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Variational Bihamiltonian Cohomologies and Integrable Hierarchies II: Virasoro Symmetries

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Abstract: We prove that for any tau-symmetric bihamiltonian deformation of the tau-cover of the Principal Hierarchy associated with a semisimple Frobenius manifold, the deformed tau-cover admits an infinite set of Virasoro symmetries.

1. Introduction

This is the second one of the series of papers devoted to the study of deformations of the Virasoro symmetries of bihamiltonian integrable hierarchies. In the first one [23], we developed a cohomology theory on the space of differential forms of the infinite jet space of a super manifold for a given bihamiltonian structure of hydrodynamic type, and we call such a cohomology theory the *variational bihamiltonian cohomology*. It can be viewed as a generalization of the bihamiltonian cohomology introduced in [12], and it provides a suitable tool for us to study deformations of the Virasoro symmetries of bihamiltonian integrable hierarchies.

The purpose of the present paper is to prove the following theorem.

Theorem 1 (Main Theorem). *For a given tau-symmetric bihamiltonian deformation of the Principal Hierarchy associated with a semisimple Frobenius manifold, there exists a unique deformation of the Virasoro symmetries of the tau-cover of the Principal Hierarchy such that they are symmetries of the tau-cover of the deformed integrable hierarchy. Moreover, the action of the Virasoro symmetries on the tau-function $\mathcal{Z}$ of the deformed integrable hierarchy can be represented in the form*

$$\frac{\partial \mathcal{Z}}{\partial s_m} = L_m \mathcal{Z} + O_m \mathcal{Z}, \quad m \geq -1, \tag{1.1}$$

where L_m are the Virasoro operators constructed in [11] and O_m are some differential polynomials, and the flows $\frac{\partial}{\partial s_m}$ satisfy the Virasoro commutation relations

$$\left[\frac{\partial}{\partial s_k}, \frac{\partial}{\partial s_l}\right] = (l-k)\frac{\partial}{\partial s_{k+l}}, \quad k, l \geq -1.$$

Let us briefly explain the basic idea for proving this theorem. Consider the following system of evolutionary PDEs with time variable t and spacial variable x:

$$\frac{\partial u^i}{\partial t} = A^i_j(u)u^j_x + \varepsilon\left(B^i_j(u)u^j_{xx} + C^i_{jk}(u)u^j_x u^k_x\right) + \dots, \quad i = 1, \dots, n. \tag{1.2}$$

We assume that this system is bihamiltonian with respect to the bihamiltonian structure (P_0, P_1) whose leading term is semisimple. We can associate with it a super extension by introducing odd unknown functions θ_i for $i = 1, \dots, n$, and by adding odd flows $\frac{\partial}{\partial \tau_0}$ and $\frac{\partial}{\partial \tau_1}$ which correspond respectively to the Hamiltonian structure P_0 and P_1 (see [22] and Sect. 2.3 given below for details). Thus the super extension of (1.2) consists of the flows $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \tau_0}$ and $\frac{\partial}{\partial \tau_1}$ for the unknown functions u^i and θ_i, and the fact that the system (1.2) is bihamiltonian with respect to (P_0, P_1) is equivalent to the following commutation relations:

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial \tau_i}\right] = 0, \quad \left[\frac{\partial}{\partial \tau_i}, \frac{\partial}{\partial \tau_j}\right] = 0, \quad i, j = 0, 1.$$

Example 1. Consider the following Korteweg-de Vries (KdV) equation:

$$\frac{\partial u}{\partial t} = u u_x + \frac{\varepsilon^2}{12} u_{xxx}.$$

It admits a bihamiltonian structure given by the following Poisson brackets:

$$\{u(x), u(y)\}_0 = \delta'(x - y),$$
$$\{u(x), u(y)\}_1 = u(x)\delta'(x - y) + \frac{u_x}{2}\delta(x - y) + \frac{\varepsilon^2}{8}\delta'''(x - y).$$

We introduce an odd unknown function θ, and construct the following super extension of the KdV equation:

$$\frac{\partial u}{\partial t} = u u_x + \frac{\varepsilon^2}{12} u_{xxx}, \quad \frac{\partial \theta}{\partial t} = u\theta_x + \frac{\varepsilon^2}{12}\theta_{xxx}, \tag{1.3}$$

$$\frac{\partial u}{\partial \tau_0} = \theta_x, \quad \frac{\partial u}{\partial \tau_1} = u\theta_x + \frac{1}{2}u_x\theta + \frac{\varepsilon^2}{8}\theta_{xxx}, \tag{1.4}$$

$$\frac{\partial \theta}{\partial \tau_0} = 0, \quad \frac{\partial \theta}{\partial \tau_1} = \frac{1}{2}\theta\theta_x. \tag{1.5}$$

It is easy to check directly that the flows in the extended system mutually commute.

Remark 1. The flow (1.3) also appeared in [1].

According to the theory of bihamiltonian cohomology developed in [7], we know that if there is another system of evolutionary PDEs given by the flow $\frac{\partial}{\partial \hat{t}}$ satisfying the commutation relation

$$\left[\frac{\partial}{\partial \hat{t}}, \frac{\partial}{\partial \tau_0}\right] = \left[\frac{\partial}{\partial \hat{t}}, \frac{\partial}{\partial \tau_1}\right] = 0, \tag{1.6}$$

then it gives a symmetry of the system (1.2), i.e.,

$$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial \hat{t}}\right] = 0.$$

We note that the above results is for a flow $\frac{\partial}{\partial \hat{t}}$ given by differential polynomials. In order to use the above results to consider symmetries of more general forms such as Virasoro symmetries, we introduce the notion of super tau-cover of a bihamiltonian integrable hierarchy and develop the theory of variational bihamiltonian cohomology [23]. Then we are able to consider the commutation relation between the Virasoro symmetries $\frac{\partial}{\partial s_m}$ and the odd flows $\frac{\partial}{\partial \tau_0}$ and $\frac{\partial}{\partial \tau_1}$. By applying the results of variational bihamiltonian cohomology proved in [23], we manage to prove the main theorem. A more detailed description of the idea for proving this theorem is given at the end of Sect. 3.2 and in Sect. 3.3.

We organize this paper as follows. In Sect. 2, we construct the super tau-cover of a given tau-symmetric bihamiltonian deformation of the Principal Hierarchy associated with a semisimple Frobenius manifold. This construction builds a bridge which relates Virasoro symmetries to bihamiltonian structures. In Sect. 3, we explain how the problem of deformations of the Virasoro symmetries can be solved via the theory of the variational bihamiltonian cohomology. In Sect. 4 we give the proof of the main theorem. Finally in Sect. 5, we make some concluding remarks.

2. Super Tau-Covers of Bihamiltonian Integrable Hierarchies

2.1. Bihamiltonian structures on infinite jet spaces. Let us start by recalling the basic construction of bihamiltonian structures as local functionals on infinite jet spaces. One may refer to [20] for a detailed introduction to this topic.

Let M be a smooth manifold of dimension n and $\hat{M}$ be the super manifold of dimension $(n|n)$ obtained by reversing the parity of the fibers of the cotangent bundle of M. In another word, if we choose a local canonical coordinate system $(u^1, \dots, u^n; \theta_1, \dots, \theta_n)$ on T^*M, then $\hat{M}$ can be described locally by the same chart while regarding the fiber coordinates as odd variables:

$$\theta_i\theta_j + \theta_j\theta_i = 0, \quad i, j = 1, \dots, n.$$

We say that the odd coordinates θ_i are dual to u^i. The transition functions between two local trivializations $(u^1, \dots, u^n; \theta_1, \dots, \theta_n)$ and $(w^1, \dots, w^n; \phi_1, \dots, \phi_n)$ are given by the same formula as those of the cotangent bundle:

$$\phi_\alpha = \frac{\partial u^\beta}{\partial w^\alpha}\theta_\beta, \quad \alpha = 1, \dots, n.$$

Here and henceforth, summation over repeated upper and lower Greek indices is assumed.

Denote by $J^\infty(\hat{M})$ the infinite jet bundle of $\hat{M}$. It is a fiber bundle over $\hat{M}$ with fiber $\mathbb{R}^\infty$. If we choose a local chart $(u^1, \dots, u^n; \theta_1, \dots, \theta_n)$ of $\hat{M}$, a trivialization can be realized by choosing the fiber coordinates being $(u^{\alpha,s}; \theta_\alpha^s)$ for $\alpha = 1, \dots, n$ and $s \geq 1$. The transition functions between different charts are given by the chain rule

$$\begin{aligned} w^{\alpha,1} &= \frac{\partial w^\alpha}{\partial u^\beta} u^{\beta,1}, \quad w^{\alpha,2} = \frac{\partial w^\alpha}{\partial u^\beta} u^{\beta,2} + \frac{\partial^2 w^\alpha}{\partial u^\beta \partial u^\gamma} u^{\beta,1} u^{\gamma,1}, \dots, \\ \phi_\alpha^1 &= \frac{\partial u^\beta}{\partial w^\alpha} \theta_\beta^1 + \frac{\partial^2 u^\beta}{\partial w^\gamma \partial w^\alpha} \frac{\partial w^\gamma}{\partial u^\lambda} u^{\lambda,1} \theta_\beta, \dots. \end{aligned}$$

Denote by $\hat{\mathcal{A}}$ the ring of differential polynomials, locally it is given by

$$\hat{\mathcal{A}} = C^\infty(u)[[u^{\alpha,s+1}, \theta_\alpha^s \mid \alpha = 1, \dots, n; s \geq 0]].$$

It is graded with respect to the super degree $\deg_\theta$ defined by

$$\deg_\theta u^{\alpha,s} = 0, \quad \deg_\theta \theta_\alpha^s = 1, \quad \alpha = 1, \dots, n, \ s \geq 0.$$

Here and henceforth we use the notation $u^{\alpha,0} = u^\alpha$, $\theta_\alpha^0 = \theta_\alpha$. The set of homogeneous elements with super degree p is denoted by $\hat{\mathcal{A}}^p$.

Introduce a global vector field ∂_x on $J^\infty(\hat{M})$ which is locally described by

$$\partial_x = \sum_{s \geq 0} u^{\alpha,s+1} \frac{\partial}{\partial u^{\alpha,s}} + \theta_\alpha^{s+1} \frac{\partial}{\partial \theta_\alpha^s}.$$

Then we have $u^{\alpha,s} = \partial_x^s u^\alpha$ and $\theta_\alpha^s = \partial_x^s \theta_\alpha$. Hence we can grade the ring $\hat{\mathcal{A}}$ with respect to the differential degree $\deg_{\partial_x}$ defined by

$$\deg_{\partial_x} u^{\alpha,s} = s, \quad \deg_{\partial_x} \theta_\alpha^s = s, \quad \alpha = 1, \dots, n, \ s \geq 0.$$

We use the notation $\hat{\mathcal{A}}_d$ to denote the set of homogeneous elements with differential degree d, and $\hat{\mathcal{A}}_d^p = \hat{\mathcal{A}}^p \cap \hat{\mathcal{A}}_d$.

Using the vector field ∂_x, one can construct the space $\hat{\mathcal{F}}$ of local functionals via the quotient $\hat{\mathcal{F}} := \hat{\mathcal{A}} / \partial_x \hat{\mathcal{A}}$. Since the vector field ∂_x is homogeneous with respect to both the super degree and the differential degree, the quotient space $\hat{\mathcal{F}}$ admits natural gradations induced from $\hat{\mathcal{A}}$ and we will use the notation $\hat{\mathcal{F}}^p$, $\hat{\mathcal{F}}_d$ and $\hat{\mathcal{F}}_d^p$ to denote the corresponding subspaces of homogeneous elements. For any element $f \in \hat{\mathcal{A}}$, we will use $\int f \in \hat{\mathcal{F}}$ to denote its image of the natural projection $\pi : \hat{\mathcal{A}} \to \hat{\mathcal{F}}$.

For a differential polynomial $f \in \hat{\mathcal{A}}$, one may define the variational derivatives by

$$\frac{\delta f}{\delta u^\alpha} = \sum_{s \geq 0} (-\partial_x)^s \frac{\partial f}{\partial u^{\alpha,s}}, \quad \frac{\delta f}{\delta \theta_\alpha} = \sum_{s \geq 0} (-\partial_x)^s \frac{\partial f}{\partial \theta_\alpha^s}.$$

It is easy to verify that the variational derivatives annihilate the elements in $\partial_x \hat{\mathcal{A}}$, hence they are also well-defined on the quotient space $\hat{\mathcal{F}}$. For any $F \in \hat{\mathcal{F}}$, we have

$$\frac{\delta F}{\delta u^\alpha} = \sum_{s \geq 0} (-\partial_x)^s \frac{\partial f}{\partial u^{\alpha,s}}, \quad \frac{\delta F}{\delta \theta_\alpha} = \sum_{s \geq 0} (-\partial_x)^s \frac{\partial f}{\partial \theta_\alpha^s},$$

with $f \in \hat{\mathcal{A}}$ being an arbitrary lift of F such that $F = \int f$. With the help of the notion of the variational derivatives, one can define the so-called Schouten-Nijenhuis bracket, which is a bilinear map $[-,-] : \hat{\mathcal{F}} \times \hat{\mathcal{F}} \to \hat{\mathcal{F}}$ defined by

$$[P, Q] = \int \left(\frac{\delta P}{\delta \theta_\alpha} \frac{\delta Q}{\delta u^\alpha} + (-1)^p \frac{\delta P}{\delta u^\alpha} \frac{\delta Q}{\delta \theta_\alpha} \right), \quad P \in \hat{\mathcal{F}}^p,\ Q \in \hat{\mathcal{F}}^q.$$

This bracket satisfies the graded commutation relation

$$[P, Q] = (-1)^{pq}\, [Q, P], \quad P \in \hat{\mathcal{F}}^p,\ Q \in \hat{\mathcal{F}}^q,$$

and the graded Jacobi identity

$$(-1)^{rp}[[P, Q], R] + (-1)^{pq}[[Q, R], P] + (-1)^{qr}[[R, P], Q] \\ = 0, \quad P \in \hat{\mathcal{F}}^p,\ Q \in \hat{\mathcal{F}}^q,\ R \in \hat{\mathcal{F}}^r.$$

Any local functional $P \in \hat{\mathcal{F}}^p$ gives rise to a graded derivation

$$D_P = \sum_{s \geq 0} \partial_x^s \left(\frac{\delta P}{\delta \theta_\alpha} \right) \frac{\partial}{\partial u^{\alpha,s}} + (-1)^p \partial_x^s \left(\frac{\delta P}{\delta u^\alpha} \right) \frac{\partial}{\partial \theta_\alpha^s} \in \mathrm{Der}(\hat{\mathcal{A}})^{p-1}. \tag{2.1}$$

Here the space $\mathrm{Der}(\hat{\mathcal{A}})^p$ ($p \in \mathbb{Z}$) is the space of linear maps

$$D : \hat{\mathcal{A}}^q \to \hat{\mathcal{A}}^{q+p}$$

satisfying the graded Leibniz rule

$$D(fg) = D(f)\, g + (-1)^{kp} f\, D(g), \quad f \in \hat{\mathcal{A}}^k,\ g \in \hat{\mathcal{A}}.$$

Denote $\mathrm{Der}(\hat{\mathcal{A}}) = \oplus_{p \in \mathbb{Z}} \mathrm{Der}(\hat{\mathcal{A}})^p$, then it is a graded Lie algebra with the graded commutator

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{kl} D_2 \circ D_1, \quad D_1 \in \mathrm{Der}(\hat{\mathcal{A}})^k,\ D_2 \in \mathrm{Der}(\hat{\mathcal{A}})^l,$$

and it is also graded by the differential degree

$$\mathrm{Der}(\hat{\mathcal{A}})_d = \{D \in \mathrm{Der}(\hat{\mathcal{A}}) \mid D(\hat{\mathcal{A}}_k) \subseteq \hat{\mathcal{A}}_{k+d}\},$$

and we denote

$$\mathrm{Der}(\hat{\mathcal{A}})_d^p = \mathrm{Der}(\hat{\mathcal{A}})^p \cap \mathrm{Der}(\hat{\mathcal{A}})_d. \tag{2.2}$$

For $P \in \hat{\mathcal{F}}^p$ and $Q \in \hat{\mathcal{F}}^q$, we have the following useful identities:

$$D_{[P,Q]} = (-1)^{p-1} \left[D_P, D_Q \right], \tag{2.3}$$

$$\frac{\delta}{\delta u^\alpha}[P, Q] = D_P \left(\frac{\delta Q}{\delta u^\alpha} \right) + (-1)^{pq} D_Q \left(\frac{\delta P}{\delta u^\alpha} \right), \tag{2.4}$$

$$\frac{\delta}{\delta \theta_\alpha}[P, Q] = (-1)^{p-1} D_P \left(\frac{\delta Q}{\delta \theta_\alpha} \right) - (-1)^{(p-1)q} D_Q \left(\frac{\delta P}{\delta \theta_\alpha} \right). \tag{2.5}$$

A Hamiltonian structure is defined as a local functional $P \in \hat{\mathcal{F}}^2$ such that $[P, P] = 0$. We can associate a matrix valued differential operator $\mathcal{P} = (\mathcal{P}^{\alpha\beta})$ with $\mathcal{P}^{\alpha\beta} = \sum_{s\geq 0} \mathcal{P}_s^{\alpha\beta} \partial_x^s$ to any bivector $P \in \hat{\mathcal{F}}^2$, where $\mathcal{P}_s^{\alpha\beta} \in \hat{\mathcal{A}}$ are defined by

$$\frac{\delta P}{\delta \theta_\alpha} = \sum_{s\geq 0} \mathcal{P}_s^{\alpha\beta} \theta_\beta^s, \quad \alpha = 1, \ldots, n.$$

If P is a Hamiltonian structure, then we call $\mathcal{P}$ the Hamiltonian operator of P.

Theorem 2 ([10]). *Let $P \in \hat{\mathcal{F}}_1^2$. Denote the differential operator associated with P by*

$$\mathcal{P}^{\alpha\beta} = g^{\alpha\beta}(u)\partial_x + \Gamma_\gamma^{\alpha\beta}(u)u^{\gamma,1}, \quad \det(g^{\alpha\beta}) \neq 0,$$

then P is a Hamiltonian structure if and only if $g = (g_{\alpha\beta}) = (g^{\alpha\beta})^{-1}$ defines a flat metric on M and the Christoffel symbols of the Levi–Civita connection of g are given by $\Gamma_{\alpha\beta}^\gamma = -g_{\alpha\lambda}\Gamma_\beta^{\lambda\gamma}$.

A Hamiltonian structure P satisfying the conditions of the above theorem is called of hydrodynamic type. It follows from the above theorem that there exists a local coordinate system $(v^\alpha; \sigma_\alpha)$ on $\hat{M}$ such that

$$P = \frac{1}{2}\int \eta^{\alpha\beta}\sigma_\alpha\sigma_\beta^1,$$

where $\eta^{\alpha\beta}$ is a constant non-degenerate matrix. The coordinates v^α and σ_α are called flat coordinates of P.

A bihamiltonian structure (P_0, P_1) is a pair of Hamiltonian structures which satisfies an additional compatibility condition $[P_0, P_1] = 0$. Assume that the bihamiltonian structure is of hydrodynamic type, then according to Theorem 2, we have two flat contravariant metrics $g_0^{\alpha\beta}$ and $g_1^{\alpha\beta}$. We say that this bihamiltonian structure is semisimple if the roots of the characteristic equation

$$\det\left(g_1^{\alpha\beta} - \lambda g_0^{\alpha\beta}\right) = 0$$

are distinct and not constant. In this case, the roots $\lambda^1(u), \ldots, \lambda^n(u)$ can serve as local coordinates of M and they are called the canonical coordinates of the semisimple bihamiltonian structure. It is proved in [14], in terms of the canonical coordinates, that

$$P_0 = \frac{1}{2}\int \sum_{i,j=1}^n \left(\delta_{i,j} f^i(\lambda)\theta_i\theta_i^1 + A^{ij}\theta_i\theta_j\right),$$

$$P_1 = \frac{1}{2}\int \sum_{i,j=1}^n \left(\delta_{i,j} g^i(\lambda)\theta_i\theta_i^1 + B^{ij}\theta_i\theta_j\right),$$

where f^i are non-vanishing functions, $g^i = \lambda^i f^i$ and the functions A^{ij} and B^{ij} are given by

$$A^{ij} = \frac{1}{2}\left(\frac{f^i}{f^j}\frac{\partial f^j}{\partial \lambda^i}\lambda^{j,1} - \frac{f^j}{f^i}\frac{\partial f^i}{\partial \lambda^j}\lambda^{i,1}\right), \quad B^{ij} = \frac{1}{2}\left(\frac{g^i}{f^j}\frac{\partial f^j}{\partial \lambda^i}\lambda^{j,1} - \frac{g^j}{f^i}\frac{\partial f^i}{\partial \lambda^j}\lambda^{i,1}\right). \tag{2.6}$$

Here by abusing the notation, we still use θ_i to denote the fiber coordinates of $\hat{M}$ dual to λ^i. We also call λ^i and θ_i the canonical coordinates of (P_0, P_1).

From now on, for a semisimple bihamiltonian structure (P_0, P_1) of hydrodynamic type, we will use $(v^\alpha; \sigma_\alpha)$ to denote flat coordinates of P_0 such that

$$P_0 = \frac{1}{2}\int \eta^{\alpha\beta}\sigma_\alpha\sigma_\beta^1,$$
$$P_1 = \frac{1}{2}\int g^{\alpha\beta}(v)\sigma_\alpha\sigma_\beta^1 + \Gamma_\gamma^{\alpha\beta}(v)v^{\gamma,1}\sigma_\alpha\sigma_\beta,$$

and we will use $(u^i; \theta_i)$ to denote the canonical coordinates for (P_0, P_1) such that

$$P_0 = \frac{1}{2}\int \sum_{i,j=1}^{n}\left(\delta_{i,j} f^i(u)\theta_i\theta_i^1 + A^{ij}\theta_i\theta_j\right),$$
$$P_1 = \frac{1}{2}\int \sum_{i,j=1}^{n}\left(\delta_{i,j} u^i f^i(u)\theta_i\theta_i^1 + B^{ij}\theta_i\theta_j\right). \tag{2.7}$$

Here and henceforth we do not assume summations over repeated upper and lower Latin indices.

In terms of the notations introduced above, a system of evolutionary PDEs

$$\frac{\partial u^\alpha}{\partial t} = X^\alpha, \quad X^\alpha \in \hat{\mathcal{A}}^0$$

can be represented by a local functional $X = \int X^\alpha\theta_\alpha$, and it is called a bihamiltonian system if there exists a bihamiltonian structure (P_0, P_1) and two Hamiltonians $G, H \in \hat{\mathcal{F}}^0$ such that

$$X = -[G, P_0] = -[H, P_1].$$

Example 2. The KdV equation

$$\frac{\partial u}{\partial t} = uu_x + \frac{\varepsilon^2}{12}u_{xxx} \tag{2.8}$$

can be represented by $X = \int (uu_x + \frac{\varepsilon^2}{12}u_{xxx})\theta$. Its bihamiltonian structure is given by

$$P_0 = \frac{1}{2}\int \theta\theta_x, \quad P_1 = \frac{1}{2}\int u\theta\theta^1 + \frac{\varepsilon^2}{8}\theta\theta^3.$$

The two Hamiltonians with respect to the bihamiltonian structure are given by

$$X = -\left[\int \frac{u^3}{6} - \frac{\varepsilon^2}{24}u_x^2, P_0\right] = -\left[\int \frac{u^2}{3}, P_1\right].$$

2.2. Frobenius manifolds and super tau-covers of the Principal Hierarchies. In this subsection, we first recall some basic facts of Frobenius manifolds and the construction of the associated Principal Hierarchies following the work of [4–6,12]. Then we recall the construction of the super tau-covers of the Principal Hierarchies given in [22].

The notion of Frobenius manifolds is a geometric description of genus zero 2D topological field theories. An n-dimensional Frobenius manifold M can be locally described by a solution $F(v^1, \dots, v^n)$ of the following Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) associativity equations [2,29]:

$$\partial_\alpha \partial_\beta \partial_\lambda F \eta^{\lambda\mu} \partial_\mu \partial_\gamma \partial_\delta F = \partial_\delta \partial_\beta \partial_\lambda F \eta^{\lambda\mu} \partial_\mu \partial_\gamma \partial_\alpha F. \tag{2.9}$$

Here $\partial_\alpha = \frac{\partial}{\partial v^\alpha}$ and we require that $(\eta_{\alpha\beta}) := (\partial_1 \partial_\alpha \partial_\beta F)$ is a constant non-degenerate matrix with inverse $(\eta^{\alpha\beta})$. The function $F(v^1, \dots, v^n)$ is called the potential of M and it defines a Frobenius algebra structure on TM:

$$\langle \partial_\alpha, \partial_\beta \rangle = \eta_{\alpha\beta}, \quad \partial_\alpha \cdot \partial_\beta = c^\gamma_{\alpha\beta} \partial_\gamma,$$

where the functions $c^\gamma_{\alpha\beta}$ are defined by

$$c^\gamma_{\alpha\beta} = \eta^{\gamma\lambda} c_{\lambda\alpha\beta}, \quad c_{\lambda\alpha\beta} = \partial_\lambda \partial_\alpha \partial_\beta F.$$

The potential F is required to be quasi-homogeneous in the sense that there exists a vector field

$$E = \sum_{\alpha=1}^n \left(\left(1 - \frac{d}{2} - \mu_\alpha \right) v^\alpha + r_\alpha \right) \partial_\alpha,$$

called the Euler vector field, such that

$$E(F) = (3-d)F + \frac{1}{2} A_{\alpha\beta} v^\alpha v^\beta + B_\alpha v^\alpha + C.$$

Here the diagonal matrix $\mu = \mathrm{diag}(\mu_1, \dots, \mu_n)$ is part of the monodromy data of M which satisfies the identity

$$(\mu_\alpha + \mu_\beta)\eta_{\alpha\beta} = 0, \quad \forall \alpha, \beta. \tag{2.10}$$

It is also assumed that $\mu_1 = -d/2$ and $r_1 = 0$.

An important property of Frobenius manifolds is that the affine connection

$$\tilde\nabla_X Y = \nabla_X Y + z X \cdot Y, \quad \forall X, Y \in \Gamma(TM), \quad z \in \mathbb{C}$$

is flat for arbitrary z, here ∇ is the Levi–Civita connection of the flat metric $(\eta_{\alpha\beta})$. It can be extended to a flat connection on $M \times \mathbb{C}^*$ by viewing z as the coordinate on $\mathbb{C}^*$ and defining

$$\tilde\nabla_{\partial_z} X = \partial_z X + E \cdot X - \frac{1}{z} \mu X, \quad \tilde\nabla_{\partial_z} \partial_z = \tilde\nabla_X \partial_z = 0.$$

The connection $\tilde\nabla$ is called the deformed flat connection or the Dubrovin connection. For such a flat connection, one can find a system of flat coordinates of the form

$$(\tilde v^1(v,z), \dots, \tilde v^n(v,z)) = (h_1(v,z), \dots, h_n(v,z)) z^\mu z^R.$$

Here R is a constant matrix. The constant matrices η, μ and R form the monodromy data of M at $z = 0$. The matrix R can be decomposed into a finite sum $R = R_1 + \cdots + R_m$, and they satisfy the conditions

$$[\mu, R_k] = kR_k, \quad \eta_{\alpha\gamma}(R_k)^\gamma_\beta = (-1)^{k+1}\eta_{\beta\gamma}(R_k)^\gamma_\alpha. \tag{2.11}$$

The functions $h_\alpha(v, z)$ are analytic at $z = 0$ and has the expansion $h_\alpha(v, z) = \sum_{p\geq 0} h_{\alpha,p}(v)z^p$. The coefficients $h_{\alpha,p}$ satisfy the recursion relations

$$h_{\alpha,0} = \eta_{\alpha\beta}v^\beta, \quad \partial_\beta\partial_\gamma h_{\alpha,p+1} = c^\lambda_{\beta\gamma}\partial_\lambda h_{\alpha,p}, \quad p \geq 0,$$

the quasi-homogeneous and normalization conditions

$$E(\partial_\beta h_{\alpha,p}) = (p + \mu_\alpha + \mu_\beta)\partial_\beta h_{\alpha,p} + \sum_{k=1}^{p}(R_k)^\gamma_\alpha \partial_\beta h_{\gamma,p-k},$$
$$\langle \nabla h_\alpha(v, z), \nabla h_\beta(v, -z)\rangle = \eta_{\alpha\beta}.$$

A choice of the functions $h_{\alpha,p}$ satisfying all the above-mentioned conditions is called a calibration of M, and a Frobenius manifold M is called calibrated if such a choice is fixed. In what follows, we assume that the Frobenius manifolds we consider are calibrated.

The Principal Hierarchy associated with a Frobenius manifold M is a bihamiltonian integrable hierarchy of hydrodynamic type. Denote by σ_α the odd variables dual to the flat coordinates v^α, then the Principal Hierarchy can be described by the local functionals $X_{\alpha,p} \in \hat{\mathcal{F}}^1$ of the form

$$X_{\alpha,p} = \int \eta^{\lambda\gamma}\partial_x(\partial_\gamma h_{\alpha,p+1})\sigma_\lambda, \quad \alpha = 1, \ldots, n, \quad p \geq 0, \tag{2.12}$$

or equivalently, we can represent the integrable hierarchy as follows:

$$\frac{\partial v^\lambda}{\partial t^{\alpha,p}} = \eta^{\lambda\gamma}\partial_x(\partial_\gamma h_{\alpha,p+1}).$$

Define two local functionals

$$P_0 = \frac{1}{2}\int \eta^{\alpha\beta}\sigma_\alpha\sigma^1_\beta, \quad P_1 = \frac{1}{2}\int g^{\alpha\beta}\sigma_\alpha\sigma^1_\beta + \Gamma^{\alpha\beta}_\gamma v^{\gamma,1}\sigma_\alpha\sigma_\beta, \tag{2.13}$$

where the functions $g^{\alpha\beta}$ and $\Gamma^{\alpha\beta}_\gamma$ are given by

$$g^{\alpha\beta} = E^\varepsilon c^{\alpha\beta}_\varepsilon, \quad \Gamma^{\alpha\beta}_\gamma = \left(\frac{1}{2} - \mu_\beta\right)c^{\alpha\beta}_\gamma$$

with $c^{\alpha\beta}_\gamma = \eta^{\alpha\lambda}c^\beta_{\lambda\gamma}$, then we have the following theorem.

Theorem 3 ([5]). *Let M be a Frobenius manifold, then*

1. *The local functionals P_0, P_1 defined in* (2.13) *form a bihamiltonian structure which is exact in the sense that*

$$P_0 = [Z, P_1], \quad Z = \int \sigma_1.$$

2. *The Principal Hierarchy $X_{\alpha,p}$ associated with M is bihamiltonian with respect to the bihamiltonian structure (P_0, P_1) and*

$$X_{\alpha,p} = -\left[H_{\alpha,p}, P_0\right], \quad H_{\alpha,p} = \int h_{\alpha,p+1}.$$

3. *The following bihamiltonian recursion relation holds true:*

$$[H_{\alpha,p-1}, P_1] = \left(p + \frac{1}{2} + \mu_\alpha\right)[H_{\alpha,p}, P_0] + \sum_{k=1}^{p} (R_k)_\alpha^\gamma \, [H_{\gamma,p-k}, P_0], \quad p \geq 0.$$

Another important property satisfied by the Principal Hierarchy is that it is tau-symmetric. Let us define the functions $\Omega_{\alpha,p;\beta,q}$ for $\alpha, \beta = 1, \dots, n$ and $p, q \geq 0$ by the generating function

$$\sum_{p\geq 0, q\geq 0} \Omega_{\alpha,p;\beta,q}(v) z_1^p z_2^q = \frac{\langle \nabla h_\alpha(v, z_1), \nabla h_\beta(v, z_2)\rangle - \eta_{\alpha\beta}}{z_1 + z_2}.$$

They have the following properties [5]:

$$\Omega_{\alpha,p;1,0} = h_{\alpha,p}, \quad \Omega_{\alpha,p;\beta,0} = \partial_\beta h_{\alpha,p+1},$$
$$\Omega_{\alpha,p;\beta,q} = \Omega_{\beta,q;\alpha,p}, \quad \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial t^{\lambda,k}} = \frac{\partial \Omega_{\lambda,k;\beta,q}}{\partial t^{\alpha,p}}.$$

It follows from these identities that one can extend the Principal Hierarchy by introducing another family of unknown functions $f_{\alpha,p}$ satisfying the following equations:

$$\frac{\partial f_{\alpha,p}}{\partial t^{\beta,q}} = \Omega_{\alpha,p;\beta,q}, \quad \frac{\partial v^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\lambda}\partial_x \Omega_{\lambda,0;\beta,q}. \tag{2.14}$$

The system (2.14) is called the tau-cover of the Principal Hierarchy.

In order to study the relation between bihamiltonian structures and Virasoro symmetries, we introduced the notion of the super tau-cover of the Principal Hierarchy of a Frobenius manifold in [22]. Let us briefly recall its construction which provides the main motivation of our work presented in the next subsection. We first introduce a family of odd unknown functions $\sigma_{\alpha,k}^s$ for $s, k \geq 0$ with $\sigma_{\alpha,0}^s = \sigma_\alpha^s$. In what follows we will also use $\sigma_{\alpha,k}$ to denote $\sigma_{\alpha,k}^0$. We extend the action of ∂_x to include these new odd variables as follows:

$$\partial_x = \sum_{s\geq 0} v^{\alpha,s+1}\frac{\partial}{\partial v^{\alpha,s}} + \sum_{k,s\geq 0} \sigma_{\alpha,k}^{s+1}\frac{\partial}{\partial \sigma_{\alpha,k}^s}.$$

These odd variables are required to satisfy the following bihamiltonian recursion relation:

$$\eta^{\alpha\beta}\sigma_{\beta,k+1}^1 = g^{\alpha\beta}\sigma_{\beta,k}^1 + \Gamma_\gamma^{\alpha\beta} v^{\gamma,1}\sigma_{\beta,k}, \quad \alpha = 1, \dots, n;\ k \geq 0. \tag{2.15}$$

We also introduce a family of odd flows $\frac{\partial}{\partial \tau_m}$ for $m \geq 0$. The first two odd flows are determined by the bihamiltonian structure (P_0, P_1) as follows:

$$\frac{\partial v^\alpha}{\partial \tau_i} = \frac{\delta P_i}{\delta \sigma_\alpha}, \quad \frac{\partial \sigma_\alpha}{\partial \tau_i} = \frac{\delta P_i}{\delta v^\alpha}, \quad i = 0, 1.$$

Note that $\frac{\partial}{\partial \tau_i} = D_{P_i}$ for $i = 0, 1$, where D_{P_i} is defined by (2.1). The actions of $\frac{\partial}{\partial \tau_i}$ can be extended to all the other odd variables $\sigma_{\alpha,k}$ such that the flows $\frac{\partial}{\partial \tau_i}$ are compatible with the recursion relation (2.15). Furthermore we can define infinitely many odd flows $\frac{\partial}{\partial \tau_m}$ for $m \geq 2$ which can be viewed as flows corresponding to certain non-local Hamiltonian structures.

We have the following theorem.

Theorem 4 ([22]). *We have the following mutually commuting flows associated with any given Frobenius manifold M:*

$$\frac{\partial v^\alpha}{\partial t^{\beta,p}} = \eta^{\alpha\gamma}(\partial_\lambda \partial_\gamma h_{\beta,p+1})v^{\lambda,1}, \quad \frac{\partial \sigma_{\alpha,k}}{\partial t^{\beta,p}} = \eta^{\gamma\varepsilon}(\partial_\alpha \partial_\varepsilon h_{\beta,p+1})\sigma^1_{\gamma,k},$$

$$\frac{\partial v^\alpha}{\partial \tau_m} = \eta^{\alpha\beta}\sigma^1_{\beta,m}, \quad \frac{\partial \sigma_{\alpha,k}}{\partial \tau_m} = -\frac{\partial \sigma_{\alpha,m}}{\partial \tau_k} = \Gamma^{\gamma\beta}_\alpha \sum_{i=0}^{m-k-1} \sigma_{\beta,k+i}\sigma^1_{\gamma,m-i-1}, \quad 0 \leq k \leq m,$$

where $\alpha, \beta = 1, \dots, n$, and $m, p \geq 0$. These flows are well-defined in the sense that they are compatible with the recursion relation (2.15).

The system described in Theorem 4 is a super extension of the Principal Hierarchy, since the reduction obtained by setting all the odd variables to be zero yields the original Principal Hierarchy. The super extension of the tau-cover (2.14) can be constructed by introducing another family of odd variables $\Phi^m_{\alpha,p}$ for $p, m \geq 0$ and we call it the super tau-cover of the Principal Hierarchy. It is given in the following theorem.

Theorem 5 ([22]). *The mutually commuting flows*

$$\frac{\partial f_{\alpha,p}}{\partial t^{\beta,q}} = \Omega_{\alpha,p;\beta,q},$$

$$\frac{\partial f_{\alpha,p}}{\partial \tau_m} = \Phi^m_{\alpha,p},$$

$$\frac{\partial \Phi^m_{\alpha,p}}{\partial t^{\beta,q}} = \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial \tau_m},$$

$$\frac{\partial \Phi^m_{\alpha,p}}{\partial \tau_k} = \Delta^{k,m}_{\alpha,p},$$

together with the ones presented in Theorem 4, give the super tau-cover of the Principal Hierarchy associated with M, where $\Delta^{k,m}_{\alpha,p}$ are defined by the formula

$$\Delta^{k,m}_{\alpha,p} = -\Delta^{m,k}_{\alpha,p} = \eta^{\gamma\lambda}\partial_\lambda h_{\alpha,p}\Gamma^{\delta\mu}_\gamma \left(\sum_{i=0}^{k-m-1} \sigma_{\mu,m+i}\sigma^1_{\delta,k-i-1}\right), \quad k \geq m.$$

It was shown in [22] that the odd variables $\Phi^m_{\alpha,p}$ satisfy the recursion relation

$$-\left(\frac{2p-1}{2} + \mu_\alpha\right)\Phi^m_{\alpha,p} = \left(\frac{1}{2} + \mu_\lambda\right)\eta^{\lambda\varepsilon}(\partial_\lambda h_{\alpha,p})\sigma_{\varepsilon,m} + \sum_{k=1}^{p}(R_k)^\xi_\alpha \Phi^m_{\xi,p-k} - \Phi^{m+1}_{\alpha,p-1}$$

with the initial condition $\Phi^m_{\alpha,0} = \sigma_{\alpha,m}$. So when the diagonal matrix μ of the Frobenius manifold M satisfies the condition $\frac{1-2k}{2} \notin \mathrm{Spec}(\mu)$ for any $k = 1, 2, \dots$, all the variables $\Phi^m_{\alpha,p}$ are linear combinations of $\sigma_{\varepsilon,k}$ with coefficients being smooth functions of $v^1, \dots, v^n$.

For an arbitrary tau-symmetric bihamiltonian deformation of the Principal Hierarchy associated with a semisimple Frobenius manifold, we are to construct in the remaining part of this section its super extension and super tau-cover by generalizing the constructions given in Theorems 4 and 5.

2.3. Super extensions of bihamiltonian integrable hierarchies. In this subsection, we construct a super extension for a given bihamiltonian integrable hierarchy with hydrodynamic limit.

We fix an n-dimensional smooth manifold M and a semisimple bihamiltonian structure $(P_0^{[0]}, P_1^{[0]})$ of hydrodynamic type defined on $J^\infty(\hat{M})$. Let us choose $(w^\alpha; \phi_\alpha)$ as local coordinates on $\hat{M}$. Recall that a Miura type transformation is a choice of n differential polynomials $\tilde{w}^1, \dots, \tilde{w}^n \in \hat{\mathcal{A}}^0_{\geq 0}$ such that

$$\det\left(\frac{\partial \tilde{w}_0^\alpha}{\partial w^\beta}\right) \neq 0,$$

where $\tilde{w}_0^\alpha$ is the differential degree zero component of $\tilde{w}^\alpha$. By defining $\tilde{w}^{\alpha,s} = \partial_x^s \tilde{w}^\alpha$, it is easy to see that we can represent any differential polynomial in $w^{\alpha,s}$ by a differential polynomial in $\tilde{w}^{\alpha,s}$. Therefore a Miura type transformation can be viewed as a special type of change of coordinates on $J^\infty(M)$. The extension of the Miura type transformations on $J^\infty(\hat{M})$ is given by the following theorem [25].

Theorem 6 ([25]). *A Miura type transformation induces a change of coordinates from $(w^{\alpha,s}; \phi_\alpha^s)$ to $(\tilde{w}^{\alpha,s}; \tilde{\phi}_\alpha^s)$ given by*

$$\phi_\alpha^s = \partial_x^s \sum_{t\geq 0} (-\partial_x)^t \left(\frac{\partial \tilde{w}^\beta}{\partial w^{\alpha,t}} \tilde{\phi}_\beta\right).$$

Now let (P_0, P_1) be any given deformation of $(P_0^{[0]}, P_1^{[0]})$. Denote by $\mathcal{P}_0$ and $\mathcal{P}_1$ the Hamiltonian operators of P_0 and P_1 in the coordinates $(w^{\alpha,s}; \phi_\alpha^s)$. We introduce another family of odd variables $\phi_{\alpha,m}^s$ for $m \geq 0$ and extend the vector field ∂_x to the following one:

$$\partial_x = \sum_{s\geq 0} w^{\alpha,s+1} \frac{\partial}{\partial w^{\alpha,s}} + \sum_{k,s\geq 0} \phi_{\alpha,k}^{s+1} \frac{\partial}{\partial \phi_{\alpha,k}^s}.$$

In what follows we also use the notations $\phi_{\alpha,0}^s = \phi_\alpha^s$ and $\phi_{\alpha,m}^0 = \phi_{\alpha,m}$. Inspired by (2.15), we require that these new odd variables satisfy the recursion relations

$$\mathcal{P}_0^{\alpha\beta} \phi_{\beta,m+1} = \mathcal{P}_1^{\alpha\beta} \phi_{\beta,m}, \quad m \geq 0. \tag{2.16}$$

We first show that (2.16) is well defined in the sense that it is invariant under Miura type transformations.

Proposition 1. *The Miura type transformation from* $(w^{\alpha,s}; \phi_\alpha^s)$ *to* $(\tilde{w}^{\alpha,s}; \tilde{\phi}_\alpha^s)$ *induces a transformation for the new odd variables* $\phi_{\alpha,m}^s$ *given by*

$$\phi_{\alpha,m}^s = \partial_x^s \sum_{t\geq 0}(-\partial_x)^t \left(\frac{\partial \tilde{w}^\beta}{\partial w^{\alpha,t}}\tilde{\phi}_{\beta,m}\right), \quad m \geq 1, \tag{2.17}$$

such that the recursion relation (2.16) *is invariant.*

Proof. Denote by $\mathcal{P}_i$ and $\tilde{\mathcal{P}}_i$ the Hamiltonian operator of P_i in the coordinates $(w^{\alpha,s}; \phi_\alpha^s)$ and $(\tilde{w}^{\alpha,s}; \tilde{\phi}_\alpha^s)$ respectively for $i = 0, 1$. Then it is well known that

$$\tilde{\mathcal{P}}_i^{\alpha\beta} = \sum_{s\geq 0}\frac{\partial \tilde{w}^\alpha}{\partial w^{\lambda,s}}\partial_x^s \circ \mathcal{P}_i^{\lambda\varepsilon} \circ \sum_{t\geq 0}(-\partial_x)^t \circ \frac{\partial \tilde{w}^\beta}{\partial w^{\varepsilon,t}}.$$

Therefore by using the relation (2.16), it is easy to see that:

$$\begin{aligned}
\tilde{\mathcal{P}}_0^{\alpha\beta}\tilde{\phi}_{\beta,m+1} &= \sum_{s\geq 0}\frac{\partial \tilde{w}^\alpha}{\partial w^{\lambda,s}}\partial_x^s \circ \mathcal{P}_0^{\lambda\varepsilon} \circ \sum_{t\geq 0}(-\partial_x)^t \circ \frac{\partial \tilde{w}^\beta}{\partial w^{\varepsilon,t}}(\tilde{\phi}_{\beta,m+1}) \\
&= \sum_{s\geq 0}\frac{\partial \tilde{w}^\alpha}{\partial w^{\lambda,s}}\partial_x^s \circ \mathcal{P}_0^{\lambda\varepsilon}\phi_{\varepsilon,m+1} \\
&= \sum_{s\geq 0}\frac{\partial \tilde{w}^\alpha}{\partial w^{\lambda,s}}\partial_x^s \circ \mathcal{P}_1^{\lambda\varepsilon}\phi_{\varepsilon,m} \\
&= \sum_{s\geq 0}\frac{\partial \tilde{w}^\alpha}{\partial w^{\lambda,s}}\partial_x^s \circ \mathcal{P}_1^{\lambda\varepsilon} \circ \sum_{t\geq 0}(-\partial_x)^t \circ \frac{\partial \tilde{w}^\beta}{\partial w^{\varepsilon,t}}(\tilde{\phi}_{\beta,m}) \\
&= \tilde{\mathcal{P}}_1^{\alpha\beta}\tilde{\phi}_{\beta,m}.
\end{aligned}$$

Thus we see that the recursion relations (2.16) are preserved under the change of coordinates (2.17). The proposition is proved. □

By using the theory of bihamiltonian cohomology [7] for $(P_0^{[0]}, P_1^{[0]})$, we can choose a coordinate system $(v^{\alpha,s}; \sigma_\alpha^s)$ such that

$$P_0 = P_0^{[0]} = \frac{1}{2}\int \eta^{\alpha\beta}\sigma_\alpha\sigma_\beta^1,$$

and the $\hat{\mathcal{F}}_2^2$ component of P_1 vanishes. From now on, we will always use $(v^{\alpha,s}; \sigma_\alpha^s)$ to denote the coordinate system described above. We use the notation $\hat{\mathcal{A}}^+$ to denote the extension of $\hat{\mathcal{A}}$ by including the odd variables $\sigma_{\alpha,m}^s$ for $m \geq 1$ satisfying the recursion relations

$$\eta^{\alpha\beta}\sigma_{\beta,m+1}^1 = \mathcal{P}_1^{\alpha\beta}\sigma_{\beta,m}, \quad m \geq 0. \tag{2.18}$$

As before we use the notation $\sigma_{\alpha,0}^s = \sigma_\alpha^s$ and $\sigma_{\alpha,m}^0 = \sigma_{\alpha,m}$. We will still use ∂_x to denote the vector field on $\hat{\mathcal{A}}^+$ defined by

$$\partial_x = \sum_{s\geq 0} v^{\alpha,s+1}\frac{\partial}{\partial v^{\alpha,s}} + \sum_{s,m\geq 0}\sigma_{\alpha,m}^{s+1}\frac{\partial}{\partial \sigma_{\alpha,m}^s}.$$

For any element $f \in \hat{\mathcal{A}}^+$, we say that f is *local* if it can be represented by an element of $\hat{\mathcal{A}}$ and we say that f is *non-local* if it is not local. Note that on the space $\hat{\mathcal{A}}^+$, the super degree is still well defined by setting the super degree of $\sigma^s_{\alpha,m}$ being 1. We will use $\hat{\mathcal{A}}^{+,p}$ to denote the set of homogeneous elements with super degree p.

Example 3. Consider the following bihamiltonian structure of the KdV equation (2.8):

$$\mathcal{P}_0 = \partial_x, \quad \mathcal{P}_1 = v\partial_x + \frac{1}{2}v_x + \frac{\varepsilon^2}{8}\partial_x^3.$$

We introduce odd variables σ^s_m for $s, m \geq 0$ such that they satisfy the recursion relations

$$\sigma^1_{m+1} = v\sigma^1_m + \frac{1}{2}v_x\sigma_m + \frac{\varepsilon^2}{8}\sigma^3_m, \quad m \geq 0.$$

Then the ring $\hat{\mathcal{A}}^+$ is given by the quotient

$$\hat{\mathcal{A}}^+ = C^\infty(v)[[v^{(s+1)}, \sigma^s_m \mid m, s \geq 0]]/J,$$

where J is the differential ideal generated by

$$v\sigma^1_m + \frac{1}{2}v_x\sigma_m + \frac{\varepsilon^2}{8}\sigma^3_m - \sigma^1_{m+1}, \quad m \geq 0.$$

Then we see that σ^1_1 is local but σ^1_2 is non-local.

Definition 1. For $k, l \geq 0$, we define the shift operators

$$T_k : \hat{\mathcal{A}}^1 \to \hat{\mathcal{A}}^{+,1}, \quad T_{k,l} : \hat{\mathcal{A}}^2 \to \hat{\mathcal{A}}^{+,2}$$

to be the linear operators given by

$$T_k(f\sigma^s_{\alpha,0}) = f\sigma^s_{\alpha,k}, \quad f \in \hat{\mathcal{A}}^0,$$
$$T_{k,l} = -T_{l,k}, \quad T_{k,l}(f\sigma^t_{\alpha,0}\sigma^s_{\beta,0}) = f\sum_{i=0}^{l-k-1}\sigma^t_{\alpha,k+i}\sigma^s_{\beta,l-i-1}, \quad k \leq l, \ f \in \hat{\mathcal{A}}^0.$$

In particular, $T_{k,k} = 0$.

The following lemmas are obvious from the above definition.

Lemma 1. *The shift operators T_k and $T_{k,l}$ commute with ∂_x.*

Lemma 2. *The shift operators T_k and $T_{k,l}$ are globally defined, i.e. they are invariant under Miura type transformations.*

Example 4. Using the shift operators, the recursion relation (2.18) can be represented by the following formula

$$T_{m+1}\frac{\delta P_0}{\delta\sigma_{\alpha,0}} = T_m\frac{\delta P_1}{\delta\sigma_{\alpha,0}}, \quad m \geq 0. \tag{2.19}$$

Example 5. The recursion relation (2.18) can also be represented by the following formula:

$$T_{m+1}\left(D_{P_0} f\right) = T_m\left(D_{P_1} f\right), \quad m \geq 0,\ f \in \hat{\mathcal{A}}^0, \tag{2.20}$$

here D_{P_i} are the derivations defined in (2.1). Indeed, when $f = v^\alpha$ we recover the relation (2.19); for general $f \in \hat{\mathcal{A}}^0$, by definition (2.1), we have

$$T_{m+1}\left(D_{P_0} f\right) = T_{m+1} \sum_{s\geq 0} \frac{\partial f}{\partial v^{\alpha,s}} \partial_x^s \frac{\delta P_0}{\delta \sigma_{\alpha,0}} = T_m \sum_{s\geq 0} \frac{\partial f}{\partial v^{\alpha,s}} \partial_x^s \frac{\delta P_1}{\delta \sigma_{\alpha,0}} = T_m\left(D_{P_1} f\right).$$

With the help of the shift operators, we can generalize the construction given in the previous subsection. We first introduce the following notation.

Definition 2. We define a family of odd derivations $\frac{\partial}{\partial \tau_m}$ on $\hat{\mathcal{A}}^+$ by

$$\frac{\partial v^\alpha}{\partial \tau_m} = T_m \frac{\delta P_0}{\delta \sigma_{\alpha,0}}, \quad \frac{\partial \sigma_{\alpha,k}}{\partial \tau_m} = T_{k,m} \frac{\delta P_1}{\delta v^\alpha}, \quad \left[\frac{\partial}{\partial \tau_m}, \partial_x\right] = 0. \tag{2.21}$$

In particular, for $f \in \hat{\mathcal{A}}$ we have

$$\frac{\partial f}{\partial \tau_0} = D_{P_0} f, \quad \frac{\partial f}{\partial \tau_1} = D_{P_1} f, \tag{2.22}$$

and for $f \in \hat{\mathcal{A}}^0$ we have

$$\frac{\partial f}{\partial \tau_m} = T_m \frac{\partial f}{\partial \tau_0}, \quad m \geq 0. \tag{2.23}$$

We need to check that that this definition is well-defined, i.e., it is compatible with the recursion relations (2.19).

Lemma 3. *The following identity holds true for any $X \in \hat{\mathcal{A}}^1$ and $m, k \geq 0$:*

$$\frac{\partial}{\partial \tau_k} T_m(X) = T_{m,k}\left(D_{P_1}(X)\right) - T_{m+1,k}\left(D_{P_0}(X)\right).$$

Proof. Since all the operators are linear, we may assume $X = f\sigma^l_{\beta,0}$ for some $f \in \hat{\mathcal{A}}^0$ and $l \geq 0$. We first assume that $k \geq m$. The case $k = m$ can be easily verified as follows:

$$\frac{\partial}{\partial \tau_m}(f\sigma^l_{\beta,m}) = T_m\left(D_{P_0} f\right)\sigma^l_{\beta,m} = T_{m,m+1}\left(D_{P_0}(f\sigma^l_{\beta,0})\right).$$

Here we use the fact that $P_0 = P_0^{[0]}$ and $D_{P_0}\sigma_{\beta,0} = 0$. Now we assume $k \geq m+1$, then by using the definition of the shift operators and the recursion relation (2.19) we obtain the following identities:

$$\begin{aligned}\frac{\partial}{\partial \tau_k}(f\sigma^l_{\beta,m}) &= T_k\left(D_{P_0} f\right)\sigma^l_{\beta,m} + T_{m,k}\left(f D_{P_1}\sigma^l_{\beta,0}\right) \\ &= T_{k-1}\left(D_{P_1} f\right)\sigma^l_{\beta,m} + T_{m,k}\left(f D_{P_1}\sigma^l_{\beta,0}\right) \\ &= T_{m,k}\left(\left(D_{P_1} f\right)\sigma^l_{\beta,0}\right)\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=0}^{k-m-2} T_{m+i} \left(D_{P_1} f\right) \sigma^l_{\beta,k-i-1} + T_{m,k} \left(f D_{P_1} \sigma^l_{\beta,0}\right) \\
&= T_{m,k} \left(D_{P_1}(f \sigma^l_{\beta,0})\right) - \sum_{i=0}^{k-m-2} T_{m+i+1} \left(D_{P_0} f\right) \sigma^l_{\beta,k-i-1} \\
&= T_{m,k} \left(D_{P_1}(f \sigma^l_{\beta,0})\right) - T_{m+1,k} \left(D_{P_0}(f \sigma^l_{\beta,0})\right).
\end{aligned}
$$

The case $k < m$ is proved in exactly the same way. The lemma is proved. □

Proposition 2. *The flows* $\frac{\partial}{\partial \tau_k}$ *are compatible with the recursion relation* (2.19)*, i.e.,*

$$
\frac{\partial}{\partial \tau_k} T_{m+1} \frac{\delta P_0}{\delta \sigma_{\alpha,0}} = \frac{\partial}{\partial \tau_k} T_m \frac{\delta P_1}{\delta \sigma_{\alpha,0}}, \quad m, k \geq 0.
$$

Proof. Using the fact that

$$
P_0 = P_0^{[0]} = \frac{1}{2} \int \eta^{\alpha\beta} \sigma_\alpha \sigma^1_\beta,
$$

it is easy to obtain the following identities:

$$
\frac{\partial}{\partial \tau_k} T_{m+1} \frac{\delta P_0}{\delta \sigma_{\alpha,0}} = \frac{\partial}{\partial \tau_k} \eta^{\alpha\beta} \sigma^1_{\beta,m+1} = T_{m+1,k} \left(\eta^{\alpha\beta} \partial_x \frac{\delta P_1}{\delta v^\beta}\right) = T_{m+1,k} \left(D_{P_1} \frac{\delta P_0}{\delta \sigma_{\alpha,0}}\right).
$$

Since $[P_0, P_1] = 0$, it follows from the identity (2.5) that

$$
\frac{\partial}{\partial \tau_k} T_{m+1} \frac{\delta P_0}{\delta \sigma_{\alpha,0}} = -T_{m+1,k} \left(D_{P_0} \frac{\delta P_1}{\delta \sigma_{\alpha,0}}\right).
$$

Thus by using $[P_1, P_1] = 0$ and Lemma 3 we finish the proof of the proposition. □

Proposition 3. *The odd flows* $\frac{\partial}{\partial \tau_m}$ *mutually commute, i.e.,*

$$
\left[\frac{\partial}{\partial \tau_m}, \frac{\partial}{\partial \tau_k}\right] = 0, \quad m,\ k \geq 0.
$$

Proof. By the definition of the flows $\frac{\partial}{\partial \tau_m}$, it is easy to see that

$$
\left[\frac{\partial}{\partial \tau_m}, \frac{\partial}{\partial \tau_k}\right] v^\alpha = \eta^{\alpha\beta} \partial_x T_{k,m} \frac{\delta P_1}{\delta v^\beta} + \eta^{\alpha\beta} \partial_x T_{m,k} \frac{\delta P_1}{\delta v^\beta} = 0.
$$

To show the commutation relation

$$
\left[\frac{\partial}{\partial \tau_l}, \frac{\partial}{\partial \tau_k}\right] \sigma_{\alpha,m} = 0,
$$

it suffices to verify the case $m = 0$ due to the recursion relations (2.18). By using the trivial relation

$$
\left[\frac{\partial}{\partial \tau_0}, \frac{\partial}{\partial \tau_0}\right] \sigma_{\alpha,0} = 0,
$$

the recursion relations (2.18), and by induction on k, we arrive at

$$\left[\frac{\partial}{\partial\tau_0},\frac{\partial}{\partial\tau_0}\right]\sigma_{\alpha,k}=0,\quad k\geq 0.$$

This commutation relation is equivalent to

$$\left[\frac{\partial}{\partial\tau_0},\frac{\partial}{\partial\tau_k}\right]\sigma_{\alpha,0}=0$$

due to the definition of the odd flows. By using induction again we arrive at the identity

$$\left[\frac{\partial}{\partial\tau_0},\frac{\partial}{\partial\tau_k}\right]\sigma_{\alpha,l}=0$$

for any $l\geq 0$. Therefore we have

$$\frac{\partial}{\partial\tau_k}\frac{\partial\sigma_{\alpha,0}}{\partial\tau_l}=-\frac{\partial}{\partial\tau_k}\frac{\partial\sigma_{\alpha,l}}{\partial\tau_0}=\frac{\partial}{\partial\tau_0}\frac{\partial\sigma_{\alpha,l}}{\partial\tau_k}.$$

It follows from the definition of the odd flows that the right hand side is anti-symmetric with respect to the indices k, l, hence we prove that

$$\left[\frac{\partial}{\partial\tau_l},\frac{\partial}{\partial\tau_k}\right]\sigma_{\alpha,0}=0.$$

The proposition is proved. □

Now let $X_i\in\hat{\mathcal{F}}^1$, $i\in I$ be a family of bihamiltonian vector fields with respect to an index set I, i.e., each X_i satisfies the equations $[X_i,P_0]=[X_i,P_1]=0$. Recall that the family $\{X_i\}$ corresponds to a bihamiltonian integrable hierarchy given by

$$\frac{\partial v^\alpha}{\partial t_i}=\frac{\delta X_i}{\delta\sigma_{\alpha,0}},\quad i\in I. \tag{2.24}$$

In what follows we will extend this integrable hierarchy such that it becomes a system of mutually commuting vector fields on $\hat{\mathcal{A}}^+$.

Definition 3. For a bihamiltonian vector field $X\in\hat{\mathcal{F}}^1$ we associate it with the following system of PDEs on $\hat{\mathcal{A}}^+$:

$$\frac{\partial v^\alpha}{\partial t_X}=D_Xv^\alpha,\quad \frac{\partial\sigma_{\alpha,m}}{\partial t_X}=T_mD_X\sigma_{\alpha,0},\quad \left[\frac{\partial}{\partial t_X},\partial_x\right]=0.$$

It is called the super extended flow associated with X.

Proposition 4. *The super extended flow $\frac{\partial}{\partial t_X}$ associated with a bihamiltonian vector field X is compatible with the recursion relation* (2.19)*, i.e.,*

$$\frac{\partial}{\partial t_X}T_{m+1}\frac{\delta P_0}{\delta\sigma_{\alpha,0}}=\frac{\partial}{\partial t_X}T_m\frac{\delta P_1}{\delta\sigma_{\alpha,0}}. \tag{2.25}$$

Proof. From Definition 3 of the flow $\frac{\partial}{\partial t_X}$, it is easy to see that

$$\frac{\partial}{\partial t_X} T_{m+1} \frac{\delta P_0}{\delta \sigma_{\alpha,0}} = T_{m+1} D_X \frac{\delta P_0}{\delta \sigma_{\alpha,0}}, \quad \frac{\partial}{\partial t_X} T_m \frac{\delta P_1}{\delta \sigma_{\alpha,0}} = T_m D_X \frac{\delta P_1}{\delta \sigma_{\alpha,0}}.$$

On the other hand from the fact that $[X, P_0] = [X, P_1] = 0$ and the identity (2.5), we see that (2.25) is equivalent to the following identity:

$$T_{m+1} D_{P_0} \frac{\delta X}{\delta \sigma_{\alpha,0}} = T_m D_{P_1} \frac{\delta X}{\delta \sigma_{\alpha,0}},$$

which holds true due to (2.20). The proposition is proved. □

Proposition 5. *Let X and Y be two bihamiltonian vector fields, then their associated super extended flows commute.*

Proof. From the theory of the bihamiltonian cohomology [7] we know that $[X, Y] = 0$, hence it follows from (2.3) that

$$\left[\frac{\partial}{\partial t_X}, \frac{\partial}{\partial t_Y}\right] v^\alpha = 0, \quad \left[\frac{\partial}{\partial t_X}, \frac{\partial}{\partial t_Y}\right] \sigma_{\alpha,0} = 0.$$

By using Definition 3 we also have

$$\left[\frac{\partial}{\partial t_X}, \frac{\partial}{\partial t_Y}\right] \sigma_{\alpha,m} = T_m \left[\frac{\partial}{\partial t_X}, \frac{\partial}{\partial t_Y}\right] \sigma_{\alpha,0} = 0.$$

The proposition is proved. □

Now let us prove that the super extended flow associated with a bihamiltonian vector field commutes with the odd flows $\frac{\partial}{\partial \tau_m}$.

Lemma 4. *For any $\mathcal{D} \in \mathrm{Der}(\hat{\mathcal{A}})^0$ satisfying the condition $[\mathcal{D}, \partial_x] = 0$, we extend its action to $\hat{\mathcal{A}}^+$ by setting*

$$\mathcal{D}\sigma_{\alpha,m} = T_m \mathcal{D}\sigma_{\alpha,0}, \quad m \geq 0.$$

Then the following identities hold true:

$$T_k \circ \left(\mathcal{D}|_{\hat{\mathcal{A}}^1}\right) = \left(\mathcal{D}|_{\hat{\mathcal{A}}^{+,1}}\right) \circ T_k; \quad T_{k,l} \circ \left(\mathcal{D}|_{\hat{\mathcal{A}}^2}\right) = \left(\mathcal{D}|_{\hat{\mathcal{A}}^{+,2}}\right) \circ T_{k,l}, \quad k, l \geq 0.$$

Proof. The first identity is obvious from the definition $\mathcal{D}\sigma_{\alpha,k} = T_k \mathcal{D}\sigma_{\alpha,0}$. The second one can also be verified by using the definition of the shift operator $T_{k,l}$. The lemma is proved. □

Proposition 6. *The odd flows $\frac{\partial}{\partial \tau_m}$ commute with the super extended flow associated with a bihamiltonian vector field X.*

Proof. Using Lemma 4 and the definition of the odd flows, it is easy to see that

$$\frac{\partial}{\partial t_X}\frac{\partial v^\alpha}{\partial \tau_m} = \frac{\partial}{\partial t_X} T_m \frac{\delta P_0}{\delta \sigma_{\alpha,0}} = T_m D_X \left(\frac{\delta P_0}{\delta \sigma_{\alpha,0}}\right), \quad \frac{\partial}{\partial \tau_m}\frac{\partial v^\alpha}{\partial t_X} = T_m D_{P_0}\left(\frac{\partial v^\alpha}{\partial t_X}\right).$$

Therefore it follows from $[X, P_0] = 0$ and the identity (2.5) that

$$\left[\frac{\partial}{\partial \tau_m}, \frac{\partial}{\partial t_X}\right] v^\alpha = 0.$$

Similarly, by using Lemma 4 again we have

$$\frac{\partial}{\partial t_X}\frac{\partial \sigma_{\alpha,k}}{\partial \tau_m} = \frac{\partial}{\partial t_X} T_{k,m}\frac{\delta P_1}{\delta v^\alpha} = T_{k,m} D_X\left(\frac{\delta P_1}{\delta v^\alpha}\right).$$

On the other hand, by using Lemma 3 and the fact that $[P_0, X] = 0$ and $\frac{\delta P_0}{\delta v^\alpha} = 0$, we obtain

$$\frac{\partial}{\partial \tau_m}\frac{\partial \sigma_{\alpha,k}}{\partial t_X} = -\frac{\partial}{\partial \tau_m} T_k \left(D_X\left(\frac{\delta X}{\delta v^\alpha}\right)\right) = -T_{k,m}\left(D_{P_1}\left(\frac{\delta X}{\delta v^\alpha}\right)\right).$$

Thus by using (2.4) and $[X, P_1] = 0$, we can conclude that

$$\left[\frac{\partial}{\partial \tau_m}, \frac{\partial}{\partial t_X}\right] \sigma_{\alpha,k} = 0.$$

The proposition is proved. □

Let us summarize the constructions given in this subsection in the following theorem.

Theorem 7. *Let (P_0, P_1) be a bihamiltonian structure with semisimple hydrodynamic leading terms and $\{X_i\}$ be a family of bihamiltonian vector fields, then we have the following super extended integrable hierarchy:*

$$\frac{\partial v^\alpha}{\partial t_i} = D_{X_i} v^\alpha, \quad \frac{\partial \sigma_{\alpha,m}}{\partial t_i} = T_m D_{X_i}\sigma_{\alpha,0},$$
$$\frac{\partial v^\alpha}{\partial \tau_m} = T_m \frac{\delta P_0}{\delta \sigma_{\alpha,0}}, \quad \frac{\partial \sigma_{\alpha,k}}{\partial \tau_m} = T_{k,m}\frac{\delta P_1}{\delta v^\alpha}.$$

The flows in this hierarchy mutually commute.

2.4. Deformations of the super tau-covers. In this subsection, we construct super tau-covers for tau-symmetric bihamiltonian deformations of the Principal Hierarchy associated with a semisimple Frobenius manifold. Let us first recall how to construct the deformations of the tau-cover (2.14) of the Principal Hierarchy following [9].

We fix a semisimple Frobenius manifold M and use $(P_0^{[0]}, P_1^{[0]})$ to denote the bihamiltonian structure (2.13). We denote the two-point functions in the tau-cover (2.14) by $\Omega^{[0]}_{\alpha,p;\beta,q}$ and denote the Hamiltonian densities of $P_0^{[0]}$ by $h^{[0]}_{\alpha,p}$, which are equal to $\Omega^{[0]}_{\alpha,p;1,0}$. Let (P_0, P_1) be a deformation of $(P_0^{[0]}, P_1^{[0]})$, then it determines a unique deformation of the Principal Hierarchy according to [7,26]. By using the results proved in [13], we know that a bihamiltonian deformation of the Principal Hierarchy is tau-symmetric if and only if the central invariants of the deformation of the bihamiltonian structure are constants. In such a case, after an appropriate Miura type transformation the Hamiltonian structure P_0 can be represented in the form

$$P_0 = P_0^{[0]} = \frac{1}{2}\int \eta^{\alpha\beta}\sigma_\alpha\sigma_\beta^1,$$

and P_1 has no $\hat{\mathcal{F}}_2^2$ components. Moreover, we can also require that the condition of exactness of the bihamiltonian structure is preserved [9,13]:

$$P_0 = [Z, P_1], \quad Z = \int \sigma_1.$$

In what follows, we will always assume that P_0 and Z take the above forms. We still use the same notation $X_{\alpha,p} \in \hat{\mathcal{F}}^1$, as we have already used in (2.12) for the flows of the Principal Hierarchy, to denote the unique deformed flows of the Principal Hierarchy, and we will also use $\frac{\partial}{\partial t^{\alpha,p}}$ to denote the vector field $D_{X_{\alpha,p}}$. Let $H_{\alpha,p} \in \hat{\mathcal{F}}^0$ be the unique deformations of the Hamiltonians of the Principal Hierarchy such that

$$X_{\alpha,p} = -\left[H_{\alpha,p}, P_0\right], \quad \alpha = 1, \dots, n, \quad p \geq 0, \tag{2.26}$$

and

$$H_{\alpha,-1} := \int \eta_{\alpha\beta} v^\beta. \tag{2.27}$$

Let us define

$$h_{\alpha,p} = D_Z H_{\alpha,p}, \quad \alpha = 1, \dots, n, \quad p \geq 0. \tag{2.28}$$

Note that we use an index convention that is different from the one used in [9].

Proposition 7 ([9]). *We have the following results:*

1. $D_{X_{1,0}} = \partial_x$.
2. *The functionals $H_{\alpha,p}$ defined in* (2.26), (2.27) *and the differential polynomials defined in* (2.28) *satisfy the relations*

$$H_{\alpha,p} = \int h_{\alpha,p+1}, \quad p \geq -1.$$

We also have the following proposition and theorem on properties of the Hamiltonians and two-point functions of the deformed Principal Hierarchy.

Proposition 8. *The following bihamiltonian recursion relation holds true:*

$$[H_{\alpha,p-1}, P_1] = \left(p + \frac{1}{2} + \mu_\alpha\right)[H_{\alpha,p}, P_0] + \sum_{k=1}^{p} (R_k)_\alpha^\gamma\, [H_{\gamma,p-k}, P_0], \quad p \geq 0. \tag{2.29}$$

Proof. Denote

$$Y_{\alpha,p} = [H_{\alpha,p-1}, P_1] - \left(p + \frac{1}{2} + \mu_\alpha\right)[H_{\alpha,p}, P_0] - \sum_{k=1}^{p} (R_k)_\alpha^\gamma \, [H_{\gamma,p-k}, P_0], \quad p \geq 0,$$

then from Theorem 3 we know that $Y_{\alpha,p} \in \hat{\mathcal{F}}^1_{\geq 2}$. Since the flows $X_{\alpha,p} = -[H_{\alpha,p}, P_0]$ are bihamiltonian, we conclude that $Y_{\alpha,p}$ is also a bihamiltonian vector field. Therefore by using the theory of bihamiltonian cohomology developed in [7] we arrive at $Y_{\alpha,p} = 0$. □

Theorem 8 ([9]). *There exist differential polynomials $\Omega_{\alpha,p;\beta,q}$ such that they are deformations of $\Omega^{[0]}_{\alpha,p;\beta,q}$, and satisfy the following properties:*

1. $\partial_x \Omega_{\alpha,p;\beta,q} = \frac{\partial h_{\beta,q}}{\partial t^{\alpha,p}}$.
2. $\Omega_{\alpha,p;\beta,q} = \Omega_{\beta,q;\alpha,p}$ *and* $\Omega_{\alpha,p;1,0} = h_{\alpha,p}$.
3. $\frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial t^{\lambda,k}} = \frac{\partial \Omega_{\lambda,k;\beta,q}}{\partial t^{\alpha,p}}$.

To construct the tau-cover of the deformed Principal Hierarchy, we introduce the following *normal coordinates* as in [12]:

$$w_\alpha = h_{\alpha,0}, \quad w^\alpha = \eta^{\alpha\beta} w_\beta, \tag{2.30}$$

then we see that the differential polynomials w^α and v^α are related by a Miura type transformation. In particular, it follows from Proposition 7 that

$$H_{\alpha,-1} = \int w_\alpha = \int \eta_{\alpha\beta} v^\beta. \tag{2.31}$$

In terms of the normal coordinates, the tau-cover of the deformed Principal Hierarchy can be represented in the form (cf. (2.14))

$$\frac{\partial f_{\alpha,p}}{\partial t^{\beta,q}} = \Omega_{\alpha,p;\beta,q}, \quad \frac{\partial w^\alpha}{\partial t^{\beta,q}} = \eta^{\alpha\lambda} \partial_x \Omega_{\lambda,0;\beta,q}. \tag{2.32}$$

Let us proceed to construct its super tau-cover. To this end we introduce odd variables $\Phi^m_{\alpha,p}$, as we do for the super tau-cover of the Principal Hierarchy, such that

$$\partial_x \Phi^m_{\alpha,p} = \frac{\partial h_{\alpha,p}}{\partial \tau_m}, \quad m \geq 0. \tag{2.33}$$

Here the odd flows $\frac{\partial}{\partial \tau_m}$ are defined by (2.21). By using Theorem 8 we obtain the following identity:

$$\frac{\partial}{\partial t^{\beta,q}} \frac{\partial h_{\alpha,p}}{\partial \tau_m} = \partial_x \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial \tau_m}.$$

Therefore we conclude that the following definitions of the evolutions of the odd variables $\Phi^m_{\alpha,p}$ along the flows $\frac{\partial}{\partial t^{\beta,q}}$ are compatible with (2.33):

$$\frac{\partial \Phi^m_{\alpha,p}}{\partial t^{\beta,q}} = \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial \tau_m}.$$

To define the evolutions of $\Phi^m_{\alpha,p}$ along the odd flows $\frac{\partial}{\partial \tau_k}$, we need the following lemma.

Lemma 5. *There exist differential polynomials* $F_{\alpha,p} \in \hat{\mathcal{A}}^2$ *such that*

$$\frac{\partial}{\partial \tau_k}\frac{\partial h_{\alpha,p}}{\partial \tau_m} = T_{m,k}\partial_x F_{\alpha,p}, \quad \alpha = 1,\dots,n;\ \ p,m,k \geq 0.$$

Proof. By using the definition (2.1) of D_{P_i} and the fact that the vector fields $X_{\alpha,p}$ are bihamiltonian, it is easy to see that

$$\int D_{P_1} D_{P_0} h_{\alpha,p} = [P_1, [P_0, H_{\alpha,p-1}]] = 0.$$

Therefore from (2.22) it follows that there exists $F_{\alpha,p} \in \hat{\mathcal{A}}^2$ such that

$$\frac{\partial}{\partial \tau_1}\frac{\partial h_{\alpha,p}}{\partial \tau_0} = \partial_x F_{\alpha,p}.$$

Now from Lemma 3 and (2.23) it follows that

$$\frac{\partial}{\partial \tau_k}\frac{\partial h_{\alpha,p}}{\partial \tau_m} = \frac{\partial}{\partial \tau_k} T_m \frac{\partial h_{\alpha,p}}{\partial \tau_0} = T_{m,k}\partial_x F_{\alpha,p}.$$

The lemma is proved. □

Now we are ready to construct the super tau-cover of the deformed Principal Hierarchy.

Theorem 9. *Let* M *be a semisimple Frobenius manifold and* (P_0, P_1) *be a deformation of the bihamiltonian structure* (2.13) *with constant central invariants, then the following flows together with the super extended flows associated with* $\frac{\partial}{\partial t^{\alpha,p}}$ *form the super tau-cover of the deformed Principal Hierarchy:*

$$\frac{\partial f_{\alpha,p}}{\partial t^{\beta,q}} = \Omega_{\alpha,p;\beta,q}, \quad \frac{\partial f_{\alpha,p}}{\partial \tau_m} = \Phi^m_{\alpha,p},$$
$$\frac{\partial \Phi^m_{\alpha,p}}{\partial t^{\beta,q}} = \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial \tau_m}, \quad \frac{\partial \Phi^m_{\alpha,p}}{\partial \tau_k} = T_{m,k} F_{\alpha,p}.$$

Recall that when the diagonal matrix μ of a Frobenius manifold M satisfies the condition $\frac{1-2k}{2} \notin \operatorname{Spec}(\mu)$ for any $k = 1, 2, \dots$, the odd variables $\Phi^m_{\alpha,p}$ for the super tau-cover of the Principal Hierarchy are redundant since they can be represented by elements in $\hat{\mathcal{A}}^+$. Proposition 9 that we are to give below shows that the deformed super tau-cover has the same property, which will play an important role in our consideration of the deformation of the Virasoro symmetries.

We start with the definition of generalized shift operators.

Definition 4. We define the shift operators $\hat{T}_k$ for $k \geq 0$ to be the linear operators from $\hat{\mathcal{A}}^{+,1}$ to $\hat{\mathcal{A}}^{+,1}$ such that

$$\hat{T}_k(f\sigma^l_{\alpha,m}) = f\sigma^l_{\alpha,m+k}, \quad f \in \hat{\mathcal{A}}^0,\ m \geq 0.$$

The following lemma is easy to verify.

Lemma 6. *The operators* $\hat{T}_k$ *commute with* ∂_x *and are compatible with the recursion relation* (2.18).

Lemma 7. *If* $\Phi^0_{\alpha,p}$ *can be represented by an element in* $\hat{\mathcal{A}}^{+,1}$*, then so does* $\Phi^m_{\alpha,p}$*. In this case* $\Phi^m_{\alpha,p} = \hat{T}_m \Phi^0_{\alpha,p}$ *for* $m \geq 1$.

Proof. By using the relations (2.33) and (2.23), it is easy to see that

$$\partial_x \hat{T}_m \Phi^0_{\alpha,p} = \hat{T}_m \frac{\partial h_{\alpha,p}}{\partial \tau_0} = \frac{\partial h_{\alpha,p}}{\partial \tau_m} = \partial_x \Phi^m_{\alpha,p}.$$

The lemma is proved. □

We will use the notation $\Phi^m_{\alpha,p} \in \hat{\mathcal{A}}^+$ to mean that $\Phi^m_{\alpha,p}$ can be represented by an element in $\hat{\mathcal{A}}^+$.

Proposition 9. *We have* $\Phi^m_{\alpha,0} \in \hat{\mathcal{A}}^+$ *and*

$$\left(\prod_{k=1}^{p}\left(k - \frac{1}{2} + \mu_\alpha\right)\right)\Phi^m_{\alpha,p} \in \hat{\mathcal{A}}^+, \quad p \geq 1. \tag{2.34}$$

Proof. It follows from Lemma 7 that we only need to prove this lemma for $m = 0$. For $\Phi^0_{\alpha,0}$, it is easy to see from (2.31) that there exist differential polynomials $g_\alpha \in \hat{\mathcal{A}}^0_{\geq 1}$ such that

$$h_{\alpha,0} = \eta_{\alpha\beta} v^\beta + \partial_x g_\alpha. \tag{2.35}$$

Therefore we arrive at

$$\Phi^0_{\alpha,0} = \sigma_{\alpha,0} + \frac{\partial g_\alpha}{\partial \tau_0} \in \hat{\mathcal{A}}. \tag{2.36}$$

We proceed to consider $\Phi^0_{\alpha,p}$ for $p \geq 1$. By using Proposition 7 we can rewrite (2.29) as follows:

$$\left[\int h_{\alpha,p}, P_1\right] = \left(p + \frac{1}{2} + \mu_\alpha\right)\left[\int h_{\alpha,p+1}, P_0\right] + \sum_{k=1}^{p} (R_k)^\gamma_\alpha \left[\int h_{\gamma,p-k+1}, P_0\right]. \tag{2.37}$$

By taking $p = 0$ in(2.37) we get

$$\left(\frac{1}{2} + \mu_\alpha\right)\int \frac{\partial h_{\alpha,1}}{\partial \tau_0} = \int \frac{\partial h_{\alpha,0}}{\partial \tau_1},$$

so there exists a differential polynomial $p_{\alpha,1} \in \hat{\mathcal{A}}^1$ such that

$$\left(\frac{1}{2} + \mu_\alpha\right)\frac{\partial h_{\alpha,1}}{\partial \tau_0} = \frac{\partial h_{\alpha,0}}{\partial \tau_1} + \partial_x p_{\alpha,1}.$$

Therefore we have

$$\left(\frac{1}{2} + \mu_\alpha\right)\Phi^0_{\alpha,1} = \Phi^1_{\alpha,0} + p_{\alpha,1} \in \hat{\mathcal{A}}^+.$$

For general $p \geq 1$, we can prove (2.34) by using (2.37) and induction on p. The proposition is proved. □

3. Deformations of Virasoro Symmetries: Formulation

In this section, we first recall the theory of variational bihamiltonian cohomology developed in [23] and then explain how to use it to study Virasoro symmetries of the deformed Principal Hierarchies. We also use the example of the deformation of the Riemann hierarchy to illustrate our approach to the study of Virasoro symmetries.

3.1. Variational bihamiltonian cohomologies. In [23], we established a cohomology theory on the space $\mathrm{Der}^\partial(\hat{\mathcal{A}})$ consisting of derivations on $\hat{\mathcal{A}}$ that commute with ∂_x. This theory provides us suitable tools to study Virasoro symmetries of deformations of the Principal Hierarchies. We recall the basic definitions and results in this subsection.

Let us define the space $\mathrm{Der}^\partial(\hat{\mathcal{A}})$ by

$$\mathrm{Der}^\partial(\hat{\mathcal{A}}) = \{X \in \mathrm{Der}(\hat{\mathcal{A}}) \mid [X, \partial_x] = 0\},$$

it admits a gradation induced from $\mathrm{Der}(\hat{\mathcal{A}})$ and we denote $\mathrm{Der}^\partial(\hat{\mathcal{A}})^p_d = \mathrm{Der}^\partial(\hat{\mathcal{A}}) \cap \mathrm{Der}(\hat{\mathcal{A}})^p_d$.

Lemma 8. $\mathrm{Der}^\partial(\hat{\mathcal{A}})^p_d = 0$ *for* $p \le -2$ *or* $d < 0$.

Proof. Let us choose a local coordinate system (w^α, ϕ_α) on $\hat{M}$. Assume $X \in \mathrm{Der}^\partial(\hat{\mathcal{A}})$ with super degree $p \le -2$ or $d < 0$. Then by definition this means

$$X(w^\alpha) = X(\phi_\alpha) = 0.$$

Since $[X, \partial_x] = 0$, we immediately see that $X(w^{\alpha,s}) = 0$ and $X(\phi^s_\alpha) = 0$ for $s \ge 0$. Hence $X = 0$ and the lemma is proved. □

Let $P^{[0]}$ be a Hamiltonian structure of hydrodynamic type and $(P^{[0]}_0, P^{[0]}_1)$ be a semisimple bihamiltonian structure of hydrodynamic type. Then by using (2.3) we have a complex $(\mathrm{Der}^\partial(\hat{\mathcal{A}}), D_{P^{[0]}})$ and a double complex $(\mathrm{Der}^\partial(\hat{\mathcal{A}}), D_{P^{[0]}_0}, D_{P^{[0]}_1})$. We define the following cohomology groups:

$$H^p_d\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P^{[0]}\big) = \frac{\mathrm{Der}^\partial(\hat{\mathcal{A}})^p_d \cap \ker D_{P^{[0]}}}{\mathrm{Der}^\partial(\hat{\mathcal{A}})^p_d \cap \mathrm{Im}\, D_{P^{[0]}}}, \quad p, d \ge 0, \tag{3.1}$$

$$BH^p_d\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P^{[0]}_0, P^{[0]}_1\big) = \frac{\mathrm{Der}^\partial(\hat{\mathcal{A}})^p_d \cap \ker D_{P^{[0]}_0} \cap \ker D_{P^{[0]}_1}}{\mathrm{Der}^\partial(\hat{\mathcal{A}})^p_d \cap \mathrm{Im}\, D_{P^{[0]}_0} D_{P^{[0]}_1}}, \quad p, d \ge 0. \tag{3.2}$$

Note that the spaces $\mathrm{Der}^\partial(\hat{\mathcal{A}})^{-1}_d \neq 0$ for $d \ge 0$ and they must be taken into account while computing the cohomology groups. For example, the space $H^0_d(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P^{[0]})$ is given by

$$H^0_d\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P^{[0]}\big) = \frac{\ker(D_{P^{[0]}} : \mathrm{Der}^\partial(\hat{\mathcal{A}})^0_d \to \mathrm{Der}^\partial(\hat{\mathcal{A}})^1_{d+1})}{\mathrm{Im}(D_{P^{[0]}} : \mathrm{Der}^\partial(\hat{\mathcal{A}})^{-1}_{d-1} \to \mathrm{Der}^\partial(\hat{\mathcal{A}})^0_d)}.$$

By using the canonical symplectic structure on $\hat{M}$, we can identify the space $\mathrm{Der}^\partial(\hat{\mathcal{A}})$ with the space $\bar{\Omega}$ of local functionals of variational 1-forms, and the vector fields $D_{P^{[0]}}$, $D_{P_0^{[0]}}$ and $D_{P_1^{[0]}}$ induce differentials on $\bar{\Omega}$ via Lie derivatives. This is the reason why we call the above cohomology groups the variational cohomology groups. In [23], the cohomology groups (3.1) and (3.2) are computed by converting them to the cohomology groups on the space $\bar{\Omega}$. The details of the computation of these cohomology groups are not used in the present paper, so we omit them and refer the readers to [23]. The following result plays an essential role in the present paper.

Theorem 10 ([23]). *We have the following results on the cohomology groups* (3.1) *and* (3.2)*:*

1. $H_d^p\left(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P^{[0]}\right) = 0$ *for* $p \geq 0$, $d > 0$.
2. $BH^0_{\geq 2}\left(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}, P_1^{[0]}\right) = 0.$
3. $BH^1_{\geq 4}\left(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}, P_1^{[0]}\right) = 0.$
4. $BH^1_3\left(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}, P_1^{[0]}\right) \cong \oplus_{i=1}^n C^\infty(\mathbb{R}).$

Moreover, if we denote the action of a cocycle $X \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^1_3$ *on the* i*-th canonical coordinate* u^i *by*

$$X(u^i) = \sum_{j=1}^n \sum_{k=0}^3 X^k_{i,j}\theta_j^{3-k}, \quad X^k_{i,j} \in \hat{\mathcal{A}}^0_k,$$

then the cohomology class $[X]$ *is determined by the following functions:*

$$c_1 = \frac{X^0_{1,1}}{(f^1)^2}, \quad c_2 = \frac{X^0_{2,2}}{(f^2)^2}, \quad \dots, \quad c_n = \frac{X^0_{n,n}}{(f^n)^2}.$$

Here each function c_i *depends only on the* i*-th canonical coordinate* u^i*, and* f^i *is the function defined in* (2.7).

3.2. Virasoro symmetries of the Principal Hierarchies. Virasoro symmetries as well as Virasoro constraints are central conceptions in the study of modern mathematical physics, see, e.g. [11,16,17,30]. In this subsection, we recall the construction of Virasoro symmetries of the super tau-cover of the Principal Hierarchy following [22]. In [11], a family of infinitely many symmetries $\frac{\partial}{\partial s_m^{even}}$ for $m \geq -1$ of the tau-cover of the Principal Hierarchy associated with a Frobenius manifold M was constructed. This family of symmetries are called the Virasoro symmetries due to the property

$$\left[\frac{\partial}{\partial s_k^{even}}, \frac{\partial}{\partial s_l^{even}}\right] = (l-k)\frac{\partial}{\partial s_{k+l}^{even}}, \quad k, l \geq -1.$$

These symmetries can be represented by a family of quadratic differential operators L_m^{even} of the form

$$\begin{aligned} L_m^{even} &= \sum_{p,q\geq 0} a_m^{\alpha,p;\beta,q}\frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,q}} + b^{\beta,q}_{m;\alpha,p}t^{\alpha,p}\frac{\partial}{\partial t^{\beta,q}} + c_{m;\alpha,p;\beta,q}t^{\alpha,p}t^{\beta,q} \\ &\quad + \frac{1}{4}\delta_{m,0}\mathrm{tr}\left(\frac{1}{4}-\mu^2\right), \end{aligned}$$

where $a_m^{\alpha,p;\beta,q}$, $b_{m;\alpha,p}^{\beta,q}$, $c_{m;\alpha,p;\beta,q}$ are some constants determined by the monodromy data of M and one may refer to [11] for details. These operators satisfy the Virasoro commutation relation

$$[L_k^{even}, L_l^{even}] = (k-l)L_{l+k}^{even}.$$

In this paper, we only need the explicit expressions for L_{-1}^{even} and L_2^{even} which are given by

$$L_{-1}^{even} = \frac{1}{2}\eta_{\alpha\beta}t^{\alpha,0}t^{\beta,0} + \sum_{p\geq 1} t^{\alpha,p}\frac{\partial}{\partial t^{\alpha,p-1}}, \tag{3.3}$$

$$L_2^{even} = a^{\alpha\beta}\frac{\partial^2}{\partial t^{\alpha,1}\partial t^{\beta,0}} + b^{\alpha\beta}\frac{\partial^2}{\partial t^{\alpha,0}\partial t^{\beta,0}} + \mathcal{L}_2^{even} + c_{2;\alpha,p;\beta,q}t^{\alpha,p}t^{\beta,q}, \tag{3.4}$$

where the constants have the expressions

$$a^{\alpha\beta} = \eta^{\alpha\beta}\left(\frac{1}{2}+\mu_\beta\right)\left(\frac{1}{2}+\mu_\alpha\right)\left(\frac{3}{2}+\mu_\alpha\right), \tag{3.5}$$

$$b^{\alpha\beta} = \frac{1}{2}\eta^{\beta\gamma}(R_1)_\gamma^\alpha\left(\frac{1}{4}+3\mu_\beta-3\mu_\beta^2\right), \tag{3.6}$$

and the operator $\mathcal{L}_2^{even}$ is given by

$$\begin{aligned}\mathcal{L}_2^{even} &= \sum_{p\geq 0}\left(p+\frac{1}{2}+\mu_\alpha\right)\left(p+\frac{3}{2}+\mu_\alpha\right)\left(p+\frac{5}{2}+\mu_\alpha\right)t^{\alpha,p}\frac{\partial}{\partial t^{\alpha,p+2}}\\ &\quad+\sum_{p\geq 0}\sum_{1\leq r\leq p+2}\left(3\left(p+\frac{1}{2}+\mu_\alpha\right)^2+6\left(p+\frac{1}{2}+\mu_\alpha\right)+2\right)\\ &\quad\times(R_r)_\alpha^\beta\, t^{\alpha,p}\frac{\partial}{\partial t^{\beta,p-r+2}}\\ &\quad+\sum_{p\geq 0}\sum_{2\leq r\leq p+2}\left(3p+\frac{9}{2}+3\mu_\alpha\right)\left(R_{r,2}\right)_\alpha^\beta\, t^{\alpha,p}\frac{\partial}{\partial t^{\beta,p-r+2}}\\ &\quad+\sum_{p\geq 1}\sum_{3\leq r\leq p+2}\left(R_{r,3}\right)_\alpha^\beta\, t^{\alpha,p}\frac{\partial}{\partial t^{\beta,p-r+2}}.\end{aligned} \tag{3.7}$$

The explicit expressions for the matrices $R_{k,l}$ and constants $c_{2;\alpha,p;\beta,q}$ are not used in this paper, so we omit them.

We have the following theorem for the Virasoro symmetries of the tau-cover of the Principal Hierarchy.

Theorem 11 ([11]). *Let us define the following time-dependent flows for $m\geq -1$:*

$$\frac{\partial f_{\lambda,k}}{\partial s_m^{even}} = \frac{\partial}{\partial t^{\lambda,k}}\left(\sum a_m^{\alpha,p;\beta,q}f_{\alpha,p}f_{\beta,q} + \sum b_{m;\alpha,p}^{\beta,q}t^{\alpha,p}f_{\beta,q} + \sum c_{m;\alpha,p;\beta,q}t^{\alpha,p}t^{\beta,q}\right),$$

$$\frac{\partial v^\lambda}{\partial s_m^{even}} = \eta^{\lambda\gamma}\frac{\partial}{\partial t^{1,0}}\frac{\partial f_{\gamma,0}}{\partial s_m^{even}}.$$

Then the following commutation relation holds true:

$$\left[\frac{\partial}{\partial s_k^{even}}, \frac{\partial}{\partial t^{\alpha,p}}\right] = 0, \quad \left[\frac{\partial}{\partial s_k^{even}}, \frac{\partial}{\partial s_m^{even}}\right] = (m-k)\frac{\partial}{\partial s_{k+m}^{even}}, \quad k, m \geq -1.$$

We also have the following theorem for the Virasoro symmetries of the super tau-cover of the Principal Hierarchy.

Theorem 12 ([22]). *Let us define*

$$\frac{\partial}{\partial s_m^{odd}} = \sum_{p\geq 0}(p+c_0)\tau_p \frac{\partial}{\partial \tau_{p+m}}, \quad m \geq -1,$$

where c_0 is an arbitrary constant, and let us define $\frac{\partial}{\partial \tau_{-1}}$ to be zero, then the following flows are symmetries of the super tau-cover of the Principal Hierarchy associated with a Frobenius manifold:

$$\frac{\partial f_{\alpha,p}}{\partial s_m} = \frac{\partial f_{\alpha,p}}{\partial s_m^{even}} + \frac{\partial f_{\alpha,p}}{\partial s_m^{odd}}, \quad \frac{\partial \Phi_{\alpha,p}^n}{\partial s_m} = \frac{\partial}{\partial \tau_n}\left(\frac{\partial f_{\alpha,p}}{\partial s_m}\right),$$
$$\frac{\partial v^\alpha}{\partial s_m} = \frac{\partial v^\alpha}{\partial s_m^{even}} + \frac{\partial v^\alpha}{\partial s_m^{odd}}, \quad \frac{\partial \sigma_{\alpha,p}}{\partial s_m} = \frac{\partial}{\partial \tau_p}\left(\frac{\partial f_{\alpha,0}}{\partial s_m}\right).$$

Moreover, these flows satisfy the commutation relation

$$\left[\frac{\partial}{\partial s_k}, \frac{\partial}{\partial s_m}\right] = (m-k)\frac{\partial}{\partial s_{k+m}}, \quad k, m \geq -1.$$

Remark 2. Let us explain why there is an arbitrary constant involved in the Virasoro symmtries of the super tau-cover of the Principal Hierarchy. If we assign the odd time variables τ_k a degree c_k, then we can modify the zeroth Virasoro symmetry of the tau-cover of the Principal Hierarchy, which is a homogeneous condition, to the following symmetry of the super tau-cover:

$$\frac{\partial}{\partial s_0} = \frac{\partial}{\partial s_0^{even}} + \sum_{k\geq 0} c_k \tau_k \frac{\partial}{\partial \tau_k}.$$

By requiring that the above flow is a symmetry of the super tau-cover of the Principal Hierarchy we arrive at $c_k = c_0 + k$, here c_0 is an arbitrary constant.

Let us explain the motivation to introduce the non-local odd variables $\sigma_{\alpha,k}$ for $k \geq 1$ and the super extension of the tau-cover of the deformed Principal Hierarchy. For a given tau-symmetric bihamiltonian deformation of the Principal Hierarchy of a semisimple Frobenius manifold, we want to deform the Virasoro symmetries given in Theorem 11, i.e., to construct the flows $\frac{\partial}{\partial \tilde{s}_m^{even}}$ as deformations of $\frac{\partial}{\partial s_m^{even}}$, such that

$$\left[\frac{\partial}{\partial \tilde{s}_k^{even}}, \frac{\partial}{\partial \tilde{t}^{\alpha,p}}\right] = 0, \quad \left[\frac{\partial}{\partial \tilde{s}_k^{even}}, \frac{\partial}{\partial \tilde{s}_m^{even}}\right] = (m-k)\frac{\partial}{\partial \tilde{s}_{k+m}^{even}}, \quad k, m \geq -1,$$

here we denote by $\frac{\partial}{\partial \tilde{t}^{\alpha,p}}$ the flows of the deformed Principal Hierarchy. Due to the Virasoro commutation relation, we only need to find the flows $\frac{\partial}{\partial \tilde{s}_{-1}^{even}}$, $\frac{\partial}{\partial \tilde{s}_2^{even}}$, and use them to generate all other flows $\frac{\partial}{\partial \tilde{s}_m^{even}}$. It is proved in [9] that the symmetry $\frac{\partial}{\partial \tilde{s}_{-1}^{even}}$ always exists and therefore it remains to construct the flow $\frac{\partial}{\partial \tilde{s}_2^{even}}$ which satisfies the following equations:

$$\left[\frac{\partial}{\partial \tilde{s}_2^{even}}, \frac{\partial}{\partial \tilde{t}^{\alpha,p}}\right] = 0, \quad \alpha = 1, \dots, n, \quad p \geq 0.$$

Since there are infinitely many equations, it is not easy to solve them. From the study of the theory of variational bihamiltonian cohomologies, it follows that the problem of solving the above equations can be converted to solve the following two equations:

$$\left[\frac{\partial}{\partial \tilde{s}_2^{even}}, \frac{\partial}{\partial \tau_0}\right] = \left[\frac{\partial}{\partial \tilde{s}_2^{even}}, \frac{\partial}{\partial \tau_1}\right] = 0.$$

Since the odd flows $\frac{\partial}{\partial \tau_0}$ and $\frac{\partial}{\partial \tau_1}$ are not contained in the original tau-cover, we need a super extension of it. However, the above equations do not hold true at the dispersionless level. In [22], we prove that one can remedy this problem by adding infinitely many odd time variables τ_m and odd flows $\frac{\partial}{\partial \tau_m}$ for $m \geq 0$ to the Virasoro operator L_k^{even}, as described in Theorem 12.

The introduction of the odd flows $\frac{\partial}{\partial \tau_m}$ is motivated by the following simple observation. Let M be an n-dimensional Frobenius manifold, then we have

$$\frac{\partial v^\alpha}{\partial \tau_1} = \mathcal{R}\frac{\partial v^\alpha}{\partial \tau_0}, \quad \mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_0^{-1},$$

here $\mathcal{P}_0$ and $\mathcal{P}_1$ are the Hamiltonian operators of the bihamiltonian structure (2.13). The non-local operator $\mathcal{R}$ is called the recursion operator and we can define the odd flows recursively as follows:

$$\frac{\partial v^\alpha}{\partial \tau_{m+1}} = \mathcal{R}\frac{\partial v^\alpha}{\partial \tau_m}, \quad m \geq 1.$$

Due to the non-local nature of the recursion operator, we see that generally the action of the flow $\frac{\partial}{\partial \tau_m}$ on v^α can not be represented by elements of $\hat{\mathcal{A}}$ for $m \geq 2$. To overcome this non-locality problem we introduce, as it is typically done in the theory of integrable system, the odd variables $\sigma_{\alpha,k}$ for $k \geq 1$ to describe the actions of the odd flows $\frac{\partial}{\partial \tau_m}$. The constructions of $\sigma_{\alpha,k}$ are given by (2.18) and the flows $\frac{\partial}{\partial \tau_m}$ are defined in (2.21).

Remark 3. The idea of introducing non-local odd variables $\sigma_{\alpha,k}$ to study the non-local Hamiltonian structures is presented and illustrated via some examples in [18], see also [27].

3.3. Formulation of the deformation problem. In this subsection, we first state the main problem of this paper, then explain the motivation and strategy of our proof of the Main Theorem 1.

From now on, we fix a semisimple Frobenius manifold M of dimension n and let (P_0, P_1) be a deformation of the bihamiltonian structure (2.13) with constant central invariants. Then (P_0, P_1) determines a unique deformation of the Principal Hierarchy associated with M. After a suitable Miura type transformation we may assume, as we do in Sect. 2.4, that

$$P_0 = \frac{1}{2}\int \eta^{\alpha\beta}\sigma_\alpha\sigma_\beta^1 = [Z, P_1], \quad Z = \int \sigma_1,$$

and P_1 has no $\hat{\mathcal{F}}_2^2$ components. We also introduce odd variables $\sigma_{\alpha,m}$ for $m \geq 0$ as we explained in Sect. 2.3.

Let us denote by $\hat{\mathcal{A}}^{++}$ the following $\hat{\mathcal{A}}^{+}$-module

$$\hat{\mathcal{A}}^{++} := \hat{\mathcal{A}}^{+}\left[\Phi_{\alpha,p}^m,\ m \geq 1\right].$$

From Proposition 9 it follows that $\hat{\mathcal{A}}^{++} = \hat{\mathcal{A}}^{+}$ if the Frobeius manifold M satisfies the condition

$$k - \frac{1}{2} + \mu_\alpha \neq 0, \quad \forall k \geq 1, \quad \forall \alpha = 1, \ldots, n.$$

Let $\hat{\mathcal{A}}^{Vir}$ be the following $\hat{\mathcal{A}}^{++}$-module:

$$\hat{\mathcal{A}}^{Vir} = \hat{\mathcal{A}}^{++}[f_{\alpha,p}][[t^{\alpha,p}, \tau_m]],$$

here the time variables τ_m are odd. We will consider the space $\mathrm{Der}^\partial(\hat{\mathcal{A}}^{Vir})$, which consists of derivations of the space $\hat{\mathcal{A}}^{Vir}$ that commute with ∂_x. Here we extend the action of ∂_x to the space $\hat{\mathcal{A}}^{Vir}$ in the following natural way:

$$\partial_x \Phi_{\alpha,p}^m = \frac{\partial h_{\alpha,p}}{\partial \tau_m}, \quad \partial_x f_{\alpha,p} = h_{\alpha,p}, \quad \partial_x t^{\alpha,p} = \delta^{\alpha,1}\delta^{p,0}, \quad \partial_x \tau_m = 0.$$

Our problem can be stated as follows: to find a unique derivation $X \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^0_{\geq 0}$ such that the flow $\frac{\partial}{\partial s_2} \in \mathrm{Der}^\partial(\hat{\mathcal{A}}^{Vir})$ defined by

$$\frac{\partial v_\lambda}{\partial s_2} = a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial v_\lambda}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right) + b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial v_\lambda}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right) + X v_\lambda + \mathcal{L}_2 v_\lambda, \tag{3.8}$$

$$\frac{\partial \sigma_{\lambda,0}}{\partial s_2} = a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right) + b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right) + X\sigma_{\lambda,0} + M_\lambda^\zeta \sigma_{\zeta,2} + N_\lambda^\zeta \sigma_{\zeta,1} + \mathcal{L}_2 \sigma_{\lambda,0}, \tag{3.9}$$

satisfies the conditions

$$\left[\frac{\partial}{\partial s_2}, \frac{\partial}{\partial \tau_0}\right] = \left[\frac{\partial}{\partial s_2}, \frac{\partial}{\partial \tau_1}\right] = 0, \tag{3.10}$$

and we require that the leading term of X is determined by the Virasoro symmetry $\frac{\partial}{\partial s_2}$ of the super tau-cover of the Principal Hierarchy given in Theorem 12. The actions of the flow $\frac{\partial}{\partial s_2}$ on $f_{\alpha,p}$ and $\Phi_{\alpha,p}^m$ are omitted here for simplicity. These actions can be derived from (3.20) and the details will be given later at the end of this subsection. We also define that

$$\frac{\partial t^{\alpha,p}}{\partial s_2} = \frac{\partial \tau_m}{\partial s_2} = 0.$$

Moreover, the operator $\mathcal{L}_2$ is given by

$$\mathcal{L}_2 = \mathcal{L}_2^{even} + \sum_{p\geq 0}(p+c_0)\tau_p \frac{\partial}{\partial \tau_{p+2}},$$

and $M_\lambda^\zeta,\ N_\lambda^\zeta \in \hat{\mathcal{A}}^0$ are some differential polynomials whose definitions will be given later. The flows $\frac{\partial}{\partial \tau_0}$ and $\frac{\partial}{\partial \tau_1}$ are also extended to the space $\hat{\mathcal{A}}^{Vir}$ naturally by using the super tau-cover of the deformed Principal Hierarchy and by defining

$$\frac{\partial t^{\alpha,p}}{\partial \tau_i} = 0, \quad \frac{\partial \tau_m}{\partial \tau_i} = \delta_{i,m}, \quad i = 0, 1.$$

If we find such a derivation X, then we can prove that the flow $\frac{\partial}{\partial s_2}$ is a symmetry of the super tau-cover of the deformed Principal Hierarchy, i.e.,

$$\left[\frac{\partial}{\partial s_2}, \frac{\partial}{\partial t^{\alpha,p}}\right] = 0, \quad \alpha = 1, \ldots, n, \quad p \geq 0.$$

The above commutator should be understood as the natural commutator defined in the space $\mathrm{Der}^\partial(\hat{\mathcal{A}}^{Vir})$ and the actions of the flows $\frac{\partial}{\partial t^{\alpha,p}}$ are naturally extended to $\hat{\mathcal{A}}^{Vir}$. Therefore a priori we have

$$\left[\frac{\partial}{\partial s_2}, \frac{\partial}{\partial t^{\alpha,p}}\right] \in \mathrm{Der}^\partial(\hat{\mathcal{A}}^{Vir}).$$

However, if we can show that

$$\left[\frac{\partial}{\partial s_2}, \frac{\partial}{\partial t^{\alpha,p}}\right] \in \mathrm{Der}(\hat{\mathcal{A}})^0_{\geq 2}, \tag{3.11}$$

i.e., the actions of the above commutator can be restricted to the space $\hat{\mathcal{A}}$, then we conclude the vanishing of (3.11) from the property that

$$BH^0_{\geq 2}\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}, P_1^{[0]}\big) = 0,$$

and the fact that the commutator (3.11) is a cocycle. Here we use the definition (3.2) and Lemma 8 to arrive at the fact that

$$BH^0_{\geq 2}\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}, P_1^{[0]}\big) = \mathrm{Der}^\partial(\hat{\mathcal{A}})^0_{\geq 2} \cap \ker D_{P_0^{[0]}} \cap \ker D_{P_1^{[0]}}.$$

By using the definition (3.8) and (3.9) of $\frac{\partial}{\partial s_2}$ and after a simple computation, we arrive at

$$\left[\frac{\partial}{\partial s_2}, \frac{\partial}{\partial t^{\alpha,p}}\right] v_\lambda \in \hat{\mathcal{A}}, \quad \left[\frac{\partial}{\partial s_2}, \frac{\partial}{\partial t^{\alpha,p}}\right] \sigma_{\lambda,0} \in \hat{\mathcal{A}}^+,$$

so the condition (3.11) is actually a *locality condition*, i.e., it is equivalent to

$$\left[\frac{\partial}{\partial s_2}, \frac{\partial}{\partial t^{\alpha,p}}\right] v_\lambda \in \hat{\mathcal{A}}, \quad \left[\frac{\partial}{\partial s_2}, \frac{\partial}{\partial t^{\alpha,p}}\right] \sigma_{\lambda,0} \in \hat{\mathcal{A}}, \tag{3.12}$$

which is the condition we actually need to check.

Let us explain how to find a unique $X \in \mathrm{Der}^\partial(\hat{\mathcal{A}})$ such that the conditions in (3.10) hold true. To this end, we first rewrite the conditions in (3.10) into the equations for X as follows:

$$\left[\frac{\partial}{\partial \tau_0}, X\right] = I_0, \quad \left[\frac{\partial}{\partial \tau_1}, X\right] = I_1, \tag{3.13}$$

where I_0 and I_1 are some derivations that will be given later. The uniqueness of X is a consequence of the equations (3.13) and the fact that $BH^0_{\geq 2}\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}, P_1^{[0]}\big) = 0$, since the leading term of X is fixed. We will prove the existence of X by taking the following steps.

Step 1. To check the *locality condition* (3.12) and to prove that

$$I_0,\ I_1 \in \mathrm{Der}^\partial(\hat{\mathcal{A}}), \tag{3.14}$$

which is also a locality condition.

Step 2. To check the *closedness condition*

$$\left[\frac{\partial}{\partial \tau_0}, I_0\right] = 0. \tag{3.15}$$

After we finish Step 2, we can find a derivation $X^\circ \in \mathrm{Der}^\partial(\hat{\mathcal{A}})$ by using the property $H^1_{>0}\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}\big) = 0$ such that

$$I_0 = \left[\frac{\partial}{\partial \tau_0}, X^\circ\right],$$

and the leading term of X° is determined by the Virasoro symmetry of the super tau-cover of the Principal Hierarchy. We define $\mathcal{C} = X - X^\circ \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^0_{\geq 2}$, then the equations (3.13) for X are transformed to the following equations for $\mathcal{C}$:

$$\left[\frac{\partial}{\partial \tau_0}, \mathcal{C}\right] = 0, \quad \left[\frac{\partial}{\partial \tau_1}, \mathcal{C}\right] = I_1 - \left[\frac{\partial}{\partial \tau_1}, X^\circ\right]. \tag{3.16}$$

Step 3. To check the closedness conditions

$$\left[\frac{\partial}{\partial \tau_0}, I_1 - \left[\frac{\partial}{\partial \tau_1}, X^\circ\right]\right] = 0, \quad \left[\frac{\partial}{\partial \tau_1}, I_1\right] = 0. \tag{3.17}$$

Step 4. To check that the differential degree 3 component of the derivation $I_1 - \left[\frac{\partial}{\partial \tau_1}, X^\circ\right]$ vanishes in the cohomology group $BH^1_3\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}, P_1^{[0]}\big)$.

We call the above fact the *vanishing of the genus one obstruction* for the following reason. By using the first equation in (3.16) and the vanishing of $H^0_{\geq 2}\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}\big)$, we see that there exists a unique $\mathcal{T} \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^{-1}_{\geq 1}$ such that

$$\mathcal{C} = \left[\frac{\partial}{\partial \tau_0}, \mathcal{T}\right].$$

The derivation $\mathcal{T}$ must also satisfy the second equation in (3.16)

$$\left[\frac{\partial}{\partial \tau_1}, \left[\frac{\partial}{\partial \tau_0}, \mathcal{T}\right]\right] = I_1 - \left[\frac{\partial}{\partial \tau_1}, X^\circ\right]. \tag{3.18}$$

If the differential degree 3 component of the derivation $I_1 - \left[\frac{\partial}{\partial \tau_1}, X^\circ\right]$ does not vanish in the cohomology group $BH^1_3\left(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}, P_1^{[0]}\right)$, then such $\mathcal{T}$ does not exist.

However if the genus one obstruction vanishes, there exists a derivation $\mathcal{T}$, whose differential degree 1 part is unique, such that the Eq. (3.18) is valid at the approximation of differential degree 3. Then by using $BH^1_{\geq 4}\left(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}, P_1^{[0]}\right) = 0$ and the closedness conditions (3.17), we can solve $\mathcal{T}$ from (3.18) degree by degree. In this way, we can find a derivation X such that the equations in (3.10) hold true.

Step 5. To lift the symmetry $\frac{\partial v_\lambda}{\partial s_2}$ to a symmetry of the tau-cover of the deformed Principal Hierarchy, and to define all the other flows $\frac{\partial}{\partial s_m}$ of the Virasoro symmetries for $m \geq 0$. Note that the symmetry $\frac{\partial}{\partial s_{-1}}$ is constructed in [9]. We remark that we can also lift the symmetry (3.8) and (3.9) to a symmetry of the super tau-cover of the deformed Principal Hierarchy, but it is not necessary for the consideration of our problem.

In the remaining part of this subsection, we explain how the equations (3.8) and (3.9) are derived from the Eq. (1.1) of the main theorem. Let $\mathcal{Z}$ be a tau-function of the tau-cover (2.32) of the deformed Principal Hierarchy, i.e.,

$$f_{\alpha,p} = \frac{\partial \log \mathcal{Z}}{\partial t^{\alpha,p}}, \quad w^\alpha = \eta^{\alpha\beta} \frac{\partial^2 \log \mathcal{Z}}{\partial t^{1,0} \partial t^{\beta,0}} \tag{3.19}$$

give a solution of the tau-cover (2.32). Our goal is to find a symmetry of the following form

$$\frac{\partial \mathcal{Z}}{\partial s_2} = L_2^{even} \mathcal{Z} + O_2 \mathcal{Z}, \tag{3.20}$$

where L_2^{even} is the operator (3.4) and O_2 is a differential polynomial. If we assume that this is indeed a symmetry, then by using (3.19) we obtain the flow

$$\begin{aligned}
\frac{\partial f_{\lambda,k}}{\partial s_2} &= a^{\alpha\beta}\left(f_{\alpha,1}\Omega_{\beta,0;\lambda,k} + f_{\beta,0}\Omega_{\alpha,1;\lambda,k} + \frac{\partial \Omega_{\alpha,1;\beta,0}}{\partial t^{\lambda,k}}\right) \\
&\quad + b^{\alpha\beta}\left(f_{\alpha,0}\Omega_{\beta,0;\lambda,k} + f_{\beta,0}\Omega_{\alpha,0;\lambda,k} + \frac{\partial \Omega_{\alpha,0;\beta,0}}{\partial t^{\lambda,k}}\right) \\
&\quad + \sum_{q\geq 0}\left(b^{\beta,q}_{2;\lambda,k} f_{\beta,q} + c_{2;\lambda,k;\beta,q} t^{\beta,q} + c_{2;\beta,q;\lambda,k} t^{\beta,q}\right) + \frac{\partial O_2}{\partial t^{\lambda,k}} + \mathcal{L}_2^{even} f_{\lambda,k} \\
\frac{\partial w_\lambda}{\partial s_2} &= a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial w_\lambda}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial w_\lambda}{\partial t^{\beta,0}}\right) + b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial w_\lambda}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial w_\lambda}{\partial t^{\beta,0}}\right) \\
&\quad + W_\lambda + \mathcal{L}_2^{even} w_\lambda,
\end{aligned}$$

here W_λ are some differential polynomials. Now recall that v_λ and w_ζ are related by a Miura type transformation, hence by using the equation

$$\frac{\partial v_\lambda}{\partial s_2} = \sum_{s\geq 0} \frac{\partial v_\lambda}{\partial w_\zeta^{(s)}} \partial_x^s \frac{\partial w_\zeta}{\partial s_2}$$

we know that there exist differential polynomials $X^0_\lambda \in \hat{\mathcal{A}}^0$ such that

$$\frac{\partial v_\lambda}{\partial s_2} = a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial v_\lambda}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right) + b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial v_\lambda}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right) + X^0_\lambda + \mathcal{L}_2^{even} v_\lambda. \tag{3.21}$$

As we have discussed at the end of Sect. 3.2, we also need to write down the actions of the flow $\frac{\partial}{\partial s_2}$ on the odd variables $\sigma_{\lambda,0}$. To this end, we must replace $\mathcal{L}_2^{even}$ by $\mathcal{L}_2$ to include the odd time variables τ_m and odd flows $\frac{\partial}{\partial \tau_m}$. By using (2.36) it is easy to see that

$$\frac{\partial \boldsymbol{f_0}}{\partial \tau_0} = \boldsymbol{A}\boldsymbol{\sigma_0},$$

where $\boldsymbol{f_0} = (f_{1,0}, \dots, f_{n,0})^T$, $\boldsymbol{\sigma_0} = (\sigma_{1,0}, \dots, \sigma_{n,0})^T$ and $\boldsymbol{A}$ is a matrix of differential operator of the form

$$\boldsymbol{A} = \sum_{g\ge 0}\sum_{k=0}^{2g} \boldsymbol{A_{g,k}}\partial_x^k, \quad \boldsymbol{A_{g,k}} \in M_n(\hat{\mathcal{A}}^0_{2g-k}).$$

Note that $\boldsymbol{A_{0,0}}$ is the identity matrix, therefore $\boldsymbol{A}$ is invertible as a differential operator, i.e., there exists $\boldsymbol{B} = \sum_{g\ge 0}\sum_{k=0}^{2g} \boldsymbol{B_{g,k}}\partial_x^k$ such that $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$, and in particular $\boldsymbol{B_{0,0}}$ is the identity matrix. Thus we can represent the odd variables $\sigma_{\lambda,0}$ in the form

$$\sigma_{\lambda,0} = \frac{\partial f_{\lambda,0}}{\partial \tau_0} + \sum_{g\ge 1}\sum_{k=0}^{2g} \left(\boldsymbol{B_{g,k}}\right)^\zeta_\lambda \partial_x^k \frac{\partial f_{\zeta,0}}{\partial \tau_0}.$$

This identity leads us to the following definition of the evolutions of the odd variables $\sigma_{\lambda,0}$ along the flow $\frac{\partial}{\partial s_2}$:

$$\frac{\partial \sigma_{\lambda,0}}{\partial s_2} = \frac{\partial}{\partial \tau_0}\frac{\partial f_{\lambda,0}}{\partial s_2} + \sum_{g\ge 1}\sum_{k=0}^{2g} \frac{\partial}{\partial s_2}\left(\boldsymbol{B_{g,k}}\right)^\zeta_\lambda \partial_x^k \frac{\partial f_{\zeta,0}}{\partial \tau_0} + \sum_{g\ge 1}\sum_{k=0}^{2g} \left(\boldsymbol{B_{g,k}}\right)^\zeta_\lambda \partial_x^k \frac{\partial}{\partial \tau_0}\frac{\partial f_{\zeta,0}}{\partial s_2}.$$

By using Proposition 9 and Lemma 7, it follows from the explicit expressions (3.5), (3.6) and (3.7) that the odd variables $\Phi^m_{\alpha,p}$ appearing in $\frac{\partial}{\partial \tau_0}\frac{\partial f_{\lambda,0}}{\partial s_2}$ can be represented by elements of $\hat{\mathcal{A}}^+$. So there exist differential polynomials $M^\zeta_\lambda, N^\zeta_\lambda \in \hat{\mathcal{A}}^0$ and $X^1_\lambda \in \hat{\mathcal{A}}^1$ such that

$$\frac{\partial \sigma_{\lambda,0}}{\partial s_2} = a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right) + b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right) + X^1_\lambda + M^\zeta_\lambda \sigma_{\zeta,2} + N^\zeta_\lambda \sigma_{\zeta,1} + \mathcal{L}_2 \sigma_{\lambda,0}.$$

Finally we define a derivation $X \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^0$ such that $X v_\lambda = X^0_\lambda$, where X^0_λ is the differential polynomial introduced in (3.21), and $X\sigma_{\lambda,0} = X^1_\lambda$. Therefore our problem of finding such a derivation X is a necessary condition of the main theorem.

3.4. Example: one-dimensional Frobenius manifold. In this subsection we present an example to illustrate how the general framework described in the previous subsection works. We consider the one-dimensional Frobenius manifold M, it has the following potential and Euler vector field:

$$F = \frac{1}{6}v^3, \quad E = v\partial_v.$$

Due to the dimension reason, we will omit the Greek indices, for example, we will use $v^{(s)}$ and σ_m^s instead of $v^{1,s}$ and $\sigma_{1,m}^s$. The Principal Hierarchy associated with M is the Riemann hierarchy

$$\frac{\partial v}{\partial t_p} = \frac{v^p}{p!}v_x, \quad p \geq 0,$$

whose bihamiltonian structure is given by

$$P_0^{[0]} = \frac{1}{2}\int \sigma_0\sigma_0^1, \quad P_1^{[0]} = \frac{1}{2}\int v\sigma_0\sigma_0^1.$$

It is proved in [7,24] that every deformation (P_0, P_1) of $(P_0^{[0]}, P_1^{[0]})$ with a constant central invariant is equivalent to the bihamiltonian structure given by

$$P_0 = \frac{1}{2}\int \sigma_0\sigma_0^1, \quad P_1 = \frac{1}{2}\int v\sigma_0\sigma_0^1 + \varepsilon^2 c\sigma_0\sigma_0^3$$

via a certain Miura type transformation. Here the dispersion parameter ε is added for clearness, and the central invariant of (P_0, P_1) is $\frac{c}{3}$. In particular, when $c = \frac{1}{8}$ the corresponding deformed Riemann hierarchy is the KdV hierarchy that controls the 2D topological gravity [19,30].

We have the following flows for the super tau-cover of the deformed Riemann hierarchy:

$$\begin{aligned}
&\frac{\partial v}{\partial t_1} = vv_x + \frac{2}{3}\varepsilon^2 cv^{(3)}, \quad \frac{\partial \sigma_0}{\partial t_1} = v\sigma_0^1 + \frac{2}{3}\varepsilon^2 c\sigma_0^3;\\
&\frac{\partial v}{\partial t_2} = \frac{1}{2}v^2 v_x + \varepsilon^2 c\left(\frac{4}{3}v_x v_{xx} + \frac{2}{3}vv^{(3)}\right) + \frac{4}{15}\varepsilon^4 c^2 v^{(5)},\\
&\frac{\partial \sigma_0}{\partial t_2} = \frac{1}{2}v^2\sigma_0^1 + \varepsilon^2 c\left(\frac{2}{3}v_{xx}\sigma_0^1 + \frac{2}{3}v_x\sigma_0^2 + \frac{2}{3}v\sigma_0^3\right) + \frac{4}{15}\varepsilon^4 c^2\sigma_0^5;\\
&\varepsilon\frac{\partial f_0}{\partial \tau_0} = \sigma_0, \quad \varepsilon\frac{\partial f_1}{\partial \tau_0} = 2\sigma_1 - v\sigma_0 - \frac{4}{3}\varepsilon^2 c\sigma_0^2;\\
&\varepsilon\frac{\partial f_2}{\partial \tau_0} = \frac{4}{3}\sigma_2 - \frac{2}{3}v\sigma_1 - \frac{1}{6}v^2\sigma_0 - \varepsilon^2 c\left(\frac{2}{3}v_{xx}\sigma_0 + \frac{4}{3}v_x\sigma_0^2 + \frac{4}{3}v\sigma_0^2\right) - \frac{16}{15}\varepsilon^4 c^2\sigma_0^4.
\end{aligned}$$

Here the odd variables σ_m satisfy the recursion relation

$$\sigma_{m+1}^1 = v\sigma_m^1 + \frac{1}{2}v_x\sigma_m + \varepsilon^2 c\sigma_m^3, \quad m \geq 0.$$

We also have the following Hamiltonian densities for the deformed Riemann hierarchy:

$$h_1 = \frac{v^2}{2} + \frac{2}{3}\varepsilon^2 cv_{xx}, \quad h_2 = \frac{v^3}{6} + \varepsilon^2 c\left(\frac{1}{3}v_x^2 + \frac{2}{3}vv_{xx}\right) + \frac{4}{15}\varepsilon^4 c^2 v^{(4)}.$$

Note that

$$L_2 = \frac{3}{8}\varepsilon^2 \frac{\partial^2}{\partial t_1 \partial t_0} + \mathcal{L}_2, \quad \mathcal{L}_2 = \sum_{p\geq 0} \frac{\Gamma\left(\frac{7}{2}+p\right)}{\Gamma\left(\frac{1}{2}+p\right)} t_p \frac{\partial}{\partial t_{p+2}} + (p+c_0)\tau_p \frac{\partial}{\partial \tau_{p+2}}.$$

Then the equations (3.8) and (3.9) for this example have the form

$$\frac{\partial v}{\partial s_2} = \frac{3}{8}\varepsilon\left(v_x f_1 + \frac{\partial v}{\partial t_1} f_0\right) + Xv + \mathcal{L}_2 v,$$
$$\frac{\partial \sigma_0}{\partial s_2} = \frac{3}{8}\varepsilon\left(\sigma_0^1 f_1 + \frac{\partial \sigma_0}{\partial t_1} f_0\right) + \left(\frac{5}{2}+c_0\right)\sigma_2 - \frac{v}{2}\sigma_1 + X\sigma_0 + \mathcal{L}_2\sigma_0.$$

We are to find the derivation $X \in \mathrm{Der}^{\partial}(\hat{\mathcal{A}})^0$ such that the flow $\frac{\partial}{\partial s_2}$ commutes with $\frac{\partial}{\partial \tau_0}$ and $\frac{\partial}{\partial \tau_1}$. These conditions yield the following equations for X:

$$\left[\frac{\partial}{\partial \tau_0}, X\right] = I_0, \quad \left[\frac{\partial}{\partial \tau_1}, X\right] = I_1,$$

where the derivations I_0 and I_1 are given by

$$I_0 v = v v_x \sigma_0 + \frac{7}{2} v^2 \sigma_0^1 + \varepsilon^2 c\left(v^{(3)}\sigma_0 + \frac{13}{2} v_{xx}\sigma_0^1 + 8 v_x \sigma_0^2 + 6 v \sigma_0^3\right) + 3\varepsilon^4 c^2 \sigma_0^5, \tag{3.22}$$

$$I_0 \sigma_0 = v\sigma_0\sigma_0^1 - \varepsilon^2 c\left(\frac{1}{2}\sigma_0^1\sigma_0^2 - \sigma_0\sigma_0^3\right), \tag{3.23}$$

$$\begin{aligned} I_1 v = {} & \frac{5}{4} v^2 v_x \sigma_0 + \frac{5}{2} v^3 \sigma_0^1 + \varepsilon^2 c\left(\frac{7}{2} v_x v_{xx}\sigma_0 + 2 v v^{(3)}\sigma_0 + \frac{45}{4} v_x^2 \sigma_0^1 + \frac{31}{2} v v_{xx}\sigma_0^1\right) \\ & + \varepsilon^2 c\left(26 v v_x \sigma_0^2 + \frac{19}{2} v^2 \sigma_0^3\right) + \varepsilon^4 c^2\left(v^{(5)}\sigma_0 + \frac{17}{2} v^{(4)}\sigma_0^1 + \frac{45}{2} v^{(3)}\sigma_0^2\right) \\ & + \varepsilon^4 c^2\left(\frac{59}{2} v_{xx}\sigma_0^3 + \frac{43}{2} v_x \sigma_0^4 + 9 v \sigma_0^5\right) + 3\varepsilon^6 c^3 \sigma_0^7, \end{aligned} \tag{3.24}$$

$$\begin{aligned} I_1 \sigma_0 = {} & \frac{5}{4} v^2 \sigma_0 \sigma_0^1 + \varepsilon^2 c\left(\frac{5}{2} v_{xx}\sigma_0\sigma_0^1 + \frac{5}{2} v_x \sigma_0 \sigma_0^2 - \frac{1}{2} v \sigma_0^1 \sigma_0^2 + 2 v \sigma_0 \sigma_0^3\right) \\ & - \varepsilon^4 c^2\left(\frac{1}{2}\sigma_0^1\sigma_0^4 - \sigma_0\sigma_0^5\right). \end{aligned} \tag{3.25}$$

It follows from (3.22) and (3.23) that we can choose $X^\circ \in \mathrm{Der}^{\partial}(\hat{\mathcal{A}})$ such that $[\frac{\partial}{\partial \tau_0}, X^\circ] = I_0$, whose actions on v and σ_0 are given by

$$X^\circ v = v^3 + \varepsilon^2 c\left(\frac{5}{4} v_x^2 + 3 v v_{xx}\right) + \varepsilon^4 c^2 v^{(4)},$$
$$X^\circ \sigma_0 = -\frac{1}{2} v^2 \sigma_0 - \varepsilon^2 c\left(v_{xx}\sigma_0 + \frac{5}{2} v_x \sigma_0^1 + 3 v \sigma_0^2\right) - 2\varepsilon^4 c^2 \sigma_0^4.$$

Then according to the general discussions given in the previous subsection, the derivation $\mathcal{C} = X - X^\circ$ satisfies the equations

$$\left[\frac{\partial}{\partial\tau_0}, \mathcal{C}\right] = 0, \quad \left[\frac{\partial}{\partial\tau_1}, \mathcal{C}\right] = I_1 - \left[\frac{\partial}{\partial\tau_1}, X^\circ\right].$$

Finally we can solve the above equations and obtain the unique derivation $\mathcal{C}$ that is defined by

$$\mathcal{C}v = \varepsilon^2 c\left(3v_x^2 + 3vv_{xx}\right) + 2\varepsilon^4 c^2 v^{(4)},$$
$$\mathcal{C}\sigma_0 = \varepsilon^2 c\left(3v_x\sigma_0^1 + v\sigma_0^2\right) + 2\varepsilon^4 c^2\sigma_0^4.$$

By forgetting all the odd variables we obtain the following symmetry for the deformed Riemann hierarchy:

$$\frac{\partial v}{\partial s_2} = \frac{3}{8}\varepsilon\left(v_x f_1 + \frac{\partial v}{\partial t_1} f_0\right) + v^3 + \varepsilon^2 c\left(\frac{17}{4}v_x^2 + 6vv_{xx}\right) + 3\varepsilon^4 c^2 v^{(4)} + \mathcal{L}_2^{even} v.$$

It is easy to check that the action of this symmetry on the tau function $\mathcal{Z}$ of the deformed Riemann hierarchy can be represented by

$$\frac{\partial \mathcal{Z}}{\partial s_2} = L_2^{even}\mathcal{Z} + \left(3c - \frac{3}{8}\right)\left(\frac{v^2}{2} + \frac{2}{3}\varepsilon^2 c v_{xx}\right)\mathcal{Z}. \tag{3.26}$$

In particular, when $c = \frac{1}{8}$, this symmetry is given by a linear action on $\mathcal{Z}$.

4. Deformation of Virasoro Symmetries: Existence and Uniqueness

In this section, we present details of the proof of the main theorem following the framework described in Sect. 3.3.

4.1. Locality conditions. We start by verifying the locality condition (3.14).

Let us first find the differential polynomials M_λ^ζ and N_λ^ζ in (3.9) to ensure that I_0 is local. By using the relation (2.36) and the equations (3.8) and (3.9), we arrive at

$$\left[\frac{\partial}{\partial\tau_0}, \frac{\partial}{\partial s_2}\right] v_\lambda = c_0\sigma_{\lambda,2}^1 + a^{\alpha\beta}\frac{\partial f_{\alpha,1}}{\partial\tau_0}\frac{\partial v_\lambda}{\partial t^{\beta,0}} - \partial_x\left(M_\lambda^\zeta\sigma_{\zeta,2} + N_\lambda^\zeta\sigma_{\zeta,1}\right) + loc. \tag{4.1}$$

Here and henceforth we will use *loc* to denote the local terms, i.e., terms belonging to $\hat{\mathcal{A}}$. We need to find M_λ^ζ and N_λ^ζ such that the right hand side of (4.1) is local.

Lemma 9. *The following identities hold true:*

$$\frac{\partial}{\partial\sigma_{\beta,0}}\frac{\delta P_1}{\delta\sigma_{\lambda,0}} = \eta^{\alpha\beta}\left(\frac{1}{2} + \mu_\alpha\right)\frac{\partial v^\lambda}{\partial t^{\alpha,0}}, \quad \frac{\partial}{\partial\sigma_{\beta,0}}\frac{\delta P_1}{\delta v^\lambda} = \eta^{\alpha\beta}\left(\frac{1}{2} + \mu_\alpha\right)\frac{\partial\sigma_{\lambda,0}}{\partial t^{\alpha,0}}.$$

Proof. By using the identity (2.29) we obtain the identity

$$\left[P_1, H_{\alpha,-1}\right] = \left(\frac{1}{2} + \mu_\alpha\right)\left[P_0, H_{\alpha,0}\right].$$

By taking the variational derivatives of both sides of the above identity and by using (2.4), (2.5) and (2.31), we arrive at the result of the lemma. □

Lemma 10. *We have the following relation:*

$$\sigma^1_{\lambda,2} = \left(\frac{1}{2} + \mu_\varepsilon\right)\eta^{\zeta\varepsilon}\frac{\partial v_\lambda}{\partial t^{\varepsilon,0}}\sigma_{\zeta,1} + loc. \tag{4.2}$$

Proof. Let us denote by $\mathcal{P}_1$ the Hamiltonian operator of P_1 and represent it in the form

$$\mathcal{P}_1 = \sum_{k\ge 0}\mathcal{P}_{1,k}\partial_x^k, \quad \mathcal{P}_{1,k} \in M_n(\hat{\mathcal{A}}).$$

Then from the recursion relation

$$\eta^{\gamma\lambda}\sigma^1_{\lambda,2} = \mathcal{P}_1^{\gamma\lambda}\sigma_{\lambda,1} = \sum_{k\ge 0}\mathcal{P}^{\gamma\lambda}_{1,k}\sigma^k_{\lambda,1},$$

it is easy to see that

$$\sigma^1_{\lambda,2} = \eta_{\lambda\gamma}\mathcal{P}^{\gamma\zeta}_{1,0}\sigma_{\zeta,1} + loc.$$

Now it follows from the definition of the Hamiltonian operator that

$$\mathcal{P}^{\gamma\zeta}_{1,0} = \frac{\partial}{\partial\sigma_{\zeta,0}}\frac{\delta P_1}{\delta\sigma_{\gamma,0}}.$$

Therefore by using Lemma 9 we arrive at the identity (4.2) and the lemma is proved. □

Proposition 10. *There exist unique differential polynomials M^ζ_λ and N^ζ_λ such that* (4.1) *is local. More explicitly, we have*

$$M^\zeta_\lambda = \left(\frac{5}{2} + \mu_\lambda + c_0\right)\delta^\zeta_\lambda, \quad \partial_x N^\zeta_\lambda = \left(\frac{1}{2} + \mu_\alpha\right)\eta^{\alpha\zeta}\left(\mu_\zeta - \mu_\lambda - 1\right)\frac{\partial v_\lambda}{\partial t^{\alpha,0}},$$

here c_0 is the arbitrary constant appearing in the operator $\mathcal{L}_2$.

Proof. To ensure the vanishing of the coefficients of $\sigma_{\zeta,2}$ in the right hand side of (4.1), M^ζ_λ should be a constant, so it is determined by the leading terms of the right hand sides of (3.8) and (3.9), which are fixed by the Virasoro symmetry $\frac{\partial}{\partial s_2}$ of the super tau-cover of the Principal Hierarchy. Hence we obtain

$$M^\zeta_\lambda = \left(\frac{5}{2} + \mu_\lambda + c_0\right)\delta^\zeta_\lambda.$$

By using Proposition 9 we have

$$\left(\frac{1}{2} + \mu_\alpha\right)\frac{\partial f_{\alpha,1}}{\partial\tau_0} = \sigma_{\alpha,1} + loc. \tag{4.3}$$

Then by using this equation, the recursion relations (2.18) and the identity (4.2), we see that the vanishing of the coefficients of $\sigma_{\zeta,1}$ in the right hand side of (4.1) gives the equation

$$\partial_x N_\lambda^\zeta = \left(\frac{1}{2} + \mu_\alpha\right) \eta^{\alpha\zeta} \left(\mu_\zeta - \mu_\lambda - 1\right) \frac{\partial v_\lambda}{\partial t^{\alpha,0}}. \tag{4.4}$$

Both sides of (4.4) are total x-derivatives, so we can integrate (4.4) to obtain N_λ^ζ upto a constant, which is uniquely determined from the Virasoro symmetry of the super tau-cover of the Principal Hierarchy. The proposition is proved. □

Now by a direct computation of the condition

$$\left[\frac{\partial}{\partial \tau_0}, \frac{\partial}{\partial s_2}\right] = 0,$$

we obtain the following explicit expressions for I_0, a derivation which is defined by (3.13):

$$\begin{aligned} I_0 v_\lambda = {} & a^{\alpha\beta}\left(f'_{\beta,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,1}} + f'_{\alpha,1}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right) - a^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial \tau_0}\frac{\partial v_\lambda}{\partial t^{\alpha,1}} + \frac{\partial f_{\alpha,1}}{\partial \tau_0}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right) \\ & + b^{\alpha\beta}\left(f'_{\beta,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,0}} + f'_{\alpha,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right) - b^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial \tau_0}\frac{\partial v_\lambda}{\partial t^{\alpha,0}} + \frac{\partial f_{\alpha,0}}{\partial \tau_0}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right) \\ & + \left(\frac{5}{2} + \mu_\lambda\right)\sigma^1_{\lambda,2} + W\sigma_{\lambda,0} + \partial_x\left(N_\lambda^\zeta \sigma_{\zeta,1}\right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} I_0 \sigma_{\lambda,0} = {} & -a^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial \tau_0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,1}} + \frac{\partial f_{\alpha,1}}{\partial \tau_0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right) - b^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial \tau_0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,0}} + \frac{\partial f_{\alpha,0}}{\partial \tau_0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right) \\ & + \left(\frac{5}{2} + \mu_\lambda\right)\frac{\partial \sigma_{\lambda,0}}{\partial \tau_2} - \frac{\partial}{\partial \tau_0}\left(N_\lambda^\zeta \sigma_{\zeta,1}\right), \end{aligned} \tag{4.6}$$

here and henceforth we will use $f'_{\alpha,p}$ to denote the differential polynomial $\frac{\partial f_{\alpha,p}}{\partial t^{1,0}}$ and use W to denote the coefficient of $t^{1,0}$ of the operator $\mathcal{L}_2$. More explicitly, we have

$$\begin{aligned} W = {} & \left(\frac{1}{2} + \mu_1\right)\left(\frac{3}{2} + \mu_1\right)\left(\frac{5}{2} + \mu_1\right)\frac{\partial}{\partial t^{1,2}} \\ & + \sum_{k=1}^{2}\left(3\left(\frac{1}{2} + \mu_1\right)^2 + 6\left(\frac{1}{2} + \mu_1\right) + 2\right)(R_k)_1^\beta \frac{\partial}{\partial t^{\beta,2-k}} \\ & + \left(\frac{9}{2} + 3\mu_1\right)\left(R_{2,2}\right)_1^\beta \frac{\partial}{\partial t^{\beta,0}}. \end{aligned} \tag{4.7}$$

Proposition 11. *The derivation I_0 given by* (4.5)*,* (4.6) *is local.*

Proof. The locality of (4.5) follows from the definition of N_λ^ζ, so we only need to check the locality of (4.6). By using the definition of the odd flows we know that

$$\frac{\partial \sigma_{\lambda,0}}{\partial \tau_2} = T_{0,2}\frac{\partial \sigma_{\lambda,0}}{\partial \tau_1} = \sum_{s\geq 0}\sigma^s_{\beta,1}\frac{\partial}{\partial \sigma^s_{\beta,0}}\frac{\partial \sigma_{\lambda,0}}{\partial \tau_1},$$

therefore it follows from Lemma 9 that

$$\frac{\partial \sigma_{\lambda,0}}{\partial \tau_2} = \sigma_{\beta,1}\eta^{\alpha\beta}\left(\frac{1}{2}+\mu_\alpha\right)\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,0}} + loc.$$

Then we arrive at the locality of (4.6) by using (4.3) and the following obvious fact:

$$\frac{\partial N_\lambda^\zeta}{\partial \tau_0} = \left(\frac{1}{2}+\mu_\alpha\right)\eta^{\alpha\zeta}\left(\mu_\zeta - \mu_\lambda - 1\right)\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,0}}.$$

The proposition is proved. □

Let us proceed to prove the locality of I_1. Similar to the expression (4.5) and (4.6), we can write down the explicit expression for I_1, which can be found in the next subsection. However the locality of I_1 can not be derived from this expression directly, so we turn to prove the following equivalent conditions:

$$\left[\frac{\partial}{\partial \tau_1}, \frac{\partial}{\partial s_2}\right] v_\lambda \in \hat{\mathcal{A}}, \quad \left[\frac{\partial}{\partial \tau_1}, \frac{\partial}{\partial s_2}\right] \sigma_{\lambda,0} \in \hat{\mathcal{A}}. \tag{4.8}$$

By a direct computation we have

$$\begin{aligned}
&\left[\frac{\partial}{\partial \tau_1}, \frac{\partial}{\partial s_2}\right] v_\lambda \\
&= a^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial \tau_1}\frac{\partial v_\lambda}{\partial t^{\alpha,1}} + \frac{\partial f_{\alpha,1}}{\partial \tau_1}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right) \\
&\quad + b^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial \tau_1}\frac{\partial v_\lambda}{\partial t^{\alpha,0}} + \frac{\partial f_{\alpha,0}}{\partial \tau_1}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right) + (1+c_0)\frac{\partial v_\lambda}{\partial \tau_3} \\
&\quad - \sum_{s\ge 0}\left(\left(\frac{5}{2}+c_0+\mu_\zeta\right)\sigma_{\zeta,2}^s + \partial_x^s\left(N_\zeta^\varepsilon \sigma_{\zeta,1}\right)\right)\frac{\partial}{\partial \sigma_{\zeta,0}^s}\frac{\partial v_\lambda}{\partial \tau_1} + loc.
\end{aligned} \tag{4.9}$$

Lemma 11. *There exists a unique differential polynomial Z_α^β, for each pair of indices $1 \le \alpha, \beta \le n$, such that*

$$\left(\frac{1}{2}+\mu_\alpha\right)\frac{\partial f_{\alpha,1}}{\partial \tau_1} = \sigma_{\alpha,2} + Z_\alpha^\beta \sigma_{\beta,1} + loc,$$

where Z_α^β satisfies the following equation

$$\partial_x Z_\alpha^\beta = -\left(\frac{1}{2}+\mu_\varepsilon\right)\eta^{\beta\varepsilon}\frac{\partial v_\alpha}{\partial t^{\varepsilon,0}}. \tag{4.10}$$

Proof. The existence and uniqueness of Z_α^β can be obtained from Proposition 9, hence we only need to derive (4.10), which can be obtained by using the identity (4.2) and the fact that $\partial_x\left(\sigma_{\alpha,2} + Z_\alpha^\beta \sigma_{\beta,1}\right)$ is local. The lemma is proved. □

From the definition of the odd flows and the recursion relation (2.18) we have the equation,

$$\frac{\partial v_\lambda}{\partial \tau_3} = \sum_{s\geq 0} \sigma^s_{\zeta,2} \frac{\partial}{\partial \sigma^s_{\zeta,0}} \frac{\partial v_\lambda}{\partial \tau_1},$$

which, together with Lemma 11 and the identity (4.2), enables us to rewrite (4.9) in the form

$$\left[\frac{\partial}{\partial \tau_1}, \frac{\partial}{\partial s_2}\right] v_\lambda = U^\zeta_\lambda \sigma_{\zeta,2} + V^\zeta_\lambda \sigma_{\zeta,1} + loc,$$

where the differential polynomials U^ζ_λ and V^ζ_λ are given by

$$U^\zeta_\lambda = \eta^{\zeta\beta}\left(\frac{1}{2}+\mu_\beta\right)\left(\frac{3}{2}+\mu_\zeta\right)\frac{\partial v_\lambda}{\partial t^{\beta,0}} - \left(\frac{3}{2}+\mu_\zeta\right)\frac{\partial}{\partial \sigma_{\zeta,0}}\frac{\partial v_\lambda}{\partial \tau_1},$$
$$V^\zeta_\lambda = a^{\alpha\zeta}\frac{\partial v_\lambda}{\partial t^{\alpha,1}} + (b^{\alpha\zeta}+b^{\zeta\alpha})\frac{\partial v_\lambda}{\partial t^{\alpha,0}} - \sum_{s\geq 0}\partial^s_x\left(N^\zeta_\gamma - \left(\frac{3}{2}+\mu_\gamma\right)Z^\zeta_\gamma\right)\frac{\partial}{\partial \sigma^s_{\gamma,0}}\frac{\partial v_\lambda}{\partial \tau_1}.$$

Lemma 12. *We have* $U^\zeta_\lambda = V^\zeta_\lambda = 0$.

Proof. The vanishing of U^ζ_λ follows directly from Lemma 9. To prove the vanishing of V^ζ_λ , let us consider the functional

$$F^\zeta = \eta^{\alpha\zeta}\left(\frac{1}{2}+\mu_\alpha\right)\left(\frac{1}{2}+\mu_\zeta\right)\int h_{\alpha,1}.$$

By using (4.4) and (4.10) we can check that there exists a constant C^ζ_λ such that

$$N^\zeta_\lambda - \left(\frac{3}{2}+\mu_\gamma\right)Z^\zeta_\lambda = \frac{\delta F^\zeta}{\delta v_\lambda} + C^\zeta_\lambda.$$

Thus from (2.5) it follows that

$$\sum_{s\geq 0}\partial^s_x\left(N^\zeta_\gamma - \left(\frac{3}{2}+\mu_\gamma\right)Z^\zeta_\gamma\right)\frac{\partial}{\partial \sigma^s_{\gamma,0}}\frac{\partial v_\lambda}{\partial \tau_1} = -\frac{\delta}{\delta \sigma_{\lambda,0}}\left[F^\zeta, P_1\right] + C^\zeta_\gamma\frac{\partial}{\partial \sigma_{\gamma,0}}\frac{\partial v_\lambda}{\partial \tau_1}.$$

By using the bihamiltonian recursion relation (2.29) and Lemma 9 we see that the right hand side of the above equation is a linear combination of the flows $\frac{\partial}{\partial t^{\alpha,p}}$ acting on v_λ, hence so is V^ζ_λ. On the other hand, the Virasoro symmetry $\frac{\partial}{\partial s_2}$ of the super tau-cover of the Principal Hierarchy implies that $V^\zeta_\lambda \in \hat{\mathcal{A}}^0_{\geq 2}$, so by using the theory of bihamiltonian cohomology [7], we know that V^ζ_λ must vanish. The lemma is proved. □

We have verified the first relation in (4.8), now let us proceed to prove the second one. We have

$$\begin{aligned}\left[\frac{\partial}{\partial\tau_1},\frac{\partial}{\partial s_2}\right]\sigma_{\lambda,0} = {}& a^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial\tau_1}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\alpha,1}}+\frac{\partial f_{\alpha,1}}{\partial\tau_1}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\beta,0}}\right)\\ &+b^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial\tau_1}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\alpha,0}}+\frac{\partial f_{\alpha,0}}{\partial\tau_1}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\beta,0}}\right)\\ &+\left(\frac{5}{2}+c_0+\mu_\lambda\right)\frac{\partial\sigma_{\lambda,2}}{\partial\tau_1}+\frac{\partial N^\zeta_\lambda}{\partial\tau_1}\sigma_{\zeta,1}+(1+c_0)\frac{\partial\sigma_{\lambda,0}}{\partial\tau_3}\\ &-\sum_{s\ge 0}\left(\left(\frac{5}{2}+c_0+\mu_\zeta\right)\sigma^s_{\zeta,2}+\partial^s_x\left(N^\varepsilon_\zeta\sigma_{\zeta,1}\right)\right)\frac{\partial}{\partial\sigma^s_{\zeta,0}}\frac{\partial\sigma_{\lambda,0}}{\partial\tau_1}+loc.\end{aligned}$$

By using the equations

$$\begin{aligned}&\frac{\partial\sigma_{\lambda,0}}{\partial\tau_3}=\sum_{s\ge0}\sigma^s_{\gamma,2}\frac{\partial}{\partial\sigma^s_{\gamma,0}}\frac{\partial\sigma_{\lambda,0}}{\partial\tau_1}+\frac{\partial\sigma_{\lambda,1}}{\partial\tau_2},\quad \frac{\partial\sigma_{\lambda,1}}{\partial t^{\alpha,p}}=\frac{\partial}{\partial\tau_1}\frac{\delta h_{\alpha,p+1}}{\delta v^\lambda},\\ &\frac{\partial\sigma_{\lambda,1}}{\partial\tau_2}=\sigma_{\zeta,1}\left(\frac{1}{2}+\mu_\varepsilon\right)\eta^{\varepsilon\zeta}\frac{\partial}{\partial\tau_1}\frac{\delta h_{\varepsilon,1}}{\delta v^\lambda}+loc,\end{aligned}$$

we obtain that

$$\left[\frac{\partial}{\partial\tau_1},\frac{\partial}{\partial s_2}\right]\sigma_{\lambda,0}=\sigma_{\zeta,2}\tilde U^\zeta_\lambda+\sigma_{\zeta,1}\tilde V^\zeta_\lambda+loc,$$

where the differential polynomials $\tilde U^\zeta_\lambda$, $\tilde V^\zeta_\lambda$ are given by

$$\begin{aligned}\tilde U^\zeta_\lambda &= \eta^{\zeta\beta}\left(\frac{1}{2}+\mu_\beta\right)\left(\frac{3}{2}+\mu_\zeta\right)\frac{\partial\sigma_{\lambda,0}}{\partial t^{\beta,0}}-\left(\frac{3}{2}+\mu_\zeta\right)\frac{\partial}{\partial\sigma_{\zeta,0}}\frac{\partial\sigma_{\lambda,0}}{\partial\tau_1},\\ \tilde V^\zeta_\lambda &= a_{\alpha\zeta}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\alpha,1}}+(b_{\alpha\zeta}+b_{\zeta\alpha})\frac{\partial\sigma_{\lambda,0}}{\partial t^{\alpha,0}}-\frac{\partial}{\partial\tau_1}\left(N^\zeta_\lambda-\left(\frac{3}{2}+\mu_\gamma\right)Z^\zeta_\lambda\right)\\ &\quad-\sum_{s\ge0}\partial^s_x\left(N^\zeta_\gamma-\left(\frac{3}{2}+\mu_\gamma\right)Z^\zeta_\gamma\right)\frac{\partial}{\partial\sigma^s_{\gamma,0}}\frac{\partial\sigma_{\lambda,0}}{\partial\tau_1}.\end{aligned}$$

These differential polynomials actually vanish, the reason is similar to the one for the vanishing of U^ζ_λ and V^ζ_λ given in the proof of Lemma 12. Hence the locality condition (3.14) is verified.

Finally let us consider the locality condition (3.12). The first condition

$$\left[\frac{\partial}{\partial t^{\delta,j}},\frac{\partial}{\partial s_2}\right]v_\lambda\in\hat{\mathcal{A}}$$

follows from the definition (3.8). To verify the second locality condition, we consider the equation

$$\begin{aligned}&\left[\frac{\partial}{\partial t^{\delta,j}},\frac{\partial}{\partial s_2}\right]\sigma_{\lambda,0}\\&=\left(\frac{5}{2}+c_0+\mu_\lambda\right)\frac{\partial\sigma_{\lambda,2}}{\partial t^{\delta,j}}+\frac{\partial N^\zeta_\lambda}{\partial t^{\delta,j}}\sigma_{\zeta,1}\\&\quad-\sum_{s\ge 0}\partial_x^s\left(N^\zeta_\gamma\sigma_{\zeta,1}+\left(\frac{5}{2}+c_0+\mu_\gamma\right)\sigma_{\gamma,2}\right)\frac{\partial}{\partial\sigma^s_{\gamma,0}}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\delta,j}}+loc.\end{aligned}\tag{4.11}$$

Proposition 12. *The right hand side of the Eq. (4.11) is local.*

Proof. By using the identity

$$\frac{\partial}{\partial\sigma_{\beta,0}}\frac{\delta}{\delta v^\alpha}=\frac{\delta}{\delta v^\alpha}\frac{\delta}{\delta\sigma_{\beta,0}}$$

that is proved in [25], we obtain

$$\frac{\partial}{\partial\sigma_{\beta,0}}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\alpha,p}}=\frac{\partial}{\partial\sigma_{\beta,0}}\frac{\delta X_{\alpha,p}}{\delta v^\lambda}=0,$$

from which it follows that the flows $\frac{\partial\sigma_{\lambda,2}}{\partial t^{\delta,j}}$ can be written as

$$\frac{\partial\sigma_{\lambda,2}}{\partial t^{\delta,j}}=T_2\frac{\partial\sigma_{\lambda,0}}{\partial t^{\delta,j}}=\sum_{s\ge 0}\sigma^s_{\beta,2}\frac{\partial}{\partial\sigma^s_{\beta,0}}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\delta,j}}=B^\zeta_\lambda\sigma_{\zeta,1}+loc,$$

where B^ζ_λ are certain differential polynomials. From the identity (4.2) we know that

$$B^\zeta_\lambda=-\frac{\partial Z^\zeta_\lambda}{\partial t^{\delta,j}},$$

and we can represent the right hand side of (4.11) in the form

$$V^\zeta_{\delta,j;\lambda}\sigma_{\zeta,1}+loc,$$

where $V^\zeta_{\delta,j;\lambda}$ is a differential polynomial given as follows:

$$\begin{aligned}V^\zeta_{\delta,j;\lambda}=&\frac{\partial}{\partial t^{\delta,j}}\left(N^\zeta_\lambda-\left(\frac{5}{2}+c_0+\mu_\gamma\right)Z^\zeta_\lambda\right)\\&-\sum_{s\ge 0}\partial_x^s\left(N^\zeta_\gamma-\left(\frac{5}{2}+c_0+\mu_\gamma\right)Z^\zeta_\gamma\right)\frac{\partial}{\partial\sigma^s_{\gamma,0}}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\delta,j}}.\end{aligned}$$

Define the following functional

$$F^\zeta=\eta^{\alpha\zeta}\left(\frac{1}{2}+\mu_\alpha\right)\left(\frac{3}{2}+c_0+\mu_\zeta\right)\int h_{\alpha,1},$$

then by applying (2.4) we obtain the following identity:

$$V^\zeta_{\delta,j;\lambda}=\frac{\delta}{\delta v^\lambda}\left[F^\zeta,X_{\delta,j}\right],$$

here $X_{\delta,j}$ is the vector field defined in (2.26). Since the functional F^ζ is a conserved quantity of the deformed Principal hierarchy, we have $\left[F^\zeta,X_{\delta,j}\right]=0$. The proposition is proved. □

4.2. Closedness conditions. In this subsection we prove the closedness condition (3.15) and (3.17). The verification of (3.15) is straightforward by using the explicit expressions (4.5) and (4.6), so we omit the details here.

Let us prove the closedness condition (3.17). We fix a choice of $X^\circ \in \mathrm{Der}^\partial(\hat{\mathcal{A}})$ such that X° satisfies the condition

$$\left[\frac{\partial}{\partial \tau_0}, X^\circ\right] = I_0,$$

and that the differential degree zero part of X° is given by the Virasoro symmetry $\frac{\partial}{\partial s_2}$ of the super tau-cover of the Principal Hierarchy.

We first write down the explicit expression for I_1 defined in (3.13). Define a derivation $\hat{I}_1 \in \mathrm{Der}(\hat{\mathcal{A}})^0$ by the formulae $\hat{I}_1 v_\gamma = \hat{I}_1 \sigma_{\gamma,0} = 0$ and

$$\begin{aligned}
\hat{I}_1 v_\gamma^{(n)} &= n\partial_x^{n-1} W(v_\gamma) + a^{\alpha\beta}\partial_x^n \left(f_{\beta,0}\frac{\partial v_\gamma}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial v_\gamma}{\partial t^{\beta,0}} \right) \\
&\quad + b^{\alpha\beta}\partial_x^n \left(f_{\beta,0}\frac{\partial v_\gamma}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial v_\gamma}{\partial t^{\beta,0}} \right) \\
&\quad - a^{\alpha\beta} \left(f_{\beta,0}\partial_x^n\frac{\partial v_\gamma}{\partial t^{\alpha,1}} + f_{\alpha,1}\partial_x^n\frac{\partial v_\gamma}{\partial t^{\beta,0}} \right) \\
&\quad - b^{\alpha\beta} \left(f_{\beta,0}\partial_x^n\frac{\partial v_\gamma}{\partial t^{\alpha,0}} + f_{\alpha,0}\partial_x^n\frac{\partial v_\gamma}{\partial t^{\beta,0}} \right), \\
\hat{I}_1 \sigma_{\gamma,0}^n &= n\partial_x^{n-1} W(\sigma_{\gamma,0}) + a^{\alpha\beta}\partial_x^n \left(f_{\beta,0}\frac{\partial \sigma_{\gamma,0}}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial \sigma_{\gamma,0}}{\partial t^{\beta,0}} \right) \\
&\quad + b^{\alpha\beta}\partial_x^n \left(f_{\beta,0}\frac{\partial \sigma_{\gamma,0}}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial \sigma_{\gamma,0}}{\partial t^{\beta,0}} \right) \\
&\quad - a^{\alpha\beta} \left(f_{\beta,0}\partial_x^n\frac{\partial \sigma_{\gamma,0}}{\partial t^{\alpha,1}} + f_{\alpha,1}\partial_x^n\frac{\partial \sigma_{\gamma,0}}{\partial t^{\beta,0}} \right) \\
&\quad - b^{\alpha\beta} \left(f_{\beta,0}\partial_x^n\frac{\partial \sigma_{\gamma,0}}{\partial t^{\alpha,0}} + f_{\alpha,0}\partial_x^n\frac{\partial \sigma_{\gamma,0}}{\partial t^{\beta,0}} \right) \\
&\quad + \left(\frac{3}{2} + \mu_\gamma\right)\left(\sigma_{\gamma,2}^n + \partial_x^n Z_\gamma^\varepsilon \sigma_{\varepsilon,1}\right) + \partial_x^n\left(N_\gamma^\varepsilon \sigma_{\varepsilon,1}\right) - \partial_x^n N_\gamma^\varepsilon \sigma_{\varepsilon,1},
\end{aligned}$$

here $n \geq 1$ and W is the derivation defined in (4.7). Note that this derivation does NOT commute with ∂_x. It is easy to see that $\hat{I}_1$ is indeed local. By a careful computation, we obtain the following expression for I_1:

$$\begin{aligned}
I_1 v_\lambda &= \hat{I}_1\left(\frac{\partial v_\lambda}{\partial \tau_1}\right) + A^\alpha \frac{\partial v_\lambda}{\partial t^{\alpha,1}} + B^\alpha \frac{\partial v_\lambda}{\partial t^{\alpha,0}}, \\
I_1 \sigma_{\lambda,0} &= \hat{I}_1\left(\frac{\partial \sigma_{\lambda,0}}{\partial \tau_1}\right) + A^\alpha \frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,1}} + B^\alpha \frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,0}} \\
&\quad + \left(\frac{3}{2} + \mu_\lambda\right)\left(\frac{\partial \sigma_{\lambda,1}}{\partial \tau_2} - \sigma_{\beta,1}\eta^{\alpha\beta}\left(\frac{1}{2} + \mu_\alpha\right)\frac{\partial \sigma_{\lambda,1}}{\partial t^{\alpha,0}}\right),
\end{aligned}$$

where A^α and B^α are local differential polynomials given by

$$\begin{aligned} A^\alpha &= -a^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial \tau_1} - \sigma_{\beta,1}\right), \\ B^\alpha &= -\eta^{\alpha\beta}\left(\frac{1}{2}+\mu_\alpha\right)\left(\frac{3}{2}+\mu_\beta\right)\left(\left(\frac{1}{2}+\mu_\beta\right)\frac{\partial f_{\beta,1}}{\partial \tau_1} - \sigma_{\beta,2} - Z^\varepsilon_\beta \sigma_{\varepsilon,1}\right) \\ &\quad - (b^{\alpha\beta}+b^{\beta\alpha})\left(\frac{\partial f_{\beta,0}}{\partial \tau_1} - \sigma_{\beta,1}\right). \end{aligned}$$

We start by proving the first identity given in (3.17), which can also be written as

$$\left[\frac{\partial}{\partial \tau_0}, I_1\right] + \left[\frac{\partial}{\partial \tau_1}, \left[\frac{\partial}{\partial \tau_0}, X^\circ\right]\right] = 0.$$

The following lemmas can be proved by a lengthy but straightforward computation.

Lemma 13. *Let $Q \in \hat{\mathcal{A}}$ be any local differential polynomial, then we have*

$$\begin{aligned} \left[\hat{I}_1, \partial_x\right] Q = \; & W(Q) + a_{\alpha\beta}\left(f'_{\beta,0}\frac{\partial Q}{\partial t^{\alpha,1}} + f'_{\alpha,1}\frac{\partial Q}{\partial t^{\beta,0}}\right) + b_{\alpha\beta}\left(f'_{\beta,0}\frac{\partial Q}{\partial t^{\alpha,0}} + f'_{\alpha,0}\frac{\partial Q}{\partial t^{\beta,0}}\right) \\ & + \sum_{n\geq 0} \sigma^1_{\zeta,1}\partial_x^n\left(Y^\zeta_\gamma - \left(\frac{3}{2}+\mu_\gamma\right)Z^\zeta_\gamma\right)\frac{\partial}{\partial \sigma^n_{\gamma,0}} Q \\ & + \left(\frac{3}{2}+\mu_\gamma\right)\partial_x\left(\sigma_{\gamma,2} + Z^\varepsilon_\gamma \sigma_{\varepsilon,1}\right)\frac{\partial}{\partial \sigma_{\gamma,0}} Q. \end{aligned}$$

Lemma 14. *The following identities hold true:*

$$\begin{aligned} \left[\frac{\partial}{\partial \tau_0}, I_1\right] v_\lambda &= \left[\frac{\partial}{\partial \tau_1}, \left[\frac{\partial}{\partial \tau_0}, \hat{I}_1\right]\right] v_\lambda + \frac{\partial A^\alpha}{\partial \tau_0}\frac{\partial v_\lambda}{\partial t^{\alpha,1}} + \frac{\partial B^\alpha}{\partial \tau_0}\frac{\partial v_\lambda}{\partial t^{\alpha,0}}, \\ \left[\frac{\partial}{\partial \tau_0}, I_1\right] \sigma_{\lambda,0} &= \left[\frac{\partial}{\partial \tau_1}, \left[\frac{\partial}{\partial \tau_0}, \hat{I}_1\right]\right] \sigma_{\lambda,0} + \frac{\partial A^\alpha}{\partial \tau_0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,1}} + \frac{\partial B^\alpha}{\partial \tau_0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,0}} \\ &\quad + \left(\frac{3}{2}+\mu_\lambda\right)\left(\frac{\partial \sigma_{\lambda,1}}{\partial \tau_2} - \sigma_{\beta,1}\eta^{\alpha\beta}\left(\frac{1}{2}+\mu_\alpha\right)\frac{\partial \sigma_{\lambda,1}}{\partial t^{\alpha,0}}\right). \end{aligned}$$

Lemma 15. *We have the following decomposition:*

$$\left[\frac{\partial}{\partial \tau_0}, \hat{I}_1 + X^\circ\right] = D + \frac{\partial}{\partial \tau_2},$$

where D is a derivation $\hat{\mathcal{A}} \to \hat{\mathcal{A}}^+$ whose actions are given by the formulae

$$\begin{aligned} D v^{(n)}_\gamma = \; & -a^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial \tau_0}\partial_x^n\frac{\partial v_\gamma}{\partial t^{\alpha,1}} + \frac{\partial f_{\alpha,1}}{\partial \tau_0}\partial_x^n\frac{\partial v_\gamma}{\partial t^{\beta,0}}\right) \\ & - b^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial \tau_0}\partial_x^n\frac{\partial v_\lambda}{\partial t^{\alpha,0}} + \frac{\partial f_{\alpha,0}}{\partial \tau_0}\partial_x^n\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right) \\ & + \sigma_{\beta,1}\eta^{\alpha\beta}\left(\frac{1}{2}+\mu_\alpha\right)\left(\frac{1}{2}+\mu_\beta\right)\partial_x^n\frac{\partial v_\gamma}{\partial t^{\alpha,0}}, \end{aligned}$$

$$
\begin{aligned}
D\sigma^n_{\gamma,0} = & -a^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial\tau_0}\partial_x^n\frac{\partial\sigma_{\gamma,0}}{\partial t^{\alpha,1}}+\frac{\partial f_{\alpha,1}}{\partial\tau_0}\partial_x^n\frac{\partial\sigma_{\gamma,0}}{\partial t^{\beta,0}}\right)\\
& - b^{\alpha\beta}\left(\frac{\partial f_{\beta,0}}{\partial\tau_0}\partial_x^n\frac{\partial\sigma_{\gamma,0}}{\partial t^{\alpha,0}}+\frac{\partial f_{\alpha,0}}{\partial\tau_0}\partial_x^n\frac{\partial\sigma_{\gamma,0}}{\partial t^{\beta,0}}\right)\\
& + \sigma_{\beta,1}\eta^{\alpha\beta}\left(\frac{1}{2}+\mu_\alpha\right)\left(\frac{1}{2}+\mu_\beta\right)\partial_x^n\frac{\partial\sigma_{\gamma,0}}{\partial t^{\alpha,0}}+\frac{\partial\sigma_{\beta,0}}{\partial\tau_1}\partial_x^n\left(N^\beta_\gamma-\left(\frac{3}{2}+\mu_\gamma\right)Z^\beta_\gamma\right)\\
& - \delta_{n,0}\left(\frac{3}{2}+\mu_\gamma\right)\frac{\partial}{\partial\tau_0}\left(\sigma_{\gamma,2}+Z^\varepsilon_\gamma\sigma_{\varepsilon,1}\right).
\end{aligned}
$$

Proposition 13. *The first identity of the closedness condition* (3.17) *holds true, i.e.,*

$$\left[\frac{\partial}{\partial\tau_0}, I_1\right]+\left[\frac{\partial}{\partial\tau_1},\left[\frac{\partial}{\partial\tau_0}, X^\circ\right]\right]=0.$$

Proof. It follows from Lemma 14 that

$$
\begin{aligned}
&\left[\frac{\partial}{\partial\tau_0}, I_1\right]v_\lambda+\left[\frac{\partial}{\partial\tau_1},\left[\frac{\partial}{\partial\tau_0}, X^\circ\right]\right]v_\lambda\\
&=\left[\frac{\partial}{\partial\tau_1},\left[\frac{\partial}{\partial\tau_0}, \hat{I}_1+X^\circ\right]\right]v_\lambda+\frac{\partial A^\alpha}{\partial\tau_0}\frac{\partial v_\lambda}{\partial t^{\alpha,1}}+\frac{\partial B^\alpha}{\partial\tau_0}\frac{\partial v_\lambda}{\partial t^{\alpha,0}}.
\end{aligned}
$$

To prove the vanishing of the right hand side of the above equation, we need to verify the following identity due to Lemma 15:

$$\left[\frac{\partial}{\partial\tau_1}, D\right]v_\lambda+\frac{\partial A^\alpha}{\partial\tau_0}\frac{\partial v_\lambda}{\partial t^{\alpha,1}}+\frac{\partial B^\alpha}{\partial\tau_0}\frac{\partial v_\lambda}{\partial t^{\alpha,0}}=0,$$

which can be checked directly by using the definition of D. Similarly we can prove that

$$\left[\frac{\partial}{\partial\tau_0}, I_1\right]\sigma_{\lambda,0}+\left[\frac{\partial}{\partial\tau_1},\left[\frac{\partial}{\partial\tau_0}, X^\circ\right]\right]\sigma_{\lambda,0}=0.$$

The proposition is proved. □

It remains to prove the second identity of (3.17).

Lemma 16. *The following identities hold true:*

$$
\begin{aligned}
\left[I_1,\frac{\partial}{\partial\tau_1}\right]v_\lambda = & \sum_{s\ge 0}G^\gamma_s\frac{\partial}{\partial v^{(s)}_\gamma}\frac{\partial v_\lambda}{\partial\tau_1}+F^\gamma_s\frac{\partial}{\partial\sigma^s_{\gamma,0}}\frac{\partial v_\lambda}{\partial\tau_1}\\
& - A^\alpha\frac{\partial}{\partial\tau_1}\frac{\partial v_\lambda}{\partial t^{\alpha,1}}-B^\alpha\frac{\partial}{\partial\tau_1}\frac{\partial v_\lambda}{\partial t^{\alpha,0}}+\frac{\partial B^\alpha}{\partial\tau_1}\frac{\partial v_\lambda}{\partial t^{\alpha,0}},\\
\left[I_1,\frac{\partial}{\partial\tau_1}\right]\sigma_{\lambda,0} = & \sum_{s\ge 0}G^\gamma_s\frac{\partial}{\partial v^{(s)}_\gamma}\frac{\partial\sigma_{\lambda,0}}{\partial\tau_1}+F^\gamma_s\frac{\partial}{\partial\sigma^s_{\gamma,0}}\frac{\partial\sigma_{\lambda,0}}{\partial\tau_1}\\
& - A^\alpha\frac{\partial}{\partial\tau_1}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\alpha,1}}-B^\alpha\frac{\partial}{\partial\tau_1}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\alpha,0}}+\frac{\partial B^\alpha}{\partial\tau_1}\frac{\partial\sigma_{\lambda,0}}{\partial t^{\alpha,0}},
\end{aligned}
$$

where G^γ_s and F^γ_s are differential polynomials defined by

$$G^\gamma_s=\left[\frac{\partial}{\partial\tau_1},\hat{I}_1\right]v^{(s)}_\gamma+\partial_x^s\left(I_1 v_\gamma\right),\quad F^\gamma_s=\left[\frac{\partial}{\partial\tau_1},\hat{I}_1\right]\sigma^s_{\gamma,0}+\partial_x^s\left(I_1\sigma_{\gamma,0}\right).$$

Lemma 17. *The differential polynomials* G_s^γ *and* F_s^γ *defined in Lemma 16 have the following expressions:*

$$\begin{aligned} G_s^\gamma &= A^\alpha \partial_x^s \frac{\partial v_\gamma}{\partial t^{\alpha,1}} + B^\alpha \partial_x^s \frac{\partial v_\gamma}{\partial t^{\alpha,0}}, \\ F_s^\gamma &= A^\alpha \partial_x^s \frac{\partial \sigma_{\gamma,0}}{\partial t^{\alpha,1}} + B^\alpha \partial_x^s \frac{\partial \sigma_{\gamma,0}}{\partial t^{\alpha,0}} \\ &\quad + \delta_{s,0}\left(\frac{3}{2}+\mu_\gamma\right)\left(\frac{\partial \sigma_{\gamma,1}}{\partial \tau_2} - \sigma_{\beta,1}\eta^{\alpha\beta}\left(\frac{1}{2}+\mu_\alpha\right)\frac{\partial \sigma_{\gamma,1}}{\partial t^{\alpha,0}}\right). \end{aligned}$$

Proof. By using the definition of G_s^γ we can obtain the identity

$$G_{s+1}^\gamma - \partial_x G_s^\gamma = \frac{\partial}{\partial \tau_1}\left(\left[\hat{I}_1, \partial_x\right] v_\gamma^{(s)}\right) + \left[\partial_x, \hat{I}_1\right] \frac{\partial v_\gamma^{(s)}}{\partial \tau_1}.$$

Therefore it follows from Lemma 13 that

$$G_{s+1}^\gamma - \partial_x G_s^\gamma = -\partial_x A^\alpha \partial_x^s \frac{\partial v_\gamma}{\partial t^{\alpha,1}} - \partial_x B^\alpha \partial_x^s \frac{\partial v_\gamma}{\partial t^{\alpha,0}}.$$

Hence G_s^γ can be solved recursively starting from the initial condition

$$G_0^\gamma = I_1 v_\gamma - \hat{I}_1 \frac{\partial v_\gamma}{\partial \tau_1} = A^\alpha \frac{\partial v_\gamma}{\partial t^{\alpha,1}} + B^\alpha \frac{\partial v_\gamma}{\partial t^{\alpha,0}}.$$

The expressions of the differential polynomials F_s^γ can be obtained similarly. The lemma is proved. □

Proposition 14. *The second identity of the closedness condition (3.17) holds true, i.e.,*

$$\left[\frac{\partial}{\partial \tau_1}, I_1\right] = 0.$$

Proof. The proof of the proposition is straightforward by combining the results of Lemmas 16 and 17. □

4.3. Vanishing of the genus one obstruction. In this subsection, we will work with the canonical coordinates of the Frobenius manifold. Let us start by recalling some useful formulae related to the canonical coordinates. For the details one may refer to the work [5,12].

We denote by $u^1, \ldots, u^n$ the local canonical coordinates of a semisimple Frobenius manifold M and denote by $(u^i; \theta_i)$ the corresponding coordinates of $\hat{M}$. Under this system of local coordinates, the bihamiltonian structure (2.13) takes the form (2.7), i.e.,

$$\begin{aligned} P_0^{[0]} &= \frac{1}{2}\int \sum_{i,j=1}^n \left(\delta_{i,j} f^i \theta_i \theta_i^1 + A^{ij}\theta_i\theta_j\right), \\ P_1^{[0]} &= \frac{1}{2}\int \sum_{i,j=1}^n \left(\delta_{i,j} u^i f^i \theta_i \theta_i^1 + B^{ij}\theta_i\theta_j\right). \end{aligned}$$

Introduce functions

$$\psi_{i1} = \frac{1}{\sqrt{f^i}}, \quad i = 1, \ldots, n, \tag{4.12}$$

where the sign of the square root can be arbitrarily chosen, and define

$$\psi_{i\alpha} = \frac{1}{\psi_{i1}} \frac{\partial v_\alpha}{\partial u^i}, \quad \gamma_{ij} = \frac{1}{\psi_{j1}} \frac{\partial \psi_{i1}}{\partial u^j}, \quad i \neq j.$$

Then it is proved in [5] that

$$\frac{\partial v_\alpha}{\partial u^i} = \psi_{i1}\psi_{i\alpha}, \quad \frac{\partial u^i}{\partial v^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}, \tag{4.13}$$

$$\frac{\partial \psi_{i\alpha}}{\partial u^k} = \gamma_{ik}\psi_{k\alpha}, \quad i \neq k, \quad \frac{\partial \psi_{i\alpha}}{\partial u^i} = -\sum_{k\neq i} \gamma_{ik}\psi_{k\alpha}. \tag{4.14}$$

Let $V_{ij} = \mu_\alpha \eta^{\alpha\beta}\psi_{i\alpha}\psi_{j\beta}$, then we have the identity

$$\gamma_{ij}(u^j - u^i) = V_{ij}. \tag{4.15}$$

Now we are going to describe the deformation problem given in Sect. 3.3 in terms of the canonical coordinates. Introduce the odd variables $\theta_{i,m}$ by the recursion relations (2.16), then it follows from the the relation (2.17) that the equations (3.8) and (3.9) can be represented in the form

$$\begin{aligned} \frac{\partial u^i}{\partial s_2} &= a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial u^i}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial u^i}{\partial t^{\beta,0}}\right) + b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial u^i}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial u^i}{\partial t^{\beta,0}}\right) \\ &\quad + Xu^i + \mathcal{L}_2 u^i, \end{aligned} \tag{4.16}$$

$$\begin{aligned} \frac{\partial \theta_{i,0}}{\partial s_2} &= a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial \theta_{i,0}}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial \theta_{i,0}}{\partial t^{\beta,0}}\right) + b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial \theta_{i,0}}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial \theta_{i,0}}{\partial t^{\beta,0}}\right) \\ &\quad + X\theta_{i,0} + \sum_j A_i^j\theta_{j,2} + B_i^j\theta_{j,1} + \mathcal{L}_2\theta_{i,0}. \end{aligned} \tag{4.17}$$

In the canonical coordinates, the Hamiltonian operator $\mathcal{P}_1$ of P_1 has the form

$$\mathcal{P}_1^{ij} = u^i f^i \delta_{ij}\partial_x + \frac{1}{2}\partial_x\left(u^i f^i\right)\delta_{ij} + B^{ij} + Q^{ij}\partial_x^3 + \ldots,$$

where B^{ij} is defined in (2.6), Q^{ij} is given by the formula (cf., e.g., [13])

$$Q^{ij} = 3c_i\left(f^i\right)^2\delta_{ij} + \frac{1}{2}\left(u^i - u^j\right)\left(f^j\partial_j f^i c_i - f^i\partial_i f^j c_j\right), \tag{4.18}$$

and c_i is the i-th central invariant. Here and henceforth we omit the terms that do not contribute to the relevant computations given later. Let us also represent the evolutions of θ_i along the flows $\frac{\partial}{\partial t^{\alpha,p}}$ as follows:

$$\frac{\partial \theta_i}{\partial t^{\alpha,p}} = T^i_{\alpha,p}\theta_i^1 + \cdots + \sum_j K^i_{\alpha,p;j}\theta_j^3 + \ldots, \quad T_{\alpha,p} \in \hat{\mathcal{A}}_0^0, \ K^i_{\alpha,p;j} \in \hat{\mathcal{A}}_3^0.$$

Note that the coefficients of θ_j^1 of the leading term of $\frac{\partial \theta_i}{\partial t^{\alpha,p}}$ are zero for $j \neq i$, this is due to the fact that the leading term of the flow $\frac{\partial}{\partial t^{\alpha,p}}$ is diagonal, one may refer to [9] for details.

Let us turn to the proof of the vanishing of the genus one obstruction. Due to the closedness condition (3.15), we can choose a derivation X° such that when we take $X = X^\circ$ in (4.16) and (4.17) we have

$$\left[\frac{\partial}{\partial \tau_0}, \frac{\partial}{\partial s_2}\right] = 0.$$

We require that the leading term of X° is given by the Virasoro symmetry $\frac{\partial}{\partial s_2}$ of the super tau-cover of the Principal Hierarchy, hence it follows from the result of [11] that the actions of X° on u^i and θ_i take the form

$$X^\circ u^i = (u^i)^3 + \sum_j L^i_j u^{j,2} + \dots, \quad L^i_j \in \hat{\mathcal{A}}^0_0,$$
$$X^\circ \theta_i = \sum_j M_i^j \theta_j + \sum_j J_i^j \theta_j^2 + \dots, \quad M_i^j, J_i^j \in \hat{\mathcal{A}}^0_0.$$

Lemma 18. *We have the following identity:*

$$J_i^i - L_i^i = \frac{2}{f^i} c_0 (u^i)^2 Q^{ii} - \sum_j \frac{A_i^j}{f^j}(u^i + u^j) Q^{ij} - \sum_j \frac{B_i^j}{f^j} Q^{ij}$$
$$- b_{2;1,0}^{\beta,q} K^i_{\beta,q;i} - a_2^{\alpha,p;\beta,q}\left(\left(f'_{\beta,q}\right)_0 K^i_{\alpha,p;i} + \left(f'_{\alpha,p}\right)_0 K^i_{\beta,q;i}\right),$$

where $b_{2;\alpha,p}^{\beta,q}$ and $a_2^{\alpha,p;\beta,q}$ are the constants that appear in the operator $\mathcal{L}_2$, and $\left(f'_{\alpha,p}\right)_0$ denotes the differential degree zero component of $f'_{\alpha,p}$.

Proof. We can prove this lemma by looking at the differential degree 3 component of the left hand side of the equation

$$\left[\frac{\partial}{\partial \tau_0}, \frac{\partial}{\partial s_2}\right] u^i = 0,$$

and by computing the coefficient of θ_i^3. The lemma is proved. □

Remark 4. Note that c_0 is the arbitrary constant that appears in the operator $\mathcal{L}_2$, which is different from central invariants $c_1, \dots, c_n$.

According to the discussion given in Sect. 3.3, we need to show the triviality of the cohomology class of the differential degree 3 component of the derivation $\left[\frac{\partial}{\partial \tau_1}, \mathcal{C}\right] = I_1 - \left[\frac{\partial}{\partial \tau_1}, X^\circ\right]$. Due to Theorem 10, it suffices to prove that in the differential degree 3 component of

$$I_1 u^i - \left[\frac{\partial}{\partial \tau_1}, X^\circ\right] u^i,$$

the coefficient of θ_i^3 vanishes. By using Lemma 18 and a straightforward computation, we obtain the following lemma.

Lemma 19. *In the differential degree 3 component of* $I_1 u^i - \left[\frac{\partial}{\partial \tau_1}, X^\circ\right] u^i$, *the coefficient of* θ_i^3 *reads*

$$\sum_j Q^{ij}\left(M^i_j + A^i_j (u^i)^2 + B^i_j u^i\right) + E^3 Q^{ii} - (6 + c_0)(u^i)^2 Q^{ii}$$
$$+ 3 Q^{ii} b^{\beta,q}_{2;1,0} T^i_{\beta,q} + 3 Q^{ii} a_2^{\alpha,p;\beta,q}\left(\left(f'_{\beta,q}\right)_0 T^i_{\alpha,p} + \left(f'_{\alpha,p}\right)_0 T^i_{\beta,q}\right), \tag{4.19}$$

here E^3 *is the cubic power of the Euler vector field which is given by*

$$E^3 = \sum_i (u^i)^3 \frac{\partial}{\partial u^i}.$$

Lemma 20. *We have the identities*

$$M^i_i + A^i_i (u^i)^2 + B^i_i u^i = (3 + c_0)(u^i)^2 - \frac{1}{f^i} E^3 f^i - b^{\beta,q}_{2;1,0} T^i_{\beta,q}$$
$$- a_2^{\alpha,p;\beta,q}\left(\left(f'_{\beta,q}\right)_0 T^i_{\alpha,p} + \left(f'_{\alpha,p}\right)_0 T^i_{\beta,q}\right),$$

and $M^j_i + A^j_i (u^j)^2 + B^j_i u^j = 0$ *for* $i \neq j$.

Proof. We can prove this lemma by considering the differential degree 1 component of the left hand side of the equation

$$\left[\frac{\partial}{\partial \tau_0}, \frac{\partial}{\partial s_2}\right] u^i = 0,$$

and by computing the coefficient of θ_j^1. The lemma is proved. □

In order to prove the vanishing of the expression (4.19), we need to check, due to Lemma 20 and the expression (4.18) for Q^{ii}, the following identity:

$$E^3 f^i + 2 f^i\left(b^{\beta,q}_{2;1,0} T^i_{\beta,q} + a_2^{\alpha,p;\beta,q}\left(\left(f'_{\beta,q}\right)_0 T^i_{\alpha,p} + \left(f'_{\alpha,p}\right)_0 T^i_{\beta,q}\right)\right) = 3 (u^i)^2 f^i.$$

Proposition 15. *For* $m \geq -1$, *we have*

$$E^{m+1} f^i + 2 f^i\left(b^{\beta,q}_{m;1,0} T^i_{\beta,q}\right.$$
$$\left. + a_m^{\alpha,p;\beta,q}\left(\left(f'_{\beta,q}\right)_0 T^i_{\alpha,p} + \left(f'_{\alpha,p}\right)_0 T^i_{\beta,q}\right)\right) = (1 + m)(u^i)^m f^i,$$

here E^{m+1} *is the* $(m+1)$*-th power of the Euler vector field* E *which is given by*

$$E^{m+1} = \sum_i (u^i)^{m+1} \frac{\partial}{\partial u^i}.$$

Proof. Let us consider the following generating functions:

$$\sum_{m\geq -1}\frac{1}{\lambda^{m+2}}E^{m+1}f^i=\sum_j\frac{1}{\lambda-u^j}\frac{\partial f^i}{\partial u^j},\quad \sum_{m\geq -1}\frac{1}{\lambda^{m+2}}(1+m)(u^i)^m f^i=\frac{f^i}{(\lambda-u^i)^2},$$

$$\sum_{m\geq -1}\frac{1}{\lambda^{m+2}}\left(b_{m;1,0}^{\beta,q}T^i_{\beta,q}+a_m^{\alpha,p;\beta,q}\left(\left(f'_{\beta,q}\right)_0T^i_{\alpha,p}+\left(f'_{\alpha,p}\right)_0T^i_{\beta,q}\right)\right)$$
$$=\frac{1}{2}\frac{1}{(\lambda-u^i)^2}+\sum_{j\neq i}\frac{\psi_{j1}}{\psi_{i1}}\frac{V_{ij}}{(\lambda-u^i)(\lambda-u^j)},$$

where the last generating function is computed in [12], one may refer to Lemma 3.10.18 and the proof of Theorem 3.10.29 of [12] for details. Then the proposition is proved by using (4.12), (4.14) and (4.15). □

Finally we have the following theorem.

Theorem 13. *For a given semisimple Frobenius manifold and a tau-symmetric bihamiltonian deformation of its Principal Hierarchy, there exists a unique deformation* $\frac{\partial}{\partial s_2}\in$ $\mathrm{Der}^\partial(\hat{\mathcal{A}}^{Vir})$ *of the Virasoro symmetry of the super tau-cover of the Principal Hierarchy such that it is a symmetry of the deformed super tau-cover. Moreover, the actions of* $\frac{\partial}{\partial s_2}$ *on the local variables are given by*

$$\frac{\partial v_\lambda}{\partial s_2}=a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial v_\lambda}{\partial t^{\alpha,1}}+f_{\alpha,1}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right)+b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial v_\lambda}{\partial t^{\alpha,0}}+f_{\alpha,0}\frac{\partial v_\lambda}{\partial t^{\beta,0}}\right)$$
$$+Xv_\lambda+\mathcal{L}_2v_\lambda, \tag{4.20}$$

$$\frac{\partial \sigma_{\lambda,0}}{\partial s_2}=a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,1}}+f_{\alpha,1}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right)+b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\alpha,0}}+f_{\alpha,0}\frac{\partial \sigma_{\lambda,0}}{\partial t^{\beta,0}}\right)$$
$$+X\sigma_{\lambda,0}+\left(\frac{5}{2}+c_0+\mu_\lambda\right)\sigma_{\lambda,2}+N_\lambda^\zeta\sigma_{\zeta,1}+\mathcal{L}_2\sigma_{\lambda,0}, \tag{4.21}$$

where $X\in\mathrm{Der}^\partial(\hat{\mathcal{A}})^0$ *and* $N^\zeta_\lambda\in\hat{\mathcal{A}}^0_{\geq 0}$ *is the differential polynomial described in Lemma 10.*

4.4. Lifting to the tau-covers. In order to lift the symmetry (4.20) to the tau-cover of the deformed Principal Hierarchy, we first need to rewrite (4.20) in terms of the normal coordinates $w^1,\ldots,w^n$ of M. We start by proving the following lemmas.

Lemma 21. *Let* $g_\lambda\in\hat{\mathcal{A}}^0_{\geq 1}$ *be the differential polynomials given in* (2.35) *which satisfy the identities*

$$h_{\lambda,0}=v_\lambda+\partial_x g_\lambda,\quad \lambda=1,\ldots,n.$$

Then we have:

1. $g_1=0$.
2. *For any* $\alpha=1,\ldots,n$ *and* $p\geq 0$,

$$\Omega_{\alpha,p;\lambda,0}=\frac{\delta h_{\alpha,p+1}}{\delta v^\lambda}+\frac{\partial g_\lambda}{\partial t^{\alpha,p}}.$$

Proof. Due to Proposition 7, we have $D_{X_{1,0}} = \partial_x$ and $X_{1,0} = -[H_{1,0}, P_0]$, from which it follows that

$$v_\alpha = \frac{\delta H_{1,0}}{\delta v^\alpha}.$$

By taking $\alpha = 1$, we obtain the first property by using the definition (2.28). The second one is obvious due to Theorem 8. The lemma is proved. □

Lemma 22. *There exists a derivation $X^\circ \in \mathrm{Der}^\partial(\hat{\mathcal{A}})$ such that its leading term is given by the Virasoro symmetry $\frac{\partial}{\partial s_2}$ of the Principal Hierarchy, and it satisfies the equations $\left[\frac{\partial}{\partial \tau_0}, X^\circ\right] = I_0$ and*

$$\begin{aligned}
X^\circ h_{\lambda,0} + \hat{I}_1 h_{\lambda,0} = {} & a^{\alpha\beta}\left(f'_{\alpha,1}\Omega_{\beta,0;\lambda,0} + f'_{\beta,0}\Omega_{\alpha,1;\lambda,0} + \frac{\partial^2\Omega_{\alpha,1;\beta,0}}{\partial t^{\lambda,0}\partial t^{1,0}}\right)\\
&+ b^{\alpha\beta}\left(f'_{\alpha,0}\Omega_{\beta,0;\lambda,0} + f'_{\beta,0}\Omega_{\alpha,0;\lambda,0} + \frac{\partial^2\Omega_{\alpha,0;\beta,0}}{\partial t^{\lambda,0}\partial t^{1,0}}\right)\\
&+ b^{\beta,q}_{2;1,0}\Omega_{\beta,q;\lambda,0} + b^{\beta,q}_{2;\lambda,0}\Omega_{\beta,q;1,0} + 2c_{2;\lambda,0;1,0}.
\end{aligned} \tag{4.22}$$

Proof. We first find a particular solution $\tilde{X}^\circ$ of the equation $\left[\frac{\partial}{\partial \tau_0}, X\right] = I_0$, then we modify it by a solution $\tilde{\mathcal{C}}$ of the homogeneous equation $\left[\frac{\partial}{\partial \tau_0}, \mathcal{C}\right] = 0$ such that $X^\circ := \tilde{X}^\circ + \tilde{\mathcal{C}}$ satisfies (4.22).

Let us define $\tilde{X}^\circ \in \mathrm{Der}^\partial(\hat{\mathcal{A}})$ as follows:

$$\begin{aligned}
\tilde{X}^\circ v_\lambda = {} & a^{\alpha\beta}\left(f'_{\alpha,1}\frac{\delta h_{\beta,1}}{\delta v^\lambda} + f'_{\beta,0}\frac{\delta h_{\alpha,2}}{\delta v^\lambda}\right) + b^{\alpha\beta}\left(f'_{\alpha,0}\frac{\delta h_{\beta,1}}{\delta v^\lambda} + f'_{\beta,0}\frac{\delta h_{\alpha,1}}{\delta v^\lambda}\right)\\
&+ b^{\beta,q}_{2;1,0}\frac{\delta h_{\beta,q+1}}{\delta v^\lambda} + b^{\beta,q}_{2;\lambda,0}\frac{\delta h_{\beta,q+1}}{\delta v^1} + 2c_{2;\lambda,0;1,0};\\
\tilde{X}^\circ \sigma_{\lambda,0} = {} & a^{\alpha\beta}\left(\frac{\partial f_{\alpha,1}}{\partial \tau_0}\frac{\delta h_{\beta,1}}{\delta v^\lambda} + \frac{\partial f_{\beta,0}}{\partial \tau_0}\frac{\delta h_{\alpha,2}}{\delta v^\lambda}\right) + b^{\alpha\beta}\left(\frac{\partial f_{\alpha,0}}{\partial \tau_0}\frac{\delta h_{\beta,1}}{\delta v^\lambda} + \frac{\partial f_{\beta,0}}{\partial \tau_0}\frac{\delta h_{\alpha,1}}{\delta v^\lambda}\right)\\
&+ b^{\beta,q}_{2;\lambda,0}\frac{\partial f_{\beta,q}}{\partial \tau_0} - \left(\frac{5}{2} + \mu_\lambda\right)\sigma_{\lambda,2} - N^\zeta_\lambda \sigma_{\zeta,1}.
\end{aligned}$$

Firstly, from the definition of N^ζ_λ it follows that $\tilde{X}^\circ$ is indeed local. We also note that the leading terms of $\tilde{X}^\circ v_\lambda$ and $\tilde{X}^\circ \sigma_{\lambda,0}$ coincide with the local terms of the Virasoro symmetry $\frac{\partial v_\lambda}{\partial s_2}$ and $\frac{\partial \sigma_{\lambda,0}}{\partial s_2}$ of the Principal Hierarchy. By a direct computation, it is easy to check that $\left[\frac{\partial}{\partial \tau_0}, \tilde{X}^\circ\right] = I_0$.

Next we want to determine $\tilde{\mathcal{C}} \in \mathrm{Der}^\partial(\hat{\mathcal{A}})$ such that $X^\circ = \tilde{X}^\circ + \tilde{\mathcal{C}}$ satisfies (4.22). By using the definition of $\tilde{X}^\circ v_\lambda$, Lemmas 13 and 21, it is straightforward to show that

$$\tilde{\mathcal{C}}(h_{\lambda,0}) = a^{\alpha\beta}\frac{\partial^2\Omega_{\alpha,1;\beta,0}}{\partial t^{\lambda,0}\partial t^{1,0}} + b^{\alpha\beta}\frac{\partial^2\Omega_{\alpha,0;\beta,0}}{\partial t^{\lambda,0}\partial t^{1,0}} - \partial_x \hat{I}_1(g_\lambda) - \partial_x \tilde{X}^\circ(g_\lambda), \tag{4.23}$$

which uniquely determines the actions of $\tilde{\mathcal{C}}$ on v_λ.

Finally we need to check such $\tilde{\mathcal{C}}$ satisfies $\left[\frac{\partial}{\partial \tau_0}, \tilde{\mathcal{C}}\right] = 0$. By using the next lemma we know that it suffices to show that $\int \tilde{\mathcal{C}} v_\lambda = 0$, which is obvious from (4.23) and (2.35). The lemma is proved. □

Lemma 23. *Let $U_1, \dots, U_n$ be differential polynomials with $U_\lambda \in \hat{\mathcal{A}}^0_{\geq 2}$. Then there exists $\mathcal{C} \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^0_{\geq 1}$ such that $\left[\frac{\partial}{\partial \tau_0}, \mathcal{C}\right] = 0$ and $\mathcal{C} v_\lambda = U_\lambda$ if and only if $\int U_\lambda = 0$.*

Proof. Let $\mathcal{C} \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^0$ be a derivation such that $\left[\frac{\partial}{\partial \tau_0}, \mathcal{C}\right] = 0$. Then by using the triviality of the variational Hamiltonian cohomology $H^0_{\geq 1}\big(\mathrm{Der}^\partial(\hat{\mathcal{A}}), P_0^{[0]}\big)$, we know the existence of a certain $\mathcal{K} \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^{-1}$ such that $\left[\frac{\partial}{\partial \tau_0}, \mathcal{K}\right] = \mathcal{C}$. Let us denote $\mathcal{K}\sigma_{\lambda,0} = V_\lambda \in \hat{\mathcal{A}}^0$. Then it is easy to see that

$$\mathcal{C} v_\lambda = \left[\frac{\partial}{\partial \tau_0}, \mathcal{K}\right] v_\lambda = \mathcal{K}\sigma^1_{\lambda,0} = \partial_x V_\lambda.$$

Therefore we have $\int U_\lambda = \int \partial_x V_\lambda = 0$.

Conversely, if $U_\lambda = \partial_x V_\lambda$ for some $V_\lambda \in \hat{\mathcal{A}}^0$, we can define a unique derivation $\mathcal{C} \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^0$ by

$$\mathcal{C} v_\lambda = \partial_x V_\lambda, \quad \mathcal{C}\sigma_{\lambda,0} = \frac{\partial V_\lambda}{\partial \tau_0},$$

then it is easy to check that $\left[\frac{\partial}{\partial \tau_0}, \mathcal{C}\right] = 0$ and the lemma is proved. □

Now let us denote $\mathcal{C} = X - X^\circ$, where the derivation X is described in Theorem 13 and X° satisfies (4.22). Then we can rewrite the Virasoro symmetry $\frac{\partial}{\partial s_2}$ in terms of the normal coordinates as follows:

$$\begin{aligned}\frac{\partial w_\lambda}{\partial s_2} = {} & a^{\alpha\beta}\left(f_{\beta,0}\frac{\partial w_\lambda}{\partial t^{\alpha,1}} + f_{\alpha,1}\frac{\partial w_\lambda}{\partial t^{\beta,0}}\right) + b^{\alpha\beta}\left(f_{\beta,0}\frac{\partial w_\lambda}{\partial t^{\alpha,0}} + f_{\alpha,0}\frac{\partial w_\lambda}{\partial t^{\beta,0}}\right)\\ & + a^{\alpha\beta}\left(f'_{\alpha,1}\Omega_{\beta,0;\lambda,0} + f'_{\beta,0}\Omega_{\alpha,1;\lambda,0} + \frac{\partial^2 \Omega_{\alpha,1;\beta,0}}{\partial t^{\lambda,0}\partial t^{1,0}}\right)\\ & + b^{\alpha\beta}\left(f'_{\alpha,0}\Omega_{\beta,0;\lambda,0} + f'_{\beta,0}\Omega_{\alpha,0;\lambda,0} + \frac{\partial^2 \Omega_{\alpha,0;\beta,0}}{\partial t^{\lambda,0}\partial t^{1,0}}\right)\\ & + b^{\beta,q}_{2;1,0}\Omega_{\beta,q;\lambda,0} + b^{\beta,q}_{2;\lambda,0}\Omega_{\beta,q;1,0} + 2c_{2;\lambda,0;1,0} + \mathcal{C}(h_{\lambda,0}) + \mathcal{L}_2 w_\lambda.\end{aligned}$$

According to Lemma 23, there exist differential polynomials $Q_\lambda \in \hat{\mathcal{A}}^0_{\geq 1}$ such that $\mathcal{C}(h_{\lambda,0}) = \partial_x Q_\lambda$. From the fact that

$$\frac{\partial}{\partial t^{\gamma,0}}\frac{\partial w_\lambda}{\partial s_2} = \frac{\partial}{\partial t^{\lambda,0}}\frac{\partial w_\gamma}{\partial s_2},$$

we have the equation

$$\frac{\partial Q_\lambda}{\partial t^{\gamma,0}} = \frac{\partial Q_\gamma}{\partial t^{\lambda,0}}, \tag{4.24}$$

which means that $\int Q_1$ is a conserved quantity of the flow $\frac{\partial}{\partial t^{\lambda,0}}$ for all $\lambda = 1, \ldots, n$. We want to find a differential polynomial $Q \in \hat{\mathcal{A}}_{\geq 0}$ such that $Q_\lambda = \frac{\partial Q}{\partial t^{\lambda,0}}$. To this end we need the following lemma.

Lemma 24 (Lemma 4.12 and Theorem A.2 of [9]). *If the Frobenius manifold with dimension $n \geq 2$ is irreducible, then there exist constants c^α for $\alpha = 1, \ldots, n$ such that the leading term of the derivation*

$$D = c^\alpha \frac{\partial}{\partial t^{\alpha,0}} \in \mathrm{Der}^\partial(\hat{\mathcal{A}})$$

is non-degenerate, i.e., in terms of the canonical coordinates u^i, its leading term $D^{[0]} \in \mathrm{Der}^\partial(\hat{\mathcal{A}})^0_1$ can be represented by

$$D^{[0]}u^i = A^i(u)u^{i,1}, \quad i = 1, \ldots, n$$

such that the condition $\frac{\partial A^i}{\partial u^i} \neq 0$ holds true for all $i = 1, \ldots, n$. Moreover, any conserved quantity $H \in \hat{\mathcal{F}}^0_{\geq 1}$ of D is trivial.

The irreducibility of a Frobenius manifold is a mild condition and without loss of generality, we may always assume that a Frobenius manifold is irreducible (see [5] for details). Therefore it follows from the above lemma that if the dimension of the Frobenius manifold $\dim M \geq 2$, there exists $Q \in \hat{\mathcal{A}}^0_{\geq 0}$ such that $Q_1 = \partial_x Q$. By setting $\gamma = 1$ in the identity (4.24), we obtain that

$$\partial_x Q_\lambda = \frac{\partial Q_1}{\partial t^{\lambda,0}} = \partial_x \frac{\partial Q}{\partial t^{\lambda,0}} \in \hat{\mathcal{A}}^0_{\geq 2},$$

from which we conclude that $Q_\lambda = \frac{\partial Q}{\partial t^{\lambda,0}}$.

Remark 5. The existence of Q can be also obtained by proving a Poincaré lemma for general semi-Hamiltonian systems [28] using the idea given in Appendix of [9], whose proof is a little bit complicated, so we use the method presented above.

Theorem 14. *Let $\mathcal{Z}$ be a tau-function of the tau-cover (2.32) of the deformed Principal Hierarchy. Then there exists a differential polynomial $O_2 \in \hat{\mathcal{A}}^0_{\geq 0}$ which yields a symmetry of the tau-cover given by*

$$\frac{\partial \mathcal{Z}}{\partial s_2} = L_2^{even}\mathcal{Z} + O_2\mathcal{Z},$$

where the evolutions of $f_{\alpha,p}$ and w_λ along the flow $\frac{\partial}{\partial s_2}$ are given by

$$\frac{\partial f_{\alpha,p}}{\partial s_2} = \frac{\partial}{\partial t^{\alpha,p}} \frac{\partial \log \mathcal{Z}}{\partial s_2}, \quad \frac{\partial w_\lambda}{\partial s_2} = \frac{\partial}{\partial t^{1,0}} \frac{\partial f_{\lambda,0}}{\partial s_2}.$$

Proof. When $\dim M \geq 2$ we take $O_2 = Q$ which is just given above. When $\dim M = 1$, O_2 is given in (3.26). The theorem is proved. ◻

Now let us proceed to determine all other Virasoro symmetries $\frac{\partial}{\partial s_m}$ of the tau-cover of the deformed Principal Hierarchy. In what follows we will denote $\mathcal{F} = \log \mathcal{Z}$ and denote

$$\mathcal{G}_m = a_m^{\alpha,p;\beta,q}\left(\frac{\partial \mathcal{F}}{\partial t^{\alpha,p}}\frac{\partial \mathcal{F}}{\partial t^{\beta,q}} + \frac{\partial^2 \mathcal{F}}{\partial t^{\alpha,p}\partial t^{\beta,q}}\right) + c_{m;\alpha,p;\beta,q}t^{\alpha,p}t^{\beta,q};$$

$$\mathcal{L}_m^{even} = b_{m;\alpha,p}^{\beta,q}t^{\alpha,p}\frac{\partial}{\partial t^{\beta,q}}, \quad m \geq -1.$$

It is proved in [9] that

$$\frac{\partial \mathcal{F}}{\partial s_{-1}} = \mathcal{G}_{-1} + \mathcal{L}_{-1}^{even}\mathcal{F}$$

induces a symmetry of the tau-cover of the deformed Principal Hierarchy. On the other hand, by using Theorem 14 we know that

$$\frac{\partial \mathcal{F}}{\partial s_2} = \mathcal{G}_2 + O_2 + \mathcal{L}_2^{even}\mathcal{F}$$

also induces a symmetry of the tau-cover of the deformed Principal Hierarchy. From the commutation relation

$$[L_{-1}^{even}, L_2^{even}] = -3L_1^{even},$$

it follows that

$$\left[\frac{\partial}{\partial s_{-1}}, \frac{\partial}{\partial s_2}\right]\mathcal{F} = 3\left(\mathcal{G}_1 + \mathcal{L}_1^{even}\mathcal{F}\right) + \frac{\partial O_2}{\partial s_{-1}} - \mathcal{L}_{-1}^{even}O_2.$$

Since $\frac{\partial O_2}{\partial s_{-1}} - \mathcal{L}_{-1}^{even}O_2$ is a differential polynomial, we can define

$$\frac{\partial \mathcal{F}}{\partial s_1} = \frac{1}{3}\left[\frac{\partial}{\partial s_{-1}}, \frac{\partial}{\partial s_2}\right]\mathcal{F} = \mathcal{G}_1 + O_1 + \mathcal{L}_1^{even}\mathcal{F}, \quad O_1 = \frac{1}{3}\left(\frac{\partial O_2}{\partial s_{-1}} - \mathcal{L}_{-1}^{even}O_2\right).$$

Then it is obvious that $\frac{\partial \mathcal{F}}{\partial s_1}$ induces a symmetry of the tau-cover of the deformed Principal Hierarchy. Similarly we define the symmetry

$$\frac{\partial \mathcal{F}}{\partial s_0} = \frac{1}{2}\left[\frac{\partial}{\partial s_{-1}}, \frac{\partial}{\partial s_1}\right]\mathcal{F} = \mathcal{G}_0 + O_0 + \mathcal{L}_0^{even}\mathcal{F}, \quad O_0 = \frac{1}{2}\left(\frac{\partial O_1}{\partial s_{-1}} - \mathcal{L}_{-1}^{even}O_1\right).$$

Now let us define $\frac{\partial \mathcal{F}}{\partial s_m}$ for $m \geq 3$ recursively in the following way. Assume we have defined

$$\frac{\partial \mathcal{F}}{\partial s_m} = \mathcal{G}_m + O_m + \mathcal{L}_m^{even}\mathcal{F}$$

such that it induces a symmetry of the tau-cover of the deformed Principal Hierarchy for $m \geq 2$, then we have

$$\frac{\partial}{\partial s_1}\frac{\partial \mathcal{F}}{\partial s_m} = \frac{\partial \mathcal{G}_m}{\partial s_1} + \frac{\partial O_m}{\partial s_1} + \mathcal{L}_m^{even}\left(\mathcal{G}_1 + O_1 + \mathcal{L}_1^{even}\mathcal{F}\right).$$

It follows from the definition of $\mathcal{G}_m$ that

$$\begin{aligned}\frac{\partial \mathcal{G}_m}{\partial s_1} &= \frac{\partial}{\partial s_1}\left(a_m^{\alpha,p;\beta,q}\left(\frac{\partial \mathcal{F}}{\partial t^{\alpha,p}}\frac{\partial \mathcal{F}}{\partial t^{\beta,q}}+\frac{\partial^2 \mathcal{F}}{\partial t^{\alpha,p}\partial t^{\beta,q}}\right)+c_{m;\alpha,p;\beta,q}t^{\alpha,p}t^{\beta,q}\right)\\ &= a_m^{\alpha,p;\beta,q}\left(f_{\alpha,p}\frac{\partial}{\partial t^{\beta,q}}(\mathcal{G}_1+O_1+\mathcal{L}_1^{even}\mathcal{F})+f_{\beta,q}\frac{\partial}{\partial t^{\alpha,p}}(\mathcal{G}_1+O_1+\mathcal{L}_1^{even}\mathcal{F})\right)\\ &\quad + a_m^{\alpha,p;\beta,q}\frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,q}}(\mathcal{G}_1+O_1+\mathcal{L}_1^{even}\mathcal{F})\\ &= a_m^{\alpha,p;\beta,q}\left(f_{\alpha,p}\frac{\partial O_1}{\partial t^{\beta,q}}+f_{\beta,q}\frac{\partial O_1}{\partial t^{\alpha,p}}+\frac{\partial^2 O_1}{\partial t^{\alpha,p}\partial t^{\beta,q}}\right)+\dots,\end{aligned}$$

here $\dots$ stands for remaining terms that are independent of O_1. In a similar way we can compute $\frac{\partial}{\partial s_m}\frac{\partial \mathcal{F}}{\partial s_1}$. By using the commutation relation

$$[L_1^{even}, L_m^{even}] = (1-m)L_{m+1}^{even}$$

we obtain the following result:

$$\begin{aligned}\left[\frac{\partial}{\partial s_1},\frac{\partial}{\partial s_m}\right]\mathcal{F} &= (m-1)\left(\mathcal{G}_{m+1}+\mathcal{L}_{m+1}^{even}\mathcal{F}\right)\\ &\quad+\frac{\partial O_m}{\partial s_1}-a_1^{\alpha,p;\beta,q}\left(f_{\alpha,p}\frac{\partial O_m}{\partial t^{\beta,q}}+f_{\beta,q}\frac{\partial O_m}{\partial t^{\alpha,p}}+\frac{\partial^2 O_m}{\partial t^{\alpha,p}\partial t^{\beta,q}}\right)-\mathcal{L}_1^{even}O_m\\ &\quad-\frac{\partial O_1}{\partial s_m}+a_m^{\alpha,p;\beta,q}\left(f_{\alpha,p}\frac{\partial O_1}{\partial t^{\beta,q}}+f_{\beta,q}\frac{\partial O_1}{\partial t^{\alpha,p}}+\frac{\partial^2 O_1}{\partial t^{\alpha,p}\partial t^{\beta,q}}\right)+\mathcal{L}_m^{even}O_1.\end{aligned}$$

Therefore we obtain the symmetry $\frac{\partial \mathcal{F}}{\partial s_{m+1}}$ by defining

$$\frac{\partial \mathcal{F}}{\partial s_{m+1}} = \frac{1}{m-1}\left[\frac{\partial}{\partial s_1},\frac{\partial}{\partial s_m}\right]\mathcal{F} = \mathcal{G}_{m+1}+O_{m+1}+\mathcal{L}_{m+1}^{even}\mathcal{F},$$

where O_{m+1} is the differential polynomial given by

$$\begin{aligned}&O_{m+1}\\ &= \frac{1}{m-1}\left(\frac{\partial O_m}{\partial s_1}-a_1^{\alpha,p;\beta,q}\left(f_{\alpha,p}\frac{\partial O_m}{\partial t^{\beta,q}}+f_{\beta,q}\frac{\partial O_m}{\partial t^{\alpha,p}}+\frac{\partial^2 O_m}{\partial t^{\alpha,p}\partial t^{\beta,q}}\right)-\mathcal{L}_1^{even}O_m\right)\\ &\quad-\frac{1}{m-1}\left(\frac{\partial O_1}{\partial s_m}-a_m^{\alpha,p;\beta,q}\left(f_{\alpha,p}\frac{\partial O_1}{\partial t^{\beta,q}}+f_{\beta,q}\frac{\partial O_1}{\partial t^{\alpha,p}}+\frac{\partial^2 O_1}{\partial t^{\alpha,p}\partial t^{\beta,q}}\right)-\mathcal{L}_m^{even}O_1\right).\end{aligned}$$

Thus we obtain recursively an infinite set of symmetries of the tau-cover of the deformed Principal Hierarchy. Their actions on $\mathcal{F}$ can be represented by

$$\frac{\partial \mathcal{F}}{\partial s_m} = \mathcal{G}_m+O_m+\mathcal{L}_m^{even}\mathcal{F}.$$

Next we show that we can further adjust O_m by adding certain constants such that these symmetries satisfy the Virasoro commutation relation

$$\left[\frac{\partial}{\partial s_k},\frac{\partial}{\partial s_l}\right] = (l-k)\frac{\partial}{\partial s_{k+l}},\quad k,l\geq -1.$$

Lemma 25. *There is a unique choice of constants κ_m for $m \geq -1$ such that the flows*

$$\frac{\partial \mathcal{F}}{\partial s_m} = \mathcal{G}_m + O_m + \kappa_m + \mathcal{L}_m^{even}\mathcal{F}$$

satisfy the Virasoro commutation relation

$$\left[\frac{\partial}{\partial s_k}, \frac{\partial}{\partial s_l}\right] = (l-k)\frac{\partial}{\partial s_{k+l}}, \quad k, l \geq -1. \tag{4.25}$$

Proof. Let us first fix an arbitrary choice of O_m and denote

$$\frac{\partial \mathcal{F}}{\partial \hat{s}_m} = \mathcal{G}_m + O_m + \mathcal{L}_m^{even}\mathcal{F}.$$

Then we obtain the differential polynomials $\tilde{O}_{k+l}$ such that

$$\left[\frac{\partial}{\partial \hat{s}_k}, \frac{\partial}{\partial \hat{s}_l}\right]\mathcal{F} = (l-k)\left(\mathcal{G}_{l+k} + \tilde{O}_{l+k} + \mathcal{L}_{l+k}^{even}\mathcal{F}\right).$$

But both $\frac{\partial}{\partial \hat{s}_{l+k}}$ and $\left[\frac{\partial}{\partial \hat{s}_k}, \frac{\partial}{\partial \hat{s}_l}\right]$ are symmetries of the tau-cover of the deformed Principal Hierarchy, so we conclude that

$$\frac{\partial \mathcal{F}}{\partial s} := \frac{\partial \mathcal{F}}{\partial \hat{s}_{k+l}} - \frac{1}{l-k}\left[\frac{\partial}{\partial \hat{s}_k}, \frac{\partial}{\partial \hat{s}_l}\right]\mathcal{F} = O_{k+l} - \tilde{O}_{k+l}$$

is also a symmetry of the tau-cover of the deformed Principal Hierarchy. The action of this symmetry on the normal coordinates has the expression

$$\frac{\partial w_\lambda}{\partial s} = \frac{\partial^2}{\partial t^{\lambda,0}\partial t^{1,0}}\left(O_{k+l} - \tilde{O}_{k+l}\right).$$

Thus $\frac{\partial w_\lambda}{\partial s} \in \hat{\mathcal{A}}_{\geq 2}$, and therefore such a symmetry must vanish due to the result of the bihamiltonian cohomology [7]. Hence we conclude that

$$O_{k+l} - \tilde{O}_{k+l} = c_{k,l}$$

for some constant $c_{k,l}$, and this means that

$$\left[\frac{\partial}{\partial \hat{s}_k}, \frac{\partial}{\partial \hat{s}_l}\right] = (l-k)\frac{\partial}{\partial \hat{s}_{k+l}} - (l-k)c_{k,l}, \quad k, l \geq -1. \tag{4.26}$$

Let us denote by $\mathfrak{W}_1$ the Lie algebra of formal vector fields on a line, which is an infinite dimensional Lie algebra with a basis

$$e_m = z^{m+1}\frac{d}{dz}, \quad m \geq -1.$$

Then the relation (4.26) implies that the Lie algebra $\{\frac{\partial}{\partial \hat{s}_m}\}$ defines a central extension of $\mathfrak{W}_1$. It is computed in [15] that $H^2(\mathfrak{W}_1, \mathbb{R}) = 0$ and hence every central extension is trivial. Therefore we can modify each O_m by adding an appropriate constant κ_m such that the modified flows

$$\frac{\partial \mathcal{F}}{\partial s_m} = \mathcal{G}_m + O_m + \kappa_m + \mathcal{L}_m^{even}\mathcal{F}$$

satisfy the commutation relations (4.25). Moreover, the choice of κ_m is unique since $H^1(\mathfrak{W}_1, \mathbb{R}) = 0$ (see [15]). The lemma is proved. □

Thus we have proved the following theorem.

Theorem 15. *For every tau-symmetric bihamiltonian deformation of the Principal Hierarchy associated with a semisimple Frobenius manifold, the deformed integrable hierarchy possesses an infinite set of Virasoro symmetries. The actions of these symmetries on the tau function* $\mathcal{Z}$ *are represented by*

$$\frac{\partial \mathcal{Z}}{\partial s_m} = L_m^{even}\mathcal{Z} + O_m\mathcal{Z}, \quad m \geq -1,$$

where $O_m \in \hat{\mathcal{A}}$ *are certain differential polynomials, and the flows* $\frac{\partial}{\partial s_m}$ *satisfy the commutation relations*

$$\left[\frac{\partial}{\partial s_k}, \frac{\partial}{\partial s_l}\right] = (l-k)\frac{\partial}{\partial s_{k+l}}, \quad k, l \geq -1.$$

Example 6. Let M be the 2-dimensional Frobenius manifold defined on the orbit space of the Weyl group of type B_2. Its potential and Euler vector field are given by

$$F = \frac{1}{2}v^2 u + \frac{4}{15}u^5, \quad E = v\partial_v + \frac{1}{2}u\partial_u.$$

Here $v = v^1$ and $u = v^2$ are the flat coordinates of M. We denote by σ_1 and σ_2 the dual coordinates of the fiber of $\hat{M}$, then the bihamiltonian structure $(P_0^{[0]}, P_1^{[0]})$ associated with M is given by

$$P_0^{[0]} = \frac{1}{2}\int \sigma_1\sigma_2^1 + \sigma_2\sigma_1^1, \quad P_1^{[0]} = \frac{1}{2}\int 8u^3\sigma_1\sigma_1^1 + \frac{1}{2}u\sigma_2\sigma_2^1 + 2v\sigma_1\sigma_2^1 - \frac{1}{2}v_x\sigma_1\sigma_2.$$

Let us first write down the Virasoro symmetry $\frac{\partial}{\partial s_1}$ of the tau-cover of the Principal Hierarchy associated with M. Here we choose the symmetry $\frac{\partial}{\partial s_1}$ instead of $\frac{\partial}{\partial s_2}$ just for simplicity. The Virasoro operator L_1^{even} has the expression

$$L_1^{even} = \frac{3}{16}\frac{\partial^2}{\partial t^{1,0}\partial t^{2,0}} + \mathcal{L}_1^{even},$$

where

$$\mathcal{L}_1^{even} = \sum_{p\geq 0}\left(p+\frac{1}{4}\right)\left(p+\frac{5}{4}\right)t^{1,p}\frac{\partial}{\partial t^{1,p+1}} + \left(p+\frac{3}{4}\right)\left(p+\frac{7}{4}\right)t^{2,p}\frac{\partial}{\partial t^{2,p+1}}.$$

Then the action of $\frac{\partial}{\partial s_1}$ on the genus zero free energy $\mathcal{F}^{[0]}$ of the tau-cover of the Principal Hierarchy is given by

$$\frac{\partial \mathcal{F}^{[0]}}{\partial s_1} = \frac{3}{16}f_{1,0}f_{2,0} + \mathcal{L}_1^{even}\mathcal{F}^{[0]}.$$

Consider the bihamiltonian structure (P_0, P_1) of the Drinfel'd–Sokolov hierarchy [3] associated with the untwisted affine Kac–Moody algebra $B_2^{(1)}$. After performing a suitable Miura type transformation we have $P_0 = P_0^{[0]}$, and the Hamiltonian operator $\mathcal{P}_1$ of P_1 has the expression

$$\mathcal{P}_1 = \begin{pmatrix} 8u^3\partial_x + 12u^2u_x & v\partial_x + \frac{1}{4}v_x \\ v\partial_x + \frac{3}{4}v_x & \frac{1}{2}u\partial_x + \frac{1}{4}u_x \end{pmatrix} + \varepsilon^2 \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} + O(\varepsilon^4),$$

where the differential operators D_i are given by $D_2 = u\partial_x^3 + \frac{3}{4}u_x\partial_x^2$, $D_4 = \frac{5}{8}\partial_x^3$ and

$$\begin{aligned} D_1 =\ & 14u^2\partial_x^3 + 42uu_x\partial_x^2 + \left(20u_x^2 + 16uu_{xx} + \frac{1}{2}v_{xx}\right)\partial_x \\ & + 12u_xu_{xx} + 6uu^{(3)} + \frac{1}{4}v^{(3)}, \\ D_3 =\ & u\partial_x^3 + \frac{9}{4}u_x\partial_x^2 + \frac{3}{2}u_{xx}\partial_x + \frac{1}{4}u^{(3)}. \end{aligned}$$

The bihamiltonian structure (P_0, P_1) is a deformation of $(P_0^{[0]}, P_1^{[0]})$ with central invariants $c_1 = \frac{1}{6}$, $c_2 = \frac{1}{12}$ (see [8,24]), and it determines a unique deformation of the Principal Hierarchy associated with M. We can find the Virasoro symmetry $\frac{\partial}{\partial s_1}$ of the tau-cover of the deformed Principal Hierarchy by using the results developed in the present paper. It turns out that the action of $\frac{\partial}{\partial s_1}$ on the tau-function $\mathcal{Z}$ can be represented by

$$\frac{\partial \mathcal{Z}}{\partial s_1} = L_1^{even}\mathcal{Z} + \left(\frac{1}{2}u^2 + \frac{1}{4}v + \frac{1}{4}\varepsilon^2 u_{xx}\right)\mathcal{Z}. \tag{4.27}$$

A similar result is also given in the Example 5.5 of [31] by using the Kac–Moody–Virasoro algebra.

5. Conclusion

In the present paper, we prove the existence of an infinite set of Virasoro symmetries for a given tau-symmetric bihamiltonian deformation of the Principal Hierarchy associated with a semisimple Frobenius manifold. These symmetries can be represented in terms of the tau-function $\mathcal{Z}$ of the integrable hierarchy in the form

$$\frac{\partial \mathcal{Z}}{\partial s_m} = L_m\mathcal{Z} + O_m\mathcal{Z}, \quad m \geq -1. \tag{5.1}$$

Note that the differential polynomials O_m depend on the choice of the representative, in the equivalence class of Miura type transformations, of the deformations of the bihamiltonian structure of hydrodynamic type $(P_0^{[0]}, P_1^{[0]})$. It is proved in [9] that, for two different choices of representatives (P_0, P_1) and $(\tilde{P}_0, \tilde{P}_1)$, the corresponding normal coordinates

$$w^\alpha = \eta^{\alpha\beta}\frac{\partial^2 \log \mathcal{Z}}{\partial t^{1,0}\partial t^{\beta,0}}, \quad \tilde{w}^\alpha = \eta^{\alpha\beta}\frac{\partial^2 \log \tilde{\mathcal{Z}}}{\partial t^{1,0}\partial t^{\beta,0}}, \quad \alpha = 1, \dots, n$$

of the deformed Principal Hierarchy are related by a Miura type transformation

$$\tilde{w}^\alpha = w^\alpha + \eta^{\alpha\beta}\frac{\partial^2 G}{\partial t^{1,0}\partial t^{\beta,0}},$$

where $G \in \hat{\mathcal{A}}$ is a differential polynomial, and the tau-functions are related by the equation

$$\tilde{\mathcal{Z}} = \exp(G)\mathcal{Z}. \tag{5.2}$$

Conversely, any differential polynomial $G \in \hat{\mathcal{A}}$ defines a Miura type transformation for the deformed bihamiltonian structure and the integrable hierarchy in the manner described above.

After a Miura type transformation induced from (5.2), the Virasoro symmetries (5.1) are transformed to the form

$$\frac{\partial \tilde{\mathcal{Z}}}{\partial s_m} = L_m \tilde{\mathcal{Z}} + \tilde{O}_m \tilde{\mathcal{Z}}, \quad m \geq -1,$$

where the differential polynomials $\tilde{O}_m$ can be computed from O_m and G.

We are going to study the problem of linearization of Virasoro symmetries in subsequent work, i.e., to study whether it is possible to find a suitable differential polynomial G such that all the functions $\tilde{O}_m$ vanish.

Let us exam the possibility of linearizing the Virasoro symmetries given in the example of Sect. 3.4 for the one-dimensional Frobenius manifold. We want to find a certain Miura type transformation given by (5.2) which linearizes the Virasoro symmetry (3.26) and leaves the expression of the Virasoro symmetry

$$\frac{\partial \mathcal{Z}}{\partial s_{-1}} = L_{-1}^{even} \mathcal{Z}$$

unchanged. It follows from these requirements that the differential degree zero component G_0 of G must satisfy the equations

$$\frac{\partial G_0}{\partial v} = 0, \quad v^3 \frac{\partial G_0}{\partial v} = -\left(3c - \frac{3}{8}\right) \frac{v^2}{2},$$

which do not possess any solution unless $c = \frac{1}{8}$. When $c = \frac{1}{8}$, the linearized Virasoro symmetries for this example are well known [30], and the central invariant of the corresponding deformed bihamiltonian structure is $\frac{1}{3}c = \frac{1}{24}$.

We can do a similar computation for Example 6. We want to find a Miura type transformation given by (5.2) to linearize the Virasoro symmetry (4.27) and to preserve the expression of the Virasoro symmetry

$$\frac{\partial \mathcal{Z}}{\partial s_{-1}} = L_{-1}^{even} \mathcal{Z}.$$

Then the differential degree zero component G_0 of G must satisfy the equations

$$\frac{\partial G_0}{\partial v} = 0, \quad uv \frac{\partial G_0}{\partial u} = -\left(\frac{1}{2}u^2 + \frac{1}{4}v\right),$$

which have no solution. Therefore the Virasoro symmetries given by the bihamiltonian structure (P_0, P_1) in this example cannot be linearized.

In general we have the following theorem, whose proof will be given in the paper [21].

Theorem 16. *The Virasoro symmetries for a given tau-symmetric bihamiltonian deformation of the Principal Hierarchy associated with a semisimple Frobenius manifold is linearizable if and only if the central invariants of the corresponding deformed bihamiltonian structure are all equal to* $\frac{1}{24}$.

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