies between the curves S' and S'' . Recall, that the gradient of the Bernoulli pressure satisfies the identity

$$\nabla\Phi \equiv -\nabla^\perp\omega + \omega \cdot \mathbf{w}^\perp.$$

Then we have

$$\begin{aligned} t'' - t' &= \Phi(B) - \Phi(A) = \int_{[A,B]} \nabla\Phi \cdot \tilde{\mathbf{e}} ds \\ &= - \int_{[A,B]} \nabla^\perp\omega \cdot \tilde{\mathbf{e}} ds + \int_{[A,B]} \omega \mathbf{w}^\perp \cdot \tilde{\mathbf{e}} dr := I + II. \end{aligned} \tag{3.44}$$

Estimate the terms I and II separately. From the above property (v) we have

$$I < 2Cm(1 + \mu)\sqrt{D}. \tag{3.45}$$

On the other hand, from (iii)–(iv) we obtain

$$II < 0. \tag{3.46}$$

Therefore,

$$t'' - t' \leq 2Cm(1 + \mu)\sqrt{D},$$

a contradiction with (i).

Of course, the above items (i)–(v) are only short description. In case of interest, a reader can find a detailed justification for all these steps in [15].

The proof of Lemma 8 is finished completely. □

Proof of Theorem 7. We start from the same simplifications as in the proof of Lemma 8. For definiteness, assume that $|\mathbf{w}(r_2)| \geq |\mathbf{w}(r_1)|$. Because of the triangle inequality, it is sufficient to consider the case

$$m = |\bar{\mathbf{w}}(r_2)| = \max_{r \in [r_1, r_2]} |\bar{\mathbf{w}}(r)|. \tag{3.47}$$

Also, it is sufficient to consider only the situation when

$$|\bar{\mathbf{w}}(r)| \geq \frac{1}{6}m \quad \forall r \in [r_1, r_2] \tag{3.48}$$

$$\sqrt{D(r_1, r_2)} = \sqrt{D} \leq \frac{1}{100}m. \tag{3.49}$$

(see the explanation in the proof of Lemma 8). Further we have to repeat all the arguments of the proof of Lemma 8 up to inequality (3.38). Then from the formulas (2.10), (3.38), and assumptions (3.48), (3.49) we obtain that

$$|\varphi(r)| \leq C \frac{1}{m^2} \left(D + m(1 + \mu)\sqrt{D} \right) \leq C' \frac{1}{m} (1 + \mu)\sqrt{D} \quad \forall r \in [\tilde{\rho}_1, \tilde{\rho}_2]. \tag{3.50}$$

The last formula and (3.4) imply easily

$$|\bar{\mathbf{w}}(\tilde{\rho}_1) - \bar{\mathbf{w}}(\tilde{\rho}_2)| \leq C(1 + \mu)\sqrt{D}. \tag{3.51}$$

Then, by virtue of elementary estimate (2.9) and inclusions $\tilde{\rho}_1 \in (12r_1, 24r_1)$, $\tilde{\rho}_2 \in (\frac{1}{16}r_2, \frac{1}{8}r_2)$, we obtain the required inequality (3.51).

The proof of Theorem 7 is finished completely. □

4. The Second Basic (Asymptotic) Estimate for the Velocity

For the limiting case (when r_i are very large and velocity is “almost constant” on the boundary circles) the general estimate of Theorem 7 can be improved essentially.

Theorem 9. *Let \mathbf{w}_k be a sequence of D -solutions to the Navier–Stokes system*

$$\begin{cases} \Delta \mathbf{w}_k - (\mathbf{w}_k \cdot \nabla) \mathbf{w}_k - \nabla p_k = \mathbf{0}, \\ \nabla \cdot \mathbf{w}_k = 0 \end{cases} \tag{4.1}$$

in the annulus type domains $\Omega_{r_{1k}, r_{2k}}$. Suppose, in addition, that

$$r_{1k} \rightarrow +\infty, \quad \frac{r_{2k}}{r_{1k}} \rightarrow +\infty, \tag{4.2}$$

and there exist two vectors $\mathbf{w}_0, \mathbf{w}_\infty \in \mathbb{R}^n$ such that

$$\max_{z \in S_{r_{1k}}} |\mathbf{w}_k(z) - \mathbf{w}_0| \rightarrow 0, \quad \max_{z \in S_{r_{2k}}} |\mathbf{w}_k(z) - \mathbf{w}_\infty| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{4.3}$$

Then

$$|\mathbf{w}_0 - \mathbf{w}_\infty| \leq C_{**} \frac{D_*}{m}, \tag{4.4}$$

where $m := \max\{|\mathbf{w}_0|, |\mathbf{w}_\infty|\}$, $D_ = \liminf_{k \rightarrow \infty} \int_{\Omega_{r_{1k}, r_{2k}}} |\nabla \mathbf{w}_k|^2$, and C_{**} is some universal positive constant (does not depend on \mathbf{w}_k , etc.)*

To prove this theorem, we have to use some fine properties of solutions to Euler system from [17], considered in the next sect. 4.1.

4.1. *On solutions to Euler equations.* In this section we consider some properties of weak solutions $\mathbf{v} \in W_{loc}^{1,2}(\Omega)$ to the Euler system

$$\begin{cases} (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v} = \mathbf{e} & \text{on } S_1 = \partial\Omega, \end{cases} \tag{4.5}$$

where $S_1 = \{z \in \mathbb{R}^2 : |z| = 1\}$ is the unit circle and $\mathbf{e} = (1, 0)$ is the unit vector. Here Ω is the unit disk or its complement, *i.e.*,

$$\Omega = B_1 \tag{4.6}$$

or

$$\Omega = \mathbb{R}^2 \setminus B_1. \tag{4.7}$$

Suppose that

$$\int_{\Omega} |\nabla \mathbf{v}|^2 < \varepsilon^2 \tag{4.8}$$

for some constant $\varepsilon \in (0, 1)$. Then from the first equation (4.5)₁ one can assume that $|p| \sim \varepsilon$. Nevertheless, surprisingly much better estimate holds.

Theorem 10. *Let $\mathbf{v} \in W_{loc}^{1,2}(\Omega)$ and $p \in W_{loc}^{1,1}(\Omega)$ satisfy the Euler equations (4.5)_{1–2} for almost all $z \in \Omega$. Suppose also that the estimate (4.8) and the boundary condition (4.5)₃ are fulfilled. Then $p \in C(\bar{\Omega})$, moreover,*

$$\sup_{z_1, z_2 \in \bar{\Omega}} |p(z_1) - p(z_2)| \leq C \varepsilon^2, \tag{4.9}$$

where C is some universal constant (does not depend on $\varepsilon, \mathbf{v}, p$).

Proof. For the case $\Omega = B_1$ the result was proved in [17]. So let us consider the case of exterior domain $\Omega = \mathbb{R}^2 \setminus B_1$.

By well-known fact concerning D -solutions to Euler and Navier–Stokes system (see, *e.g.*, [9, Lemma 4.1]), the averages of the pressure $\bar{p}(r)$ are uniformly bounded and have some limit at infinity, without loss of generality we may assume that this limit is zero:

$$\bar{p}(r) \rightarrow 0 \quad \text{as } r \rightarrow \infty. \tag{4.10}$$

Moreover, since $\int_{\Omega} |\nabla \mathbf{v}|^2 dx < \infty$, there exists an increasing sequence $r_i \rightarrow +\infty$ such that

$$\int_{S_{r_i}} |\nabla \mathbf{v}| ds \leq \frac{\varepsilon_i}{\sqrt{\ln r_i}} \quad \text{with } \varepsilon_i \rightarrow 0 \quad \text{as } i \rightarrow \infty \tag{4.11}$$

and

$$\sup_{z \in S_{r_i}} |\mathbf{v}(z)| \leq \varepsilon_i \sqrt{\ln r_i} \tag{4.12}$$

(see [9, Lemmas 2.1–2.2]). From (4.10)–(4.12) and from the equation (4.5)₁ it follows that

$$\sup_{z \in S_{r_i}} |p(z)| \rightarrow 0 \quad \text{as } i \rightarrow \infty. \tag{4.13}$$

Indeed,

$$|p(r_i, \theta) - \bar{p}(r_i)| \leq \int_{S_{r_i}} |\nabla p| \, ds \leq \int_{S_{r_i}} |\mathbf{v}| \cdot |\nabla \mathbf{v}| \, ds \leq \varepsilon_i \sqrt{\ln r_i} \int_{S_{r_i}} |\nabla \mathbf{v}| \, ds \leq \varepsilon_i^2.$$

Taking divergence on the first equation in (4.5) gives

$$\Delta p = -\nabla \mathbf{v} \cdot (\nabla \mathbf{v})^\top \tag{4.14}$$

We can extend \mathbf{v} outside Ω by the constant vector \mathbf{e} so that \mathbf{v} is globally defined and divergence free in \mathbb{R}^2 . By the classical div-curl lemma (see, e.g., [3]), $\nabla \mathbf{v} \cdot (\nabla \mathbf{v})^\top$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$. Put

$$G(x) = -\frac{1}{2\pi} \int_{\Omega} \log|x - y| (\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top)(y) \, dy.$$

By Calderón–Zygmund theorem for Hardy’s spaces [23], $G \in D^{2,1}(\mathbb{R}^2) \cap D^{1,2}(\mathbb{R}^2)$, where $D^{k,q}(\mathbb{R}^2)$ means the space of measurable functions, whose distributional derivatives of k -th order belong to $L^q(\mathbb{R}^2)$. By classical facts from the theory of Sobolev spaces (see, e.g., [19]), the last inclusion implies that G is continuous and convergent to zero at infinity, in particular,

$$\|\nabla G\|_{L^2(\mathbb{R}^2)} + \|\nabla^2 G\|_{L^1(\mathbb{R}^2)} \leq C\varepsilon^2, \tag{4.15}$$

$$\sup_{z \in \mathbb{R}^2} |G(z)| < C\varepsilon^2, \tag{4.16}$$

$$G(z) \rightarrow 0, \quad \text{as } z \rightarrow \infty. \tag{4.17}$$

Consider the decomposition

$$p = p_* + G. \tag{4.18}$$

By construction, $\Delta p_* = 0$ in Ω , i.e., p_* is a harmonic function, and by (4.13) we have

$$\sup_{z \in S_{r_i}} |p_*(z)| \rightarrow 0 \quad \text{as } i \rightarrow \infty. \tag{4.19}$$

Of course, it implies

$$|p_*(z)| \rightarrow 0 \quad \text{as } |z| \rightarrow \infty. \tag{4.20}$$

Since $p = G + p_*$, we have $p(z) \rightarrow 0$ as $z \rightarrow \infty$. In [17, Lemma 9], it was proved that p satisfies the Neumann boundary conditions (in weak sense) on S_1 :

$$\partial_n p = 0 \quad \text{on } S_1.$$

Therefore, p can be solved from (4.14) using the Green function for the Poisson problem in Ω with Neumann boundary conditions:

$$p(x) = -\frac{1}{2\pi} \int_{\Omega} \left(\log|x - y| + \log \left| \frac{x}{|x|^2} - y \right| - \log|y| \right) (\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top)(y) dy. \tag{4.21}$$

It is important to observe that in the above formula, the integral domain Ω can be replaced by \mathbb{R}^2 , since $\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top = 0$ in B_1 due to our extension of \mathbf{v} by the unit constant vector in B_1 . Using the above definition of the function $G(x)$, we can rewrite identity (4.21) as

$$p(x) = G(x) - G(0) + G\left(\frac{x}{|x|^2}\right).$$

So the the required bound (4.9) follows from (4.16). □

The proof of the central Theorem 10 is finished. In [17, §2] it was shown, that the established estimates imply

Corollary 1. *Let the assumptions of Theorem 10 be fulfilled. Then there exists a set $\mathcal{F}_0 \subset (\frac{1}{2}, \frac{3}{2})$ of positive measure (having one-sided density 1 at 1) such that*

$$S_r \subset \Omega \quad \text{and} \quad \max_{z \in S_r} \left(|\mathbf{v}(z) - \mathbf{e}| + \left| p(z) - \int_{S_r} p \right| \right) \leq C \varepsilon^2 \quad \forall r \in \mathcal{F}_0, \tag{4.22}$$

where C is some universal constant (does not depend on $\varepsilon, \mathbf{v}, p$, etc.).

Sometimes solutions to Euler system can be obtained as a limit of solutions to Navier–Stokes equations when viscosity coefficient tends to zero. We can use this fact in the following useful statement.

Theorem 11. *Let the assumptions of Theorem 9 be fulfilled. Suppose, in addition, that $\bar{p}_k(r_{1k}) = 0$. Then there exists $\delta \in (0, \frac{1}{2})$ such that*

$$m_0 \cdot |\mathbf{w}_k(z) - \mathbf{w}_0| + |p_k(z)| \leq C D_* + \varepsilon_k \quad \forall z \in S_{(1+\delta)r_{1k}}, \tag{4.23}$$

$$m_\infty \cdot |\mathbf{w}_k(z) - \mathbf{w}_\infty| + |p_k(z)| \leq C D_* + \varepsilon_k \quad \forall z \in S_{(1-\delta)r_{2k}}, \tag{4.24}$$

where $m_\infty = |\mathbf{w}_\infty|$, $m_0 = |\mathbf{w}_0|$, $D_* = \lim_{k \rightarrow \infty} \int_{\Omega_{r_{1k}, r_{2k}}} |\nabla \mathbf{w}_k|^2$, $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$, and C is some universal positive constant (does not depend on \mathbf{w}_k, r_{ik} , etc.)

Proof. The assertion of the last theorem was deduced (by corresponding scaling and limiting procedure) from Theorem 10 and Corollary 1 in [17, Section 4] (see, e.g., the formula (4.10) in [17] and commentaries). □

4.2. Proof of Theorem 9. Let the conditions of Theorem 9 be fulfilled. As before, we have to make several simplifications. For definiteness, assume that $m = |\mathbf{w}_\infty| \geq |\mathbf{w}_0|$. Denote $m_0 = |\mathbf{w}_0|$. Now take and fix some small number $\varepsilon_0 \in (1, \frac{1}{100})$ (the exact value of ε_0 will be specified below). Obviously, it is sufficient to consider only the case when

$$D_* < \varepsilon_0 m^2. \tag{4.25}$$

Indeed, if the opposite inequality valid, then the required estimate (4.4) is fulfilled automatically with the fixed constant $C = \frac{2}{\varepsilon_0}$, and there are nothing to prove. So below we assume that (4.25) is fulfilled as well, therefore,

$$D(r_{1k}, r_{2k}) < \varepsilon_0 m^2 \tag{4.26}$$

for k large enough. Applying Theorem 7 (with μ arbitrary small as $k \rightarrow \infty$ because of assumption $r_{1k} \rightarrow +\infty$), we obtain

$$\frac{1}{2}m < |\bar{\mathbf{w}}_k(r)| < \frac{3}{2}m \quad \forall r \in [r_{1k}, r_{2k}]. \tag{4.27}$$

Take $\delta \in (0, \frac{1}{2})$ from Theorem 11 and denote $\rho_{1k} = (1 - \delta)r_{1k}$, $\rho_{2k} = (1 - \delta)r_{2k}$. Repeating Steps 1–6 of the proof of Lemma 8 (with evident modifications and arbitrary small μ as $k \rightarrow +\infty$), we obtain that

$$\int_{\rho_{1k} \leq r \leq \rho_{2k}} \frac{1}{r} |\nabla \omega_k| \leq \varepsilon_k \rightarrow 0 \quad \text{as } k \rightarrow +\infty, \tag{4.28}$$

consequently, from the formula (2.10) and assumptions (4.27), (4.26) we obtain that

$$|\varphi_k(r)| \leq \frac{1}{4\pi m^2} (D_* + \varepsilon_k) \leq \frac{1}{4\pi} \varepsilon_0 \quad \forall r \in [\rho_{1k}, \rho_{2k}] \tag{4.29}$$

for all sufficiently large k .

Recall, that by Theorem 11,

$$m_0 |\mathbf{w}_k(z) - \mathbf{w}_0| + |p_k(z)| \leq CD_* + \varepsilon_k \quad \forall z \in S_{\rho_{1k}}, \tag{4.30}$$

$$m |\mathbf{w}_k(z) - \mathbf{w}_\infty| + |p_k(z)| \leq CD_* + \varepsilon_k \quad \forall z \in S_{\rho_{2k}}. \tag{4.31}$$

Consequently, using $m_0 \geq \frac{1}{2}m$, we obtain:

$$\min_{z \in S_{\rho_{2k}}} \Phi_k(z) \geq \max_{z \in S_{\rho_{1k}}} \Phi_k(z) + \frac{1}{2} \left(m^2 - m_0^2 - 12CD_* - \varepsilon_k \right), \tag{4.32}$$

where C is the same as in (4.30)–(4.31).

Below we consider the case $D_* > 0$ (the case $D_* = 0$ can be considered similarly, with some evident simplifications). Now we claim that the inequality

$$m - m_0 \leq \frac{10}{m} CD_* \tag{4.33}$$

holds for sufficiently small ε_0 , where C is the same as in (4.32). Indeed, suppose that the opposite inequality is valid, then from (4.32) we have

$$\min_{z \in S_{\rho_{2k}}} \Phi_k(z) > \max_{z \in S_{\rho_{1k}}} \Phi_k(z) + CD_*. \tag{4.34}$$

Now we can apply the methods of [15] (based on level set structures of Φ_k) in the same way as on the Step 7 of the proof of previous Lemma 8, and to obtain the desired contradiction. So the claim (4.33) is proved.

The formulas (4.30)–(4.31), (4.33) imply that

$$|\bar{\mathbf{w}}_k(\rho_{1k}) - \mathbf{w}_0| \leq C \frac{1}{m} D_* \tag{4.35}$$

$$|\bar{\mathbf{w}}_k(\rho_{2k}) - \mathbf{w}_\infty| \leq C \frac{1}{m} D_* \tag{4.36}$$

$$\left| |\bar{\mathbf{w}}_k(\rho_{2k})| - |\bar{\mathbf{w}}_k(\rho_{1k})| \right| \leq C \frac{1}{m} D_* \tag{4.37}$$

with some universal constant C . We need the following elementary fact: for any pair of vectors $\mathbf{a} = a\mathbf{e}_a$, $\mathbf{b} = b\mathbf{e}_b$, where \mathbf{e}_a and \mathbf{e}_b are the corresponding unit vectors, one has:

$$|\mathbf{a} - \mathbf{b}| \leq |a - b| + b|\mathbf{e}_a - \mathbf{e}_b|.$$

This elementary formula together with established estimates (4.37), (4.29) imply

$$|\bar{\mathbf{w}}_k(\rho_{2k}) - \bar{\mathbf{w}}_k(\rho_{1k})| \leq C \frac{1}{m} D_*. \tag{4.38}$$

Thus by (4.35)–(4.36) the proof of Theorem 9 is finished completely. □

5. Uniform Bounds for the Invading Domain Solutions \mathbf{w}_k

The main result of this section is the following k -uniform bounds.

Theorem 12. *Suppose that $R > 0$, $\text{supp } \mathbf{f} \subset B_R$, and $A = \|\mathbf{f}\|_{H^{-1}(B_{2R})} < \infty$, and consider $R_k \geq 2R$. Then the following uniform estimates for the invading domain solutions \mathbf{w}_k to (1.4) are valid:*

$$D_k := \int_{B_{R_k}} |\nabla \mathbf{w}_k|^2 \leq C_1 A \left(A + \lambda + \frac{A^{\frac{1}{3}}}{R^{\frac{2}{3}}} \right), \tag{5.1}$$

$$\max_{R \leq r \leq R_k} |\bar{\mathbf{w}}_k(r)| \leq C_2 \left(A + \lambda + \frac{A^{\frac{1}{3}}}{R^{\frac{2}{3}}} \right), \tag{5.2}$$

where C_i are some universal constants (do not depend on \mathbf{w}_k and parameters A, R, λ , etc.).

Remark 13. The considered problem (1.4) has the following scaling property: if \mathbf{w}_k is a solution to (1.4) with parameters (A, λ, R) , then for any $\tau > 0$ the mapping $\mathbf{w}_{\tau k}(z) := \tau \mathbf{w}_k(\tau z)$ is a solution to (1.4) with the corresponding parameters $(\tau A, \tau \lambda, \tau^{-1} R)$. Note, that basic estimates in theorems 7, 9, 12, 15–17 are scaling invariant.

Proof. Let χ be a smooth cut-off function satisfying $\chi \equiv 1$ on B_R , $\chi = 0$ outside B_{2R} , and $|\nabla\chi| \leq \frac{C}{R}$. By the system (1.4) and the assumptions on \mathbf{f} , we have the following direct estimate of the Dirichlet integral

$$\begin{aligned}
 D_k &= \int_{B_{R_k}} |\nabla \mathbf{w}_k|^2 = \int_{B_{R_k}} \mathbf{f} \cdot (\mathbf{w}_k - \mathbf{w}_\infty) \\
 &= \int_{B_{R_k}} \chi \mathbf{f} \cdot (\mathbf{w}_k - \mathbf{w}_\infty) \\
 &\leq \|\mathbf{f}\|_{H^{-1}(B_{2R})} \|\chi(\mathbf{w}_k - \mathbf{w}_\infty)\|_{H_0^1(B_{2R})} \\
 &\leq C A \left(\|\nabla \mathbf{w}_k\|_{L^2(B_{2R})} + \frac{1}{R} \|\mathbf{w}_k - \mathbf{w}_\infty\|_{L^2(B_{2R})} \right) \\
 &\leq C A (D_k^{\frac{1}{2}} + |\bar{\mathbf{w}}_k(2R)| + \lambda), \tag{5.3}
 \end{aligned}$$

here we used (2.2) for the last inequality (the exact values of C may change from line to line). Using (5.3) it is easy to deduce

$$D_k \leq C A (A + |\bar{\mathbf{w}}_k(2R)| + \lambda). \tag{5.4}$$

Therefore,

$$D_k \leq C A (A + \Lambda_k + \lambda), \tag{5.5}$$

where we denote $\Lambda_k = \max_{R \leq r \leq R_k} |\bar{\mathbf{w}}_k(r)|$. Let $\Lambda_k = |\bar{\mathbf{w}}_k(r_k)|$ with some $r_k \in [R, R_k]$. Applying Theorem 7 in the annulus type domain Ω_{r_k, R_k} and inserting (5.5), we obtain

$$\begin{aligned}
 |\Lambda_k - \lambda| &\leq C_* \left(1 + \frac{1}{R\Lambda_k}\right) \sqrt{D_k} \\
 &\leq C \left(1 + \frac{1}{R\Lambda_k}\right) \sqrt{A(A + \Lambda_k + \lambda)}. \tag{5.6}
 \end{aligned}$$

Now we have to consider two different cases:

- CASE 1. $\Lambda_k \leq 2(A + \lambda)$. Then the required estimates (5.1)–(5.2) follow immediately.
- CASE 2. Now suppose that $\Lambda_k > 2(A + \lambda)$ Then from (5.6) we obtain:

$$\Lambda_k \leq C \left(1 + \frac{1}{R\Lambda_k}\right) \sqrt{A\Lambda_k}. \tag{5.7}$$

If $\frac{1}{R\Lambda_k} \leq 2$, then, similarly to the Case 1, we have $\Lambda_k \leq C A$, and the required estimate (5.1) follow easily from (5.5). So we can assume in addition that $\frac{1}{R\Lambda_k} > 2$. Then

$$\Lambda_k \leq C \frac{1}{R\Lambda_k} \sqrt{A\Lambda_k} \leq C \frac{1}{R} \sqrt{\frac{A}{\Lambda_k}}. \tag{5.8}$$

Therefore

$$\Lambda_k \leq C \frac{A^{\frac{1}{3}}}{R^{\frac{2}{3}}}, \tag{5.9}$$

and finally the required estimates (5.1)–(5.2) are valid in this case as well. □

Corollary 2. *Under the hypothesis of Theorem 12, for any sequence $R_k \rightarrow \infty$, there exists a subsequence of the invading domain solutions \mathbf{w}_k to (1.4) which converges weakly to a D -solution \mathbf{w}_L to the Navier–Stokes equations (1.1)_{1,2}. Moreover, the convergence is strong in the C^m_{loc} -topology in $\mathbb{R}^2 \setminus \bar{B}_R$ for any $m \geq 0$, and also strong in $L^2(B_{2R})$.*

Proof. Theorem 12 implies that

$$\|\mathbf{w}_k\|_{H^1(B_\rho)} \leq C(A, R, \lambda, \rho) \tag{5.10}$$

uniformly in k , for any $\rho > 0$. Hence, we can extract a subsequence of \mathbf{w}_k which converges weakly to some function $\mathbf{w}_L \in H^1_{loc}(\mathbb{R}^2)$ satisfying the equations (1.1)_{1,2} and

$$\int_{\mathbb{R}^2} |\nabla \mathbf{w}_L|^2 \leq \liminf_{k \rightarrow \infty} \int_{B_{R_k}} |\nabla \mathbf{w}_k|^2 \leq C(A, R, \lambda). \tag{5.11}$$

By the standard local regularity theory for stationary Navier–Stokes equations (see, e.g., [16, Section 2.3]), we can obtain uniform local C^m bounds for \mathbf{w}_k in $\mathbb{R}^2 \setminus \bar{B}_R$ (outside the support of \mathbf{f}) for any $m > 0$. This together with Arzela-Ascoli theorem implies the local strong C^m convergence in $\mathbb{R}^2 \setminus \bar{B}_R$. Finally, (5.10) and the compact embedding $H^1(B_{2R}) \subset L^2(B_{2R})$ implies the strong convergence in $L^2(B_{2R})$. \square

6. The Limiting Velocity of \mathbf{w}_L

Let \mathbf{w}_k be the invading domain solutions that we studied in the last section. By passing to a subsequence, we assume that \mathbf{w}_k converges weakly to \mathbf{w}_L in the sense described in Corollary 2. In this section, we study the fundamental question: *does \mathbf{w}_L achieve the prescribed limiting velocity $\mathbf{w}_\infty = \lambda \mathbf{e}_1$ at spatial infinity?*

Since \mathbf{w}_L is a D -solution to the Navier–Stokes equations (without force) in the exterior domain $\mathbb{R}^2 \setminus B_{2R}$, it has some finite uniform limit at spatial infinity (this important fact is established through the papers [1, 9, 13]). Let us denote $\mathbf{w}_0 = \lim_{r \rightarrow \infty} \mathbf{w}_L$. We state a simple but crucial identity for the tail Dirichlet integral

$$D_* = \lim_{r \rightarrow \infty} \overline{\lim}_{k \rightarrow \infty} \int_{\Omega_k \cap \{|z| \geq r\}} |\nabla \mathbf{w}_k|^2. \tag{6.1}$$

Lemma 14. *If $\mathbf{w}_0 \neq 0$, then $D_* = -\mathcal{F} \cdot (\mathbf{w}_\infty - \mathbf{w}_0)$.*

Proof. Recall the following energy equalities for the stationary Navier–Stokes solutions \mathbf{w}_k and \mathbf{w}_L respectively,

$$D_k = \int_{B_{R_k}} |\nabla \mathbf{w}_k|^2 = \int_{B_{R_k}} \mathbf{f} \cdot (\mathbf{w}_k - \mathbf{w}_\infty) = \int_{B_{R_k}} \mathbf{f} \cdot \mathbf{w}_k - \mathcal{F} \cdot \mathbf{w}_\infty, \tag{6.2}$$

$$D_L = \int_{\mathbb{R}^2} |\nabla \mathbf{w}_L|^2 = \int_{\mathbb{R}^2} \mathbf{f} \cdot (\mathbf{w}_L - \mathbf{w}_0) = \int_{\mathbb{R}^2} \mathbf{f} \cdot \mathbf{w}_L - \mathcal{F} \cdot \mathbf{w}_0. \tag{6.3}$$

The last line relies on the asymptotic behaviour of \mathbf{w}_L at spatial infinity when $\mathbf{w}_0 \neq 0$. By the result of [20], \mathbf{w}_L is *physically reasonable* in the sense of [22, page 350]. In [22, Theorem 5] it is proved that \mathbf{w}_L exhibits the wake region behaviour, and has the polynomial convergence rates at spatial infinity indicated by the Oseen fundamental

tensor (with some logarithmic correction). Thus (6.3) can be obtained with multiplying (1.1)₁ by $\mathbf{w}_L - \mathbf{w}_0$ and integrating on B_ρ , then sending $\rho \rightarrow \infty$. Observe that $\int \mathbf{f} \cdot \mathbf{w}_k \rightarrow \int \mathbf{f} \cdot \mathbf{w}_L$ since $\mathbf{w}_k \rightharpoonup \mathbf{w}_L$ in $H^1(B_{2R})$. It remains to conclude using the simple fact that

$$\begin{aligned} D_* &= \lim_{r \rightarrow \infty} \overline{\lim}_{k \rightarrow \infty} \left\{ D_k - \int_{\Omega_k \cap \{|z| \leq r\}} |\nabla \mathbf{w}_k|^2 \right\} \\ &= \overline{\lim}_{k \rightarrow \infty} D_k - D_L. \end{aligned} \tag{6.4}$$

□

Next, we justify the equality $\mathbf{w}_0 = \mathbf{w}_\infty$ in two different scenarios.

6.1. Scenario I: λ is much larger than force.

Theorem 15. *There exists an universal constant $\varepsilon > 0$ such that, under the hypothesis of Theorem 12, if*

$$A \leq \frac{\varepsilon^2}{\ln^{\frac{1}{2}} \left(\frac{1}{\lambda R} + 2 \right)} \lambda, \tag{6.5}$$

then we have

- (i) $\int_{B_{R_k}} |\nabla \mathbf{w}_k|^2 \leq C_1 \varepsilon^2 \lambda^2$, for $R_k \geq 2\hat{R} := 2 \max\{R, \frac{1}{\lambda}\}$,
- (ii) $\max_{\hat{R} \leq r \leq R_k} |\bar{\mathbf{w}}_k(r)| \leq C_2 \lambda$,
- (iii) The limiting solution \mathbf{w}_L satisfies the condition $\mathbf{w}_0 = \mathbf{w}_\infty = \lambda \mathbf{e}_1$.

Here again C_i are some universal constants (do not depend on $\mathbf{w}_k, A, \lambda, \text{etc.}$). If, in addition, the total force $\mathcal{F} = 0$, then the factor $\ln^{\frac{1}{2}} \left(2 + \frac{1}{\lambda R} \right)$ in (6.5) can be removed.

Proof. To prove Claims 1–2, we consider two separate cases:

Case I: $\lambda R \geq 1$. In this case, $\ln^{\frac{1}{2}} \left(\frac{1}{\lambda R} + 2 \right) \sim 1$. The term $\frac{A^{\frac{1}{3}}}{R^{\frac{2}{3}}}$ in (5.1)–(5.2) can be estimated as

$$\frac{A^{\frac{1}{3}}}{R^{\frac{2}{3}}} = \frac{A^{\frac{1}{3}} \lambda^{\frac{2}{3}}}{R^{\frac{2}{3}} \lambda^{\frac{2}{3}}} \leq A^{\frac{1}{3}} \lambda^{\frac{2}{3}} \stackrel{(6.6)}{\leq} \varepsilon^{\frac{2}{3}} \lambda. \tag{6.6}$$

Then Claims 1–2 follows immediately from (6.5) and Theorem 12.

Case II: $\lambda R < 1$. By Lemma 3, we have

$$\|\mathbf{f}\|_{H^{-1}(B_{\frac{2}{\lambda}})} \leq C \left(1 + \ln^{\frac{1}{2}} \left(\frac{1}{\lambda R} \right) \right) \|\mathbf{f}\|_{H^{-1}(B_{2R})}. \tag{6.7}$$

It is easy to check that, in this case, $1 + \ln^{\frac{1}{2}} \left(\frac{1}{\lambda R} \right) \sim \ln^{\frac{1}{2}} \left(\frac{1}{\lambda R} + 2 \right)$. Since $\frac{1}{\lambda} > R$, \mathbf{f} is of course compactly supported in $B_{\frac{1}{\lambda}}$. Hence, we can reduce to Case I with the new parameters

$$R' = \frac{1}{\lambda}$$

and

$$A' \leq C \ln^{\frac{1}{2}} \left(\frac{1}{\lambda R} + 2 \right) A.$$

Hence, Claims 1–2 are again valid in this case.

Note that when $\mathcal{F} = 0$, the logarithmic factors can be removed due to the last statement in Lemma 3.

To prove Claim 3, notice that we are now in a position similar to [17]. More precisely, if $\varepsilon < \varepsilon_0$ with ε_0 small enough, then we have

- (a) $D_k = \int |\nabla \mathbf{w}_k|^2 \leq \varepsilon \lambda^2$,
- (b) The total force $|\mathcal{F}| = \left| \int_{\mathbb{R}^2} \mathbf{f} \right| \leq \varepsilon \lambda$.

These are the key ingredients to invoke the proof of [17, Theorem 1] (see Section 6 there). Here we can simplify the proof essentially using our second basic estimate in Theorem 9. Namely, using Corollary 2, we can find $r_{1k} \rightarrow \infty$ such that (4.2) and (4.3) are valid. By Theorem 9 we have

$$|\mathbf{w}_0 - \mathbf{w}_\infty| \leq C_{**} \frac{D_*}{\lambda}, \tag{6.8}$$

where D_* is the tail Dirichlet energy defined in (6.1) which controls the limit $\lim_{k \rightarrow \infty} \int_{\Omega_{r_{1k}, r_{2k}}} |\nabla \mathbf{w}_k|^2$. By Lemma 14 and (a), (b), we have

$$D_* \leq \begin{cases} |\mathcal{F}| |\mathbf{w}_\infty - \mathbf{w}_0| \leq \varepsilon \lambda |\mathbf{w}_\infty - \mathbf{w}_0|, & \text{if } \mathbf{w}_0 \neq 0, \\ \lim_{k \rightarrow \infty} D_k \leq \varepsilon \lambda^2, & \text{if } \mathbf{w}_0 = 0. \end{cases} \tag{6.9}$$

By (6.8)–(6.9) we obtain $|\mathbf{w}_\infty - \mathbf{w}_0| \leq C_{**} \varepsilon |\mathbf{w}_\infty - \mathbf{w}_0|$ which, by taking ε small, immediately implies $\mathbf{w}_\infty = \mathbf{w}_0$. □

6.2. Scenario II : zero total force, and $\lambda = 0$.

Theorem 16. *Under the hypothesis of Theorem 12, assume in addition that the total force vanishes, i.e., $\mathcal{F} = \int_{\mathbb{R}^2} \mathbf{f} = \mathbf{0}$. Moreover, we consider zero prescribed limiting velocity, i.e., $\mathbf{w}_\infty = \mathbf{0}$. Then the limiting solution \mathbf{w}_L satisfies the condition $\mathbf{w}_0 = \mathbf{w}_\infty = \mathbf{0}$.*

Proof. We assume that $\mathbf{w}_0 \neq 0$ and obtain a contradiction. By Lemma 14 we have

$$D_* = -\mathcal{F} \cdot (0 - \mathbf{w}_0) = 0. \tag{6.10}$$

Using Corollary 2, we can find $r_{1k} \rightarrow \infty$ such that (4.2) and (4.3) are valid. By Theorem 9 we have

$$|\mathbf{w}_0 - \mathbf{w}_\infty| \leq C_{**} \frac{D_*}{|\mathbf{w}_0|}, \tag{6.11}$$

where D_* is the tail Dirichlet energy defined in (6.1) which controls the limit $\lim_{k \rightarrow \infty} \int_{\Omega_{r_{1k}, r_{2k}}} |\nabla \mathbf{w}_k|^2$. (6.10)–(6.11) immediately imply that $\mathbf{w}_0 = \mathbf{w}_\infty$, a contradiction with the initial assumption. □

7. A Uniqueness Theorem in the Case of Small Force

Theorem 17. *There exists a universal constant $\varepsilon > 0$ such that, under the hypothesis of Theorem 12, if*

$$A < \frac{\varepsilon^2}{(1 + \lambda R)^3 \ln^{\frac{1}{2}} \left(2 + \frac{1}{\lambda R}\right)} \lambda, \tag{7.1}$$

then the problem (1.1) is uniquely solvable in the class of D -solutions. If, in addition, the total force $\mathcal{F} = 0$, then the factor $\ln^{\frac{1}{2}} \left(2 + \frac{1}{\lambda R}\right)$ in (7.1) can be removed.

Remark 18. In fact, the essence of the proof below can be best understood by setting $R = 1, \lambda = 1$.

Proof. The existence of a D -solution \mathbf{w}_L to (1.1) when f is sufficiently small was already proved in Theorem 15 based on the invading domains method. We just need to show uniqueness. Using scaling, we may assume, without loss of generality, that $\lambda = 1$, i.e., $\mathbf{w}_\infty = \mathbf{e}_1$. In particular, the assumption (7.1) will now read

$$A < \frac{\varepsilon^2}{(1 + R)^3 \ln^{\frac{1}{2}} \left(2 + \frac{1}{R}\right)}. \tag{7.2}$$

By the results of [20,22], \mathbf{w} is physically reasonable in the sense of [22, page 350]. In [22, Theorem 5], it is proved that \mathbf{w} exhibits the wake region behaviour, and has the polynomial convergence rates at spatial infinity indicated by the Oseen fundamental tensor (with some logarithmic correction). Hence, the following energy equality is valid in whole space,

$$D = \int_{\mathbb{R}^2} |\nabla \mathbf{w}|^2 = \int \mathbf{f} \cdot (\mathbf{w} - \mathbf{e}_1). \tag{7.3}$$

Using the ideas of proving Theorems 12, 15, we obtain the same type of estimate, that is,

$$D \leq C A \lambda \leq C \frac{\varepsilon^2}{(1 + R)^3 \ln^{\frac{1}{2}} \left(2 + \frac{1}{R}\right)}. \tag{7.4}$$

Below we have to consider two different cases:

Case I: $R \geq 1$. Then by Claim 2 of Theorem 15 we obtain

$$\sup_{r \geq R} |\bar{\mathbf{w}}(r)| \leq C. \tag{7.5}$$

More precisely, using the first basic estimate of Theorem 7, from (7.4) we obtain immediately

$$\sup_{r \geq R} |\bar{\mathbf{w}}(r) - \mathbf{e}_1| \leq C \frac{\varepsilon}{(1 + R)^{3/2}}. \tag{7.6}$$

As a by-product in the proof of Theorem 7, from the estimate (3.34) we also have

$$\int_{r \geq \frac{3}{2}R} r |\nabla \omega|^2 \leq C \frac{\varepsilon^2}{(1 + R)^3}, \tag{7.7}$$

and consequently,

$$\int_{r \geq \frac{3}{2}R} |\nabla \omega|^2 \leq C \frac{\varepsilon^2}{(1+R)^4}. \tag{7.8}$$

Next, the proof will be divided into a few steps.

STEP 1. For simplicity, we denote

$$\tilde{\varepsilon} = R^{-3/2} \varepsilon, \tag{7.9}$$

so that

$$D + \int_{r \geq \frac{3}{2}R} r |\nabla \omega|^2 \leq C \tilde{\varepsilon}^2, \tag{7.10}$$

$$\sup_{r \geq R} |\bar{\mathbf{w}}(r) - \mathbf{e}_1| \leq C \tilde{\varepsilon}. \tag{7.11}$$

By (7.10)–(7.11), we can find a good circle $S_{r_*}, r_* \in [\frac{3}{2}R, 2R]$, on which

$$\max_{z \in S_{r_*}} |\mathbf{w}(z) - \mathbf{e}_1| \leq C \tilde{\varepsilon} \tag{7.12}$$

$$\max_{z \in S_{r_*}} |p(z) - \bar{p}(r_*)| \leq C \tilde{\varepsilon}. \tag{7.13}$$

Without loss of generality, let us assume that $p \rightarrow 0$ as $r \rightarrow \infty$. Due to Lemma 5, we have

$$|\bar{p}(r_*)| \leq C \tilde{\varepsilon}^2. \tag{7.14}$$

Using the standard decomposition of pressure (see, e.g., (4.18) or [17, Corollary 16]), we deduce that

$$|p(z)| \leq C \tilde{\varepsilon} \quad \text{for } |z| \geq 2R. \tag{7.15}$$

Following [16, Section 3],⁴ we can find a sequence of good radii $r_k \in [2^k R, 2^{k+1} R), k = 1, 2, 3, \dots$ such that

$$|\mathbf{w} - \mathbf{e}_1| \leq C \tilde{\varepsilon}, \tag{7.16}$$

on each circle S_{r_k} . Locating these “good circles” is a necessary preparation for Step 2.

STEP 2. Recall Amick’s auxiliary function $\gamma = \frac{|\mathbf{w}|^2}{2} + p - \omega \psi$. Here ψ is the stream function satisfying $\nabla \psi = \mathbf{w}^\perp = -w_2 \mathbf{e}_1 + w_1 \mathbf{e}_2$. Let $\psi(2R, 0) = 0$ to be definite. It is known that γ satisfies the two-sided maximum principle and converges to $\frac{1}{2}$ at infinity, see [1]. Using (7.12)–(7.13), (7.15), and the maximum principle for Bernoulli pressure $\Phi = \frac{|\mathbf{w}|^2}{2} + p$, we have

$$|\mathbf{w}(z)| \leq C, \tag{7.17}$$

⁴ Amick’s special ω -level sets as described in [16, Lemma 13] are still available in the whole plane setting outside the support of \mathbf{f} .

for all $|z| \geq r_*$. Similarly, using (7.10)–(7.13) and the two-sided maximum principle for γ we can deduce that

$$|\gamma(z) - \frac{1}{2}| \leq C R^{\frac{1}{2}} \tilde{\varepsilon} \tag{7.18}$$

for all $|z| \geq 2R$. The $R^{\frac{1}{2}}$ factor shows up here because, on a typical circle S_{r_*} , $R < r_* < 2R$, the term $\omega\psi$ in the definition of γ is bounded by $R^{-\frac{1}{2}}\tilde{\varepsilon} \cdot R = R^{\frac{1}{2}}\tilde{\varepsilon}$. Following [16, Appendix I], and using the information from Step 1 we obtain the key pointwise estimate

$$\|\mathbf{w}(z) - \mathbf{e}_1\| \leq C R^{\frac{1}{2}} \tilde{\varepsilon} = C R^{-1} \varepsilon, \tag{7.19}$$

for all $|z| \geq 2R$. Also, from (2.2) and (7.10)–(7.11) we have

$$\|\mathbf{w} - \mathbf{e}_1\|_{L^q(B_{3R})} \leq C R^{\frac{2}{q} - \frac{3}{2}} \varepsilon, \tag{7.20}$$

in particular,

$$\|\mathbf{w} - \mathbf{e}_1\|_{L^5(B_{3R})} \leq C R^{-\frac{11}{10}} \varepsilon. \tag{7.21}$$

STEP 3. Following [16, Section 5],⁵ and setting ε sufficiently small, one can obtain the pointwise decay estimate

$$|(\mathbf{w} - \mathbf{e}_1)_i(z)| \leq C\varepsilon h_i(z), \quad i = 1, 2, \tag{7.22}$$

for $|z| \geq 2R$, where $h_i(z)$ is the majorant function given by ⁶

$$|z| > 1 : \begin{cases} h_1(z) = |z|^{-\frac{1}{2}}, \\ h_2(z) = |z|^{-\frac{3}{5}}. \end{cases} \tag{7.23}$$

STEP 4. Let

$$I_i(z)(\mathbf{u}, \mathbf{v}) = \int_{\mathbb{R}^2} \partial_l E_{ij}(z - z') u_j(z') v_l(z') dz'_1 dz'_2 := I_i^R(z)(\mathbf{u}, \mathbf{v}) + I_i^\infty(z)(\mathbf{u}, \mathbf{v}), \tag{7.24}$$

where

$$I_i^R(z)(\mathbf{u}, \mathbf{v}) := \int_{B_{2R}} \partial_l E_{ij}(z - z') u_j(z') v_l(z') dz'_1 dz'_2, \tag{7.25}$$

$$I_i^\infty(z)(\mathbf{u}, \mathbf{v}) := \int_{\mathbb{R}^2 \setminus B_{2R}} \partial_l E_{ij}(z - z') u_j(z') v_l(z') dz'_1 dz'_2. \tag{7.26}$$

⁵ The rescaling procedure in [16] is not needed here since λ is fixed to be 1. However we need to deal with the R -dependence in the estimates here while $R = 1$ in [16]. The big denominator $(1 + R)^3$ in (7.4) is useful to absorb all the R -dependence that shows up.

⁶ In [4, 16], $h_i(z)$ is defined as $\log \frac{2}{|z|}$ inside the unit disk. Such a logarithmic singularity plays an crucial role there as it accounts for the delicate boundary effects in the flow around obstacle problem. For the whole plane problem, we only use the definition of h outside B_1 .

Here \mathbf{E} is the fundamental Oseen tensor, i.e, the fundamental solution to the Oseen system:

$$\begin{cases} \Delta E_{ij} - \partial_1 E_{ij} - \partial_i e_j = \delta_{ij} \delta_0, & i, j = 1, 2, \\ \sum_{i=1,2} \partial_i E_{ij} = 0, & j = 1, 2, \end{cases} \tag{7.27}$$

where δ_0 is the standard δ -function supported at the origin. For more information on \mathbf{E} , see for instance [16, Section 2.4] or [11, Section 2].

Recall, that $\partial_l E_{ij} \in L^q(\mathbb{R}^2)$ for any $q \in (3/2, 2)$, in particular,

$$\partial_l E_{ij} \in L^{5/3}(\mathbb{R}^2) \tag{7.28}$$

(for the list of integrability properties for E_{ij} , see, e.g., in [20]). Define the auxiliary norm

$$\|\mathbf{u}\|_Y := \sup_{|z| \geq 2R; i=1,2} \frac{|u_i(z)|}{h_i(z)}. \tag{7.29}$$

We will need a crucial bilinear estimate (see [4, Lemmas 3.1–3.2]): there exists an absolute constant C such that

$$\sup_{|z| \geq 1; i=1,2} \frac{|I_i^\infty(z)(\mathbf{u}, \mathbf{v})|}{h_i(z)} \leq C \|\mathbf{u}\|_Y \|\mathbf{v}\|_Y. \tag{7.30}$$

In particular,

$$|\mathbf{I}^\infty(z)(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\|_Y \|\mathbf{v}\|_Y |z|^{-\frac{1}{2}}, \quad |z| \geq 1. \tag{7.31}$$

Further, from (7.28) and the elementary estimate $\|\mathbf{u}\|_{L^5(\mathbb{R}^2 \setminus B_{2R})} \leq C R^{-\frac{1}{10}} \|\mathbf{u}\|_Y$ we have:

$$|\mathbf{I}^\infty(z)(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\|_Y \|\mathbf{v}\|_Y R^{-\frac{1}{5}}, \quad |z| \leq 1 \tag{7.32}$$

(really, the last pointwise estimate is valid for all $z \in \mathbb{R}^2$, but we will use it in the unit ball only). Therefore, from (7.31)–(7.32) we conclude:

$$\|\mathbf{I}^\infty\|_{L^5(B_{2R})} \leq C \|\mathbf{u}\|_Y \|\mathbf{v}\|_Y. \tag{7.33}$$

$$\|\mathbf{I}^\infty\|_Y \leq C \|\mathbf{u}\|_Y \|\mathbf{v}\|_Y. \tag{7.34}$$

Similarly, from (7.28) and Hölder’s inequality we have:

$$|\mathbf{I}^R(z)(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\|_{L^5(B_{2R})} \|\mathbf{v}\|_{L^5(B_{2R})}, \quad \forall z \in \mathbb{R}^2. \tag{7.35}$$

In particular,

$$\|\mathbf{I}^R\|_{L^5(B_{2R})} \leq C \|\mathbf{u}\|_{L^5(B_{2R})} \|\mathbf{v}\|_{L^5(B_{2R})} R^{\frac{2}{5}}. \tag{7.36}$$

It is well-known that $|\nabla \mathbf{E}(z)| \leq C \frac{1}{|z|}$ for $|z| \geq 1$ (see, e.g., [22, §2]). This implies

$$|\mathbf{I}^R(z)(\mathbf{u}, \mathbf{v})| \leq C \|\mathbf{u}\|_{L^5(B_{2R})} \|\mathbf{v}\|_{L^5(B_{2R})} \frac{R^{\frac{6}{5}}}{|z|}, \quad \forall z \in \mathbb{R}^2 \setminus B_{2R}. \quad (7.37)$$

Therefore,

$$\|\mathbf{I}^R(\mathbf{u}, \mathbf{v})\|_Y \leq C \|\mathbf{u}\|_{L^5(B_{2R})} \|\mathbf{v}\|_{L^5(B_{2R})} R^{\frac{4}{5}}. \quad (7.38)$$

Finally, we can summarize all previous estimates in the following form:

$$\|\mathbf{I}(\mathbf{u}, \mathbf{v})\|_X \leq C \|\mathbf{u}\|_X \left(\|\mathbf{v}\|_Y + \|\mathbf{v}\|_{L^5(B_{2R})} R^{\frac{4}{5}} \right), \quad (7.39)$$

and also

$$\|\mathbf{I}(\mathbf{u}, \mathbf{v})\|_X \leq C \|\mathbf{v}\|_X \left(\|\mathbf{u}\|_Y + \|\mathbf{u}\|_{L^5(B_{2R})} R^{\frac{4}{5}} \right), \quad (7.40)$$

where we denote

$$\|\mathbf{u}\|_X := \|\mathbf{u}\|_Y + \|\mathbf{u}\|_{L^5(B_{2R})}. \quad (7.41)$$

STEP 5. The local uniqueness argument we are going to invoke next is similar to that in [4, Section 7]. Suppose $\mathbf{w}^{(1)}, \mathbf{w}^{(2)}$ are two D -solution to (1.1). Denote $\mathbf{u}^{(k)} := \mathbf{w}^{(k)} - \mathbf{e}_1$, $k = 1, 2$, then by Steps 2–3 we have

$$\|\mathbf{u}^{(k)}\|_X \leq C\varepsilon, \quad k = 1, 2, \quad (7.42)$$

$$\|\mathbf{u}^{(k)}\|_{L^5(B_{2R})} \leq C\varepsilon R^{-\frac{11}{10}}, \quad k = 1, 2. \quad (7.43)$$

Since $\mathbf{u}^{(k)}$ satisfies the system

$$\begin{cases} \Delta \mathbf{u}^{(k)} - \partial_1 \mathbf{u}^{(k)} - \nabla p^{(k)} = \mathbf{u}^{(k)} \cdot \nabla \mathbf{u}^{(k)} + \mathbf{f}, \\ \nabla \cdot \mathbf{u}^{(k)} = 0, \end{cases} \quad (7.44)$$

and the decay estimate (7.42), we obtain the following standard representation formula,

$$\mathbf{u}^{(k)}(z) = \int_{\mathbb{R}^2} \mathbf{E}(z - z') \cdot \left\{ (\mathbf{u}^{(k)} \cdot \nabla) \mathbf{u}^{(k)} + \mathbf{f} \right\} (z') dz'_1 dz'_2, \quad k = 1, 2.$$

From this representation we deduce

$$\begin{aligned} (u_i^{(1)} - u_i^{(2)})(z) &= \int E_{ij}(z - z') (\mathbf{u}^{(1)} \cdot \nabla \mathbf{u}^{(1)} - \mathbf{u}^{(2)} \cdot \nabla \mathbf{u}^{(2)})_j(z') dz'_1 dz'_2 \\ &= - \int \partial_l E_{ij}(z - z') (u_i^{(1)} u_j^{(1)} - u_i^{(2)} u_j^{(2)})(z') dz'_1 dz'_2 \\ &= -I_i(\mathbf{u}^{(1)} - \mathbf{u}^{(2)}, \mathbf{u}^{(2)}) - I_i(\mathbf{u}^{(1)}, \mathbf{u}^{(1)} - \mathbf{u}^{(2)}). \end{aligned} \quad (7.45)$$

Using (7.42)–(7.43) and (7.39)–(7.40) we obtain

$$\begin{aligned} & \| \mathbf{u}^{(1)} - \mathbf{u}^{(2)} \|_X \\ & \leq C \| \mathbf{u}^{(1)} - \mathbf{u}^{(2)} \|_X \left(\| \mathbf{u}^{(1)} \|_Y + \| \mathbf{u}^{(2)} \|_Y + \| \mathbf{u}^{(1)} \|_{L^5(B_{2R})} R^{\frac{4}{5}} + \| \mathbf{u}^{(2)} \|_{L^5(B_{2R})} R^{\frac{4}{5}} \right) \\ & \leq C' \varepsilon \| \mathbf{u}^{(1)} - \mathbf{u}^{(2)} \|_X, \end{aligned}$$

which immediately implies that $\mathbf{u}^{(1)} \equiv \mathbf{u}^{(2)}$ when ε is sufficiently small.

Case II: $R < 1$.

The proof here is similar to Case II in the proof of Theorem 15. By Lemma 3, we have

$$\| \mathbf{f} \|_{H^{-1}(B_2)} \leq C \left(1 + \ln^{\frac{1}{2}} \left(\frac{1}{R} \right) \right) \| \mathbf{f} \|_{H^{-1}(B_{2R})}. \tag{7.46}$$

Since $R < 1$, \mathbf{f} is of course compactly supported in B_1 . Hence, we can reduce to Case I with the new parameters

$$R' = 1$$

and

$$A' \leq C \ln^{\frac{1}{2}} \left(2 + \frac{1}{R} \right) A.$$

Finally, in the case that $\mathcal{F} = 0$, the logarithmic factors here can be removed due to the last statement in Lemma 3. □

Declarations

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A Uniform Estimates for the Pressure

In this section we have to prove estimates (3.10) and (3.13) concerning pressure. The main idea of “good circles” method is quite simple: if we have a nonnegative integrable function $g : \Omega \rightarrow \mathbb{R}_+$ on the annulus domain $\Omega = \Omega(r_0, \beta r_0)$ with

$$\int_{\Omega} g = \int_{r_0}^{\beta r_0} \left(\int_{S_r} g \, ds \right) dr = M,$$

then there exists a ‘good’ circle S_{r_*} with $r_* \in [r_0, \beta r_0]$ such that

$$\int_{S_{r_*}} g \leq \frac{M}{(\beta - 1)r_0}.$$

So the assertion (3.10) follows from (3.9) easily (the role of g is played by $|\nabla p|^{\frac{4}{3}}$ there). Further, following exactly the same argument, from the assertion (3.9) we can deduce the existence of sequence of good radii

$$\rho_1 = \tilde{r}_1, \rho_2, \rho_3, \dots, \rho_k = \tilde{r}_2,$$

such that

$$\left\{ \begin{array}{l} \rho_{i+1} \in (2\rho_i, 3\rho_i), \quad i = 1, \dots, k - 1; \\ \int_{\rho_i < r < \rho_{i+1}} |\nabla p|^{\frac{4}{3}} ds \leq C \rho_i^{\frac{2}{3}} m^{\frac{4}{3}} D^{\frac{2}{3}} (1 + \mu)^{\frac{4}{3}}, \\ \int_{S_{\rho_i}} |\nabla p|^{\frac{4}{3}} ds \leq C \frac{1}{\rho_i^{\frac{1}{3}}} m^{\frac{4}{3}} D^{\frac{2}{3}} (1 + \mu)^{\frac{4}{3}}, \\ \int_{S_{\rho_i}} |\nabla p| ds \leq (2\pi\rho_i)^{\frac{1}{4}} \left(\int_{S_{\rho_i}} |\nabla p|^{\frac{4}{3}} ds \right)^{\frac{3}{4}} \leq C m D^{\frac{1}{2}} (1 + \mu). \end{array} \right. \tag{A.1}$$

By (3.11) we have

$$|\bar{p}(r)| \leq CD \leq Cm\sqrt{D} \quad \text{for any } r \in [\rho_1, \rho_k], \tag{A.2}$$

thus the estimates (A.1₄) and the trivial inequality $|p(z)| \leq \bar{p}(r) + \int_{S_r} |\nabla p| ds$ for $z \in S_r$, imply the corresponding pointwise bound for the pressure:

$$|p| \leq Cm(1 + \mu)\sqrt{D} \quad \text{on } S_{\rho_i}, \quad i = 1, \dots, k. \tag{A.3}$$

Now take arbitrary and fix a point $z_0 \in \Omega(\rho_i, \rho_{i+1})$ with $i \in \{2, \dots, k - 2\}$. Put $R_0 = |z_0| - \rho_{i-1}$ and denote by $B(z_0, R_0)$ the disk centered at z_0 with radius R_0 . Then by construction and by the triangle inequality,

$$\begin{aligned} B(z_0, R_0) &\subset \Omega(\rho_{i-1}, \rho_{i+2}), \\ \frac{1}{2}\rho_i &< R_0 < \frac{8}{3}\rho_i, \\ B(z_0, R_0) \cap S_{\rho_i} &\neq \emptyset \neq B(z_0, \frac{1}{2}R_0) \cap S_{\rho_i}. \end{aligned} \tag{A.4}$$

From (A.1₂) and (A.4₁₋₂) we have

$$\int_{B(z_0, R_0)} |\nabla p|^{\frac{4}{3}} ds \leq C R_0^{\frac{2}{3}} m^{\frac{4}{3}} D^{\frac{2}{3}} (1 + \mu)^{\frac{4}{3}}. \tag{A.5}$$

Therefore, there is a “good” circle S_{z_0, R_*} centered at z_0 with radius $R_* \in [\frac{1}{2}R_0, R_0]$ such that

$$\begin{cases} \int_{S_{z_0, R_*}} |\nabla p|^{\frac{4}{3}} ds \leq C \frac{1}{R_*^{\frac{1}{3}}} m^{\frac{4}{3}} D^{\frac{2}{3}} (1 + \mu)^{\frac{4}{3}}, \\ \int_{S_{z_0, R_*}} |\nabla p| ds \leq (2\pi R_*)^{\frac{1}{4}} \left(\int_{S_{z_0, R_*}} |\nabla p|^{\frac{4}{3}} ds \right)^{\frac{3}{4}} \leq C m D^{\frac{1}{2}} (1 + \mu). \end{cases} \tag{A.6}$$

Since $S_{z_0, R_*} \cap S_{\rho_i} \neq \emptyset$ (see (A.4₃)), from (A.3) and (A.6₂) we obtain

$$|p| \leq C m D^{\frac{1}{2}} (1 + \mu) \quad \text{on } S_{z_0, R_*}, \tag{A.7}$$

consequently,

$$|\bar{p}(z_0, R_*)| \leq C m D^{\frac{1}{2}} (1 + \mu), \tag{A.8}$$

where left hand side denotes, as usual, the mean value of p over the circle S_{z_0, R_*} . Finally, from the last inequality (A.8) and from the basic pressure estimate (2.8) we obtain immediately

$$|p(z_0)| \leq C m D^{\frac{1}{2}} (1 + \mu). \tag{A.9}$$

Because of arbitrariness of z_0 we have proved the estimate

$$|p| \leq C m D^{\frac{1}{2}} (1 + \mu) \quad \text{in } \Omega(\rho_2, \rho_{k-1}) \supset \Omega(3\tilde{r}_1, \frac{1}{3}\tilde{r}_2) \supset \Omega(6r_1, \frac{1}{6}r_2) \tag{A.10}$$

as required.

B Estimates for the Gradient of Vorticity

In this section we have to prove that for any $\rho \in [6r_1, \frac{1}{30}r_2]$ and $D_\rho := D(\rho, 5\rho)$ there exist two “good” radii $\rho_1 \in (\rho, 2\rho)$, $\rho_2 \in (4\rho, 5\rho)$ satisfying estimates (3.30)–(3.31). Denote $f(r) = \int_{S_r} \omega^2 ds$. Then by construction $0 < f \in C^\infty([\rho, 5\rho])$,

$$\int_{\rho}^{5\rho} f(r) dr \leq 2D_\rho. \tag{B.1}$$

Denote

$$T = \left\{ r \in [\rho, 5\rho] : f(r) \leq \frac{8}{\rho} D_\rho \right\}.$$

Then by Chebyshev inequality

$$\text{meas}\left([\rho, 5\rho] \setminus T\right) \leq \frac{1}{4}\rho. \quad (\text{B.2})$$

Therefore, there exist points

$$t_1 \in T \cap \left[\rho, \frac{4}{3}\rho\right], \quad t_2 \in T \cap \left[\frac{14}{3}\rho, 5\rho\right].$$

Denote

$$\tau_1 = \max\{r \in [t_1, 2\rho] : r \in T\}.$$

We have to consider two different subcases:

Case 1: $\tau_1 < 2\rho$. Then by construction $f'(\tau_1) \geq 0$. So we put $\rho_1 = \tau_1$ and by construction we have

$$\rho_1 \in [\rho, 2\rho], \quad f(\rho_1) \leq \frac{8}{\rho}D\rho, \quad f'(\rho_1) \geq 0. \quad (\text{B.3})$$

Case 2: $\tau_1 = 2\rho$. Then $\tau_1 - t_1 \geq \frac{2}{3}\rho$, $|f(\tau_1) - f(t_1)| \leq \frac{8}{\rho}D\rho$. Therefore by Lagrange mean value theorem there exists $\rho_1 \in [t_1, \tau_1]$ with $f'(\rho_1)(\tau_1 - t_1) = f(\tau_1) - f(t_1)$, in particular,

$$|f'(\rho_1)| \leq \frac{12}{\rho^2}D\rho.$$

So for both cases 1–2 we find $\rho_1 \in [\rho, 2\rho]$ satisfying

$$\rho_1 \in [\rho, 2\rho], \quad f(\rho_1) \leq \frac{8}{\rho}D\rho, \quad -f'(\rho_1) \leq \frac{12}{\rho^2}D\rho. \quad (\text{B.4})$$

Similarly, we can find $\rho_2 \in [4\rho, 5\rho]$ satisfying

$$\rho_2 \in [4\rho, 5\rho], \quad f(\rho_2) \leq \frac{8}{\rho}D\rho, \quad f'(\rho_2) \leq \frac{12}{\rho^2}D\rho. \quad (\text{B.5})$$

So the required estimates (3.30)–(3.31) are proved completely. \square

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